Mobile Robot Kinematics

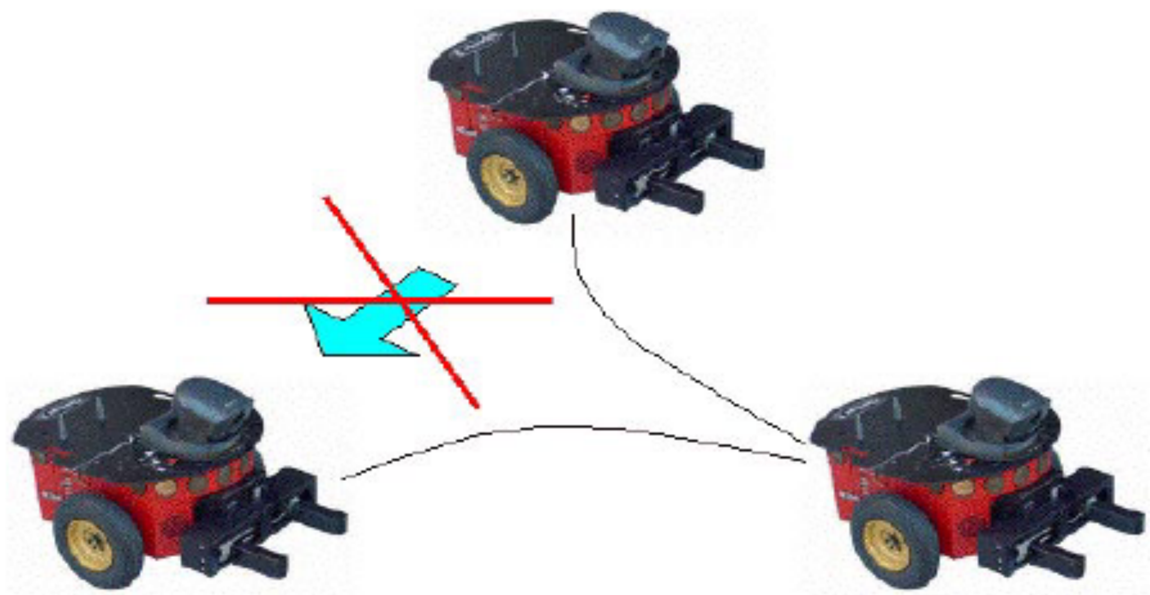
Alfredo Weitzenfeld

Dynamics Versus Kinematics

- **Dynamics** – study of motion where forces are modeled
  - The effect of all forces (internal and external) on a robot's motion.
- **Kinematics** – study of the mathematics of motion without considering the forces that affect the motion
  - The effect of a robot's geometry on its motion.

Non-holonomic Robots

- Robot can move in some directions (forwards and backwards), but not others (side to side).
- For a two-wheeled **differential drive**, the robot can instantly move forward and back, but it can not move to the right or left without the wheels slipping.



Holonomic Robots

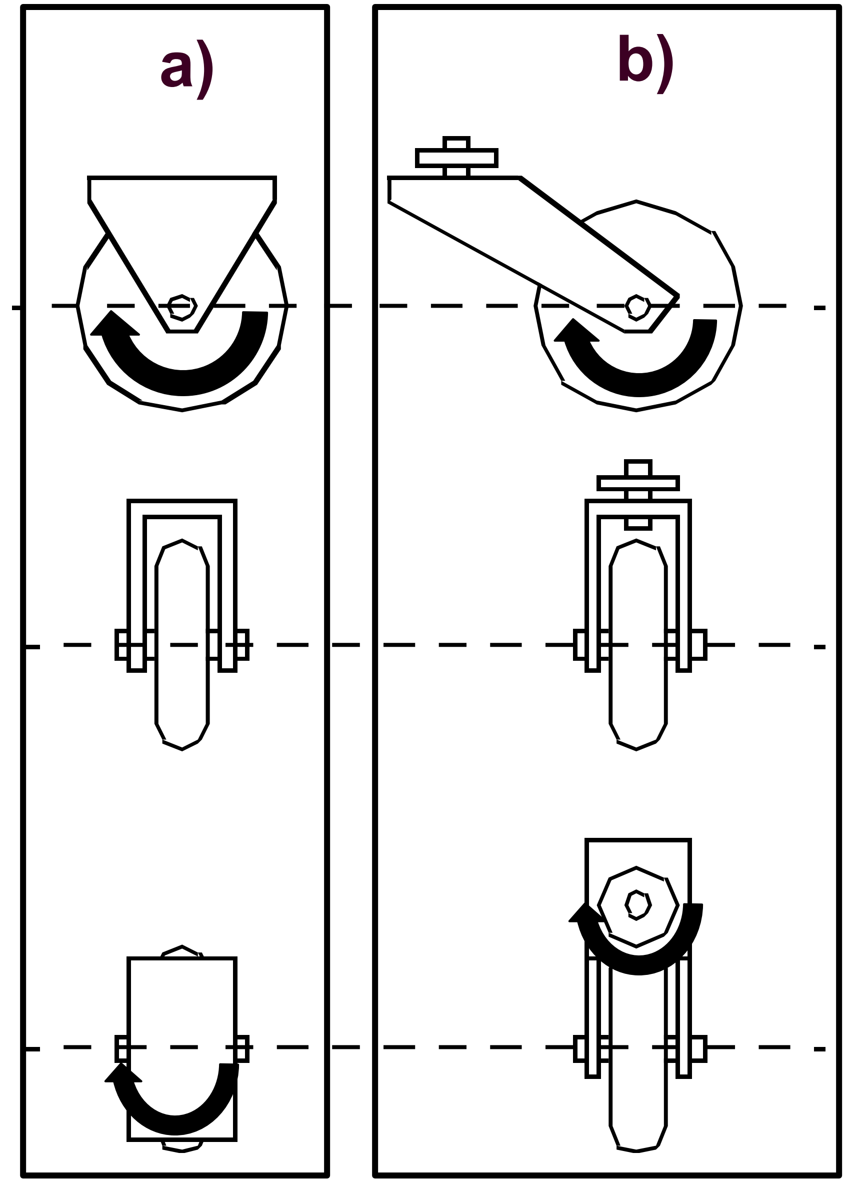
- Navigation is simplified considerably if a robot can move instantaneously in any direction, i.e., **holonomic** or **omnidirectional.**

Motorized Versus Castor Wheels

- **Motorized Wheels**
  - Wheel rotation controlled by a motor
- **Castor Wheels**
  - Wheel rotation controlled by ground friction (not by motor).

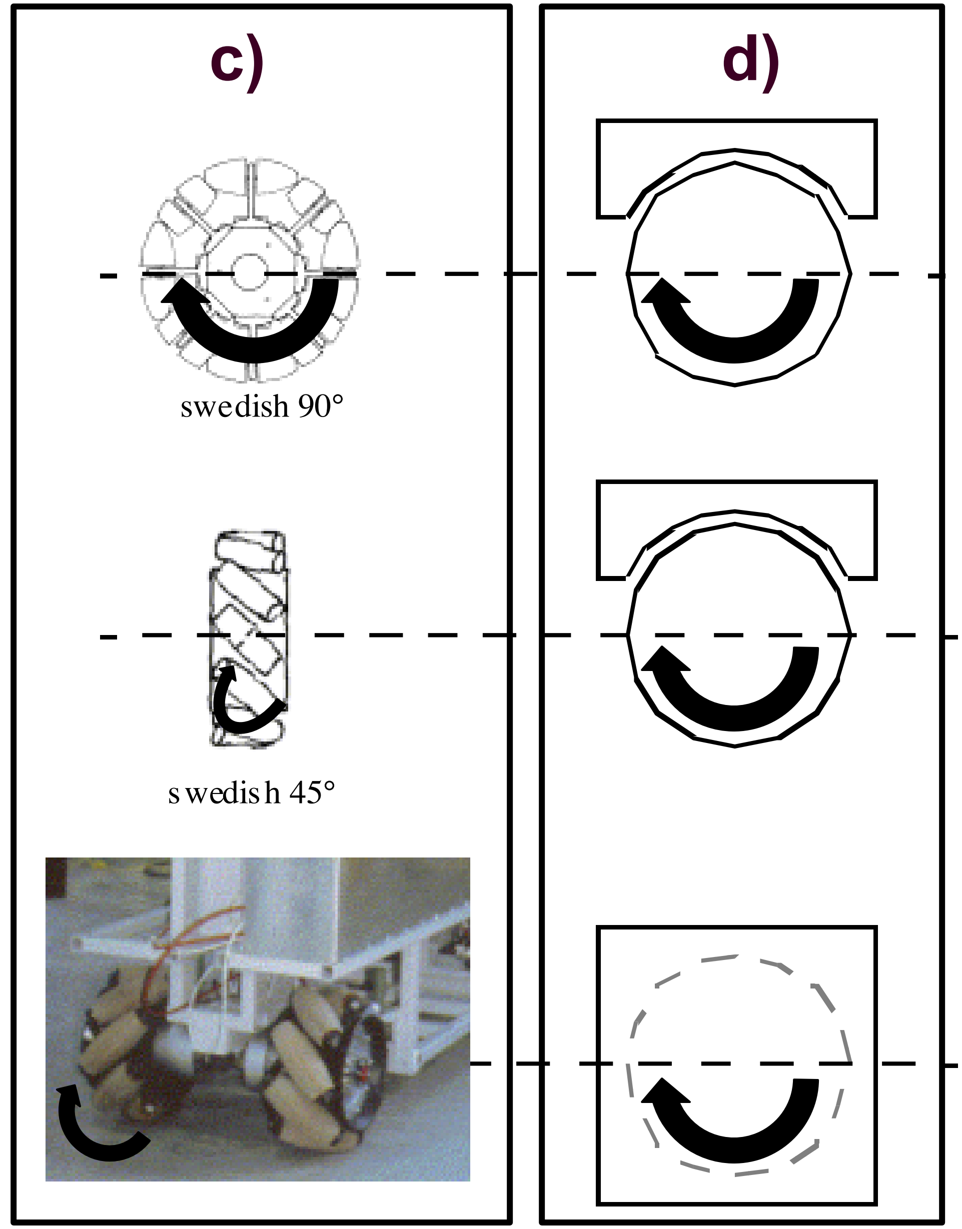
Basic Types of Wheels

- a) **Standard wheel**: 2-degrees of freedom:
  - Rotation around the wheel axle
  - Rotation around the ground steering point
- b) **Offset wheel**: 2-degrees of freedom:
  - Rotation around the wheel axle
  - Rotation around an offset steering joint



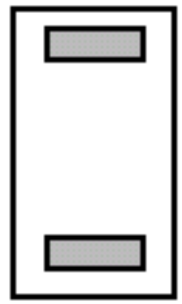
Basic Types of Wheels

- c) **Swedish wheel**: 3-degrees of freedom:
  - Rotation around the wheel axle
  - Rotation around the rollers
  - Rotation around the ground steering point
- d) **Spherical wheel**: 3-degrees of freedom:
  - Rotations in 360 degrees

