pr 5. A white piece of chalk is thrown onto a black horizontal board moving at constant velocity. Initially, th

velocity was perpendicular to the board's direction of motion.
What is the shape of the chalk's trace on the board?

To solve the next problem, in addition to the previous idea we also need to use 2, which can be rephrased in a slightly more general (but less specific) way: some minima and maxima can be found without taking any derivatives, in fact the solution without a derivative can turn out to be much simpler. For this problem, an even more narrowed down formulation would be the following.

idea 8: If one of two vectors is constant and the direction of the other is fixed then the modulus of their sum is minimal if they form a right-angled triangle.

pr 6. A block is pushed onto a conveyor belt. The belt is moving at velocity $v0 = 1$ m/s, the block's initial v

with respect to the ground? The coefficient of friction is large
enough to prevent the block from falling off the belt.

The next problem is slightly unusual, specific comments will be given after the problem. To tackle such situation one can give seemingly trivial but very often an overlooked advice.

idea 9: Read carefully the problem text, try to understand the meaning of every statement, don't make hasty assumptions by yourself.

For a well-written problem, there are no redundant sentences. Things become more troublesome if that is not the case. Sometimes the problem author wants to educate you more than just by giving you the very problem, and tells you many things (such as historical background) which are definitely interesting but unrelated to the solution of the problem. It is OK if you are solving the problem as an exercise at home and you have plenty of time. However, you need to develop skills of parsing fast through such paragraphs at competitions under time pressure: you need to make sure that there are really no important hints hidden inside.

pr 7. After being kicked by a footballer, a ball started to fly straight towards the goal at velocity $v = 25$ m/

an angle $= \arccos 0.8$ with the horizontal. Due to side wind blowing at $u = 10$ m/s perpendicular the initial velocity of the ball, the ball had deviated from its initial course by $s = 2$ m by the time it reached the plane of the goal. Find the time that it took the ball to reach the plane of the goal, if the goal was situated at distance $L = 32$ m from the footballer.
A typical problem gives all the parameter values describing a system and then asks about its behaviour. Here, the system might seem to be over-described: why do we need the value of s, couldn't we just use the initial velocity to determine the flight time to deduce $t = v \cos$? Such a question might arise, first of all, because you are used to ignoring air fricti
no-one mentioned that you can neglect it here! Furthermore, it is even evident that the air drag cannot be neglected, because otherwise the ball would not depart from its free-fall trajectory!

2. VELOCITIES

It would be a very difficult task (requiring a numerical integration of a differential equation) to estimate the trajectory of the ball subject to a turbulent air drag. However, this is not what you need to do, because the air drag is not described by a formula for the drag force, but instead, by the final departure from the corresponding free-fall-trajectory.

So, with the help of idea 9 we conclude that the air drag cannot be neglected here. Once we have understood that, it becomes evident that we need to apply the idea 7. However, even when equipped with this knowledge, you might run into mathematical difficulties as there is no direct way of expressing the flight time t in terms of the given quantities. Instead, you are advised to write down an equation containing t as an unknown, and then to solve it.

idea 10: It is often useful first to write down an equation (or a system of equations) containing the required quantity as an unknown, instead of trying to express it directly (sometimes it is necessary to include additional unknowns that later get eliminated).

Furthermore, unlike the problems we had thus far, this problem deals with a 3-dimensional geometry, which makes it difficult to draw sketches on a 2-dimensional sheet of paper. Thus we need one more simple idea.

idea 11: It is difficult to analyse three-dimensional motion as a whole, so whenever possible, it should be reduced to two dimensions (projecting on a plane, looking at planes of intersection).

The next problem illustrates

idea 12: An elastic collision is analysed most conveniently in the centre of mass frame of the process.

Let us derive from this idea a ready-to-use recipe when a ball collides with a moving wall. First, since the wall is heavy, the system's centre of mass coincides with that of the wall, hence we'll use the wall's frame. In the frame of the centre of mass, if the collision is elastic and there is no friction then due to the energy and momentum conservation, the bodies will depart with the same speed as they approached, i.e. the normal component of the ball's velocity is reversed. If we apply the addition of velocities twice (when we move to the wall's frame, and when we switch back to the lab frame), we arrive at the following conclusion.

idea 13: For an elastic bouncing of a ball from a wall which moves with a velocity u in the direction of the surface normal, the normal component vn of the ball's velocity v is increased by $2u$, i.e. $vn= vn+ 2u$.

