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Abstract

Please use no more than 300 words and avoid mathematics or complex script.

"These violent delights have violent ends"
(Romeo and Juliet: Act 2, Scene 6, Line 9)

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I S. Milovanović and L. von Sydow. *Radial Basis Function generated Finite Differences for Option Pricing Problems.* Comp. Math. Appl., 75(4):1462–1481, 2017.

II Slobodan Milovanović. *Pricing Financial Derivatives using Radial Basis Function generated Finite Differences with Polyharmonic Splines on Smoothly Varying Node Layouts.* arXiv preprint, arXiv:1808.02365[q-fin.CP], 2018.

III S. Milovanović and L. von Sydow. *A High Order Method for Pricing of Financial Derivatives using Radial Basis Function fenerated Finite Differences.* arXiv preprint, arXiv:xxxx.yyyyy[q-fin.CP], 2018.

IV S. Milovanović and V. Shcherbakov. *Pricing Derivatives under Multiple Stochastic Factors by Localized Radial Basis Function Methods.* Journ. Comp. Fin., (in review), 2018.

V L. von Sydow, L. J. Höök, E. Larsson, E. Lindström, S. Milovanović, J. Persson, V. Shcherbakov, Y. Shpolyanskiy, S. Sirén, J. Toivanen, J. Waldén, M. Wiktorsson, J. Levesley, J. Li, C. W. Oosterlee, M. J. Ruijter, A. Toropov, and Y. Zhao. *BENCHOP — The BENCHmarking Project in Option Pricing.* Int. Journ. Comp. Math., 92(12): 2361–2379, 2015.

VI L. von Sydow, S. Milovanović, E. Larsson, K. in 't Hout, M. Wiktorsson, C. W. Oosterlee, V. Shcherbakov, M. Wyns, A. Leitao, S. Jain, T. Haentjens, and J. Waldén. *BENCHOP: The BENCHmarking Project in Option Pricing — Stochastic and local volatility problems.* Int. Journ. Comp. Math., (in review), 2018.

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Related Work

The following ongoing project, although not included, is related to the contents of the present thesis.

L. von Sydow, E. Larsson, S. Milovanović, V. Shcherbakov, et al. *BENCHOP: The BENCHmarking Project in Option Pricing — Basket Options,* manuscript in preparation, 2018.

Contents

1. Introduction

The purpose of this thesis is to report on state of the art in Radial Basis Function generated Finite Difference (RBF-FD) methods for pricing of financial derivatives. Based on the six appended papers which are referred to by their Roman numerals, this work provides a detailed overview of RBF-FD properties and challenges that arise when the RBF-FD methods are used in financial applications. Moreover, with this manuscript, we aim to motivate further development of RBF-FD for finance.

Across the financial markets of the world, financial derivatives such as futures, options, and others, are traded in substantial volumes. The value of all assets that underly outstanding derivatives transactions is several times larger than the gross world product (GWP). Financial derivatives are the most commonly used instruments when it comes to hedging risks, speculation based investing, and performing arbitrage. Therefore, knowing the prices of those financial instruments is of utmost importance at any given time. In order to make that possible in practice, it is often required to employ a set of skills incorporating knowledge in financial theory, engineering methods, mathematical tools, and programming practice — which altogether constitute the field known as *financial engineering*.

Many of theoretical pricing models for financial derivatives can be represented using partial differential equations (PDEs). In many cases, those equations are time-dependent, of high spatial dimension, and with challenging boundary conditions — which most often makes them analytically unsolvable. In those cases, we need to utilize numerical approximation as a mean of estimating their solution. The fields of *numerical analysis* and *scientific computing* are concerned with obtaining approximate solutions while maintaining reasonable bounds on errors. Unfortunately, there is no universal numerical method which can be used to solve all problems of this type efficiently. In fact, there are tremendously many numerical methods for solving different types of differential equations, and all those methods are featured with their own limitations in performance, stability, and accuracy — mostly dependent on details of the problems they aim to solve. Therefore, carefully selecting and developing numerical methods for particular applications has been the only way to build efficient PDE solvers in ongoing practice.

RBF-FD is a recent numerical method with potential to efficiently approximate solutions of PDEs in finance. Over the past years, besides the purely academic development and research of its numerical properties, the method has been mainly applied for simulations of atmospheric phenomena. As its name suggests, the RBF-FD method is of a finite difference type, from the radial basis function family. As a finite difference method, RBF-FD approximates differential equations by linear systems of algebraic equations, known as difference equations. Radial basis functions (RBFs) are used as interpolants that enable local approximations of differential operators that are necessary for constructing the difference equations. Constructed like that, the method is featured with a sparse matrix of the linear system of difference equations, and it is relatively simple to implement like the standard finite difference methods. Moreover, the method is mesh-free, meaning that it does not require a structured discretization of the computational domain which makes it equally easy to use in spaces of different dimensions, and it is of a customizable order of accuracy — which are the features it inherits from the global radial basis function approximations. It is those properties that led us to recognize RBF-FD as a method with high potential for efficiently solving some analytically unfeasible and computationally challenging pricing problems in finance.

Nevertheless, being a young method, RBF-FD is still under intense development and we face many challenges when moving from simple theoretical cases toward more complex real-world applications. The core of this thesis deals with finding solutions for overcoming obstacles when financial derivatives are priced using RBF-FD to solve PDEs with several spatial dimensions. Thus, it represents a contribution to making the RBF-FD methods more reliable and efficient for use in financial applications.

The rest of this manuscript is organized as follows. We introduce and define financial derivatives in Chapter 2. An overview of some popular financial models and techniques for the pricing of options are presented in Chapter 3. We present the features and properties of RBF-FD methods for solving PDEs in finance in Chapter 4. Finally, we conclude with some unsolved challenges and suggestions for further development of the RBF-FD method for financial applications in Chapter 5.

2. Financial Derivatives

A *financial derivative* is a market instrument whose value depends on the values of some other underlying variables. Most often, those underlying variables are the prices of another traded asset (e.g., a stock underlying stock options), but they may as well be almost any variables of stochastic nature (e.g., air temperatures underlying weather derivatives). There are numerous financial derivatives in existence, available for almost every type of investment asset, ranging from agricultural grains to cryptocurrencies. Futures and options are best known as *exchange-traded* derivatives, standardized to be bought and sold on derivatives exchanges (e.g., Chicago Mercantile Exchange for futures and Chicago Board Options Exchange for options). On the other hand, much greater volumes of financial derivatives are traded bilaterally *over-the-counter* in a highly customizable fashion. This gave birth to many contracts with tailored properties such as forward contracts, swaps, exotic options, and other custom financial instruments.

