maximizing cross-correlation

Consider $X = (X_1, \dots, X_n)^\top$, $Y = (Y_1, \dots, Y_n)^\top$ such that $\mathbb{E}X_i = \mathbb{E}Y_i = 0$ and $\mathbb{E}X_i^2 = \sigma_X^2$, $\mathbb{E}Y_i^2 = \sigma_Y^2$, $\mathbb{E}X_iY_i = \sigma_{XY}$ for all $i = 1, \dots, n$.

Problem:

Assuming $\sigma_{XY} \neq 0$, find matrices A, B such that

$$A, B = \arg\max_{A,B} \frac{|\mathbb{E}[(AX)^\top BY]|}{\sqrt{\mathbb{E}[(AX)^\top AX)]\mathbb{E}[(BY)^\top BY]}}.$$

Partial Solution:

Writing the i, jth entries for A and B respectively as a_{ij}, b_{ij}, we have $AX = (\sum_j a_{1j}X_j, \dots, \sum_j a_{nj}X_j)^\top$ and $BY = (\sum_k b_{1k}Y_k, \dots, \sum_k b_{nk}Y_k)^\top$. Then

$$\mathbb{E}[(AX)^\top BY] = \mathbb{E}\sum_i (\sum_j a_{ij}X_j)(\sum_k b_{ik}Y_k) = \sigma_{XY}\sum_{ijk} a_{ij}b_{ik},$$

$$\mathbb{E}[(AX)^\top AX] = \sigma_X^2 \sum_{ijk} a_{ij}a_{ik},$$

$$\mathbb{E}[(BY)^\top BY] = \sigma_Y^2 \sum_{ijk} b_{ij}b_{ik}.$$

Setting $e = (1, \dots, 1)^\top$, the problem can be restated as finding A, B such that

$$A, B = \arg\max_{A,B} \frac{|eA^\top Be|}{\sqrt{(eA^\top Ae)(eB^\top Be)}}$$

which does not depend on σ_{XY}.

Set α and β equal to the row sums of A and B represented as column vectors (that is, $\alpha_i = \sum_j a_{ij}$ and $\beta_k = \sum_k b_{ik}$). From this we can see that any A, B that solve the above problem must have row sum vectors that solve

$$\alpha, \beta = \arg\max_{\alpha,\beta} \frac{|\alpha^\top \beta|}{\sqrt{(\alpha^\top \alpha)(\beta^\top \beta)}},$$

and the solution here is any α, β such that $\alpha = c\beta$ for some $c \in \mathbb{R}$. In particular, we can set $A = B = I$ with $c = 1$. This suggests applying linear transformations to X and Y will at best preserve the known correlation, but never amplify it.