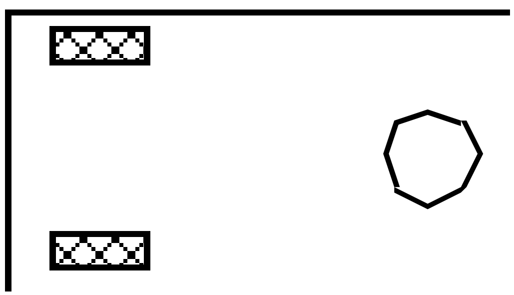


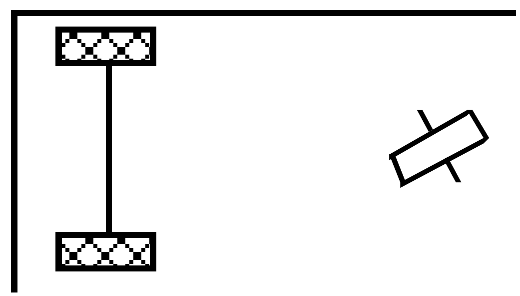
Arrangements of Wheels

- Two wheels

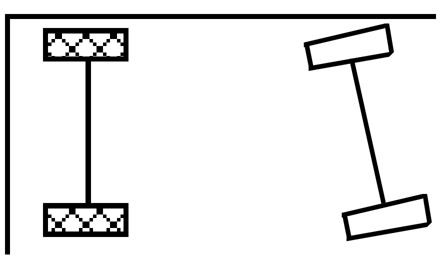


- Three wheels

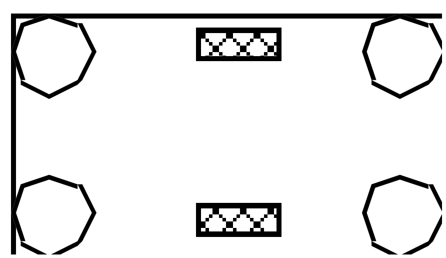




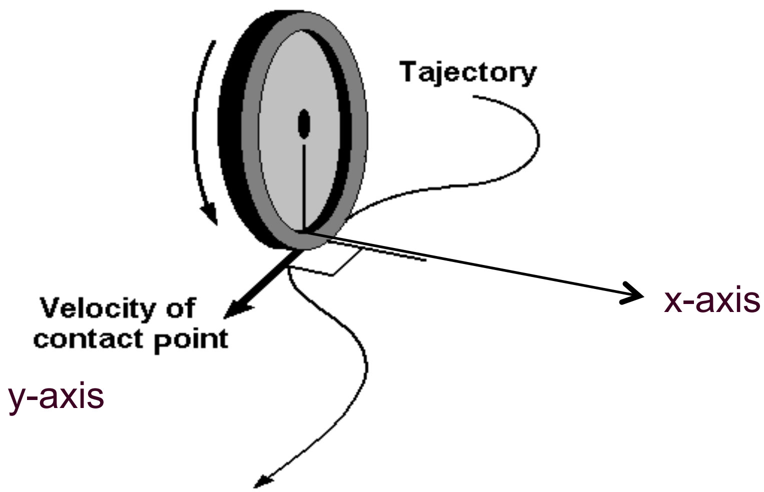
- Four wheels



Six wheels



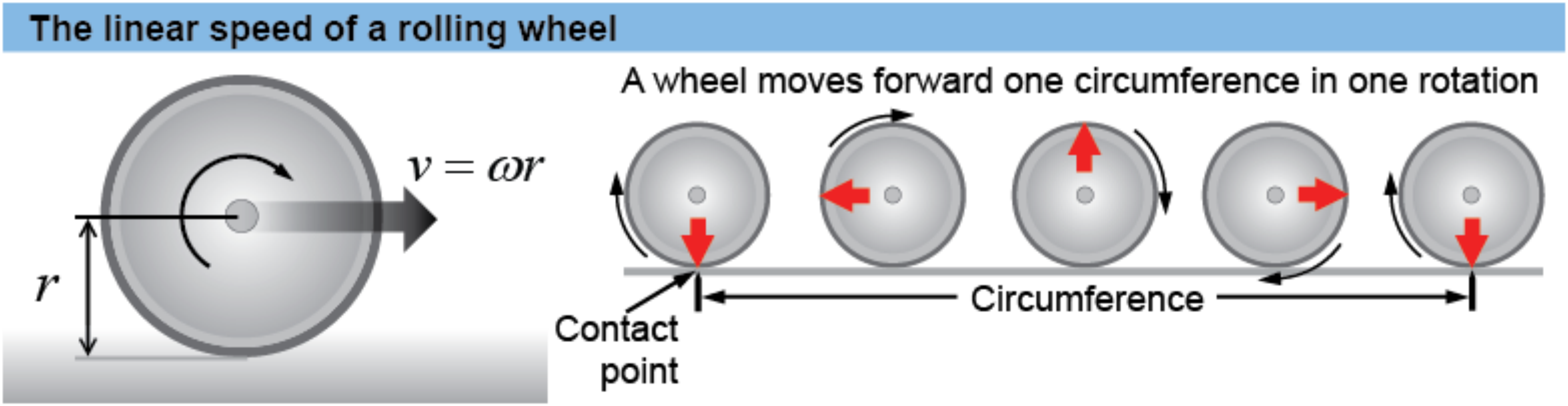
Idealized Rolling Wheel



Non-slippage and pure rolling

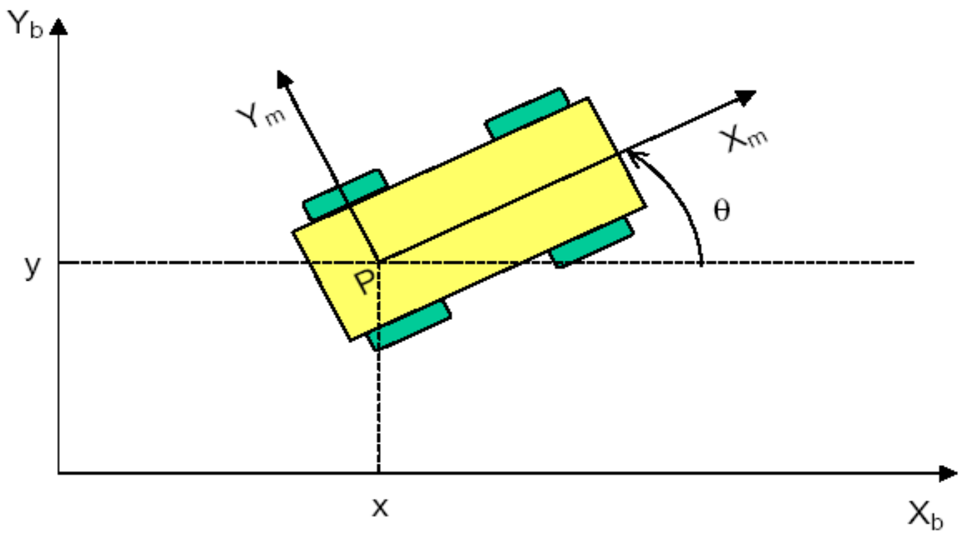
- Assumptions:
  - No slippage occurs in the orthogonal direction of rolling (no side slippage in x-axis).
  - Pure rolling between the wheel and the floor (no translation slippage in y-axis).
  - At most one steering link per wheel with steering axis perpendicular to the floor (steering in x-axis).

Wheel Velocity



- Wheel parameters:
  - r = wheel radius
  - v = wheel forward linear velocity
  - ω = wheel angular velocity
- Wheel forward linear velocity:
  - $v = \omega r$

Robot Pose



Wheeled Robot Pose:
- Position *P(x, y)*
- Orientation θ

$\{X_M, Y_M\}$ – Moving Frame (Local Frame)

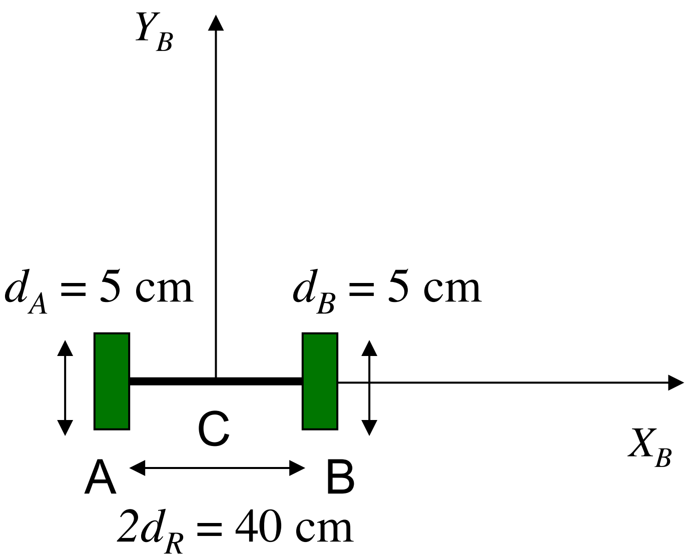
$\{X_B, Y_B\}$ – Base Frame (Global Frame for Robot Pose)

Forward Versus Inverse Kinematics

- ***Forward Kinematics***
  - Where will be the final robot pose after a sequence of motor controls starting from a start robot pose ?
  - Where will be the final robot pose ?

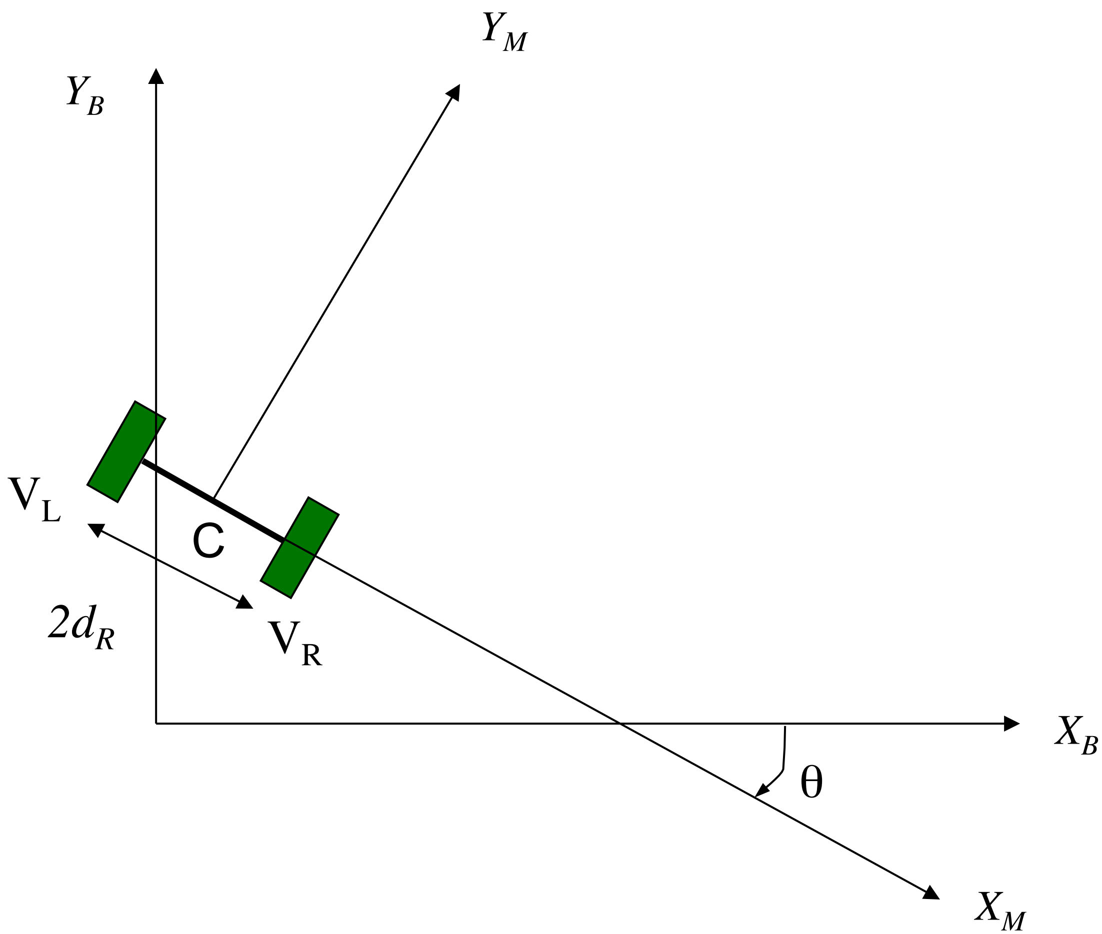
- ***Inverse Kinematics***
  - What should be the sequence of motor controls in order to get to a final robot pose from a start robot pose ?
  - What should be the motor control sequence ?

Forward Kinematics



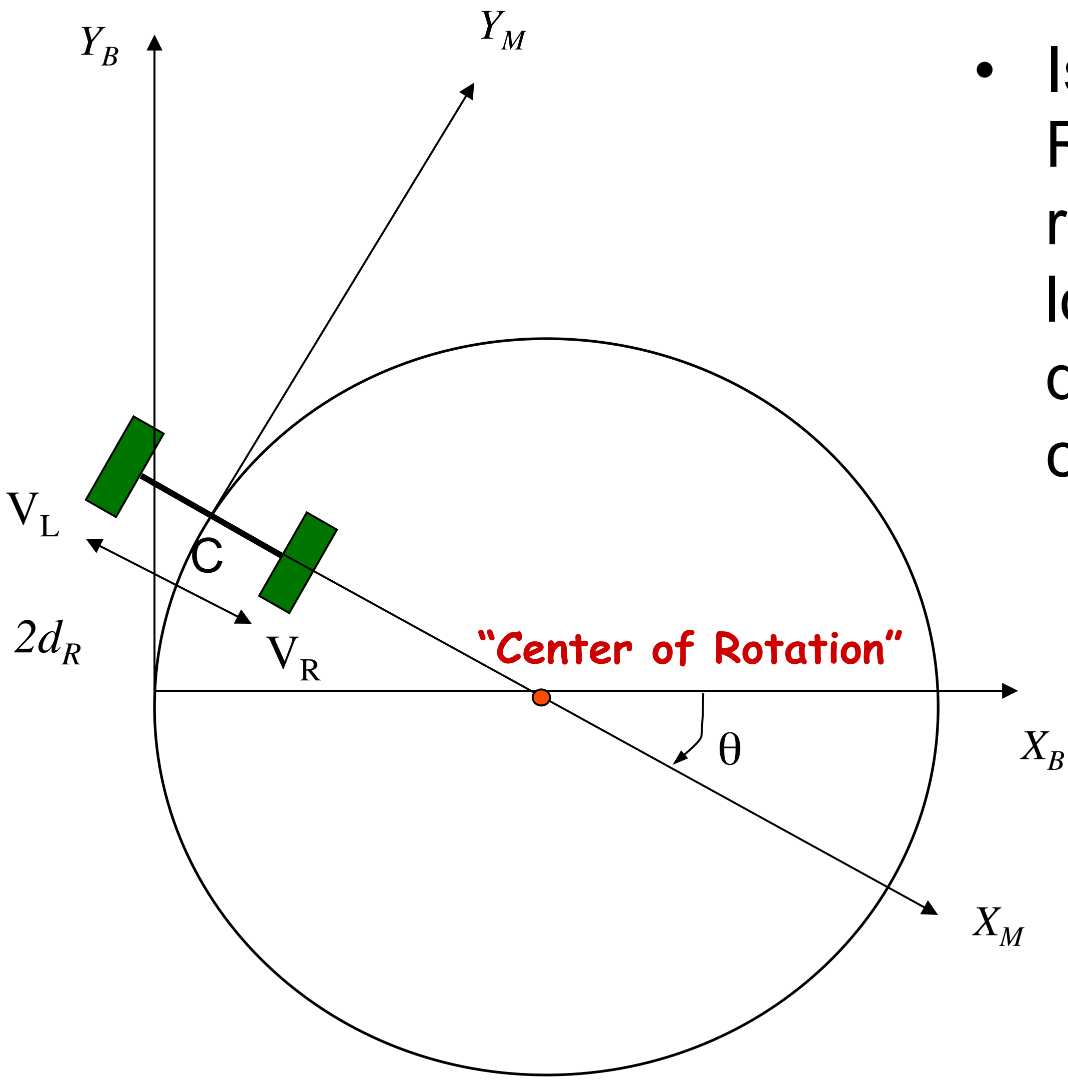
- Assume a robot configuration as shown in the figure.
- Start robot pose computed at center **C** = (0, 0, 90^o)
- Wheels **A** and **B** have a diameter d_A and d_B of 5 cm each.
- Robot axis distance, $2d_R$, between **A** and **B** wheels is 40 cm (distance d_R is computed between center **C** and each wheel).
- The robot rotates its wheels as follows:
  - Wheel **A** makes 8 full rotations
  - Wheel **B** makes 6 full rotations
- What is the final robot pose for **C**?

