Contact Hamiltonians on the Hopf Bundle: Templates + Worked Examples

Setup. We work on the Hopf contact manifold $(S^3, \alpha_{\text{Hopf}})$, locally contactomorphic to the 1-jet space

$$J^1(\mathbb{S}^2) \;\cong\; \{(q,p,s) : q \in \mathbb{S}^2,\ p \in T_q^*\mathbb{S}^2,\ s \in \mathbb{R}\}, \qquad \alpha = ds - p\cdot dq.$$

A Legendrian graph generated by a potential $\psi : \mathbb{S}^2 \to \mathbb{R}$ is

$$L_\psi = \{(q,p,s) :\ p = \nabla_{S^2}\psi(q),\ s = \psi(q)\}.$$

Given a contact Hamiltonian $H(q,p,s,t)$, the contact Hamilton–Jacobi (HJ) condition for a stationary Legendrian invariant under X_H is

$$H\big(q, \nabla_{S^2}\psi(q), \psi(q)\big) = 0.$$

Along characteristics $(q(t), p(t), s(t))$,

$$\dot q = \partial_p H, \qquad \dot p = -\partial_q H - (\partial_s H)\, p, \qquad \dot s = p\cdot\partial_p H - H, \tag{1}$$

and on the Legendrian

$$\boxed{\nabla_{S^2}\psi\big(q(t)\big) = p(t)}.$$

A probability field on the *screen* $\mathbb{S}^2$ is the (contact) softmax of ψ:

$$\mu(\Omega)\ d\Omega \;=\; \frac{e^{\psi(\Omega)}}{\int_{\mathbb{S}^2} e^{\psi(\Omega)(\Omega)}\ d\Omega}\ d\Omega \quad (\text{discrete: } p_i = \frac{e^{\psi_i}}{\sum_j e^{\psi_j}}).$$

Below: (A) a minimal template for each domain, then (B) a worked example with explicit choices and consequences.

1. Optics / Geodesic Focusing (Hopf baseline)

A. Template

$$\begin{aligned} H(q,p,s) &= \tfrac{1}{2}\,\|p\|^2 + V(q) - E, \\ \text{HJ:}\quad \tfrac{1}{2}\,\|\nabla_{S^2}\psi(q)\|^2 + V(q) &= E, \\ \dot q = \partial_p H = p &= \nabla_{S^2}\psi(q), \\ \mu(\Omega) &\propto e^{\psi(\Omega)}. \end{aligned}$$

Interpretation: V encodes focusing/defocusing; ψ is an eikonal. Peaks of ψ bias μ (intensity).

B. Worked Example (local chart)

Work in a small chart around the north pole, identify $q \simeq (x,y)$ with the tangent plane. Take $V(x,y) = \kappa x$ (unidirectional lensing), $E > \max V$. Seek $\psi = \psi(x)$. Then $\frac{1}{2}(\psi_x)^2 + \kappa x = E \implies \psi_x = \sqrt{2(E-\kappa x)}$ (choose + branch). Integrating,

$$\psi(x) = -\frac{2}{3\kappa}\big(2(E-\kappa x)\big)^{3/2} + C, \qquad \mu(x,y)\ \propto\ \exp\Big[\psi(x)\Big].$$

As $\kappa > 0$ increases, rays tilt and μ skews toward $+x$, encoding *focusing bias.*

