Optimal Streaming Feature Selection

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August 31, 2012

Abstract

Streaming feature selection identifies variables to add to a model by testing a sequence of proposed variables. Rather than specify a set of explanatory variables as in the typical stepwise regression, streaming selection allows the search for predictors to adapt to the success of prior choices. The choice of the next variable can depend on results of prior tests. To avoid over-fitting, alpha-investing controls the expected false discovery rate (mFDR) of these sequential tests. Alpha investing rules, however, provide the modeler with considerable flexibility in both the selection of features to test as well as the alpha-level committed to each test. Our work here identifies optimal strategies for alpha investing that perform within a tolerance of the best possible. The modeler is then free to focus on strategies for choosing the next explanatory variable rather than setting the level of the next test. We show that a spending rule based on a universal prior is competitive with an oracle that knows the strategy of the modeler and sets the signal of the tested hypothesis. We also consider the risk-ratio of the selected model vis-a-vis that of the oracle model.

Key Phrases:

- note that we find the convex hull of the feasible set of solutions
- we start the universal at 1

*Research supported by NSF grant DMS-1106743

- universal is above, below, then ultimately above others
- try things with uniform since it saves the most among monotone
- bulge solved by rotating gamma?
- Why the up down for the fits with gamma 0.025?

1 Testing Game

The choice of an optimal spending plan is difficult to formulate in total generality, but a special case related to information theory suggests a general plan. We call this case the 'testing game' to remind us all that it is contrived. In the game, a player (the statistician) is confronted by a sequence of null hypotheses H_1, H_2, ... chosen by Nature. The objective of the player is to detect a false null hypothesis positioned by Nature within this sequence. The false null is H_K, where $0 < K$ is a random variable with monotone decreasing probabilities, $P(K = k) = p_k$, $k = 1, 2, \ldots$. The player begins the game with alpha-wealth $0 \leq W$, and *at the start* of the game and without knowing the distribution p, the player must announce a plan for how this wealth will be invested in testing the sequence of hypotheses. That is, the player commits to invest $q_1 W$ to the test of H_1, $q_2 W$ to the test of H_2 and so forth, with $\sum q_j = 1$ with $0 \leq q_j$ (q is a discrete probablity distribution). The player wins the amount $g(q_K W)$ where g is a monotone non-decreasing function. The more the player commits to testing the false null, the more the player wins. The question is then "How should the player allocate the alpha-wealth W?"

If the player knows Nature's probability distribution p and the payoff function $g = \log$, then the optimal allocation is an immediate property of the divergence (or Jensen's inequality). The divergence (or relative entropy) between two discrete probability distributions is

$$D(p \parallel q) = \mathbb{E}_p \log \frac{p}{q} = \sum_j \log \left(\frac{p(j)}{q(j)} \right) p(j) \geq 0 \, . \tag{1}$$

In this special version of the Testing Game, the expected amount won is $\mathbb{E}\, g(q_K W) = \mathbb{E}\, \log q_K W \leq \mathbb{E}\, \log p_K W$ because the divergence is non-negative. The best strategy is to allocate the alpha-wealth according to the probabilities used by nature to pick the location of the false null. In this contrived setting, the Testing Game is equivalent to finding the binary code for K with the shortest expected length. The solution is well known to be the binary code with length determined by the probability distribution of K. {**ras:** *Can there be any further generality, even here, such as to other functions than log? I thought so, but log is special in that $\log x - \log y = \log(x/y)$ allows one to play the Jensen game in showing the divergence is positive.*}

Information theory also supplies a solution to the more general problem of allocating W when p is unknown. Assume now that Nature picks the distribution p after the player announces the strategy q, with the goal of minimizing the expected winnings of the player. The player now needs a strategy q which solves the minimax problem

$$\max_q \min_p \mathbb{E}\, g(q_K) \tag{2}$$

Provided we limit nature to monotone decreasing distributions for which $p_{k+1} \le p_k$ and keep $g = \log$, the optimal strategty of the player is to use the universal distribution of Elias (1975); Rissanen (1983). A simple version of the universal distribution is given by the distribution

$$u_j = \frac{c_u}{j + \log(1+j)} \ , \quad j = 1, 2 \ldots, \tag{3}$$

where the constant $c_j \approx 3.388$ normalizes the sum so that $\sum u_j = 1$.

