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Cheat Sheet

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Cheat Sheet

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Abstract

Physics documentation for the BPS temperature equilibration in the code Clog.

Contents

I. FORMULAE

$$F(v) \;=\; -\int_{-\infty}^{\infty} du\, \frac{\rho_{\rm tot}(u)}{v - u + i\eta} \quad \text{with} \;\; \rho_{\rm tot}(u) = \sum_b \rho_b(u) \tag{1.1}$$

$$\rho_b(v) \;=\; \kappa_b^2 \sqrt{\frac{\beta_b m_b}{2\pi}}\, v\, \exp\!\left\{-\frac{1}{2}\,\beta_b m_b\, v^2\right\} \;, \tag{1.2}$$

and its relation to the dielectric function is $k^2\,\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_p) = k^2 + F(\hat{\mathbf{k}}\cdot\mathbf{v}_p)$. The Dawson integral is defined by

$$\text{daw}(x) = \int_0^x dy\, e^{y^2 - x^2} = \frac{\sqrt{\pi}}{2}\, e^{-x^2}\text{erfi}(x) \;, \tag{1.3}$$

$$F_{\rm Re}(v) \;=\; \sum_b \kappa_b^2 \left[1 - 2\sqrt{\frac{\beta_b m_b}{2}}\; v\; \text{daw}\left(\sqrt{\frac{\beta_b m_b}{2}}\; v\right)\right] \tag{1.4}$$

$$F_{\rm Im}(v) \;=\; \sqrt{\pi} \sum_b \kappa_b^2 \sqrt{\frac{\beta_b m_b}{2}}\;\; v\; \exp\left[-\frac{\beta_b m_b}{2}\, v^2\right] = \pi \rho_{\rm tot}(v) \;. \tag{1.5}$$

$F^*(v) = F(-v)$

$$H(v) \;\equiv\; -i\left[F(v)\ln\!\left\{\frac{F(v)}{\kappa_e^2}\right\} - F^*(v)\ln\!\left\{\frac{F^*(v)}{\kappa_e^2}\right\}\right] \tag{1.6}$$

$$\;=\; 2\left[F_{\rm Re}\arg\{F\} + F_{\rm Im}\ln\!\left\{\frac{|F|}{\kappa_e^2}\right\}\right] \;. \tag{1.7}$$

We can write the dielectric function as a sum over plasma components,

$$F(v) = \sum\nolimits_b F_b(v) \;, \tag{1.8}$$

where we express the contribution from plasma species b as

$$F_b(v) \;=\; -\int_{-\infty}^{\infty} du\, \frac{\rho_b(v)}{v - u + i\eta} \tag{1.9}$$

$$\rho_b(v) \;=\; \kappa_b^2 \sqrt{\frac{\beta_b m_b}{2\pi}}\, v\, \exp\!\left\{-\frac{1}{2}\,\beta_b m_b\, v^2\right\} \;. \tag{1.10}$$

The functions $F_b(v)$ have units of wave-number-squared $[1/L^2]$ and their argument v has units of velocity. We can express the functions $F_b(v)$ in terms of a single dimensionless function $\mathbb{F}(x)$ as follows:

$$F_b(v) \;=\; \kappa_b^2\, \mathbb{F}\left(\sqrt{\frac{\beta_b m_b}{2}}\, v\right) \tag{1.11}$$

with

$$\mathbb{F}(x) \;=\; -\int_{-\infty}^{\infty} dy \, \frac{\bar{\rho}(y)}{x - y + i\eta} \tag{1.12}$$

$$\bar{\rho}(y) \;=\; \frac{y}{\sqrt{\pi}}\, e^{-y^2} \;. \tag{1.13}$$

Relation (1.11) holds because $\rho_b(u) = \kappa_b^2\, \bar{\rho}(y)$ for $u = (2/\beta_b m_b)^{1/2}\, y$. Note the reflection property

$$\mathbb{F}(-x) = \mathbb{F}^*(x) \;, \tag{1.14}$$

which means that the real part is even in x and the imaginary part is odd,

$$\mathbb{F}_{\rm Re}(-x) \;=\; \mathbb{F}_{\rm Re}(x) \tag{1.15}$$

$$\mathbb{F}_{\rm Im}(-x) \;=\; -\mathbb{F}_{\rm Im}(x) \;. \tag{1.16}$$

As with expressions (1.4) and (1.5), the real and imaginary parts can be written

$$\mathbb{F}_{\rm Re}(x) \;=\; 1 - 2x\, {\rm daw}\,(x) \tag{1.17}$$

$$\mathbb{F}_{\rm Im}(x) \;=\; \pi\, \bar{\rho}(x) = \sqrt{\pi}\, x\, e^{-x^2} \;. \tag{1.18}$$

