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Physics Notes

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Contents

1 Mechanics

1.1 Relating velocities

Always remember to draw vector triangles to try to relate velocities of objects.

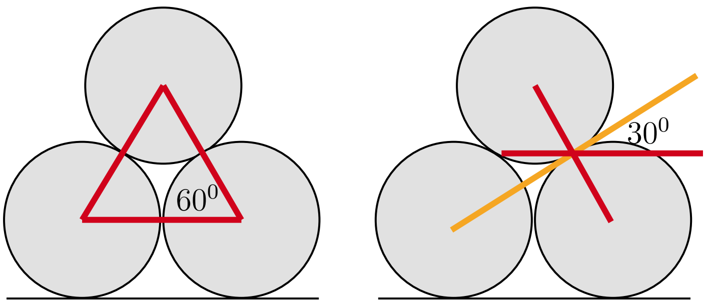
Example from **Morin 8.16**

Basically part of the question requires us to relate the relative velocity between the 3 identical cylinders (2 below, 1 on top)

In this question, we don't really have to care about whether the circles are rolling, or sliding. As long as the surfaces are always in contact with each other, rolling and sliding will be the same.

The top cylinder (A) is moving vertically downwards. Its **instantaneous** resultant velocity can be seen as a vector summation of its motion of sliding down one of the bottom cylinder, B (slide down along the tangent of the contact point) and moving together with the bottom cylinder to the right.

$$\mathbf{v}_B = \mathbf{v}_{B,A} + \mathbf{v}_A \tag{1.1}$$



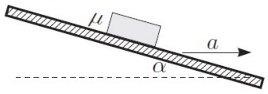
1.2 Friction

1.2.1 Friction on inclined slope

If a body is on the verge of slipping (or already slipping), then the sum of the friction force and the reaction force is angled by $\arctan \mu$ from the surface normal. This is actually quite useful in certain questions. *Sometimes, we don't even have to resolve forces by projecting their* *components onto axes. We can instead add them up vectorially by drawing vector diagram (head to tail).* This can be shown in the example below from *Jaan Kalda's Mechanics handout*.

Mechanics **pr 5.**

pr 5. A block rests on an inclined surface with slope angle α. The surface moves with a horizontal acceleration a which lies in the same vertical plane as a normal vector to the surface. Determine the values of the coefficient of friction μ that allow the block to remain still.



Here we are helped by the very universal

Method 1 To solve this problem, the common method would be to resolve forces Resolving forces along the slope gives us:

$$mg\sin\alpha - f - ma\cos\alpha = 0$$

Resolving forces perpendicular to the slope gives us:

$$N - mg\cos\alpha - ma\sin\alpha = 0$$

Since N must be positive for the block to stay on the plane, we get the first inequality

$$g\cos\alpha + a\sin\alpha > 0$$

Our second inequality comes from the force balance along the slope, where $\mid f \mid \leq \mu N$ So we get

$$\mu \geq \frac{\mid a\cos\alpha - g\sin\alpha \mid}{g\cos\alpha + a\sin\alpha} \tag{1.2}$$

Both inequalities have to be satisfied for the block to remain on the slope.

Method 2 heheheh

1.3 Tension

1.3.1 Rubber band around cylinder

In fact, we have to

1. Analyse an infinitesimal string element which subtends an angle of $d\theta$.
2. It is not hard to see that the radial component is $T\sin\frac{d\theta}{2}$. By small angle approximation, which states that $sin\theta \approx \theta$ for small θ, we get: $Td\theta$ (2 tensions acting on both ends)
3. Integrate this tension from 0 to 2π, the total tension would then be (look abit weird but I think it's correct) $2\pi T$.

1.3.2 Hanging Chain

Example from **Morin 2.8**

1. Realise that T_x is constant throughout the chain, and $\frac{T_y}{T_x} = y'$. Hence, $T_y = T_x y' = Cy'$

2. And also realised that the mass of each infinitesimal chain segments can be expressed as such

$$\rho \mathbf{g} dl = \rho \mathbf{g} \sqrt{dx^2 + dy^2} = \rho \mathbf{g} \sqrt{1 + y'^2} dx \tag{1.3}$$

3. However, it is the **difference in tension** that balance out the wright. We can then write down the final relationships and solve the equation using substitution and separation of variables.

