# For Whom the Bell Tunnels

T ‘n’ T

November 11, 2013

## Todo list



## 1 Notes on Expectation Values

Before launching into a discussion of [1], we should review a few points about expectation values.[1] If we have a system in a *pure* state $|\psi\rangle$ and we seek to measure a quantity with associated operator $\hat{A}$, then the expectation value of $\hat{A}$ *in that state* is

$$\langle \hat{A} \rangle_\psi := \langle \psi | \hat{A} | \psi \rangle.$$

Of course, if have some complete orthogonal basis of states $\{\phi_n\}$, we may expand the pure state $\psi$ in this basis as

$$|\psi\rangle = \sum_n a_n |\phi_n\rangle,$$

and then the expectation value becomes

$$\langle \psi | \hat{A} | \psi \rangle = \langle \psi | \hat{A} \sum_n a_n |\phi_n\rangle = \sum_n a_n \langle \psi | \hat{A} | \phi_n \rangle.$$

[1] See the Wikipedia article for a quick reference.

The situation is slightly different if we do not have a pure state, but rather a *mixed* state. In such cases we employ a density operator $\hat{\rho}$ composed of the various individual pure states $|\psi_k\rangle$, expressed as follows:

$$\hat{\rho} := \sum_k \rho_k |\psi_k\rangle\langle\psi_k|.$$

Then the expectation value of $\hat{A}$ in this collection of mixed states becomes

$$\langle\hat{A}\rangle_\rho := \text{trace}(\hat{\rho}\hat{A}) = \sum_k \rho_k \langle\psi_k|\hat{A}|\psi_k\rangle = \sum_k \rho_k \langle\hat{A}\rangle_{\psi_k}.$$

## 2 Assumption, and a simple example

In [1, p.448] we find the discussion of a spin-$\frac{1}{2}$ system. In particular we look at the action of the operator

$$\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}$$

on a spin–$\frac{1}{2}$ vector $\psi$. Here $\mathbb{1}$ is the 2–dimensional unit matrix, and $\boldsymbol{\sigma}$ is the vector whose components comprise the individual Pauli spin matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\boldsymbol{\beta}$ is some 3–dimensional vector of coefficients.

**Proposition 1** (Eigenvalues)**.** *The operator $\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ operating on a spin–$\frac{1}{2}$ vector $\psi$ has eigenvalues*

$$\alpha \pm |\boldsymbol{\beta}|.$$

*Calculation of eigenvalues.* We seek to solve the following equation:

$$(\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma})\psi = \lambda\psi$$

for some complex number $\lambda$. That is, we look for

$$(\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma} - \lambda\mathbb{1})\psi = 0.$$

Thus we are looking for the nullity of the operator in parentheses. The corresponding eigenvalues are given by the zeroes of the determinant. Thus we look for $\lambda$ satisfying

$$\det[(\alpha-\lambda)\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}] \overset{\heartsuit}{=} 0.$$

In components, the given operator is

$$\begin{pmatrix} \alpha + \beta_3 - \lambda & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & \alpha - \beta_3 - \lambda \end{pmatrix}.$$

The characteristic equation therefore yields

$$\begin{aligned} 0 &\overset{\heartsuit}{=} (\alpha + \beta_3 - \lambda)(\alpha - \beta_3 - \lambda) - (\beta_1 + i\beta_2)(\beta_1 - i\beta_2) \\ &= (\alpha + \beta_3)(\alpha - \beta_3) - (\alpha + \beta_3)\lambda - (\alpha - \beta_3)\lambda + \lambda^2 - \left[\beta_1^2 - (i\beta_2)^2\right] \\ &= \lambda^2 - 2\alpha\lambda + \left(\alpha^2 - |\boldsymbol{\beta}|^2\right). \end{aligned}$$

