Bob The Legend

fisik is phunn

Physics Notes

August 2021

Contents

1 Mechanics

1.1 Relating velocities

Always remember to draw vector triangles to try to relate velocities of objects.

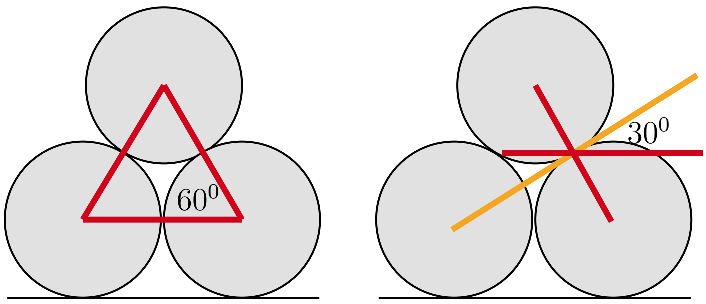
Example from **Morin 8.16**

Basically part of the question requires us to relate the relative velocity between the 3 identical cylinders (2 below, 1 on top)

In this question, we don't really have to care about whether the circles are rolling, or sliding. As long as the surfaces are always in contact with each other, rolling and sliding will be the same.

The top cylinder (A) is moving vertically downwards. Its **instantaneous** resultant velocity can be seen as a vector summation of its motion of sliding down one of the bottom cylinder, B (slide down along the tangent of the contact point) and moving together with the bottom cylinder to the right.

$$\mathbf{v}_B = \mathbf{v}_{B,A} + \mathbf{v}_A \tag{1.1}$$



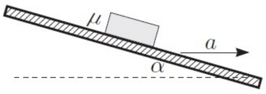
1.2 Friction

1.2.1 Friction on inclined slope

If a body is on the verge of slipping (or already slipping), then the sum of the friction force and the reaction force is angled by $\arctan \mu$ from the surface normal. This is actually quite useful in certain questions. *Sometimes, we don't even have to resolve forces by projecting their* *components onto axes. We can instead add them up vectorially by drawing vector diagram (head to tail).* This can be shown in the example below from *Jaan Kalda's Mechanics handout*.

Mechanics **pr 5.**

pr 5. A block rests on an inclined surface with slope angle α. The surface moves with a horizontal acceleration a which lies in the same vertical plane as a normal vector to the surface. Determine the values of the coefficient of friction μ that allow the block to remain still.



Here we are helped by the very universal

Method 1 To solve this problem, the common method would be to resolve forces Resolving forces along the slope gives us:

$$mg\sin\alpha - f - ma\cos\alpha = 0$$

Resolving forces perpendicular to the slope gives us:

$$N - mg\cos\alpha - ma\sin\alpha = 0$$

Since N must be positive for the block to stay on the plane, we get the first inequality

$$g\cos\alpha + a\sin\alpha > 0$$

Our second inequality comes from the force balance along the slope, where $\mid f \mid \leq \mu N$ So we get

$$\mu \geq \frac{\mid a\cos\alpha - g\sin\alpha \mid}{g\cos\alpha + a\sin\alpha} \tag{1.2}$$

Both inequalities have to be satisfied for the block to remain on the slope.

Method 2 heheheh

1.3 Tension

1.3.1 Rubber band around cylinder

In fact, we have to

1. Analyse an infinitesimal string element which subtends an angle of $d\theta$.
2. It is not hard to see that the radial component is $T\sin\frac{d\theta}{2}$. By small angle approximation, which states that $sin\theta \approx \theta$ for small θ, we get: $Td\theta$ (2 tensions acting on both ends)
3. Integrate this tension from 0 to 2π, the total tension would then be (look abit weird but I think it's correct) $2\pi T$.

1.3.2 Hanging Chain

Example from **Morin 2.8**

1. Realise that T_x is constant throughout the chain, and $\frac{T_y}{T_x} = y'$. Hence, $T_y = T_x y' = Cy'$

2. And also realised that the mass of each infinitesimal chain segments can be expressed as such

$$\rho \mathbf{g} dl = \rho \mathbf{g} \sqrt{dx^2 + dy^2} = \rho \mathbf{g} \sqrt{1 + y'^2} dx \tag{1.3}$$

3. However, it is the **difference in tension** that balance out the wright. We can then write down the final relationships and solve the equation using substitution and separation of variables.

$$d(Cy') = \rho \mathbf{g} \sqrt{1 + y'^2} dx \tag{1.4}$$

$$Cy'' = \rho \mathbf{g} \sqrt{1 + y'^2} \tag{1.5}$$

1.4 SHM

1.4.1 Translational perturbation

1. Find $\sum F$, and find all the equilibrium positions where $\sum F = 0$ (these are the stable equilibriums)
2. Use: $k = -\frac{dF}{dx}$. (negative sign depends on the direction of F. In this case, F is defined as positive in the direction that points away from equilibrium)
3. Then do smth like $\frac{dF}{dx}\Big|_{x=x_{\text{eqm}}}$, and substitute the values (Hookes constant k) into $\omega = \sqrt{\frac{k}{m}}$

1.4.2 Radial perturbation

I am not sure about this, but this is what i see people do on PhOD to solve questions (physics olympiad discord)

1. Let the small perturbation "displacement" be δ.
2. If the mass is revolving around some bigger object and the only force acting on it is in the radial direction, then we can use **conservation of angular momentum**

$$r^2 \omega = (r + \delta)^2 \omega' \tag{1.6}$$

Through simple algebraic manipulation, we get

$$\omega' = \left(\frac{r}{r + \delta} \right)^2 \omega \tag{1.7}$$

3. In the rotating frame, we can write

$$m \frac{d^2 \delta}{dt^2} = m(r + \delta)\omega'^2 - F_{\text{central}} \tag{1.8}$$

4. Try to fit that to the classic SHM equation (maybe can simplify RHS and expand to first order in δ): $\ddot{x} + \omega^2 x = 0$, and use $\omega = \sqrt{\frac{k}{m}}$

1.5 Non-inertial reference frames

I think fictitious forces are just there to make sure that everything works on in non-inertial reference frames as well.

1.5.1 Translational

Let the lab frame be S, and the non inertial frame moving at $\mathbf{a_0}$ be S'. If the object's acceleration is $\mathbf{a}$ in the lab frame, its acceleration in the non-inertial frame would be $\mathbf{a}-\mathbf{a_0}=\mathbf{a}'$. The fictitious force present is thus $-m\mathbf{a_0}$. (translational force)

1.5.2 Rotational

Consider a system of reference, which rotates around the origin O with an angular velocity $\boldsymbol{\omega}$. Consider a point P, which is motionless in the ortating system, and let us denote $\mathbf{r}=\mathbf{OP}$. In the lab system of reference, the point P moves with velocity $\mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{r}\times\boldsymbol{\omega}$. Now, if the point P moves in the rotating frame of refernece with velocity $\mathbf{u}=\frac{d\mathbf{r}}{d\tau}$ (τ is used to emasure the time in the rotating system), then this additional velocity needs to be added to what should have been for a motionless point:

$$\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{d\tau}+\boldsymbol{\omega}\times\mathbf{r}$$