For this problem we must also remember

fact 1: Angle between velocity vectors depends on the frame of reference!

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pr 8. A tennis ball falls at velocity v onto a heavy racket and bounces back elastically. What does the racket's velocity

u have to be to make the ball bounce back at a right angle to its initial trajectory and not start spinning if it did not spin before the bounce? What is the angle between u and the normal of the racket's plane, if the corresponding angle for v is

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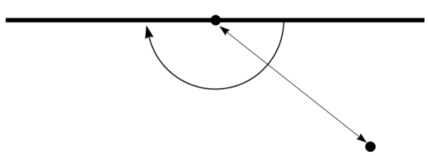 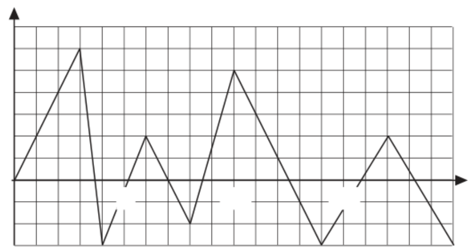

3. ACCELERATIONS, DISPLACEMENTS

?	axis, the direction being given by the screw rule (if the screw is

3. ACCELERATIONS, DISPLACEMENTS	
If we keep in mind the idea 9 and read the text carefully, we notice that the racket is *heavy* so that we can use the idea 13.	rotated in the same way as the body, the vector points in the direction of the screw movement).

Also, pay attention that the ball will not rotate after the collision — this is important for finding the parallel (to the racket's plane) component of the velocity.

idea 17: When switching between rotating frames of reference, angular velocities are to be added in the same as transla

Earlier we mentioned that vectors can be dealt with either geometrically (e.g. by applying the triangle rule for a sum tional velocities in the case of translationally moving frames of reference. NB! This remains valid even if the angular velocit ies are not parallel (although non-small rotation angles can be added only as long as the rotation axis remains unchanged). This idea is illustrated by a relatively simple problem below.

the following recommendation.

idea 14: For vectorial calculations, prefer geometrical approach, but if it seems unreasonable (e.g. some of the conditions

pr 9. Vertical mirror with two reflecting surfaces (front and back) rotates around a vertical axis as shown in figure, with

3. ACCELERATIONS, DISPLACEMENTS		
	are formulated through the projections of the vectors) switch to the algebraic approach and write expressions down in terms	angular speed . There is an unmoving point source of light S at a distance a from the rotation axis. Find the speed of the image of the point source as a function of time.

of components. a For the algebraic approach, **optimal choice of axes** is very

S important. "Optimal" means that the conditions are written in the simplest possible way. Sometimes it may happen that the most useful coordinate axes are not even at right angles.

3 ACCELERATIONS, DISPLACEMENTS

For the problem 8, geometrical solution turns out to be simpler, but more difficult to come up with. This is quite typical: algebraic approach leads to a brute-force-solution when it is

3. ACCELERATIONS, DISPLACEMENTS

Thus far we dealt with instantaneous or constant velocities, and in few cases we applied a simple formula $s = vt$ for displacements. In general, when the velocity v is not constant, the dis placement is found as the curve under the graph of the velocity

difficulties and you just need to execute it.

As long as the

mathematical part will not be *unreasonably long* or leading to fundamental difficulties (such as unsolvable equations), brute force approach is still OK: figuring out an elegant solution can

as a function of time.

For instance, the displacement along

x-coordinate x is surface area under the graph $vx = vx(t)$; mathematically we can write it via integral $x = vx(t)\,dt$. You don't need to know more about integrals right now, just that it represents surface areas under graphs.

resent very simple geometrical tasks and hence, finding these

idea 18: Calculation of many physical quantities can be reduced (sometimes not in an obvious way) to the calculation of

shorter-than-algebraic solutions is also quite easy. In this case, however, the geometrical task turns out to be quite a tricky

surface areas under a graph (i.e. to an integral). In particular:

problem.