Applying the quadratic formula, this leaves us with

$$\lambda = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4 \cdot 1 \cdot (\alpha^2 - |\boldsymbol{\beta}|^2)}}{2 \cdot 1} = \alpha \pm |\boldsymbol{\beta}|,$$

as desired. □

**Proposition 2** (Eigenvectors)**.** *The normalized eigenvectors of the operator* $\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$ *are*

$$\phi_+ := \frac{1}{\sqrt{2|\boldsymbol{\beta}|(|\boldsymbol{\beta}| + \beta_z)}} \begin{pmatrix} |\boldsymbol{\beta}| + \beta_z \\ \beta_x + i\beta_y \end{pmatrix}, \quad \phi_- := \frac{1}{\sqrt{2|\boldsymbol{\beta}|(|\boldsymbol{\beta}| - \beta_z)}} \begin{pmatrix} \beta_z - |\boldsymbol{\beta}| \\ \beta_x + i\beta_y \end{pmatrix}.$$

*Proof.* We know the operator has eigenvalues $\lambda_\pm := \alpha \pm |\boldsymbol{\beta}|$. We seek to solve the equation

$$(\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma})\,\phi_\pm = \lambda_\pm \phi_\pm$$

for the vector $\phi_\pm$ associated with each eigenvalue... □

The section following the above calculation in [1] then goes on to introduce a "hidden variable" $\lambda$ (unrelated to the $\lambda$ used above). By design, this variable removes any indeterminacy as to which state the spinor $\psi$ is actually in. That is, without the use of the ancillary variable $\lambda$, all we can say is that when we measure the quantity corresponding to the operator $\alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$, the measurement must return an eigenvalue of the operator: either $\alpha + |\boldsymbol{\beta}|$ or $\alpha - |\boldsymbol{\beta}|$, but we don't know which of these will result before making the measurement. However, by incorporating the information of the hidden variable $\lambda$, by virtue of what we mean by the phrase "hidden variable", this information should be sufficient to tell us beforehand which of the two possible eigenvalues will result from the measurement. Equation (3) of [1] is the *specification* of just such a rule: the eigenvalue that results from any measurement will be determined by the value of $\lambda$ according to the rule

$$\alpha + |\boldsymbol{\beta}| \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) \operatorname{sgn} X, \tag{1}$$

where

$$X = \begin{cases} \beta_z & \text{if } \beta_z \neq 0, \\ \beta_x & \text{if } \beta_z = 0, \text{ but } \beta_x \neq 0, \\ \beta_y & \text{if } \beta_z = 0 \text{ and } \beta_x = 0. \end{cases}$$

That is, $X$ is the first non-zero component of $\boldsymbol{\beta}$, taken in the order $z, x, y$.

How do we know that $\lambda$ specifies the eigenvalue in the way given by eqn. (1)? We don't. But we're talking completely generally at this point: we're creating a spin–$\frac{1}{2}$ system and asserting that it behaves in the manner given by eqn. (1). The proper question is in fact, is the stipulation provided by eqn. (1) an *allowable* stipulation according to the framework of quantum mechanics? To answer that, we must ask a different question: what constraints must this specification satisfy? This seems to be the point behind the calculation that follows eqn. (3) in [1]: an average over all variables, including the hidden variables, must give us the proper expectation value.

To calculate the expectation value, we recall that this particular model specifies $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$. Then we seek to calculate the average of the quantity in parentheses in eqn. (1):

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) d\lambda.$$

We note that

$$\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z| \geq 0 \qquad \text{when} \qquad \lambda \geq -\frac{|\beta_z|}{2|\boldsymbol{\beta}|}.$$

Then we have

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} \operatorname{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) d\lambda &= \int_{-1/2}^{-\frac{|\beta_z|}{2|\boldsymbol{\beta}|}} (-1) d\lambda + \int_{-\frac{|\beta_z|}{2|\boldsymbol{\beta}|}}^{\frac{1}{2}} (+1) d\lambda \\ &= \left(\frac{1}{2} + \frac{|\beta_z|}{2|\boldsymbol{\beta}|}\right) - \left(-\frac{|\beta_z|}{2|\boldsymbol{\beta}|} + \frac{1}{2}\right) \\ &= \frac{|\beta_z|}{|\boldsymbol{\beta}|}. \end{aligned}$$

Now, given our specification in eqn. (1), the expectation of the operator $\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ is given by

$$
\begin{aligned}
\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{\alpha + |\boldsymbol{\beta}|\,\mathrm{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right)\mathrm{sgn}\,X\right\} d\lambda \\
&= \alpha + |\boldsymbol{\beta}|\,\mathrm{sgn}\,X \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{sgn}\left(\lambda|\boldsymbol{\beta}| + \frac{1}{2}|\beta_z|\right) d\lambda \\
&= \alpha + |\boldsymbol{\beta}|\,\mathrm{sgn}\,X \frac{|\beta_z|}{|\boldsymbol{\beta}|} \\
&= \alpha + \beta_z,
\end{aligned}
$$

precisely because [1] has chosen coordinates where $\psi$ lies along the $z$-axis, so that $X = \beta_z \neq 0$.