So, we can conclude that the time-derivatives of vectors in rotating and lab frames of refernce are related via equality

$$\frac{d}{dt}=\frac{d\mathbf{r}}{d\tau}+\boldsymbol{\omega}\times.$$

This is written in the form of an operator, which means that we can write any vector (e.g. $\mathbf{r}$,$\mathbf{v}$) rightwards of all the three terms. In particular, we can apply this formula to the right and left hand sides of the equality $\mathbf{v}=\mathbf{u}+\boldsymbol{\omega}\times\mathbf{r}$

$$\frac{d\mathbf{v}}{dt}=\left(\frac{d}{d\tau}+\boldsymbol{\omega}\times\right)\left(\mathbf{u}+\boldsymbol{\omega}\times\mathbf{r}\right)=\frac{d\mathbf{u}}{d\tau}+\boldsymbol{\omega}\times\mathbf{u}+\frac{d(\boldsymbol{\omega}\times\mathbf{r})}{d\tau}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

We obtain

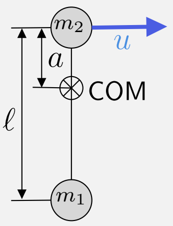
$$\frac{d\mathbf{v}}{dt}=\frac{d\mathbf{u}}{d\tau}+2\boldsymbol{\omega}\times\mathbf{u}-\boldsymbol{\omega}^2\mathbf{r}$$

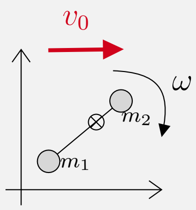
Recall that $\frac{d\mathbf{v}}{dt}$ is the acceleration of the point P as seen in the lab frame of refernce, and $\frac{d\mathbf{u}}{d\tau}$ is the same as seen in the rotating frame of reference. now, if P is a point mass m, and thee is an external force $\mathbf{F}$ acting on P, then $\mathbf{F}=m\frac{d\mathbf{v}}{dt}$ and hence,

$$m\frac{d\mathbf{u}}{d\tau}=\mathbf{F}-2\boldsymbol{\omega}\times\mathbf{u}m+\boldsymbol{\omega}^2\mathbf{r}m$$

1.6 SJPO2021 Special Round Qn

Da question





Basically the question asks you to find the equation for the trajectory of the thing shown in the diagram when m_2 starts with an initial velocity u perpendicular to the rod. (masses on frictionless ground). I think using the centre of mass frame would help in solving this problem.

Position of centre of mass with respect to m_2,

$$a = \frac{m_1 \ell}{m_1 + m_2} \tag{1.9}$$

Velocity of centre of mass of the system can be calculated through conservation of momentum:

$$v_{\text{COM}} = \frac{m_2 u}{m_1 + m_2} \tag{1.10}$$

The moment of inertia of the system with respect to its centre of mass is

$$I = m_2 \left(\frac{m_1 \ell}{m_1 + m_2} \right)^2 + m_1 \left(\frac{m_2 \ell}{m_1 + m_2} \right)^2 = \frac{m_1 m_2 \ell^2}{m_1 + m_2} \tag{1.11}$$

Angular velocity could be either determined using **angular momentum**:

$$\omega = \frac{L}{I} \tag{1.12}$$

$$\therefore \omega = \frac{u}{\ell} \tag{1.13}$$

Or determined by **finding velocity of m_2 in the COM frame** and then **divide by the radius of rotation to find angular velocity** as $v = r\omega$.

$$v' = u - \frac{m_2 u}{m_1 + m_2} = \frac{m_1 u}{m_1 + m_2} \tag{1.14}$$

Dividing this by the radius of rotation (distance from m_2 to COM), we obtain the same expression for ω

$$\omega = \frac{v'}{a} = \frac{u}{\ell} \tag{1.15}$$

Formulating the equation for trajectory:

$$x(t) = a \sin(-\omega t) + v_{\text{COM}} t \tag{1.16}$$

$$y(t) = a \cos(\omega t) \tag{1.17}$$

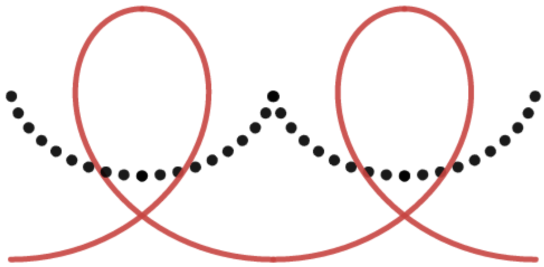
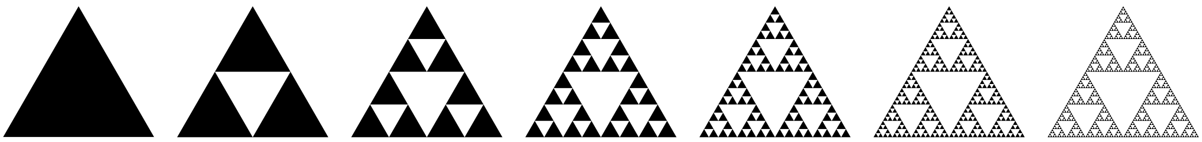


Figure 1.1: Trajectory plotted with Desmos

1.7 Gravitation

1.8 Scaling and Parallel Axis theorem

1.8.1 MOI of a sierpinski triangle



Building up

We will build the triangle from the smallest triangle possible, and slowly work backwards. Let the smallest triangle be the basic repeating unit with mass m and side length ℓ.

It is obvious that $m_{n+1} = 3m_n$ and $\ell_{n+1} = 2\ell_n$ with $I_{n+1} = 3(I_n + ma^2) = 3I_n + m\ell_n^2$ where a is the distance from the COM of the small triangle to that of the big triangle and is $\ell_n/\sqrt{3}$

The moment inertia can then be found through iterations,

$$\begin{aligned}
I_1 &= 3Cm_0\ell_0^2 + m_0^2\ell_0^2 = (3C+1)m_0\ell_0^2 \\
I_2 &= 3I_1 + m_1\ell_1^2 = (3^2C + 3 + 12)m_0\ell_0^2 \\
I_3 &= 3I_2 + m_2\ell_2^2 = (3^2C + 3^2 + 3 + 12^2)m_0\ell_0^2 \\
I_n &= 3I_{n-1} + m_{n-1}\ell_{n-1}^2 = (12^{n-1} + 12^{n-2}3 + ... + 3^{n-1} + 3^nC)m_0\ell_0^2
\end{aligned} \tag{1.18}$$

where $Cm_0\ell_0^2$ is the moment of inertia of the basic building block

Taking the limits as $n \to \infty$

$$\lim_{n\to\infty} I_n = 12^{n-1}m_0\ell_0^2\left(1 + \frac{1}{4} + \frac{1}{4^2} + ... + \frac{1}{4^{n-1}} + \frac{1}{4^n}C\right) \tag{1.19}$$

We note that the last term with C plays no role in this infinite geometric series. After defining $\lim_{n\to\infty} 3^n m_0 = M$ and $\lim_{n\to\infty} 2^n\ell_0 = \ell$, and simplifying the above expression, we get

$$\lim_{n\to\infty} I_n = \frac{1}{9}M\ell^2 \tag{1.20}$$

Repeating pattern

For a 2D-planar object (this is also from Ryan's lesson), the moment of inertia is calculated using

$$I = \int r^2 dm = \int r^2 \sigma dx dy \tag{1.21}$$

where σ is the surface mass density. We see that if we were to scale the side length of the planar object by 2 (or just scale it up by 2 times), the moment of inertia will be scaled up by $2^2 \times 2 \times 2 = 16$, due to the $r^2 dx dy$ in the above integral.

