Folded Symplectic Origins of Replicator Dynamics.

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Abstract

We show that conservative Hamiltonian motion in a folded symplectic bulk induces a canonical contact Hamiltonian system on the fold, and that Legendrian-invariant graphs of this contact flow descend to normalized dissipative dynamics on a screen. Globally, under a mild cohomological hypothesis on the fold, the pullback form is exact and furnishes a global contact form. We give a canonical model on S^4 with equatorial fold S^3 whose Hopf projection to S^2 realizes the bulk–fold–screen mechanism end to end. The framework provides a compact geometric origin for softmax/replicator dynamics and unifies models for geodesic focusing, gauge-coupled interference, LLM attention, population dynamics with mutation, multiscale morphology, and materials textures.

1 Introduction

Replicator dynamics provide a unifying perspective on systems where populations of competing strategies, particles, or signals evolve according to relative advantage. Originating in evolutionary game theory Taylor and Jonker [1978], Hofbauer and Sigmund [1998], the replicator equation,

$$\dot{p}_i = p_i\Big(f_i(p,t) - \sum_j p_j f_j(p,t)\Big),$$

governs how probabilities adjust under fitness fields, ensuring that advantageous types grow in frequency. This normalized, positivity-preserving flow has since found applications far beyond its origins, linking statistical inference, optimization, and geometry Sandholm [2010], Weibull [1995].

The geometric perspective on evolutionary dynamics has deep roots in symplectic geometry and Hamiltonian mechanics Arnold [1989], Libermann and Marle [1987]. Developments in folded symplectic geometry da Silva et al. [2000] provide new tools for understanding systems where conservative dynamics transition to dissipative behavior.

A particularly compelling modern instance arises in large language models (LLMs). Transformer heads Vaswani et al. [2017] compute unnormalized scores, or *logits*, for possible tokens. The softmax map,

$$\sigma(z)_i = \frac{e^{z_i}}{\sum_{j=1}^n e^{z_j}},$$

projects these scores onto a probability distributions over vocabularies. The symmetry $\sigma(z + c1) = \sigma(z)$ highlights that only relative logit values matter. Training or inference induces a

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time-dependent fitness field $f(p,t)$ from gradients, attention weights, or reinforcement signals. Projected through softmax, the downstream probability dynamics are precisely of replicator form:

$$\dot{p}_i = p_i\big(f_i(p,t) - \langle f(p,t), p\rangle\big).$$

From an information-geometric perspective ichi Amari and Nagaoka [2000], these operations naturally live on the probability simplex, suggesting deep connections to evolutionary dynamics.

This paper builds upon the author's prior work on folded symplectic four-manifolds Lee [2018] and the foundational theory of symplectic unfolding da Silva et al. [2000] to establish a bulk–fold–screen mechanism. We show that conservative Hamiltonian motion in a folded symplectic bulk induces contact dynamics on the fold and descends, via projection, to normalized dissipative flows on the screen.

This framework provides both a geometric explanation of the replicator equation and a conceptual bridge between optimization, information geometry, and physical dynamics. It unifies examples ranging from classical evolutionary games to modern LLM heads, showing that what appear to be disparate phenomena share a single geometric underpinning. By situating replicator dynamics within contact Hamiltonian systems, we clarify why softmax-based flows exhibit both the gradient-like behavior familiar from optimization and the dissipative normalization that aligns with probability dynamics.

2 Mathematical Background

The theory of folded symplectic manifolds was developed by Cannas da Silva, Guillemin, and Woodward da Silva et al. [2000] as a natural generalization of symplectic geometry. A folded symplectic form on a $2n$-manifold X is a closed 2-form ω that is nondegenerate away from a hypersurface $Z \subset X$ and has a specific degeneracy along Z. Near the fold Z, the form has local normal form

$$\omega = x\,dx \wedge dy + \sum_{i=2}^{n} dq_i \wedge dp_i, \tag{2.1}$$

with $x = 0$ defining the fold. The hypersurface Z is co-oriented, and the change of sign of ω across Z is an intrinsic geometric datum da Silva [2001].

Contact geometry studies odd-dimensional manifolds equipped with a maximally non-integrable hyperplane distribution Geiges [2008], Arnold [1989]. A contact form α on a $(2n+1)$-manifold satisfies $\alpha \wedge (d\alpha)^n \neq 0$ everywhere. Unlike symplectic geometry, which preserves volume, contact structures naturally accommodate the normalization and dissipation inherent in probability dynamics. The Darboux theorem for contact manifolds guarantees local equivalence to the standard model $\mathbb{R}^{2n+1}$ with $\alpha = dz - \sum_{i=1}^{n} p_i\,dx_i$.

Classical Hamilton-Jacobi theory Arnold [1989], Evans [2010] seeks complete integrals of the Hamilton-Jacobi equation $H(x, \nabla S(x), t) = 0$. In the contact setting, this becomes the contact Hamilton-Jacobi equation, where the generating function defines Legendrian submanifolds preserved by the contact flow.