$$d(Cy') = \rho \mathbf{g} \sqrt{1 + y'^2} dx \tag{1.4}$$

$$Cy'' = \rho \mathbf{g} \sqrt{1 + y'^2} \tag{1.5}$$

1.4 SHM

1.4.1 Translational perturbation

1. Find $\sum F$, and find all the equilibrium positions where $\sum F = 0$ (these are the stable equilibriums)
2. Use: $k = -\frac{dF}{dx}$. (negative sign depends on the direction of F. In this case, F is defined as positive in the direction that points away from equilibrium)
3. Then do smth like $\frac{dF}{dx}\Big|_{x=x_{\text{eqm}}}$, and substitute the values (Hookes constant k) into $\omega = \sqrt{\frac{k}{m}}$

1.4.2 Radial perturbation

I am not sure about this, but this is what i see people do on PhOD to solve questions (physics olympiad discord)

1. Let the small perturbation "displacement" be δ.
2. If the mass is revolving around some bigger object and the only force acting on it is in the radial direction, then we can use **conservation of angular momentum**

$$r^2\omega = (r + \delta)^2\omega' \tag{1.6}$$

Through simple algebraic manipulation, we get

$$\omega' = \left(\frac{r}{r + \delta}\right)^2 \omega \tag{1.7}$$

3. In the rotating frame, we can write

$$m\frac{d^2\delta}{dt^2} = m(r + \delta)\omega'^2 - F_{\text{central}} \tag{1.8}$$

4. Try to fit that to the classic SHM equation (maybe can simplify RHS and expand to first order in δ): $\ddot{x} + \omega^2 x = 0$, and use $\omega = \sqrt{\frac{k}{m}}$

1.5 Non-inertial reference frames

I think fictitious forces are just there to make sure that everything works on in non-inertial reference frames as well.

1.5.1 Translational

Let the lab frame be S, and the non inertial frame moving at $\mathbf{a_0}$ be S'. If the object's acceleration is $\mathbf{a}$ in the lab frame, its acceleration in the non-inertial frame would be $\mathbf{a} - \mathbf{a_0} = \mathbf{a}'$. The fictitious force present is thus $-m\mathbf{a_0}$. (translational force)

1.5.2 Rotational

Consider a system of reference, which rotates around the origin O with an angular velocity $\boldsymbol{\omega}$. Consider a point P, which is motionless in the ortating system, and let us denote $\mathbf{r} = \mathbf{OP}$. In the lab system of reference, the point P moves with velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{r} \times \boldsymbol{\omega}$. Now, if the point P moves in the rotating frame of refernece with velocity $\mathbf{u} = \frac{d\mathbf{r}}{d\tau}$ (τ is used to emasure the time in the rotating system), then this additional velocity needs to be added to what should have been for a motionless point:

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\tau} + \boldsymbol{\omega} \times \mathbf{r}$$

So, we can conclude that the time-derivatives of vectors in rotating and lab frames of refernce are related via equality

$$\frac{d}{dt} = \frac{d\mathbf{r}}{d\tau} + \boldsymbol{\omega} \times .$$

This is written in the form of an operator, which means that we can write any vector (e.g. $\mathbf{r}$,$\mathbf{v}$) rightwards of all the three terms. In particular, we can apply this formula to the right and left hand sides of the equality $\mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}$

$$\frac{d\mathbf{v}}{dt} = \left(\frac{d}{d\tau} + \boldsymbol{\omega} \times\right)\left(\mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}\right) = \frac{d\mathbf{u}}{d\tau} + \boldsymbol{\omega} \times \mathbf{u} + \frac{d(\boldsymbol{\omega} \times \mathbf{r})}{d\tau} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

We obtain

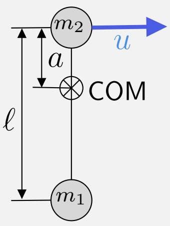
$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{u}}{d\tau} + 2\boldsymbol{\omega} \times \mathbf{u} - \boldsymbol{\omega}^2\mathbf{r}$$

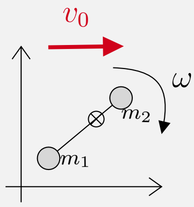
Recall that $\frac{d\mathbf{v}}{dt}$ is the acceleration of the point P as seen in the lab frame of refernce, and $\frac{d\mathbf{u}}{d\tau}$ is the same as seen in the rotating frame of reference. now, if P is a point mass m, and thee is an external force $\mathbf{F}$ acting on P, then $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$ and hence,

$$m\frac{d\mathbf{u}}{d\tau} = \mathbf{F} - 2\boldsymbol{\omega} \times \mathbf{u}m + \boldsymbol{\omega}^2\mathbf{r}m$$

1.6 SJPO2021 Special Round Qn

Da question





Basically the question asks you to find the equation for the trajectory of the thing shown in the diagram when m_2 starts with an initial velocity u perpendicular to the rod. (masses on frictionless ground). I think using the centre of mass frame would help in solving this problem.

