

Ring on a rod

Bob the legend

January 3, 2022

1 Steady state motion (threadless rod) with no slip

I think this whole theory is **extremely questionable** because

- I am ignoring friction, which is a key reason why the ring will reach a terminal velocity. (energy gained from GPE dissipated through friction, so KE and RKE remains the same)
- The ring's motion has been decomposed into its rotation about COM and the rotation of COM about the rod is slightly questionable.
- The way I calculated torque from angular momentum is very questionable. Normally, when we visualise the angular momentum vector, it should have an origin (e.g. angular momentum of pendulum points out from the pivot). But in this case, I don't think it is possible to find such an origin. So I don't know whether I should calculate torque about contact point? COM? Or some other points. I asked around online and the people said that angular momentum is not in fact an vector; but it can be put **in bijection with a vector**.

Each axis has its own angular momentum. The angular momentum has magnitude, and the axis has direction, so this combination of magnitude and direction looks like a vector, but it isn't really, and it doesn't have an origin in physical space. It makes sense to depict it as having its origin on the corresponding axis, and often the axis being considered goes through the center of mass, making the center of mass a natural choice for the origin, but the choice to depict its origin there doesn't really correspond to anything physical.

Even angular momentum "vectors" with the same "origin" can't be added like normal vectors if they have different axes (you get things like precession that wouldn't appear if they were truly vectors), and it's even more complicated to combine angular momenta from different axes.

so uhm this thing is kinda confusing.

2 Overview of theory

Just referred to a part in Morin. Basically the goal of this theory is to be able to find the terminal frequency of motion of the ring about the rod.

1. Calculate the principal moments of the ring
2. Calculate angular momentum
3. Calculate $\frac{d\mathbf{L}}{dt}$
4. Calculate torque
5. Equate torque to $\frac{d\mathbf{L}}{dt}$

Let's visualise the motion as a rotation of the ring about its centre of mass (Ω), and the rotation of the centre of mass about the centre of the rod (ω), as illustrated in the figure below.

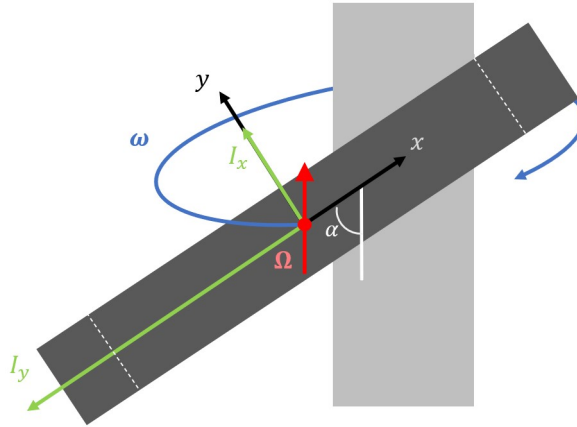


Figure 1: Coordinate system (in rotating frame), x_3 and I_3 are pointing out of the page

2.1 Moment of inertia

The principal axes are any 2 orthogonal axes (y and z) parallel to the ring, along with 1 axis perpendicular to the ring (x). The positive z axis points out of the page. The principle moments (relative to the object's centre of mass) are

$$\mathbf{I}_1 = \left(\frac{1}{12}m(3(R_1^2 + R_2^2) + d^2), \frac{1}{2}m(R_1^2 + R_2^2), \frac{1}{12}m(3(R_1^2 + R_2^2) + d^2) \right) \quad (1)$$

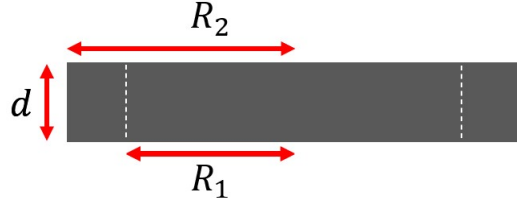


Figure 2: Geometry of the ring

The moments (relative to the centre of the rod using parallel axis theorem) are

$$\mathbf{I}_2 = \begin{pmatrix} \frac{1}{12}m(3(R_1^2 + R_2^2) + d^2) + m(R_1 \sin \alpha - r)^2 \cos^2 \alpha, \\ \frac{1}{2}m(R_1^2 + R_2^2) + m(R_1 \sin \alpha - r)^2 \sin^2 \alpha, \\ \frac{1}{12}m(3(R_1^2 + R_2^2) + d^2) + m(R_1 \sin \alpha - r)^2 \cos^2 \alpha \end{pmatrix} \quad (2)$$

In the frame that rotates together with the ring's COM, the Ω vector points vertically upwards. In the basis of the principal axes of the rings (x, y, z) :

$$\Omega = (\Omega \cos \alpha, \Omega \sin \alpha, 0) \quad (3)$$

The same applies for ω .