3 Replicator Dynamics

The *replicator equation* and its softmax lift reveal a natural geometric structure: probability flows arise as projections of logit dynamics through the softmax map. The equation evolves a probability vector $p(t) = (p_1(t), \dots, p_n(t))$ on the open simplex according to

$$\dot{p}_i \;=\; p_i\Bigg(f_i(p,t) - \sum_{j=1}^{n} p_j f_j(p,t)\Bigg), \tag{3.1}$$

where $f : M \times \mathbb{R} \to \mathbb{R}^n$ is a (possibly time–dependent) *fitness field*; $f_i(p,t)$ is the instantaneous growth rate of type i when the current mix is p. The form (3.1) simply says: each coordinate grows in proportion to how much its own fitness exceeds the population's average fitness. The flow preserves normalization ($\sum_i p_i = 1$) and positivity ($p_i > 0$ stays > 0 in finite time), and it is invariant under adding the same constant to all fitnesses (only relative fitness matters). One may view f as coming from a game payoff map, a statistical score, or a physical "intensity" field; in all cases (3.1) is the normalized dynamics on M. In this section we adopt this equation as the basic object and use it as the bridge to the lifted (z, p, ψ) description developed later.

On the open probability simplex

$$\operatorname{Int}\Delta^{n-1} \;=\; \left\{p \in \mathbb{R}^n : p_i > 0,\ \sum_{i=1}^{n} p_i = 1\right\},$$

introduce the *softmax potential with temperature* $S > 0$:

$$\psi_S(z) \;=\; S \log \sum_{j=1}^{n} e^{z_j/S}, \qquad \sigma_S(z) \;=\; \nabla\psi_S(z), \qquad p \;=\; \sigma_S(z). \tag{3.2}$$

By *logits* we mean unconstrained scores $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ whose softmax produces a probability vector: $p = \sigma_S(z)$. Softmax is invariant under adding a constant to all coordinates,

$$\sigma_S(z + c\,\mathbf{1}) = \sigma_S(z) \quad \forall\, c \in \mathbb{R},$$

so the physically relevant space of logits is the *quotient*

$$\bar{Z} \;:=\; \mathbb{R}^n/\mathbb{R}\mathbf{1},$$

i.e. only the differences $z_i - z_j$ matter.

Lemma 3.1 (Jacobian of the softmax)**.** *Let $p = \sigma_S(z)$ with*

$$p_i \;=\; \frac{e^{z_i/S}}{\sum_{j=1}^{n} e^{z_j/S}}, \qquad Z := \sum_j e^{z_j/S}.$$

Then

$$\frac{\partial p_i}{\partial z_k} \;=\; \frac{1}{S}\, p_i\big(\delta_{ik} - p_k\big), \qquad \textit{so} \qquad D_z p \;=\; \frac{1}{S}\big(\operatorname{Diag}(p) - pp^{\top}\big).$$

Proof. Differentiate $p_i = (e^{z_i/S})/Z$:

$$\frac{\partial p_i}{\partial z_k} = \frac{1}{S}\frac{e^{z_i/S}}{Z}\,\delta_{ik} \;-\; \frac{e^{z_i/S}}{Z^2}\,\frac{1}{S}\,e^{z_k/S} = \frac{1}{S}\,p_i\,\delta_{ik} - \frac{1}{S}\,p_i p_k = \frac{1}{S}\,p_i(\delta_{ik} - p_k).$$

Stacking the partials gives the matrix form. □

For a path, $z(t)$, of logits, applying the chain rule to $p(t) = \sigma_S(z(t))$,

$$\dot{p} \;=\; D_z p\, \dot{z} = \frac{1}{S}\big(\operatorname{Diag}(p) - pp^\top\big)\dot{z}.$$

It is convenient to write this as

$$\dot{p} \;=\; J_\psi(p)\, \dot{z}, \qquad J_\psi(p) := \frac{1}{S}\big(\operatorname{Diag}(p) - pp^\top\big).$$

Given a fitness field $f(p,t) \in \mathbb{R}^n$, we choose $\dot{z} = f(p,t)$, so that

$$\dot{p} \;=\; J_\psi(p)\, f(p,t), \qquad \Longleftrightarrow \qquad \dot{p}_i \;=\; p_i\Big(f_i(p,t) - \sum_j p_j f_j(p,t)\Big), \tag{3.3}$$

the replicator equation (3.1). Since $J_\psi(p)\,\mathbf{1} = 0$, we have $\sum_i \dot{p}_i = 0$; and $p_i > 0$ follows from $p = \sigma(z)$. Hence normalization and positivity are automatic. (When $\psi = \psi_S$, $J_\psi = \frac{1}{S}(\operatorname{Diag}(p) - p\,p^\top)$.)

4 Local Contact Geometry on $\mathbb{R}^{2n+1}$

The replicator equation and its softmax lift reveal a natural geometric structure: probability flows arise as projections of logit dynamics through the softmax map. To place this relationship on solid footing and to extend it beyond specific games or examples, we require a general framework that unifies logits, probabilities, and their generating potentials. Contact geometry provides exactly such a setting. It augments phase space with an additional coordinate, yielding a $(2n+1)$-dimensional manifold equipped with a canonical one-form. Within this space, softmax submanifolds emerge as Legendrian, and Hamiltonian flows on the contact manifold descend to replicator dynamics on the simplex. In this section we develop the local Darboux model that underpins this construction.