However, the tricky part is that for a sierpinski triangle, it is a bit different. Because the σ is always changing. Instead, we can rewrite I as

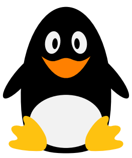
$$I = \int r^2 dm \tag{1.22}$$

If we were to scale up the triangle by 2 (go up by 1 layer), $r^2 dm$ will be equivalent to scaling I up by $2^2 \times 3 = 12$ times.

so let the moment of inertia of one infinite siepinski triangle be I with side length ℓ. We know that we can form a bigger siepinski triangle (side length: 2ℓ) with 3 smaller ones. With this method, we arrive at the same answer

$$(2^2 \times 3)I = 3I + m\ell^2 \tag{1.23}$$

$$I = \frac{1}{9} m\ell^2 \tag{1.24}$$



2 Electromagnetism

2.1 Electric field and potential

Potential and field are related by the following equation

$$-\int_A^B E \cdot ds = V(B) - V(A) \tag{2.1}$$

I think the intuition is that a **positive workdone by E field** always results in a decrease in **potential**.

Intuition

when the E field is pointing away from a **positive charge**, to move the **negative test charge** away from the positive charge, you need a force with magnitude of at least qE and opposite in direction of the force of attraction. Workdone by that force results in an increase in potential (less negative) by conservation of energy. Since Eq always points in the opposite direction of that force, qE always points in direction of decreasing potential.

2.1.1 Faraday's law of induction

2.1.2 More about electromotive force

In the circuit, the flowing electrons experience 2 forces, namely the force provided by the external agent, such as the battery ($\mathbf{f_s}$), and the force provided by E-field due to the build-up of charges around the circuit ($\mathbf{E}$).

Hence, the net force driving the current is

$$\mathbf{f} = \mathbf{f_s} + \mathbf{E} \tag{2.2}$$

The emf (electromotive force) is defined as

$$\text{emf} = \oint \mathbf{f} \cdot d\mathbf{s} = \oint \mathbf{f_s} \cdot d\mathbf{s} \tag{2.3}$$

This is because $\oint \mathbf{E} \cdot d\mathbf{s} = 0$.

2.2 Different kinds of "Current"

linear current

$$\mathbf{I} = \lambda \mathbf{v} \tag{2.4}$$

The unit for λ is $[A][m]^{-1}[s]$. This seemed a bit weird to be initially. I guess can think about it this way:

I guess this makes sense...

$$\mathbf{I} = \frac{Q}{t} \tag{2.5}$$

$$\mathbf{I} = \frac{\lambda d}{t} = \lambda \mathbf{v} \tag{2.6}$$

surface current

$$\mathbf{K} = \frac{d\mathbf{I}}{d\ell_{\perp}} = \sigma \mathbf{v} \tag{2.7}$$

The unit for σ is $[A][m]^{-2}[s]$

volume current

$$\mathbf{J} = \frac{d\mathbf{I}}{da_{\perp}} = \rho \mathbf{v} \tag{2.8}$$

2.3 Vector potential is pretty gae

Vector potential. In electrostatics, we have $-\nabla \mathbf{V} = \mathbf{E}$, thanks to the fact that $\nabla \times \mathbf{V} = 0$. Apparently, we can also define such a potential $(\mathbf{V})$ for magnostatics $(\mathbf{A})$. Since $\nabla \cdot \mathbf{B} = 0$, it allows for $\nabla \times \mathbf{A} = \mathbf{B}$ (divergence of curl: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$)

We can substitute $\nabla \times \mathbf{A} = \mathbf{B}$ into $\nabla \times \mathbf{B}$, and get

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla \cdot (\nabla \mathbf{A}) && (2.9) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} && (2.10) \end{aligned}$$

But why are we doing this? Well, if we can make $\nabla(\nabla \cdot \mathbf{A}) = 0$, then we get $-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$ (just **Poisson's equation**), making the magnetic vector potential a very useful tool, from which the B field can then be easily calculated.

To show that we can indeed obtain **Poisson's equation** for magnetism, we now have to prove that it is possible to let $\nabla \cdot \mathbf{A} = 0$.

Supposed the original $\mathbf{A_0}$ is not divergenceless. But we can make it divergenceless by adding the **gradient** of something else, maybe $\nabla \mathbf{\Gamma}$ ($\mathbf{\Gamma}$ looks cool), turning $\mathbf{A_0}$ into $\mathbf{A}$.We are adding the gradient of something else because we don't want it to affect $\nabla \times \mathbf{A}$, since the curl of gradient is 0 (so the curl is still $\mathbf{B}$). We then get

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2 \mathbf{\Gamma} \tag{2.11}$$

Setting $\nabla \cdot \mathbf{A} = 0$, we get

$$\nabla \cdot \mathbf{A_0} = -\nabla^2 \mathbf{\Gamma} \tag{2.12}$$

This is again, just Poisson's equation, and we know that there will always be a solution for $\mathbf{\Gamma}$. Hence, it is always possible to make $\mathbf{A}$ divergenceless while still keeping $\nabla \times \mathbf{A} = \mathbf{B}$.

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \tag{2.13}$$

■

2.4 Dipoles

2.4.1 Electric dipole

Torque experienced by electric dipole in a uniform electric field is

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \tag{2.14}$$

The equation for force experienced by an electric dipole is given by $mathbfF = \nabla(\mathbf{p} \cdot \mathbf{E})$ (used for non uniform $\mathbf{E}$ field), where $\mathbf{p} = qd$. It the E field is uniform, the net force would be 0, but the net torque would not be 0.

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}) = (\mathbf{p} \cdot \nabla)\mathbf{E}. \tag{2.15}$$

2.4.2 Magnetic dipole

Torque experienced by a magnetic dipole in a uniform magnetic field is

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \tag{2.16}$$

The equation for force experienced by a magnetic dipole in **uniform magnetic field** is always 0.

Proof

The loop can be arranged/positioned in any manner in the uniform $\mathbf{B}$ field. The force on an infinitesimal part of the loop of length $d\ell$ is

$$d\mathbf{F} = (\mathbf{B} \times d\mathbf{l})I \tag{2.17}$$

If we integrate the equation above around the loop, we get

$$\mathbf{F} = \mathbf{B}I \times \oint d\mathbf{l} \tag{2.18}$$

where $\oint d\mathbf{l} = 0$

In a non-uniform magnetic field, the force experienced would be

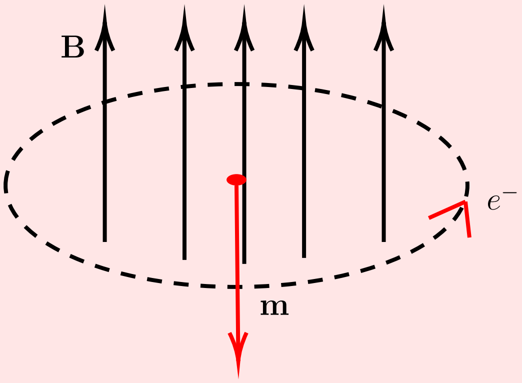
$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{2.19}$$

Unlike the equation for electric dipole, $\mathbf{F} \neq (\mathbf{m} \cdot \nabla)\mathbf{B}$ (not sure why yet).

2.4.3 Diamagnetism and Paramagnetism

Paramagnets acquire a magnetisation parallel to $\mathbf{B}$ while diamagnets acquire a magnetisation opposite to $\mathbf{B}$. We will have to study the effect of magnetic field at the sub-atomic level...