Position of centre of mass with respect to m_2,

$$a = \frac{m_1 \ell}{m_1 + m_2} \tag{1.9}$$

Velocity of centre of mass of the system can be calculated through conservation of momentum:

$$v_{\text{COM}} = \frac{m_2 u}{m_1 + m_2} \tag{1.10}$$

The moment of inertia of the system with respect to its centre of mass is

$$I = m_2 \left(\frac{m_1 \ell}{m_1 + m_2} \right)^2 + m_1 \left(\frac{m_2 \ell}{m_1 + m_2} \right)^2 = \frac{m_1 m_2 \ell^2}{m_1 + m_2} \tag{1.11}$$

Angular velocity could be either determined using **angular momentum**:

$$\omega = \frac{L}{I} \tag{1.12}$$

$$\therefore \omega = \frac{u}{\ell} \tag{1.13}$$

Or determined by **finding velocity of m_2 in the COM frame** and then **divide by the radius of rotation to find angular velocity** as $v = r\omega$.

$$v' = u - \frac{m_2 u}{m_1 + m_2} = \frac{m_1 u}{m_1 + m_2} \tag{1.14}$$

Dividing this by the radius of rotation (distance from m_2 to COM), we obtain the same expression for ω

$$\omega = \frac{v'}{a} = \frac{u}{\ell} \tag{1.15}$$

Formulating the equation for trajectory:

$$x(t) = a \sin(-\omega t) + v_{\text{COM}} t \tag{1.16}$$

$$y(t) = a \cos(\omega t) \tag{1.17}$$

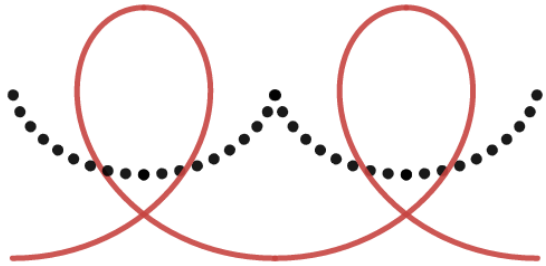


Figure 1.1: Trajectory plotted with Desmos

2 Electromagnetism

2.1 Electric field and potential

Potential and field are related by the following equation

$$-\int_A^B E \cdot ds = V(B) - V(A) \tag{2.1}$$

I think the intuition is that a **positive workdone by E field** always results in a decrease in **potential**.

Intuition

when the E field is pointing away from a **positive charge**, to move the **negative test charge** away from the positive charge, you need a force with magnitude of at least qE and opposite in direction of the force of attraction. Workdone by that force results in an increase in potential (less negative) by conservation of energy. Since Eq always points in the opposite direction of that force, qE always points in direction of decreasing potential.

2.1.1 Faraday's law of induction

2.1.2 More about electromotive force

In the circuit, the flowing electrons experience 2 forces, namely the force provided by the external agent, such as the battery ($\mathbf{f_s}$), and the force provided by E-field due to the build-up of charges around the circuit ($\mathbf{E}$).

Hence, the net force driving the current is

$$\mathbf{f} = \mathbf{f_s} + \mathbf{E} \tag{2.2}$$

The emf (electromotive force) is defined as

$$\text{emf} = \oint \mathbf{f} \cdot d\mathbf{s} = \oint \mathbf{f_s} \cdot d\mathbf{s} \tag{2.3}$$

This is because $\oint \mathbf{E} \cdot d\mathbf{s} = 0$.

2.2 Different kinds of "Current"

linear current

$$\mathbf{I} = \lambda \mathbf{v} \tag{2.4}$$

The unit for λ is $[A][m]^{-1}[s]$. This seemed a bit weird to be initially. I guess can think about it this way:

I guess this makes sense...

$$\mathbf{I} = \frac{Q}{t} \tag{2.5}$$

$$\mathbf{I} = \frac{\lambda d}{t} = \lambda \mathbf{v} \tag{2.6}$$

surface current

$$\mathbf{K} = \frac{d\mathbf{I}}{d\ell_{\perp}} = \sigma \mathbf{v} \tag{2.7}$$

The unit for σ is $[A][m]^{-2}[s]$

volume current

$$\mathbf{J} = \frac{d\mathbf{I}}{da_{\perp}} = \rho \mathbf{v} \tag{2.8}$$

2.3 Vector potential is pretty gae

Vector potential. In electrostatics, we have $-\nabla \mathbf{V} = \mathbf{E}$, thanks to the fact that $\nabla \times \mathbf{V} = 0$. Apparently, we can also define such a potential $(\mathbf{V})$ for magnostatics $(\mathbf{A})$. Since $\nabla \cdot \mathbf{B} = 0$, it allows for $\nabla \times \mathbf{A} = \mathbf{B}$ (divergence of curl: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$)