Let $M = \mathbb{R}^n_z \times \mathbb{R}^n_p \times \mathbb{R}_\psi$ and define

$$\alpha \;=\; d\psi \;-\; \sum_{i=1}^n p_i\, dz_i. \tag{4.1}$$

Then

$$d\alpha \;=\; -\sum_{i=1}^n dp_i \wedge dz_i, \qquad \alpha \wedge (d\alpha)^n \;=\; d\psi \wedge \big(\textstyle\sum_i dz_i \wedge dp_i\big)^n \neq 0,$$

so (4.1) is a *contact form*. Darboux's theorem states that any contact manifold is locally equivalent to this model. The *Reeb vector field* R is defined by

$$\alpha(R) = 1, \qquad \iota_R d\alpha = 0.$$

From (4.1), we conclude $R = \partial_\psi$.

A smooth n–dimensional submanifold $L \subset M$ is *Legendrian* if α restricted to the tangent bundle TL is identically zero. A basic class of Legendrian submanifolds comes from *graphs of differentials*: given $U \subset \mathbb{R}^n$ and $F \in C^2(U)$,

$$L_F \;=\; \{(z,p,\psi) \in U \times \mathbb{R}^n \times \mathbb{R} : \; p = \nabla F(z),\; \psi = F(z)\}. \tag{4.2}$$

Then $dF(z) = \sum_i \partial_{z_i} F\, dz_i = \sum_i p_i\, dz_i$, so $\alpha|_{L_F} = d\psi - \sum_i p_i\, dz_i = 0$ on L_F. Hence L_F is Legendrian.

Given a smooth function $\mathcal{H}(z, p, \psi, t)$ (the *Hamiltonian*), the associated vector field $X_{\mathcal{H}}$ is defined by

$$\iota_{X_{\mathcal{H}}} d\alpha \;=\; d\mathcal{H} - R(\mathcal{H})\,\alpha, \qquad \alpha(X_{\mathcal{H}}) \;=\; -\,\mathcal{H}. \tag{4.3}$$

Let $X_{\mathcal{H}} = (\dot z, \dot p, \dot\psi)$. Then,

$$\alpha \;=\; d\psi - p\cdot dz, \qquad R = \partial_\psi, \qquad \iota_{X_{\mathcal{H}}} d\alpha \;=\; d\mathcal{H} - (R\mathcal{H})\,\alpha, \quad \alpha(X_{\mathcal{H}}) = -\mathcal{H}.$$

Substituting and expanding (4.3) we get

$$\begin{aligned} d\mathcal{H} - (\partial_\psi\mathcal{H})(d\psi - p\cdot dz) &= \partial_z\mathcal{H}\cdot dz + \partial_p\mathcal{H}\cdot dp + \partial_\psi\mathcal{H}\,d\psi \;-\; (\partial_\psi\mathcal{H})\,d\psi \;+\; (\partial_\psi\mathcal{H})\,p\cdot dz \\ &= \partial_z\mathcal{H}\cdot dz \;+\; \partial_p\mathcal{H}\cdot dp \;+\; (\partial_\psi\mathcal{H})\,p\cdot dz. \end{aligned}$$

Therefore,

$$d\alpha \;=\; -\,dp\wedge dz \quad\Longrightarrow\quad \iota_{X_{\mathcal{H}}} d\alpha \;=\; -(\dot p\cdot dz - \dot z\cdot dp).$$

Comparing coefficients of dz and dp gives

$$-\dot p \;=\; \partial_z\mathcal{H} \;+\; p\,\partial_\psi\mathcal{H}, \qquad \dot z \;=\; \partial_p\mathcal{H}.$$

The constraint $\alpha(X_{\mathcal{H}}) = \dot\psi - p\cdot\dot z = -\mathcal{H}$ yields

$$\dot\psi \;=\; p\cdot\partial_p\mathcal{H} \;-\; \mathcal{H}.$$

Thus,

$$\boxed{\dot z \;=\; \partial_p\mathcal{H}, \qquad \dot p \;=\; -\,\partial_z\mathcal{H} \;-\; p\,\partial_\psi\mathcal{H}, \qquad \dot\psi \;=\; p\cdot\partial_p\mathcal{H} \;-\; \mathcal{H}.} \tag{4.4}$$

In matrix form, (4.4) is

$$\begin{bmatrix} \dot z \\ \dot p \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & -p \\ 0 & p^\top & -1 \end{bmatrix} \begin{bmatrix} \partial_z\mathcal{H} \\ \partial_p\mathcal{H} \\ \partial_\psi\mathcal{H} \end{bmatrix}.$$

We are now in a position to see how these general equations project onto more familiar dynamical laws. In particular, when the Hamiltonian is coupled to a fitness field, the flow of the system naturally reduces onto the probability simplex, yielding the replicator dynamics of evolutionary game theory. This descent illustrates how abstract Hamiltonian structure, when constrained to invariant submanifolds, produces concrete models of population-level behavior.

