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**C-Symmetry Notes**

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## C-Symmetry Notes

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### Abstract

Physics documentation for the BPS temperature equilibration in the code Clog.

**Contents**

# I. THE SYMMETRY CONDITION

## A. Statement of the Problem

We consider a plasma with component species labeled by $a$ and $b$, each in local thermodynamic equilibrium among themselves (but not each other) with respective temperatures $T_a$ and $T_b$. The rate of energy exchange per unit volume *from* the $a$-subsystem *to* the $b$-subsystem takes the form

$$\frac{d\mathcal{E}_{ab}}{dt} = -\,\mathcal{C}_{ab}\Big(T_a - T_b\Big)\ , \tag{1.1}$$

which serves to define the rate coefficients $\mathcal{C}_{ab}$. Energy conservation, which can be written in the form $d\mathcal{E}_{ab}/dt = -d\mathcal{E}_{ba}/dt$, implies the symmetry of the rate coefficient itself

$$\mathcal{C}_{ab} = \mathcal{C}_{ba}\ . \tag{1.2}$$

On the other hand, from Eq. (12.3) of BPS, the rate coefficients can be expressed as

$$\mathcal{C}_{ab} \;=\; \int \frac{d^3 p_a}{(2\pi\hbar)^3}\, f_a(\mathbf{p}_a)\, \beta_a v_a\, \mathcal{A}_{ab}(\mathbf{p}_a) \tag{1.3}$$

$$=\; \frac{2 n_a\, c\, \beta_a^{5/2}}{\sqrt{m_a c^2}} \sqrt{\frac{2}{\pi}} \int_0^\infty dE\, E\, e^{-\beta_a E}\, \mathcal{A}_{ab}(E)\ . \tag{1.4}$$

We will derive this latter form in another section, but for now it suffices to note that we have only used the Maxwell-Boltzmann distribution (1.11), along with spherical symmetry in momentum to express $f_a$ and $\mathcal{A}_{ab}$ in terms of energy $E = p_a^2/2m_a$. The symmetry of (1.3) or (1.4) under the interchange $a \leftrightarrow b$ is hardly evident, and the purpose of these notes is to explain this seeming puzzle.

We can express the problem in another form. Suppose the ions are in equilibrium among themselves at some temperature $T_{\rm I}$, and suppose the electron temperature is $T_e$. Letting the first index of (1.1) correspond to the electron, and upon summing over the ions in the second index, the rate equation becomes

$$\frac{d\mathcal{E}_{e\rm I}}{dt} = -\,\mathcal{C}_{e\rm I}\Big(T_e - T_{\rm I}\Big)\ , \tag{1.5}$$

where $\mathcal{C}_{e\rm I} = \sum_i \mathcal{C}_{ei}$ and $d\mathcal{E}_{e\rm I}/dt = \sum_i d\mathcal{E}_{ei}/dt$. As before, energy conservation (treating the ions as a single subsystem) implies that $\mathcal{C}_{e\rm I}$ is symmetric under $e \leftrightarrow {\rm I}$. The advantage of the form (1.5), however, is that the rate coefficient greatly simplifies in the extreme quantum limit ($\eta \to 0$). Since the electron is so light, the calculational trick is to employ a sum-rule that emerges from the $m_e/m_{\rm I} \to 0$ limit, from which a quite simple result emerges,

$$m_e/m_{\rm I} \to 0\,,\ \eta \to 0:$$

$$\mathcal{C}_{e\rm I} = \underbrace{\frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\omega_{\rm I}^2}_{\text{prefactor}} \cdot \underbrace{\frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\} - \gamma - 1\right]}_{\text{Coulomb Log: }\ln\Lambda_{\rm BPS}^{\rm QM}}\ , \tag{1.6}$$

where $\omega_{\rm I}^2 = \sum_i \omega_i^2$. We will use rationalized cgs units (the Coulomb potential is $V = e^2/4\pi r$) for which the Debye wavenumber and the plasma frequency take the form