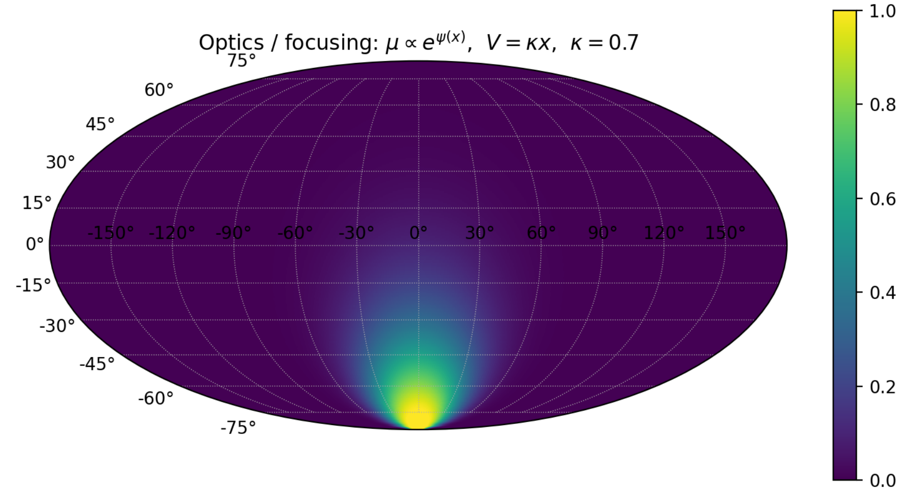

Figure 1: Optics / Geodesic focusing. Screen distribution $\mu \propto e^{\psi(x)}$ with potential $V = \kappa x$. Rays skew toward $+x$ as focusing strength κ increases.

2. Thermo / Replicator (screen dynamics)

A. Template

$$\begin{aligned} H(q,p,s) &= \Phi(q) - s + \tfrac{\lambda}{2}\,\|p\|^2, \\ \text{HJ:}\quad \Phi(q) - \psi(q) &+ \tfrac{\lambda}{2}\,\|\nabla_{S^2}\psi(q)\|^2 = 0, \\ \dot q = \partial_p H &= \lambda\, p = \lambda\, \nabla_{S^2}\psi(q), \\ \mu(\Omega) &\propto e^{\psi(\Omega)}. \end{aligned}$$

Discrete softmax of ψ yields classic replicator $\dot p_i = p_i(f_i - \bar f)$, with $f_i = \Phi_i$.

B. Worked Example (small-λ asymptotics)

Formally expand $\psi = \Phi + \lambda\psi_1 + O(\lambda^2)$. Plugging into HJ:

$$\Phi - (\Phi + \lambda\psi_1) + \tfrac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2 + O(\lambda^2) = 0 \;\Rightarrow\; \psi_1 = \tfrac{1}{2}\|\nabla_{S^2}\Phi\|^2.$$

Hence

$$\psi \;=\; \Phi + \frac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2 + O(\lambda^2), \qquad \mu \;\propto\; \exp\Big(\Phi + \tfrac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2\Big).$$

Interpretation: selection (Φ) amplified by curvature of the landscape.

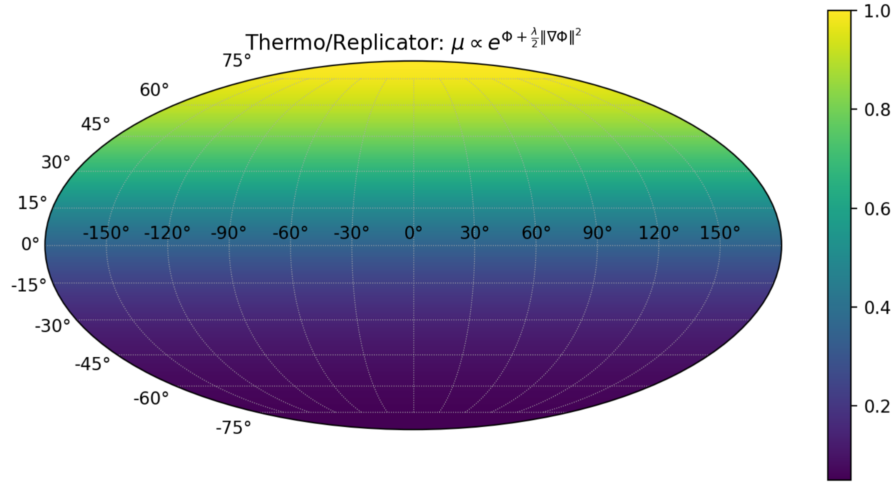

Figure 2: Thermo / Replicator dynamics. Probability field $\mu \propto e^{\Phi+\frac{\lambda}{2}\|\nabla\Phi\|^2}$ for directional selection $\Phi = \alpha(u\cdot q)$. The curvature correction amplifies selection gradients.