Differential Drive



- Difference in wheel speeds determines robot turning angle θ.
- Will the robot turn left or right ?
- The robot wheels' velocities are as follows:
    - V_A = 8 rotations in time T
    - V_B = 6 rotations in time T

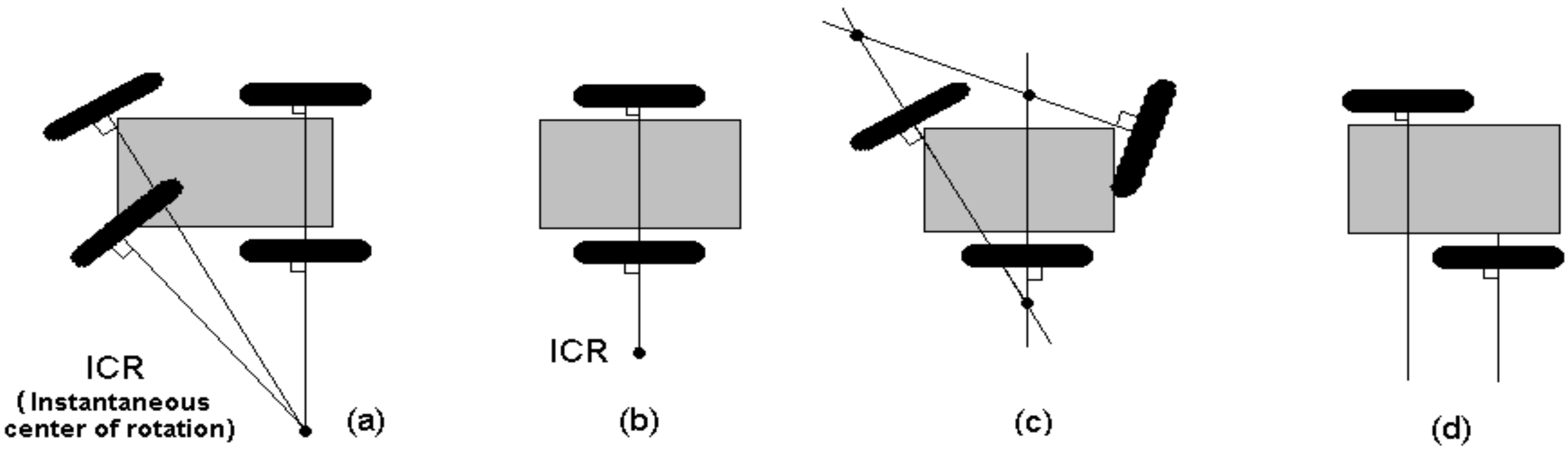
Differential Drive



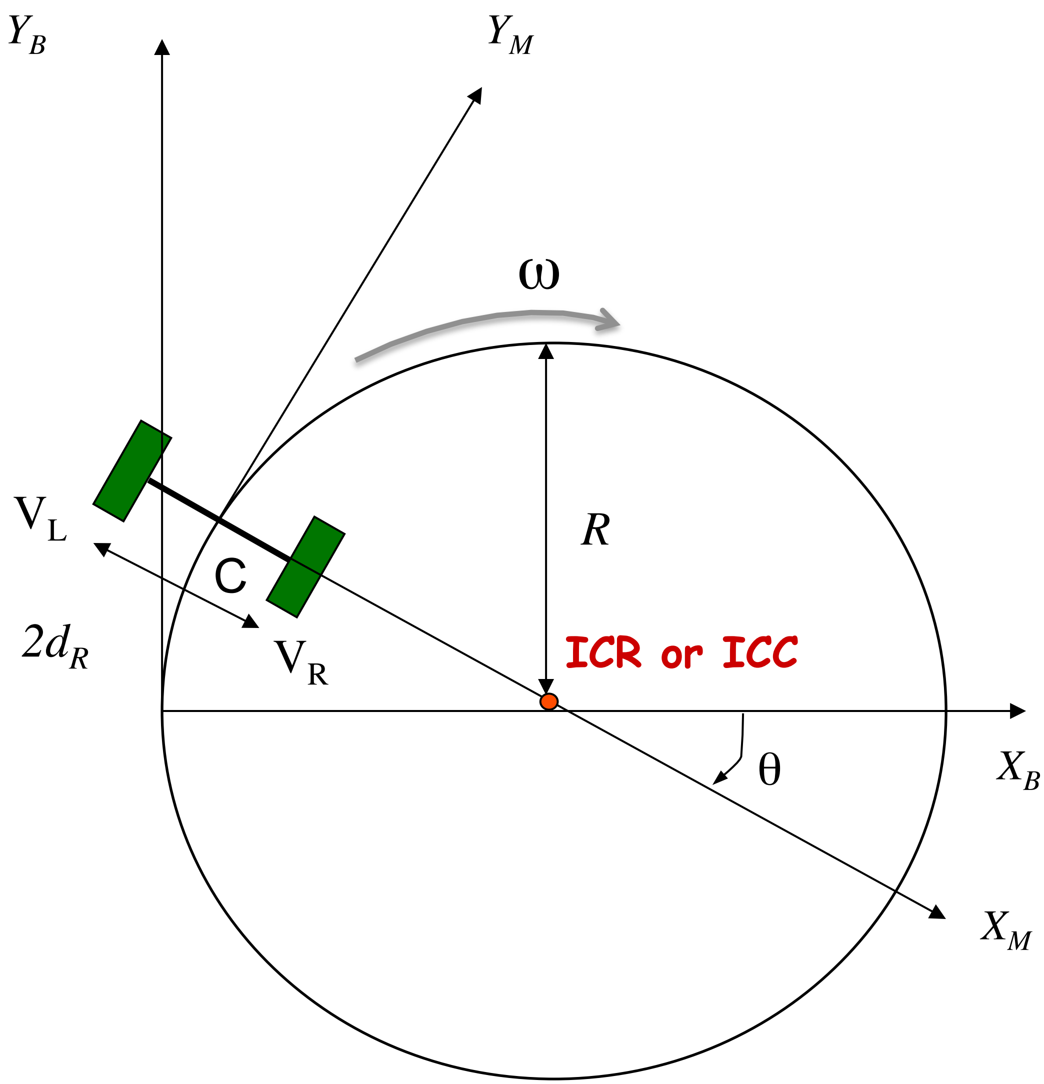
- Is there a point, or "Center of Rotation", around which the robot is always rotating (as long as wheels relative differential drive remains constant) ?

Instantaneous Center of Curvature

- Instantaneous center of rotation (ICR) or Instantaneous center of curvature (ICC)
  - *A cross point of all axes of the wheels*

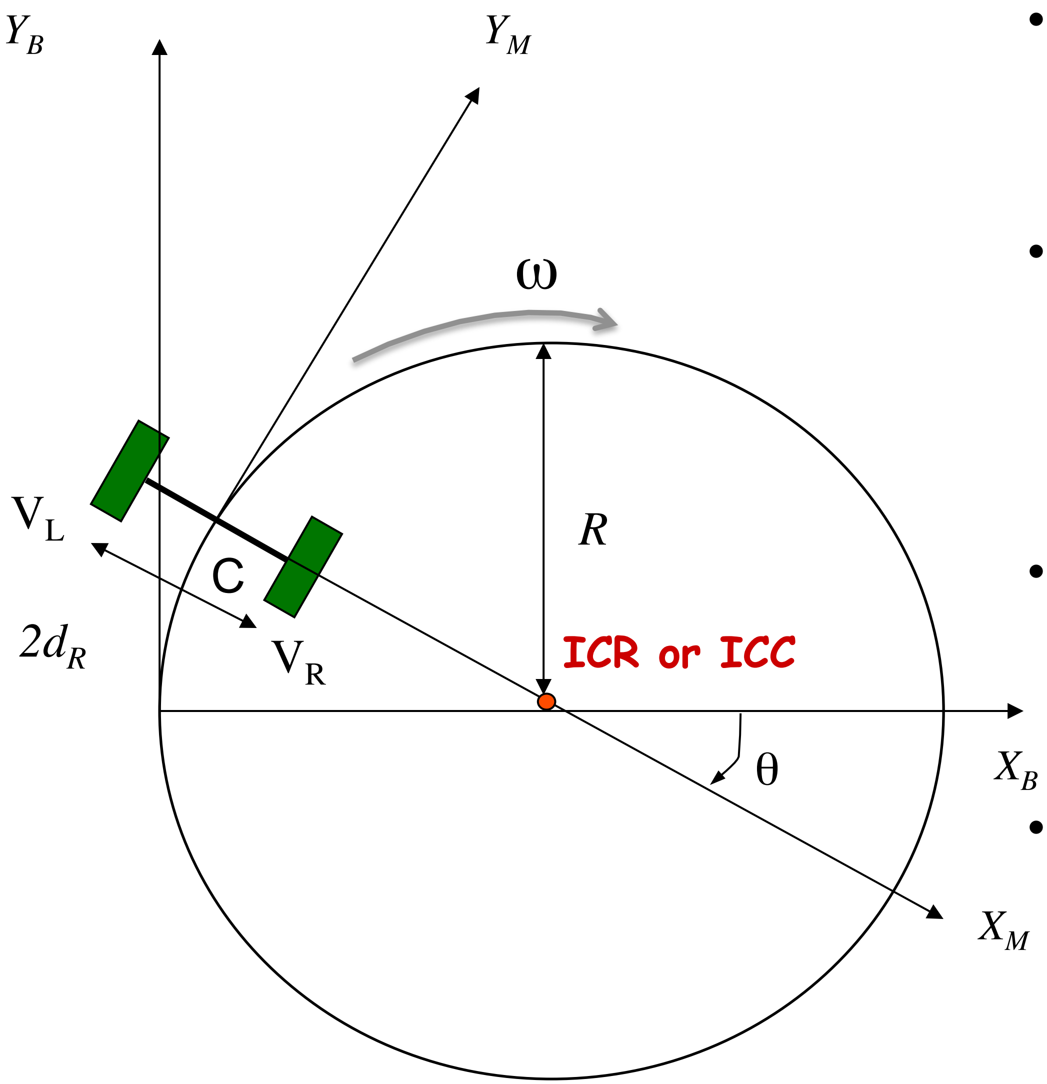


Differential Drive



- To minimize wheel slippage, the instantaneous center of curvature (**ICC**) must lie at the intersection of the wheels' axles.
- Need to determine the point (**ICC**) and radius *R* around which the robot is turning.
- Each wheel must be traveling at the same angular velocity ω with respect to ICC.
- Need to determine the linear velocities of the robot, V_L and V_R

Differential Drive



- Constant ω around ICC.
- Determine linear velocities V_L and V_R

$$V_L = \omega(R+d_R)$$
$$V_R = \omega(R-d_R)$$

- By subtracting the two equations:

$$V_R - V_L = (\omega R - \omega d_R) - (\omega R + \omega d_R)$$
$$V_R - V_L = -2\omega d_R$$
$$\omega = (V_L - V_R) / 2d_R$$

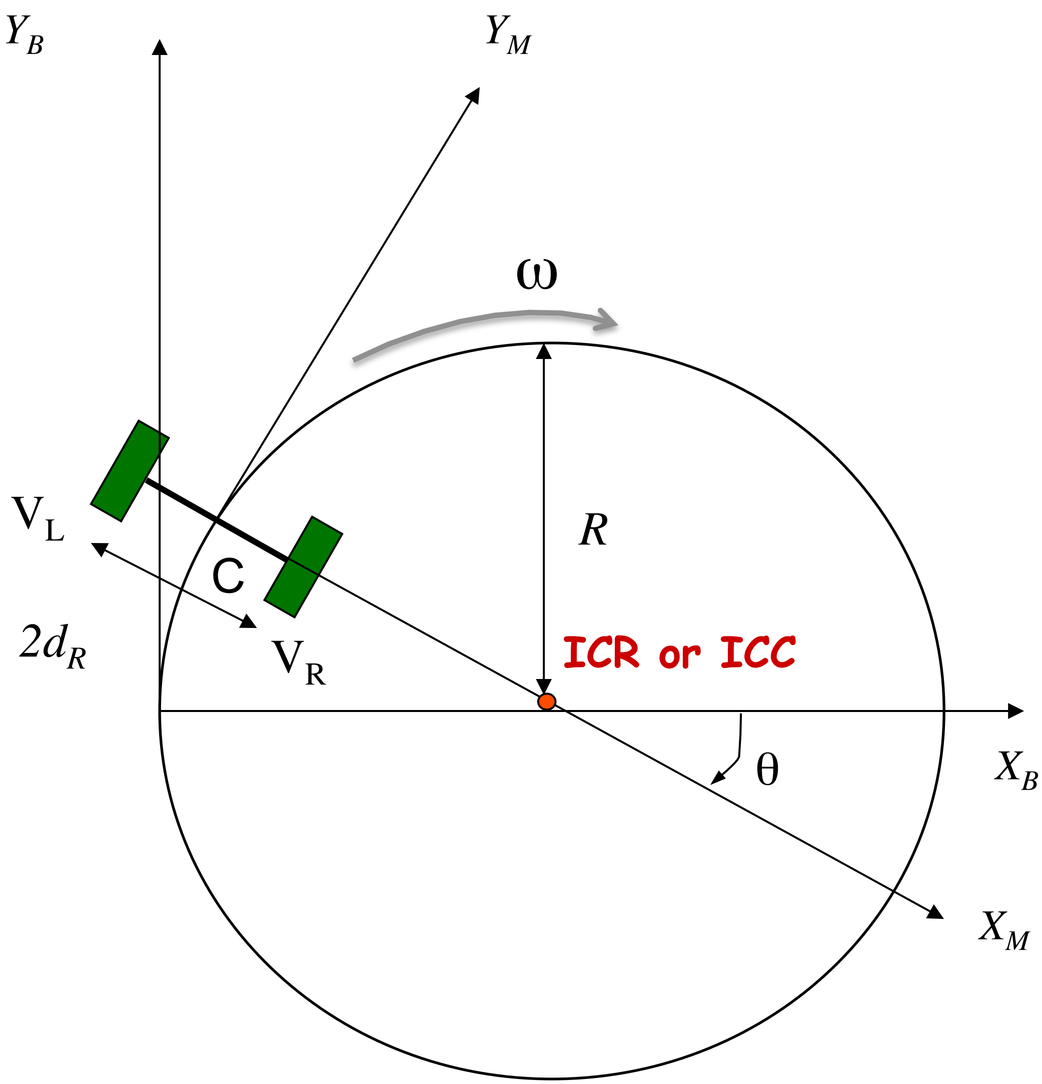
- By adding the two equations:

$$V_R + V_L = (\omega R - \omega d_R) + (\omega R + \omega d_R)$$
$$V_R + V_L = 2\omega R$$