$$\kappa_a^2 = \frac{e_a n_a}{T_a} \tag{1.7}$$

$$\omega_a^2 = \frac{e_a n_a}{m_a} \ ; \tag{1.8}$$

the electron plasma coupling is given by

$$g_e = \frac{e^2 \kappa_e}{4\pi\, T_e} \ . \tag{1.9}$$

### B. Realizing a Two-Temperature Plasma in Nature

In these notes we consider a two-temperature plasma for which the electrons and ions have temperatures $T_e$ and $T_{\rm I}$, respectively. The most direct way to realize this in nature is a two-component plasma, that is to say, a plasma with with electrons of mass $m_e$ and a *single* species of ion with mass $m_{\rm I}$. Because the electron is thousands of times lighter than an ion, collisions between the electrons will rapidly bring the electron sub-system into equilibrium with itself at some temperature $T_e$; soon thereafter the ions will equilibrate to different common temperature $T_{\rm I}$ (at a rate $\sqrt{m_e/m_{\rm I}}$ slower than electron equilibration). Finally, the electron-ion system begins to exchange Coulomb energy and starts to equilibrate according to (1.1)

In fact, the form (1.1) expresses the Coulomb energy exchange between any number of species in a multi-component plasma $[a, b \in \{1, 2, \cdots, N\}\,]$, provided each species maintains local thermodynamic equilibrium with itself at a well defined temperature. This hypothetical scenario is physically realized in a plasma with multiple, well-separated, mass scales. For example, consider a Deuterium-Tritium (DT) plasma: the electrons will equilibrate to a temperature $T_e$ and the DT to a common temperature $T_{\rm I}$. If this plasma also contained a heavy large-$Z$ species, such as iron, then this component will equilibrate to some temperature $T_{\rm Fe}$ soon after the D-T equilibration, although we will not consider the three-temperature case.

### C. The Rate Coefficient

Let's now use the fact that $f_a$ and $\mathcal{A}_{ab}$ are radially symmetric in their momentum argument to change variables in (1.3) to energy $E_a = p_a^2/2m_a$:

$$\begin{aligned}
\mathcal{C}_{ab} &= \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty p_a^2 dp_a\, f_a(E_a)\,\frac{\beta_a p_a}{m_a}\,\mathcal{A}_{ab}(E_a) = \frac{4\pi\,\beta_a}{(2\pi\hbar)^3}\int_0^\infty (2m_a E_a) d(p_a^2/2m_a) f_a(E_a)\,\mathcal{A}_{ab}(E_a) \\
&= \frac{8\pi\,\beta_a m_a}{(2\pi\hbar)^3}\int_0^\infty dE_a\, E_a\, f_a(E_a)\mathcal{A}_{ab}(E_a) \ . 
\end{aligned} \tag{1.10}$$

The distribution function is expressed by Eq. (7.4) of BPS, and in terms of energy $E_a$ we can write

$$f_a(E_a) = n_a \left(\frac{2\pi\hbar^2\beta_a}{m_a}\right)^{3/2} e^{-\beta_a E_a} \ . \tag{1.11}$$

This gives the rate coefficient,

$$\mathcal{C}_{ab} \ = \ \frac{8\pi\, \beta_a m_a}{(2\pi\hbar)^3} \int_0^\infty dE_a\, E_a \cdot n_a \left(\frac{2\pi\hbar^2\beta_a}{m_a}\right)^{3/2} e^{-\beta_a E_a} \cdot \mathcal{A}_{ab}(E_a) \ ; \tag{1.12}$$

changing the dummy integration variable from $E_a$ to $E$, and combining terms in the prefactor gives

$$\mathcal{C}_{ab} \ = \ \frac{2n_a\, \beta_a^{5/2}}{\sqrt{m_a}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty dE\, E\, e^{-\beta_a E}\, \mathcal{A}_{ab}(E) \ . \tag{1.13}$$

It is useful to change the integration variable to the dimensionless quantity $x = \beta_a E$, thereby giving

$$\int_0^\infty dE\, E\, e^{-\beta_a E}\, \mathcal{A}_{ab}(E) = \frac{1}{\beta_a^2} \int_0^\infty dx\, x\, e^{-x}\, \mathcal{A}_{ab}(x/\beta_a) \ . \tag{1.14}$$

The coefficient can now be written

$$\boxed{\mathcal{C}_{ab} \ = \ \frac{2\, n_a\, c}{\sqrt{T_a\, m_a c^2}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, x\, e^{-x}\, \mathcal{A}_{ab}(T_a\, x) \ .} \tag{1.15}$$

Units: $[nc/\text{Energy}] \cdot [\mathcal{A}] = \text{cm}^{-3} \cdot (\text{cm/s}) \cdot \text{Energy}^{-1} \cdot (\text{Energy/cm}) = \text{cm}^{-3} \cdot s^{-1}$.