When it comes to traders, three categories can be readily identified: *hedgers*, *speculators*, and *arbitrageurs* [1]. Hedgers use derivatives to reduce risks from potential future movements in a market variable, speculators use them to bet on the future outcome of a market variable, and arbitrageurs aim at making riskless profit by exploiting discrepancies in values of the same underlying variable traded under different derivatives or across different markets. Thanks to them, derivatives markets have been highly liquid over the past decades as many of the traders find trading derivatives more attractive compared to trading their underlying assets.

Financial derivatives are traded in extremely large volumes across the planet. The estimated total notional value of these financial instruments has been above half a quadrillion of USD during the current decade [2]. That is about an order of magnitude larger than GWP [3]. Moreover, derivatives markets have received great criticism due to their role in the most recent global financial crisis. As a result of the crisis, strict regulations in trading of derivatives have been introduced in order to increase transparency on the markets, improve market efficiency, and reduce systemic risk. Now, in the post-crisis period, methods for valuation of financial derivatives are still under the spotlight of financial institutions, as they look for the most efficient ways to solve the mathematical problems stemming from the regulations.

In order to bring financial derivatives closer to the mathematical framework, it is useful for us to define several of their features. We assume that the contract representing a particular financial derivative is signed at time $t = t_0 \equiv 0$ and expires at $t = T$, where T is also known as the time of *maturity* of the contract. The contract is issued on the underlying stochastic variable $S(t)$. At the expiration of the contract, the holder receives payoff $g(S(T))$, which is equivalent to the value of the financial derivative at the time of maturity T, i.e., $u(S(T), T) = g(S(T))$. The value of the contract is represented by a function $u(t, S(t))$.

When it comes to hierarchy of financial derivatives, we can see most of them either as a type of a forward/futures contract, or as a type of an option. Therefore, it is common to study forwards and futures as binding contracts (i.e., $-\infty < g(S(T)) < \infty$), and options as non-obligatory contracts towards their holders (i.e., $0 \leq g(S(T)) < \infty$). In the following sections, we consider them in more detail.

2.1 Forwards and Futures

A *forward* contract is an agreement between two parties signed at $t = t_0$ to buy or sell an underlying $S(t)$ at a certain future time T for a certain price $K(t_0) = K_0$. The price $K(t)$ is called the *forward price* of the contract, and it is determined at time t_0 in such a way that the value of the forward contract at the time of signing is equal to zero, i.e., $u(t_0, S(t_0)) = u_0 = 0$. One of the parties in the contract takes a *long* position and agrees to the payoff

$$g_l(S(T)) = S(T) - K_0.$$

The other party assumes a *short* position and agrees to sell $S(t)$ at the same time T for the stipulated forward price K_0, effectively obliging to the payoff

$$g_s(S(T)) = K_0 - S(T).$$

Forward contracts are traded in over-the-counter markets and may be further customized according to the preferences of the signing parties.

A *futures* contract is an exchange-traded, and thus standardized financial derivative, that is very similar to a forward contract. It is in agreement signed at no cost between two parties at $t = t_0$ to buy or sell an underlying $S(t)$ at a certain time T. The principal difference from the forward contract lies in the way in which the payments are realized. Namely, at every point in time $t_0 \leq t \leq T$, there exists a price $K(t)$, now called the *futures price* of the contract, that is quoted on the exchange. At time T, the long position holder of the contract is entitled to the payoff

$$g_l(S(T)) = S(T) - K(T),$$

while the short position holder gets

$$g_s(S(T)) = K(T) - S(T).$$

Moreover, during an arbitrary time interval $(t_i, t_j]$, where $t_0 \leq t_i < t_j < T$, the long holder of the contract receives the amount $K(t_j) - K(t_i)$, and the short holder receives $K(t_i) - K(t_j)$. The futures price $K(t)$ evolves in such way that obtaining the futures contract at any time $t_0 \leq t \leq T$ incurs a zero cost, i.e., $u(t, S(t)) = 0$.

As far as the pricing of forwards and futures is concerned, it is clear that these contracts are designed in such a way that their prices are equal to zero at the signing. Thus, computational problems of interest here are related to fairly determining the defined forward and futures prices. For more details on forwards and futures, it is wise to turn to [4, 1].

2.2 Options

An *option* is a contract that gives its holder the right, but not the obligation to buy or sell an underlying $S(t)$ by a certain time of maturity T for a certain price K. The price K in the contract is known as the strike price. If the contract gives the buying right to its owner, then it is called a *call* option, and if it gives the selling right, it is called a *put* option. Call options are characterized with

$$g_c(S(T)) = \max(S(T) - K,\ 0), \tag{2.1}$$

and put options with

$$g_p(S(T)) = \max(K - S(T),\ 0), \tag{2.2}$$

as their respective payoff functions. Options that can be exercised at any time $t_0 < t \leq T$ are called *American* options, and options that can be exercised only at time $t = T$ are known as *European* options. Since options are traded both on exchanges and in over-the-counter markets, there are many more types of them (e.g., binary options, barrier options, Asian options, Bermudan options, and other exotic options) — as the ways of customizing them are limitless. For instance, *rainbow* options are defined in such a way that their payoffs may depend on more than one underlying asset, consequently requiring a multi-dimensional pricing model in order to estimate their value. An example of such a multi-asset derivative is an arithmetic European call basket option issued on D underlying assets $S_1, \ldots, S_D$, whose payoff function is

$$g_{bc}(S_1(T), \ldots, S_D(T)) = \max\left(\frac{1}{D}\sum_{d=1}^{D} S_d(T) - K,\ 0\right). \tag{2.3}$$

Moreover, for a given underlying S, there may be a large number of options with different dates of expiration T, and different strike prices K.

Due to their versatility, options have been among the most popular financial derivatives on the financial markets, and many examples of their applications can be found in [1].