- By inserting the value of ω:

$$V_R + V_L = 2((V_L - V_R) / 2d_R)R$$
$$(V_R + V_L) / (V_L - V_R) = R/d_R$$
$$R = d_R (V_R + V_L) / (V_L - V_R)$$

Differential Drive



- Determine velocity V at C

$$\mathbf{V = \omega R}$$

$$\mathbf{\omega = (V_L - V_R) / 2d_R}$$

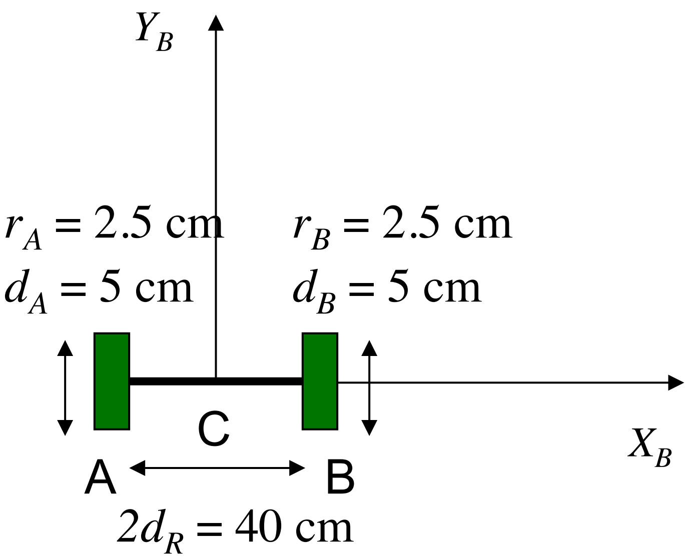
$$\mathbf{R = d_R (V_R + V_L) / (V_L - V_R)}$$

$$\mathbf{V = ((V_L - V_R) / 2d_R)^* (d_R (V_R + V_L) / (V_L - V_R))}$$

$$\mathbf{V = (d_R (V_L - V_R) (V_R + V_L) / (2d_R (V_L - V_R))}$$

$$\mathbf{V = (V_R + V_L) / 2}$$

Differential Drive



- Start robot pose computed at center **C** = (0, 0, 90°)
- Wheels **A** and **B** have a diameter d_A and d_B of 5 cm each (r_A and r_B = 2.5 cm each).
- The robot rotates its wheels as follows:
  - Wheel **A** makes 8 full rotations
  - Wheel **B** makes 6 full rotations

$V_L = V_A = 8 \text{ rotations} = 8 * 2\pi r_A = 8 * 2\pi * 2.5 = 8 * \pi * 5 = 40\pi$

$V_R = V_B = 6 \text{ rotations} = 6 * 2\pi r_B = 6 * 2\pi * 2.5 = 6 * \pi * 5 = 30\pi$

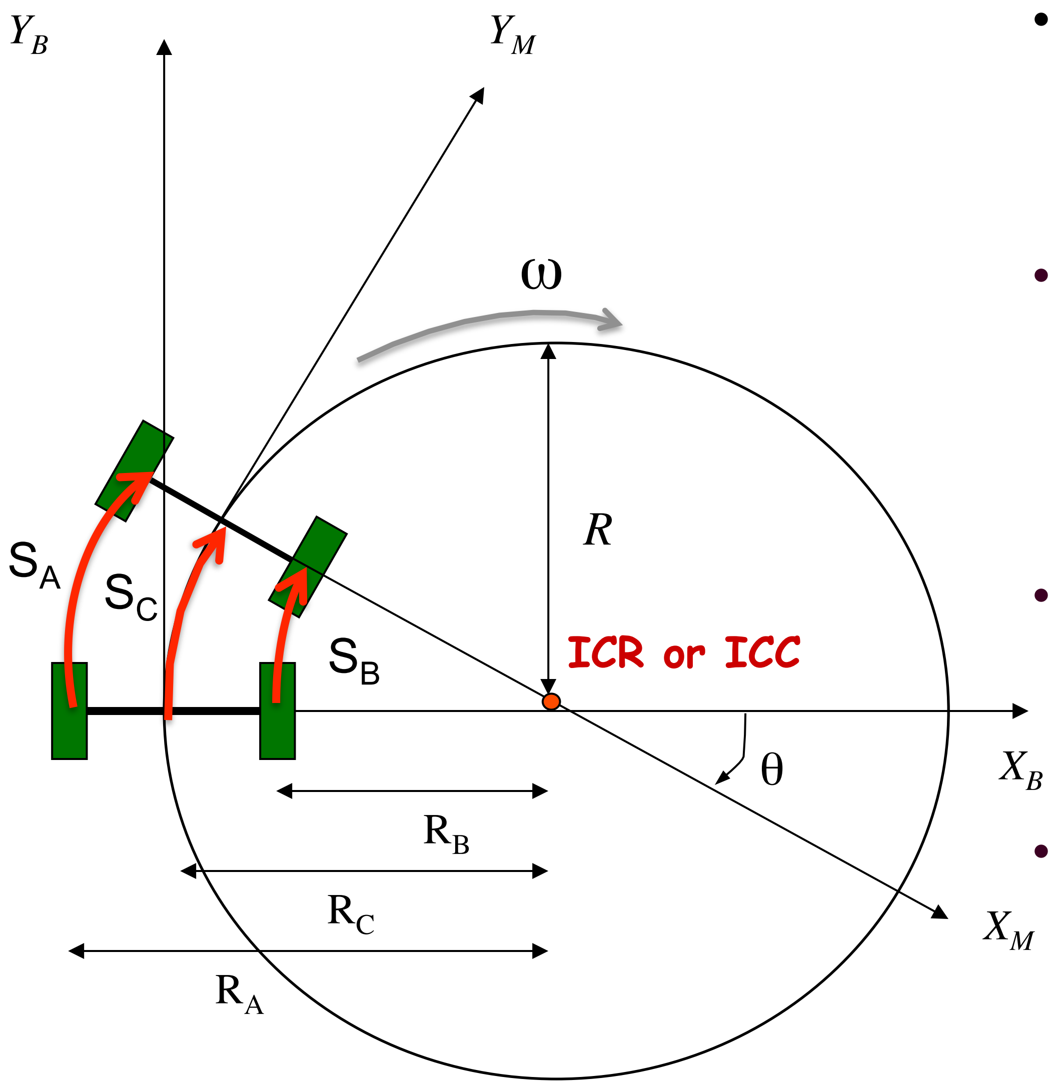
$V = V_C = (V_R + V_L) / 2 = (40\pi+30\pi) / 2 = 35\pi$

$\omega = (V_L - V_R) / 2d_R = (40\pi-30\pi) / 40 = 10\pi / 40 = \pi / 4$

$R = d_R (V_R+V_L) / (V_L-V_R) = 20 (40\pi+30\pi) / (40\pi-30\pi) = 20*70/10 = 140$

$V = \omega R = 140\pi / 4 = 35\pi$

Differential Drive



- Determine final robot pose at C
- S is arc length and R is the circle radius, both vary depending on their distance to ICC.

$$S = R * \theta$$

- The arc lengths S_A, S_B, S_C were already computed.

$$S_A = R_A * \theta = 40\pi$$
$$S_B = R_B * \theta = 30\pi$$
$$S_C = R_C * \theta = 35\pi$$

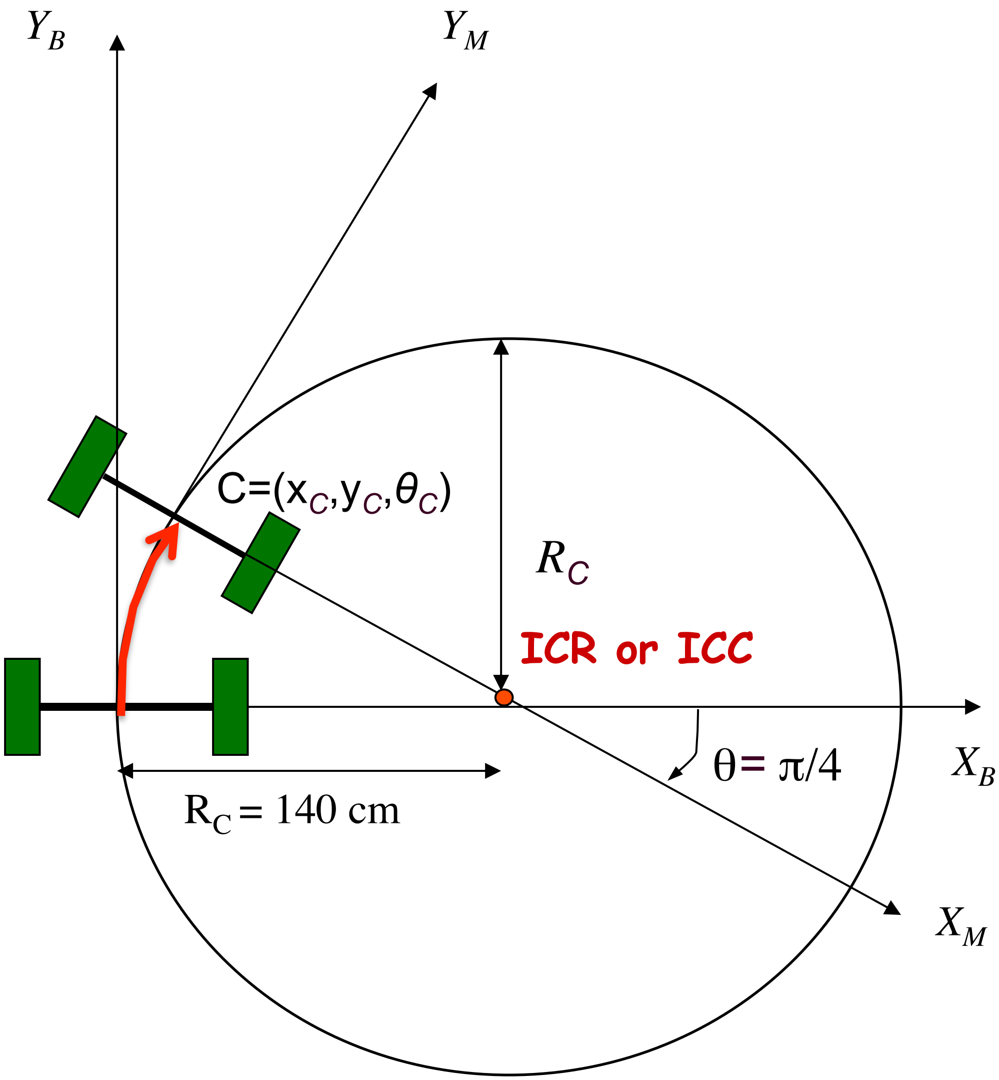
- The radius R was already computed

$$R_A = R+d_R = 140+20 = 160$$
$$R_B = R-d_R = 140-20 = 120$$
$$R_C = R = 140$$

- θ is the angle of rotation between the two frames, the same for all arcs.

$$\theta = S_A / R_A = S_B / R_B = S_C / R_C$$
$$\theta = 40\pi/160 = 30\pi/120 = 35\pi/140$$
$$\theta = \pi/4 = 45^o$$

Differential Drive



- Determine final robot pose at C(x_C, y_C, θ_C).