The Testing Game motivates the use of the universal spending rule u, but the risk of the estimator implied by this approach are unknown. Suppose that the universal distribution is used to define an alpha-investing rule. Then, since tests produced by following an alpha-investing rule control mFDR, we know that the sequence of tests defined by the universal distribution control false positives in this sense as well. Optimality in the Testing Game, however, does not imply optimality in the false discovery rate. We further do not know the risk associated with this procedure. To describe the risk, assume that the hypotheses being tested are of the form $H_j : \mu_j = 0$ with available test statistics $\overline{y}_1$, $\overline{y}_2$, ... The tests then define estimators (testimators)

$$\hat{\mu}_j = \begin{cases} 0 & \text{if } H_j \text{ is not rejected,} \\ \overline{y}_j & \text{otherwise.} \end{cases} \tag{4}$$

Such testimators are equivalent to hard thresholding with the threshold set adaptively by the allocated alpha-level.

Our objective here is to understand the risk of these estimators.

2 Introduction

Emphasize that we have a computational approach that determines an attainable upper bound on the risk rather than a mathematical procedure that gives an inequality that obtains asymptotically with error terms (stat) or with some high probability (mach learning style). We obtain nonasymptotic results in moderate size problems, with the scope of the results limited only by computing.

Note that there are not results (?) that connect FDR to the risk of the associated estimators. We might be able to do that, though not clear to me how we can keep track of the growing/necessary state.

Our results are in a different style in that not just the nearly black sparse setting, but rather more general.

Convexity. A simple randomization argument shows that any linear combination of the mean points that we found is attainable. That's not the same, however, as showing that the actual set is convex. Each calculation we do (for some gamma or direction angle) finds the maximum value of the function in that direction. Hence, you cannot get further out in that direction. That removes half of the space; a collection of these leaves a convex interior. That convex interior holds the collection of solutions. The smaller the angle between direction vectors, the closer we approximate the feasible set. Rather than get the intersection of *all* half-spaces that hold the feasible region, we git a finite number of them.

Streaming feature selection is a sequential method for identifying predictive features for a model. Rather than choose the best feature from a fixed, predetermined collection, streaming feature selection evaluates candidate predictors one-at-a-time, following an order specified by an exogenous source. Unlike stepwise regression, streaming selection evaluates each predictor in the context of the current model, without knowledge of others that might follow. Sequential feature selection allows the evaluation of initial features to generate others, allowing the model to consider infinite sequences of explanatory variables. Alpha-investing provides control of over-fitting in this context without the sacrifice of data to cross-validation.

As an example for comparison, consider the familiar setting in which one observes p vectors of features $x_j \in \mathbb{R}^n$ to consider using as explanatory variables in a model to predict a response $y \in \mathbb{R}^n$. The classical version of this problem is the choice of features for the homoscedastic linear regression model

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i, \qquad \mathbb{E}\,\epsilon_i = 0,\ \mathrm{Var}(\epsilon_i) = \sigma^2, \tag{5}$$

in which many $\beta_j \approx 0$. Forward stepwise regression is perhaps the earliest algorithmic approach to identifying the variables with statistically interesting, non-zero coefficients. Stepwise regression begins by selecting the variable, say $x_{(1)}$, that has the largest correlation with y, $x_{(1)} = \arg\max_{x_j} \text{corr}(x_j, y)$. The algorithm continues adding variables $x_{(2)}, x_{(3)}, \ldots$ until some stopping criterion is reached, such as AIC or BIC. In the somewhat utopian case in which the observed vectors x_j are orthonormal ($x_j' x_k = I_{\{j=k\}}$) and one has an independent estimate s^2 of σ^2, one can use Bonferroni (hard thresholding). The least squares estimators are $b_j = x_j' y$. This criterion picks those variables for which the usual t-statistic $t_j = b_j/s$ exceeds the threshold $\tau \approx \sqrt{2 \log p}$. Further improvements related to FDR reduce the threshold adaptively.