It seems that the point here is that, without the hidden variable, the expectation value of the operator $\alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}$ should be $\alpha$, since its eigenvalues are equally probable, and we have

I think I'm assuming here that the system is in a mixed state of $\psi_+$ & $\psi_-$, with $\rho_\pm = \frac{1}{2}$. See $\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle_{\psi_\pm}$.

$$
\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle = \frac{1}{2}\left(\alpha + |\boldsymbol{\beta}|\right) + \frac{1}{2}\left(\alpha - |\boldsymbol{\beta}|\right) = \alpha.
$$

However, by incorporating the hidden variable $\lambda$, not only can we decisively say which eigenvalue will result, but even the expectation value itself is skewed in the direction of the proper state:

$$
\langle \alpha\mathbb{1} + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle_\lambda = \alpha + \beta_z.
$$

That is, within the standard quantum mechanical formalism (referred to in [1] as von Neumann's formulation), we can construct a spin–$\frac{1}{2}$ system whereby knowledge of the hidden variable allows the formalism itself to chose unambiguously the *actual* state of the system.

Let us try to unpack a little more the statements made leading up to equation (3) in [1, p.448]. In particular Bell states

> The dispersion free states are specified by a real number $\lambda$, in the interval $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$ as well as the spinor $\psi$. To describe how $\lambda$ determines which eigenvalue the measurement gives, we note that by a rotation of coordinates $\psi$ can be brought to the form
>
> $$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

> Let $\beta_x, \beta_y, \beta_z$ be the components of $\boldsymbol{\beta}$ in the new coordinate system. Then measurement of $\alpha + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$ on the state specified by $\psi$ and $\lambda$ results with certainty in the eigenvalue...

and there follows the expression given above in eqn. (1). It seems that part of what's being left unsaid is that, from the standpoint of an adherent of a hidden variable theory, the physical system is always *in one particular state* or the other. The notion of being in a superposition of states, e.g. $\phi = a_+\psi_+ + a_-\psi_-$, is a mathematical construct but not supposed to be a reflection of the underlying reality.

Given that perspective on the physical reality, then to decide which state the physical system is *really* in, we need simply to know what eigenvalue the measurement will return. If the measurement returns $\alpha + |\boldsymbol{\beta}|$, then the system is in the state $\psi_+$; if $\alpha - |\boldsymbol{\beta}|$, then the state $\psi_-$. We can see this directly from the calculation of the expectation value of the operator $\alpha + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$ on the particular form of the state $\psi$ in Bell's quote above:

Maybe we need to see what this calculation gives for all values of $X$.

**Proposition 3.** *The expectation values of the operator* $\alpha \mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}$ *in the states* $\psi_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\psi_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *are* $\alpha + \beta_z$ *and* $\alpha - \beta_z$*, respectively.*

*Proof.* In the state $\psi_+$, the expectation value yields

$$\begin{aligned}
\langle \alpha \mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \rangle_{\psi_+} &= \langle \psi_+ | \alpha \mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} | \psi_+ \rangle \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha + \beta_z & \beta_x - i\beta_y \\ \beta_x + i\beta_y & \alpha - \beta_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha + \beta_z \\ \beta_x + i\beta_y \end{pmatrix} \\
&= \alpha + \beta_z.
\end{aligned}$$

By contrast, for a system in the state $\psi_-$ we would have

$$\begin{aligned}
\langle \alpha \mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \rangle_{\psi_-} &= \langle \psi_- | \alpha \mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} | \psi_- \rangle \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha + \beta_z & \beta_x - i\beta_y \\ \beta_x + i\beta_y & \alpha - \beta_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_x - i\beta_y \\ \alpha - \beta_z \end{pmatrix} \\
&= \alpha - \beta_z.
\end{aligned}$$

