Collatz Dynamics Through Energy, Entropy, and 2-Adic Structure

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Abstract

The Collatz map, defined by the simple rules $T(n) = n/2$ if n is even and $T(n) = 3n + 1$ if n is odd, produces sequences whose convergence remains an unsolved problem in mathematics. In this paper, we recast the Collatz map as a discrete dynamical system and explore its structure through multiple lenses: energy and information theory, 2-adic valuation, geometric embeddings, and modular recursion.

We define energy-like and entropy-like quantities to describe the system's compressive dynamics and use phase space and delay embeddings to visualize their evolution. Applying UMAP and HDBSCAN to fused embeddings of sequence data, we discover clusters of Collatz trajectories that correspond to distinct dynamical regimes — including chaotic oscillators, rapid collapsers, and valuation-stable sequences.

Crucially, these clusters align with modular arithmetic towers in the reverse Collatz tree: vertical chains generated by repeated doubling, selectively connected through valuation spikes governed by residue class constraints. We find that the tree's average spacing contracts exponentially and appears to converge to $1 - \frac{1}{e}$, revealing a probabilistic backbone to its combinatorial growth.

Contents

1 Introduction

The **Collatz map** is a deceptively simple recursive function on the positive integers. Let $T : \mathbb{N} \to \mathbb{N}$ be defined by:

$$T(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$

Starting from any seed $n_0 \in \mathbb{N}$, one iteratively applies T to generate a sequence:

$$n_0,\, n_1 = T(n_0),\, n_2 = T(n_1),\, \ldots$$

This defines the *Collatz sequence* for n_0. The famous **Collatz conjecture** asserts that every such sequence eventually reaches the cycle

$$1 \to 4 \to 2 \to 1.$$

Despite extensive computational verification, the conjecture remains unproven. Beneath its simple rules lies complex, chaotic behavior — especially for small odd seeds — and an intricate structure that resists classification.

Examples. Consider two small seeds:

- For $n_0 = 5$, the sequence is short and quickly converges:

$$5 \to 16 \to 8 \to 4 \to 2 \to 1$$

- For $n_0 = 7$, the sequence is longer and more erratic:

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to \cdots$$

These examples illustrate a central theme: *although all trajectories appear to converge, their paths vary wildly in shape, length, and behavior*. Some collapse quickly, others flare upward before falling, and many oscillate in complex patterns before settling. This tension between deterministic rules and unpredictable outcomes is the hallmark of a dynamical system.

In this paper, we approach the Collatz map not as an isolated conjecture, but as a rich and structured dynamical system. Through the lenses of physics, topology, information theory, and modular arithmetic, we examine the hidden regularities that emerge from its apparent chaos. We explore:

1. an **energy-theoretic** model, where odd steps inject energy and even steps dissipate it;
2. an **information-theoretic** framework, tracking entropy and complexity across the sequence;
3. a **topological and geometric** analysis, using delay embeddings and UMAP to uncover global shapes;
4. a **recursive tree** structure that organizes all possible trajectories through modular and 2-adic constraints.

Each perspective contributes a piece to the puzzle: how do simple arithmetic rules generate such intricate dynamics? And what does that complexity reveal about the underlying structure of the integers themselves?

Along the way, we discover that the Collatz map is not merely chaotic — it is recursive, modular, and topologically rich. Its trajectories trace paths through valuation space, spiral in energy and entropy, and ultimately align into families organized by deep arithmetic logic. In what follows, we trace these paths and uncover the structures they encode.

2 Mathematical Background

The Collatz map defines a discrete dynamical system on the natural numbers, but its behavior invites reinterpretation through many mathematical lenses. In this section, we develop key conceptual tools — energy and entropy analogies, 2-adic valuation, and recursive backward structure — that underlie our later analysis of Collatz trajectories.

2.1 Forward and Backward Dynamics

In the standard (forward) iteration,

$$T(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd,} \end{cases}$$

we apply T repeatedly to generate a trajectory starting from an initial seed. However, we can also reverse the map, asking which integers could have mapped to a given value.

This leads to the construction of the **reverse Collatz tree**, where each node represents a value n, and edges connect it to its possible predecessors. Every n has at least one reverse path via doubling: $2n$. Some values also admit an odd predecessor:

$$(n-1)/3 \in \mathbb{N} \quad \text{if and only if } n \equiv 4 \pmod 6.$$

This recursive structure generates a branching tree rooted at 1, and — as we will later explore — its level-wise distribution reveals remarkable regularities, including exponential decay in average spacing.

2.2 2-Adic Valuation

A central organizing tool is the **2-adic valuation** $\nu_2(n)$, defined as:

$$\nu_2(n) = k \quad \text{if } 2^k \mid n \text{ and } 2^{k+1} \nmid n.$$

This measures the number of times n can be halved before it becomes odd. For example:

$$\nu_2(8) = 3, \quad \nu_2(12) = 2, \quad \nu_2(7) = 0.$$

The valuation provides a kind of dynamical metric: even steps increase it, odd steps reset it. Seen this way, the Collatz map defines a sequence that oscillates through layers of compressibility. As we will see, sequences with characteristic valuation patterns often correspond to distinct families of behavior — a connection that will surface again in our topological clustering analysis.

2.3 Energy-Theoretic Quantities

Inspired by physics, we define energy-like measures to interpret Collatz dynamics as energetic motion. For a value x_n in a trajectory:

- **Potential energy:** $V(x_n) = \log(x_n)$
- **Kinetic energy:** $p(x_n) = |x_{n+1} - x_n|$
- **Total energy:** $H(x_n) = V(x_n) + p(x_n)$

Odd steps inject energy (via $3n+1$), resulting in sharp increases in p, while even steps dissipate energy smoothly through halving. These injections and decays trace a jagged but structured flow — an energy landscape we analyze in detail in the next section.

2.4 Information-Theoretic Quantities

The same structure can be reinterpreted through the lens of information theory. We define:

- **Entropy:** $H(x_n) = \log(x_n)$, reflecting informational complexity.
- **Entropy rate:** $\Delta H_n = \log(x_{n+1}) - \log(x_n)$, capturing the directional flow of complexity.
- **Mutual information:** $I(x_{n+1}; x_n) = H(x_n)$, since the system is deterministic.

These quantities parallel the energy definitions: entropy behaves like potential, and entropy rate mirrors kinetic energy. Together, they capture how the system alternates between injecting and compressing uncertainty.

2.5 Connecting the Perspectives

Energy, entropy, and valuation align into a coherent framework:

$$\text{Potential energy} \sim \text{Entropy} \sim \log(n), \qquad \text{Kinetic energy} \sim \text{Entropy rate}.$$

Meanwhile, the 2-adic valuation adds structure beneath the surface: it captures not just the magnitude of a state, but its divisibility architecture — a deeper measure of compression potential. Many of the most interesting Collatz behaviors — from early spikes to long collapses — are best understood by following valuation dynamics.

Finally, the recursive tree formed by the reverse map provides a global skeleton for this behavior. It organizes integers by their backward trajectories, revealing patterns not only in local dynamics but in the modular arithmetic that shapes possible paths. In later sections, we will see how this structure echoes in valuation-based embeddings, in the shapes of UMAP projections, and in the spacing and growth of the reverse tree itself.

Together, these tools frame our approach. We will use them not only to measure Collatz sequences, but to see their shape, uncover their symmetries, and classify their behavior — from high-energy outliers to deeply compressible orbits.

3 Energy-Theoretic Analysis

Building on the conceptual tools from the previous section, we now examine the Collatz map through the lens of energy. Instead of viewing sequences as mere lists of numbers, we treat them as the evolving states of a discrete dynamical system — one in which values rise and fall, expand and compress, reflecting an energetic flow with structure and tension.

3.1 Energy as a Lens on Discrete Dynamics

In classical mechanics, potential energy reflects position and kinetic energy reflects change. Translating this to the Collatz setting, we define the following quantities for the n-th element of a sequence x_n:

- **Potential energy:** $V(x_n) = \log(x_n)$ — representing the "height" or size of the state.
- **Kinetic energy:** $p(x_n) = |x_{n+1} - x_n|$ — the magnitude of change from one step to the next.
- **Total energy:** $H(x_n) = V(x_n) + p(x_n)$ — combining positional and dynamical content.

This framework captures how the map alternates between sharp energetic bursts and smoother contractions. Energy is not conserved — it is rhythmically injected and dissipated, reflecting the core asymmetry of the Collatz rules.

3.2 Interpreting Odd and Even Steps

Odd steps, which apply $3n+1$, typically cause large upward jumps in value — moments of expansion that produce spikes in kinetic energy. We interpret these as *energy injection events*.

Even steps halve the current value, reducing both its magnitude and the change to the next step. These are *dissipative events*, compressing the system toward stability.

Together, these operations generate an oscillatory tension: growth followed by collapse, injection followed by decay. It is this interplay that makes Collatz trajectories feel chaotic yet somehow convergent — a system always climbing and collapsing in jagged cycles.

3.3 Visualizing Energy for $x_0 = 27$

The sequence beginning with $x_0 = 27$ is a well-known archetype: long, erratic, and full of energetic pulses before it ultimately settles into the trivial cycle $4 \to 2 \to 1$. Below, we visualize the evolution of its energy over time:

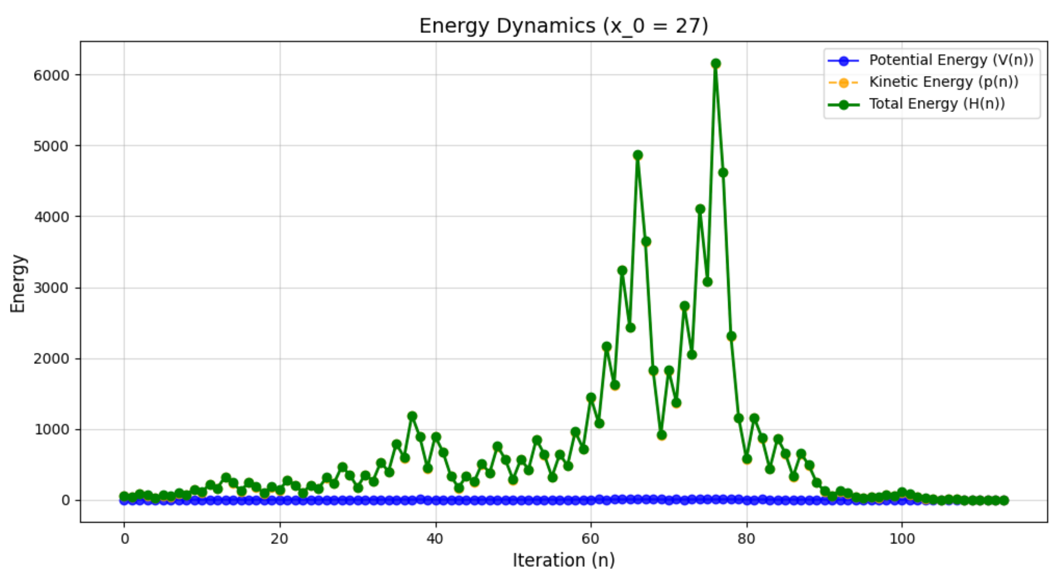

Figure 1: Energy dynamics for $x_0 = 27$, showing potential energy $V(x_n)$, kinetic energy $p(x_n)$, and total energy $H(x_n)$. Odd steps inject energy via large jumps; even steps dissipate it. The total energy exhibits jagged decay, spiraling downward as the sequence compresses.