While the idea 14 suggests that the algebraic ap

3. ACCELERATIONS, DISPLACEMENTS

distance is the area under a vt curve (velocity-time), velocity the area below an a t curve etc.

proach is good for problem 8 (the no-rotation-requirement gives us a condition for the parallel component of the velocity), it is recommended that you try both methods here. In both cases

Note that drawing a graph is not always absolutely necessary (if

3. ACCELERATIONS, DISPLACEMENTS	
you need one more mathematical idea.	you are skilled with integrals, formulae can be derived analytic

idea 15: Two vectors $a = (ax, ay, az)$ and $b = (bx, by, bz)$ are perpendicular if their scalar product is zero, $axbx + ayby + azbz = 0$. (This assumes that the axes x, y and z are perpen-

ally, without drawing graphs), but doing it helps to imagine the process. Visualisation of this kind is always beneficial, it simplifies finding the solution and reduces the chances of making mistakes.

pr 10. A particle starts from the origin of coordinates; the figure shows its velocity as a function of time. What is its

idea 16: For trigonometric problems involving right triangles keep in mind that the circumcentre of a right triangle is at the

maximum shift from the origin?

v(m/s)

centre of the hypotenuse, hence the median drawn from the

6

right angle divides the triangle into two isosceles triangles, and

4 5 10 15

t(s)

the right angle into the angles equal to the acute angles of the triangle.

2

0

Idea 1 told us to make use of switching between different

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-2

frames of reference. This idea can be also used when dealing with rotational motion.

3. ACCELERATIONS, DISPLACEMENTS	
def. 3: Angular velocity equals by modulus to the rotation angle (in radians) per unit time, and is parallel to the rotation	The next problem is much more difficult, although it is also reduced to finding a surface area; due to difficulty, the full

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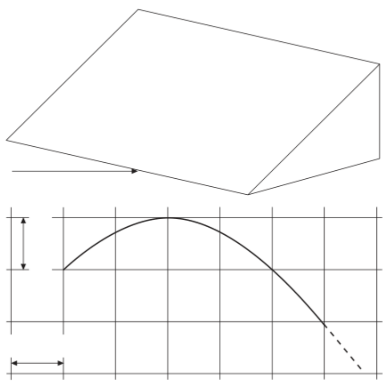


3. ACCELERATIONS, DISPLACEMENTS

spective perpendicular components, $g = gx{+}gy$ with $gx =$ sin and gy = cos , being the angle between the surface and the horizon.

of the two surfaces and y-axis lays on the inclined surface, x, y, and z-motions are all separated; at the impact, vz goes to zero, and due to the absence of friction, vx and vy are preserved. In this problem, however, the transition from one surface

During a collision, as there is no friction force, vx (parallel to the surface velocity component) does not change, i.e. it does

to the other is smooth: around the line separating the two flat surfaces, there is a narrow region where the surface has a not “notice” that there was a collision: in order to analyse the evolution of the x-coordinate, we can completely forget about the changes of the y-coordinate (and vice versa).

If the surface is curved, generally such a separation is no

curvature. Within this narrow region, the motion in y- and z-directions cannot be separated from each other, and we need one more idea.

idea 21: If a force is perpendicular to the direction of motion (normal force when sliding along a curved surface, tension in a longer possible. Indeed, previously x was independent of y be cause the dependence of the acceleration on the coordinates is introduced by the normal force, which is a function of y only, and has no x-component. If the surface is curved, it is im possible to have the x-axis to be everywhere parallel to the surface: the acceleration due to the normal force has both non vanishing x and y components, and depends both on x and y coordinates. However, in the case of side surfaces of cylinders,

rope when a moving body is attached to an un-stretchable rope fixed at the other end, force on a charge in magnetic field) then the velocity vector can only turn, its modulus will not change.4

pr 15. Three turtles are initially situated in the corners of an equilateral triangle at distances 1 m from one another. They prisms and other generalized cylinders3, it is still possible to find one axis x which is everywhere parallel to the surface and hence, motion along x can be separated from the motion in y z-plane.

move at constant velocity 10 cm/s in such a way that the first always heading towards the second, the second towards the third and the third towards the first. After what time will they meet?

pr 13. An elastic ball is released above an inclined plane (inclination angle) at distance d from the plane. What is the distance between the first bouncing point and the second?

