The BPS Fokker Planck Equation

Robert L Singleton Jr

University of Leeds

School of Mathematics

LS2 9JT

(Dated: 3/24/2020)

Abstract

These are notes on the BPS Fokker-Planck equation.

I. INTRODUCTION

I would like to start these notes by addressing some potential misconceptions of the Brown-Preston-Singleton (BPS) paper [1], and to clarify some issues with the presentation of the original manuscript. Given the way in which the paper is structured, one might be led to think that the BPS stopping power omits small angle scattering, and is only valid in the Fokker-Planck limit. I should emphasize, however, that the BPS method employs the full Boltzmann equation, and indeed includes the scattering effects from all angles. The Boltzmann short-distance physics, joined together with the long-distance physics of the Lenard-Balescu equation, provides an exact result to leading and next-to-leading order in the plasma coupling g. The misconception that BPS only takes into account small angle collisions could well arise from our premature emphasis on the Fokker-Planck equation in Section IV of the BPS paper. In fact, the name of Section IV is *General Formalism*, and this is a misleading title. The section should more properly be called *The BPS Fokker-Planck Equation*, and it should have appeared later in the text. The location and title of this section gives the misleading impression that the BPS stopping power and temperature equilibration are performed only within the Fokker-Planck approximation. This misconception is reinforced by the way in which we present our results, namely, in terms of coefficients called $\mathcal{A}$ and $\mathcal{B}$ (or actually $\mathcal{A}$ and the trace $\sum_\ell C^{\ell\ell}$), which were introduced within the context of a Fokker-Planck description.

To be clear, BPS uses the method of dimensional continuation to calculate the rate of Coulomb energy exchange dE/dt and the rate of momentum exchange $\mathbf{v}\cdot d\mathbf{P}/dt$ for a beam of particles with incident velocity $\mathbf{v}$ in a weakly coupled and fully ionized plasma. These two independent (scalar) quantities are calculated within the BPS framework, using the short-distance physics of the Boltzmann equation and the long distance-physics of the Lenard-Balescu equation, and the results are exact to leading order ($g^2 \ln g$) and to next-to-leading order (g^2) in the plasma coupling. After these calculates were completed, we used them to *define* the coefficient functions $\mathcal{A}(v)$ and $\mathcal{B}(v)$ of a Fokker-Planck (FP) equation, which can then be used to study phenomena such as straggling and the angular separation of the incident beam. The BPS Fokker-Planck equation is an improved equation in that it gives dE/dt and $\mathbf{v}\cdot d\mathbf{P}/dt$ *exactly* to order $g^2 \ln g + g^2$, including *large angle* effects from the Boltzmann equation. However, all other quantities calculated from this FP equation are (of course) only accurate to leading order and do not include large-angle effects. The order of our presentation does not make this strategy entirely clear.

To summarize, we do four things in the BPS paper of Ref. [1]: (i) we calculate dE/dt and $\mathbf{v}\cdot d\mathbf{P}/dt$ for a projectile in a plasma, exactly to leading and next-to-leading order in g, (ii) we calculate the electron-ion temperature equilibration rate dE/dt exactly to leading and next-to-leading order in g, (iii) we use these calculations to define a Fokker-Planck equation, and (iv) we calculate the quantity $dE_\perp/dt$ to leading and next-to-leading order in g. Note that (i) and (ii) use the full Boltzmann equation, and consequently include large and small angle collisions. The resulting Fokker-Planck equation in (iii) is meant to be used in the small-angle limit in $\nu = 3$ dimensions, and, in general, is accurate to leading order. However, since the coefficients $\mathcal{A}$ and $\mathcal{B}$ have been defined in terms of dE/dt and $\mathbf{v}\cdot d\mathbf{P}/dt$, the BPS FP equation gives these quantities to leading and next-to-leading order, including the effects from all angles. One can think of this as a "corrected" Fokker-Plank equation in which the large-angle contributions to the stopping power (and the momentum deposition) have been included.