We can substitute $\nabla \times \mathbf{A} = \mathbf{B}$ into $\nabla \times \mathbf{B}$, and get

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla \cdot (\nabla \mathbf{A}) \tag{2.9}$$

$$= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{2.10}$$

But why are we doing this? Well, if we can make $\nabla(\nabla \cdot \mathbf{A}) = 0$, then we get $-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$ (just **Poisson's equation**), making the magnetic vector potential a very useful tool, from which the B field can then be easily calculated.

To show that we can indeed obtain **Poisson's equation** for magnetism, we now have to prove that it is possible to let $\nabla \cdot \mathbf{A} = 0$.

Supposed the original $\mathbf{A_0}$ is not divergenceless. But we can make it divergenceless by adding the **gradient** of something else, maybe $\nabla \mathbf{\Gamma}$ ($\mathbf{\Gamma}$ looks cool), turning $\mathbf{A_0}$ into $\mathbf{A}$.We are adding the gradient of something else because we don't want it to affect $\nabla \times \mathbf{A}$, since the curl of gradient is 0 (so the curl is still $\mathbf{B}$). We then get

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2 \mathbf{\Gamma} \tag{2.11}$$

Setting $\nabla \cdot \mathbf{A} = 0$, we get

$$\nabla \cdot \mathbf{A_0} = -\nabla^2 \mathbf{\Gamma} \tag{2.12}$$

This is again, just Poisson's equation, and we know that there will always be a solution for $\mathbf{\Gamma}$. Hence, it is always possible to make $\mathbf{A}$ divergenceless while still keeping $\nabla \times \mathbf{A} = \mathbf{B}$.

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \tag{2.13}$$

■

2.4 Dipoles

2.4.1 Electric dipole

Torque experienced by electric dipole in a uniform electric field is

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \tag{2.14}$$

The equation for force experienced by an electric dipole is given by $mathbfF = \nabla(\mathbf{p} \cdot \mathbf{E})$ (used for non uniform $\mathbf{E}$ field), where $\mathbf{p} = qd$. It the E field is uniform, the net force would be 0, but the net torque would not be 0.

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}) = (\mathbf{p} \cdot \nabla)\mathbf{E}. \tag{2.15}$$

2.4.2 Magnetic dipole

Torque experienced by a magnetic dipole in a uniform magnetic field is

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \tag{2.16}$$

The equation for force experienced by a magnetic dipole in **uniform magnetic field** is always 0.

Proof

The loop can be arranged/positioned in any manner in the uniform $\mathbf{B}$ field. The force on an infinitesimal part of the loop of length $d\ell$ is

$$d\mathbf{F} = (\mathbf{B} \times d\mathbf{l})I \tag{2.17}$$

If we integrate the equation above around the loop, we get

$$\mathbf{F} = \mathbf{B}I \times \oint d\mathbf{l} \tag{2.18}$$

where $\oint d\mathbf{l} = 0$

In a non-uniform magnetic field, the force experienced would be

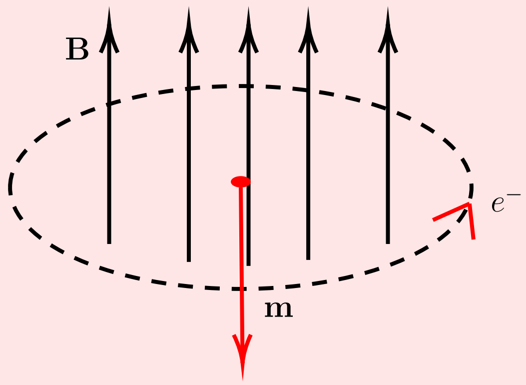
$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{2.19}$$

Unlike the equation for electric dipole, $\mathbf{F} \neq (\mathbf{m} \cdot \nabla)\mathbf{B}$ (not sure why yet).

2.4.3 Diamagnetism and Paramagnetism

Paramagnets acquire a magnetisation parallel to $\mathbf{B}$ while diamagnets acquire a magnetisation opposite to $\mathbf{B}$. We will have to study the effect of magnetic field at the sub-atomic level...