The pattern is striking: an early chaotic phase with energetic surges, followed by gradual damping. Kinetic energy spikes mark the odd steps — visible as vertical jumps — while the even steps form smoother slopes between them. The total energy traces a spiral of decay that foreshadows the geometric structures we will soon examine in phase space.

3.4 From Arithmetic to Dynamics

This energetic framing transforms the Collatz map from arithmetic into motion. A number like 27, though small, encodes high initial "tension" — requiring many energetic transformations before it stabilizes. These dynamics are not random but shaped by structure: even steps compress the system consistently, and odd steps inject new volatility.

As we'll see, this energy landscape reflects deeper patterns. The structure of the reverse tree, the behavior of valuation, and the modular tower framework all leave their imprint on the energy profile of a trajectory. These connections will surface again in our geometric, topological, and clustering analyses — where energetic behavior maps onto visible shape.

4 Information-Theoretic Analysis

Just as the Collatz map can be interpreted in terms of energy, it also lends itself to an information-theoretic framework. Here, we shift from physical intuition to informational flow, treating each step in the sequence as carrying a kind of entropy — a measure of uncertainty, complexity, or surprise.

4.1 Entropy as Logarithmic Size

We define the entropy at step n as

$$H(x_n) = \log(x_n).$$

While not Shannon entropy in the formal probabilistic sense, this logarithmic measure reflects the amount of information needed to represent or describe the state x_n. Larger values correspond to more complex or unresolved states — those requiring more bits to describe and more steps to reduce.

4.2 Entropy Rate as Dynamical Signal

To capture the local change in informational complexity, we define the *entropy rate*:

$$\Delta H_n = \log(x_{n+1}) - \log(x_n).$$

This measures the informational "shock" between states — how much entropy is gained or lost from one step to the next. As with energy, odd steps tend to increase entropy (positive rate), while even steps compress it (negative rate). The result is a signal that oscillates with the parity of the steps, encoding a rhythm of expansion and collapse.

4.3 Mutual Information and Determinism

In a probabilistic system, mutual information quantifies how much knowledge of one variable reduces uncertainty about another. But the Collatz map is deterministic: each x_n determines x_{n+1} exactly. Therefore, the mutual information satisfies:

$$I(x_{n+1}; x_n) = H(x_n),$$

indicating that all information in the next state is fully contained in the current one. Still, tracking this reinforces the notion that each state holds the system's complexity in compressed form — awaiting expansion or resolution through the map.

4.4 Informational Profile for $x_0 = 27$

To illustrate these dynamics, we return to the seed $x_0 = 27$, whose trajectory is long, oscillatory, and chaotic before converging. We visualize how its entropy, entropy rate, and mutual information evolve:

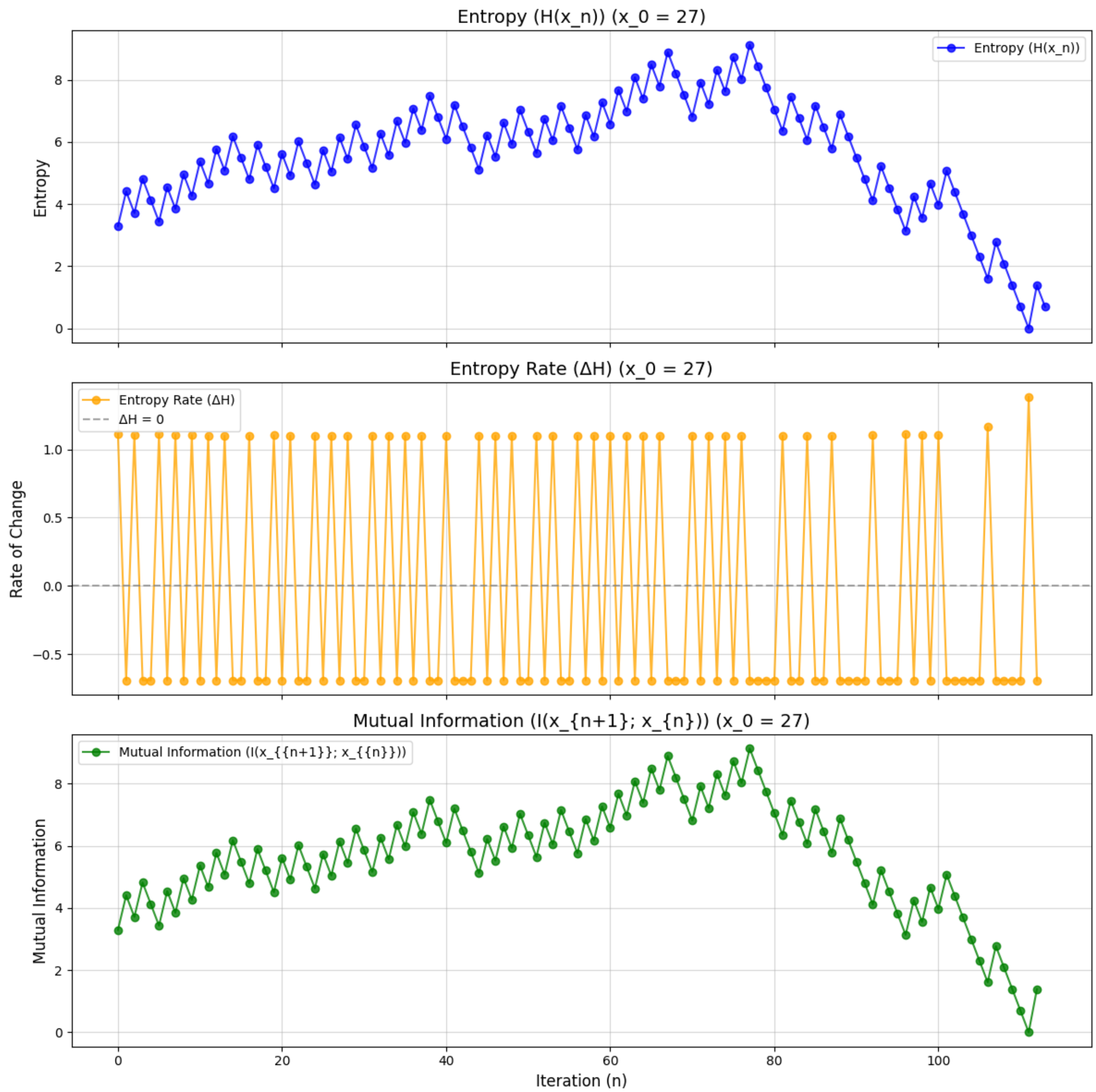

Figure 2: Information-theoretic quantities for $x_0 = 27$. Top: entropy $H(x_n)$ decreases overall as the sequence approaches the cycle. Middle: entropy rate ΔH_n oscillates with each step, spiking on odd transitions. Bottom: mutual information $I(x_{n+1}; x_n) = H(x_n)$ tracks the total complexity present at each point.

These plots mirror the energy analysis: entropy decays unevenly, through repeated injections and compressions. The early phases exhibit high entropy rate oscillations, reflecting the system's volatile attempts to compress complexity into a convergent form.

4.5 Energy and Entropy: Parallel Stories

The energy and information frameworks mirror one another:

$$\text{Entropy} \sim \text{Potential Energy}, \qquad \text{Entropy Rate} \sim \text{Kinetic Energy}.$$

Where energy rises and falls, entropy expands and contracts. In both cases, the system undergoes a jagged descent — not smooth, but structured. These metrics reveal that the Collatz map is not merely compressive: it is *alternating*, balancing growth and collapse.

But entropy, like energy, is only part of the story. While it captures complexity in terms of size and rate of change, it does not yet account for the number-theoretic structure beneath — how values decompose into powers of 2, or how recursive towers emerge in the reverse tree. For that, we turn next to geometric and topological tools: delay embeddings, clustering, and valuation-based projections that reveal the hidden shapes of Collatz dynamics.

5 Phase Space Representation

The energy and information frameworks reveal that Collatz trajectories oscillate and decay — not linearly, but in sharp bursts and smooth collapses. To visualize this dynamic behavior, we now turn to **phase space**, a classical tool from dynamical systems that plots state variables against their rates of change to reveal motion in geometric form.

5.1 Phase Space for Collatz Dynamics

In physics, phase space often represents position versus momentum. Here, we construct an analogous two-dimensional phase space using potential and kinetic energy:

$$(V(x_n),\ p(x_n)) = (\log(x_n),\ |x_{n+1} - x_n|)\,,$$

where $V(x_n)$ reflects the size of the current state, and $p(x_n)$ captures the volatility of its change. Each point corresponds to a transition in the sequence, and the full trajectory traces a path through this energy landscape.

This phase space makes visible the alternation of forces: odd steps produce sharp vertical leaps in kinetic energy, while even steps produce steady horizontal contractions. The shape of the trajectory becomes a kind of geometric signature — a fingerprint of the sequence's internal rhythm.

5.2 Spiraling Toward the Attractor

For sequences like $x_0 = 27$, the resulting path spirals inward. The system starts with wild energetic surges, but as it progresses, both potential and kinetic energy decline. Eventually, the orbit converges to the attractor cycle $4 \to 2 \to 1$, and the motion collapses into a low-energy loop.

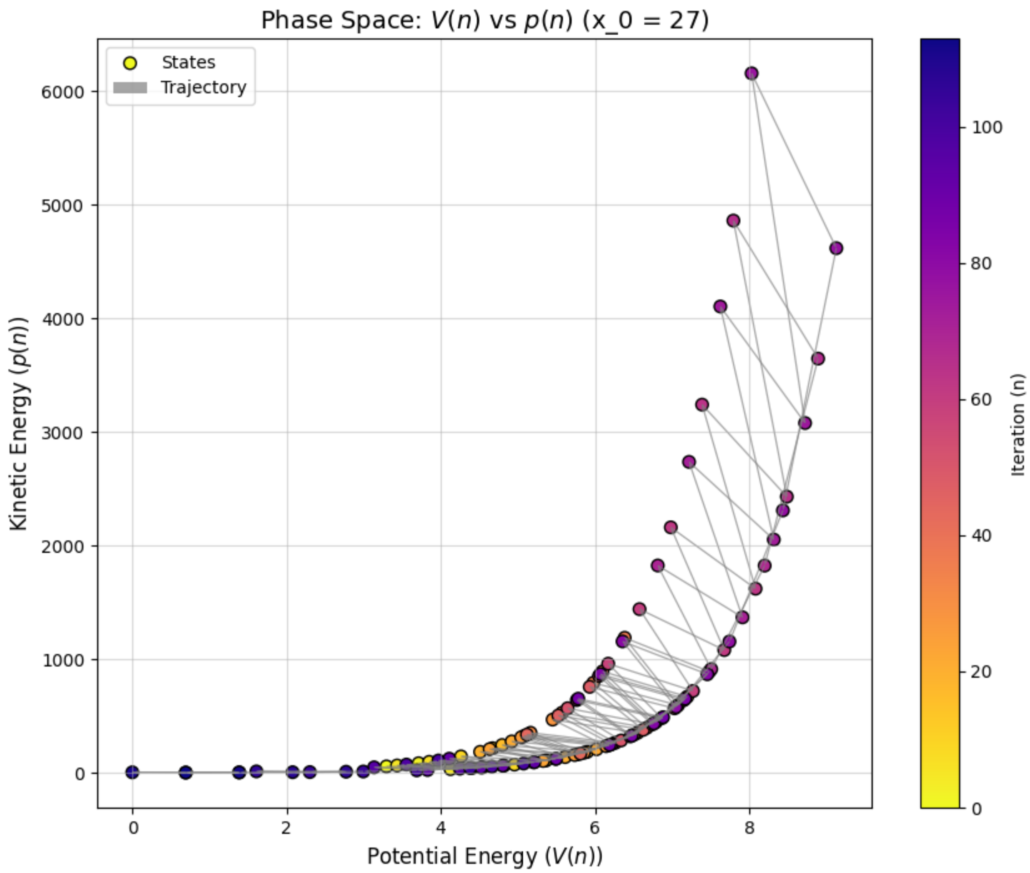

Figure 3: Phase space trajectory for $x_0 = 27$. Horizontal axis: potential energy $\log(x_n)$. Vertical axis: kinetic energy $|x_{n+1} - x_n|$. Color encodes iteration order. The path spirals inward through alternating bursts and contractions, eventually settling into the stable cycle.

