For Whom the Bell Tunnels

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In [1, p.448] we find the discussion of a spin-$\frac{1}{2}$ system. In particular we look at the action of the operator

$$\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$$

on a spin–$\frac{1}{2}$ vector ψ. Here $\mathbb{1}$ is the 2–dimensional unit matrix, and $\boldsymbol{\sigma}$ is the vector whose components comprise the individual Pauli spin matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\boldsymbol{\beta}$ is some 3–dimensional vector of coefficients.

Proposition 1 (Eigenvalues)**.** *The operator $\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ operating on a spin–$\frac{1}{2}$ vector ψ has eigenvalues*

$$\alpha \pm |\boldsymbol{\beta}|.$$

Calculation of eigenvalues. We seek to solve the following equation:

$$(\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma})\psi = \lambda\psi$$

for some complex number λ. That is, we look for

$$(\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} - \lambda\mathbb{1})\psi = 0.$$

Thus we are looking for the nullity of the operator in parentheses. The corresponding eigenvalues are given by the zeroes of the determinant. Thus we look for λ satisfying

$$\det[(\alpha-\lambda)\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}] \overset{\heartsuit}{=} 0.$$

In components, the given operator is

$$\begin{pmatrix} \alpha+\beta_3-\lambda & \beta_1-i\beta_2 \\ \beta_1+i\beta_2 & \alpha-\beta_3-\lambda \end{pmatrix}.$$

The characteristic equation therefore yields

$$\begin{aligned} 0 &\overset{\heartsuit}{=} (\alpha+\beta_3-\lambda)(\alpha-\beta_3-\lambda)-(\beta_1+i\beta_2)(\beta_1-i\beta_2) \\ &= (\alpha+\beta_3)(\alpha-\beta_3)-(\alpha+\beta_3)\lambda-(\alpha-\beta_3)\lambda+\lambda^2-\left[\beta_1^2-(i\beta_2)^2\right] \\ &= \lambda^2-2\alpha\lambda+\left(\alpha^2-|\boldsymbol{\beta}|^2\right). \end{aligned}$$

Applying the quadratic formula, this leaves us with

$$\lambda=\frac{2\alpha\pm\sqrt{4\alpha^2-4\cdot 1\cdot(\alpha^2-|\boldsymbol{\beta}|^2)}}{2\cdot 1}=\alpha\pm|\boldsymbol{\beta}|,$$

as desired. □

The section following the above calculation in [1] then goes on to introduce a "hidden variable" λ (unrelated to the λ used above). By design, this variable removes any indeterminacy as to which state the spinor ψ is actually in. That is, without the use of the ancillary variable λ, all we can say is that when we measure the quantity corresponding to the operator $\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}$, the measurement must return an eigenvalue of the operator: either $\alpha+|\boldsymbol{\beta}|$ or $\alpha-|\boldsymbol{\beta}|$, but we don't know which of these will result before making the measurement. However, by incorporating the information of the hidden variable λ, by virtue of what we mean by the phrase "hidden variable", this information should be sufficient to tell us beforehand which of the two possible eigenvalues will result from the measurement. Equation (3) of [1] is the *specification* of just such a rule: the eigenvalue that results from any measurement will be determined by the value of λ according to the rule

$$\alpha+|\boldsymbol{\beta}|\operatorname{sgn}\left(\lambda|\boldsymbol{\beta}|+\frac{1}{2}|\beta_z|\right)\operatorname{sgn}X, \tag{1}$$

where

$$X = \begin{cases} \beta_z & \text{if } \beta_z \neq 0, \\ \beta_x & \text{if } \beta_z = 0, \text{ but } \beta_x \neq 0, \\ \beta_y & \text{if } \beta_z = 0 \text{ and } \beta_x = 0. \end{cases}$$

That is, X is the first non-zero component of $\boldsymbol{\beta}$, taken in the order z, x, y.

How do we know that λ specifies the eigenvalue in the way given by eqn. (1)? We don't. But we're talking completely generally at this point: we're creating a spin–$\frac{1}{2}$ system and asserting that it behaves in the manner given by eqn. (1). The proper question is in fact, is the stipulation provided by eqn. (1) an *allowable* stipulation according to the framework of quantum mechanics? To answer that, we must ask a different question: what constraints must this specification satisfy? This seems to be the point behind the calculation that follows eqn. (3) in [1]: an average over all variables, including the hidden variables, must give us the proper expectation value.

To calculate the expectation value, we recall that this particular model specifies $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$. Then we seek to calculate the average of the quantity in parentheses in eqn. (1):

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) d\lambda.$$

We note that

$$\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z| \geq 0 \qquad \text{when} \qquad \lambda \geq \frac{|\beta_z|}{2|\boldsymbol{\beta}|}.$$

Then we have

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) d\lambda &= \int_{-1/2}^{\frac{|\beta_z|}{2|\boldsymbol{\beta}|}} (-1) d\lambda + \int_{\frac{|\beta_z|}{2|\boldsymbol{\beta}|}}^{\frac{1}{2}} (+1) d\lambda \\ &= \left(\frac{1}{2} - \frac{|\beta_z|}{2|\boldsymbol{\beta}|}\right) - \left(\frac{|\beta_z|}{2|\boldsymbol{\beta}|} + \frac{1}{2}\right) \\ &= \frac{|\beta_z|}{|\boldsymbol{\beta}|}. \end{aligned}$$

Now, given our specification in eqn. (1), the expectation of the operator

$\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ is given by

$$\begin{aligned}
\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \alpha + |\boldsymbol{\beta}| \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) \operatorname{sgn} X \right\} d\lambda \\
&= \alpha + |\boldsymbol{\beta}| \operatorname{sgn} X \int_{-\frac{1}{2}}^{\frac{1}{2}} \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) d\lambda \\
&= \alpha + |\boldsymbol{\beta}| \operatorname{sgn} X \frac{|\beta_z|}{|\boldsymbol{\beta}|} \\
&= \alpha + \beta_z,
\end{aligned}$$

precisely because [1] has chosen coordinates where ψ lies along the z-axis, so that $X = \beta_z \neq 0$.

It seems that the point here is that, without the hidden variable, the expectation value of the operator $\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ should be α, since its eigenvalues are equally probable, and we have

$$\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle = \frac{1}{2}\left(\alpha + |\boldsymbol{\beta}|\right) + \frac{1}{2}\left(\alpha - |\boldsymbol{\beta}|\right) = \alpha.$$

However, by incorporating the hidden variable λ, not only can we decisively say which eigenvalue will result, but even the expectation value itself is skewed in the direction of the proper state:

$$\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle_\lambda = \alpha + \beta_z.$$

That is, within the standard quantum mechanical formalism (referred to in [1] as von Neumann's formulation), we can construct a spin–$\frac{1}{2}$ system whereby knowledge of the hidden variable allows the formalism itself to chose unambiguously the *actual* state of the system.

References

[1] John S Bell. On the Problem of Hidden Variables in Quantum Mechanics. *Reviews of Modern Physics*, 38(3):447–452, 1966.