Magnetic field and electron orbital



Paramagnetism arises when the dipole moment[1] of the electron orbital align with the external magnetic field, while diamagnetism is a result of the change in the speed of electron due to the external magnetic field. The orbital contribution to paramagnetism is a lot smaller as it is much harder to tilt the orbital then to change its spin.
The motion of the electron can be approximated as a steady current (because the orbital period is extremely short), then the magnetic dipole can be written as

$$\mathbf{m} = I\pi R^2\hat{\mathbf{z}} = \frac{-e}{T}\pi R^2\hat{\mathbf{z}} = \frac{-ev}{2\pi R}\pi R^2\hat{\mathbf{z}} = \frac{-evR}{2}\hat{\mathbf{z}} \tag{2.20}$$

where R is the radius of the orbit (the negative sign accounts for the negative charge of the electron, as current is always a positive quantity)(v is defined as positive when it is anti clockwise).
This shows that the electron orbital will always be subject to a torque ($\mathbf{m}\times\mathbf{B}$). However, it is hard to tilt the orbital. A more significant effect would be the change in speed of the electron due to the external $\mathbf{B}$ field.
Assuming uniform circular motion, the initial speed (v_0) is given by

$$m\frac{v_0^2}{R} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} \tag{2.21}$$

Let's assume that the $\mathbf{B}$ field is perpendicular to the orbit, the new speed of the electron would be given by

$$m\frac{v_1^2}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + ev_1B \tag{2.22}$$

$$ev_1B = \frac{m}{R}(v_1^2 - v_0^2) = \frac{m}{R}(v_1 - v_0)(v_1 + v_0) = \frac{m}{R}(\Delta v)(v_1 + v_0) \tag{2.23}$$

Assuming difference between v_1 and v_0 is small, making Δv the subject

$$\Delta v = \frac{eRB}{2m} \tag{2.24}$$

$$\boxed{\therefore \Delta\mathbf{m} = \frac{-e^2R^2}{4m}\mathbf{B}} \tag{2.25}$$

2.5 Vector potential of a rotating, charged sphere

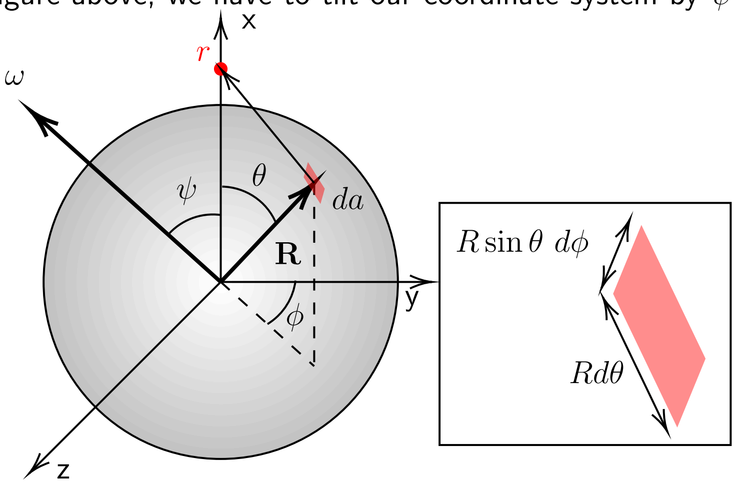
Problem Statement

A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential it produces at point r.

Firstly, we will have to recall the equation for vector potential from a surface current density

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{r}da$$

As shown in the figure above, we have to tilt our coordinate system by ψ so as to make it



easier to integrate. We can make our point of interest lie on the x axis, and the ω vector lie on the x-z plane to make our lives easier. The first step to find surface current is to find the velocity of da.

Power of cross product: $\mathbf{v} = \omega \times \mathbf{r}$ (not the other way round) [2]

$$\mathbf{v} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\cos\psi & 0 & \omega\sin\psi \\ R\cos\theta & R\sin\theta\cos\phi & R\sin\theta\sin\phi \end{pmatrix}$$

$$\mathbf{v} = -R\omega\sin\psi\sin\theta\cos\phi\hat{\mathbf{x}} + (\omega\sin\psi R\cos\theta - \omega\cos\psi R\sin\theta\sin\phi)\hat{\mathbf{y}} + \omega\cos\psi R\sin\theta\cos\phi\hat{\mathbf{z}}$$

$$\mathbf{K} = \sigma\mathbf{v}$$

The integral would then be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\mathbf{K}}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}R^2\sin\theta d\theta d\phi$$

[1] Fingers grip around direction of current (movement of **positive** charges, direction of thumb is the direction of dipole moment) - Right Hand Grip Rule