This spiral is not smooth — it jitters and snaps, reflecting the discrete, rule-based nature of the map. Still, the overall direction is unmistakable: a collapse through contraction, interspersed with chaotic thrusts. The geometry encodes the system's long-term behavior far more vividly than the numeric sequence alone.

5.3 Phase Space vs. Delay Embedding

At first glance, this trajectory resembles a delay embedding, but the two are conceptually distinct. In phase space, each point depends only on a single step — it plots state versus immediate change:

$$\text{Phase space: } (x_n,\, x_{n+1} - x_n)$$

In contrast, delay embeddings unfold across multiple steps, capturing the temporal structure of a sequence:

$$\text{Delay embedding: } (x_n,\, x_{n+1},\, x_{n+2},\, \dots).$$

Phase space is instantaneous — a snapshot of motion. Delay embeddings are temporal — a reconstruction of trajectory shape. Both approaches reveal hidden structure, but from different angles.

Phase space gives us a first geometric intuition of convergence as motion — a collapsing spiral traced by alternating energy injections and dissipation. In the next section, we extend this view through delay embeddings, where we will reconstruct full trajectory shapes and examine how they cluster into coherent dynamical regimes.

6 Delay Embedding and Clustering

We now move from local energy profiles and instantaneous dynamics to a geometric, data-driven view of Collatz behavior over time. By interpreting each trajectory as a discrete time series, we construct high-dimensional representations — delay embeddings — and analyze them using modern tools from dynamical systems and unsupervised learning. The result is a rich geometric landscape of Collatz behavior, shaped by parity, valuation, and convergence.

6.1 Delay Embedding: Reconstructing Hidden Geometry

Delay embeddings are a classic technique for recovering the geometry of dynamical systems. For a time series $\{x_n\}$, we embed each trajectory as a point in $\mathbb{R}^d$ using time-lagged coordinates:

$$\mathbf{v}_n = (x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+d-1}).$$

This transformation unfolds the temporal structure into a higher-dimensional space, where similar dynamics trace similar geometric paths.

We apply this method to two complementary features:

- The raw values x_n, which encode numerical growth and collapse;
- The 2-adic valuations $\nu_2(x_n)$, which emphasize divisibility, parity, and depth of compression.

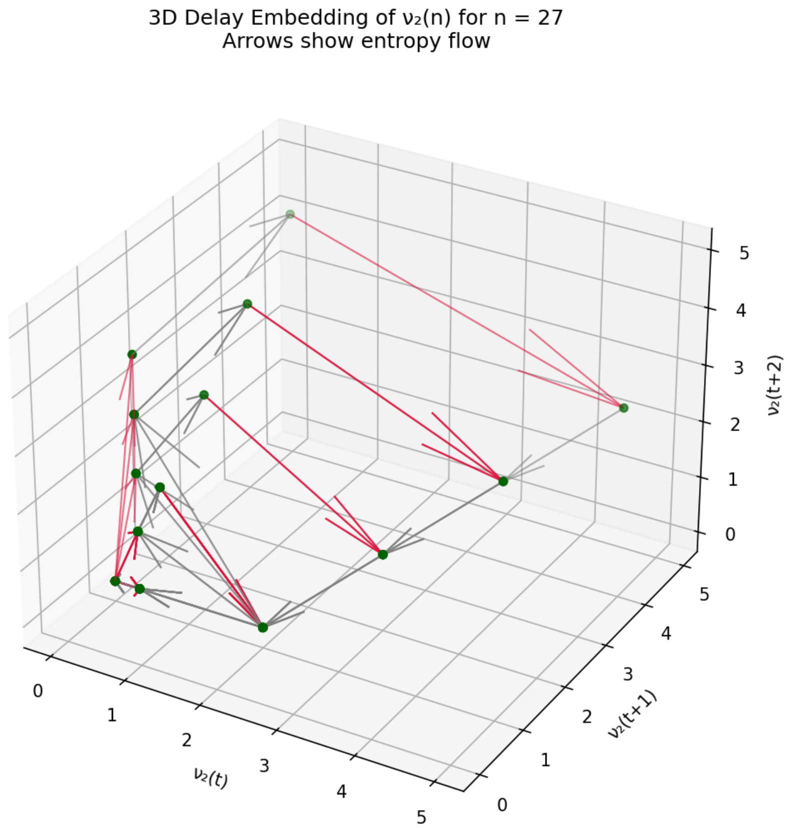

Figure 4: 3D delay embedding of $\nu_2(x_n)$ for $x_0 = 27$. The geometry reflects bursts of valuation and compression. Valuation spikes appear as structural bends in trajectory shape.

This embedding reveals hidden structure in the valuation sequence — sequences don't just collapse, they spiral, snap, and stall in characteristic patterns that are visible in this lifted space.

6.2 UMAP: Discovering Shape in Sequence Space

To compare embeddings across many sequences, we must project them into a common lower-dimensional space. We use **UMAP** (Uniform Manifold Approximation and Projection), a nonlinear method designed to preserve local and global topological structure.

UMAP constructs a high-dimensional graph of the data's neighborhood structure, then optimizes a 2D or 3D embedding that preserves this connectivity as faithfully as possible. In our context:

- Each point in UMAP space represents a delay-embedded Collatz sequence.
- Proximity corresponds to similar dynamical behavior — not in raw value, but in geometric shape.

We apply UMAP to three types of embeddings:

1. Raw values x_n, capturing magnitude-based evolution;
2. 2-adic valuations $\nu_2(x_n)$, capturing parity and divisibility structure;

3. Fused embeddings, combining both to capture full dynamical character.

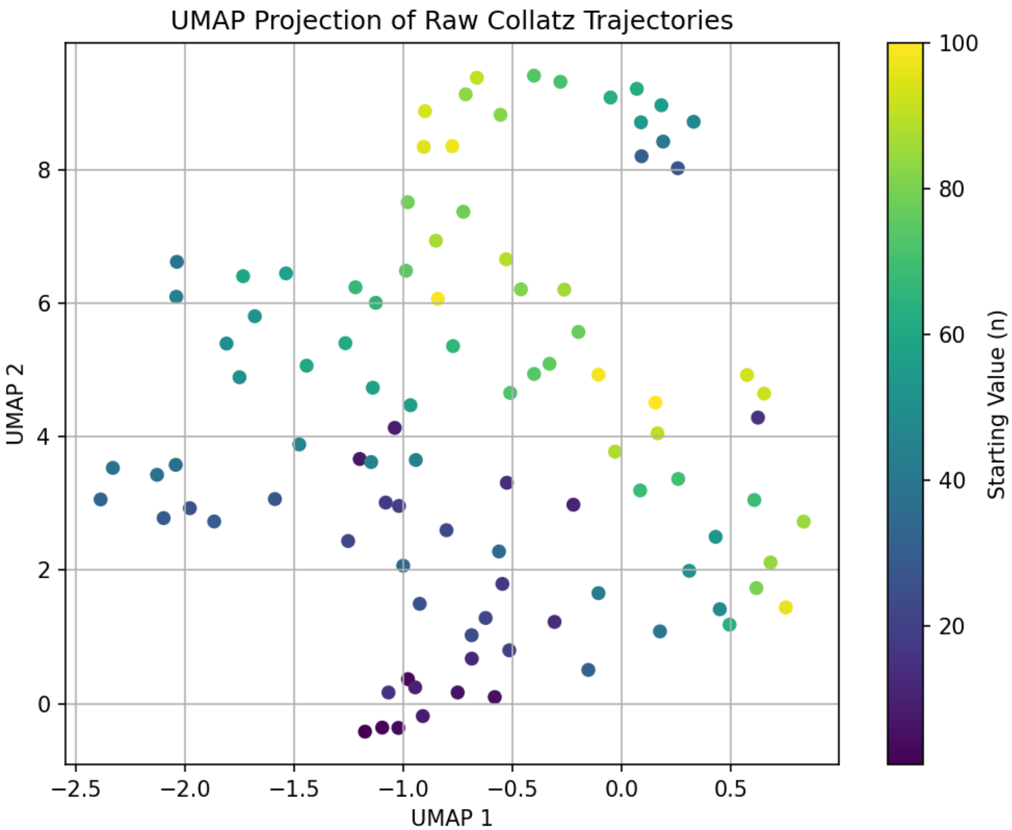

Figure 5: UMAP projection of delay embeddings from raw Collatz values. Some structural grouping emerges, but key valuation-based distinctions remain entangled.