Magnetic field and electron orbital



Paramagnetism arises when the dipole moment[1] of the electron orbital align with the external magnetic field, while diamagnetism is a result of the change in the speed of electron due to the external magnetic field. The orbital contribution to paramagnetism is a lot smaller as it is much harder to tilt the orbital then to change its spin.
The motion of the electron can be approximated as a steady current (because the orbital period is extremely short), then the magnetic dipole can be written as

$$\mathbf{m} = I\pi R^2\hat{\mathbf{z}} = \frac{-e}{T}\pi R^2\hat{\mathbf{z}} = \frac{-ev}{2\pi R}\pi R^2\hat{\mathbf{z}} = \frac{-evR}{2}\hat{\mathbf{z}} \tag{2.20}$$

where R is the radius of the orbit (the negative sign accounts for the negative charge of the electron, as current is always a positive quantity)(v is defined as positive when it is anti clockwise).
This shows that the electron orbital will always be subject to a torque ($\mathbf{m}\times\mathbf{B}$). However, it is hard to tilt the orbital. A more significant effect would be the change in speed of the electron due to the external $\mathbf{B}$ field.
Assuming uniform circular motion, the initial speed (v_0) is given by

$$m\frac{v_0^2}{R} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} \tag{2.21}$$

Let's assume that the $\mathbf{B}$ field is perpendicular to the orbit, the new speed of the electron would be given by

$$m\frac{v_1^2}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + ev_1B \tag{2.22}$$

$$ev_1B = \frac{m}{R}(v_1^2 - v_0^2) = \frac{m}{R}(v_1 - v_0)(v_1 + v_0) = \frac{m}{R}(\Delta v)(v_1 + v_0) \tag{2.23}$$

Assuming difference between v_1 and v_0 is small, making Δv the subject

$$\Delta v = \frac{eRB}{2m} \tag{2.24}$$

$$\boxed{\therefore \Delta\mathbf{m} = \frac{-e^2R^2}{4m}\mathbf{B}} \tag{2.25}$$

2.5 Vector potential of a rotating, charged sphere

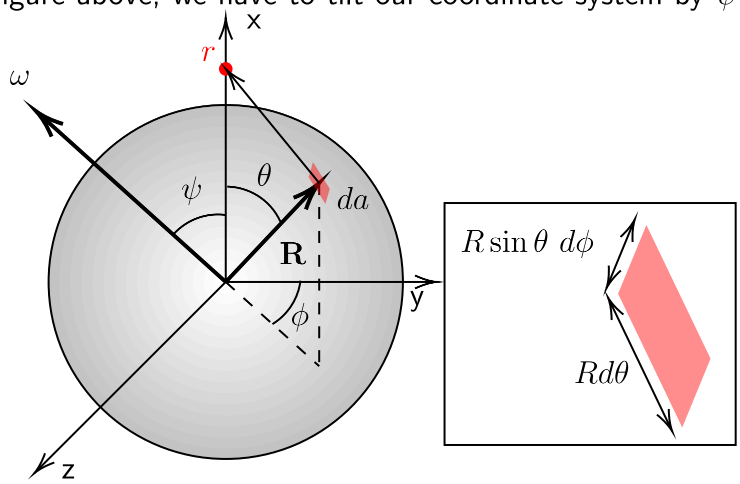
Problem Statement

A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential it produces at point r.

Firstly, we will have to recall the equation for vector potential from a surface current density

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{r}da$$

As shown in the figure above, we have to tilt our coordinate system by ψ so as to make it



easier to integrate. We can make our point of interest lie on the x axis, and the ω vector lie on the x-z plane to make our lives easier. The first step to find surface current is to find the velocity of da.

Power of cross product: $\mathbf{v} = \omega \times \mathbf{r}$ (not the other way round) [2]

$$\mathbf{v} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\cos\psi & 0 & \omega\sin\psi \\ R\cos\theta & R\sin\theta\cos\phi & R\sin\theta\sin\phi \end{pmatrix}$$

$$\mathbf{v} = -R\omega\sin\psi\sin\theta\cos\phi\hat{\mathbf{x}} + (\omega\sin\psi R\cos\theta - \omega\cos\psi R\sin\theta\sin\phi)\hat{\mathbf{y}} + \omega\cos\psi R\sin\theta\cos\phi\hat{\mathbf{z}}$$

$$\mathbf{K} = \sigma\mathbf{v}$$

The integral would then be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\mathbf{K}}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} R^2\sin\theta d\theta d\phi$$

[1] Fingers grip around direction of current (movement of **positive** charges, direction of thumb is the direction of dipole moment) - Right Hand Grip Rule

[2] $\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2b_3 - a_3b_2)\hat{\mathbf{x}} + (a_3b_1 - a_1b_3)\hat{\mathbf{y}} + (a_1b_2 - a_2b_1)\hat{\mathbf{z}}$

Since the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components of $\mathbf{K}$ contain $\cos\phi$, after integration, those terms will vanish since

$$\int_0^{2\pi} \cos\phi d\phi = \int_0^{2\pi} \sin\phi d\phi = 0$$

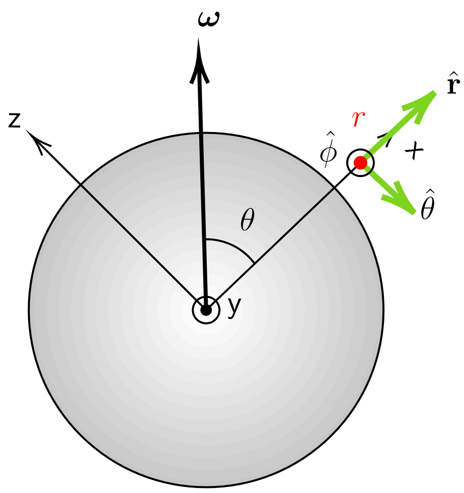
Therefore,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \sigma R^3 \omega \sin\psi}{2}\left(\int_0^{\pi} \frac{\sin\theta\cos\theta}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} d\theta\right)\hat{\mathbf{y}}$$

After doing some crazy math and integration (which includes making the substitution: $u = \cos\theta$), we get

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 \sigma R r \omega \sin\psi}{3}\hat{\mathbf{y}} & \text{inside the sphere} \\ \dfrac{\mu_0 \sigma R^4 \omega \sin\psi}{3r^2}\hat{\mathbf{y}} & \text{outside the sphere} \end{cases}$$

To convert the solutions back to the orignal coordinate system, where the angular velocity vector coincides with the x axis, where our point of interest is located at (r, θ, ϕ), we can do the conversion as such



Just use a bit of imagination (or refer to this diagram shown here). Also take note that $\boldsymbol{\omega}$ lie in the xz plane

Hence, we obtain the final solution in spherical coordinates

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 \sigma R r \omega \sin\theta}{3}\hat{\boldsymbol{\phi}} & \text{inside the sphere} \\ \dfrac{\mu_0 \sigma R^4 \omega \sin\theta}{3r^2}\hat{\boldsymbol{\phi}} & \text{outside the sphere} \end{cases}$$