[2] $\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2b_3 - a_3b_2)\hat{\mathbf{x}} + (a_3b_1 - a_1b_3)\hat{\mathbf{y}} + (a_1b_2 - a_2b_1)\hat{\mathbf{z}}$

Since the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components of $\mathbf{K}$ contain $\cos\phi$, after integration, those terms will vanish since

$$\int_0^{2\pi} \cos\phi d\phi = \int_0^{2\pi} \sin\phi d\phi = 0$$

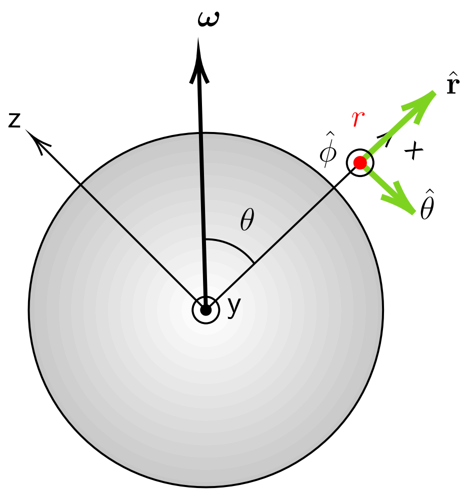
Therefore,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \sigma R^3 \omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\sin\theta\cos\theta}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} d\theta \right) \hat{\mathbf{y}}$$

After doing some crazy math and integration (which includes making the substitution: $u = \cos\theta$), we get

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 \sigma R r \omega \sin\psi}{3} \hat{\mathbf{y}} & \text{inside the sphere} \\ \dfrac{\mu_0 \sigma R^4 \omega \sin\psi}{3r^2} \hat{\mathbf{y}} & \text{outside the sphere} \end{cases}$$

To convert the solutions back to the orignal coordinate system, where the angular velocity vector coincides with the x axis, where our point of interest is located at (r, θ, ϕ), we can do the conversion as such



Just use a bit of imagination (or refer to this diagram shown here). Also take note that $\boldsymbol{\omega}$ lie in the xz plane

Hence, we obtain the final solution in spherical coordinates

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 \sigma R r \omega \sin\theta}{3} \hat{\boldsymbol{\phi}} & \text{inside the sphere} \\ \dfrac{\mu_0 \sigma R^4 \omega \sin\theta}{3r^2} \hat{\boldsymbol{\phi}} & \text{outside the sphere} \end{cases}$$

To derive magnetic field from vector potential, we have to calculate it with $\mathbf{B} = \nabla \times \mathbf{A}$ (in spherical coordinates)

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{pmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{pmatrix} = \frac{1}{r^2 \sin\theta} \begin{pmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta A_\phi \end{pmatrix}$$
$$= \frac{1}{r^2\sin\theta} \left(\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)\hat{\mathbf{r}} - \frac{\partial}{\partial r}(r\sin\theta A_\phi) r\hat{\boldsymbol{\theta}} \right) \quad (2.26)$$

$$\mathbf{B} = \begin{cases} \frac{2}{3}\mu_0 R\omega\sigma(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) & \text{inside the sphere} \\ \frac{\mu_0 R^4\omega\sigma}{3r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) & \text{outside the sphere} \end{cases} \tag{2.27}$$

2.6 Magnetisation

The magnetisation, $\mathbf{M}$ is defined as the magnetic dipole moment per unit volume. Since $\nabla \times \mathbf{M} = \mathbf{J_b}$, where $\mathbf{J_b}$ refers to the **bound current**. Bound current refers to *invisible* current in the magnet due to magnetisation. There is another type of current called **free current** ($\mathbf{J_f}$). This is the current due to a battery, or a voltage source etc. In the case where both $\mathbf{J_f}$ and $\mathbf{J_b}$ are steady currents, we can apply the equation:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J_f} + \mathbf{J_b}) \tag{2.28}$$

After replacing $\mathbf{J_b}$ with $\nabla \times \mathbf{M}$, and manipulation the equation, we get

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J_f} \tag{2.29}$$

Now comes the cool part, we can replace $\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right)$ with a quantity called $\mathbf{H}$, whose name is **Auxiliary Field**:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J_f}} \quad \text{or} \quad \boxed{\oint \mathbf{H} \cdot ds = I_{\text{f,enc}}} \tag{2.30}$$

This equation is very elegant as it allows to express **Ampere's Law** in terms of free current alone, which is what we are able to control and measure.