For a single-ion plasma such as a hydrogen plasma, this becomes

$$\mathcal{C}_{e\text{I}}^{\text{dist}} \ = \ \underbrace{\frac{2\, n_e\, c}{\sqrt{T_e\, m_e c^2}}}_{\text{prefactor } N_e} \cdot \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty dx\, x\, e^{-x}\, \mathcal{A}_{e\text{I}}(T_e\, x)}_{\text{integral } \bar{C}_{e\text{I}}} \ , \tag{1.16}$$

where the superscript "dist" now indicates that the rate coefficient has been calculated using the distribution function relation. The rescaled integrand $\bar{C}_{e\text{I}}(x) = \sqrt{\frac{2}{\pi}}\, x\, e^{-x} \mathcal{A}_{e\text{I}}(T_e\, x)$ is plotted below for the temperatures $T_e = T_\text{I} = 10\,\text{keV}$ and $T_e = T_\text{I} = 100\,\text{keV}$.

FIG. 1: The integrand $\bar{C}_{e\mathrm{I}}(x) = \sqrt{2/\pi}\, x\, e^{-x} \mathcal{A}_{e\mathrm{I}}(T_e x)$ for a hydrogen plasma with $n_e = 10^{25}\,\mathrm{cm}^{-3}$ for electron-ion temperatures of $T_e = T_\mathrm{I} = 10\,\mathrm{keV}$ (red solid) and $T_e = T_\mathrm{I} = 100\,\mathrm{keV}$ (blue dashed). See the table for numerical values. [main.symmetry.f90, main.symmetry.sm, main.symmetry.010kev.dat, main.symmetry.100kev.dat, main.symmetry.eps]

This integrand turns out to be difficult to numerically integrate, and the problem with the integral will turn out to lie with the small mass ratio. The next figure plots the rescaled integrand $\bar{C}_{e\mathrm{I}}(x) = \sqrt{\frac{2}{\pi}}\, x\, e^{-x} \mathcal{A}_{e\mathrm{I}}(T_e\, x)$ in the region of small-$x$:

FIG. 2: Same as Fig. 1, except $x$-axis runs over $[0, 0.01]$.

The dip in the 10 keV case suggest that we will have to break the interval into small regions, and we may even be required to use the small-E asymptotic form for $\mathcal{A}_{ab}$.
In the following table we illustrate the integral $\bar{C}_{e\mathrm{I}} = \int_0^{x_\mathrm{max}} dx\, \bar{C}_{e\mathrm{I}}(x)$. The integration problem crops up immediately, as errors of order 15–20% are far too large.

TABLE I: Rate coefficients with $x_\mathrm{max} = 30$ with $N = 1000$

| $\mathcal{C}_{e\mathrm{I}}$ $[\mathrm{cm}^{-3}\,\mathrm{s}^{-1}]$ | $T_e = T_\mathrm{I} = 10\,\mathrm{keV}$ | $T_e = T_\mathrm{I} = 100\,\mathrm{keV}$ |
|---|---|---|
| $\mathcal{C}_{e\mathrm{I}}^{\,\text{sum-rule}}$ | $2.228 \times 10^{36}$ | $1.050 \times 10^{35}$ |
| $\mathcal{C}_{e\mathrm{I}}^{\,\text{exact}}$ | $2.220 \times 10^{36}$ | $1.048 \times 10^{35}$ |
| $\mathcal{C}_{e\mathrm{I}}^{\,\text{dist}}$ | $2.612 \times 10^{36}$ | $1.194 \times 10^{35}$ |
| $\bar{\mathcal{C}}_{e\mathrm{I}}$ | | |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\,\text{sum-rule}}$ | 265.7 | 39.6 |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\,\text{exact}}$ | 264.7 | 39.5 |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\,\text{dist}}$ | 311.5 | 45.0 |
| % error | 18% | 14% |

Note that the sum-rule and the exact results are almost the same, and therefore the sum rule is a good approximation in these plasma regimes. The distribution function result is about 20% too high; however, the next table shows that we can do much better and the large error is a numerical artifact.

Let's check the extent to which the accuracy problem arises from the extremely small mass ratio $m_e/m_\mathrm{I}$ by setting the electron mass to half the proton mass, $m_e = \frac{1}{2}\, m_p$. In this case, the sum-run results should not be accurate, but the other two methods (exact and distribution function) should agree.

