TheCALT Complex Numbers Part 1

Rohan Garg

October 2020

§1 Introduction

In this handout, we will learn the basics of complex numbers. This is Part 1 of the Complex Number Handout Series that we will be writing! Complex numbers involve using imaginary units which we will be getting into later. They are "complex" in more ways than one! Complex numbers might seem simple at first, but these examples require for us to understand them much more deeply.

Before we start learning, we need to know a few terms that we will be using throughout this handout.

Definition 1.1. For a long time, the square root of negative numbers was undefined. However, we now have a way to define the square roots of negative numbers - using the imaginary unit "i". It has the unique property that $i^2 = -1$. Multiples and variations of "i" form other imaginary numbers and complex.

Definition 1.2. A *complex number* is any number that can be written in the form $a + bi$, where a and b are real numbers. Note that a and b can be 0. The set of all imaginary numbers is the union of the set of real numbers and the set of imaginary numbers, and also the results of all of the operations between their elements. The set of complex numbers is denoted as $\mathbb{C}$.

Definition 1.3. A polynomial is a function that's the sum of several terms, each of which is the product of variables raised to integer powers and constants. All polynomials in one variable (which are our main focus here) are of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

The *coefficients* of a polynomial are the constants $a_n, a_{n-1}, \ldots, a_1, a_0$. The *degree* of a polynomial is the value of n in the above expression, where $a_n \neq 0$. The degree can be denoted as $\deg P$ and it **must** be a nonnegative integer for P to be a polynomial.

Definition 1.4. The *absolute value* or *magnitude* of a complex number z can be written as $|z|$. If z is in the form $a + bi$, then the magnitude will be $\sqrt{a^2 + b^2}$.

Definition 1.5. The *conjugate* of a complex number z can be expressed as $\overline{z}$. If z is in the form $a + bi$, the conjugate will be $a - bi$. The pair of these two values is called a *conjugate pair*.

§2 Arithmetic of Complex Numbers

The arithmetic of complex numbers is very similar to the one we do in real numbers. The key idea is to combine similar terms and always if possible, put the result in the form $a + bi$. Arithmetic operations can be performed on complex numbers with the simple techniques of algebra.

§2.1 Addition

Example 2.1

Find $(3 + 2i) + (4 + 7i)$.

Adding complex numbers is very similar to adding variables. We can think of i as a variable, even though it has a numerical value. In doing so, we group together terms with an i and without an i. We get $(3 + 4) + (2 + 7)i$ or $\boxed{7 + 9i}$.

§2.2 Subtraction

Example 2.2

Find $(4 + 7i) - (3 + 2i)$.

We use the same technique used in addition. We let i be a variable and we group together like terms. We get $(4 - 3) + (7 - 2)i$. Simplifying, we get our answer $\boxed{1 + 5i}$.

§2.3 Multiplication

Multiplication gets a bit trickier. The reason is because if you assume that i is a variable, the result will often leave you with a term with i^2 in it. Here you have to remember that $i^2 = -1$.

Example 2.3

Find $(4 + 3i)(2 - 7i)$.

We can expand this using the distributive property. Broken up into two parts, we have $2(4 + 3i) - 7i(4 + 3i)$. Multiplying everything out, we get $8 + 6i - 28i - 21i^2$. What is $-21i^2$? Since $i = \sqrt{-1}$, we can square it to find $i^2 = -1$! Our new expression is $8 + 6i - 28i + 21$. Adding like terms, we get our final answer as $\boxed{29 - 22i}$.

Here is an important lemma.

Lemma 2.4

The product of a complex number and its conjugate is always real.

Proof: Let the complex number be $a + bi$ and its conjugate be $a - bi$. This tactic, setting the complex number to a certain value, will be useful in many problems. Instead of expanding it and distributing like we did in Example 2.3, we can work smarter. We recall that $x^2 - y^2$ can always be factored as $(x + y)(x - y)$ (this is the difference of squares formula). If we plug in $x = a$ and $y = bi$, then $(a + bi)(a - bi)$ will be equal to

$a^2 - (bi)^2$ or $a^2 + b^2$. a and b are real coefficients by the definition of a complex number, so $a^2 + b^2$ must be real as well.