2.2 Angular momentum

The angular momentum in the lab frame can be calculated by adding the "spin angular momentum" (angular momentum ring about its COM) to the orbital angular momentum (angular momentum due to rotation of COM about centre of rod). The total angular momentum in the lab frame is thus:

$$\sum \mathbf{L} = \mathbf{I}_1 \Omega + \mathbf{I}_2 \omega \quad (4)$$

The spin angular momentum is:

$$L_s = (I_{1,x} \Omega \cos \alpha, I_{1,y} \Omega \sin \alpha, 0) \quad (5)$$

The orbital angular momentum is:

$$L_o = (I_{2,x} \omega \cos \alpha, I_{2,y} \omega \sin \alpha, 0) \quad (6)$$

Adding up the 2 components gives us ¹:

$$L = ((I_{1,x} \Omega + I_{2,x} \omega) \cos \alpha, (I_{1,y} \Omega + I_{2,y} \omega) \sin \alpha, 0) \quad (7)$$

¹This part is very questionable.

2.3 Torque

The final angular momentum in the lab frame (labelled as L) will precess about the vertical axis (dotted line) due to torque by gravity (about contact point??? yea this is really questionable, since the "origin" of the angular momentum vector is unknown, I have no idea about which point I should calculate torque)²

I will explicitly calculate the horizontal component of the angular momentum.

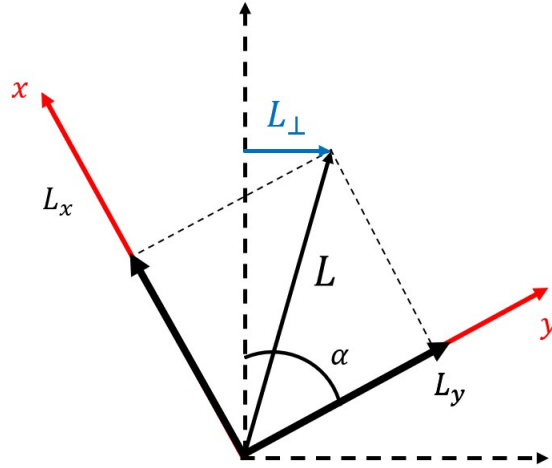


Figure 3: How I calculated torque

$$\begin{aligned} L_{\perp} &= L_y \sin \alpha - L_x \cos \alpha \\ &= (I_{1,y}\Omega + I_{2,y}\omega) \sin \alpha \cos \alpha - (I_{1,x}\Omega + I_{2,x}\omega) \sin \alpha \cos \alpha \end{aligned} \quad (8)$$

Magnitude for rate of change of L is

$$\begin{aligned} \left| \frac{d\mathbf{L}}{dt} \right| &= \omega L_{\perp} \\ &= \omega \sin \alpha \cos \alpha ((I_{1,y}\Omega + I_{2,y}\omega) - (I_{1,x}\Omega + I_{2,x}\omega)) \end{aligned} \quad (9)$$

Expression for torque by gravity

$$\tau_g = mg\ell \sin(\alpha - \beta) \quad (10)$$

where $\ell = \sqrt{R_1^2 + (d/2)^2}$ and $\beta = \arctan(d/2R_1)$

Equating torque to this expression gives us:

$$mg\ell \sin(\alpha - \beta) = \omega \sin \alpha \cos \alpha ((I_{1,y}\Omega + I_{2,y}\omega) - (I_{1,x}\Omega + I_{2,x}\omega)) \quad (11)$$

²In most diagrams, if an object is rotating about the contact point (maybe a falling stick with one end pivoted to the floor), τ_g and L vectors will both be drawn as arrows pointing away from the pivot.

2.4 no slip constraint to relate Ω with ω

We still need to introduce a no-slip constraint to relate Ω with ω . I am not so sure about this but maybe it is

$$\begin{aligned}\omega r &= \Omega_y R_1 \\ &= \Omega R_1 \sin \alpha\end{aligned}\tag{12}$$

2.5 Terminal velocity (more questionable stuff)

If we observe the motion of the ring in its COM frame, we will realise that the contact point will move downwards by $R_1 \cos \alpha$ every half a rotation (assuming no slip). The terminal velocity of the ring can then be calculated as

$$v = \frac{d}{t} = \frac{\Omega R_1 \sin \alpha}{\pi}\tag{13}$$

3 References

<https://physics.stackexchange.com/questions/686439/physics-of-mad-tea-party>
<https://amesweb.info/inertia/hollow-cylinder-moment-of-inertia.aspx>