$$\theta = \pi/4, R_C = 140$$

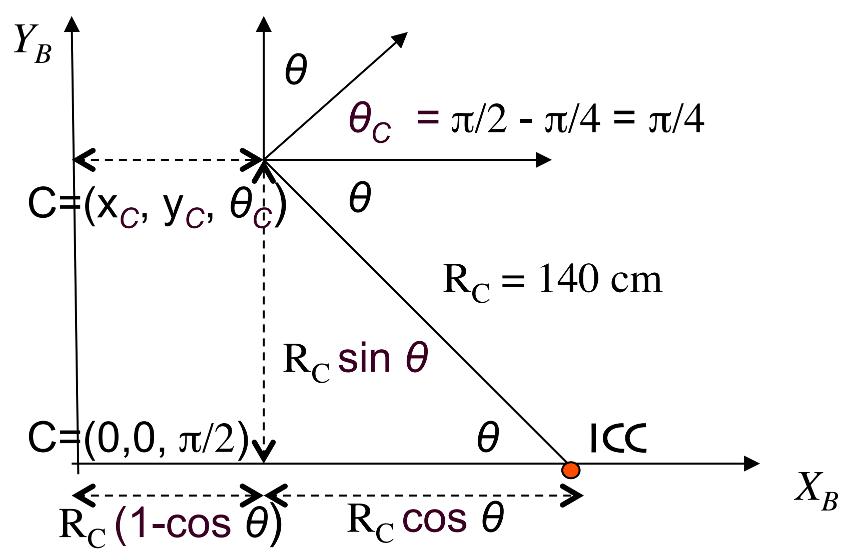
$$x_C = R_C (1 - \cos \theta_C) = 140 (1 - \cos \pi/4)$$

$$x_C = 140 (1 - \cos \pi/4)$$

$$y_C = R_C \sin \theta_C = R_C \sin \pi/4$$

$$y = 140 \sin \pi/4$$

$$C = (140 (1 - \cos \pi/4), 140 \sin \pi/4, \pi/4)$$



Differential Drive

Forward Kinematics

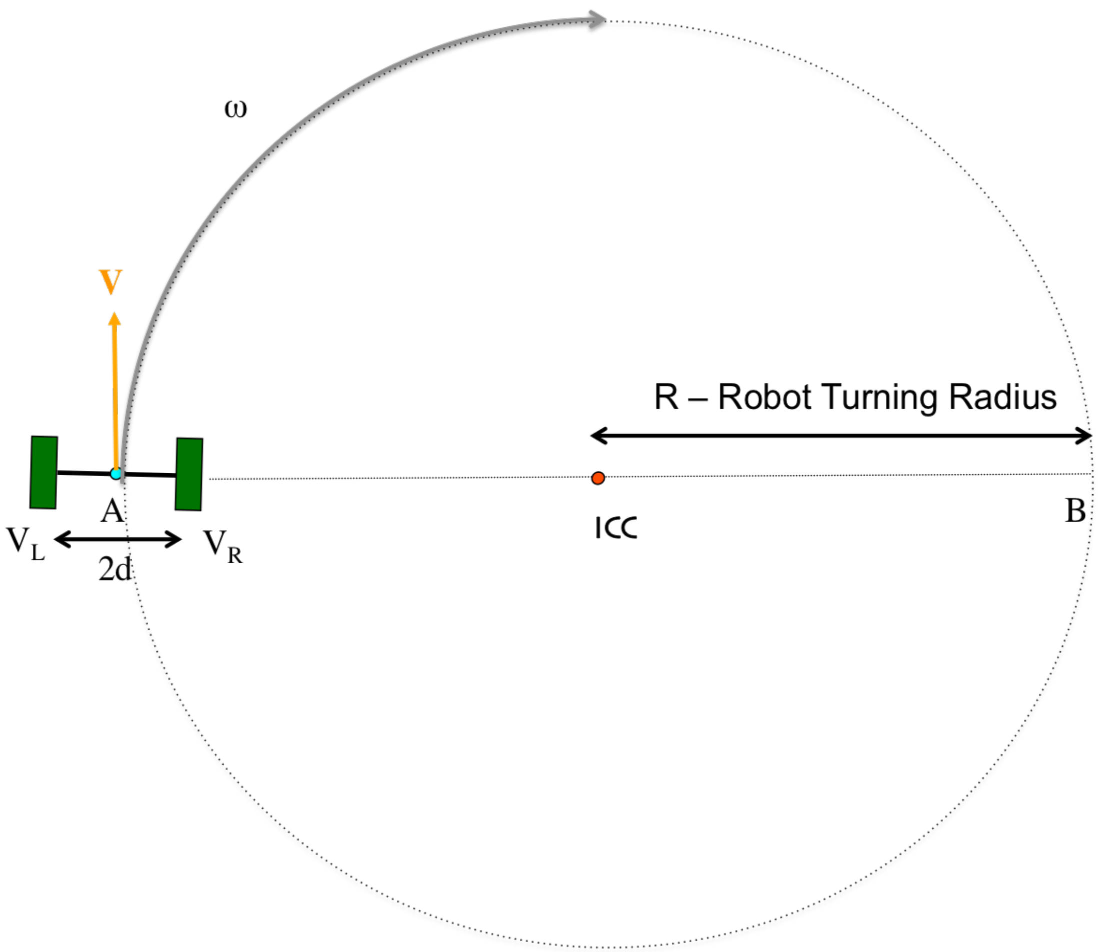


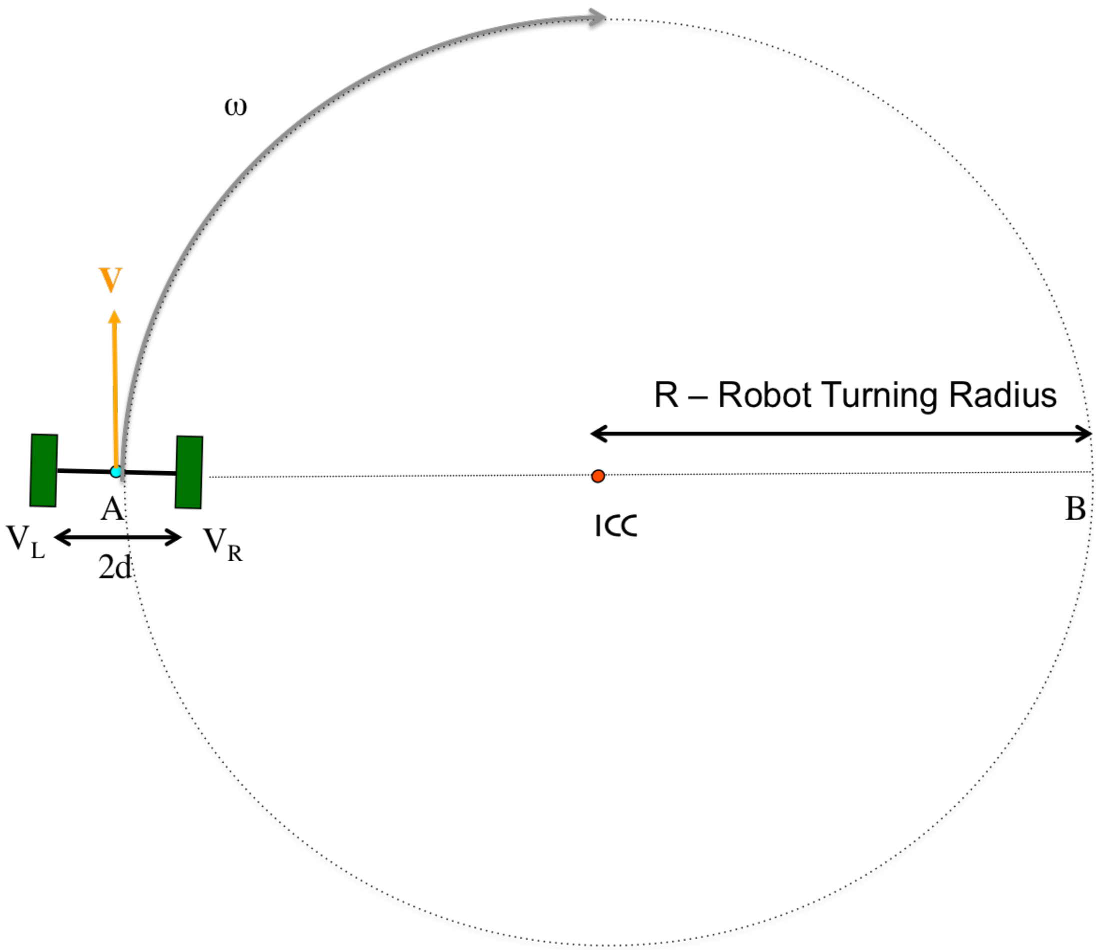
Figure shows:

- 2-wheel differential drive robot traveling from A to B in a circle around ICC
- robot turning radius R,
- constant robot velocity V,
- constant right wheel velocity V_R
- constant left wheel velocity V_L
- wheel axis distance 2d
- angular robot velocity ω.

Assume there is no wheel slippage.

Differential Drive

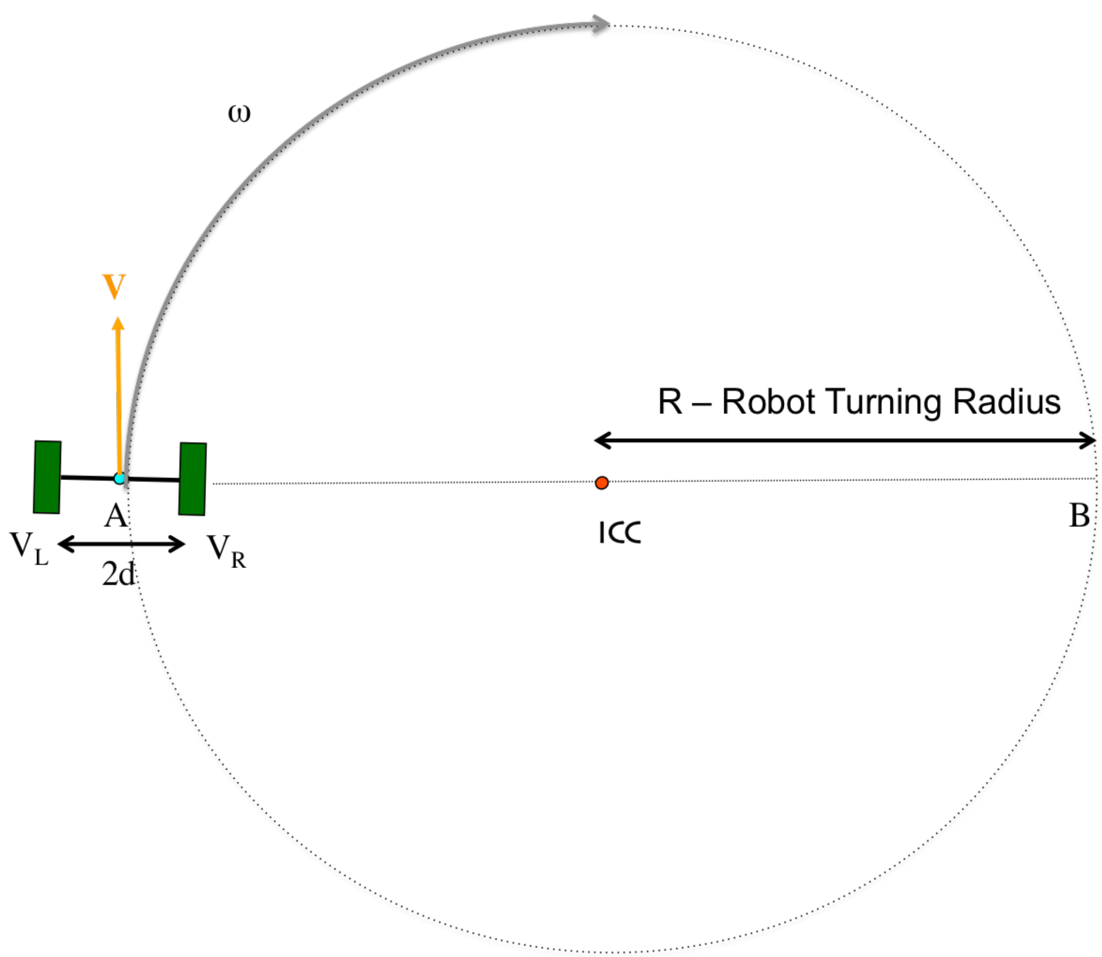
Forward Kinematics



Problem1 (Midterm Fall 2016)
Determine the Equations and Solutions for V and R.
Consider:
d = 2 inch, V_L = 8 inch per sec and V_R = 4 inch per sec.

Problem2 (Midterm Fall 2016)
Determine the Equations and Solutions for V_R and V_L
Consider:
d = 2 inch, V = 10 inch per sec and R = 10 inch.

Differential Drive



Problem1

Determine the Equations and Solutions for V and R.

Consider:

$d = 2$ inch, $V_L = 8$ inch per sec and $V_R = 4$ inch per sec.