§2.4 Division

Division is even trickier than multiplication. Dividing complex numbers requires using Lemma 2.4, and we'll see why with our first example.

Example 2.5
Find $\frac{4+3i}{3-2i}$.

We can't start the division right away. The denominator makes it nearly impossible to find an easy way to divide the two terms. So, let's try to get rid of the imaginary terms! In the beginning of this section, we stated that Lemma 2.4 would come in handy. To clear the imaginary terms, we can multiply the numerator and denominator of the fraction by the conjugate of the denominator!

The conjugate of the denominator is $3 + 2i$. Multiplying, we get

$$\frac{(4+3i)(3+2i)}{(3-2i)(3+2i)} \to \frac{(4+3i)(3+2i)}{3^2+2^2} \to \frac{6+17i}{13}.$$

But wait! We want to get it in terms of $a + bi$, so the answer is actually $\boxed{\frac{6}{13} + \frac{17}{13}i}$.

§2.5 Generalizations

Since any of these operations are just forms of algebraic simplification, we can generalize these operations to those done on any two complex numbers.

For any two complex numbers such that $z = a + bi$ and $w = c + di$, each of these properties holds true::

1. $z + w = (a + c) + (b + d)i$.
2. $z - w = (a - c) + (b - d)i$.
3. $z \cdot w = (ac - bd) + (ad + bc)i$.
4. $\frac{z}{w} = \frac{ac+bd}{c^2+d^2} + \frac{-ad+bc}{c^2+d^2}i$.

It is important to note that you should not memorize these formulas; instead you should understand why they are true.

§2.6 Exercises

Exercise 2.6. Find the value of z given that $\overline{z} = z + 4i$. Write your answer in the form $a + bi$.

Exercise 2.7. Evaluate the expression $(7 - 2i)^2 - (3 + i)^2 - (4 - 3i)^2$ and express your answer in the form $a + bi$.

Exercise 2.8. Let $z = a + bi$ be a complex number for real a, b. If $z\overline{z} = 28$, find the area of the locus determined by all ordered pairs of solutions (a, b).

§3 Parts of a Complex Number

A complex number can be written as $a + bi$ for real numbers a and b. a is the real part and b is the imaginary part of the complex number. The notation that we can use for the real and imaginary parts are $Re(z)$ and $Im(z)$, respectively.

Note that the imaginary part of a complex number is real. For example, $Im(3 + 2i) = 2$, not $2i$.

Definition 3.1. Two complex numbers are equal if and only if their real parts and complex parts are simultaneously equal.

With this, let us jump into our first example.

> **Example 3.2** (Classic)
> Find the two values of $\sqrt{i}$. Express your answer in terms of $a + bi$.

The square root of any number, say n, equals a value such that the square of that value is n. Similarly, we can let the square root of i be $a + bi$. We get the equation

$$\sqrt{i} = a + bi \rightarrow i = (a + bi)^2.$$

Again, we use our previous definitions to solve the problem. Expanding the right hand side (RHS), we get $a^2 - b^2 + 2abi$. From this, we get two equations to work with:

$$a^2 - b^2 = 0$$
$$2ab = 1$$

From the first equation, we get $a = \pm b$. We look at the two cases separately. If $a = b$, we have

$$2a^2 = 1 \rightarrow a = \pm\frac{\sqrt{2}}{2}.$$

Since $a = b$, we also have $b = \pm\frac{\sqrt{2}}{2}$.

The second equation gives us $a = -b$. Before even trying to solve it, we realize that a and b will be complex, which is something that we don't want.

Now we get that our two values of $\sqrt{i}$ are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

> **Example 3.3** (1985 AIME Problem 3)
> Find c if a, b, and c are positive integers that satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$.