FIG. 3: The integrand $\bar{C}_{e\mathrm{I}}(x) = \sqrt{2/\pi}\, x\, e^{-x} \mathcal{A}_{e\mathrm{I}}(T_e x)$ with a heavy electron. [main.symmetry.f90, main.symmetry.heavyMe.sm, main.symmetry.heavyMe.010kev.dat, main.symmetry.heavyMe.100 kev.dat, main.symmetry.heavyMe.eps]

TABLE II: Rate coefficients using four smaller integration intervals: $[0, 0.2]$, $[0.2, 1]$, $[1, 3]$, and $[3, 30]$, with $N = 1000$ in each interval.

| $\mathcal{C}_{e\mathrm{I}}$ $[\mathrm{cm}^{-3}\,\mathrm{s}^{-1}]$ | $T_e = T_\mathrm{I} = 10\,\mathrm{keV}$ | $T_e = T_\mathrm{I} = 100\,\mathrm{keV}$ |
|---|---|---|
| $\mathcal{C}_{e\mathrm{I}}^{\text{sum-rule}}$ | $2.228 \times 10^{36}$ | $1.050 \times 10^{35}$ |
| $\mathcal{C}_{e\mathrm{I}}^{\text{exact}}$ | $2.220 \times 10^{36}$ | $1.049 \times 10^{35}$ |
| $\mathcal{C}_{e\mathrm{I}}^{\text{dist}}$ | $2.248 \times 10^{36}$ | $1.013 \times 10^{35}$ |
| $\bar{\mathcal{C}}_{e\mathrm{I}}$ | | |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\text{sum-rule}}$ | 265.7 | 39.6 |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\text{exact}}$ | 264.7 | 39.5 |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\text{dist}}$ | 268.0 | 38.2 |
| % error | 1.3% | 3.4% |

TABLE III: Rate coefficients setting $m_e = \frac{1}{2}\, m_p$ with an interval $[1, 30]$ and $N = 1000$.

| $\mathcal{C}_{e\mathrm{I}}$ $[\mathrm{cm}^{-3}\,\mathrm{s}^{-1}]$ | $T_e = T_\mathrm{I} = 10\,\mathrm{keV}$ | $T_e = T_\mathrm{I} = 100\,\mathrm{keV}$ |
|---|---|---|
| $\mathcal{C}_{e\mathrm{I}}^{\text{sum-rule}}$ | $11.65 \times 10^{37}$ | $4.733 \times 10^{36}$ |
| $\mathcal{C}_{e\mathrm{I}}^{\text{exact}}$ | $5.676 \times 10^{37}$ | $2.523 \times 10^{36}$ |
| $\mathcal{C}_{e\mathrm{I}}^{\text{dist}}$ | $5.729 \times 10^{37}$ | $2.545 \times 10^{36}$ |
| $\bar{\mathcal{C}}_{e\mathrm{I}}$ | | |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\text{sum-rule}}$ | 421094 | 54069 |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\text{exact}}$ | 205068 | 28826 |
| $\bar{\mathcal{C}}_{e\mathrm{I}}^{\text{dist}}$ | 206980 | 29073 |
| % error | 0.9% | 0.86% |

## II. LOW ENERGY CLASSICAL RESULT

In A_asymptotic1.7.tex we calculated the classical and quantum rate coefficient analytically using the low energy forms. As we see from Figs. 4 and 5, the classical contribution dominates the low-energy asymptotic form for the ions. We will therefore calculate the rate using the classical form

$$\mathcal{A}_{\text{CL},ab}(v_p) = \frac{e_p^2\,\kappa_b^2}{2\pi}\left(\frac{\beta_b m_b}{2\pi}\right)^{1/2} v_p\,\bar{A}_b^{\text{CL}} \tag{2.1}$$

$$\bar{A}_b^{\text{CL}} = -\frac{2}{3}\left[\ln\left\{\frac{e_p e_b \beta_b \kappa_{\text{D}}}{16\pi}\,\frac{m_b}{m_{pb}}\right\} + \frac{1}{2} + 2\gamma\right] , \tag{2.2}$$

or specializing to ions (and electron projectiles) we have

$$\mathcal{A}_{\text{CL},e\text{I}}(v_e) = \frac{e^2\,\omega_{\text{I}}^2}{2\pi}\,\beta_{\text{I}} m_{\text{I}}\left(\frac{\beta_{\text{I}} m_{\text{I}}}{2\pi}\right)^{1/2} \bar{A}_{\text{I}}^{\text{CL}}\,v_e \tag{2.3}$$

$$\bar{A}_{\text{I}}^{\text{CL}} = -\frac{2}{3}\left[\ln\left\{\frac{Z_{\text{I}} e^2 \beta_{\text{I}} \kappa_{\text{D}}}{16\pi}\,\frac{m_{\text{I}}}{m_{e\text{I}}}\right\} + \frac{1}{2} + 2\gamma\right] . \tag{2.4}$$