3. Option Pricing

We emphasize that an option gives the right to the holder to do something, and that the holder does not need to use that right. This is the main difference between options and other financial derivatives. Whereas it costs nothing to buy a forward or futures contract, there is always a non-negative price to acquiring an option. This very detail is the cornerstone of one of the most involving fundamental problems in financial markets, known as *option pricing*. Depending on the option characteristics, the pricing problem can be as trivial as deriving an analytical pricing formula — such is the case for the standard European call option with certain market assumptions. Nevertheless, in many other cases of option valuation, we are faced against an eternal struggle of balancing between reasonable market assumptions for deriving delicate mathematical models and developing efficient numerical solvers that are able to estimate the solutions of the equations posed by those models.

As the option gives stipulated rights, but not the obligations to their holder, it is natural to assume that this contract must have some objective non-negative value at any time. The central task of option pricing is to objectively determine the fair value of an option at any given time $t \leq T$. The fundamental mathematical framework for approaching this problem is the *arbitrage theory*. In order to model option prices, the theory heavily relies on carefully argued assumptions about the market and mathematical ingredients such as martingale measures, stochastic differential equations (SDEs), Itô calculus, Feynman–Kac representations, and PDEs. We refer to these topics throughout the manuscript in limited capacity, as the detailed definitions and proofs can be found elegantly presented in [5].

3.1 Market Models

In order to be able to price an option, we need a set of assumptions that can be used to build a financial market model. The models range from the simple ones capturing a rough approximate picture of reality to extremely advanced ones aimed at capturing very fine details of the market. Once the model is defined, we should be able to set up an option pricing problem that needs to be solved in order to estimate the option value. Difficulty of such pricing problems strongly depends on the complexity of the chosen market model as well as on the complexity of the specifics in the option contract that we want to price.

3.1.1 Black–Scholes–Merton Model

We start by considering a plain European option on a stock that does not pay dividends, under the famous Black–Scholes–Merton model [6, 7]. Creation of this model in 1973 is considered as one of the most successful quantitative breakthroughs in social sciences, initiating a pricing framework that still keeps occupied thousands of researchers across financial institutions and universities of the world. This was recognized by the Royal Swedish Academy of Sciences, when the *Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel* was awarded to Robert C. Merton and Myron S. Scholes in 1997, while Fischer S. Black was credited with equal contribution since he had passed away two years before the prize was awarded.

The main feature of this model is that it allows the prices of European call and put options to be calculated analytically using variables that are either directly observable on the market or can be easily estimated. It is still widely used as a benchmark, although more advanced models have been developed over the years to take into account more realistic features of asset price dynamics, such as jumps and stochastic volatility.

The model consists of two assets, a riskless bond $B(t)$ and a risky stock $S(t)$, with dynamics given by the following SDEs

$$\mathrm{d}B(t) = rB(t)\,\mathrm{d}t, \tag{3.1}$$

$$\mathrm{d}S(t) = \mu S(t) + \sigma S(t)\,\mathrm{d}W(t), \tag{3.2}$$

where r is the risk-neutral interest rate, μ is the drift coefficient, and σ is the volatility of the stock — all three being constant in the model. Moreover, $W(t)$ is the Wiener process.

The Black–Scholes–Merton model stands on several important assumptions. The main assumption is that the considered financial market is arbitrage free, meaning that it is not possible to make positive earnings on the market without being exposed to risk. The next assumption states that the market is complete and efficient, which means that every contract on the market can be hedged and that the market prices fully reflect all available information. Those assumptions allow us to determine a unique price of the option whose payoff function is $g(S(T))$, using the following valuation under the risk-neutral measure $\mathbb{Q}$,

$$u(S(t),\ t) = \exp\left(-r(T-t)\right)\mathbb{E}^{\mathbb{Q}}\left[g(S(T))\right]. \tag{3.3}$$

That effectively means that the expected value is calculated on an adapted dynamics by using r instead of μ as the drift constant of the stochastic process $S(t)$ defined in (3.2).

Moreover, using the Itô's lemma and the Feynman–Kac theorem, we can equivalently express the option price as the solution of the following PDE, known as the Black–Scholes–Merton equation

$$\begin{aligned} \frac{\partial u}{\partial t} + rs\frac{\partial u}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 u}{\partial s^2} - ru &= 0, \\ u(s,T) &= g(s), \end{aligned} \tag{3.4}$$

where s is the deterministic representation of the stochastic asset price S. The equation (3.4) is a parabolic PDE that has an analytical solution $u = u(s,t)$ in case of European call and put options. In order to have a fully defined PDE problem, formulation in (3.4) requires appropriate boundary conditions. We omit stating the boundary conditions on purpose in this chapter for readability, and discuss them with particular problems examples in the subsequent chapters as they may vary from case to case.

In order to make better trading decisions, investors often look at the hedging parameters, which are also known as the *greeks*. The most commonly used ones are *delta* $\Delta = \frac{\partial u}{\partial s}$, *gamma* $\Gamma = \frac{\partial^2 u}{\partial s^2}$, and *vega* $\nu = \frac{\partial u}{\partial \sigma}$. As these hedging parameters represent risk sensitivities, being able to compute them is of great importance.

We can use this basic framework to price financial derivatives with different payoffs or extend it in order to be able to valuate options with different underlying assets (e.g., stocks that pay discrete dividends). Also, we can further adapt the model to capture different market features more accurately. (e.g., introduce local volatility instead of the constant one). Moreover, it is sometimes beneficial to use the Merton model [8] to model underlying assets with jumps. On the other hand, stochastic volatility models, such as the Heston model presented in Section 3.1.3, are useful when there are prominent volatility smiles in the underlying asset. To push things even further, it is not uncommon to have a stochastic volatility model with jumps — the most known representative is the Bates model [9]. An overview of such extensions of the Black–Scholes–Merton framework can be seen in **Paper V**.