II. A FOKKER-PLANCK FORMULATION

A. Basic Formalism and Definitions

Let us now consider a background plasma consisting of multiple species b with temperature $T_b \equiv 1/\beta_b$, charge e_b, and mass m_b. Having calculated the basic quantities dE/dt and $\mathbf{v}\cdot d\mathbf{P}/dt$ using the method of dimensional continuation, we now return to $\nu = 3$. Consider an universalized swarm (or a beam) of test particles within the plasma. Suppose the test particles are identical and have mass m and distribution function f. Then the Fokker-Planck (FP) equation for the beam particles takes the form

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right] f(\mathbf{r},\mathbf{p},t) = \sum\nolimits_b \sum\nolimits_{k\ell} \frac{\partial}{\partial p^k}\, C_b^{k\ell}(\mathbf{r},\mathbf{p},t)\left[\beta_b v^\ell + \frac{\partial}{\partial p^\ell}\right] f(\mathbf{r},\mathbf{p},t)\ , \tag{2.1}$$

where $\mathbf{v} = \mathbf{p}/m$ and $\boldsymbol{\nabla} = \partial/\partial\mathbf{r}$. Given a physical quantity $q(\mathbf{p})$ associated with an individual particle, we define the corresponding spatial density and the flux vector by

$$\mathscr{Q}(\mathbf{r},t) = \int \frac{d^3p}{(2\pi\hbar)^3}\, q(\mathbf{p})\, f(\mathbf{r},\mathbf{p},t) \tag{2.2}$$

$$\mathscr{F}^k(\mathbf{r},t) = \int \frac{d^3p}{(2\pi\hbar)^3}\, q(\mathbf{p})\, v^k\, f(\mathbf{r},\mathbf{p},t)\ . \tag{2.3}$$

After integrating the momentum variable by parts, the FP equation (2.1) implies

$$\frac{\partial}{\partial t}\,\mathscr{Q}(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\boldsymbol{\mathscr{F}}(\mathbf{r},t) = -\sum\nolimits_b \int \frac{d^3p}{(2\pi\hbar)^3}\,\frac{dQ_b}{dt}(\mathbf{r},\mathbf{p},t)\, f(\mathbf{r},\mathbf{p},t) \tag{2.4}$$

where

$$\frac{dQ_b}{dt}(\mathbf{r},\mathbf{p},t) \equiv \sum\nolimits_{k\ell}\left[\beta_b v^\ell - \frac{\partial}{\partial p^\ell}\right] C_b^{k\ell}(\mathbf{r},\mathbf{p},t)\,\frac{\partial}{\partial p^k}\, q(\mathbf{p})\ . \tag{2.5}$$

Note that dQ_b/dt is the average rate of loss of the quantity $q(\mathbf{p})$ from the interaction of the beam with plasma component b.

B. Application to Energy and Momentum Loss

The energy density and energy flux take the form

$$\mathscr{U} = \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{p^2}{2m}\, f(\mathbf{r},\mathbf{p},t) \tag{2.6}$$

$$\mathscr{S}^k = \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{p^2}{2m}\, \frac{p^k}{m}\, f(\mathbf{r},\mathbf{p},t) ~, \tag{2.7}$$

where $E(\mathbf{p}) \equiv p^2/2m$ is the kinetic energy of a single projectile within the beam. The energy continuity equation is

$$\frac{\partial \mathscr{U}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathscr{S}} = -\sum\nolimits_b \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{dE_b}{dt}\, f(\mathbf{r},\mathbf{p},t) ~. \tag{2.8}$$

The momentum density and momentum flux are

$$\mathscr{P}^k = \int \frac{d^3p}{(2\pi\hbar)^3}\, p^k\, f(\mathbf{r},\mathbf{p},t) \tag{2.9}$$

$$\mathscr{T}^{k\ell} = \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{p^k p^\ell}{m}\, f(\mathbf{r},\mathbf{p},t) ~, \tag{2.10}$$

and the continuity equation is

$$\frac{\partial \mathscr{P}^k}{\partial t} + \nabla^\ell \mathscr{T}^{k\ell} = -\sum\nolimits_b \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{dP_b}{dt}\, f(\mathbf{r},\mathbf{p},t) ~. \tag{2.11}$$

The rate of change of energy and momentum takes the form

$$\frac{dE_b}{dt} = \sum\nolimits_{k\ell} \left[\beta_b v^\ell - \frac{\partial}{\partial p^\ell}\right] C_b^{k\ell}\, v^k ~, \tag{2.12}$$

$$\frac{dP_b^k}{dt} = \sum\nolimits_{\ell} \left[\beta_b v^\ell - \frac{\partial}{\partial p^\ell}\right] C_b^{k\ell} ~. \tag{2.13}$$

We have used $\partial E/\partial p^k = p^k/m = v^k$ to express (2.12), and $\partial p^k/\partial p^r = \delta^{kr}$ in deriving (2.13). We can also write (2.13) in the form

$$v^k\, \frac{dP_b^k}{dt} = \frac{dE_b}{dt} + \frac{1}{m} \sum\nolimits_\ell C_b^{\ell\ell} \,, \tag{2.14}$$

which is BPS Eq. (4.22). By calculating dE_b/dt and dP^k/dt (or actually $v^k\, dP^k/dt$) using the BPS method, we can define an FP tensor $C_b^{k\ell}$. As calculated by the FP equation, the energy and momentum exchange in the plasma are given by (2.12) and (2.13). However, since these quantities were calculated using the BPS method, we see that the BPS Fokker-Planck equation gives exact (including large angles) results for the rates of energy and momentum exchange. For all other quantities, the BPS Fokker-Planck equation is only accurate to leading order in the small angle limit.