Streaming feature selection operates differently. An exogenous source determines the first feature to examine. Rather than find the feature that maximizes the correlation with y among all p possible variables, streaming selection begins with the first variable that a researcher selects. For example, a scientist might order the genes to consider when modeling their impact on disease in a micro-array experiment, or an economist might have a particular pet variable or theory that explains unemployment. In any case, we assume that the indexing of the explanatory variables reflects this precedence. A streaming selection algorithm then tests this first variable x_1 for statistical significance using some variation on the usual t-statistic, say t_1. For this paper, we assume that this statistic is computed conservatively in the manner of Foster and Stine (2004). If the initial t-statistic exceeds a threshold, $|t_1| > \tau_1$, the algorithm adds x_1 to the regression model. Otherwise, the model is left empty. Whether x_1 joins the model or not, the algorithm continues on to examine the second variable, with the choice of x_2 possibly influenced by the outcome of the initial test. The geneticist, for instance, might alter the choice of the gene represented by x_2 depending on whether x_1 was selected into the model. Alternatively, a scientist might decide to use the variable $x_2 = x_1^2$ or some other transformation of x_1 depending on the outcome of the first test. Although one begins with a predefined set of p features, this ability to combine and transform variables dynamically expands the scope of the search. If the t-statistic exceeds the next threshold, $|t_2| > \tau_2$, x_2 joins the model. The search then moves on to consider the next variable. Our results here consider methods for setting the sequence of thresholds $\tau_1, \tau_2, \ldots$.

This illustration highlights two characteristics that separate streaming feature selection from stepwise selection. The first characteristic collects the advantages of streaming selection, and the second implies the need for a different means to control the selection of variables.

One feature is considered at each step of streaming feature selection This step leads to an evident speed advantage over stepwise selection and means that one evaluates the statistical significance of x_j quite differently from the method applied to $x_{(j)}$. The stepwise choice $x_{(j)}$ must exceed a threshold for significance derived from the maximum of a collection of test statistics, whereas the test of x_j does not. Consequently the streaming threshold is possibly much smaller ($\tau_1 < \tau_{(1)}$), offering gains in power — provided the exogenous ordering is reasonable.

The scope of the streaming search is not initially known In particular, the number of possible features considered in the search is allowed to change as the search proceeds. The initial collection is allowed to expand to include powers x_j^2, interactions $x_j\, x_k$, functional transformations $g(x_j)$, and so forth.

The sequential, unbounded nature of the search requires an adaptive means of setting the sequence of thresholds $\tau_,\tau_2,\ldots$. Because one does not know the size of the search space or the full collection of features to be considered, methods such as AIC or Bonferroni are not suited to controlling the search in streaming feature selection.

3 Alpha Investing

Alpha investing (Foster and Stine, 2008) is a method for testing a possibly infinite sequence of hypotheses, designed with this application to model selection in mind. Alpha investing begins with an initial allocation W_1 of alpha wealth; alpha wealth is the current total alpha level that can be spent to test subsequent hypotheses. Label a sequence of null hypotheses $H_1, H_2, \ldots$, such as the sequence $H_j : \beta_j = 0$ in regression. An alpha investing rule can test H_1 at any level up to the total available alpha wealth, $0 \le \alpha_1 \le W_1$. The level α_1 is 'spent' and cannot be used for subsequent tests. Let p_1 denote the p-value of the test of H_1. If $p_1 \le \alpha_1$, the initial test rejects H_1. In this case, the alpha investing rule earns an additional contribution $\omega > 0$ to its alpha wealth; ω
otherwise, the alpha wealth available for subsequent tests falls to $W_2 = W_1 - \alpha_1$. We typically set $\omega = 0.05$ in applications. In general, the alpha wealth available for the test of H_{j+1} is W_j

$$W_{j+1} = W_j - \alpha_j + \omega\, I_{\{p_j < \alpha_j\}} \tag{6}$$

Alpha investing resembles alpha spending used in clinical trials, with the key distinction that rejecting a hypothesis earns an additional allocation ω of alpha wealth for subsequent testing. An alpha spending rule, such as the Bonferroni rule, controls the family wide error rate (FWER). FWER is the probability of falsely rejecting any null hypothesis. An important special case is control of FWER under the so-called complete null hypothesis for which H_j holds for all tests. We refer to controlling FWER under the complete null hypothesis as controlling FWER in the weak sense.

With the possible injection of additional alpha wealth, alpha investing weakly controls FWER and a version of the false discovery rate (FDR). Let $R(j)$ count the number of hypothesis rejected in the first j tests, and let $V(j) \leq R(j)$ denote the number of false rejections through the first j tests. Then we can define the criterion

$$\mathrm{mFDR}(j) = \frac{\mathbb{E}\, V(j)}{1 + \mathbb{E}\, R(j)} \ . \tag{7}$$

FDR uses the expected value of the ratio whereas mFDR uses the ratio of expected values. Foster and Stine (2008) show that alpha investing controls $\mathrm{mFDR}(j) \leq \omega$, and this result implies weak control of the FWER. The index j is allowed to be an arbitrary stopping time, such as the occurrence of the kth rejection.