3.1.2 Multi-Asset Options

To price multi-asset financial derivatives, such as rainbow options issued on D underlying assets $S_1, S_2, \ldots, S_D$, we consider a multi-dimensional analogue to (3.1) and (3.2)

$$\begin{aligned} \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t, \\ \mathrm{d}S_1(t) &= \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t), \\ &\vdots \\ \mathrm{d}S_D(t) &= \mu_D S_D(t)\,\mathrm{d}t + \sigma_D S_D(t)\,\mathrm{d}W_D(t), \end{aligned} \tag{3.5}$$

where the Wiener processes are correlated such that $\mathrm{d}W_i(t)\,\mathrm{d}W_j(t) = \rho_{ij}\,\mathrm{d}t$. In this high-dimensional setting, an option with the payoff function $g(S_1(T), \ldots, S_D(T))$, can be priced by solving the corresponding high-dimensional Black–Scholes–Merton equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}_b u &= 0, \\ u(s_1, \ldots, s_D,\, T) &= g(s_1, \ldots, s_D), \end{aligned} \tag{3.6}$$

where

$$\mathcal{L}_b u \equiv r \sum_{i}^{D} s_i \frac{\partial u}{\partial s_i} + \frac{1}{2} \sum_{i,j}^{D} \rho_{i,j} \sigma_i \sigma_j s_i s_j \frac{\partial^2 u}{\partial s_i \partial s_j} - ru. \tag{3.7}$$

We observe (3.6) as a time-dependent PDE with D spatial dimensions.

When it comes to American options, since these financial derivatives can be exercised at any $t \leq T$, as opposed to the European options (that can only be exercised at $t = T$), instead of using a PDE as a model, we formulate the pricing task as a linear complementarity problem (LCP)

$$\frac{\partial u}{\partial t} + \mathcal{L}_b u \geq 0,$$
$$u(s_1, \ldots, s_D,\ t) \geq g(s_1, \ldots, s_D),$$
$$\left(\frac{\partial u}{\partial t} + \mathcal{L}_b u\right)\left(u(s_1, \ldots, s_D,\ t) - g(s_1, \ldots, s_D)\right) = 0, \tag{3.8}$$

where the initial data is given by the terminal condition $u(s_1, \ldots, s_D,\ T) = g(s_1, \ldots, s_D)$. This formulation also applies to pricing of a single-asset American option by choosing $D = 1$.

3.1.3 Multi-Factor Models

Another direction in development of pricing models is to include more stochastic factors. Models with multiple stochastic factors allow for better reproduction of market features compared to the standard Black–Scholes–Merton formulation, which is known to fall short in capturing heavy tails of return distributions and volatility skews. Therefore, various models with local volatilities, local stochastic volatilities, stochastic interest rates, and their combinations have been getting popular. In this section, we present two models with multiple stochastic factors that are used for pricing options.

The attention to local volatility models started with [10]. The first multi-factor model that we introduce is the Heston model [11], featured with a stochastic volatility. The adapted dynamics for this model is as follows

$$\mathrm{d}S(t) = rS(t)\,\mathrm{d}t + \sqrt{V(t)}S(t)\,\mathrm{d}W_s(t), \tag{3.9}$$
$$\mathrm{d}V(t) = \kappa\left(\eta - V(t)\right)\mathrm{d}t + \sigma\sqrt{V(t)}\,\mathrm{d}W_v(t), \tag{3.10}$$

where $V(t)$ is the stochastic volatility, σ is the constant volatility of volatility, κ is the speed of mean reversion of the volatility process, η is the mean reversion level, r is the risk-free interest rate, $W_s(t)$ and $W_v(t)$ are correlated Wiener processes with constant correlation ρ, i.e., $\mathrm{d}W_s(t)\,\mathrm{d}W_v(t) = \rho\,\mathrm{d}t$. After using the Itô's lemma and the Feynman–

Kac theorem, the PDE for the Heston model reads as

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \mathcal{L}_h u = 0, \\
&u(s, v,\ T) = g(s),
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
\mathcal{L}_h u \equiv \frac{1}{2} v s^2 \frac{\partial^2 u}{\partial s^2} + \rho \sigma v s \frac{\partial^2 u}{\partial s \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 u}{\partial v^2} \\
+ r s \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} - r u,
\end{aligned} \tag{3.12}$$

s and v are deterministic representations of the stochastic asset price and volatility processes, respectively.

The Heston–Hull–White model is an enhancement of the Heston stochastic volatility model. The improvement consists of adding a stochastic interest rate that follows the Hull–White process [12], as the interest rates on the market are not constant. The model is useful when pricing long-term derivatives in which we observe an implied volatility smile in the underlying asset. Another notable property of the Hull–White model is that the interest rates can be negative, as nowadays happens in some economies. The adapted dynamics for this model is as follows

$$\mathrm{d}S(t) = R(t) S(t)\,\mathrm{d}t + \sqrt{V(t)} S(t)\,\mathrm{d}W_s(t), \tag{3.13}$$

$$\mathrm{d}V(t) = \kappa\Big(\eta - V(t)\Big)\,\mathrm{d}t + \sigma_v \sqrt{V(t)}\,\mathrm{d}W_v(t), \tag{3.14}$$

$$\mathrm{d}R(t) = a\Big(b - R(t)\Big)\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_r(t), \tag{3.15}$$

where R_t is the stochastic interest rate, a is the speed of mean reversion of the interest rate process, b is its mean reversion level, σ_r is its volatility, $W_s(t)$, $W_v(t)$, and $W_r(t)$ are correlated Wiener processes.

We can apply the Itô's lemma and the Feynman–Kac theorem to derive the pricing PDE

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \mathcal{L}_w u = 0, \\
&u(s, v, r,\ T) = g(s),
\end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
\mathcal{L}_w u \equiv \frac{1}{2} v s^2 \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 u}{\partial r^2} + \\
\rho_{sv} \sigma_v v s \frac{\partial^2 u}{\partial s \partial v} + \rho_{sr} \sigma_r \sqrt{v} s \frac{\partial^2 u}{\partial s \partial r} + \rho_{vr} \sigma_v \sigma_r \sqrt{v} \frac{\partial^2 u}{\partial v \partial r} + \\
r s \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b - r) \frac{\partial u}{\partial r} - r u,
\end{aligned}
\tag{3.17}
$$

Here, it becomes clear how advanced models easily grow in complexity, which in turn makes it difficult to calibrate and valuate them in practice. Several other multi-factor models are discussed in more detail in **Papers IV** and **VI**.