Solution1

$V = (V_L + V_R) / 2 = (8 + 4) / 2 = 6$

$V = \omega R$

$V_L = \omega (R + d), \omega = V_L / (R + d)$

$V_R = \omega (R - d), \omega = V_R / (R - d)$

$V_L / (R + d) = V_R / (R - d)$

$V_L (R - d) = V_R (R + d)$

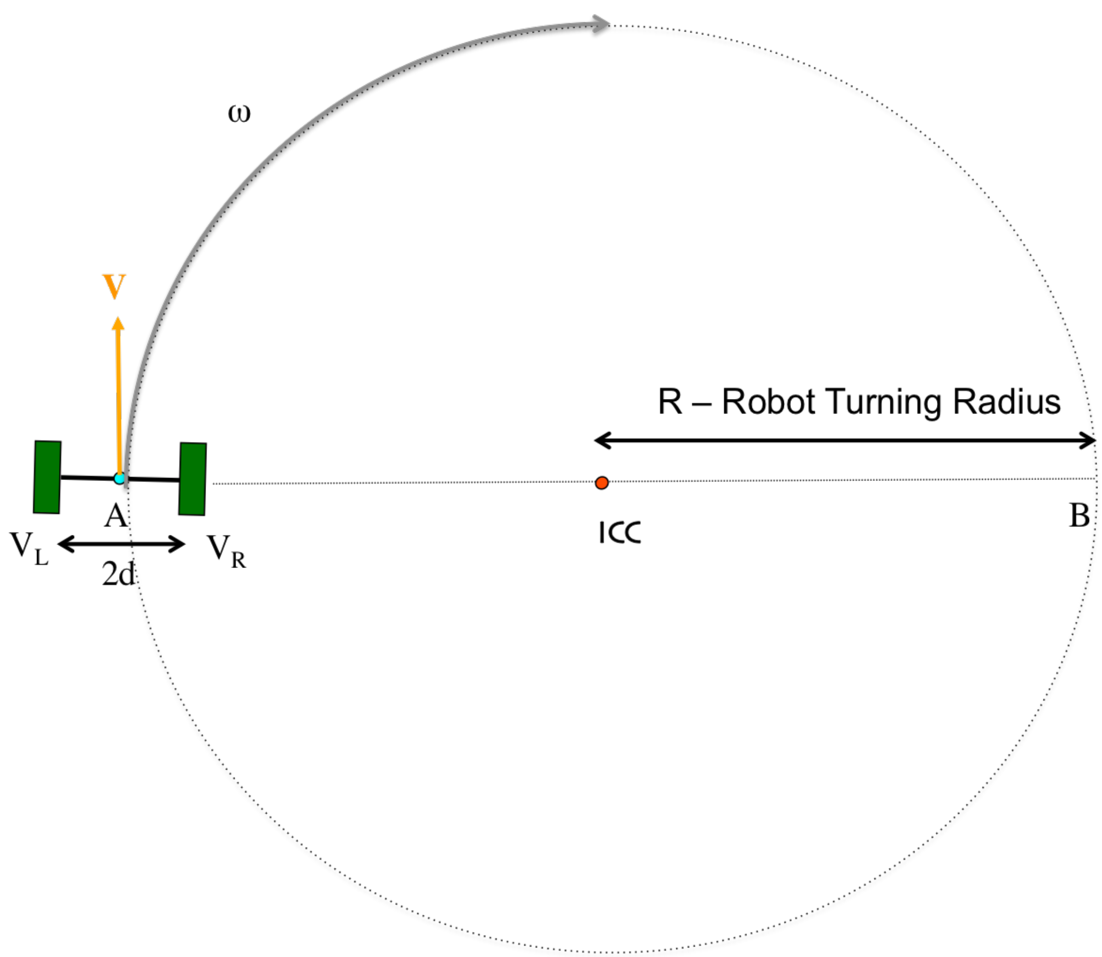
$V_L R - V_L d = V_R R + V_R d$

$R (V_L - V_R) = d (V_L + V_R)$

$R = d (V_L + V_R) / (V_L - V_R)$

$R = 2 (8 + 4) / (8 - 4) = 6$

Differential Drive



Problem2

Determine the Equations and Solutions for V_R and V_L

Consider:

d = 2 inch, V = 10 inch per sec and R = 10 inch.

Solution2

$V = \omega R = (V_L + V_R) / 2$

$\omega = V_L / (R + d)$

$\omega = V_R / (R - d)$

$V_R = V_L (R - d) / (R + d)$

$V_L = 2V - V_R$

$V_R(R + d) = (2V - V_R)(R - d)$

$V_R R + V_R d = 2V(R - d) - V_R R + V_R d$

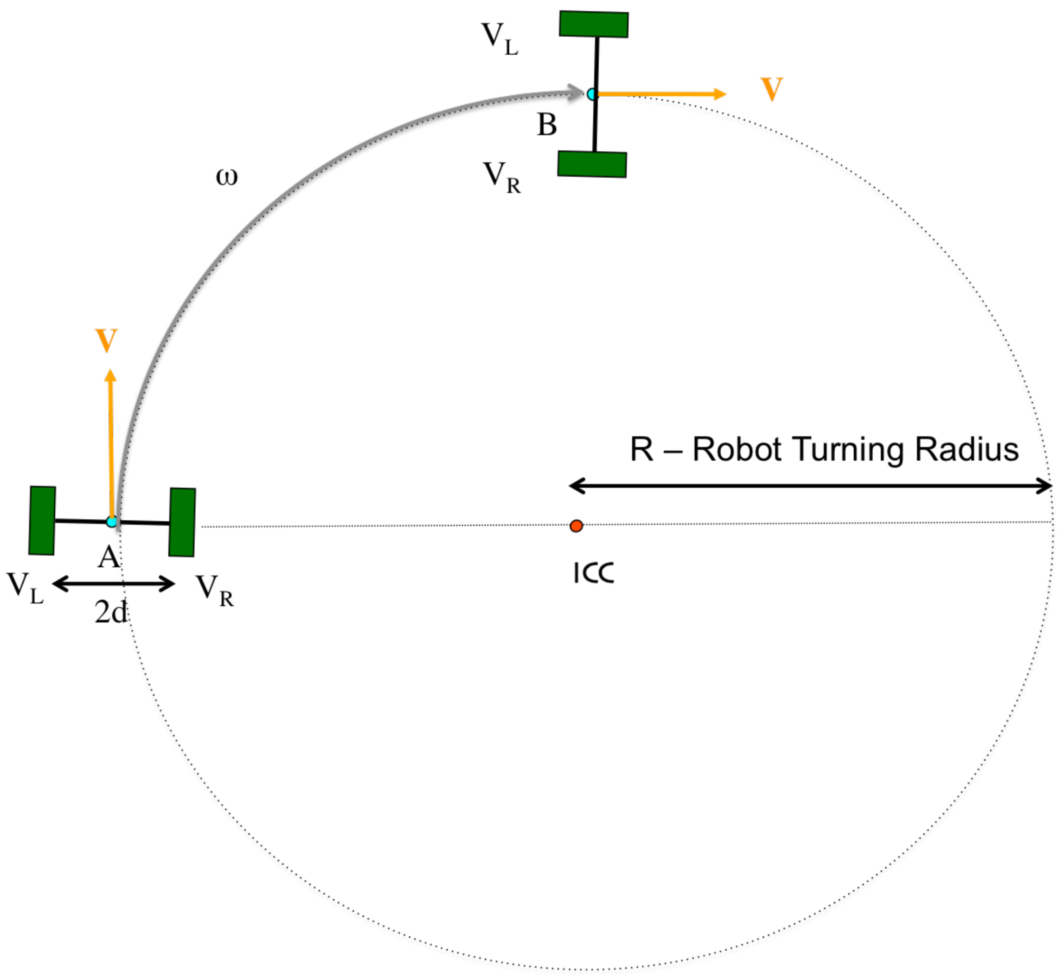
$2V_R R = 2V(R - d)$

$V_R = V(R - d)/R = 10(10\text{-}2)/10 = 8$

$V_L = 2V - V_R = 20\text{-}8 = 12$

Differential Drive

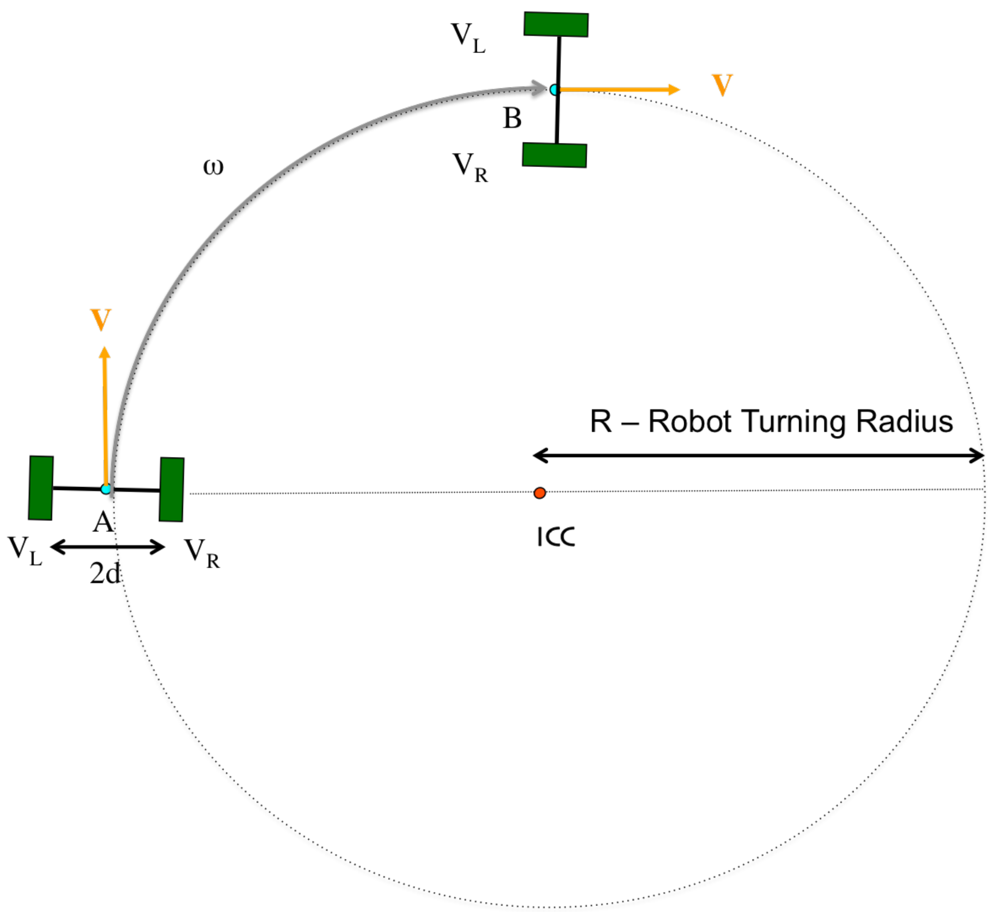
Forward Kinematics



Problem 1 - Midterm Spring 2017
The figure describes a 2-wheel differential drive robot traveling from A to B in a circle around ICC as the center of the robot rotates from point A to point B. The robot turning radius around ICC is given by R = 8 inch. The robot moves with constant velocity V = 8 inch, and angular velocity ω. The wheel axis distance is given by 2d, where d = 2 inch. Consider there is no wheel slippage. Show all equations and determine the robot constant left wheel velocity V_L and the constant right wheel velocity V_R around ICC.
Consider that $V = \omega * R$