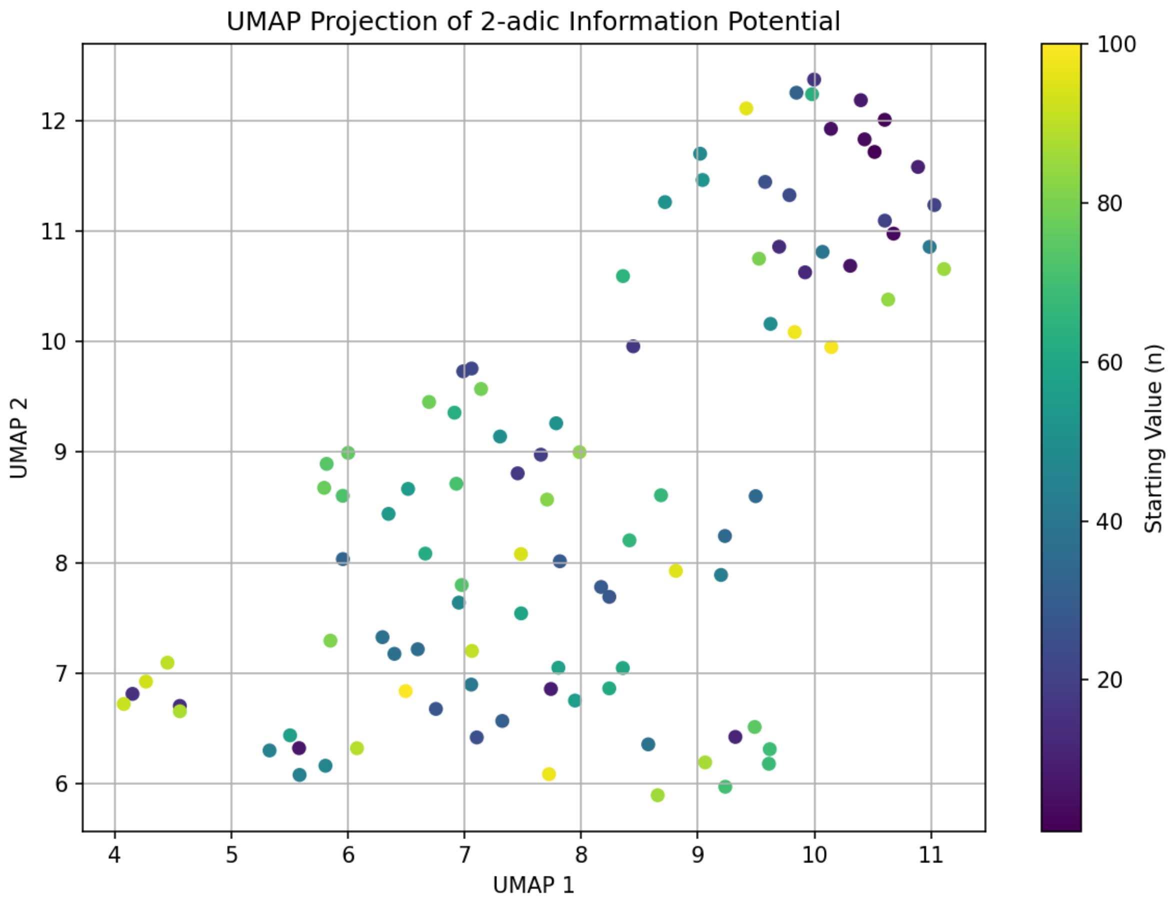

Figure 6: UMAP projection of delay embeddings from 2-adic valuation $\nu_2(x_n)$. Topological bands emerge, shaped by parity dynamics and valuation collapses.

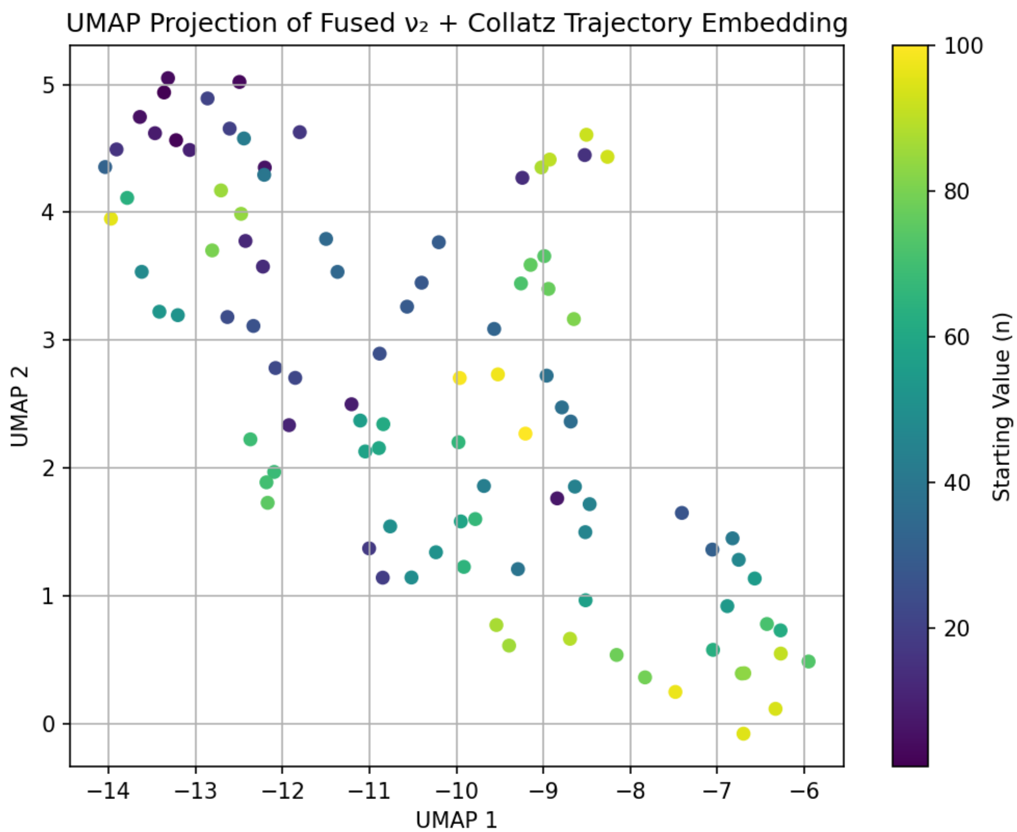

Figure 7: UMAP projection of fused embeddings (raw values + valuations). The result is a richly structured manifold of Collatz behavior, balancing numerical growth with valuation-driven compression.

The fused space best reflects the complexity of the system: sequences with similar energetic rhythms and valuation signatures group together, even when their raw values differ significantly.

6.3 HDBSCAN: Letting Structure Speak

To discover families of behavior within UMAP space, we apply **HDBSCAN**, a density-based clustering algorithm. Unlike k-means or spectral methods, HDBSCAN does not require us to choose the number of clusters. It identifies coherent regions of density and labels low-density points as noise.

In our setting, clusters correspond to families of Collatz sequences with:

- Similar delay-embedded trajectories,
- Comparable valuation oscillations and energy flow,
- Related convergence times or dynamical complexity.

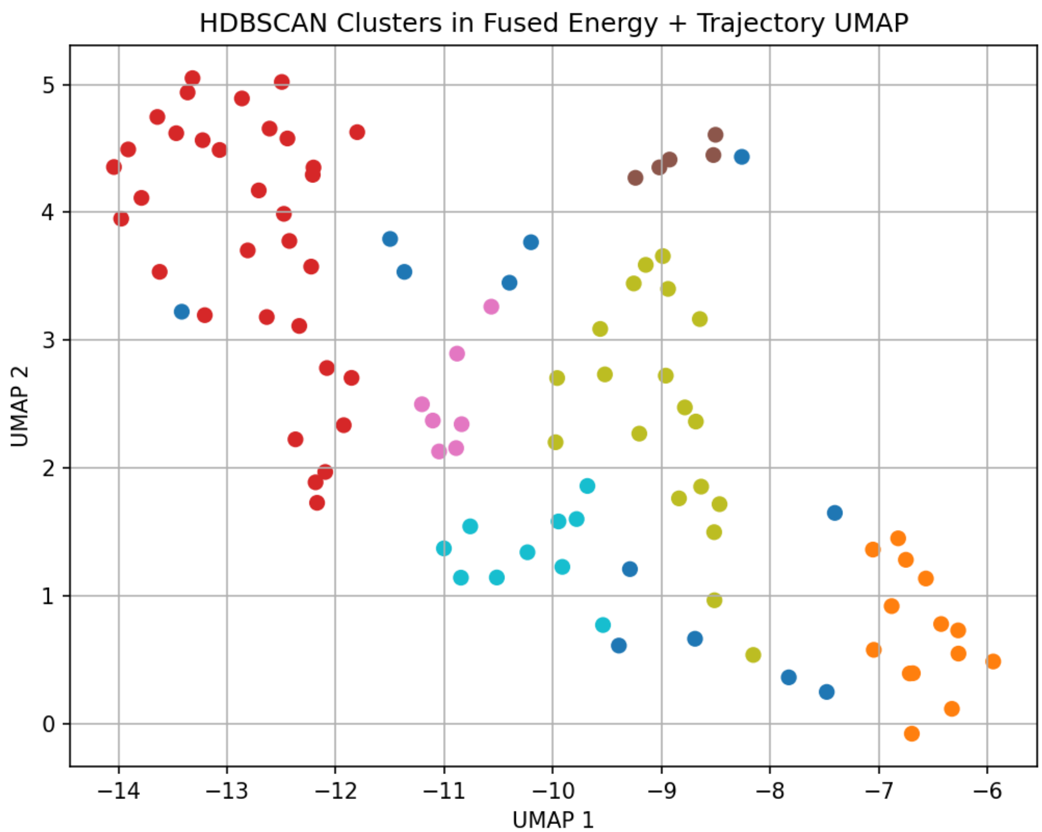

Figure 8: HDBSCAN cluster assignments over the fused UMAP projection. Each color represents a discovered dynamical regime — emerging naturally from the geometry of valuation and value dynamics.

These clusters are not artificial partitions — they reflect deep structure in the Collatz map. Sequences that oscillate for a long time form coherent groups, while quickly converging trajectories fall into distinct, tight bands. Transitional behaviors sit at the edges, revealing the gradient between dynamical regimes.

In the next section, we explore representative trajectories from each cluster to understand how their internal structure — particularly in 2-adic valuation — reflects their position in this geometric taxonomy. As we will see, these families align closely with the modular tower structure defined by the reverse Collatz tree.

7 Representative Cluster Trajectories

After identifying clusters in UMAP space using HDBSCAN, we now examine the internal structure of each group by selecting a representative seed x_0 from each cluster and analyzing its 2-adic valuation trajectory $\nu_2(x_n)$.