To start off, we expand $(a + bi)^3$. We do this to separate it into the real and imaginary parts. This can help us compare and thus find the required values. After doing so, our new equation is

$$c = (a^3 - 3ab^2) + (3a^2b - b^3)i - 107i.$$

Moving the $107i$ to the other side, and factoring out a b from $(3a^2b - b^3)i$, our new equation is

$$c + 107i = (a^3 - 3ab^2) + (3a^2 - b^2)bi.$$

. We defined that two complex numbers are equal if both parts are equal. This means that $b(3a^2 - b^2) = 107$ and $a^3 - 3ab^2 = c$. Since a and b are positive integers, b must be a factor of 107. If b is 107, this means that $3a^2 - 107^2 = 1$. $107^2 + 1$ is not divisible by 3, and if the above equation is true, then a will not be an integer. If b is 1, this means that $3a^2 = 108$, resulting in $a = 6$. To find c, we plug a and b into $a^3 - 3ab^2$.

$$c = 6^3 - 3 \times 6 \times 1^2 = 198 \to \boxed{198}.$$

§3.1 Exercises

Exercise 3.4 (2009 AIME I Problem 2)**.** There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

Exercise 3.5 (Classic)**.** Find all complex numbers z such that $\overline{z} = z^2$. Express your answer(s) in terms of $a + bi$.

§4 Conjugate Properties

Lemma 4.1 (Conjugate Properties)

For any two complex numbers z and w, these two properties always hold true:

a) $\overline{z+w} = \overline{z} + \overline{w}$
b) $\overline{zw} = \overline{z} \times \overline{w}$

Proof: For both properties, we can let $z = a + bi$ and $w = c + di$. Using conjugate properties, we know that $\overline{z} = a - bi$ and $\overline{w} = c - di$.

a) The LHS is $\overline{a + bi + c + di} = \overline{(a + c) + (b + d)i}$. The conjugate of this is $(a + c) - (b + d)i = a + c - bi - di = a - bi + c - di$. The RHS is $\overline{a + bi} + \overline{c + di} = a - bi + c - di$. These are equal, therefore we are done.

b) Again, we use the values defined in the beginning. The LHS is $\overline{(a + bi)(c + di)}$. Expanding, we get $\overline{(ac - bd) + (bc + ad)i}$. The conjugate of this is $(ac - bd) - (bc + ad)i$. The RHS can be simplified to $(a - bi)(c - di)$. Expanding, we get $(ac - bd) - (bc + ad)i$, the same as the LHS. We are done.

Lemma 4.2

For any complex number z, $\overline{z^n} = (\overline{z})^n$.

This lemma is trivial by induction and reapplying b) in Lemma 3.1.

Example 4.3 (1995 AIME Problem 5)

For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

We don't immediately see a way to start. To solve this, we need to utilize this following theorem.

Theorem 4.4 (Complex Conjugate Root Theorem)

Let polynomial P have real coefficients. If complex number z is a root of P, then $\overline{z}$ is also a root of P.

Proof of Theorem: A polynomial can essentially be written as

$$\sum_{i=0}^{n} a_i x^i,$$

where a_i is real for all $0 \leq i \leq n$. By the Factor Theorem, we have $P(z) = 0$. Plugging $\overline{z}$ into our newly written polynomial, we get

$$P(\overline{z}) = \sum_{i=0}^{n} a_i \overline{z}^i.$$

Since the conjugate of a power is equal to the power of the conjugate, we can rewrite this as

$$\sum_{i=0}^{n} a_i \overline{z^i}.$$

The conjugate of a real number is real so we can rewrite this again as

$$\sum_{i=0}^{n} \overline{a_i}\,\overline{z^i}.$$

Using b) from Lemma 3.1, we know that this is equal to

$$\sum_{i=0}^{n} \overline{a_i z^i}.$$

This is just the conjugate of $P(z)$, which we said was equal to 0 from the factor theorem. The conjugate of 0 is 0, so $P(\overline{z}) = 0$. By the Factor Theorem, we know that $\overline{z}$ is also a root of P, and thus we are done.