Let $f : M\times\mathbb{R}\to\mathbb{R}^n$ be a smooth fitness field . Define the *contact Hamiltonian*

$$\mathcal{H}(z, p, t) \;=\; (p - \nabla\psi(z))\cdot f(\sigma_\psi(z), t). \tag{4.5}$$

By construction $\mathcal{H}|_{\mathcal{L}_\psi} \equiv 0$.

Theorem 4.1 (Hamiltonian dynamics descend to replicator dynamics)**.** *The flow of $X_{\mathcal{H}}$ leaves $\mathcal{L}_\psi$ invariant. Its projection $\pi_* X_{\mathcal{H}}$ on M is*

$$\dot p \;=\; J_\psi(p)\, f(p, t). \tag{4.6}$$

where $J_\psi(p) := \nabla^2\psi(z)\big|_{p=\nabla\psi(z)}$

Proof. From (4.5), $\partial_p\mathcal{H} = f(\sigma_\psi(z), t)$ and $\partial_\psi\mathcal{H} = 0$, so on $\mathcal{L}_\psi$ (where $p = \sigma_\psi(z)$)

$$\dot{z} \;=\; \partial_p\mathcal{H} \;=\; f(p,t). \tag{4.7}$$

Differentiate the Legendre constraint $p - \nabla\psi(z) = 0$ along the flow:

$$\frac{d}{dt}(p - \nabla\psi(z)) = \dot{p} - \nabla^2\psi(z)\,\dot{z} = \dot{p} - J_\psi(p)\,\dot{z}.$$

Meanwhile,

$$\mathcal{H}(z,p,t) = (p - \nabla\psi(z))\cdot f(\sigma_\psi(z),t), \qquad \sigma_\psi(z) = \nabla\psi(z),$$

and so

$$\begin{aligned}\partial_z\mathcal{H} &= \big(\partial_z(p - \nabla\psi(z))\big)^\top f(\sigma_\psi(z),t) \;+\; \big(\partial_z f(\sigma_\psi(z),t)\big)^\top (p - \nabla\psi(z))\\ &= \big(-\nabla^2\psi(z)\big)^\top f(\sigma_\psi(z),t) \;+\; \Big(D_p f(\sigma_\psi(z),t)\,\nabla^2\psi(z)\Big)^\top (p - \nabla\psi(z))\\ &= -\nabla^2\psi(z)\,f(\sigma_\psi(z),t) \;+\; \Big(D_p f(\sigma_\psi(z),t)\,\nabla^2\psi(z)\Big)^\top (p - \nabla\psi(z)).\end{aligned} \tag{4.8}$$

Hence on $\mathcal{L}_\psi = \{p = \nabla\psi(z)\}$:

$$\partial_z\mathcal{H}\big|_{\mathcal{L}_\psi} = -\nabla^2\psi(z)\,f(\sigma_\psi(z),t) = -J_\psi(p)\,f(p,t), \quad J_\psi(p) := \nabla^2\psi(z)\big|_{p=\nabla\psi(z)}.$$

Using (4.4) with $\partial_\psi\mathcal{H} = 0$ and (4.8) gives $\dot{p} = J_\psi(p)\,f(p,t)$. $\square$

Theorem 4.1 identifies the precise contact–to–replicator descent at the Darboux level: with $H(z,p,\psi,t) = (p - \nabla\psi(z))\cdot f(\sigma_\psi(z),t)$ the flow preserves the Legendrian graph L_ψ and projects to the probability space as $\dot{p} = J_\psi(p)\,f(p,t)$ (normalization and positivity coming for free). Invariance of L_ψ and the projected law follow directly from the Hamiltonian choice and the Legendre constraint $p = \nabla\psi(z)$, so the mechanism is intrinsic (coordinate-free up to contactomorphism) and does not assume any special structure on f beyond smoothness. The Hessian J_ψ acts as the canonical preconditioner set by the generating potential ψ. In the next section, we couple to a folded symplectic bulk and promote the projection to a global statement on the fold.

5 Global Dynamics: Folded Symplectic and Contact Descent

A *folded symplectic form* on a $2n$–manifold X is a closed 2–form ω which is nondegenerate away from a hypersurface $Z \subset X$, and such that ω^n vanishes transversely along Z. Near Z, the form has local normal form

$$\omega = x\,dx \wedge dy + \sum_{i=2}^{n} dq_i \wedge dp_i,$$

with $x = 0$ the fold. The hypersurface Z is itself co–oriented, and the neighborhood theorem for folds states that any two folded symplectic forms are locally equivalent near the fold up to a diffeomorphism preserving Z. Thus the change of sign of ω across Z is an intrinsic geometric datum.

The connection between folded symplectic geometry and contact structures was established in the foundational work of Cannas da Silva, Guillemin, and Woodward da Silva et al. [2000]. Their unfolding theory shows that the fold hypersurface of any folded symplectic manifold carries a canonical contact structure.

We record their basic structural facts in the following theorem, which provides the theoretical foundation for our construction. Our contribution is recognizing that replicator dynamics naturally fit this geometric framework, with the softmax map providing the bridge between contact geometry and probability theory.