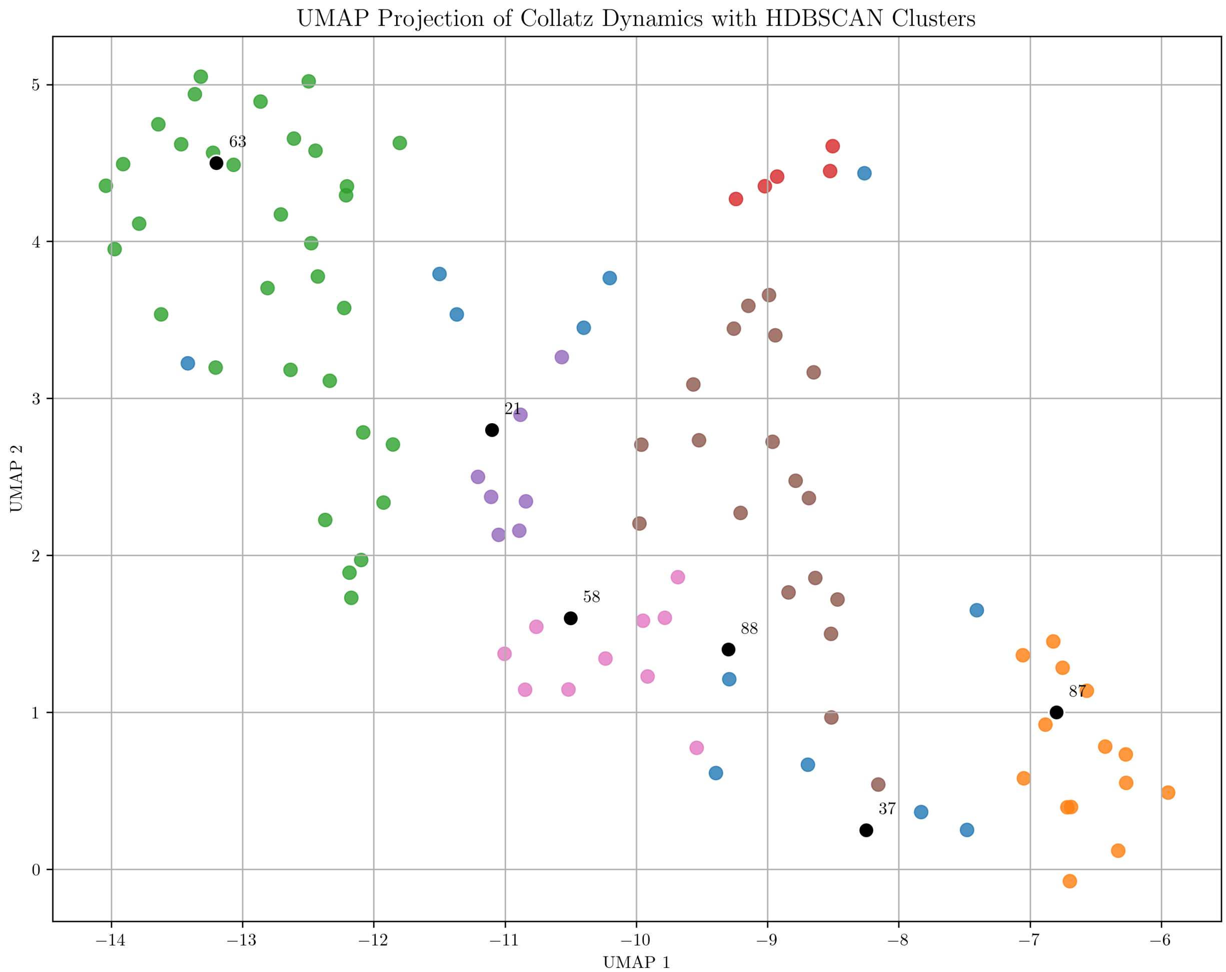

Figure 9: UMAP projection of fused delay embeddings, color-coded by HDBSCAN cluster. Each labeled point represents a seed value (e.g., $x_0 = 63, 21, 88$) whose valuation dynamics are examined in detail below. Clusters reflect families of Collatz behavior discovered via unsupervised learning.

These clusters emerge from dynamical similarity in delay embeddings — they are not imposed by predefined rules, but arise from valuation structure, energy flow, and convergence rhythm. The 2-adic valuation $\nu_2(x_n)$ provides a window into each sequence's internal cadence: when and how it compresses, how it resists, and where it collapses.

Each representative trajectory illustrates a distinct dynamical regime. Below, we unpack the behavioral signatures of each cluster and explain how they reflect modular and recursive structure seeded by the reverse Collatz map.

Cluster 0: Long Chaotic Delay Before Collapse — $x_0 = 63$

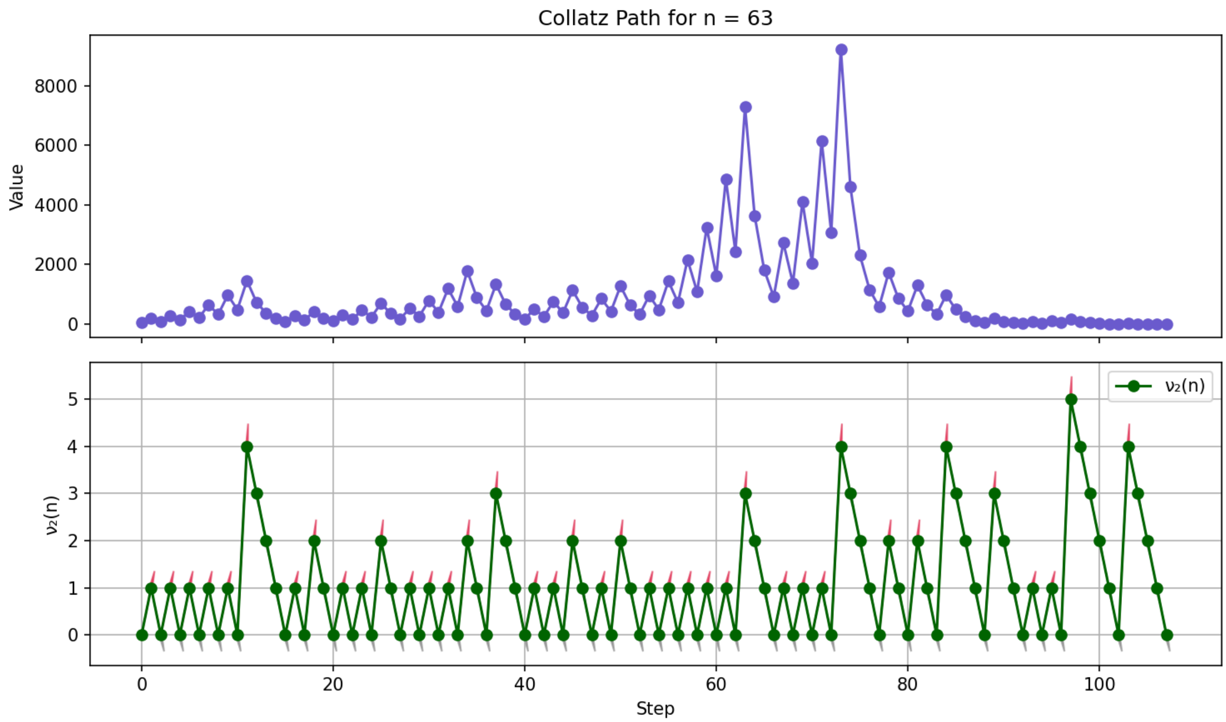

Figure 10: Cluster 0 — Trajectory for $x_0 = 63$. Prolonged high-energy phase with frequent valuation resets and delayed compression.

This cluster is the "chaotic core." Sequences like $x_0 = 63$ display frequent odd steps and valuation drops, indicating repeated energetic injection. These trajectories resist compression and often climb far in value before descending. They correspond to shallow towers or nodes far from compressive predecessors in the reverse tree.

Cluster 1: Early Collapse After Sharp Injection — $x_0 = 21$

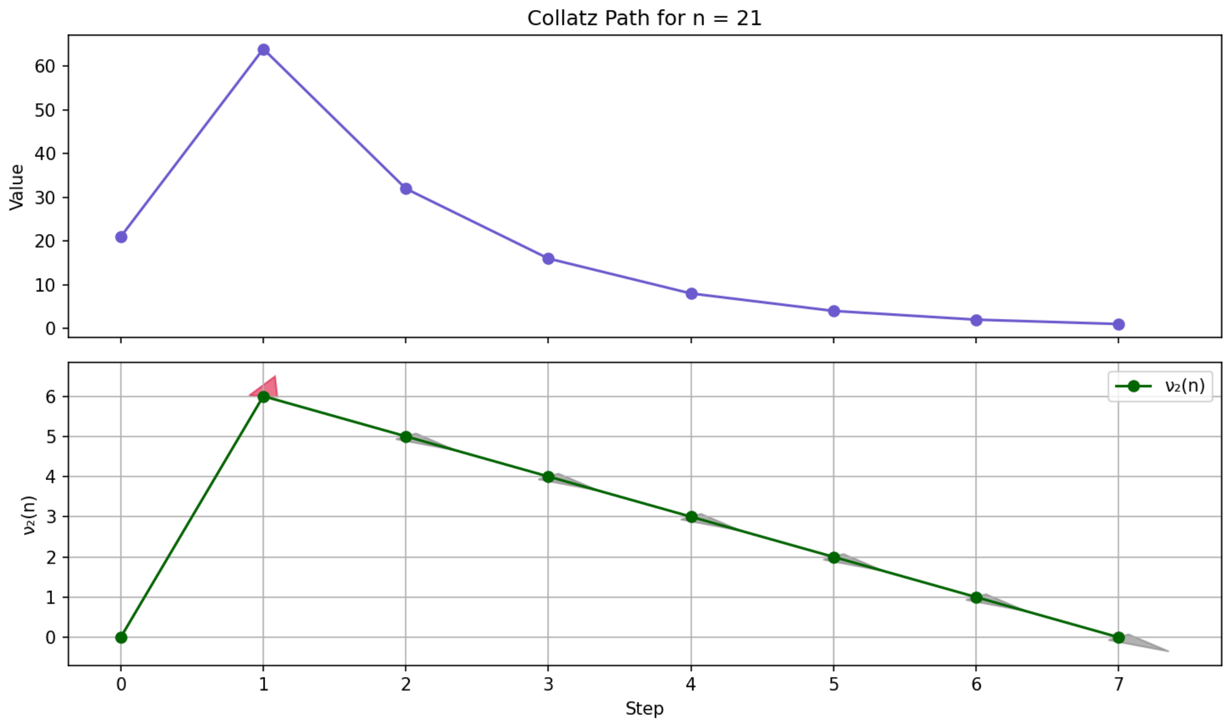

Figure 11: Cluster 1 — Trajectory for $x_0 = 21$. Sharp early valuation spike followed by clean, monotonic compression.

These trajectories exhibit a brief energetic phase followed by rapid collapse. The early odd step injects complexity, but the system quickly stabilizes into long even-step sequences. These seeds likely lie in shallow towers that admit compressive branches early, bypassing chaos.

Cluster 2: Flat and Even-Dominated — $x_0 = 88$

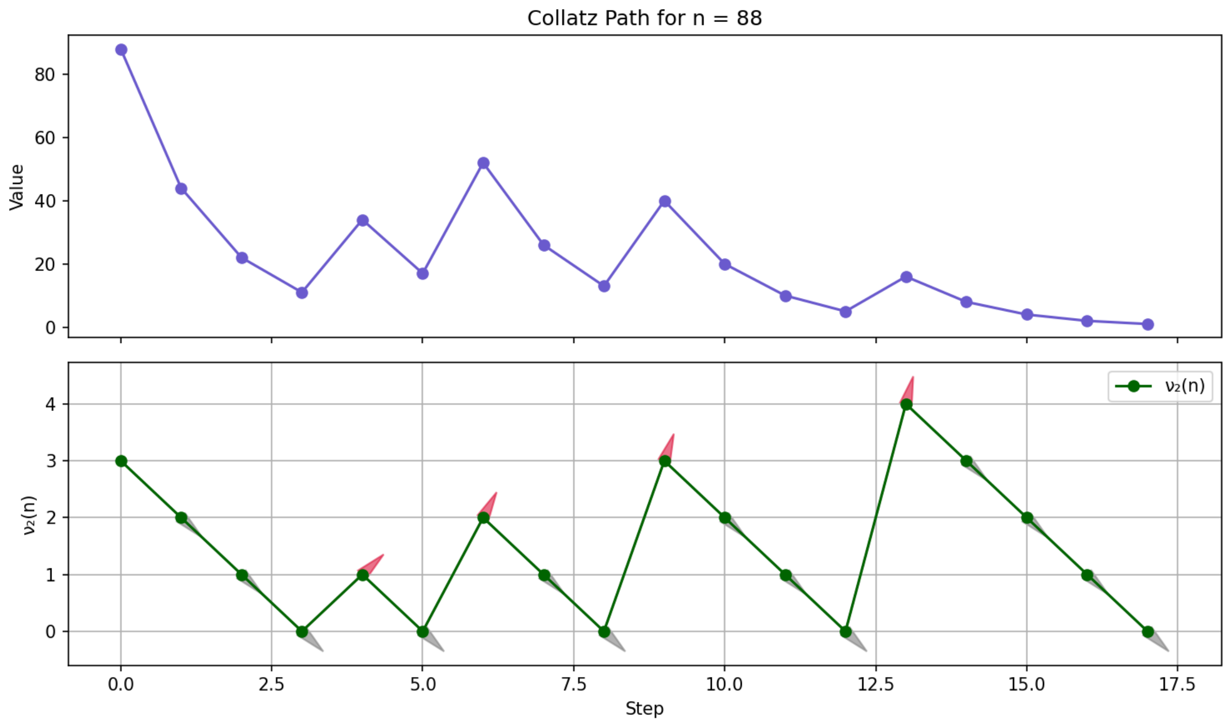

Figure 12: Cluster 2 — Trajectory for $x_0 = 88$. High valuation maintained for many steps, dominated by even transitions.