Since the Dawson function ${\rm daw}(x)$ is odd we see that $\mathbb{F}_{\rm Re}(x)$ is even. The real (blue) and the imaginary (red) parts of $\mathbb{F}(x)$ are graphed in Fig. ** For a proof of (1.17) and (1.18), see physics/research/Tei/BPS-classical/notes/cei_reg.tex

$$\mathbb{F}(x) \;=\; \int_{-\infty}^{\infty} dy \, \frac{\bar{\rho}(y)}{y - x - i\eta} \;, \tag{1.19}$$

A. Asymptotic forms of F

From expressions (1.11), (1.17) and (1.18), the real and imaginary parts of $F_b(v)$ can be expressed as

$$F_b^{\rm Re}(v) \;=\; \kappa_b^2 \left[1 - 2\sqrt{\frac{\beta_b m_b}{2}}\; v\; {\rm daw} \left\{ \sqrt{\frac{\beta_b m_b}{2}}\; v \right\} \right] \tag{1.20}$$

$$F_b^{\rm Im}(v) \;=\; \kappa_b^2 \sqrt{\frac{\beta_b m_b\, \pi}{2}}\;\; v\; \exp\left\{ -\frac{\beta_b m_b}{2}\, v^2 \right\} \;. \tag{1.21}$$

The limits of small and large arguments of the Dawson function are

$${\rm daw}(x) \;=\; x + \frac{2x^3}{3} + \frac{4x^5}{15} + \mathcal{O}(x^7) \tag{1.22}$$

$${\rm daw}(x) \;=\; \frac{1}{2x} + \frac{1}{4x^3} + \frac{3}{8x^5} + \mathcal{O}(x^{-7}) \;. \tag{1.23}$$

II. NUMERICAL CHECKS

For a single component plasma note that (1.5) reduces to

$$F_{\rm Re}(v) \;=\; \kappa^2 \left[1 - 2\sqrt{\frac{\beta m}{2}}\; v \;\mathrm{daw}\left(\sqrt{\frac{\beta m}{2}}\; v\right)\right] \tag{2.1}$$

$$F_{\rm Im}(v) \;=\; \sqrt{\pi}\kappa^2\sqrt{\frac{\beta m}{2}}\;\; v\;\exp\left[-\frac{\beta m}{2}\, v^2\right]\;\; ; \tag{2.2}$$

and therefore $F(0) = \kappa^2$, with $\kappa^2 = e^2 n/T$. For large values of the argument, the Dawson function is [1]

[1] For completeness, small values of the argument give $\mathrm{daw}(x) = x + 2x^3/3 + 4x^5/15$.

III. EXTRA

A. Numerical Forms

The following numerical forms have been used in the code:

$$\kappa_b \cdot \mathrm{cm} \;=\; 4.25390 \times 10^{-5}\, |Z_b| \left(\frac{n_b \cdot \mathrm{cm}^3}{T_b/\mathrm{keV}}\right)^{1/2} \tag{3.1}$$

$$\omega_b \cdot \mathrm{s} \;=\; 1.32155 \times 10^{3}\, \big|Z_b\big| \Big(n_b \cdot \mathrm{cm}^3\Big)^{1/2} \Big(\frac{m_{\mathrm{AMU}}}{m_b}\Big)^{1/2} \tag{3.2}$$

$$\omega_e \cdot \mathrm{s} \;=\; 5.62016 \times 10^{4}\, \big(n_e/\mathrm{cm}^3\big)^{1/2} \tag{3.3}$$

where

$$m_{\mathrm{AMU}} c^2 \;=\; 931.19 \times 10^{3}\, \mathrm{keV}\ . \tag{3.4}$$