To derive magnetic field from vector potential, we have to calculate it with $\mathbf{B} = \nabla \times \mathbf{A}$ (in spherical coordinates)

$$\nabla \times \mathbf{A} = \frac{1}{r^2\sin\theta}\begin{pmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{pmatrix} = \frac{1}{r^2\sin\theta}\begin{pmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta A_\phi \end{pmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)\hat{\mathbf{r}} - \frac{\partial}{\partial r}(r\sin\theta A_\phi)r\hat{\boldsymbol{\theta}}\right) \quad (2.26)$$

$$\mathbf{B} = \begin{cases} \frac{2}{3}\mu_0 R\omega\sigma(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) & \text{inside the sphere} \\ \frac{\mu_0 R^4\omega\sigma}{3r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) & \text{outside the sphere} \end{cases} \tag{2.27}$$

2.6 Magnetisation

The magnetisation, $\mathbf{M}$ is defined as the magnetic dipole moment per unit volume. Since $\nabla \times \mathbf{M} = \mathbf{J_b}$, where $\mathbf{J_b}$ refers to the **bound current**. Bound current refers to *invisible* current in the magnet due to magnetisation. There is another type of current called **free current** ($\mathbf{J_f}$). This is the current due to a battery, or a voltage source etc. In the case where both $\mathbf{J_f}$ and $\mathbf{J_b}$ are steady currents, we can apply the equation:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J_f} + \mathbf{J_b}) \tag{2.28}$$

After replacing $\mathbf{J_b}$ with $\nabla \times \mathbf{M}$, and manipulation the equation, we get

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J_f} \tag{2.29}$$

Now comes the cool part, we can replace $\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right)$ with a quantity called $\mathbf{H}$, whose name is **Auxiliary Field**:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J_f}} \quad \text{or} \quad \boxed{\oint \mathbf{H} \cdot ds = I_{\text{f,enc}}} \tag{2.30}$$

This equation is very elegant as it allows to express **Ampere's Law** in terms of free current alone, which is what we are able to control and measure.

Under certain circumstances however, we cannot use Ampere's Law methods to find $\mathbf{H}$ field. I asked this on stack exchange (my question)

When can we use Ampere's Law methods to find $\mathbf{H}$ field

Suppose that the H-field was composed of two parts. One of which had a curl of zero and the other which had a divergence of zero. Call them $\mathbf{H_{c=0}}$ and $\mathbf{H_{d=0}}$ respectively, such that

$$\mathbf{H} = \mathbf{H_{c=0}} + \mathbf{H_{d=0}} \ .$$

In which case Ampere's law

$$\mathbf{J_f} = \nabla \times (\mathbf{H_{c=0}} + \mathbf{H_{d=0}}) = \nabla \times \mathbf{H_{d=0}}$$

The Helmholtz decomposition theorem tells us that any vector field can be represented by two such fields, so Amperes law will only tell us about the total H-field when it consists only of a divergence-free component. i.e. when $\mathbf{H} = \mathbf{H_{d=0}}$.
In what circumstances is the H-field not entirely divergence free? When the magnetisation has a divergence.

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

$$\nabla \cdot \mathbf{H_{c=0}} = -\nabla \cdot \mathbf{M}$$

In an infinitely long cylinder it is safe to assume the divergence of the magnetisation [3], and hence the H-field, is zero, and so the H-field only has a curl and is given by Ampere's law. (I always thought that divergence means gradient, but it is actually more like flux, see foonote).

A short cylinder has a divergence in magnetisation at the ends (and possibly also at parts of the curved boundary, if there is any component of the magnetisation unaligned with the cylinder), so the H-field has an additional curl-free term that isn't given by Ampere's law.

2.7 Circuit

2021 nodes connected to each other, find equivalent resistance between any 2 nodes.

$$\boxed{\text{Ans} = \frac{2R}{2021}}$$

[3] If the magnetisation is along the axis of the cylinder, then there is no magnetisation flux that enters or leaves the cylinder at the curved boundary. If the field varies along a coordinate perpendicular to the field direction then there is no divergence.

3

Thermodynamics

3.1 Kinetic theory - Ryan's lesson transcript

If KE is dependent on temperature, whats the difference between a cold moving ball, and a hot stationary ball. This is because the former is about **microscopic**, but the latter is about **macroscopic**.

Because it is not feasible to model the motion of every single molecules with newton's laws, it is more convenient to use probability.

3.1.1 Probability distribution

We are working with **continuous** probability distribution $P(x)$where $0 \leq P(x) \leq 1$. The area under the curve (let's say from a to b, $b > a$) would be the probability of the number n to be $a \leq n \leq b$

3.1.2 velocity distribution

let the velocity probability distribution be $g(v_x)$

$$g(v_x) \propto \exp\left(\frac{-mv_x^2}{2k_BT}\right) \tag{3.1}$$

To find the proportionality constant, we have to normalise the distribution by integrating the distribution from -1 to 1, and setting the integral to 1.

$$\int_{-\infty}^{\infty} A\exp\left(\frac{-mv_x^2}{2k_BT}\right)dv_x = 1 \tag{3.2}$$

This is the case where $\alpha = m/2k_BT$ (see Math section). Hence,

$$g(v_x) = \sqrt{\frac{m}{2\pi k_BT}}\exp\left(\frac{-mv_x^2}{2k_BT}\right) \tag{3.3}$$

To find the average velocity, we just need to integrate

$$\langle v_x \rangle = \int_{-\infty}^{\infty} v_x g(v_x)dv_x = 0 \tag{3.4}$$

HAHA sike it is 0 cos its velocity not speed.

3.1.3 speed distribution

Speed $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ can be represented as the distance between the origin and a point in the velocity space.

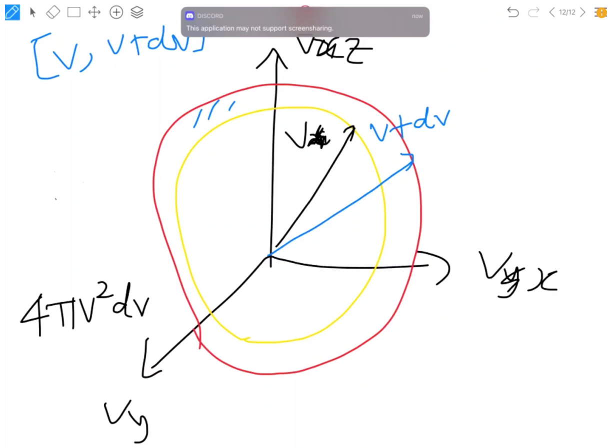


Figure 3.1: beautiful drawing by Ryan

The volume enclosed, $V = 4\pi v^2 dv$, between the red and yellow circle, multipled by the probability of the particles moving in that region, denotes the particles with speed $v \leq v_p \leq v + dv$. The speed probability distribution is then given by

$$f(v)dv \propto v^2 \exp\left(\frac{-mv^2}{2k_BT}\right)dv \tag{3.5}$$

Integrating, we get

$$f(v) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{\frac{3}{2}} v^2 \exp\left(\frac{-mv^2}{2k_BT}\right) \tag{3.6}$$

Average speed:

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}} \tag{3.7}$$

Root mean squared velocity:

$$\langle v^2 \rangle = \frac{3k_BT}{m} = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle \tag{3.8}$$

Most probable speed: $df/dv = 0$

$$v = \sqrt{\frac{2k_BT}{m}} \tag{3.9}$$

Note: m stand for the **molar mass** lmao not the actual mass of all the gas molecules.