Although the ordering of the variables to consider for selection in a model is key to the success of streaming feature selection, we show in this paper that the investigator need not be so concerned about how to set the sequence of levels α_j. Basically, we demonstrate alpha investing rules that perform as well as competitors that know the underlying parameters. The use of such a rule allows the investigator to focus on the search strategy rather than nuances of the choice of α_j. For our analysis, we consider alpha investing rules that are defined by a monotone discrete distribution on non-negative integers. Let $\mathcal{F} = \{f : \{0, 1, \ldots\} \mapsto \mathbb{R}^+,\ f(j) \geq f(j+1),\ \sum_j f(j) = 1\}$ denote the collection of monotone, non-increasing probability distributions on the non-negative integers. Each $f \in \mathcal{F}$ defines an alpha investing rule that 'resets' after rejecting a null hypothesis. Given the wealth after rejecting some hypothesis H_{k-1} is W_k, then the levels for testing subsequent hypotheses H_{k+j}, $j = 0, 1, \ldots$, are $\alpha_{k+j} = W_k\ f(j)$, until the next rejection. Monotonocity implies that the alpha investing rule spends more heavily after rejecting a hypothesis than otherwise; such rules are suitable in applications in which significant factors cluster, as if non-zero μ_j occur in bundles. Foster and Stine (2008) offer further motivation for this approach.

We focus on two members of this class of alpha investing rules: those identified by a geometric distribution and by a universal distribution. Geometric alpha investing rules spend a fixed fraction ψ of the current wealth on each round. Let $g_\psi(j) = \psi(1-\psi)^j$, $j = 0, 1, \ldots$, denote the geometric distribution with parameter $0 < \psi < 1$. For example, g_ψ, ψ the geometric rule with $\psi = 0.25$ invests one-fourth of the current alpha wealth in the test of H_j, $\alpha_j = W_j/4$. In general, given wealth W_k after rejecting H_{k-1}, say, the amount invested in testing H_{k+j} is $\alpha_{k+j} = W_k\, g_\psi(j)$. Large values for ψ rapidly spend down the alpha wealth available after a rejected hypothesis. Geometric investing is natural and related to entropic approximations to distributions. That is, given a fixed set of initial probabilities, assigning a geometric tail to the distribution minimizes the relative entropy of the approximation (cite).

The second alpha investing rule uses a version the universal prior for integers defined by Rissanen (1983). The universal prior arises in the context of encoding a sequence of positive integers using a prefix code. A geometric rule spends a constant fraction of the wealth on each test. The universal rule instead invests a diminishing proportion of the available alpha wealth. Of the wealth W_k available after a rejecting H_{k-1}, say, the universal rule invests $\alpha_{k+j} = W_k\, u(j)$ with

$$u_\delta(j) = \frac{c}{(j+\delta)(\log(1+j+\delta))^2}, \qquad j = 0,\, 1,\, \ldots, \quad 1 \le \delta. \tag{8}$$

in the tests of H_{k+j} until the next rejection. (c_δ is a normalizing constant so that the discrete probabilities $u_\delta(j)$ add to 1; for example, $c_1 \approx 3.388$ and $c_{20} \approx 0.3346$.) The constant δ serves as an offset that slows the initial spending rate; our examples fix $\delta = 20$ and we abbreviate $u_{20}(j) = u(j)$. (For instance, $u_1(0) \approx 0.614$ so that the rule spends about 60% of the available wealth on the first test.) More elaborate forms of the universal distribution make use of the so-called log-star function, defined as $\log^* x = \log x + \log\log x + \cdots$, where the sum accumulates only positive terms. For example, $\log^* 8 = \log 8 + \log\log 8$. The version (8) simply uses the first two summands of the $\log^*$ function. δ

4 Optimal Alpha Investing

To evaluate an alpha investing rule, we consider its performance in the following context. Consider testing a sequence of T one-sided null hypotheses $H_j : \mu_j \le 0$, $j = 1, 2, \ldots, \mu_T$ versus the alternatives $H_{j,a} : \mu_j > 0$. Denote the collection of mean parameters $\mu_{1:T} = \{\mu_1, \mu_2, \ldots, \mu_T\}$. The test statistics are $Z_j \sim N(\mu_j, 1)$. The Z_j are independent and observed one at a time; the test of H_k is made in the knowledge of prior Z_j for $j < k$, but future $Z_j, j > k$ are unknown. The number of tests T is fixed and known at the start of testing. For the purpose of calculating risks, a sequence of tests defines a sequence of 'testimators' by setting $\hat{\mu}_j = Z_j$ if $p_j < \alpha_j$ and zero otherwise. $\mu_{1:T}$