Differential Drive

Forward Kinematics



Solution 1 - Midterm Spring 2017

$V = 8$, $R = 8$, $d = 2$

$V = \omega R = (V_L + V_R) / 2$

$\omega = V_L / (R + d)$

$\omega = V_R / (R - d)$

$V_R = V_L (R - d) / (R + d)$

$V_L = 2V - V_R$

$V_R(R + d) = (2V - V_R)(R - d)$

$V_R R + V_R d = 2V(R - d) - V_R R + V_R d$

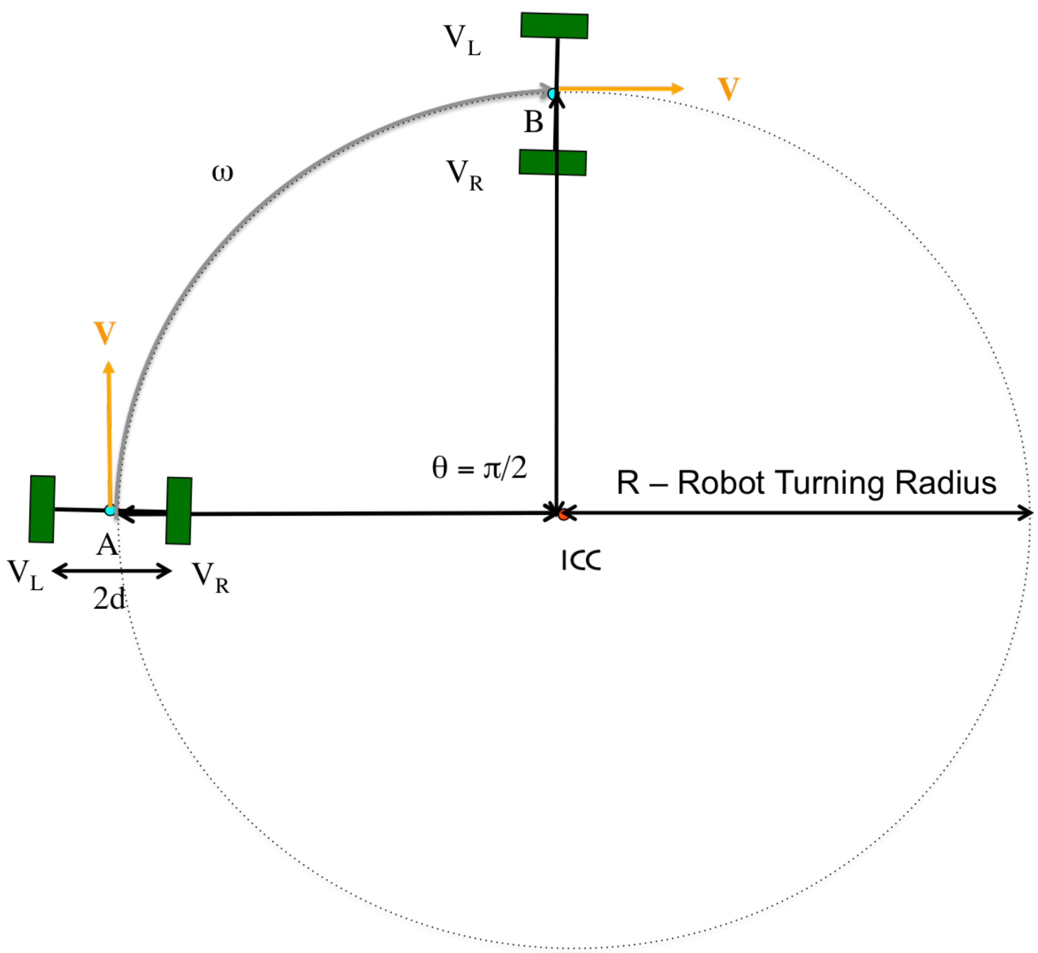
$2V_R R = 2V(R - d)$

$V_R = V(R - d)/R = 8(8-2)/8 = 6$

$V_L = 2V - V_R = 16-6 = 10$

Differential Drive

Inverse Kinematics

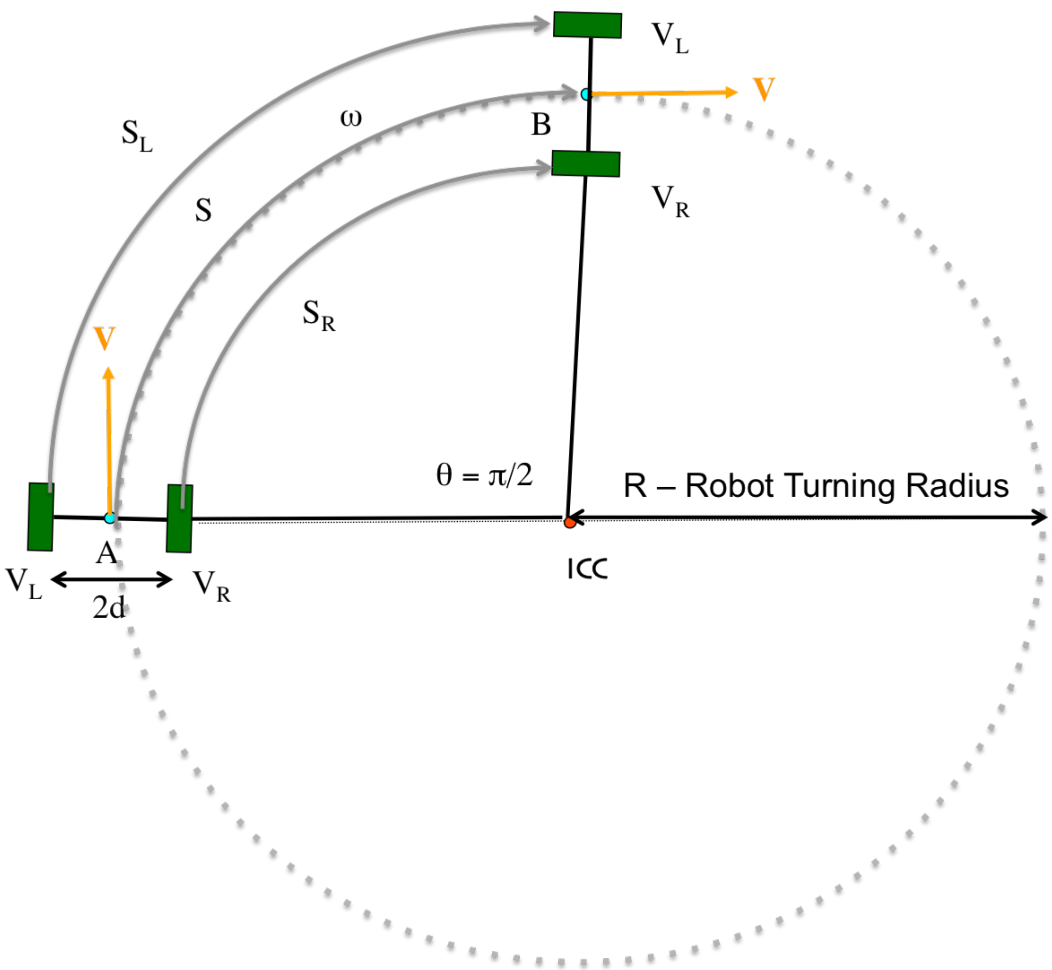


Problem 2 - Midterm Spring 2017

The figure describes a 2-wheel differential drive robot traveling from A to B in a circle around ICC given by R = 8 inch. The robot moves around ICC with constant velocity V = 8 inch, constant velocity V_L, constant velocity V_R, and constant angular velocity ω, until it reaches $\theta = \pi/2$. The wheel axis distance is given by 2d, where d = 2 inch. Consider there is no wheel slippage. Show all equations and determine how many rotations the Left and Right wheels make from point A to point B. Consider that $V = \omega*R$ and $S = \theta*R$, where S is the arc of the circle from point A to point B. Each robot wheel radius = 1/8 inch.

Differential Drive

Inverse Kinematics



Solution 2 - Midterm Spring 2017

$V = 8,\ R = 8,\ d = 2,\ \theta = \pi / 2,\ r = 1/8$

$S = \theta R = (\pi / 2) R = (\pi / 2) 8 = 4 \pi$

$S_R = \pi (R - d) = \pi (8-2) / 2 = 3 \pi$

$S_L = \pi (R + d) = \pi (8+2) / 2 = 5 \pi$

$Rot_R\ 2 \pi r = 3 \pi$

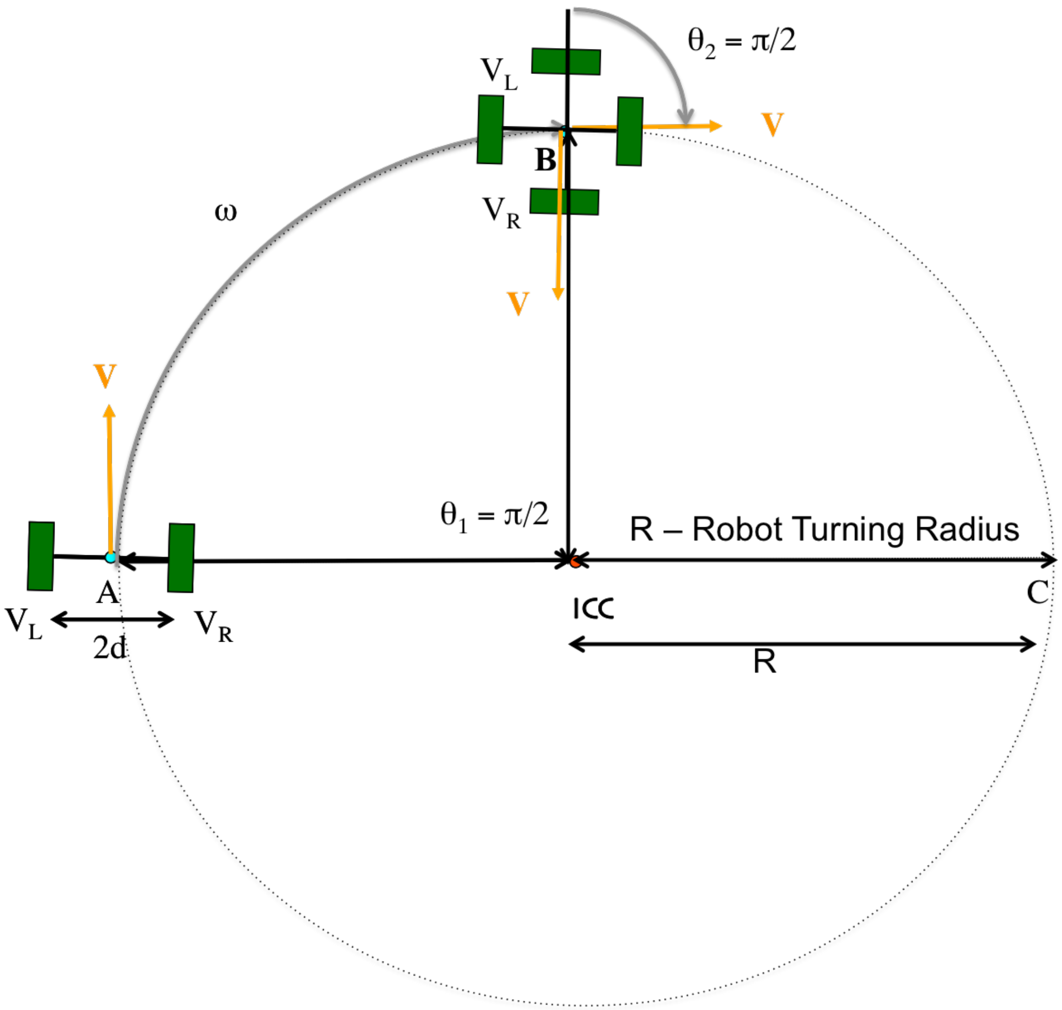
$Rot_R = 3 \pi / (2 \pi\ 1 / 8) = 12$ rotations

$Rot_L\ 2 \pi r = 5 \pi$

$Rot_L = 5 \pi / (2 \pi\ 1 / 8) = 20$ rotations

Differential Drive

Inverse Kinematics

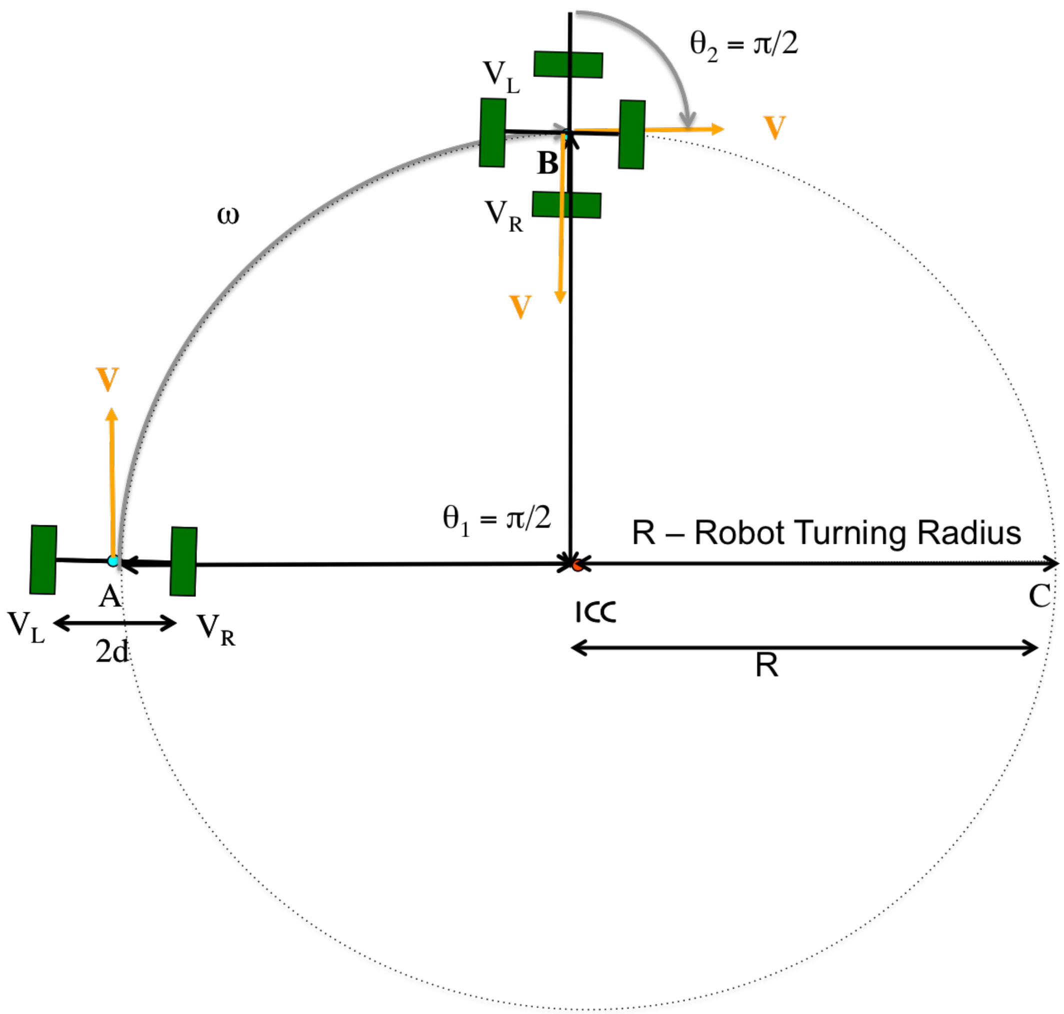


Problem 3 - Midterm Spring 2017

The figure describes a 2-wheel differential drive robot initially traveling from A to B in a circle around ICC. Then, the center of the robot rotates by $\theta_2 = \pi/2$ around point B in order to point straight down towards ICC. Show all equations and determine how many wheel rotations, Rot_L and Rot_R, need to be made around point B. The wheel axis distance is given by 2d, where d = 2 inch. Each robot wheel radius r = 1/8 inch. Consider there is no wheel slippage. Compute turning velocity V around point B, as well as wheel velocities V_R and V_L.