Going back to the problem, since the polynomial has real coefficients, the roots must come up in conjugate pairs. Let us name the roots $m, n, \overline{m}$, and $\overline{n}$. Since the product and sum of a complex number and its conjugate is always real, the pairs that are multiplied and added can't be conjugate pairs. WLOG, we assume that $mn = 13 + i$ and $\overline{m} + \overline{n} = 3 + 4i$. We can take the conjugate of both equations using a) and b) of Lemma 3.1 to get $\overline{mn} = 13 - i$ and $m + n = 3 - 4i$. Using Vieta's Formulas, we get that $b = mn + m\overline{n} + m\overline{m} + \overline{m}n + n\overline{n} + \overline{mn}$.

We can factor b as $(m+n)(\overline{m}+\overline{n})+mn+\overline{mn}$. We have these values from the equations we found! $m+n=3-4i$, $\overline{m}+\overline{n}=3+4i$, $mn=13+i$, and $\overline{mn}=13-i$. Plugging them in, we get

$$(3-4i)(3+4i)+(13+i)+(13-i).$$

After multiplying and adding, we get our answer, which is $\boxed{51}$.

§4.1 Exercises

Exercise 4.5 (Classic)**.** Let $f(x)=x^3-8x^2+29x-52$. If one of the roots of $f(x)$ is $2-3i$, find the sum of the other two roots.

Exercise 4.6 (2007 AMC 12A Problem 18)**.** The polynomial $f(x)=x^4+ax^3+bx^2+cx+d$ has real coefficients, and $f(2i)=f(2+i)=0$. What is $a+b+c+d$?

§5 Problems

Problem 5.1. Compute $Re(z)+Im(z)$ if

$$z=(1+i)^5.$$

Problem 5.2. Find the value of $\sum_{n=0}^{600} i^n$, if $i=\sqrt{-1}$.

Problem 5.3. Simplify $(i+1)^{3200}-(i-1)^{3200}$.

Problem 5.4 (1992 AHSME Problem 15)**.** Let $i=\sqrt{-1}$. Define a sequence of complex numbers by

$$z_1=0, \quad z_{n+1}=z_n^2+i \text{ for } n\geq 1.$$

In the complex plane, how far from the origin is z_{111}?

Problem 5.5.

Problem 5.6 (AoPS Intermediate Algebra)**.** The value

$$\left(\frac{1+\sqrt{3}}{2\sqrt{2}}+\frac{\sqrt{3}-1}{2\sqrt{2}}i\right)^{72}$$

is a positive real number. What real number is it?

Problem 5.7 (1988 AHSME Problem 21)**.** The complex number z satisfies $z+|z|=2+8i$. What is $|z|^2$? Note: if $z=a+bi$, then $|z|=\sqrt{a^2+b^2}$.

Problem 5.8 (HMMT (Source Unknown))**.** Complex number w satisfies $\omega^5=2$. Find the sum of all possible values of $\omega^4+\omega^3+\omega^2+\omega+1$.

Problem 5.9 (1988 AIME Problem 11)**.** Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n}(z_k-w_k)=0.$$

For the numbers $w_1=32+170i$, $w_2=-7+64i$, $w_3=-9+200i$, $w_4=1+27i$, and $w_5=-14+43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line.

Problem 5.10 (2013 HMMT Algebra Problem 9)**.** Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}.$$

§6 Questions

If you have any concerns or questions about this handout, make sure to join TheCALT Discord Server. In here, we release daily POTDs which can challenge you. If you are having trouble understanding concepts used in the handout, you can ask questions in our server as well! To join click this link: TheCALT Discord Server.

If you don't have Discord, PM bobthegod78 on AoPS to ask questions.