Two approaches are possible here: first, we may go into the frame of reference rotating with the turtles, in which case we apply the following idea.

idea 22: Sometimes even a reference frame undergoing very complex motion can be useful.

Collisions occur without friction.

The next problem makes also use of the idea 20; however, one more idea is needed, see below.

pr 14. A puck slides onto an icy inclined plane with inclination angle . The angle between the plane's edge and the puck's initial velocity $v0 = 10$ m/s is $= 60\circ$. The trace left by the puck on the plane is given in the figure (this is only a part of the trajectory). Find under the assumption that

Alternatively, we can use

idea 23: Instead of calculating physical velocities, it is sometimes wise to look at the rate of change of some distance, the ratio of two lengths, etc.

The following problem requires integration5, so it can be skipped by those who are not familiar with it.

friction can be neglected and that transition onto the slope was smooth.

pr 16. An ant is moving along a rubber band at velocity $v = 1$ cm/s. One end of the rubber band (the one from which the ant started) is fixed to a wall, the other (initially at distance $L = 1$ m from the wall) is pulled at $u = 1$ m/s. Will the ant reach the other end of the band? If yes then when will it happen?

Here we need to apply the

idea 24: For some problems, optimal choice of parametrization can simplify mathematical calculations significantly. An

2.5m	incomplete list of options: Cartesian, polar, cylindrical, and spherical coordinates; travel distance; *Lagrangian coordinates*
(i.e. for fluids flow using the initial coordinate of a fluid particle	
The last sentence here is very important: if the transition is *sharp*, the puck approaches the inclined plane by sliding along	instead of its current coordinate); relative position of a particle according to a certain ranking scheme, etc.
the horizontal and collides with it — either elastically in which	Here, the problem itself contains a hint about which type of
case it jumps up, or plastically. In particular, if the collision	parametrization is to be used. It is clear that the Cartesian
is perfectly plastic then that part of the kinetic energy which	coordinate of the ant is not good: it does not reflect the pro

is associated with the motion along the surface normal of the inclined plane is lost. More specifically, if we introduce perpen dicular coordinates so that the x-axis is along the contact line

gress of the ant in advancing along the rubber band. In order to describe such a progress, we can use the relative position on the band: which fraction k of the rubber is left behind; the ant

3Surfaces with constant cross-sections.

4This is the energy conservation law using the fact that forces perpendicular to the velocity will not perform work.

5You may find helpful to know that $ax+b= a1 \ln(ax + b) + C$—

time will the string be fully wound around the cylinder again,

this time the other way round? [This problem leads to a very simple differential equation; if you don't know how to solve it, the following equality can be helpful: $ldldt$= 1

v

r

pr 47. A heavy box is being pulled using two tractors. One of these has velocity $v1$, the other $v2$, the angle l

the ropes are parallel to velocity vectors?

v2

v1

pr 48. A boy is running on a large field of ice with velocity $v = 5$ m/s toward the north. The coefficient of friction between

his feet and the ice is $\mu = 0.1$. Assume as a simplification that the reaction force between the boy and the ice stays constant (in reality it varies with every push, but the assumption is jus tified by the fact that the value averaged over one step stays constant).

i) What is the minimum time necessary for him to change his moving direction to point towards the east so that the final speed is also $v = 5$ m/s?

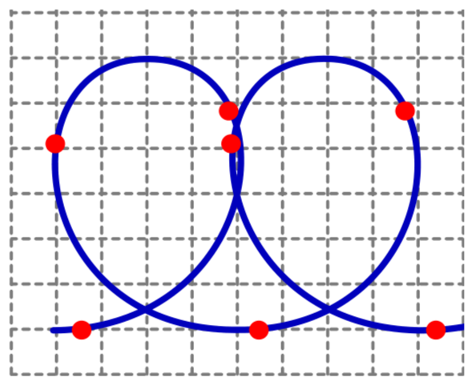
i) What is the shape of the optimal trajectory called?

pr 49. A ball thrown with an initial speed $v0$ moves in a homogeneous gravitational field of strength g; neglect the air drag. The throwing point can be freely selected on the ground level $z = 0$, and the launching angle can be adjusted as needed; the aim is to hit the topmost point of a spherical building of radius R (see fig.) with as small as possible initial speed $v0$ (prior hitting the target, bouncing off the roof is not allowed).