Trajectories in Cluster 2 are highly compressive from the start. Long stretches of even steps yield smooth valuation plateaus or gentle decay. These sequences may correspond to deep initial placement in a doubling tower, requiring few if any odd interventions before collapse.

Cluster 3: Rhythmic Oscillation and Convergence — $x_0 = 58$

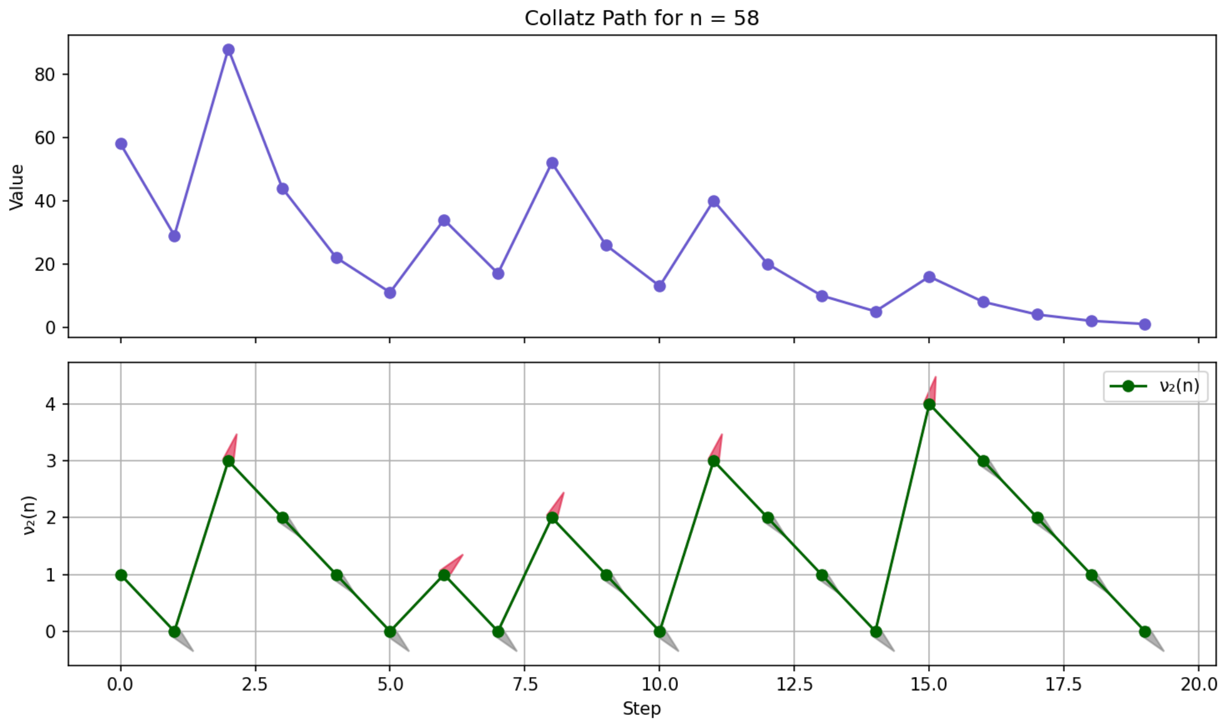

Figure 13: Cluster 3 — Trajectory for $x_0 = 58$. Regular alternation of injection and compression, leading to convergence.

These sequences show structured oscillations in valuation: periodic dips and recoveries that eventually trend downward. This suggests a trajectory moving along a tower that connects periodically to compressive branches. The result is orderly convergence through repeated cycles.

Cluster 4: Brief Resistance, Then Collapse — $x_0 = 37$

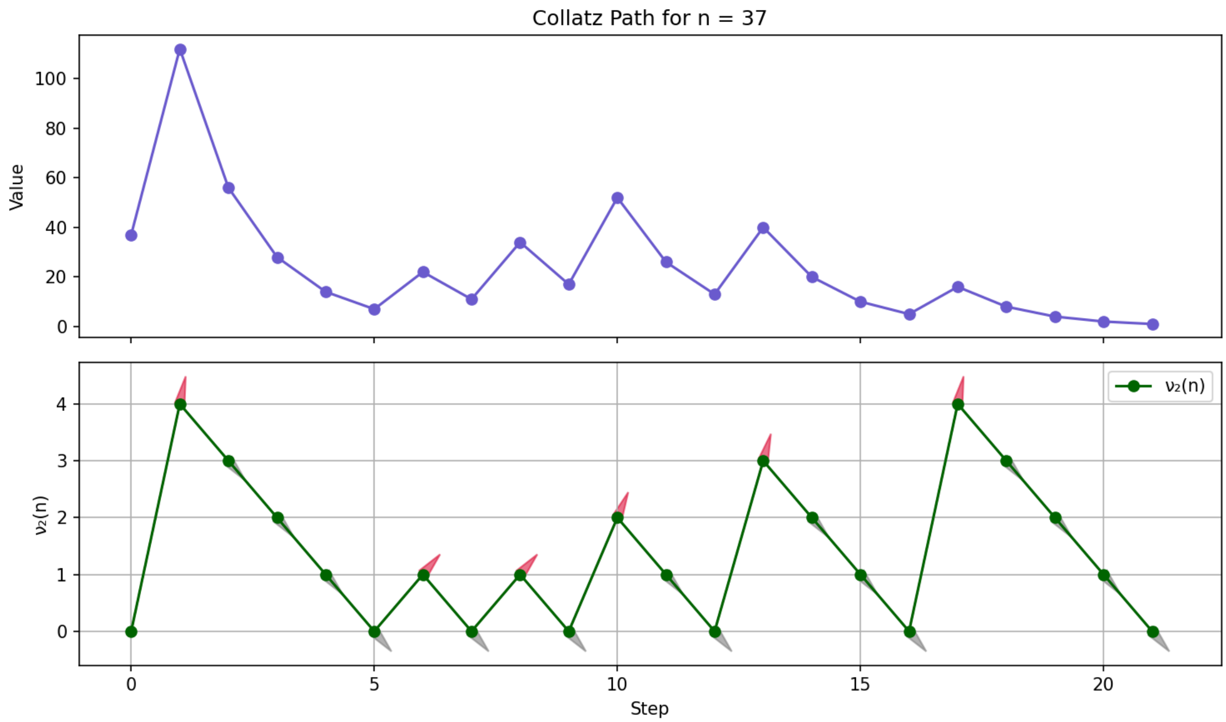

Figure 14: Cluster 4 — Trajectory for $x_0 = 37$. Small initial burst, followed by steady decline.

These trajectories resemble "soft collapses." An initial odd-step spike may occur, but valuation stabilizes quickly and descends cleanly. These may originate near the top of towers with valuation spikes, but quickly branch into strongly compressive regions of the tree.

Cluster 5: Transitional and Mixed Dynamics — $x_0 = 87$

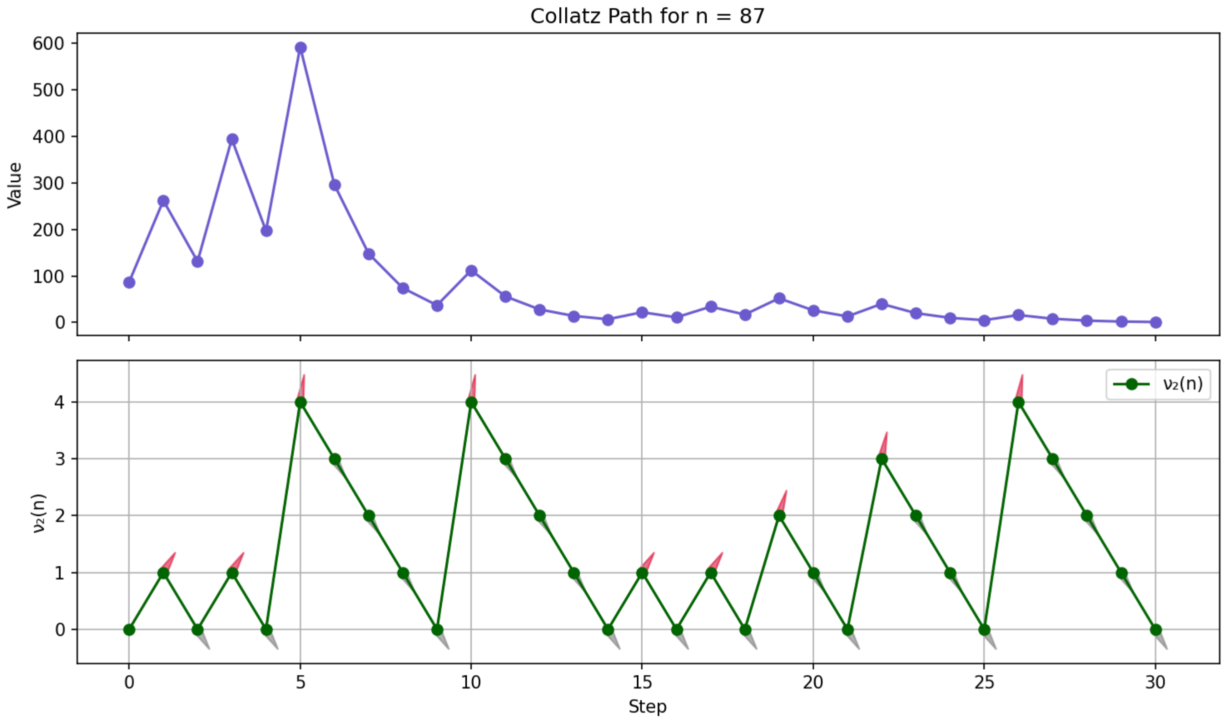

Figure 15: Cluster 5 — Trajectory for $x_0 = 87$. Valuation dynamics are irregular, combining features of multiple regimes.

Cluster 5 sits between worlds — sometimes chaotic, sometimes flat. Its valuation sequences include both energetic resets and long compressive plateaus. These trajectories likely emerge near bifurcations in the reverse tree, where multiple tower structures converge or cross-connect.

Clusters as Compression Signatures

These clusters provide compelling evidence that Collatz dynamics are not arbitrary. Valuation sequences, shaped by parity and divisibility, organize into discrete families. These families:

- Reflect modular and recursive structure in the reverse Collatz tree,
- Capture differences in how early a sequence enters a compressive regime,
- Reveal distinct dynamical paths seeded by tower height and residue class.

That this structure emerges purely from delay embeddings and unsupervised learning highlights the depth of organization in the Collatz map. In the next section, we return to the reverse tree to understand these trajectories from the inside out — as the recursive unfolding of valuation towers and modular constraints.

8 Recursive Structure and the Reverse Collatz Tree

While forward trajectories describe how sequences collapse toward 1, the inverse view — tracing how numbers can flow into 1 — reveals a recursive and modular skeleton beneath the Collatz universe. This structure is encoded in the **reverse Collatz tree**, where each node represents a number, and edges represent valid backward steps under the map.