C. Decomposition of the FP Tensor

We can decompose the FP tensor $C_b^{k\ell}(\mathbf{p})$ into longitudinal and transverse components along $\mathbf{p}$ (or equivalently along $\mathbf{v} = \mathbf{p}/m$),

$$C_b^{k\ell}(\mathbf{p}) = \mathcal{A}_b(v)\, \frac{\hat{v}^k\, \hat{v}^\ell}{\beta_b v} + \mathcal{B}_b(v)\, \frac{1}{2}\big(\delta^{k\ell} - \hat{v}^k\, \hat{v}^\ell\big) \ , \tag{2.15}$$

where $\hat{\mathbf{v}} = \mathbf{v}/v$ is the unit vector in the $\mathbf{v}$ direction. This is BPS Eq. (4.18). The time dependence has been left implicit, as has the spatial dependence (we are in fact only interested in a spatially uniform and isotropic plasma). It follows from (2.12) that

$$\frac{dE_b}{dt} = \left[v - \sum\nolimits_\ell \frac{1}{\beta_b m}\, \frac{\partial}{\partial v^\ell}\, \hat{v}_\ell\right] \mathcal{A}_b(v) \tag{2.16}$$

$$= \left[v - \frac{2}{\beta_b m v} - \frac{1}{\beta_b m}\frac{\partial}{\partial v}\right] \mathcal{A}_b(v)\,, \tag{2.17}$$

which is BPS Eq. (4.21). Note that the (three dimensional) trace of (2.15) is

$$\sum\nolimits_\ell C_b^{\ell\ell} = \frac{\mathcal{A}_b}{\beta_b v} + \mathcal{B}_b \ . \tag{2.18}$$

D. A Columnated Beam

Now consider a columated beam with indicdent velocity $\mathbf{v}_p$ and particle mass m_p. We can define the stopping power and momentum loss per unit distance as

$$\frac{dE_b}{dx}(v_p) = \frac{1}{v_p}\, \frac{dE_b}{dt}(v_p) \tag{2.19}$$

$$\frac{d\mathbf{P}_b}{dx}(v_p) = \frac{1}{v_p}\, \frac{d\mathbf{P}_b}{dt}(v_p) \ , \tag{2.20}$$

so that

$$\frac{dE_b}{dx}(v_p) = \left[1 - \frac{2}{\beta_b m_p v_p^2} - \frac{1}{\beta_b m_p v_p}\frac{\partial}{\partial v_p}\right] \mathcal{A}_b(v_p) \tag{2.21}$$

$$\mathbf{v_p} \cdot \frac{d\mathbf{P}_b^k}{dx}(v_p) = \frac{dE_b}{dx}(v_p) + \frac{1}{m_p v_p} \sum\nolimits_\ell C_b^{\ell\ell}(v_p) \tag{2.22}$$

$$= \left[1 - \frac{1}{\beta_b m_p v_p^2} - \frac{1}{\beta_b m_p v_p}\frac{\partial}{\partial v_p}\right] \mathcal{A}_b(v_p) + \frac{1}{m_p v_p}\, \mathcal{B}_b(v_p) \ . \tag{2.23}$$

Define the *transverse energy* by

$$E_\perp(\mathbf{p}) = \frac{1}{2}\, m_p \Big[\mathbf{v}^2 - (\mathbf{v}\cdot\hat{\mathbf{v}}_p)^2\Big] ~, \tag{2.24}$$

where $\mathbf{p} = m_p\mathbf{v}$, and $\hat{\mathbf{v}}_p = \mathbf{v}_p/v_p$ is the unit vector in the direction of the projectile velocity $\mathbf{v}_p$. We can also write the transverse energy as

$$E_\perp(\mathbf{p}) = \frac{1}{2m_p} \sum\nolimits_{k\ell} p^k p^\ell \left[\delta^{k\ell} - \hat{v}_p^k\, \hat{v}_p^\ell\right] ~. \tag{2.25}$$