Within this context, how does the choice of an investing rule influence the performance of alpha investing? Let U denote a figure of merit, or utility, provided by using an alpha investing rule. For example, the utility might be the expected number of hypothesis $H_j : \mu_j = 0$ rejected by the alpha investing rule defined by the distribution

$f \in \mathcal{F}$:

$$U_r(\mu_{1:T},\, f) \;=\; \mathbb{E}_{\mu_{1:T}} \sum_{j=1}^{T} r_{\mu_j}(\alpha_j) \tag{9}$$

$$= \; r_{\mu_1}(\alpha_1)\,(1 + r_{\mu_2}(W_1 - \alpha_1 + \omega) + \cdots) + (1 - r_{\mu_1}(\alpha_1))\,(r_{\mu_2}(W_1 - \alpha_1) + \cdots)\ , \tag{10}$$

where $r_\mu(\alpha) = \Phi(\mu - z_\alpha)$ is the probability of rejecting, $z_\alpha = \Phi^{-1}(1-\alpha)$ is the normal quantile, and Φ is the cumulative standard normal distribution. In (9), α_j denotes the amount invested in the test of H_j by following the investing rule defined by the distribution f; this notation suppresses the detail of how prior rejections influence this random variable as suggested by (10) that shows how α_2 depends on the prior outcome. Alternatively, we also measure the utility in the sense of accumulated (negative) risk. If $Z \sim N(\mu, 1)$, then the risk of the testimator $\hat{\mu} = Z\, I_{\{Z<z_\alpha\}}$ is

$$R_\mu(\alpha) = (1 - r_\mu(\alpha))\mu^2 + (z_\alpha - \mu)\phi(z_\alpha - \mu) + \Phi(\mu - z_\alpha) \tag{11}$$

The associated cumulative utility is then

$$U_R(\mu_{1:T},\, f) = \mathbb{E}_{\mu_{1:T}} \sum_{j=1}^{T} R_{\mu_j}(\alpha_j) \tag{12}$$

To compare two alpha investing rules defined by $f,\, g \in \mathcal{F}$, define the *performance envelope* of $(f,\, g)$ to be the region

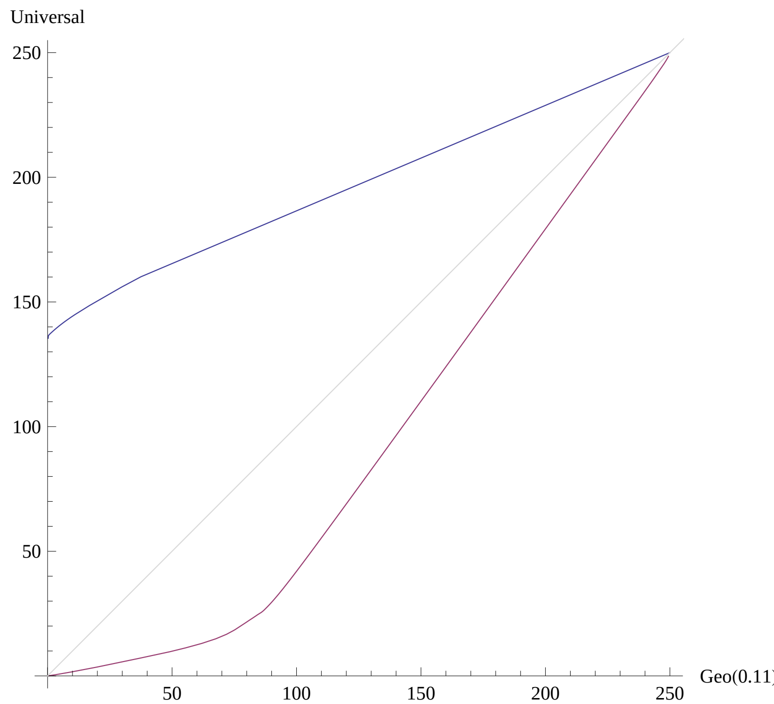
$$\mathrm{PE}(f, g; \rho) = \{(x,\, y) \in \mathbb{R}^2 : \exists\ \mu_{1:T} \in \mathbb{R}^T\ s.t.\ x = U_\rho(\mu_{1:T}, f),\, y = U_\rho(\mu_{1:T}, g))\}\ . \tag{13}$$

