For Whom the Bell Tunnels

Just Us

November 4, 2013

In [1, p.448] we find the discussion of a spin-$\frac{1}{2}$ system. In particular we look at the action of the operator

$$\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$$

on a spin–$\frac{1}{2}$ vector ψ. Here $\mathbb{1}$ is the 2–dimensional unit matrix, and $\boldsymbol{\sigma}$ is the vector whose components comprise the individual Pauli spin matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\boldsymbol{\beta}$ is some 3–dimensional vector of coefficients.

Proposition 1 (Eigenvalues)**.** *The operator* $\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ *operating on a spin–*$\frac{1}{2}$ *vector* ψ *has eigenvalues*

$$\alpha \pm |\boldsymbol{\beta}|.$$

Calculation of eigenvalues. We seek to solve the following equation:

$$(\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma})\psi = \lambda\psi$$

for some complex number λ. That is, we look for

$$(\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} - \lambda\mathbb{1})\psi = 0.$$

Thus we are looking for the nullity of the operator in parentheses. The corresponding eigenvalues are given by the zeroes of the determinant. Thus we look for λ satisfying

$$\det[(\alpha - \lambda)\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}] \overset{\heartsuit}{=} 0.$$

In components, the given operator is

$$\begin{pmatrix} \alpha + \beta_3 - \lambda & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & \alpha - \beta_3 - \lambda \end{pmatrix}.$$

The characteristic equation therefore yields

$$\begin{aligned} 0 &\overset{\heartsuit}{=} (\alpha + \beta_3 - \lambda)(\alpha - \beta_3 - \lambda) - (\beta_1 + i\beta_2)(\beta_1 - i\beta_2) \\ &= (\alpha + \beta_3)(\alpha - \beta_3) - (\alpha + \beta_3)\lambda - (\alpha - \beta_3)\lambda + \lambda^2 - \left[\beta_1^2 - (i\beta_2)^2\right] \\ &= \lambda^2 - 2\alpha\lambda + \left(\alpha^2 - |\boldsymbol{\beta}|^2\right). \end{aligned}$$

Applying the quadratic formula, this leaves us with

$$\lambda = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4 \cdot 1 \cdot (\alpha^2 - |\boldsymbol{\beta}|^2)}}{2 \cdot 1} = \alpha \pm |\boldsymbol{\beta}|,$$

as desired. □

References

[1] John S Bell. On the Problem of Hidden Variables in Quantum Mechanics. *Reviews of Modern Physics*, 38(3):447–452, 1966.