3. LLM Heads (continuous attention logits)

A. Template

$$\begin{gathered} H(q,p,s) = \Phi_{\text{attn}}(q;\text{ctx}) - s + \tfrac{\lambda}{2}\|p\|^2, \\ \text{HJ:} \quad \Phi_{\text{attn}} - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 = 0, \\ \mu(\Omega) \propto e^{\psi(\Omega)} \quad \text{(attention weights)}. \end{gathered}$$

B. Worked Example (query–key cosine score)

Let $\Phi_{\text{attn}}(q) = \beta + \kappa\,\langle u, q\rangle$ (unit $u \in \mathbb{S}^2$). From Section 2,

$$\psi \approx \Phi_{\text{attn}} + \frac{\lambda}{2}\|\nabla_{S^2}\Phi_{\text{attn}}\|^2.$$

On $\mathbb{S}^2$, $\nabla_{S^2}\langle u, q\rangle = u - (\langle u, q\rangle)\,q$, hence $\|\nabla_{S^2}\Phi_{\text{attn}}\|^2 = \kappa^2\|u-(u\cdot q)q\|^2 = \kappa^2(1-(u\cdot q)^2)$. Therefore

$$\psi(q) \;\approx\; \beta + \kappa(u\cdot q) + \frac{\lambda\kappa^2}{2}\big(1-(u\cdot q)^2\big), \qquad \mu(q) \;\propto\; \exp\big[\psi(q)\big].$$

This produces attention lobe(s) around u, sharpened by the quadratic correction.

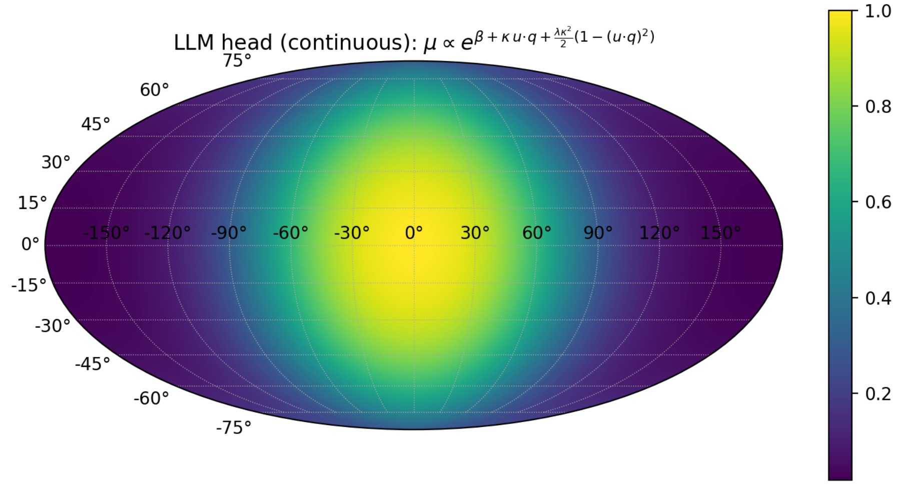

Figure 3: LLM heads (continuous attention). Distribution $\mu \propto e^{\psi}$ with attention potential $\Phi_{\mathrm{attn}} = \beta + \kappa(u \cdot q)$. The quadratic correction sharpens the attention lobe around u.

4. Electromagnetism (gauge-coupled optics)

A. Template

$$
\begin{aligned}
H(q,p,s) &= \tfrac{1}{2}\,\|p - A(q)\|^2 + \phi(q) - E, \\
\text{HJ:}\quad & \tfrac{1}{2}\,\|\nabla_{S^2}\psi(q) - A(q)\|^2 + \phi(q) = E, \\
\dot{q} &= \nabla_{S^2}\psi(q) - A(q), \qquad \mu(\Omega) \propto e^{\psi(\Omega)}.
\end{aligned}
$$

Gauge shifts change ψ by a scalar generating function; loop holonomy encodes Aharonov–Bohm type structure.

B. Worked Example (two-slit superposition on the screen)

Place two *effective* source lobes at angles $\theta = \pm\theta_0$ along a great circle. Model ray actions $S_{\pm}(\theta) = \kappa\cos(\theta \mp \theta_0) \pm \varphi_{\mathrm{AB}}$, where φ_{AB} encodes loop holonomy of A. Aggregate by log-sum-exp (contact superposition):

$$
\psi(\theta) = \log\!\Big(e^{S_+(\theta)} + e^{S_-(\theta)}\Big) = \kappa \log\!\Big(e^{\cos(\theta-\theta_0)} + e^{\cos(\theta+\theta_0)+2\varphi_{\mathrm{AB}}/\kappa}\Big).
$$

Then $\mu(\theta) \propto e^{\psi(\theta)}$ displays alternating dominance as φ_{AB} varies, yielding fringe-like modulation via gauge-controlled phase offset.

