

Likelihood of Heirarchical Matern Gaussian Random Field

We model trait values of individuals as independent normally distributed variables around a mean \bar{z} with variance v . In turn the mean \bar{z} is a function of location $\mathbf{x} = (x_1, x_2)^\top$ and follows a stationary Gaussian random field with spatially homogeneous mean \bar{z} and Matern covariance $C(d)$ with collocated variance $V = C(0)$, characteristic length ξ and smoothness parameter $\nu = 1$. Hence, observed individual trait values z_1 and z_2 at locations $\mathbf{x}_1, \mathbf{x}_2$ separated at a distance d will be bivariate normally distributed with the following statistical properties:

$$\mathbb{E}z_1 = \mathbb{E}z_2 = \bar{z}, \quad \mathbb{V}z_1 = \mathbb{V}z_2 = v + V, \quad \mathbb{C}(z_1, z_2) = C(d),$$

$$\mathbb{E}(z_i|\bar{z}) = \bar{z}(\mathbf{x}_i), \quad \mathbb{V}(z_i|\bar{z}) = v, \quad \mathbb{C}(z_1, z_2|\bar{z}) = 0.$$

Since we only get a single sample of this random field, the likelihood function is just

$$L(\bar{z}, v, V, \xi|\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{z} - \bar{\mathbf{z}})\Sigma^{-1}(\mathbf{z} - \bar{\mathbf{z}})^\top\right),$$

where $\mathbf{z} = (z_1, \dots, z_n)^\top$ is the vector of observed individual trait values (here there are n individuals) observed at locations $\mathbf{x}_1, \dots, \mathbf{x}_n$, $\bar{\mathbf{z}}$ is the vector of expected trait values $(\bar{z}, \dots, \bar{z})^\top$ and Σ is a covariance matrix with $v + V$ on the diagonal and $C(\|\mathbf{x}_i - \mathbf{x}_j\|)$ on the i, j th off-diagonal entry.

To maximize likelihood, we first compute gradients of the log-likelihood $\ell = \ln L$ with respect to model parameters. We then solve for the parameter values when those gradients are zero:

$$\begin{aligned} \frac{\partial \ell}{\partial \bar{z}} &= -\frac{1}{2} \frac{\partial}{\partial \bar{z}} [(\mathbf{z} - \bar{\mathbf{z}})\Sigma^{-1}(\mathbf{z} - \bar{\mathbf{z}})^\top] = -\frac{1}{2} \frac{\partial}{\partial \bar{z}} \sum_{ij} (\sigma^{-1})_{ij} (z_i - \bar{z})(z_j - \bar{z}) \\ &= \sum_{ij} (\sigma^{-1})_{ij} \left(\frac{z_i + z_j}{2} - \bar{z} \right) = \sum_{ij} \frac{z_i + z_j}{2} (\sigma^{-1})_{ij} - \bar{z} \sum_{ij} (\sigma^{-1})_{ij}, \\ \implies \hat{\bar{z}} &= \frac{\sum_{ij} \frac{z_i + z_j}{2} (\sigma^{-1})_{ij}}{\sum_{ij} (\sigma^{-1})_{ij}}. \end{aligned}$$

Unfortunately, the rest is analytically intractable. So we proceed via numerical optimization to maximize likelihood... in julia... then import the results below.