Under certain circumstances however, we cannot use Ampere's Law methods to find $\mathbf{H}$ field. I asked this on stack exchange (my question)

When can we use Ampere's Law methods to find $\mathbf{H}$ field

Suppose that the H-field was composed of two parts. One of which had a curl of zero and the other which had a divergence of zero. Call them $\mathbf{H_{c=0}}$ and $\mathbf{H_{d=0}}$ respectively, such that

$$\mathbf{H} = \mathbf{H_{c=0}} + \mathbf{H_{d=0}} \ .$$

In which case Ampere's law

$$\mathbf{J_f} = \nabla \times (\mathbf{H_{c=0}} + \mathbf{H_{d=0}}) = \nabla \times \mathbf{H_{d=0}}$$

The Helmholtz decomposition theorem tells us that any vector field can be represented by two such fields, so Amperes law will only tell us about the total H-field when it consists only of a divergence-free component. i.e. when $\mathbf{H} = \mathbf{H_{d=0}}$.
In what circumstances is the H-field not entirely divergence free? When the magnetisation has a divergence.

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

$$\nabla \cdot \mathbf{H_{c=0}} = -\nabla \cdot \mathbf{M}$$

In an infinitely long cylinder it is safe to assume the divergence of the magnetisation [3], and hence the H-field, is zero, and so the H-field only has a curl and is given by Ampere's law. (I always thought that divergence means gradient, but it is actually more like flux, see foonote).

A short cylinder has a divergence in magnetisation at the ends (and possibly also at parts of the curved boundary, if there is any component of the magnetisation unaligned with the cylinder), so the H-field has an additional curl-free term that isn't given by Ampere's law.

[3] If the magnetisation is along the axis of the cylinder, then there is no magnetisation flux that enters or leaves the cylinder at the curved boundary. If the field varies along a coordinate perpendicular to the field direction then there is no divergence.

3 Thermodynamics

3.1 First Law of thermodynamics

The first law of thermodynamics state that:

$$\Delta E = \Delta Q + W \tag{3.1}$$

Since there is a plus sign in front of W, it must mean **work done on the gas**.

Molar specific heat at constant **volume** and **pressure**

Molar specific heat:

- at constant **volume**

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V \tag{3.2}$$

  Proportionality factor between heat supplied and temperature change of a system at constant volume

- at constant **pressure**

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p = \left(\frac{\partial (E - W)}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p \tag{3.3}$$

(*the plus sign is because $W = -pV$ as W means work done **on** the gas*)

Since $\left(\frac{\partial V}{\partial T}\right)$ at constant pressure is essentially $\frac{nR}{p}$, the second term reduces to just nR.

$$\boxed{\therefore C_p = C_V + nR}$$

This shows that it takes more heat to increase temperature for an isobaric process. This is because extra energy is needed to offset those used when the gas does work by expanding.

3.2 Quasi-static process

Definition: A thermodynamic process that happens slowly enough for the system to remain in internal **thermodynamic equilibrium** (no tendency for the state of a system to change spontaneously).

Assuming a quasi-static process

I think for most questions that talk about isochoric, isothermal and isobaric processes, the process is assumed to be quasi-static. This is because to be isothermal, isochoric or isobaric throughout the processes, you must be able to state the temperature, pressure and volume of the system at each step, which is possible only if the system is in equilibrium continuously.

If process is not quasistatic

If the question is about adiabatic process (according to Blundell, if the process if both adiathermal [without the flow of heat] and reversible, it's called adiabatic), the process can be carried out both quasi-statically and non quasi-statically. If the process is not quasi-static , we can no longer use the equation for various thermodynamic processes to solve the question. However, it is still possible to solve such questions using **conservation of energy.**

Quasi-static (reversible) process (my own interpretation)

If a gas changes state from A to B, it is only quasi-static if the intermediate states are all in thermodynamic equilibrium. Work done by isothermal / isochoric / isobaric processes are calculated with the assumption that processes are quasi-static (because infinitely small increments of change represent a continuous function).

E.g. If the process involves adding or removing weights from a piston (with gas beneath it), and the piston settles into a new equilibrium position, such a process is not quasi-static because the temperature / pressure / volume of the intermediate state is unknown (not in equilibrium).

3.3 Heat engines lol

A Carnot engine (engine is defined as something that converts heat into work, 2 adiabats + 2 isotherms). We can prove by analysing each individual process that:

$$\frac{Q_h}{Q_l} = \frac{T_h}{T_l} \tag{3.4}$$

Efficiency a carnot engine (turns heat into work) is defined as the ratio of work done to the heat input, hence $\eta = \frac{W}{Q_h}$, where $W = Q_h - Q_l$, as the process is cyclic and there is no change in internal energy.