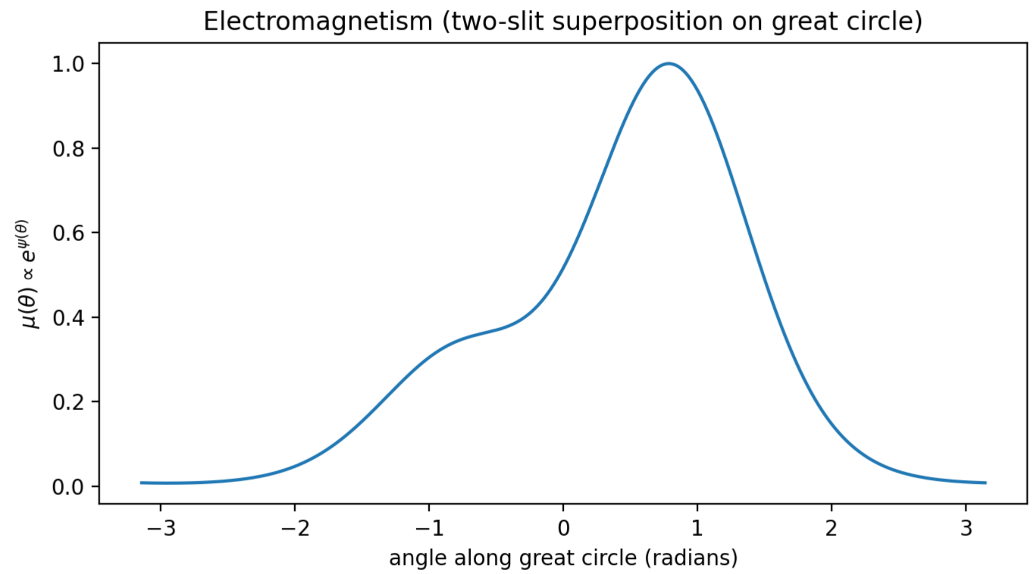

Figure 4: Electromagnetism (two-slit gauge interference). Superposition $\psi(\theta) = \log(e^{S_+} + e^{S_-})$ with Aharonov–Bohm phase offset φ_{AB}. The resulting $\mu(\theta)$ displays fringe-like modulation along a great circle.

5. Population Biology (replicator–mutator)

A. Template

Same H as Section 2 with Φ a fitness landscape on traits $q \in \mathbb{S}^2$. Optionally allow small viscosity $\nu > 0$ in the HJ equation to model mutation:

$$\Phi - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 - \nu\,\Delta_{S^2}\psi = 0.$$

B. Worked Example (directional selection)

Let $\Phi(q) = \alpha\langle u, q\rangle$ with unit u. For small λ, ν, expand:

$$\psi \approx \Phi + \frac{\lambda}{2}\|\nabla_{S^2}\Phi\|^2 + \nu\,(\text{harmonic correction}) = \alpha(u\cdot q) + \frac{\lambda\alpha^2}{2}\big(1 - (u\cdot q)^2\big) + O(\nu).$$

Hence $\mu \propto \exp(\alpha(u\cdot q) + \cdots)$ concentrates along u; ν spreads mass (mutation).

Discrete check (3 types). Let $f = (f_1, f_2, f_3)$ and softmax $p_i \propto e^{\psi_i}$ with $\psi_i \approx f_i$. Then $\dot{p}_i = p_i(f_i - \bar{f})$, $\bar{f} = \sum_j p_j f_j$, reproducing the classic replicator.