Theorem 5.1 (Cannas da Silva-Guillemin-Woodward da Silva et al. [2000])**.** *Let (X^{2n}, ω) be a folded symplectic manifold with fold $Z = \{x = 0\}$, so that in a neighborhood of Z there are coordinates $(x, y, q_2, p_2, \dots, q_n, p_n)$ with*

$$\omega = x\, dx \wedge dy + \sum_{i=2}^{n} dq_i \wedge dp_i. \tag{5.1}$$

Set $\beta := \iota^*\omega$ on Z; then β has rank $2n-2$ and $\ker\beta$ is spanned by ∂_y.

(i) (Stable Hamiltonian structure.) The pair (α_0, β) with $\alpha_0 := dy$ on Z is a stable Hamiltonian structure: β is closed, $\alpha_0 \wedge \beta^{n-1}$ is a volume form on Z, and the Reeb field R defined by $\alpha_0(R) = 1$, $\iota_R\beta = 0$ is $R = \partial_y$.

(ii) (Local contact form.) Locally on Z, there exists a 1–form θ with $d\theta = \beta$. Then

$$\alpha \ := \ \alpha_0 + \theta$$

is a contact form on Z, satisfying $d\alpha = \beta$ and $\alpha \wedge (d\alpha)^{n-1} = \alpha_0 \wedge \beta^{n-1} \neq 0$. Hence (Z, α) is a (cooriented) contact manifold whose contact structure depends only on the germ of ω near Z.

(iii) (Global contact, cohomological condition.) If β is exact globally on Z (e.g. $[\iota^*\omega] = 0$ in $H^2(Z)$), one can choose θ globally with $d\theta = \beta$, yielding a global contact form $\alpha = \alpha_0 + \theta$ on Z.

(iv) (Induced contact Hamiltonian flow.) For any smooth function $H : Z \times \mathbb{R} \to \mathbb{R}$, the contact Hamiltonian vector field X_H on (Z, α) is defined by

$$\iota_{X_H} d\alpha \ = \ dH - (RH)\,\alpha, \qquad \alpha(X_H) \ = \ -H.$$

In particular, with $d\alpha = \beta = \iota^*\omega$ as above, the contact flow X_H is canonically induced by the folded symplectic structure, and its Legendrian-invariant graphs descend to the screen dynamics as in Sections 3–4.

When Z is simply connected the closed form β is automatically exact, so that the contact form $\alpha = \alpha_0 + \theta$ extends globally. This yields a globally defined contact Hamiltonian flow canonically induced by the folded symplectic structure, whose Legendrian-invariant graphs descend coherently to replicator dynamics.

Theorem 5.2. *Let (X^{2n}, ω) be a folded symplectic manifold with fold $Z = \{x = 0\}$ and set $\beta := \iota^*\omega$ on Z. Assume that Z is simply connected and that $[\beta] = 0$ in $H^2(Z)$. Then, for every smooth Hamiltonian $H : Z \times \mathbb{R} \to \mathbb{R}$, the contact Hamiltonian vector field X_H on (Z, α) is globally defined. Moreover, any Legendrian-invariant graph $\Gamma \subset Z \times \mathbb{R}$ of X_H projects* globally *and the projected flow coincides with the replicator dynamics.*

$$\pi_* X_H \ = \ \dot{p} \ = \ J_\psi(p)\, f(p, t).$$

Proof. By hypothesis $[\beta] = 0$ in $H^2(Z)$, so there exists a global 1–form θ with $d\theta = \beta$. Part (ii) of Theorem 5.1 then yields that $\alpha = \alpha_0 + \theta$ is a contact form on Z, and it is global because θ is global. Part (iv) of the theorem gives the defining equations for the contact Hamiltonian vector field X_H on (Z, α); since α and $d\alpha = \beta$ are global, so is X_H.

Finally, with $d\alpha = \beta = \iota^*\omega$ as in the theorem, the construction of Section 4 shows that Legendrian-invariant graphs for X_H are preserved by the flow and project along π to the screen. The Hamiltonian descent calculation identifies this projected vector field with the replicator dynamics $\dot{p} = J_\psi(p) f(p,t)$; because X_H is global, the descent is global as well. $\square$

Application: the S^4 bulk, equatorial Hopf fold, and screen dynamics

We now illustrate the general theory with a canonical global example developed in the author's previous work Lee [2018]. Let S^4 carry a folded symplectic form ω whose fold locus is the equatorial $S^3 \subset S^4$. As shown in Lee [2018], such structures arise naturally and provide concrete realizations of the abstract unfolding theory da Silva et al. [2000]. Across the equator, ω changes sign, so the two hemispheres of S^4 push against each other with opposite orientations. This global sign reversal can be heuristically viewed as a "pressure differential" across the fold: orientation reversal across the fold induces a co-oriented stable Hamiltonian structure. It is precisely this bulk effect that forces the fold $Z = S^3$ to carry a contact structure. Restricting ω to Z and contracting with a transverse vector field V produces

$$\alpha \;=\; \iota_V \, \omega|_Z,$$

so that the contact form α is coupled with the bulk form. Since $Z = S^3$ is simply connected, Theorem 5.2 applies: $\beta = \iota^*\omega$ is globally exact, and therefore α extends to a *global* contact form on Z. Thus, the folded symplectic *bulk* drives a globally defined contact Hamiltonian system (Z, α).