3.1.4 Equipartition theorem

Every degree of freedom will contribute $k_BT/2$ to the internal energy.

$$U = \frac{f}{2}k_BT \tag{3.10}$$

More rigorous form: Each quadratic mode with energy in the form of $E = \alpha x^2$, where x can be position or velocity, contributes $k_BT/2$ to the internal energy (so its not really about degree of freedom). It just so happens that both rotational and translational energy are $\propto v^2, \omega^2$

3.2 Ideal gas law

$$p = \frac{1}{3} nm\langle v^2 \rangle \tag{3.11}$$

where n is the number of molecules per unit volume. and m is the mass of the molecules.

3.3 First Law of thermodynamics

The first law of thermodynamics state that:

$$\Delta E = \Delta Q + W \tag{3.12}$$

Since there is a plus sign in front of W, it must mean **work done on the gas**. The differntial form of the the equation above is

$$dE = dQ + dW \tag{3.13}$$

where dQ and dW are called **inexact differential** since they are path dependent (and not state dependent)quantities.

On the other hand, E is a state dependent quantity (depends on both temperature, T and volume, V). Hence, a small change in E (dE) can also be written as

$$dE(T, V) = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV \tag{3.14}$$

Molar specific heat at constant **volume** and **pressure**

Molar specific heat:

- at constant **volume**

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V \tag{3.15}$$

  Proportionality factor between heat supplied and temperature change of a system at constant volume

- at constant **pressure**

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p = \left(\frac{\partial (E - W)}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p \tag{3.16}$$

(the plus sign in front of the work is because $dQ = dE - dW$, and $dW = -pdV$. This is before when volume of gas decreases, work must be done **on** it, increasing its internal energy).

Since $\left(\frac{\partial V}{\partial T}\right)$ at constant pressure is essentially $\frac{nR}{p}$, the second term reduces to just nR.

$$\boxed{\therefore C_p = C_V + nR}$$

This shows that it takes more heat to increase temperature for an isobaric process. This is because extra energy is needed to offset those used when the gas does work by expanding.

This relationship can actually be derived through many other methods as well. But just need to remember **first law of thermodynamics**, **internal energy's dependence on both temperature and volume**, as well as the fact that **heat capacity is a proportionality factor between heat input and change in temperature (internal energy)**, then shld be okay :)

3.4 Quasi-static process

Definition: A thermodynamic process that happens slowly enough for the system to remain in internal **thermodynamic equilibrium** (no tendency for the state of a system to change spontaneously).

Assuming a quasi-static process

I think for most questions that talk about isochoric, isothermal and isobaric processes, the process is assumed to be quasi-static. This is because to be isothermal, isochoric or isobaric throughout the processes, you must be able to state the temperature, pressure and volume of the system at each step, which is possible only if the system is in equilibrium continuously.

If process is not quasistatic

If the question is about adiathermal process (according to Blundell, if the process is both adiathermal [without the flow of heat] and reversible, it's called adiabatic), the process can be carried out both quasi-statically and non quasi-statically. If the process is not quasi-static , we can no longer use the equation for various thermodynamic processes to solve the question. However, it is still possible to solve such questions using **conservation of energy.**

Quasi-static (reversible) process (my own interpretation)

If a gas changes state from A to B, it is only quasi-static if the intermediate states are all in thermodynamic equilibrium. Work done by isothermal / isochoric / isobaric processes are calculated with the assumption that processes are quasi-static (because infinitely small increments of change represent a continuous function).

E.g. If the process involves adding or removing weights from a piston (with gas beneath it), and the piston settles into a new equilibrium position, such a process is not quasi-static because the temperature / pressure / volume of the intermediate state is unknown (not in equilibrium). (Or when the process happens too fast such that there's insufficient time for the gas to settle into equilibrium)

Also err i am not exactly sure why but for an adiabatic process,

$$dQ = \left(\frac{\partial Q}{\partial V}\right)_p dV + \left(\frac{\partial Q}{\partial p}\right)_V dp = 0 \tag{3.17}$$

3.5 Heat engines lol

A Carnot engine (engine is defined as something that converts heat into work, 2 adiabats + 2 isotherms). We can prove by analysing each individual process that:

$$\frac{Q_h}{Q_l} = \frac{T_h}{T_l} \tag{3.18}$$

Efficiency a carnot engine (turns heat into work) is defined as the ratio of work done to the heat input, hence $\eta = \frac{W}{Q_h}$, where $W = Q_h - Q_l$, as the process is cyclic and there is no change in internal energy.

However, for a refrigerator (engine run in reverse), the efficiency is defined as a different way. It is instead

$$\eta = \frac{Q_l}{W} \tag{3.19}$$

It makes sense because efficiency in this case means the amount of heat you can **remove** from a refrigerator when the engine does a certain amount of work. (similar for a heat pump, where the engine does work to pump heat instead of remove heat).

Clausius' inequality

Clausius' inequality states that for any closed loop,

$$\oint \frac{dQ}{T} \leq 0 \tag{3.20}$$

and equality must hold for reversible reactions.

3.6 Second law of thermodynamics

Entropy is a state function, and it is defined as such

$$S(B) - S(A) = \int_A^B \frac{dQ}{T} \tag{3.21}$$

This is because the closed loop integral of $\frac{dQ}{T} = 0$ for ideal, reversible process, showing that this quantity is path independent (state dependent just like electric potential.)

For an adiabatic process, $dQ = 0$. Hence, there is no change in entropy for an adiabaticprocess (**isentropic**)

3.7 Modes of heat transfer

4 Fluid for sypt 2023 :)

Stuff from Fox and McDonald's introduction from fluid dynamics in the

4.1 Eulerian

In fluid dynamics, it is common to use the Eulerian method of description, which focuses attention on the properties of a flow at a given point in space as a function of time. For example $v(x, y, z, t)$.

5 Special Relativity

There are 3 fundamental effects of special relativity

- Loss of simultaneity
- Time dilation
- Length contraction

5.1 kinematics

Let the frame S' be moving at speed v relative to a stationary frame S. Denote all quantities (for e.g. time) in the S' frame with a prime ($'$).

5.1.1 basics

Time dilation

$$t' = \frac{t}{\gamma} = \gamma t' \tag{5.1}$$

Length contraction

$$\ell' = \gamma \ell = \frac{\ell'}{\gamma} \tag{5.2}$$

where $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$ Since c is the maximum possible speed a thing can reach, γ will always be **greater than or equal to** 1.

5.1.2 Lorentz transformation

$$\begin{cases} \Delta x = \gamma(\Delta x' + v\Delta t') \\ \Delta t = \gamma(\Delta t' + v\frac{\Delta x'}{c^2}) \end{cases} \tag{5.3}$$

or

$$\begin{cases} \Delta x' = \gamma(\Delta x - v\Delta t) \\ \Delta t' = \gamma(\Delta t - v\frac{\Delta x}{c^2}) \end{cases} \tag{5.4}$$

To derive the equation for time dilation, we note that the person measuring it (either in S or S') has to be stationary (i.e. $v = 0$).

From the equation above and setting $\Delta x'$ to 0, we see that $\Delta t = \gamma \Delta t'$. Since $\gamma \geq 1$, the time elapsed measured by the stationary observer in the S' frame is always shorter than the time elapsed in S.

However, if we set Δx to 0 and use the second set of equation, we get $\Delta t' = \gamma \Delta t$. This seems to be a contradiction. But in fact, this scenario is very different as the stationary observer is now in the S frame (and not the S' frame).

5.1.3 relativistic velocity addition

Let v_1 be the speed of the object measured in the frame: S' moving at speed v_2 relative to the ground.