The transerve energy density and energy flux take the form

$$\mathscr{U}_\perp = \int \frac{d^3p}{(2\pi\hbar)^3}\, E_\perp(\mathbf{p})\, f(\mathbf{r},\mathbf{p},t) \tag{2.26}$$

$$\mathscr{S}_\perp^k = \int \frac{d^3p}{(2\pi\hbar)^3}\, E_\perp(\mathbf{p})\, \frac{p^k}{m}\, f(\mathbf{r},\mathbf{p},t) ~, \tag{2.27}$$

and the continuity equation is

$$\frac{\partial \mathscr{U}_\perp}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathscr{S}}_\perp = -\sum\nolimits_b \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{dE_{b\,\perp}}{dt}\, f(\mathbf{r},\mathbf{p},t) ~. \tag{2.28}$$

Acknowledgments

I am particularly indebted to Jean-Etienne Sauvestre for reading the BPS manuscript so thoroughly, and for helping clarify a number of potential problems with the structure of the text.

Appendix A: Straggling and Angular Spreading

1. Exact Definitions and Relations

There are three independent quantities that one can calculate: dE/dt, $\mathbf{v}_p \cdot d\mathbf{P}/dt$, and $dE_\perp/dt$. Since $dE/dt = dE_\perp/dt + dE_\parallel/dt$, we can trade off $dE_\parallel/dt$ for $dE_\perp/dt$. Every time t that $\mathbf{v}_p$ is specified must be considered as an initial time, and could write $\mathbf{v}_p(t)$ to emphasize that the velocity is given at the initial time t. Furthermore, we must specify all ensemble averages $\langle X \rangle$ at the initial time. Let us form the following. Since the beam is sharply peaked about $\mathbf{v}_p$, we find

$$\langle v^k(t_0)\rangle = v_p^k \qquad\qquad T^{kl} = \langle v^k(t_0) v^l(t_0)\rangle = v_p^k v_p^l \tag{A1}$$

$$P^k = m\langle v^k(t_0)\rangle = m v_p^k \qquad E = \frac{1}{2} m \mathbf{v}_p^2 = \frac{1}{2} m T^{kk}\,. \tag{A2}$$

These are fixed, given by the initial conditions; however, the time derivative of a quantity is is determined by the dynamics,[1]

$$\frac{d}{dt}\langle v^k(t)\rangle = \Big\langle \frac{dv^k(t)}{dt} \Big\rangle \qquad\qquad \frac{d}{dt} T^{kl}(t) = \Big\langle \Big[\frac{dv^k(t)}{dt} v^l(t) + v^k(t)\frac{dv^l(t)}{dt}\Big]\Big\rangle \tag{A3}$$

$$v_p^k\, \frac{dP^k(t)}{dt} = v_p^k\, m \,\Big\langle \frac{dv^k(t)}{dt}\Big\rangle \qquad \frac{dE}{dt} = \frac{1}{2}\, m\, \frac{dT^{kk}(t)}{dt}\ . \tag{A4}$$

Using the tensor $T^{kl}(t)$, we can define the quantities $dE_\parallel/dt$ and $dE_\perp/dt$ by taking moments with respect to the direction of motion,

$$\hat{\mathbf{v}}_p = \frac{\mathbf{v}_p}{|\mathbf{v}_p|} = \frac{\mathbf{v}_p}{v_p}\ , \tag{A5}$$

[1] Look at footnote 18 in BPS. I am not sure Eq. (4.33) is correct to order g^2 under the log. We should also look at Refs. [15] and [16] in BPS.

namely

$$\frac{dE_\parallel}{dt} = \hat{v}_p^k\,\hat{v}_p^l\,\frac{1}{2}\,m\,\frac{d}{dt}T^{kl}(t) = \hat{v}_p^k\,\hat{v}_p^l\,m\left\langle\frac{dv^k(t)}{dt}\,v^l(t)\right\rangle \tag{A6}$$

$$\frac{dE_\perp}{dt} = \Big[\delta^{kl} - \hat{v}_p^k\,\hat{v}_p^l\Big]\frac{1}{2}\,m\,\frac{d}{dt}T^{kl}(t) = \Big[\delta^{kl} - \hat{v}_p^k\,\hat{v}_p^l\Big]m\left\langle\frac{dv^k(t)}{dt}v^l(t)\right\rangle\,. \tag{A7}$$