The equatorial S^3 admits the Hopf fibration $\pi : S^3 \to S^2$ with S^1 fibers. Under the dynamics, Legendrian submanifolds of (S^3, α), generated by potentials ψ, are preserved by the contact Hamiltonian flow X_H. The Hopf map then projects these Legendrians onto distributions on the *screen* S^2. In this way, the causal chain is complete: folded symplectic dynamics in the S^4 bulk induce contact flows on the fold S^3, and by Theorem 5.2 these flows are globally defined. The Hopf projection projects them to normalized, dissipative probability dynamics on the S^2 screen.

Different Hamiltonians in the bulk produce distinct classes of screen behavior (see Appendix A):

- Quadratic H induce geodesic focusing distributions on S^2, modeling optical bias through the Hopf projection of Legendrian rays.
- Linear or thermodynamic H yield replicator equations, with the fitness field f descending directly from the Hamiltonian potential.
- Gauge–coupled H transmit holonomy and phase information to S^2, producing interference and Aharonov–Bohm type phenomena.

This work illustrates how established geometric theory da Silva et al. [2000], Lee [2018] can provide unexpected insights into modern machine learning Vaswani et al. [2017], demonstrating the continuing relevance of classical differential geometry. The bulk–fold–screen mechanism unifies phenomena across evolutionary biology Hofbauer and Sigmund [1998], information geometry ichi Amari and Nagaoka [2000], and neural computation, suggesting that geometric methods will play an increasingly important role in understanding complex adaptive systems.

A Proposed Models for Bulk–Fold–Screen Examples

The examples in this appendix demonstrate how the abstract geometric machinery of da Silva et al. [2000], Lee [2018], and this paper provides concrete insights into diverse physical and computational systems. Each model instantiates the contact–descent mechanism: a contact Hamiltonian on the fold induces characteristics that preserve Legendrian submanifolds, which then project to probability distributions on the screen. They may be read as *consistent illustrative examples* rather than domain-specific derivations.

Remarks (i) The choices of H are not unique; many lead to equivalent screen phenomenology up to gauge and reparametrization. (ii) Constitutive details and units are normalized for clarity; Φ, F, A, ϕ are placeholders to be calibrated to data in each application. (iii) Claims here concern structural compatibility with the theorems; empirical validation and system-specific boundary conditions are outside this paper's scope. (iv) Throughout, $q \in \mathbb{S}^2$, $p \in T_q\mathbb{S}^2$, $s \in \mathbb{R}$; $\nabla_{S^2}, \Delta_{S^2}$ are the spherical gradient/Laplacian; small parameters $\lambda, \nu \geq 0$ encode mobility and viscosity/mutation.

We record: (1) Contact Hamiltonian; (2) contact Hamilton-Jacobi (H-J) for ψ; (3) characteristics $\dot{q}$; (4) screen readout $\mu \propto e^{\psi}$ (and discrete softmax $\Rightarrow$ replicator when applicable).

A.1 Optics / Geodesic Focusing (Hopf baseline)

$$\begin{aligned}
&\textbf{(1)}\ H(q,p,s) = \tfrac{1}{2}\|p\|^2 + V(q) - E,\\
&\textbf{(2)}\ \text{(H-J):}\quad \tfrac{1}{2}\|\nabla_{S^2}\psi(q)\|^2 + V(q) = E,\\
&\textbf{(3)}\ \dot{q} = \partial_p H = p = \nabla_{S^2}\psi(q),\\
&\textbf{(4)}\ \mu(\Omega) \propto e^{\psi(\Omega)}.
\end{aligned}$$

Interpretation: V controls focusing/defocusing; peaks of ψ bias intensity μ.

Explicit chart (north-pole plane): with $V(x,y) = \kappa x$, $E > \max V$, seek $\psi = \psi(x)$. Then $\frac{1}{2}(\psi_x)^2 + \kappa x = E \Rightarrow \psi_x = \sqrt{2(E-\kappa x)}$ (choose + branch) and

$$\psi(x) = -\frac{2}{3\kappa}\big(2(E-\kappa x)\big)^{3/2} + C, \qquad \mu(x,y) \propto e^{\psi(x)}.$$

As $\kappa > 0$ increases, rays tilt and μ skews toward $+x$ (focusing bias).

A.2 Thermo / Replicator (screen dynamics)

$$\begin{aligned}
&\textbf{(1)}\ H(q,p,s) = \Phi(q) - s + \tfrac{\lambda}{2}\|p\|^2,\\
&\textbf{(2)}\ \text{(H-J):}\quad \Phi(q) - \psi(q) + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi(q)\|^2 = 0,\\
&\textbf{(3)}\ \dot{q} = \partial_p H = \lambda p = \lambda\, \nabla_{S^2}\psi(q),\\
&\textbf{(4)}\ \mu(\Omega) \propto e^{\psi(\Omega)}.
\end{aligned}$$

Small-λ expansion: $\psi = \Phi + \frac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2 + O(\lambda^2)$, so $\mu \propto \exp(\Phi + \frac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2)$: selection amplified by curvature. *Discrete screen:* softmax of ψ_i yields $\dot{p}_i = p_i(f_i - \bar{f})$ with $f_i = \Phi_i$.