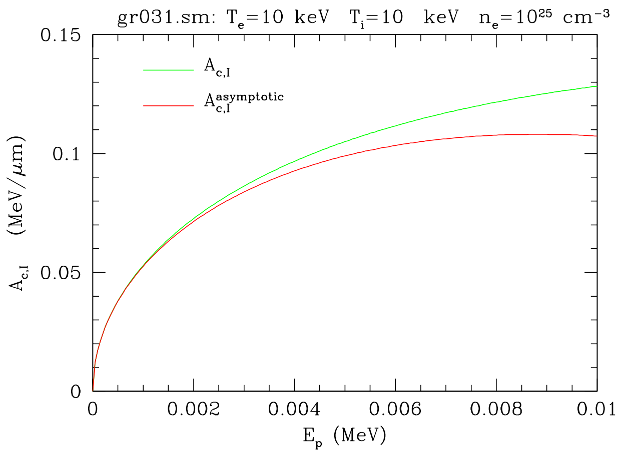


FIG. 4: Asymptotic classical ion contribution at low energies. [gr001.f90, gr031.sm, gr001.dat, gr001.smallE.dat, gr031.eps]

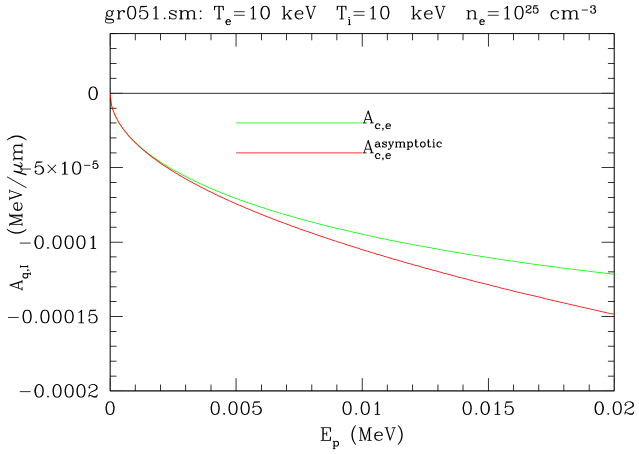


FIG. 5: Asymptotic quantum ion contribution at low energies. [gr001.f90, gr051.sm, gr001.dat, gr001.smallE.dat, gr051.eps]

We need to separate the classical small-E result into its singular and regular contributions. Equation (9.8) on p. 300 of BPS gives the small energy asymptotic form of the singular piece:

$$\mathcal{A}^{\rm CL}_{ab,{\rm S}} = \underbrace{\frac{e_p^2\,\kappa_b^2}{4\pi}}_{c_1}\,\underbrace{\left(\frac{\beta_b m_b}{2\pi}\right)^{1/2}}_{c_2} v_p\cdot\bar{A}^{\rm CL}_{ab,{\rm S}} \tag{2.5}$$

$$\bar{A}^{\rm CL}_{ab,{\rm S}} = -\left(\frac{2}{3}-\frac{1}{5}\,\beta_b m_b\,v_p^2\right)\left[\ln\left\{\frac{e_p\,e_b\beta_b\,K}{16\pi}\,\frac{m_b}{m_{pb}}\right\}+2\gamma\right]+\frac{2}{15}\,\beta_b m_b\,v_p^2+\mathcal{O}(v_p^5)\ ; \tag{2.6}$$

while Eq. (7.34) on p (287) of BPS gives the regular piece:

$$\mathcal{A}^{<}_{ab,{\rm R}} = \underbrace{\frac{e_p^2\,\kappa_b^2}{4\pi}}_{c_1}\,\underbrace{\left(\frac{\beta_b m_b}{2\pi}\right)^{1/2}}_{c_2} v_p\cdot\bar{A}^{<}_{ab,{\rm R}} \tag{2.7}$$