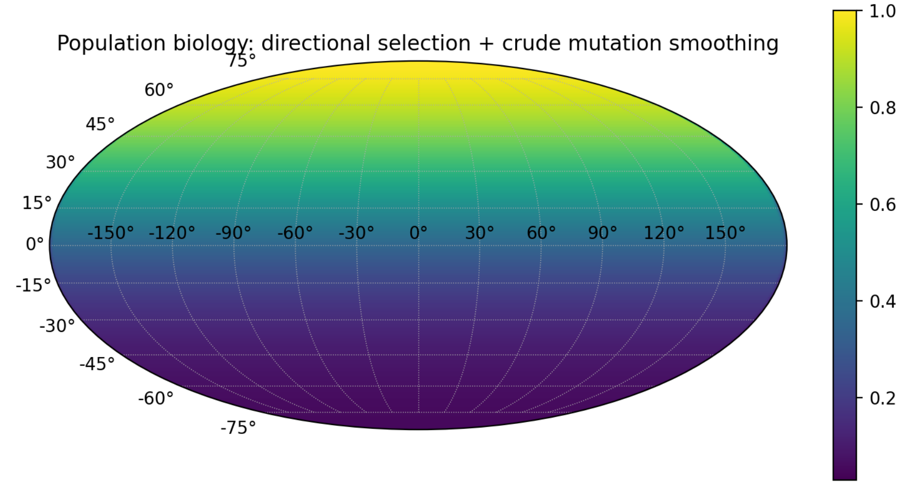

Figure 5: Population biology. Directional selection plus mutation smoothing produces a broadened probability field μ. Mutation spreads mass away from the fittest trait direction.

6. Biological Morphology (multiscale competency)

A. Template

$$H(q,p,s) = \Big(\Phi_{\text{morph}}(q) - s\Big) + \tfrac{\lambda}{2}\|p\|^2 + \sum_k \gamma_k \left(\psi(q) - \psi_k(q)\right),$$
$$\text{HJ:} \quad \Phi_{\text{morph}} - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 + \sum_k \gamma_k(\psi - \psi_k) = 0.$$

Here ψ_k are mesoscale controllers (cells/tissues/organs); $\gamma_k > 0$ measure cross-scale alignment pressure.

B. Worked Example (two-scale compromise, small gradients)

Let one cellular controller ψ_1 and one tissue controller ψ_2 with weights γ_1, γ_2. Neglect the gradient term (or take $\lambda \to 0$) to get an explicit closed form:

$$(1+\gamma_1+\gamma_2)\,\psi \;=\; \Phi_{\text{morph}} + \gamma_1\psi_1 + \gamma_2\psi_2 \quad \Rightarrow \quad \psi \;=\; \frac{\Phi_{\text{morph}} + \gamma_1\psi_1 + \gamma_2\psi_2}{1+\gamma_1+\gamma_2}.$$

Thus $\mu \propto e^{\psi}$ realizes a consensus morphology as a *soft* weighted average across scales; turning on small λ adds curvature corrections that penalize sharp spatial disagreements.

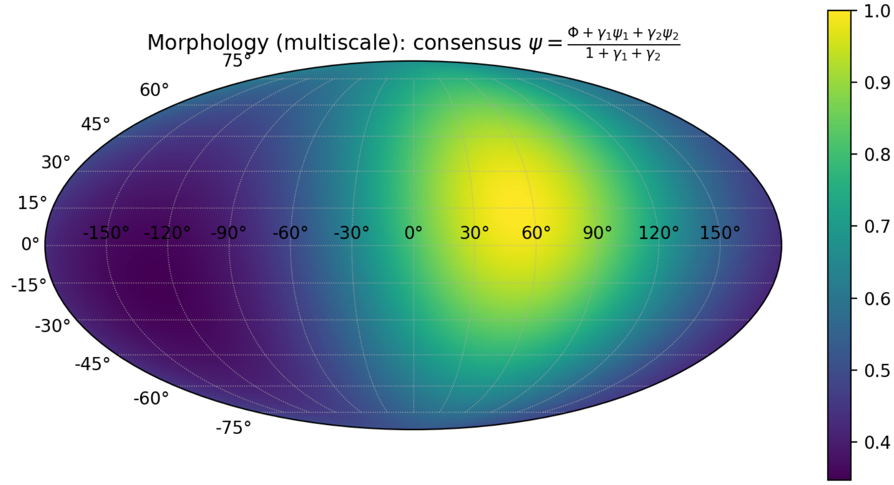

Figure 6: Biological morphology (multiscale competency). Consensus potential $\psi = (\Phi_{\text{morph}} + \gamma_1\psi_1 + \gamma_2\psi_2)/(1+\gamma_1+\gamma_2)$ yields a probability field μ that balances across scales (cell, tissue, whole form).