Writing out the lorentz transformation for S', we get

$$\begin{cases} \Delta x = \gamma_2(\Delta x' + v_2 \Delta t') \\ \Delta t = \gamma_2(\Delta t' + v_2 \frac{\Delta x'}{c^2}) \end{cases} \tag{5.5}$$

where γ_2 is the Lorentz factor associated with v_2. As the speed measured in the lab frame (S) is defined as $\Delta x/\Delta t$, we get

$$\frac{\Delta x}{\Delta t} = \frac{\gamma_2(\Delta x' + v_2 \Delta t')}{\gamma_2(\Delta t' + v_2 \frac{\Delta x'}{c^2})} = \frac{v_1 + v_2}{1 + (v_1 v_2)/c^2} \tag{5.6}$$

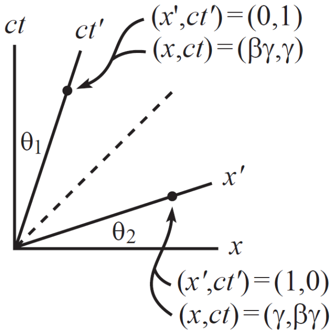
The derivation **above** only applies to velocity addition in the **longitudinal direction**. For **transverse velcoity addition**, we can use a similar method (Lorentz transformation: $\Delta y' = \Delta y = v \Delta t$)

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma(\Delta t' + v\frac{\Delta x'}{c^2})} = \frac{\Delta y'/\Delta t'}{\gamma(1 + v\frac{\Delta x'/\Delta t'}{c^2})} = \frac{v_y'}{\gamma(1 + v_x' \frac{v}{c^2})} \tag{5.7}$$

I think this could be easily derived using 4-vectors, but becuase i havent learnt that yet, so idk LMAO.

5.1.4 invariance and minkowski diagram

Consider the quantity $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2$, where s (dropping the Δ from now on) is known as the **invariant interval** because this quantity is invariant to coordinates (i.e. $ct - x = ct' - x'$, (ct, x) is usually known as the **spacetime interval**).



This could be derived with **lorentz transformation** with $\beta = v/c$. 1 unit on the ct axis would correspond to $\gamma\sqrt{(1+\beta^2)}$ on the ct' axis. Hence,

$$\frac{\text{one } ct' \text{ unit}}{\text{one } ct \text{ unit}} = \frac{\sqrt{1+\beta^2}}{\sqrt{1-\beta^2}} \tag{5.8}$$

Simultaenous measurements can be represented by drawing a line perpendicular to the ct/ct' axes and seeing the points of intersection of the line with the ct/ct' axes.

5.2 dynamics

5.2.1 Energy and momentum

Energy

$$E = \gamma mc^2 \tag{5.9}$$

Momentum

$$\mathbf{p} = \gamma m\mathbf{v} \tag{5.10}$$

To justify these expressions, I think it is quite elegant to perform Taylor series expansion in the limit of $v \ll c$ for $1/\sqrt{1-x^2}$ [1], you obtain

$$\begin{cases} E = mc^2(1 + \frac{1}{2}(\frac{v}{c})^2 + \frac{3}{8}(\frac{v}{c})^3 + \frac{5}{16}(\frac{v}{c})^4 + ...) = mc^2 + \frac{1}{2}mv^2 + ... \\ \mathbf{p} = m\mathbf{v}(1 + \frac{1}{2}(\frac{v}{c})^2 + \frac{3}{8}(\frac{v}{c})^3 + \frac{5}{16}(\frac{v}{c})^4 + ...) = m\mathbf{v} + ... \end{cases} \tag{5.11}$$

There is this very important equation

$$E^2 = p^2c^2 + m^2c^4 \tag{5.12}$$

Another pretty useful equation is from the fundamental equations describing relativistic momentum and energy,

$$\frac{\mathbf{p}}{E} = \frac{\mathbf{v}}{c^2} \tag{5.13}$$

Moreover, the equations to convert energy and momentum between frames are also quite elegant. Let the particle be travelling at speed v in frame S', which is moving at speed u relative to lab frame S, then the energy and momentum in the 2 frames are related through

$$\begin{cases} E = \gamma_u(E' + vp') \\ p = \gamma_u(p' + vE') \end{cases} \tag{5.14}$$

[2]

Through this equation, we note that E and p behave exactly the same way as ct and x. Hence, we obtain the following equation

$$m^2 = E^2 - p^2 = E'^2 - p'^2 \tag{5.15}$$

where the mass of the particle, m, is the **invariant quantity**.

5.2.2 Relativistic Collisions

It is important to note that this $E = \gamma mc^2$ here is not equivalent to kinetic energy($E - mc^2$). During collision, it is the **total energy** that is conserved, so always use E and not KE.

To solve collisions problems, it is nice to define a **4- momentum**, or $P = (E, p_x, p_y, p_z)$ [3]. The inner product of this "vector" is defined to be $a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4$. It is defined this way so that we can concisely write $m^2 = E^2 - p^2$ as

$$P \cdot P = m^2 \tag{5.16}$$

[1] $(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 ...$

[2] $E' = \gamma_v m$ and $p' = \gamma_v mv$, note that it is γ_v and not γ_u

[3] Note that it shld be E/c to make it dimensionally consistent but since $c = 1$, we can just write it like this

5.2.3 Force

5.3 4-vector

The most basic 4-vector is the **4-displacement** between two points in spacetime, which we call events. The 4-displacement is a fundamentally invariant geoemtric object, which you can think of as an arrow pointing between these 2 events.

Consider two inertial frame S and S', such that S' is moving at speed v in the positive x-direction relative to S. Their coordinates are calibrated such that their origins coincide at $t = t' = 0$. The frames are said to be in **standard configuration**, and their change-of-coordinates transformation can be summarised by the following

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{5.17}$$

This is lorentz transformation in the matrix form, where the transformation matrix is known as the **lorentz boost** to boost from the S frame to the S' frame.

6 Math

6.1 Single variable calculus

6.2 Multi-variable calculus

6.2.1 Why does gradient points in direction of greatest ascent

Intuition - Explanation 1

Easy explanation (imo): "Gradient" is defined as the vector summation of

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + ... \tag{6.1}$$

the maximum rate of change along the individual axes. So naturally, the summation of these rate of changes would give the maximum rate of change as well.

Intuition/Proof - Explanation 2 (by **Yi Fan**)

Single variable tangent line approximation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{6.2}$$

Multi-variable (2 in this case) tangent **plane** approximation:

$$f(x, y) - f(x_0, y_0) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{6.3}$$

where f_x and f_y denotes partial derivatives with respect to x and y respectively.

Locally if we zoom in near (x_0, y_0) this shows that any nice function is just a plane.

Then now suppose our plane is given by $z = ax + by$, where we take reference point to be the origin (otherwise we can just translate the function).

Now you can think of it as $z = (a, b) \cdot (x, y)$ where we have a dot product, and we want to find the direction (x, y) that maximises this. ($a = f_x(x_0, y_0)$ and $b = f_y(x_0, y_0)$)

But dot product is maximised when the vector points in parallel direction. So we must have $(x, y) = k(a, b)$ for some constant k.

