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Abstract

This note explains the origins of the martingale used in the JRSS paper and how it can be "simplified" as worked out in the correlated test manuscript.

JRSS version

The martingale of interest in the paper is motivated by showing that the mFDR is bounded above by α,

$$\frac{\mathbb{E}\, V(j)}{\mathbb{E}\, R(j) + \eta} \le \alpha \;, \tag{1}$$

where $V(j)$ is the cumulative number of incorrectly rejected hypotheses, $R(j)$ is the cumulative number of rejected hypotheses, and $\eta > 0$ is a constant used to control the bound when the complete null holds. It does not appear in the martingale. The bound (1) holds if

$$0 < \alpha(\mathbb{E}\, R(j) + \eta) - \mathbb{E}\, V(j) \tag{2}$$

We can prove this result if we can show that the expected value of increments that go into the cumulative sum are non-negative. (Since we study the increments, η drops out.) In the end, the martingale that we study is one that looks harder to work with since it subtracts a further positive term, the cumulative wealth $W(j)$ of the process,

$$A(j) = \alpha(\mathbb{E}\, R(j) + \eta) - \mathbb{E}\, V(j) - W(j) \;.$$

It looks rather puzzling that it is easier to show that this inequality holds, even though we have subtracted a further positive term.

The sum in (2) has increment

$$\alpha\, R_j - V_j \tag{3}$$

Consider this term first if the tested null hypothesis H_j is false, then when H_j is true. The latter is the harder situation. If H_j is false, then we cannot falsely reject the null hypothesis, so

$$H_j \text{ is false} \quad \Rightarrow \quad V_j \equiv 0$$

The increment is then

$$H_j \text{ false: } \alpha\, R_j \geq 0 \text{ a.s.}$$

If however the null is true, then $V_j \equiv R_j$, the level of the test is $\mathbb{E}\, R_j = \alpha_j$, and the expected value of the increment is negative:

$$H_j \text{ true: } \mathbb{E}\,(\alpha\, R_j - V_j) = \mathbb{E}\,(\alpha\, R_j - R_j) = \alpha_j(\alpha - 1) < 0.$$

The martingale that works comes from "giving up" some of the fat cushion we have when H_j is false to compensate for what happens when H_j is true. For the compensator, we use the change in the wealth of the process. The wealth typically goes up when H_j is false and goes down on average when H_j is true, so we subtract this from the increment. Its a bit like an antithetic variable in a simulation. The increment is then ($\omega \leq \alpha$ is the payout when a hypothesis is rejected.)

$$\begin{aligned}
\alpha\, R_j - V_j - W_j &= \alpha\, R_j - V_j - [\omega R_j - (1 - R_j)\alpha_j] \\
&= (\alpha - \omega)\, R_j - V_j + (1 - R_j)\alpha_j \\
&\geq (1 - R_j)\alpha_j - V_j
\end{aligned}$$

We have so much cushion that we can give up a bit right there at the start. (Since $\omega \leq \alpha$, we might not have had this part anyway!) This increment is again positive if H_j is false,

$$H_j \text{ false: } (1 - R_j)\alpha_j - V_j \equiv (1 - R_j)\alpha_j \geq 0$$

If H_j holds,

$$H_j \text{ true: } (1 - R_j)\alpha_j - V_j \equiv (1 - R_j)\alpha_j - R_j$$

In expectation, the increment is (p-values are uniform when H_j holds)

$$\mathbb{E}\ ((1 - R_j)\alpha_j - R_j) = (1 - \alpha_j)\alpha_j - \alpha_j = \alpha_j^2 - \alpha_j < 0$$

That's not quite enough, so we increased the cost for bidding to $\alpha_j/(1 - \alpha_j)$ and the expected value becomes

$$\mathbb{E}\left((1 - R_j)\frac{\alpha_j}{1 - \alpha_j} - R_j\right) = \alpha_j - \alpha_j = 0$$

Simplified Version in Multiple Endpoints

This version of the martingale removes the tuning parameters ω and η and simplifies the wealth process. Set the payout to the initial wealth $\omega = \alpha$ and set the offset in the denominator $\eta = 1$. The wealth process in the JRSS version increases by ω when rejecting H_0 and pays $\alpha_j/(1-\alpha_j)$ when it does not reject. Notice that it does not "pay" the bid amount if the hypothesis is rejected. The version used in this paper operates differently in that it *always* pays the bid α_j. (Note: The C++ implementation works this way.) These changes simplify the increment used in the JRSS paper from

$$\omega R_j - (1 - R_j)\alpha_j/(1 - \alpha_j)$$

to

$$\alpha\, R_j - \alpha_j\ .$$

That is, the expert always pay its bid α_j and earns α if the hypothesis is rejected. The increment in the martingale is then (compare to (4))

$$\begin{aligned} \alpha\, R_j - V_j - W_j &= \alpha\, R_j - V_j - [\alpha R_j - \alpha_j] \\ &= \alpha_j - V_j \end{aligned}$$

Obviously, the increment of the process $A(j)$ is positive when H_j is false since then $V_j \equiv 0$. If the null H_j is true, we again have the needed result since then $\mathbb{E}\, V_j = \mathbb{E}\, R_j = \alpha_j$.