However, for a refrigerator (engine run in reverse), the efficiency is defined as a different way. It is instead

$$\eta = \frac{Q_l}{W} \tag{3.5}$$

It makes sense because efficiency in this case means the amount of heat you can **remove** from a refrigerator when the engine does a certain amount of work. (similar for a heat pump, where the engine does work to pump heat instead of remove heat).

Clausius' inequality

Clausius' inequality states that for any closed loop,

$$\oint \frac{dQ}{T} \leq 0 \tag{3.6}$$

and equality must hold for reversible reactions.

3.4 Second law of thermodynamics

Entropy is a state function, and it is defined as such

$$S(B) - S(A) = \int_A^B \frac{dQ}{T} \tag{3.7}$$

This is because the closed loop integral of $\frac{dQ}{T} = 0$ for ideal, reversible process, showing that this quantity is path independent (state dependent just like electric potential.)

For an adiabatic process, $dQ = 0$. Hence, there is no change in entropy for an adiabatic process (**isentropic**)

3.5 Modes of heat transfer

4

Fluid for sypt 2022 :)

5 Math

5.1 Single variable calculus

5.2 Multi-variable calculus

5.2.1 Why does gradient points in direction of greatest ascent

Intuition - Explanation 1

Easy explanation (imo): "Gradient" is defined as the vector summation of

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + ... \tag{5.1}$$

the maximum rate of change along the individual axes. So naturally, the summation of these rate of changes would give the maximum rate of change as well.

Intuition/Proof - Explanation 2 (by **Yi Fan**)

Single variable tangent line approximation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{5.2}$$

Multi-variable (2 in this case) tangent **plane** approximation:

$$f(x,y) - f(x_0,y_0) \approx f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \tag{5.3}$$

where f_x and f_y denotes partial derivatives with respect to x and y respectively.

Locally if we zoom in near (x_0, y_0) this shows that any nice function is just a plane.

Then now suppose our plane is given by $z = ax + by$, where we take reference point to be the origin (otherwise we can just translate the function).

Now you can think of it as $z = (a, b) \cdot (x, y)$ where we have a dot product, and we want to find the direction (x, y) that maximises this. ($a = f_x(x_0, y_0)$ and $b = f_y(x_0, y_0)$)

But dot product is maximised when the vector points in parallel direction. So we must have $(x, y) = k(a, b)$ for some constant k.

So essentially, we are trying to move (x, y) around such that we can get the maximum z value. And we realised that when (x, y) is aligned with the gradient, the maximum z value is achieved. Hence, **gradient always points in direction of greatest ascent.**

5.2.2 Divergence

5.2.3 Curl

5.3 Differential equations - from Hexiang's slides

If you want to solve something simple like

$$\frac{dy}{dx} = ay,$$

one could use separation of variables

$$\int dx = \frac{1}{a} \int \frac{1}{y} dy$$

$$x + d = \frac{\ln(y)}{a} + c$$

$$ax + c = \ln(y)$$

$$y = e^{ax+c} = e^{ax} e^{c}$$

which can be generalised to

$$y = Ae^{ax}.$$

Alternatively, we can also guess some solution, maybe $y = e^{cx}$ will do. $\frac{dy}{dx} = ce^{c}x$. Hence, $c = a$. This solution wil still hold true when we multiply it by an arbitrary constant ($y = Ae^{ax}$).

5.3.1 Solution to $\ddot{x} = -\omega^2 x$

Let's try subbing in $x = Ae^{ct}$ again and see what we get,

$$\ddot{x} = c^2 Ae^{ct}$$

$$c^2 Ae^{ct} = -\omega^2 Ae^{ct}$$

$$c = \pm i\omega$$

So there are 2 possible values for c, hence 2 solutions for x ($Ae^{i\omega t}$ or $Ae^{-i\omega t}$) For a linear differential equation (in this case it is), the most general solution would be the **linear combination** of the individual solutions

$$\boxed{x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}}$$

Using **Euler's identity**

$$x = A_1(\cos(\omega t) + i\sin(\omega t)) + A_2(\cos(-\omega t) + i\sin(-\omega t))$$

$$x = (A_1 + A_2)\cos(\omega t) + i(A_1 - A_2)\sin(\omega t)$$
$$x = B\cos(\omega t) + C\sin(\omega t)$$

which can then be converted to the form which we are more familiar with (r-formula):

$$\boxed{x = A\sin(\omega t + \phi) \quad \text{or} \quad x = A\cos(\omega t + \phi).}$$

Since we absorbed i into the constants, A and ϕ are both complex right now. However, we can just impose the conditions that they must always be real for physical purposes.