8.1 Constructing the Tree in Reverse

Each integer n has at least one predecessor under the Collatz map:

- The **even predecessor** $2n$, which corresponds to the forward halving operation.
- A conditional **odd predecessor**: if $n \equiv 4 \pmod 6$, then $(n-1)/3$ is a valid odd preimage under the $3n+1$ rule.

Using these rules, we recursively build a tree rooted at 1. Each level of the tree corresponds to a fixed number of backward steps. The even path (doubling) produces uniform, exponential growth; odd predecessors introduce rare, modular branches that give the tree its complexity.

8.2 Forward Staircases and Reverse Towers

When we view a forward trajectory in logarithmic coordinates, it forms a staircase:

- Downward linear segments: compressive even steps.
- Discontinuous upward jumps: energetic odd steps.

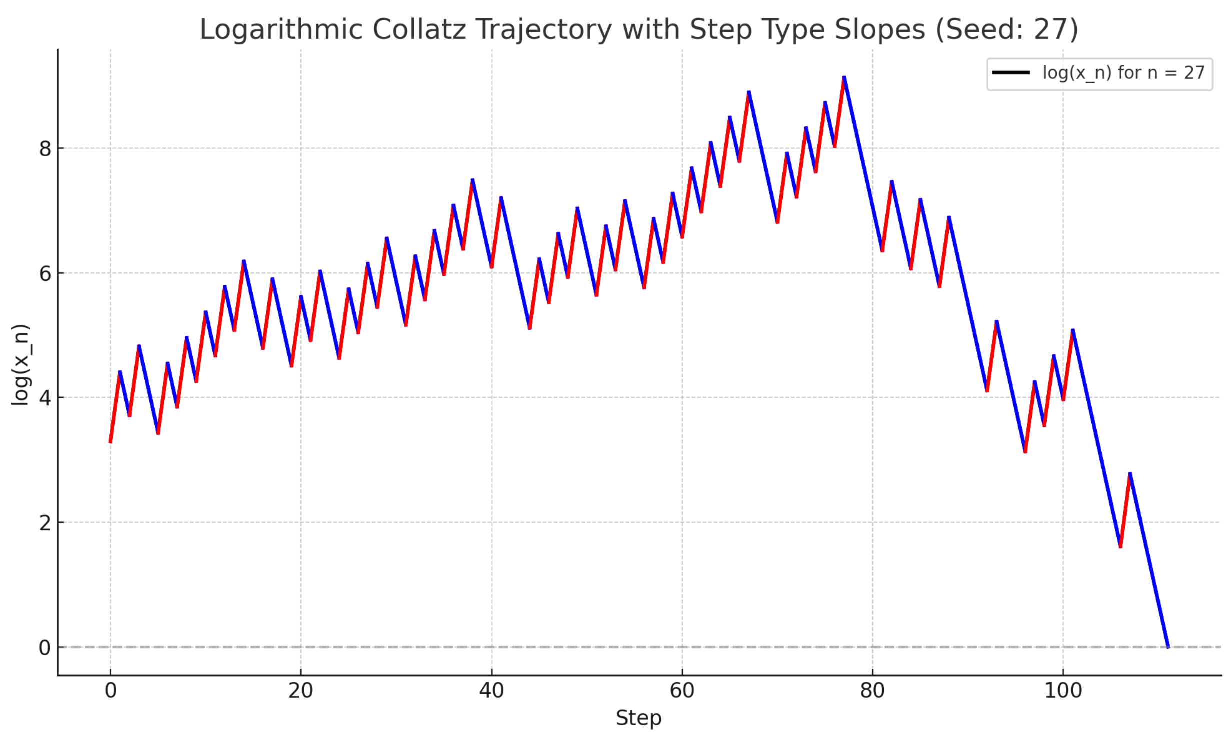

Figure 16: Logarithmic trajectory for $x_0 = 27$. Blue: even steps; red: odd. The consistent slope of even steps forms staircase "flights," reflecting predictable compression.

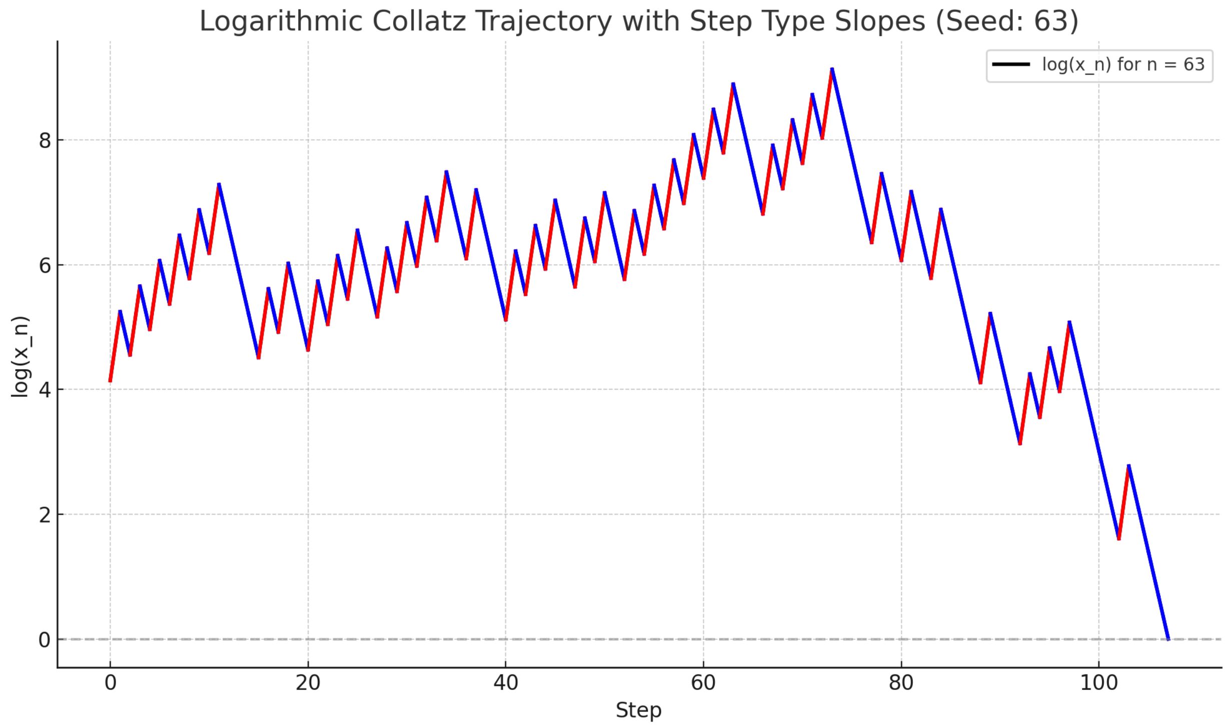

Figure 17: Logarithmic trajectory for $x_0 = 63$. Longer chaotic regime, but even-step slopes remain fixed. The system resists compression longer than 27.

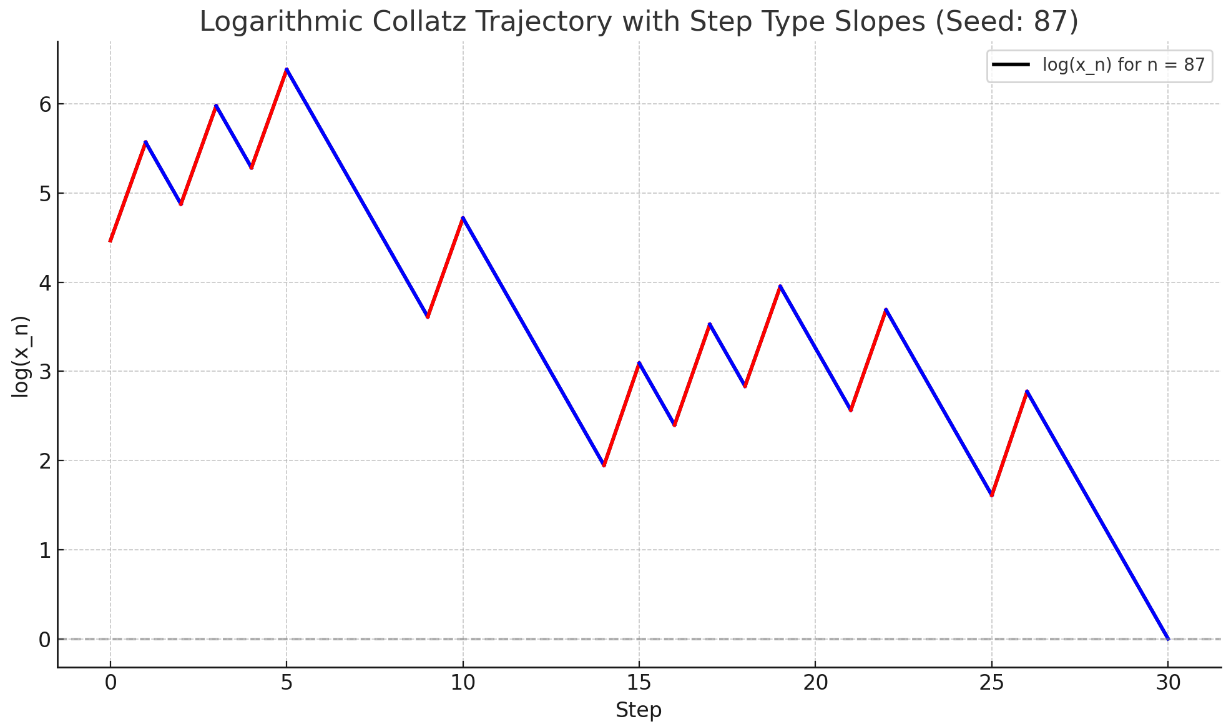

Figure 18: Logarithmic trajectory for $x_0 = 87$. Mixed regime with alternating compression lengths and chaotic resets.

These plots show that forward trajectories collapse predictably, interrupted by chaotic odd-step resets. When viewed in reverse, these become recursive towers: chains of doubling (even-step backtracking) interrupted by modular odd-step "branch points."

8.3 Valuation as Tree Height

Each doubling in the reverse tree increases $\nu_2(n)$ by 1. Thus, valuation measures depth within these towers. Sequences that spend many steps in a fixed valuation plateau in forward time — such as Cluster 2 — correspond to deep towers in reverse. Seeds like 63, on the other hand, which exhibit chaotic valuation resets, arise from shallow towers with more complex predecessor branches.