7. Materials Science (free-energy & phase fractions)

A. Template

$$H(q,p,s) = F(q) - s + \tfrac{\lambda}{2}\,\|p\|^2,$$
$$\text{HJ:}\quad F - \psi + \tfrac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 = 0, \qquad \mu \propto e^{\psi}.$$

F is an effective free energy over variant/orientation states $q \in \mathbb{S}^2$. Adding viscosity $\nu\,\Delta_{S^2}\psi$ produces Allen–Cahn/Cahn–Hilliard-type relaxations on the screen after projection.

B. Worked Example (two-variant texture)

Let $u, v \in \mathbb{S}^2$ be two variant axes with

$$F(q) = \alpha\,(1 - (u\cdot q)^2) + \beta\,(1 - (v\cdot q)^2), \qquad \alpha, \beta > 0.$$

Small-λ expansion gives

$$\psi \approx F + \frac{\lambda}{2}\|\nabla_{S^2}F\|^2, \quad \nabla_{S^2}(u\cdot q) = u - (u\cdot q)q,$$

so $\mu \propto \exp(\psi)$ concentrates near $q \parallel u$ and $q \parallel v$ with weights controlled by (α, β) and by curvature corrections from $\|\nabla_{S^2}F\|^2$. Including a small viscosity term smooths grain boundaries and yields screen-level coarsening dynamics.

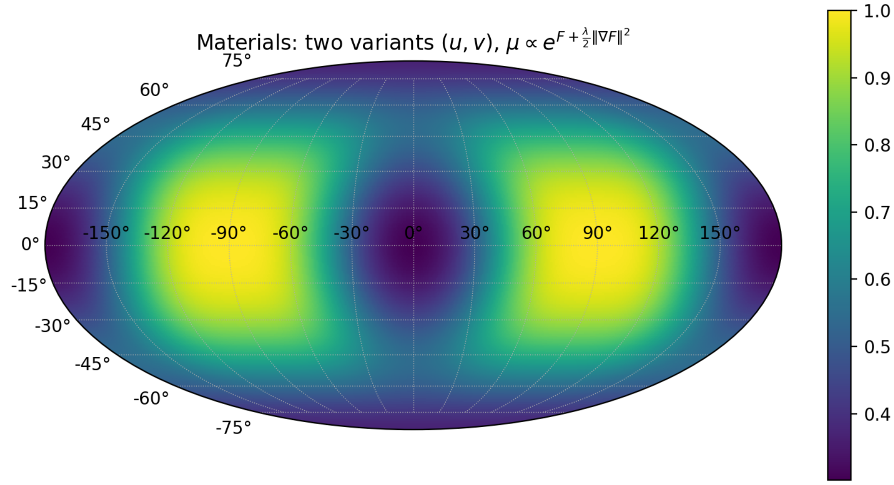

Figure 7: Materials science (two-variant texture). Free-energy potential $F(q) = \alpha(1 - (u\cdot q)^2) + \beta(1 - (v\cdot q)^2)$ produces $\mu \propto e^{F+\frac{\lambda}{2}\|\nabla F\|^2}$ concentrated near the variant axes u, v.

Implementation Notes and Variants

(i) Vanishing-viscosity regularization. For convex Hamiltonians, the stationary HJ equations with a small $\nu\,\Delta_{S^2}\psi$ admit unique viscosity solutions; taking $\nu \downarrow 0$ selects physically relevant weak solutions.

(ii) Log-sum-exp composition for multi-source scenes. When multiple Legendrian branches (paths/controllers) contribute with actions S_k, a natural contact composition is $\psi = \log \sum_k e^{S_k}$, which preserves convexity and yields mixture-like μ.

(iii) From ψ to dynamics of μ. Given $\mu \propto e^{\psi}$, ψ evolving by a relaxation $\partial_t \psi = \Phi - \psi + \frac{\lambda}{2}\|\nabla_{S^2}\psi\|^2 + \nu\,\Delta_{S^2}\psi$ induces a transport-diffusion on μ; in discrete settings this reduces to replicator–mutator.

(iv) Geometry on $\mathbb{S}^2$. All gradients/Laplacians are w.r.t. the round metric on $\mathbb{S}^2$. In small charts, replace ∇_{S^2} by Euclidean ∇ for local derivations; for global analysis, expand in spherical harmonics.