□

We may view the same calculation from the perspective of the eigenvectors of the operator $\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}$. In terms of the eigenstates of the operator $\psi$ may be expanded as follows:

$$\psi := \begin{pmatrix}1\\0\end{pmatrix} = \sqrt{\frac{|\boldsymbol{\beta}|+\beta_z}{2|\boldsymbol{\beta}|}}\phi_+ - \sqrt{\frac{|\boldsymbol{\beta}|-\beta_z}{2|\boldsymbol{\beta}|}}\phi_-.$$

Therefore, the probability that we get $\alpha+|\boldsymbol{\beta}|$ is $\dfrac{|\boldsymbol{\beta}|+\beta_z}{2|\boldsymbol{\beta}|}$, while we get $\alpha-|\boldsymbol{\beta}|$ with probability $\dfrac{|\boldsymbol{\beta}|-\beta_z}{2|\boldsymbol{\beta}|}$. Thus the expectation value is

$$\langle\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle_\psi = (\alpha+|\boldsymbol{\beta}|)\,\frac{|\boldsymbol{\beta}|+\beta_z}{2|\boldsymbol{\beta}|} + (\alpha-|\boldsymbol{\beta}|)\,\frac{|\boldsymbol{\beta}|-\beta_z}{2|\boldsymbol{\beta}|} = \alpha+\beta_z.$$

So in some sense this isn't really anything new, until we add the stipulation that we should be able to say which state the system is in *before* we make a measurement, solely based on our knowledge of the value of the hidden variable $\lambda$.[2] The preceding calculation assumes we know the state $\psi$ before we take the expectation value, whereas the calculation of $\langle\alpha\mathbb{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\rangle_\lambda$ does not.

Is this really right?

Eqn. (1) then provides one particular instance for how one could arrange such a hidden variable $\lambda$. Given the value of $\lambda$, then we know from eqn. (1) which eigenvalue we *will* measure, and so we know what state the system is in *before* we make any measurement. That's how a hidden variable should work. The interesting feature, and the focus of the discussion, is that this particular setup seems permissible within the framework of standard quantum mechanics.

An example todo note in another color.

We might also take a moment to note some features of $\lambda$. It seems that Bell falls into the typical practice of physicists which assumes that any random variable is uniformly distributed unless otherwise stated.[3] This agrees

[2]Really what we achieve by this construction is that we avoid the necessity of imposing a magical mechanism of "collapsing the wave function". Such a collapse is necessitated by the view that the system can be *in* a state which is a superposition of base states. The construction here says that, no, the system was in one of the base states all along, and was never in a superposition.

[3]Really there are only two distributions that ever come up in standard presentations of physics: the uniform and the gaussian. Since Bell makes no mention of the gaussian, we fall back to the uniform distribution.

with the calculation that follows, since the integral expression presented in the calculation for $\langle \alpha \mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \rangle_\lambda$ shows no particular probability density other than the implied constant density.

## 3 von Neumann

Let $|\psi\rangle = \sum_i a_i |q_i\rangle$, where $\hat{Q}|q_i\rangle = q_i|q_i\rangle$.

| | **Expectation** Any number | **Observation** An eigenvalue of $\hat{Q}$ |
|---|---|---|
| Quantum Mechanics | $\dfrac{\langle\psi\|\hat{Q}\psi\rangle}{\langle\psi\|\psi\rangle}$ | $q_i$ with probability $\|a_i\|^2$ |
| Hidden Variable | Average over $\lambda$ | $\lambda$ determines which $q_i$ |

von Neumann's argument compares the upper left and the lower right. The expectation value and the observed values should not be mixed up. The Quantum mechanical *expectation value* is linear in $\alpha$ and $\beta_i$, while the *observed value*, $\alpha \pm |\boldsymbol{\beta}|$ is not linear in them, but there is nothing wrong. What should be compared are the quantum mechanical expectation value and the averaged value over $\lambda$. The hidden variable theory is built up so that gives the same results as standard quantum mechanics.

## References


[1] John S Bell. On the Problem of Hidden Variables in Quantum Mechanics. *Reviews of Modern Physics*, 38(3):447–452, 1966. 1, 2, 3, 4, 5