The point $(x,\, y)$ lies in the performance envelope if there exists a sequence of means for which these coordinates identify the utilities obtained by the two alpha investing rules. As an example, Figure 1 shows $\mathrm{PE}(g_{0.11}, u;\ r)$, the rejection performance envelope of alpha investing with a geometric distribution having $\psi = 0.11$ versus the universal distribution u defined in (8). Points within the performance envelope of $(g_{0.11}, u)$ that lie below the diagonal indicate parameters $\mu_{1:T}$ for which $g_{0.11}$ produces higher utility than u; those above the diagonal (the larger portion of Figure 1) indicate u dominates $g_{0.11}$. In this example, the universal distribution dominates the geometric almost everywhere. The advantage is particularly stark near the origin; in this 'nearly black' situation, few hypotheses are rejected and the universal rule produces far better performance.

5 Finding the Performance Envelope

We identify the boundary of the performance envelope by solving a collection of one-dimensional optimizations. Figure 2 illustrates the method used to identify a boundary

Figure 1: Performance envelope $\mathrm{PE}(g_{0.11}, u;\ r)$ shows the expected number of rejected hypotheses obtained by the geometric alpha investing rule $g_{0.11}$ versus the universal investing rule ($T = 250$)



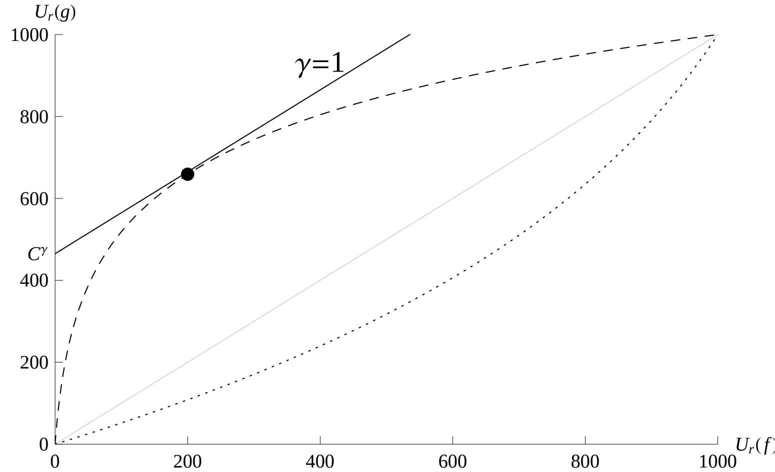
point of $\mathrm{PE}(f, g; r)$ that lies above the diagonal. Pick some value $\gamma > 0$; $\gamma = 1$ in the figure. The intercept C^γ of the tangent line identifies the boundary value of the performance envelope at the point of tangency: γ

$$C^\gamma = \max_{\mu \in \mathbb{R}^T} U_r(\mu, g) - \gamma\, U_r(\mu, f) \tag{14}$$

The solution is obtained recursively as in a Bellman equation. To express the recursion, expand the notation and let $C^\gamma = C_1^\gamma(A_1, 0; B_1, 0)$ where $A_1 = B_1 = W_1$ denote the initial alpha wealths associated with the alpha investing rules defined by f and g. The zeros indicate that no tests have occurred since the last rejection so that $f(0)$ and $g(0)$ determine the amount to invest in the test of H_1.

Now consider the general case of the test of H_j. Assume that the alpha wealth available to the two investing rules is A_j and B_j, respectively, at this stage, and that it has been $\ell \le j$ tests since the last rejection by the first rule and $m \le j$ tests since the last rejection by the second. Assume also for ease of presentation that the level $\alpha_j = A_j f(\ell)$ invested in the test of H_j by the first investing rule is less than the level $\beta_j = B_j g(m)$ invested by the second ($\alpha_j < \beta_j$). It follows that, when utility is measured by the number of rejections, that we only need a one-dimensional optimization at each

Figure 2: The intercept of the tangent line with slope $\gamma = 1$ identifies a point on the boundary of the performance envelope.