Differential Drive

Inverse Kinematics



Solution 3 - Midterm Spring 2017

d = 2 (axis), r = 1/8 (wheel)

$S = S_R = S_L = \theta_2\, d = (\pi / 2)\, d$

$= (\pi / 2) * 2 = \pi$

$Rot_L = Rot_R$

$Rot_L\, 2\, \pi\, r = \pi$

$Rot_L = \pi / (2\, \pi\, r) = \pi / (2\, \pi\, 1 / 8)$

= 4 rotations

Rot_R = 4 rotations

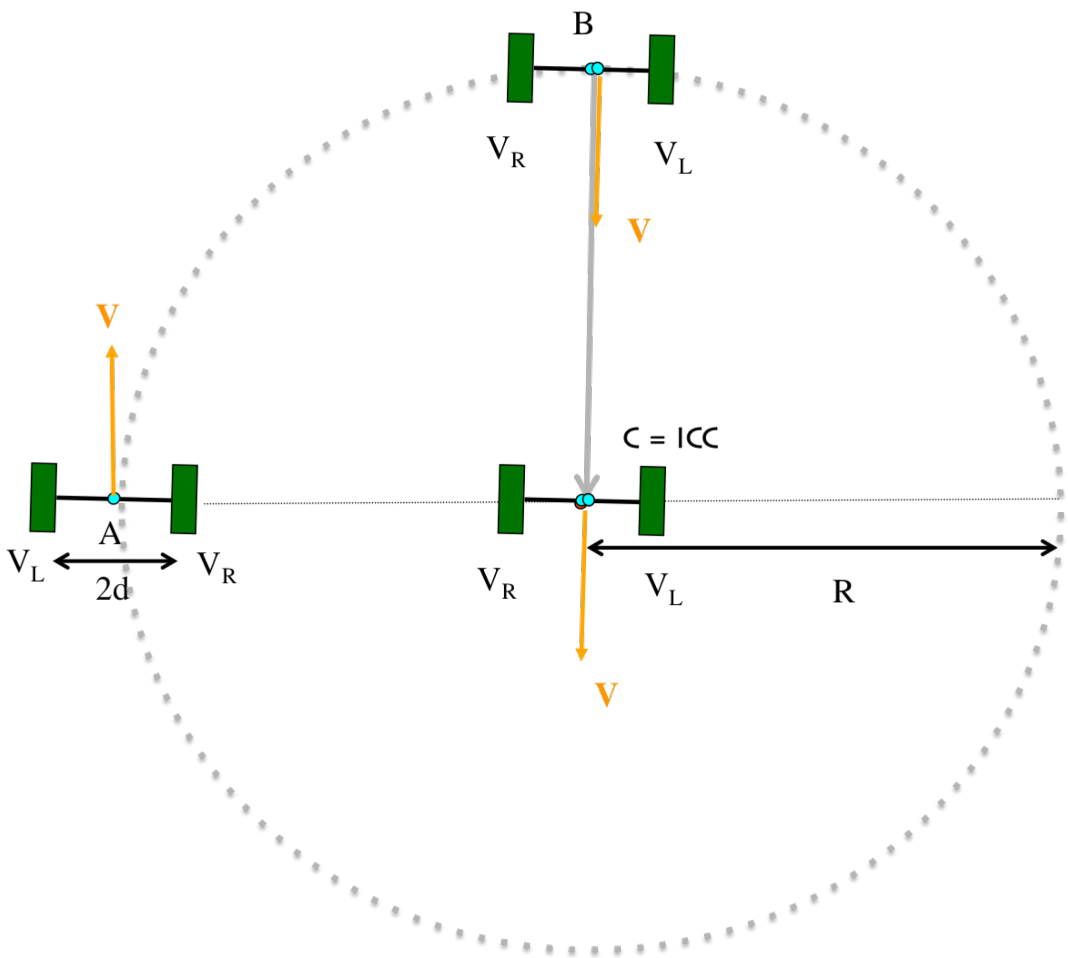
V_L = any velocity

$V_R = - V_L$

$V = (V_L + V_R) / 2 = 0$

Differential Drive

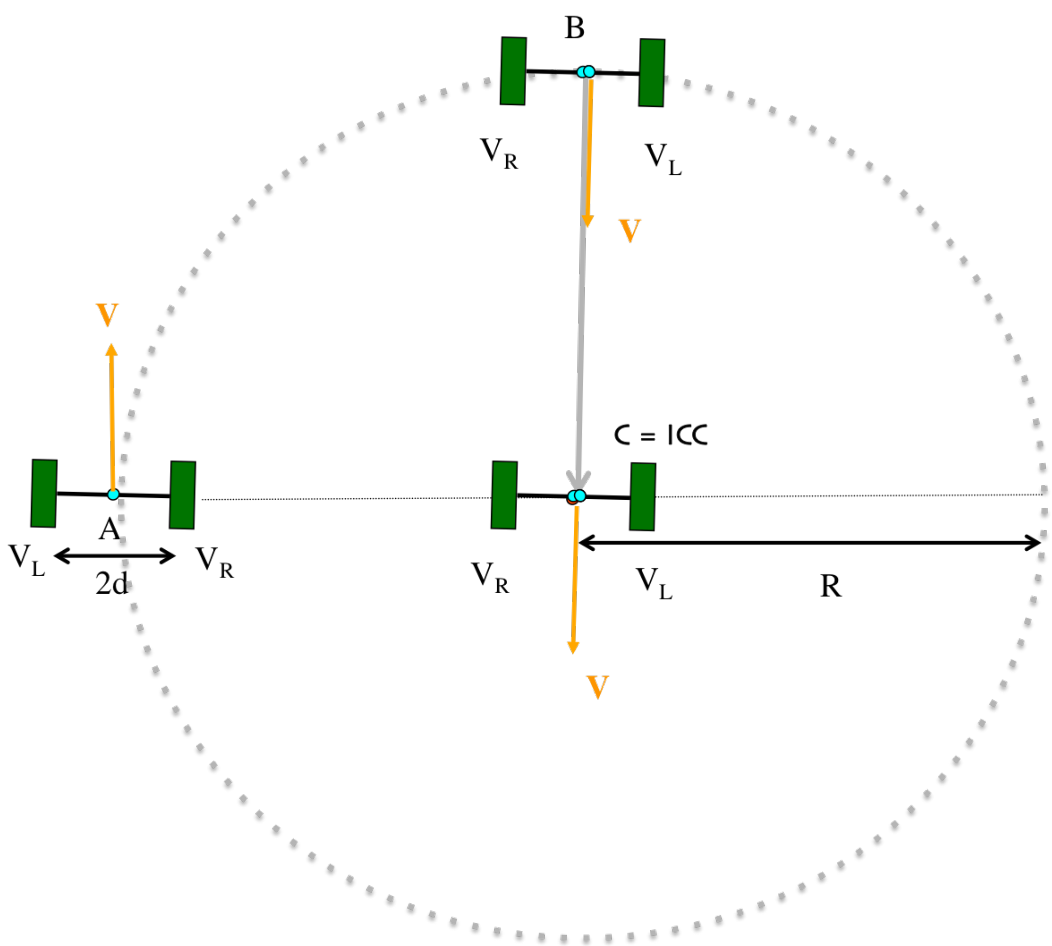
Inverse Kinematics



Problem 4 - Midterm Spring 2017
The figure describes a 2-wheel differential drive robot traveling straight down from B towards point C = ICC, where R = 8 inch, without changing its orientation. Show all equations and determine how many wheel rotations, Rot_L and Rot_R, need to be made from point B to reach point C = ICC. The wheel axis distance is given by 2d, where d = 2 inch. Each robot wheel radius r = 1/8 inch. Consider there is no wheel slippage.

Differential Drive

Inverse Kinematics



Solution 4 - Midterm Spring 2017

R = 8, d = 2, r = 1/8

$Rot_L = Rot_R$

$Rot_L \, 2 \pi r = R = 8$

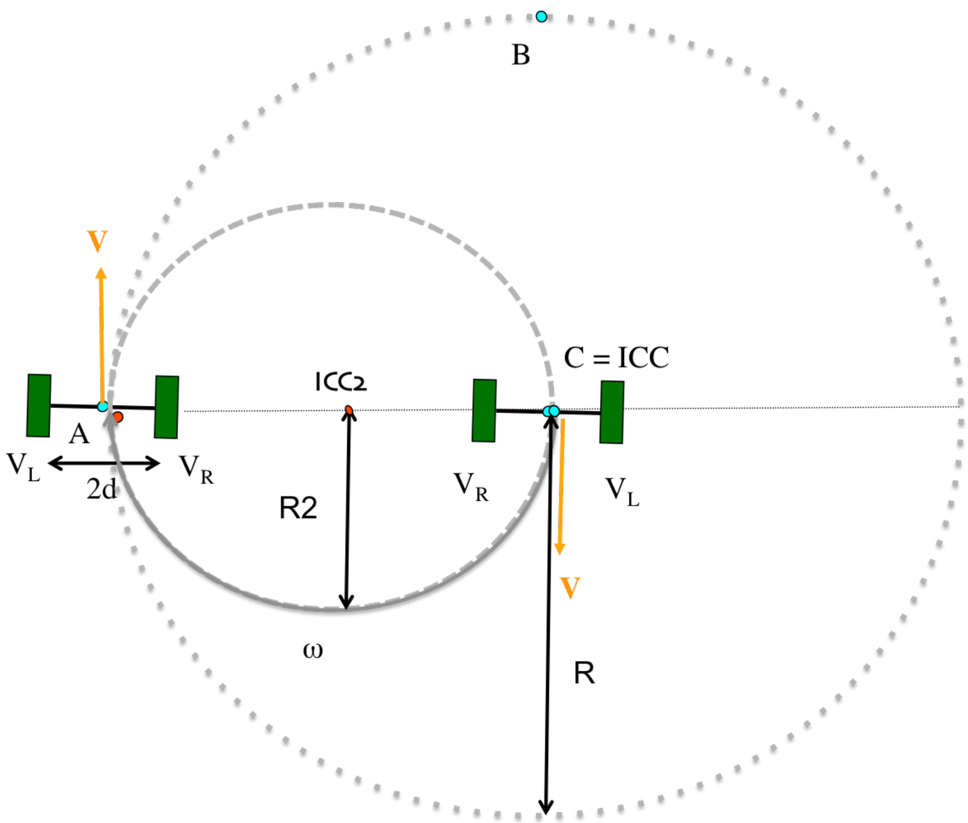
$Rot_L = Rot_R = 8 / (2 \pi r) = 8 / (2 \pi 1 / 8) =$

$64 / 2 \pi = 32 / \pi$ rotations

$Rot_L = Rot_R = 32 / \pi$ rotations

Differential Drive

Inverse Kinematics

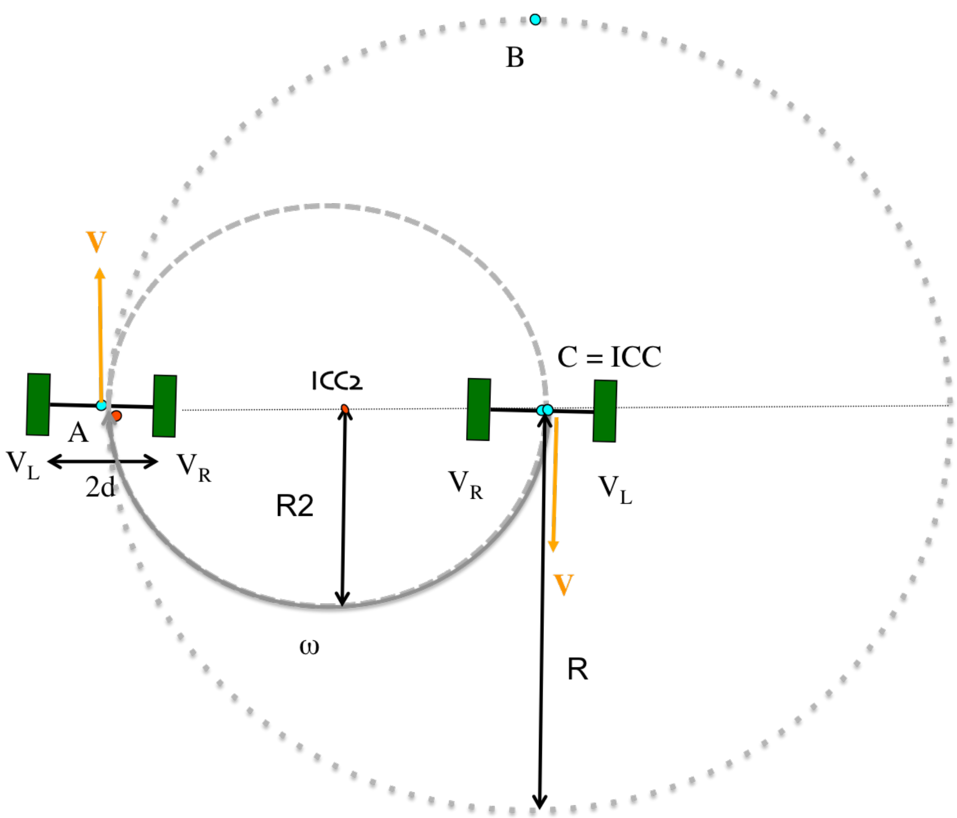


Problem 5 - Midterm Spring 2017

The figure describes a 2-wheel differential drive robot traveling from C to A to generate a similar robot pose as it started. The robot rotates around ICC_2 with angular velocity ω. The turning circle around ICC_2 is given by R_2 = 4 inch, where the robot moves with constant velocity V = 8 inch. The wheel axis distance is given by 2d, where d = 2 inch. Consider there is no wheel slippage. Show all equations and determine the robot constant left wheel velocity V_L and the constant right wheel velocity V_R around ICC_2.

Differential Drive

Inverse Kinematics



Solution 5 - Midterm Spring 2017

$V = 8,\ R_2 = 4,\ d = 2$

$V = \omega R_2 = (V_L + V_R) / 2$

$\omega = V_L / (R_2 + d)$

$\omega = V_R / (R_2 - d)$

$V_R = V_L (R_2 - d) / (R_2 + d)$

$V_L = 2V - V_R$

$V_R(R_2 + d) = (2V - V_R)(R_2 - d)$

$V_R R_2 + V_R d = 2V(R_2 - d) - V_R R_2 + V_R d$

$2V_R R_2 = 2V(R_2 - d)$

$V_R = V(R_2 - d)/R_2 = 8(4 - 2)/4 = 4$

$V_L = 2V - V_R = 16\text{-}4 = 12$