$$\begin{aligned}\bar{A}^{<}_{ab,{\rm R}} &= -\left(\frac{1}{3}-\frac{1}{10}\,\beta_b m_b\,v_p^2\right)\left(1+\ln\left\{\frac{\kappa_{\rm D}^2}{K^2}\right\}\right)+\frac{1}{5}\sum\nolimits_c\frac{\kappa_c^2}{\kappa_{\rm D}^2}\,\beta_c m_c\,v_p^2 - \\ &\quad \frac{\pi}{36}\left[\sum\nolimits_c\frac{\kappa_c^2}{\kappa_{\rm D}^2}\left(\beta_c m_c\,v_p^2\right)^{1/2}\right]^2+\mathcal{O}(v_p^5)\ .\end{aligned} \tag{2.8}$$

Let us work to leading order:

$$\mathcal{A}^{\rm CL}_{ab} \;=\; \underbrace{\frac{e_p^2\,\kappa_b^2}{4\pi}}_{c_1}\;\underbrace{\left(\frac{\beta_b m_b}{2\pi}\right)^{1/2} v_p}_{c_2}\cdot\left[\bar{A}^{\rm CL}_{ab,{\rm S}} + \bar{A}^{<}_{ab,{\rm R}}\right] \tag{2.9}$$

$$\bar{A}^{\rm CL}_{ab,{\rm S}} \;=\; -\frac{2}{3}\left[\ln\left\{\frac{e_p\,e_b\beta_b\,K}{16\pi}\,\frac{m_b}{m_{pb}}\right\} + 2\gamma\right] \tag{2.10}$$

$$\bar{A}^{<}_{ab,{\rm R}} \;=\; -\frac{2}{3}\left(\frac{1}{2} + \ln\left\{\frac{\kappa_{\rm D}}{K}\right\}\right) + \mathcal{O}(v_p)\;, \tag{2.11}$$

and note that for any value of $K$ we have

$$\bar{A}^{\rm CL}_{ab} \;=\; \bar{A}^{\rm CL}_{ab,{\rm S}} + \bar{A}^{<}_{ab,{\rm R}} = -\frac{2}{3}\left[\ln\left\{\frac{e_p e_b\beta_b\kappa_{\rm D}}{16\pi}\,\frac{m_b}{m_{pb}}\right\} + \frac{1}{2} + 2\gamma\right]\;. \tag{2.12}$$

### Appendix A: BPS Rate Coefficients

We assume a plasma with any number of species $b$, each with its own private temperature $T_b$ (we need a mass hierarchy for this assumption to be realized in nature). The other plasma parameters are the masses $m_b$, number densities $n_b$, and the charges $e_b = Z_b\, e$ (we also assume full ionization). The rate of energy density exchange from species $a$ to species $b$ is given by

$$\frac{d\mathcal{E}_{ab}}{dt} = -\mathcal{C}_{ab}(T,m,n,e)\Big(T_a - T_b\Big) \ , \tag{A1}$$

where $\mathcal{C}_{ab}(T,m,n,e)$ is the rate coefficient (in principle it can depend upon every plasma parameter in a non-separable manner). The rate coefficients as calculated by BPS can be decomposed into three parts

$$\mathcal{C}_{ab}^{\rm BPS} = \underbrace{\Big(\mathcal{C}_{ab,\rm R}^{\rm C} + \mathcal{C}_{ab,\rm S}^{\rm C}\Big)}_{\text{classical}} + \mathcal{C}_{ab}^{\rm QM} + \mathcal{O}(g^3) \ , \tag{A2}$$

where the classical contribution is given by

$$\mathcal{C}_{ab,\rm R}^{\rm C} = \frac{\kappa_a^2\,\kappa_b^2}{2\pi}\left(\frac{\beta_a m_a}{2\pi}\right)^{1/2}\left(\frac{\beta_b m_b}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} dv\, v^2 e^{-\frac{1}{2}(\beta_a m_a+\beta_b m_b)v^2}\,\frac{i}{2\pi}\,\frac{F(v)}{\rho_{\rm tot}(v)}\ln\left\{\frac{F(v)}{\kappa_e^2}\right\} \tag{A3}$$

$$\mathcal{C}_{ab,\rm S}^{\rm C} = -\kappa_a^2\,\kappa_b^2\,\frac{(\beta_a m_a \beta_b m_b)^{1/2}}{(\beta_a m_a + \beta_b m_b)^{3/2}}\left(\frac{1}{2\pi}\right)^{3/2}\left[\ln\left\{\frac{e_a\,e_b}{4\pi}\,\frac{\kappa_e}{4\,m_{ab}\,V_{ab}^2}\right\} + 2\gamma\right], \tag{A4}$$

