Likelihood of Heirarchical Matern Gaussian Random Field

We model trait values of individuals as independent normally distributed variables around a mean $\bar{z}$ with variance v. In turn the mean $\bar{z}$ is a function of location $\boldsymbol{x} = (x_1, x_2)^\top$ and follows a stationary Gaussian random field with spatially homogeneous mean $\tilde{z}$ and Matern covariance $C(d)$ with collocated variance $V = C(0)$, characteristic length ξ and smoothness parameter $\nu = 1$. Hence, observed individual trait values z_1 and z_2 at locations $\boldsymbol{x}_1, \boldsymbol{x}_2$ separated at a distance d will be bivariate normally distributed with the following statistical properties:

$$\mathbb{E}z_1 = \mathbb{E}z_2 = \tilde{z},\ \mathbb{V}z_1 = \mathbb{V}z_2 = v + V, \mathbb{C}(z_1, z_2) = C(d),$$

$$\mathbb{E}(z_i|\bar{z}) = \bar{z}(\boldsymbol{x}_i),\ \mathbb{V}(z_i|\bar{z}) = v,\ \mathbb{C}(z_1, z_2|\bar{z}) = 0.$$

Since we only get a single sample of this random field, the likelihood function is just

$$L(\tilde{z}, v, V, \xi|\boldsymbol{z}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left(-\frac{1}{2}(\boldsymbol{z} - \tilde{\boldsymbol{z}})\Sigma^{-1}(\boldsymbol{z} - \tilde{\boldsymbol{z}})^\top\right),$$

where $\boldsymbol{z} = (z_1, \ldots, z_n)^\top$ is the vector of observed individual trait values (here there are n individuals) observed at locations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, $\tilde{\boldsymbol{z}}$ is the vector of expected trait values $(\tilde{z}, \ldots, \tilde{z})^\top$ and Σ is a covariance matrix with $v + V$ on the diagonal and $C(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)$ on the i, jth off-diagonal entry.

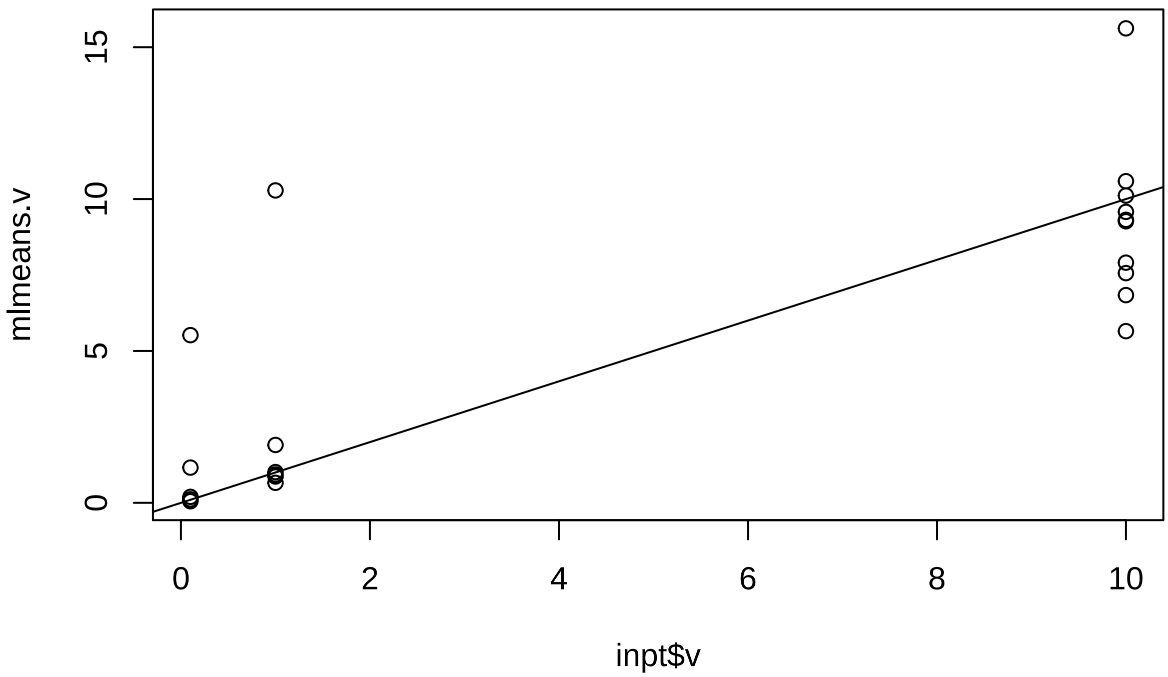
To maximize likelihood, we first compute gradients of the log-likelihood $\ell = \ln L$ with respect to model parameters. We then solve for the parameter values when those gradients are zero:

$$\frac{\partial \ell}{\partial \tilde{z}} = -\frac{1}{2}\frac{\partial}{\partial \tilde{z}}\left[(\boldsymbol{z} - \tilde{\boldsymbol{z}})\Sigma^{-1}(\boldsymbol{z} - \tilde{\boldsymbol{z}})^\top\right] = -\frac{1}{2}\frac{\partial}{\partial \tilde{z}}\sum_{ij}(\sigma^{-1})_{ij}(z_i - \tilde{z})(z_j - \tilde{z})$$

$$= \sum_{ij}(\sigma^{-1})_{ij}\left(\tfrac{z_i + z_j}{2} - \tilde{z}\right) = \sum_{ij}\tfrac{z_i + z_j}{2}(\sigma^{-1})_{ij} - \tilde{z}\sum_{ij}(\sigma^{-1})_{ij},$$

$$\implies \hat{\tilde{z}} = \frac{\sum_{ij}\frac{z_i + z_j}{2}(\sigma^{-1})_{ij}}{\sum_{ij}(\sigma^{-1})_{ij}}.$$