test,

$$
\begin{aligned}
C_j^{\gamma}(A_j,\ell;B_j,m) \;=\; \max_{\mu\in\mathbb{R}} \Big[& r_\mu(\alpha_j) - \gamma\, r_\mu(\beta_j) \\
& + \quad r_\mu(\alpha_j) \qquad\quad C_{j+1}^{\gamma}(A_j+\omega-\alpha_j,0;\; B_j+\omega-\beta_j,0) \\
& + (r_\mu(\beta_j) - r_\mu(\alpha_j))\; C_{j+1}^{\gamma}(A_j-\alpha_j,\ell+1;\; B_j+\omega-\beta_j,0) \\
& + (1-r_\mu(\beta_j))\; C_{j+1}^{\gamma}(A_j-\alpha_j,\ell+1;\; B_j-\beta_j,m+1) \Big] \;, \quad (15)
\end{aligned}
$$

with the boundary condition $C_{T+1}^{\gamma} = 0$. The successive lines identify the expected differential in the number of rejections produced by the test of H_j, and following summands denote the subsequent expected values if both reject, if only the rule with the larger alpha level rejects, and if neither rejects.

Practical solution of the recursion for C_1^{γ} requires a discrete approximation. Notice in (15) that the state of the recursion depends on the wealths of the two investing rules. Feasible calculation requires that we restrict the possible wealths to a discrete grid. If the wealths are allowed to vary over any $W \geq 0$, then solving this recursion for any sizable T is intractable. Our approach discretizes the wealth functions so that the optimization occurs over a grid for each test j rather than the positive quadrant of $\mathbb{R}^2$. For each investing rule, we initialize a grid of $T+M+1$ wealth values w_j, indexed from $j = M, M-1, \ldots, 1, 0, -1, \ldots, -T+1, -T$. This grid holds the state of the wealth at each test, and the differences in adjacent wealths determine the amounts used to test the next hypothesis. For the rule defined by the distribution $f \in \mathcal{F}$, we set $w_0 = W_1$, $w_{-1} = w_0(1-f(0))$, $w_{-2} = w_{-1}(1-f(1)), \ldots$. If the investing rule does not reject any hypotheses, these wealths are exact. If the rule does reject, we accumulate the utility as though performing a randomized test that tosses a biased coin to decide which of the nearby wealths to spend. Suppose that the alpha wealth when rejecting is $x = w_j + \omega$. It is unlikely that x lies at one of the grid of wealth values, so assume that $x = c\,w_k + (1-c)w_{k+1}$ for some $0 < c < 1$. In this case, we treat the next test as a randomized test. The test earns the expected utility from wealth w_k with probability $(1 - c)$ and from wealth w_{k+1} with probability c. Basically, this approximation adds a second expectation to the sum in (15). We set w_j for $j > 0$ somewhat arbitrarily in a manner that prevents the accumulation of excess wealth. In our examples, $M = 5$ with $w_i = W_1 + i\,\omega/3$, $i = 1, 2, 3$, and $w_i = w_{i-1} + \omega$ for $i = 4$, 5. Should the wealth reach w_4, then the bid for the next test is ω, the amount earned by a rejection. Hence, the testing does not increment the wealth beyond this boundary.

We obtain a performance envelope by varying the competitive factor γ. To find the boundary points below the diagonal, we reverse the roles of the alpha investing rules and repeat the optimization. As the optimization proceeds, we accumulate the component utilities that identify the boundary point.

6 More results, mean choices

7 Discussion

Alpha investing can mimic regular testing procedure by revisiting the test of prior hypotheses.

One might also use accumulated alpha wealth as a measure of the performance, and this provides a more useful metric of performance, particularly when competing against the oracle. If maximizing alpha wealth, then the oracle loses the amount bid and chooses μ_j to maximize $r_{\mu_j}(\alpha) - \alpha$ rather than $r_{\mu_j}(\alpha)$ alone. This perspective would not only capture aspects of rejecting hypotheses, it also anticipates having resources to test future hypotheses. Such consideration is appropriate, however, only in the context of testing a larger collection of hypotheses than considered here.

{**ras:** *Things left to do:*

1. *Graphs of envelope that suggest that alpha wealth is a decent proxy for risk, at least better than something like FDR, number rejects - constant times number false rejects.*
2. *What does the steady state look like. If take the envelope for 250 and double to get for 500, is that close to correct for the risk?*
3. *Comment on the value of saving if hope to compete with a universal bidder.*

}

Acknowledgement

The authors thank ...

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