3.2 Pricing Methods

For a small number of cases, such as plain European call or put options under the Black–Scholes–Merton model, calculating the option price can be done by closed form solutions, derived using analytical methods. In some other cases, it is possible to approximate the solutions using semi-analytical schemes. Commonly used methods for pricing of financial derivatives in the absence of analytical or semi-analytical solutions can be split in three main groups: stochastic methods, methods based on Fourier transform, and deterministic methods. Performance of these methods when pricing several option types across different market models is presented in Papers V and VI.

3.2.1 Stochastic Methods

Stochastic methods, such as Monte Carlo (MC), aim at approximating option prices using the form showed in (3.3). The idea of estimating expectations by repeated random sampling was used in different forms for centuries, but it was officially defined in [13]. The first application of an MC method in option pricing was reported in 1977 for European options [14]. A least square MC method for pricing American options was introduced in 2001 [15], and more recently, a new regression based MC method, named stochastic grid bundling method (SGBM), has been developed for efficient pricing of early-exercise options and their hedging parameters [16]. Furthermore, quasi-MC [17] — methods that use deterministic sequences of numbers to boost convergence — became successful at efficiently tackling problems in hundreds of dimensions [18]. More recently, many advanced versions of MC methods have been developed, of which some of the most notable are multilevel MC methods [19], which are inspired by the multigrid ideas for the iterative solution of PDEs. Interestingly, in the time of publishing of this thesis, some pioneering approaches in development of quantum computing MC algorithms for pricing of financial derivatives have been made [20].

Discrete models like binomial trees that appeared in 1979 [21, 22], also fall in the group of stochastic methods. These models work by simulating stochastic trajectories of the underlying dynamics on predefined discrete lattices, and are among the simplest nontrivial models of financial markets.

Stochastic methods are mostly suitable for multi-asset derivatives and/or multi-factor models, which result in problems of high dimensionality. The classical versions of these methods are arguably easy to implement and use. MC methods are significantly less efficient than other methods when used for problems in smaller dimensions, as their convergence rate is much slower in comparison. This can be observed on several pricing examples in Paper V.

3.2.2 Fourier Methods

This group consists of methods based on Fourier transform such as Carr–Madan fast Fourier transform method [23]. More recently, Fourier-cosine series expansions (COS) for European options [24] and early-exercise options [25], showed to be extremely efficient in pricing. In 2012, the COS method has been extended to higher dimensions [26]. The methods from this category are very fast and accurate, but they typically require existence of the characteristic function for the price process of the underlying asset in closed form, or at least its approximation — which is available for a fairly large class of the models, but not all.

3.2.3 Deterministic Methods

Deterministic methods are used to solve pricing problems in PDE form such as (3.4), by discretizing its differential operators.

The main methods in this category are the finite differences (FD). The first time an FD method was used for pricing of a contract was in 1976 [27], to solve a one-dimensional Black–Scholes–Merton equation. A few years later, FD schemes, together with MC methods, were established as a standard numerical approach for pricing financial derivatives when analytical solutions are not available [28]. Moreover, a notable operator splitting scheme was introduced in [29], enabling FD methods to efficiently price American options. Over the years, FD methods have been used to solve mostly one-dimensional and two-dimensional pricing problems. More recently, hierarchical approximation using sparse grids [30] and asymptotic expansions [31] of high-dimensional option pricing problems have been developed — enabling state of the art FD [32, 33] to be used for pricing high-dimensional options by solving a sequence of lower dimensional problems. Apparently, many high-dimensional pricing problems have such a configuration of volatilities and correlations that their effective dimensionality is low, and as such can be represented by a small number of lower dimensional components [34].

Although they are used less often, the finite element methods can excel in certain cases [35, 36, 37], and the same applies for finite volumes [38], which finds its use in convection dominated or degenerate cases.

Finally, RBF methods are a more recent group of deterministic methods to be used for option pricing — first time applied in 1999 for one-dimensional European options [39]. Ever since, these methods have been becoming popular, as they possess potential to cope with PDEs of moderately high dimensions.

3.2.4 Method Selection

Based on the presented details and the results reported in **Papers V** and **VI**, a basic guide for selecting an appropriate option pricing method is to first check if it is possible to analytically calculate the solution to the pricing problem. In case that is not possible, the next best option is a Fourier transform based method. Deterministic methods come into play as robust numerical schemes when Fourier methods are not applicable. Nevertheless, they often suffer from the curse of dimensionality as the degrees of freedom in the resulting approximations grow exponentially with the dimensionality of the problem. Therefore, if the pricing problem is of a higher dimensionality that cannot be reduced, Monte Carlo methods are the most common alternative.

Typically, deterministic methods are used to solve pricing problems of up to no more than three dimensions. In the following chapter we present a localized RBF method that might become an alternative to Monte Carlo methods for moderately high-dimensional problems, i.e., of dimensionality three to five.

4. Radial Basis Function generated Finite Differences

Using the RBF methods for approximating solutions of PDEs dates back to the beginning of the nineties in the previous century [40, 41]. Ever since, these methods have been used in different fields, including financial engineering [39, 42, 43].

In order to apply an RBF method, we observe option pricing problems on the truncated computational domain $\Omega \subset \mathbb{R}^d$ in the following PDE form

$$\frac{\partial}{\partial t}u(\underline{x},t) + \mathcal{L}u(\underline{x},t) = 0, \qquad \underline{x} \in \Omega \tag{4.1}$$

$$\mathcal{B}u(\underline{x},t) = f(\underline{x},t), \quad \underline{x} \in \partial\Omega, \tag{4.2}$$

$$u(\underline{x},T) = g(\underline{x}), \qquad \underline{x} \in \Omega, \tag{4.3}$$

where $u(\underline{x},t)$ is the option price, $\underline{x}$ is the spatial variable representing underlying assets and/or stochastic factors, with $\mathcal{L}$ as the differential operator of the pricing model; $\mathcal{B}$ is the boundary operator which together with the function $f(\underline{x},t)$ models the boundary conditions; initial data are defined by the terminal condition $g(\underline{x})$.