Unfortunately, the rest is analytically intractable. So we proceed via numerical optimization to maximize likelihood... in julia... then import the results below.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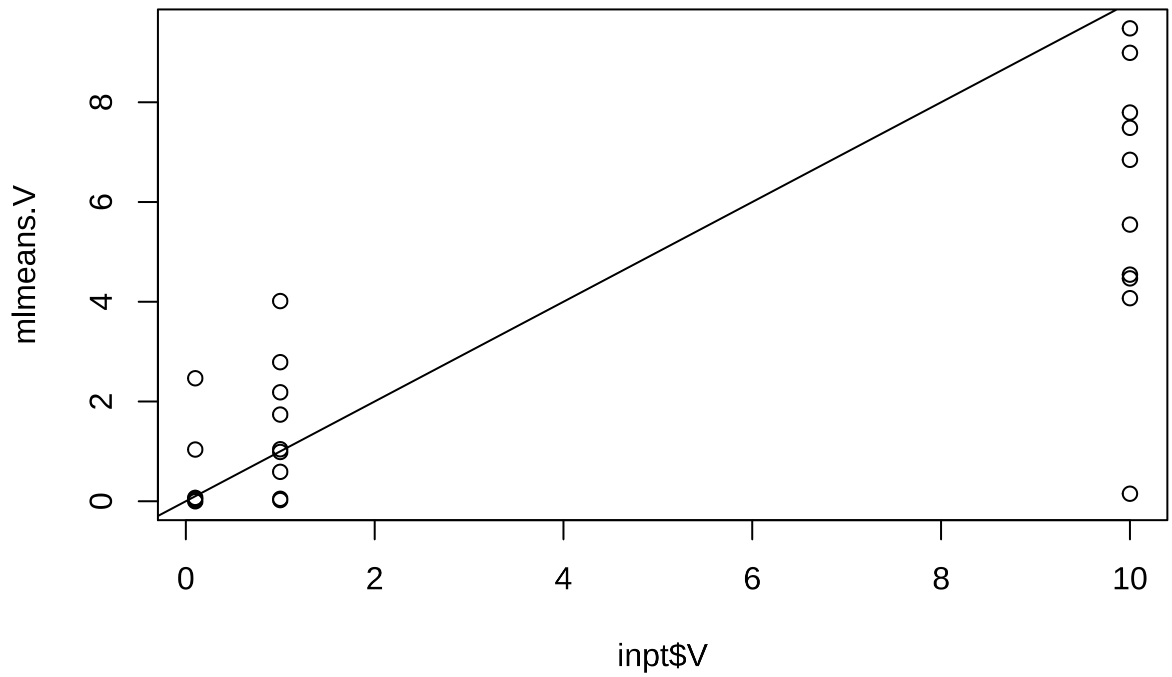
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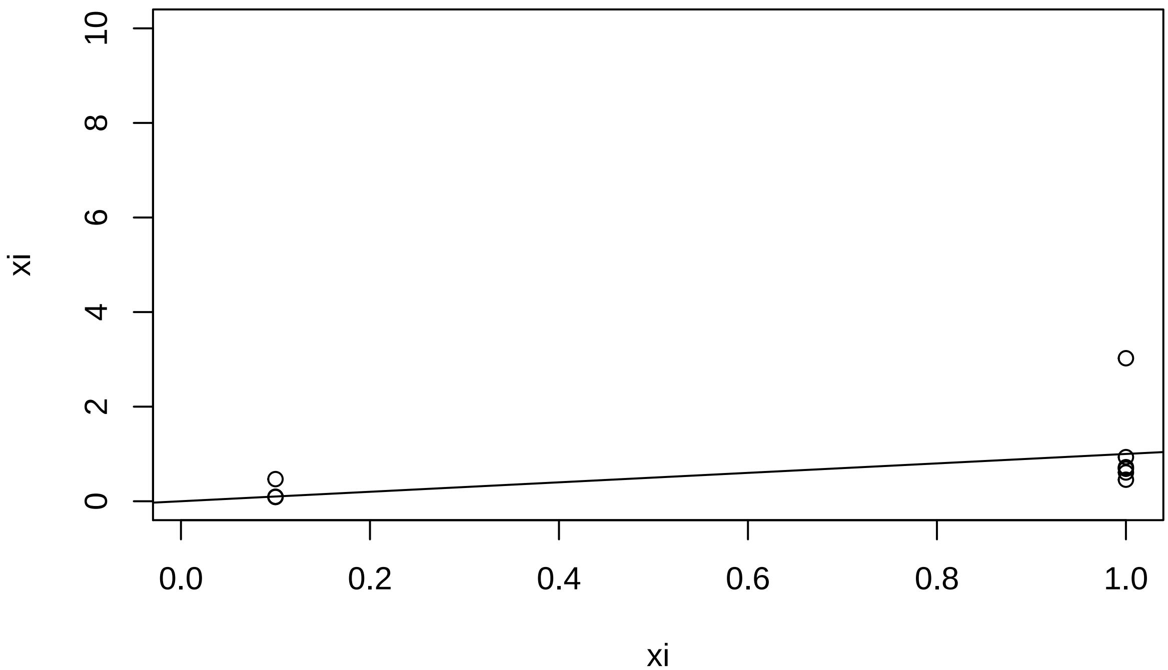

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