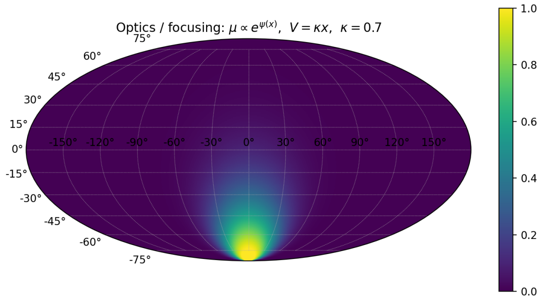

Figure 1: Optics / Geodesic focusing: $\mu \propto e^{\psi}$ for $V = \kappa x$. Rays skew toward $+x$ as κ grows.

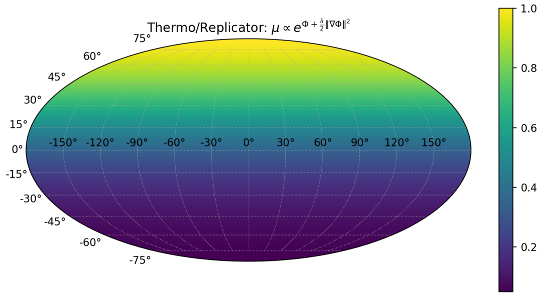

Figure 2: Thermo/Replicator: $\mu \propto e^{\Phi + \frac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2}$ for $\Phi = \alpha(u \cdot q)$.

A.3 LLM Heads (continuous attention logits)

$$\begin{aligned}
&\textbf{(1)}\ H(q,p,s) = \Phi_{\text{attn}}(q;\text{ctx}) - s + \tfrac{\lambda}{2}\|p\|^2,\\
&\textbf{(2)}\ \text{HJ:}\quad \Phi_{\text{attn}} - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 = 0,\\
&\textbf{(3)}\ \dot q = \lambda\,\nabla_{S^2}\psi(q), \qquad \textbf{(4)}\ \ \mu(\Omega) \propto e^{\psi(\Omega)}\ \text{(attention weights)}.
\end{aligned}$$

With $\Phi_{\text{attn}}(q) = \beta + \kappa\langle u, q\rangle$ and $\|\nabla_{S^2}\langle u, q\rangle\|^2 = 1 - (u\cdot q)^2$,

$$\psi(q) \approx \beta + \kappa(u\cdot q) + \frac{\lambda\kappa^2}{2}\big(1 - (u\cdot q)^2\big), \quad \mu \propto e^{\psi(q)}.$$

The quadratic correction sharpens the lobe around u.

A.4 Electromagnetism (gauge-coupled optics)

$$\begin{aligned}
&\textbf{(1)}\ H(q,p,s) = \tfrac{1}{2}\|p - A(q)\|^2 + \phi(q) - E,\\
&\textbf{(2)}\ \text{(H-J):}\quad \tfrac{1}{2}\|\nabla_{S^2}\psi(q) - A(q)\|^2 + \phi(q) = E,\\
&\textbf{(3)}\ \dot q = \nabla_{S^2}\psi(q) - A(q), \qquad \textbf{(4)}\ \ \mu(\Omega) \propto e^{\psi(\Omega)}.
\end{aligned}$$

Gauge shifts change ψ by a scalar generating function; loop holonomy (Aharonov–Bohm) enters via $\oint A$. Two effective lobes at $\theta = \pm\theta_0$ with actions $S_\pm(\theta) = \kappa\cos(\theta \mp \theta_0) \pm \varphi_{\text{AB}}$ aggregate by log-sum-exp:

$$\psi(\theta) = \log\big(e^{S_+(\theta)} + e^{S_-(\theta)}\big) = \kappa\log\Big(e^{\cos(\theta-\theta_0)} + e^{\cos(\theta+\theta_0)+2\varphi_{\text{AB}}/\kappa}\Big),$$

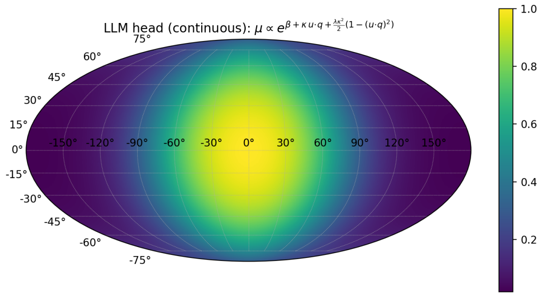

Figure 3: LLM attention: $\Phi_{\text{attn}} = \beta + \kappa(u \cdot q)$ with curvature sharpening.

producing fringe-like modulation as φ_{AB} varies.

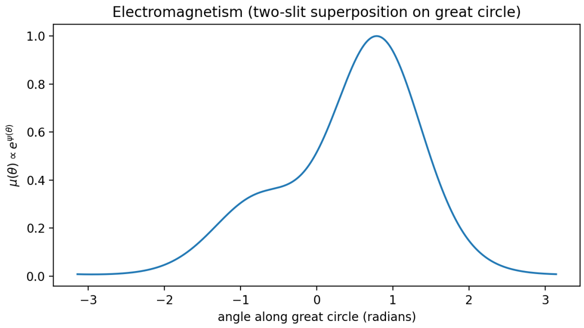

Figure 4: Gauge-coupled two-slit superposition with AB phase offset φ_{AB}.