So essentially, we are trying to move (x, y) around such that we can get the maximum z value. And we realised that when (x, y) is aligned with the gradient, the maximum z value is achieved. Hence, **gradient always points in direction of greatest ascent.**

6.2.2 Divergence

6.2.3 Curl

6.3 Differential equations

6.3.1 Linear differential equations - from Hexiang's slides

If you want to solve something simple like

$$\frac{dy}{dx} = ay,$$

one could use separation of variables

$$\int dx = \frac{1}{a} \int \frac{1}{y} dy$$

$$x + d = \frac{\ln(y)}{a} + c$$

$$ax + c = \ln(y)$$

$$y = e^{ax+c} = e^{ax} e^{c}$$

which can be generalised to

$$y = Ae^{ax}.$$

Alternatively, we can also guess some solution, maybe $y = e^{cx}$ will do. $\frac{dy}{dx} = ce^{c}x$. Hence, $c = a$. This solution wil still hold true when we multiply it by an arbitrary constant ($y = Ae^{ax}$).

6.3.2 Solution to $\ddot{x} = -\omega^2 x$

Let's try subbing in $x = Ae^{ct}$ again and see what we get,

$$\ddot{x} = c^2 Ae^{ct}$$

$$c^2 Ae^{ct} = -\omega^2 Ae^{ct}$$

$$c = \pm i\omega$$

So there are 2 possible values for c, hence 2 solutions for x ($Ae^{i\omega t}$ or $Ae^{-i\omega t}$) For a linear differential equation (in this case it is), the most general solution would be the **linear combination** of the individual solutions

$$\boxed{x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}}$$

Using **Euler's identity**

$$x = A_1(\cos(\omega t) + i\sin(\omega t)) + A_2(\cos(-\omega t) + i\sin(-\omega t))$$
$$x = (A_1 + A_2)\cos(\omega t) + i(A_1 - A_2)\sin(\omega t)$$
$$x = B\cos(\omega t) + C\sin(\omega t)$$

which can then be converted to the form which we are more familiar with (r-formula):

$$\boxed{x = A\sin(\omega t + \phi) \quad \text{or} \quad x = A\cos(\omega t + \phi).}$$

Since we absorbed i into the constants, A and ϕ are both complex right now. However, we can just impose the conditions that they must always be real for physical purposes.

6.3.3 Integrating fractor

If you have an equation in the form of

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{6.4}$$
$$M(x)\frac{dy}{dx} + M(x)P(x)y = M(x)Q(x) \tag{6.5}$$
$$\tag{6.6}$$

You can multiply both side by an integrating factor $M(x)$, such that the LHS becomes the result of a product rule. This is useful because once we can get it to be like product rule, we can integrate both side and solve for y.

$$M(x)y = \int Q(x)M(x)dx \tag{6.7}$$
$$y = \frac{\int Q(x)M(x)dx}{y} \tag{6.8}$$

How do we find this integrating factor?

$$M'(x) = M(x)P(x) \tag{6.9}$$

$$\boxed{M(x) = \exp\left(\int P(x)dx\right)} \tag{6.10}$$

6.4 Cool integrals

6.4.1 Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \tag{6.11}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}} \tag{6.12}$$

Very useful in thermodynamics.

7

PO training

7.1 18/08/22

7.1.1 Gallilean Invariance

All things in newtonian physics has to obey **gallilean invariance**, meaning that the laws have to hold true in every inertial reference frame (i.e. stationary frame/frame moving at constant v)

In school, we are taught that since $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ where $\mathbf{p} = m\mathbf{v}$, by chain rule, we will get

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} \tag{7.1}$$

Now, we shall verify the galliean invariance of the equation above. Let's have a frame S' moving at constant velocity $\mathbf{u}$. The force, mass,velocity and time in S' are denoted with a prime ($'$). Note that this frame is **inertial** and **all newtonian laws should hold in inertial frames**.

$$\mathbf{F}' = m'\frac{d\mathbf{v}'}{dt'} + \mathbf{v}'\frac{dm'}{dt'} \tag{7.2}$$

Since mass time and force are invariant, and $\mathbf{v}' = \mathbf{v} - \mathbf{u}$, we can rewrite it as

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\frac{dm}{dt} \tag{7.3}$$

which is definitely not the same as the equation above. Hence, it is proven that the equation 7.1 is **not** gallilean invariant.

7.1.2 rocket momentum

However, in some special cases, the "wrong" equation can still be applied. Let the mass and velocity of a rocket be $m(t)$ and $v(t)$ respectively. At time $t' = t + \Delta t$

$$\begin{align} m(t)v(t) &= m(t')v(t') + (m(t') - m(t))(v(t') - u) \tag{7.4} \\ &= (m(t) + \Delta m)(v(t) + \Delta v) + (-\Delta m)(v(t) + \Delta v - u) \tag{7.5} \end{align}$$

where u is the speed of the fuel (particle) emitted.

Expanding the equation, we get

$$m(t)v(t) = m(t)v(t) + m(t)\Delta v + \Delta m v(t) + \Delta m \Delta v - \Delta m v(t) - \Delta m \Delta v - \Delta m u \tag{7.6}$$

Simplifying everything, we get

$$mdv + udm = 0 \tag{7.7}$$

7.2 25/08/22

7.2.1 linear independence of solution

From wikipedia: A sequence of vectors $\mathbf{v_1}, \mathbf{v_2}, ...\mathbf{v_k}$ from a vector space V is said to be **linearly dependent** if there exist scalars $a_1, a_2, ..., a_k$, not all zero, such that

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_k\mathbf{v_k} = \mathbf{0} \tag{7.8}$$

where $\mathbf{0}$ denotes the zero vector. basically, as long as the vectors are not parallel, they are **linearly independent**

7.3 27/08/22

7.3.1 Newton-Raphson method

for example if you wanna solve

$$y = f(x) = 0 \tag{7.9}$$

you can kinda use

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{7.10}$$

but its kinda useless

7.3.2 PS 2 Q 7

how to decide the direction of static friction? do torque about the contact point (where static friction acts), and realise that torque by F has to result in a clockwise rotation (into the page). This means that torque about the COM also has to be clockwise, meaning that static frction points in the opposite direction of F.

Kenneth Hong's solution Translational motion of the center of mass

$$F\cos\theta - f_s = ma_{CM,x} \tag{7.11}$$

Rotational mostion about the centre of mass $I = M(a^2 + b^2)/2$

$$f_s a - Fb = I\alpha \tag{7.12}$$

with the additional no slip condition ($a_{CM,x} = a\alpha$) and solving for pure rolling, you get

$$\begin{cases} a_{CM,x} = \frac{F/M}{1+I/Ma^2}(\cos\theta - \frac{b}{a}) \\ f_s = \frac{F}{1+I/Ma^2}(\frac{I}{Ma^2}\cos\theta + \frac{b}{a}) \end{cases} \tag{7.13}$$

Maximum tension:

$$f_s \leq \mu_s(Mg - F\sin\theta) \tag{7.14}$$

$$T_{max} = \frac{\mu_s(1 + I/Ma^2)/Mg}{(I/Ma^2)\cos\theta + \mu_s(1 + I/Ma^2)\sin\theta + b/a} \tag{7.15}$$

7.3.3 PS 2 Q 8

Workdone by friction on a rolling and **slipping** object is not just $W = f_k \cdot d$, where d is the displacement of the centre of mass.

It is actually

$$W_{f_k} = W_{\text{trans}} + W_{\text{rot}} \tag{7.16}$$

$$= \int_0^{\Delta x} f_k dx - \int_0^{\Delta\theta} \tau d\theta \tag{7.17}$$

Note that it is a minus sign because in the question, friction is adding energy in the translational motion, but removing energy in the rotational motion.