Sketch qualitatively the shape of the optimal trajectory of the ball. What is the minimal launching speed $v$min needed to hit the topmost point of a spherical building of radius R?

z

axis was vertical, and the disk slid and rotated freely on a ho rizontal smooth ice surface. What was the speed of the centre of the disk? You can take measurements from the figure using a ruler.



pr 51. There is a capital O and three cities A, B and C, connected with the capital via roads 1, 2, and 3 as shown in the left figure. Each road has length $2a$. Two cars travel from one city to another: they depart from their respective starting points simultaneously, and travel with a constant speed v. The figure on the right depicts the increasing rate of the distance between the cars (negative values means that the distance decreases) as measured by the GPS devices of the cars. The turns are taken by the cars so fast that the GPS devices will not record the

behaviour during these periods.

i) Which cities were the starting and destination points of the cars? Motivate your answer.

ii) What is the surface area between the $v$dist -graph and the t-axis for the interval from $t = 0$ to $t = a/v$?

iii) Now let us consider a case when three cars (denoted by A, B, and C) depart simultaneously from their cities (A, B, and C, respectively) towards the capital; all the cars travel with a constant speed v. Sketch the graphs for the distance changing rate for the following car pairs: A B and B C. **iv)** Suppose that now the GPS-devices are good enough to re cord the periods of taking the turns. Sketch a new appropriate graph for the pair of cars B C. The curvature of the turns is small enough so that the cars can still keep theirs speed v.A B vdist

road 1

a a

road 2

90o a O a 90o 0 av 2av t

90o

C road 3 a - v0

a 90o - 0v

2

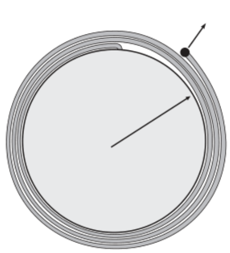
x

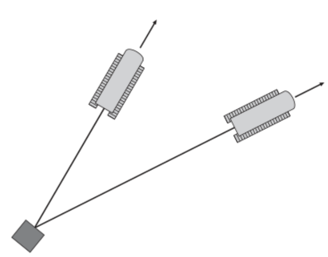
pr 50. The figure represents a photo which was taken using a very long exposure time (camera was pointing directly down).

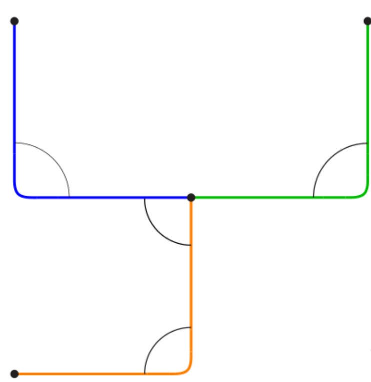
What you can see is a trace of a blue lamp which burned con tinuously, but also flashed periodically with a red light (after each $t = 0.1$ s. The lamp was fixed to the surface of a solid disk, at the distance $a = 4.5$ cm from its symmetry axis. The

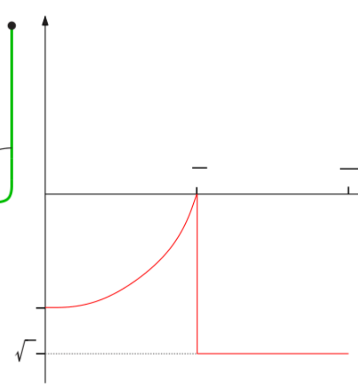
pr 52. Consider two rings with radius r as depicted in the figure: the blue ring is at rest, and the yellow ring rotates

around the point O (which is one of the intersection points of the two rings) with a constant angular speed . Find the minimal and maximal speeds $v$min and $v$max of the other intersection point of the two rings.









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