A.5 Population Biology (replicator–mutator)

$$
\begin{aligned}
&\textbf{(1)}\ H(q,p,s) = \Phi(q) - s + \tfrac{\lambda}{2}\|p\|^2,\\
&\textbf{(2)}\ \text{(H-J)}: \quad \Phi - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 - \nu\,\Delta_{S^2}\psi = 0,\\
&\textbf{(3)}\ \dot{q} = \lambda\,\nabla_{S^2}\psi(q),\\
&\textbf{(4)}\ \mu \propto e^{\psi}, \quad \text{discrete: } \dot{p}_i = p_i(f_i - \bar{f}) + \text{(mutation)}.
\end{aligned}
$$

Example $\Phi(q) = \alpha(u \cdot q)$ gives $\psi \approx \alpha(u \cdot q) + \frac{\lambda\alpha^2}{2}(1 - (u \cdot q)^2) + O(\nu)$; ν spreads mass away from the fittest direction.

A.6 Biological Morphology (multiscale competency)

$$
\begin{aligned}
&\textbf{(1)}\ H(q,p,s) = (\Phi_{\text{morph}}(q) - s) + \tfrac{\lambda}{2}\|p\|^2 + \sum_k \gamma_k(\psi(q) - \psi_k(q)),\\
&\textbf{(2)}\ \text{(H-J)}: \quad \Phi_{\text{morph}} - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 + \sum_k \gamma_k(\psi - \psi_k) = 0,\\
&\textbf{(3)}\ \dot{q} = \lambda\,\nabla_{S^2}\psi(q),\\
&\textbf{(4)}\ \mu \propto e^{\psi} \quad \text{(consensus across scales)}.
\end{aligned}
$$

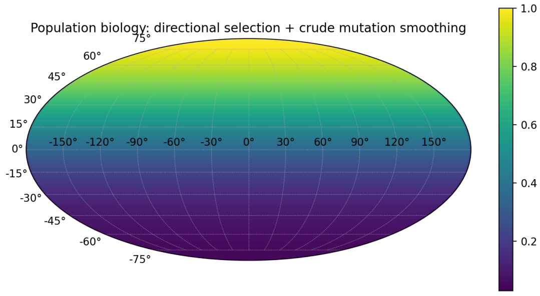

Figure 5: Directional selection with mutation smoothing.

In the quasistatic limit $\lambda \to 0$,

$$\left(1 + \sum_k \gamma_k\right) \psi = \Phi_{\text{morph}} + \sum_k \gamma_k \psi_k \quad \Rightarrow \quad \psi = \frac{\Phi_{\text{morph}} + \sum_k \gamma_k \psi_k}{1 + \sum_k \gamma_k},$$

with small λ adding curvature penalties to sharp cross-scale disagreement.

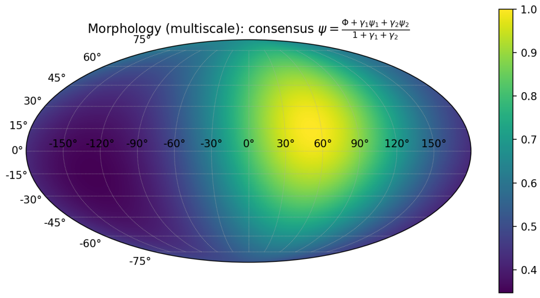

Figure 6: Consensus morphology as a soft cross-scale average.

A.7 Materials Science (free energy & phase fractions)

(1) $H(q, p, s) = F(q) - s + \frac{\lambda}{2}\|p\|^2$,

(2) (H-J): $\quad F - \psi + \frac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 = 0 \quad$ (optionally $\; - \nu\, \Delta_{S^2}\psi$),

(3) $\dot{q} = \lambda\, \nabla_{S^2}\psi(q)$,

(4) $\mu \propto e^{\psi}$ (texture/variant weights).

Two variants along $u, v \in \mathbb{S}^2$ with $F(q) = \alpha(1 - (u \cdot q)^2) + \beta(1 - (v \cdot q)^2)$ yield

$$\psi \approx F + \frac{\lambda}{2}\|\nabla_{S^2} F\|^2, \qquad \nabla_{S^2}(u \cdot q) = u - (u \cdot q)\, q,$$

so μ concentrates near $q \parallel u$ and $q \parallel v$; $\nu > 0$ smooths boundaries.

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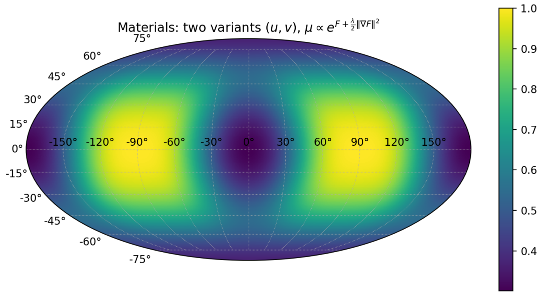

Figure 7: Two-variant texture: curvature term sharpens variant alignment.

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