Note that

$$\frac{dE}{dt} = \frac{dE_\parallel}{dt} + \frac{dE_\perp}{dt}\,, \tag{A8}$$

illustrating that $dE_\parallel/dt$ and $dE_\perp/dt$ are not independent in our formulation. We close this section by examining the velocity fluctuations,

$$\begin{aligned}\frac{d}{dt}\widetilde{T}^{kl}(t) &\equiv \frac{d}{dt}\Big\langle\Big[v^k(t) - \langle v^k(t)\rangle\Big]\Big[v^l(t) - \langle v^l(t)\rangle\Big]\Big\rangle \\ &= \frac{d}{dt}T^{kl}(t) - \Big\langle\frac{dv^k(t)}{dt}\Big\rangle v_p^l - v_p^k\Big\langle\frac{dv^l(t)}{dt}\Big\rangle\,,\end{aligned} \tag{A9}$$

and thus

$$\frac{d\widetilde{E}_\parallel}{dt} = \frac{dE_\parallel}{dt} - v_p^k\,\frac{dP^k}{dt}\,. \tag{A10}$$

And independent calculation gives

$$\frac{d\widetilde{E}_\perp}{dt} = \frac{dE_\perp}{dt}\,. \tag{A11}$$

At high velocities, we expect that $d\widetilde{E}_\parallel/dt \approx 0$. Then $dE_\perp/dt \approx dE/dt - \mathbf{v}_p\cdot d\mathbf{P}/dt$.

2. Isotropic Plasma

In an isotropic plasma, $d\mathbf{P}/dt$ points in the direction $\mathbf{v}_p$, which means that

$$\frac{dP^k}{dt} = \hat{v}_p^k\,\dot{P}(t)\,, \tag{A12}$$

where $P(t)$ is a scalar. BPS calculates the quantity $\dot{P} = dP/dt$ to leading to next-to-leading order in g. BPS also calculates dE/dt to the same order. This determines the $\mathcal{A}$ and $\mathcal{B}$ coefficients defined above. Also note the exact relation

$$\frac{d}{dt}\,\frac{1}{2}\,mT^{kl}(t) = \hat{v}_p^k\,\hat{v}_p^k\,\frac{dE_\parallel}{dt} + \frac{1}{2}\Big[\delta^{kl} - \hat{v}_p^k\,\hat{v}_p^l\Big]\frac{dE_\perp}{dt}\,, \tag{A13}$$

which gives a physical interpretation to $dE_\parallel/dt$ and $dE_\perp/dt$.

3. Transverse Energy Loss

There is an ambiguity in the definition of the BPS Fokker-Planck equation. We have chosen the coefficients $\mathcal{A}$ and $\mathcal{B}$ in such a way as to give exact large-angle corrections to dE/dt and $\mathbf{v}_p \cdot d\mathbf{P}/dt$. As such, when we evaluate $dE_\perp/dt$ by this FP equation, the result is only accurate to order $g^2 \ln g$. Equation (4.30) of BPS gives the Fokker-Planck evaluation

$$\frac{dE_\perp}{dt} = \frac{1}{m}\mathcal{B}\,. \tag{A14}$$

In Section X of BPS, we calculate the g^2 correction, thereby providing all three independent quantities exactly to leading and next-to-leading order.

** On the other hand, the discussion after (A11) suggests that

$$\frac{dE_\perp}{dt} \approx \frac{dE}{dt} - \mathbf{v}_p \cdot \frac{d\mathbf{P}}{dt}\,. \tag{A15}$$

But Eqs.(4.23) and (4.24) of BPS yield

$$\begin{aligned}\frac{dE}{dt} - \mathbf{v}_p \cdot \frac{d\mathbf{P}}{dt} &= \frac{1}{m}C^{ll} \\ &= \frac{1}{\beta m v_p}\mathcal{A} + \frac{1}{m}\mathcal{B}\end{aligned} \tag{A16}$$

which is far away from BPS Eq. (4.30) which we quoted in our Eq. (A14) except for such high projectile velocities that $(m_p/m_e)(T/E_p) \ll 1$, in which case Eqs.(10.42) and (10.45) of BPS show that the $\mathcal{A}$ term in Eq. (A16) can be neglected.

At any rate, as described in BPS, the correction (11.21) added to the approximation (4.30) quoted above gives $dE_\perp/dt$ to order $\ln g^2 + g^2$.

---

[1] Lowell S Brown, Dean L Preston, and Robert L Singleton, Jr, *Charged Particle Motion in a Highly Ionized Plasma*, Phys. Rep. **410** (2005) 237-333.