To construct a global RBF approximation in space, we scatter N nodes $\underline{x}_j$, where $j = 1, \ldots, N$, across the computational domain Ω. Then, we consider an interpolant

$$\tilde{u}(\underline{x},t) = \sum_{j=1}^{N} \lambda_j(t)\phi(\|\underline{x} - \underline{x}_j\|), \tag{4.4}$$

where ϕ is the RBF, and $\lambda_j(t)$ are the time-dependent interpolation coefficients. At any time t, the value of the interpolant in every point $\underline{x}$ only depends on the distance to the nodes and this expression is valid for any number of dimensions.

Some examples of commonly used RBFs are listed in **Table 4.1**, split into two groups. The first group in the table consists of infinitely smooth

RBFs that can provide spectral accuracy for interpolation and are featured with a shape parameter ε. The second group contains a piecewise smooth RBF that can give algebraic convergence for interpolation.

Table 4.1. Commonly used RBFs, where $\varepsilon \in \mathbb{R}^+$ is the shape parameter for the infinitely smooth RBFs, and $q \in \{2m-1,\ m \in \mathbb{N}\}$ is the degree of the polyharmonic spline as a piecewise smooth RBF.

RBF	$\phi(r)$
Gaussian (GS)	$\exp(-\varepsilon^2 r^2)$
Multiquadric (MQ)	$\sqrt{1+\varepsilon^2 r^2}$
Inverse Multiquadric (IMQ)	$1/\sqrt{1+\varepsilon^2 r^2}$
Inverse Quadratic (IQ)	$1/(1+\varepsilon^2 r^2)$
Polyharmonic Spline (PHS)	r^q

In this thesis, we consider GS and PHS functions for approximating solutions of the pricing equations. Those two RBFs are shown plotted on a unit domain in **Figure 4.1**.

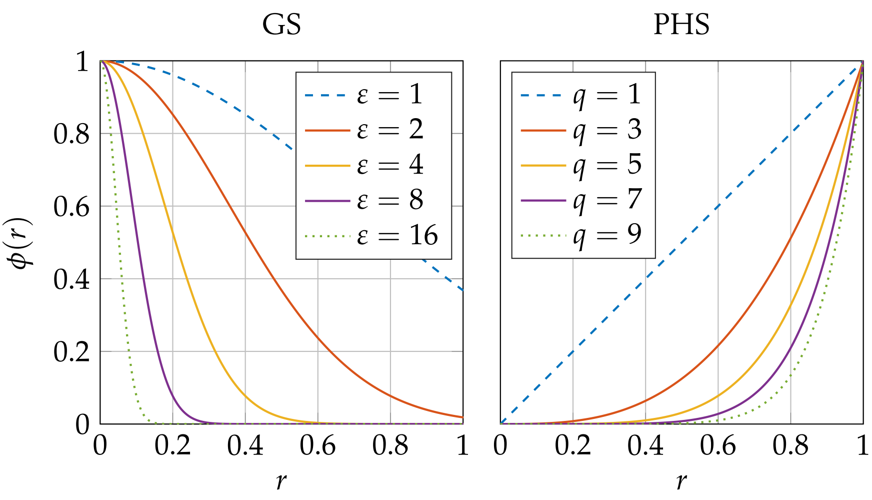

Figure 4.1. Examples of Gaussian RBFs with different shape parameter values (left) and polyharmonic splines of different degrees (right).

We can apply the global RBF method by collocating at the same $\underline{x}_j$ points through substituting (4.4) into (4.1). Thus, we obtain a dense linear system of ordinary differential equations (ODEs) of size N, where $\lambda_j(t)$ are the unknowns. Starting from the terminal condition (4.3), we can use a backward time integration method of our choice to compute the coefficients $\lambda_j(t)$, and therefore evaluate the interpolant $\tilde{u}$ which approximates the option price.

Even though the global RBF methods possess desirable properties such as spectral convergence and mesh-free domain discretization, they are featured with dense system matrices which makes the method very computationally demanding. To overcome this weakness, several localized RBF approaches were introduced, among which radial basis function partition of unity (RBF-PU) methods [44] and RBF-FD [45, 46], are the most popular, and still actively developed. These localized RBF methods are featured with sparser system matrices while still maintaining great properties from the global RBF methods, such as being mesh-free and of high-order.

The RBF-PU method has been used in finance for pricing multi-asset derivatives [47, 48, 49], and its performance when pricing one-dimensional options and their hedging parameters is also documented in **Paper V**. Moreover, RBF-PU is extensively compared against the RBF-FD method at solving multiple stochastic factors problems, which is reported in **Paper IV**. While in that paper both methods performed similarly, on a more objective study with stochastic and local volatility problems, presented in **Paper VI** — RBF-FD showed as a robust method that performs more efficiently in most of the considered cases. As RBF-FD and RBF-PU are still in development, it is hard to say which method has better potential for the future. Therefore, it is very important for the field of computational finance that both of these methods continue developing.

4.1 Method Definition

In this thesis, we focus on development of the RBF-FD methods. RBF-FD can be seen as a kind of an FD method that belongs to the family of RBF methods. To construct an RBF-FD approximation, as firstly introduced by Andrei I. Tolstykh in 2000 [45], we can reuse the same N scattered nodes across the computational domain Ω. For each node $\underline{x}_j$, we define an array of nodes $\mathbf{x}_j$ consisting of $n_j - 1$ neighboring nodes and $\underline{x}_j$ itself, and consider it as a stencil of size n_j centered at $\underline{x}_j$. The differential operator $\mathcal{L}$ defined in (4.1) is approximated in every node $\underline{x}_j$ as

$$\mathcal{L}u(\underline{x}_j, t) \approx \sum_{i=1}^{n_j} w_j^i u_j^i \equiv \mathbf{w}_j \cdot u(\mathbf{x}_j, t), \quad j = 1, \ldots, N, \tag{4.5}$$

where $u_j^i = u(\underline{x}_j^i, t)$ and $\underline{x}_j^i$ is a locally indexed node in $\mathbf{x}_j$, while $\mathbf{w}_j$ is the array of differentiation weights for the stencil centered at $\underline{x}_j$. In the standard RBF-FD methods, the weights w_j^i are calculated by enforcing (4.5) to be exact for RBFs centered at each of the nodes in $\mathbf{x}_j$, yielding

$$\underbrace{\begin{bmatrix} \phi(\|\underline{x}_j^1 - \underline{x}_j^1\|) & \ldots & \phi(\|\underline{x}_j^1 - \underline{x}_j^{n_j}\|) \\ \vdots & \ddots & \vdots \\ \phi(\|\underline{x}_j^{n_j} - \underline{x}_j^1\|) & \ldots & \phi(\|\underline{x}_j^{n_j} - \underline{x}_j^{n_j}\|) \end{bmatrix}}_{\mathbf{A}_j} \cdot \underbrace{\begin{bmatrix} w_j^1 \\ \vdots \\ w_j^{n_j} \end{bmatrix}}_{\mathbf{w}_j} = \underbrace{\begin{bmatrix} \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^1\|) \\ \vdots \\ \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^{n_j}\|) \end{bmatrix}}_{\mathbf{l}_j}. \tag{4.6}$$