8.4 Exponential Spacing and the Gap Ratio

To understand how nodes spread as the tree grows, we compute the average spacing between sorted values at each level. Let G_L be the mean gap between nodes at level L, and define the ratio:

$$R(L) = \frac{G_L}{G_{L+1}}.$$

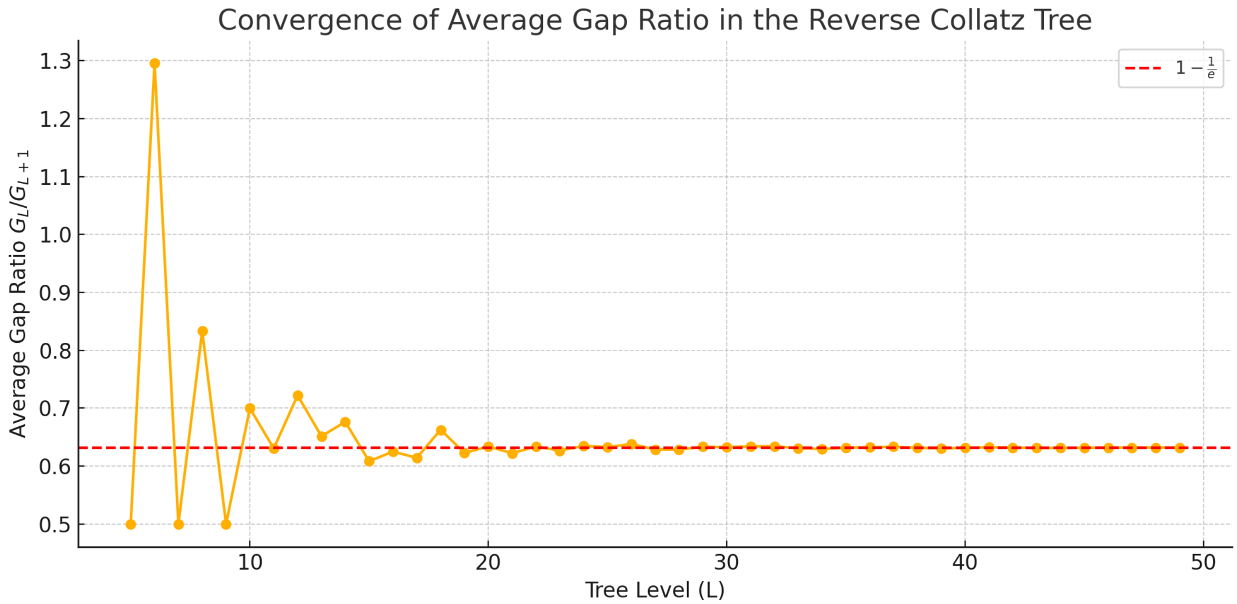

Figure 19: Average gap ratio G_L/G_{L+1} by tree depth L. The ratio converges toward $1 - \frac{1}{e}$, suggesting exponential spacing in recursive growth.

This convergence hints at a universal growth pattern beneath the tree's arithmetic definition. The emergence of $1 - \frac{1}{e}$ — common in entropy decay and information theory — suggests that the reverse tree organizes not just values, but dynamic potential.

8.5 Recursive Geometry and Cluster Origins

The modular structure of the reverse tree helps explain the clusters discovered earlier. For instance:

- Seeds like $x_0 = 5$, 7, and 63 belong to shallow towers such as T_5, which delay entry into compressive zones.
- Seeds like $x_0 = 21$ branch quickly into deep valuation layers — their forward collapse is rapid, and their UMAP embeddings cluster accordingly.
- Transitional seeds like $x_0 = 87$ lie at junctions between multiple towers, producing hybrid valuation signatures.

These tower identities are not merely arithmetic curiosities — they are structural, defining the energetic rhythm and valuation geometry of the resulting trajectories.

8.6 The Tree as an Entropic Scaffold

Just as forward dynamics compress entropy, the reverse tree unfolds it. It describes a space of possible pasts: an expanding scaffold constrained by parity, valuation, and modular congruence. Each level of the tree reflects not just what is arithmetically reachable, but what is dynamically plausible.

The convergence of the gap ratio, the predictability of slope geometry, and the modular branching rules all point to an underlying recursive logic. The Collatz universe is not random — it is an entropic machine driven by valuation structure, expanding backward even as it collapses forward.

In the next section, we build on this recursive insight by examining explicit examples of tower structures and how they control the evolution of entire Collatz families.

9 Modular Towers and Valuation Constraints in Reverse Dynamics

The recursive structure of the reverse Collatz tree, explored in the previous section, hints at a deeper arithmetic scaffold. In this section, we examine how that structure emerges from modular and valuation constraints. Each tower of doubled values is governed by residue class behavior and conditional connectivity, producing branching rules that shape the entire tree. These towers — and the valuation spikes they carry — form the modular skeleton underlying both the recursive dynamics and the empirical clusters we observed earlier.

9.1 Towers of the Form $T_m = \{2^k m\}$

Every odd number m generates a tower of the form:

$$T_m = \{2^k m \mid k \in \mathbb{N}\}.$$

These towers correspond to vertical paths in the reverse Collatz tree created by repeated doubling — they are guaranteed branches formed by reversing the halving operation.

9.2 Backwards Connectivity and the Filter Condition

The more subtle structure comes from identifying which nodes in T_m admit an odd predecessor. If a tower contains a number n such that:

$$\frac{n-1}{3} \in \mathbb{N} \quad \text{and} \quad \frac{n-1}{3} \equiv 1 \pmod 2,$$

then that node admits a valid predecessor under the $3n + 1$ rule. This happens if and only if:

$$n \equiv 4 \pmod 6.$$

We define a backwards filter to characterize where these connections occur:

$$3m + 1 = 2^k n \quad \Rightarrow \quad \nu_2(3m + 1) = k.$$

That is, the value $3m+1$ must be divisible by 2^k for some k, and the resulting quotient n must be odd and satisfy the modulo condition. When such a k exists, the tower T_m connects upward to n via an odd branch; otherwise, the tower grows by doubling alone.

9.3 Examples of the Backwards Filter

1. $m = 5$: we compute $3 \cdot 5 + 1 = 16 = 2^4$. Hence, $\nu_2(3m + 1) = 4$, and $n = 1$. This tower has an odd connection at the top.
2. $m = 3$: $3 \cdot 3 + 1 = 10 = 2 \cdot 5 \Rightarrow \nu_2 = 1$, not sufficient for deep branching.
3. $m = 1$: $3 \cdot 1 + 1 = 4 = 2^2$. Here, $\nu_2 = 2$, and we recover the classic cycle.

The filter imposes a strict modular logic on where odd predecessors are possible, segmenting towers into "branching" and "non-branching" types.

9.4 Periodicity in Valuation

The 2-adic valuation of $3m+1$ follows a periodic pattern based on $m \mod 8$. Specifically:

$$m \equiv 1 \pmod 8 \Rightarrow \nu_2(3m+1) = 2,$$
$$m \equiv 3,5,7 \pmod 8 \Rightarrow \nu_2(3m+1) = 1.$$

Most values fall into these baseline categories, but certain values of m generate unusually high valuations — these are the **valuation spikes**, which define nodes of deep connectivity in the tree.

These spikes play a visible role in delay embeddings of valuation and cluster structure. Sequences with unusually high initial $\nu_2(x_n)$ often correspond to towers that spike early — a fingerprint of deep preimage connectivity.

9.5 Growth Classes and Tree Shape

Towers fall into broad growth classes:

1. Towers with no odd connections (growing only by doubling).
2. Towers that eventually admit a valid odd step, but only after many doublings.
3. Towers that connect quickly to a predecessor (small ν_2 of $3m+1$).

These classes define the architecture of the reverse tree. Rapidly-connecting towers (like T_5) play central roles, forming "spines" that reach back to the root early. Shallow or isolated towers form peripheral structures.

9.6 Recursive Well-Ordering and Absence of Counterexamples

If one were to construct a non-converging trajectory, it must include a minimal element m under the natural order. But every m that satisfies the backwards filter connects to a smaller m', due to the filter's structure:

$$3m+1 = 2^k n \Rightarrow n < m.$$

Thus, no such minimal counterexample can exist — a recursive collapse is built into the modular and valuation logic itself. The tree flows inward toward 1 not by chance, but by structural necessity.

9.7 Toward a Modular Skeleton of Dynamics

This valuation and modular framework doesn't just explain tree shape — it also unites our major themes:

1. The reverse tree expands via powers of 2, constrained by residue classes.
2. Valuation spikes govern branching and delay embedding behavior.
3. UMAP clustering reflects these structural differences across trajectories.
4. The staircase geometry of even steps mirrors doubling in log-space — a visual cue to tower structure.

In this view, the Collatz dynamics are not merely chaotic or number-theoretic curiosities, but the unfolding of a modular recursive system with deep valuation geometry. The towers show us where complexity begins — and how compression, growth, and connection are all guided by structure at the arithmetic level.

10 Conclusion

The Collatz map, with its deceptively simple rules, has long been viewed as an arithmetic curiosity — a singular riddle rather than a coherent system. But through the lenses of energy, information, valuation, geometry, and recursion, we have uncovered a layered dynamical architecture: one in which order arises not despite chaos, but because of it.

We began with energy-theoretic and information-theoretic frameworks, reinterpreting the Collatz sequence as a flow of dynamics — shaped by the tension between explosive injection (odd steps) and steady dissipation (even steps). Total energy and entropy decay not smoothly, but in sharp, rhythmic pulses: a staircase structure encoding collapse through compression.

Phase space representations and delay embeddings made this structure geometric. Through UMAP, we revealed the latent topology of Collatz trajectories — families of sequences unfolding in similar shapes. HDBSCAN clustering uncovered distinct dynamical regimes: delayed collapses, quick compressions, oscillatory convergences — each arising purely from internal structure, not from external categorization.

The 2-adic valuation emerged as a central dynamical coordinate. It encoded the "depth" of compressibility within a sequence, measuring how often and how sharply a trajectory could descend. Early valuation spikes corresponded to chaotic orbits; smooth valuations marked stable collapse. These patterns echoed across energy plots, delay embeddings, and UMAP cluster geometry.

We then turned to the reverse Collatz tree — not merely a mirror of forward dynamics, but the recursive scaffold of the entire system. Each level of the tree grows predictably through doubling, with occasional odd predecessors admitted via strict modular filters. Remarkably, the average spacing between nodes converges toward $1 - \frac{1}{e}$, suggesting a hidden statistical logic beneath the tree's arithmetic surface.

Finally, we examined the modular tower structure: families $T_m = \{2^k m\}$ that grow vertically in the tree, and whose connectivity is governed by residue classes and valuation spikes. These towers do more than structure the tree — they control the dynamics of entire families of trajectories.

Towers as Origins of Cluster Geometry

This modular framework gives rise to the observed UMAP clusters. Each tower acts as a dynamical seed, generating trajectories whose structure reflects how — and whether — the tower connects to smaller values:

- Towers like T_5, with high $\nu_2(3m+1)$, produce long, energetic orbits — the chaotic sequences of Cluster 0.
- Towers like T_{21}, with low valuation and shallow growth, yield fast-converging sequences — the clean collapses of Cluster 1.
- Transitional towers like T_7 give rise to oscillatory yet stabilizing behavior — the spiral geometries of Clusters 3 and beyond.

Thus, the clusters in UMAP space are not emergent artifacts — they are shadows cast by deeper modular structure. The recursive tree, the valuation spikes, and the residue filters form a coherent geometric grammar: a way to predict and interpret Collatz behavior from its arithmetic DNA.

From Curiosity to Dynamical System

In this light, the Collatz map is more than an open conjecture. It is a deterministic universe of compression and expansion, an entropic engine wrapped in recursive syntax. The tools of physics, information theory, and topology are not incidental — they are essential to seeing how the system breathes.

There remains much to explore: topological invariants in valuation space, algebraic interpretations of spike periodicity, probabilistic models of tree growth, and generalizations to other integer maps. But already, we glimpse a profound shift in perspective: the Collatz map is not random, not trivial, and not singular. It is recursively rich — a dynamical object of surprising depth and astonishing beauty.