and the quantum correction by

$$\mathcal{C}_{ab}^{\rm QM} = -\frac{1}{2}\,\kappa_a^2\,\kappa_b^2\,\frac{(\beta_a m_a\,\beta_b m_b)^{1/2}}{(\beta_a m_a+\beta_b m_b)^{3/2}}\left(\frac{1}{2\pi}\right)^{3/2}\int_0^{\infty} d\zeta\, e^{-\zeta/2}\left[{\rm Re}\,\psi\left(1+i\frac{\bar\eta_{ab}}{\zeta^{1/2}}\right) - \ln\left\{\frac{\bar\eta_{ab}}{\zeta^{1/2}}\right\}\right] \ . \tag{A5}$$

The reduced mass of species $a$ and $b$ is determined from

$$\frac{1}{m_{ab}} = \frac{1}{m_a} + \frac{1}{m_b} \ , \tag{A6}$$

while the thermal velocity and the quantum parameter are determined by

$$V_{ab}^2 = \frac{1}{\beta_a m_a} + \frac{1}{\beta_b m_b} \tag{A7}$$

$$\bar\eta_{ab} = \frac{e_a e_b}{4\pi\,\hbar V_{ab}} \ . \tag{A8}$$

The function $F(v)$ takes the form

$$F(v) = -\int_{-\infty}^{\infty} du\,\frac{\rho_{\rm tot}(u)}{v-u+i\eta} \quad \text{with} \quad \rho_{\rm tot}(u) = \sum_b \rho_b(u) \tag{A9}$$

$$\rho_b(v) = \kappa_b^2\sqrt{\frac{\beta_b m_b}{2\pi}}\; v\, \exp\left\{-\frac{1}{2}\,\beta_b m_b\, v^2\right\} \ , \tag{A10}$$

where its relation to the dielectric function is $k^2\,\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}) = k^2 + F(\hat{\mathbf{k}}\cdot\mathbf{v})$. The first term $\mathcal{C}^{\rm C}_{ab,{\rm R}}$ arises from long-distance collective effects from the dielectric function, and it involves *all* plasma species (even species $c$ different from $a$ and $b$). This is the term I call non-separable, meaning that it cannot be written as a sum of individual plasma components. The second term $\mathcal{C}^{\rm C}_{ab,{\rm S}}$ arises from short-distance two-body classical scattering,[1] and the third term $\mathcal{C}^{\rm QM}_{ab}$ is the two-body quantum scattering correction to all orders in the quantum parameters $\bar\eta_{ab}$.

A dramatic simplification occurs under the following conditions: (i) the extreme quantum limit is realized, *i.e.* $\bar\eta_{ab} \ll 1$, (ii) the ions have the same temperature $T_{\rm I}$ [large mass hierarchy, $m_e/m_{\rm I} \ll 1$], and (iii) sum over the ions to construct

$$\mathcal{C}_{e{\rm I}} = \sum\nolimits_i \mathcal{C}_{ei} \ . \tag{A11}$$

The rate equation now takes the form

$$\frac{d\mathcal{E}_{e{\rm I}}}{dt} = -\,\mathcal{C}^{\rm BPS}_{e{\rm I}}\Big(T_e - T_{\rm I}\Big) \ , \tag{A12}$$

where $d\mathcal{E}_{e{\rm I}}/dt = \sum_i d\mathcal{E}_{ei}/dt$. Because of the sum over ions and the extreme quantum limit, a sum-rule arises and the result simplifies to

$$\mathcal{C}^{\rm BPS}_{e{\rm I}} = \frac{\omega_{\rm I}^2}{2\pi}\,\kappa_e^2\,\sqrt{\frac{m_e}{2\pi\,T_e}}\,\ln\Lambda_{\rm BPS} \ , \quad \text{with} \quad \ln\Lambda_{\rm BPS} = \frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\} - \gamma - 1\right] \ , \tag{A13}$$

where $\omega_{\rm I} = \sum_i \omega_i$. As opposed to the model of Lee-More, there is no ion dependence inside the logarithm. The lack of ion dependence has also been observed by Diamante and Daligault in their MD simulations.

[1] For calculational convenience we have set the arbitrary wave number to the value $K = \kappa_e$ in both the regular and singular contributions.