In theory on RBF interpolation, it is known that (4.6) forms a nonsingular system of equations. Therefore, a unique set of weights can be computed for each node $\underline{x}_j$ by solving N linear systems of size $n_j \times n_j$. We arrange those weights in a differentiation matrix $\mathbf{L}$, which now represents a discrete approximation of the spatial operator $\mathcal{L}$ on the chosen set of nodes $\{\underline{x}_j\}_{j=1}^N$. Since $n_j \ll N$, the resulting differentiation matrix is sparse, having n_j non-zero elements per row. It is important to note that the linear systems (4.6) can be solved in parallel, which significantly reduces the weights computation overhead that RBF-FD has compared to FD. Moreover, when it comes to the boundary nodes and the nodes that are close to the boundary, the nearest neighbor based stencils automatically form according to the shape of the boundary and require no special treatment for computing the differentiation weights — which can be seen in **Figure 4.2**. The only data that is required for approximation of differential operators are Euclidian distances between the nodes. This means that (4.6) represents a way to approximate a differential operator in any number of dimensions.

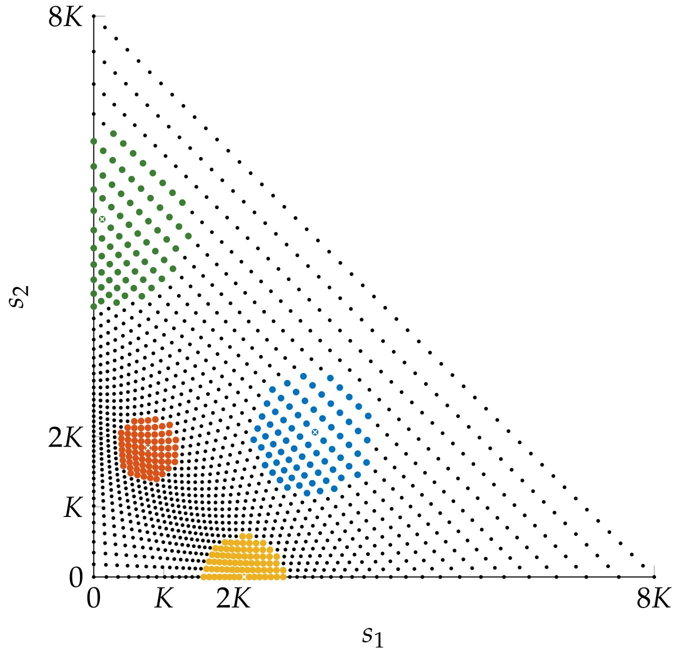

Figure 4.2. An example of nearest neighbor based stencils, used for approximating the differential operator $\mathcal{L}$ on a nonuniform node layout adapted for pricing of two-dimensional basket options with the underlying assets s_1 and s_2, and strike price K. The central node of each displayed stencil is denoted by a white cross mark. All stencils are of the size $n_j = n = 75$.

After the weights are computed and stored in the differentiation matrix $\mathbf{L}$, an approximation of (4.1) can be presented in the form of the following semi-discrete equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t) = \mathbf{L}\mathbf{u}(t), \tag{4.7}$$

$$\mathbf{u}(T) = \mathbf{g}, \tag{4.8}$$

where $\mathbf{u} \equiv u(\mathbf{x})$ is the discrete numerical solution of the pricing equation, $\mathbf{g} \equiv g(\mathbf{x})$, while $\mathbf{x}$ is the array of all nodes in the computational domain. To compute the option price $\mathbf{u}$, we need to integrate (4.7) in time.

For the time discretization, in all of our reported research, we use the unconditionally stable second order backward differentiation method (BDF2). The BDF2 scheme requires two known previous states in order to compute the current one. To initiate the method, the Euler backward method (BDF1) is often used for the first time step. In order to avoid factoring two different matrices, we use BDF2 with BDF1 as described in [50], so that we get a single differentiation matrix with nonuniform time steps.

We split the time interval $[0, T]$ into M non-uniform steps of length $\tau^k = t^{M-k} - t^{M-k+1}$, $k = 1, \ldots, M$ and define the BDF2 weights as

$$\beta_0^k = \tau^k \frac{1+\omega_k}{1+2\omega_k}, \quad \beta_1^k = \frac{(1+\omega_k)^2}{1+2\omega_k}, \quad \beta_2^k = \frac{\omega_k^2}{1+2\omega_k}, \tag{4.9}$$

where $\omega_k = \tau^k/\tau^{k-1}$, $k = 2, \ldots, M$. In [50], it is shown how the time steps can be chosen in such a way that $\beta_0^k \equiv \beta_0$. Therefore, the coefficient matrix is the same in all time steps, and only one matrix factorization is needed. Applying the BDF2 scheme to (4.7) we obtain a fully discretized system of equation

$$\underbrace{(\mathbf{E} - \beta_0 \mathbf{L})}_{\mathbf{C}} \mathbf{u}^k = \beta_1^k \mathbf{u}^{k-1} - \beta_2^k \mathbf{u}^{k-2} \tag{4.10}$$

where $\mathbf{E}$ is the identity matrix of the appropriate size. To solve this system, we employ the iterative GMRES method with an incomplete LU factorization as the preconditioner.

RBF-FD may be seen as a generalization of classical FD methods — where a polynomial interpolant is used instead of an RBF. Ever since its introduction, the RBF-FD methods have been successfully applied for solving convection-diffusion equations [51, 52], incompressible Navier–Stokes [53, 54, 55], and elliptic equations [56, 46]. RBF-FD methods were introduced to finance with a master thesis supervised by Lina von Sydow at Uppsala University in 2013 [57], and the results reported in this thesis represent the continuation of that work. In parallel with our research, classical versions of RBF-FD using infinitely smooth RBFs with constant shape parameters have been applied on equidistant Cartesian grids for pricing of different contracts [58, 59, 60, 61]. Although those articles noted the importance of RBF shape parameters for the RBF-FD approximation stability and accuracy, no special attention was payed to choosing them appropriately. Moreover, the RBF-FD examples in those articles were not exploiting the great RBF advantage of being mesh-free, as the method was applied to pricing problems using node layouts that correspond to standard equidistant FD grids. The first results of option pricing with the RBF-FD method using nonuniform node layouts and recommendations for choosing the shape parameter for GA RBFs, were reported in **Paper I**.

4.2 Scattering Nodes

In general, pricing PDEs are defined on infinite real domains. In many cases, the domain may be limited from one side, e.g., because a stock as an underlying cannot be negative, but in most cases the domain remains open towards $+\infty$. Since we want to use a numerical discretization scheme, it is first required to truncate the domain and assign appropriate boundary conditions at the boundaries. When it comes to pricing multi-asset options under single-factor models, we truncate the far positive side of each dimension at $s_{\text{max}} = aKD$, $a \in \mathbb{N}$. In practice, most of the time, using $a = 4$ keeps the approximation in the area around the strike price K safe from the artificial boundary effects. In that case, the close-field boundary is usually set as $s_{\text{min}} = 0$. The details about the domain truncation for the multi-factor models, as well as the specifics about the boundary conditions can be found in the appended papers, next to each problem used in the numerical experiments.

Once the computational domain boundaries are defined, we discretize the domain by scattering nodes across it. In order to study RBF-FD approximation and be able to compare it with the standard FD methods, we started by using equidistant Cartesian grid based node layouts. **Figure 4.3** shows equidistant Cartesian based node layouts for arithmetic basket option pricing problems of up to three dimensions in space.

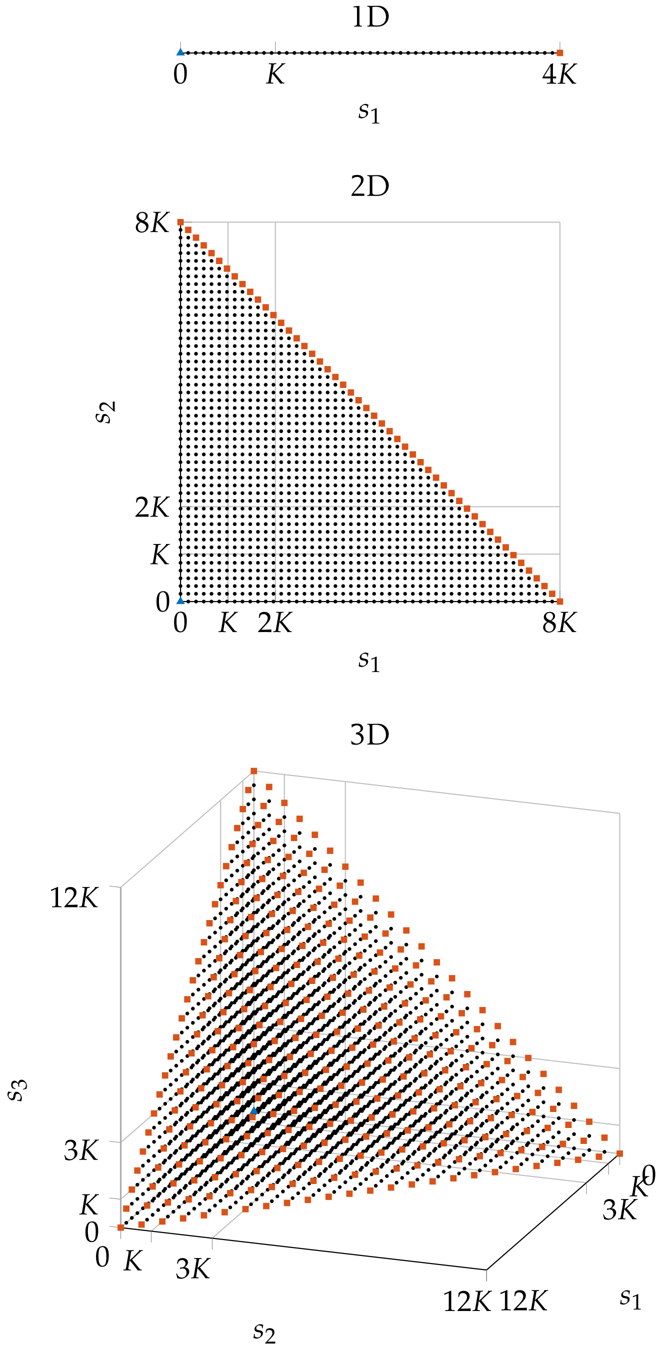

Figure 4.3. Equidistant Cartesian grid based node layouts for pricing options with different number of underlying assets. The close-field boundary conditions are enforced in the blue triangle node, and in the far-field boundary conditions are enforced in the red square nodes.

Although not fully exploiting the mesh-free feature of the RBF-FD methods, the presented node layouts for $D \geq 2$ are truncated diagonally via a hyperplane that is parallel to the hyperplane of discontinuity in the first derivative of the payoff function for call and put basket options — as it is not necessary to have computations in the parts of the domain that are far away from the strike value.

For many contracts, it is common to have discontinuities in the first derivative of the payoff function, and for some instruments even in the payoff function itself. Those discontinuities pose an obstacle for accurate numerical approximations and can often limit the order of convergence of the numerical methods. Knowing the location of a discontinuity allows as to scatter the nodes such that their density is higher around the discontinuity, and therefore the accuracy improved in those area. One way to do this is by using a one-dimensional nonuniform node scattering scheme from [32]. This scheme in the case of arithmetic basket options can be expanded to higher dimensions by generating one-dimensional layouts on the axes of the domain, and then connecting them via equidistant node scattering diagonally across the domain, as seen in **Figure 4.4**. In **Papers I**, **II**, **III**, we show the advantages of this adapted node layout over the standard equidistant Cartesian based node layouts.

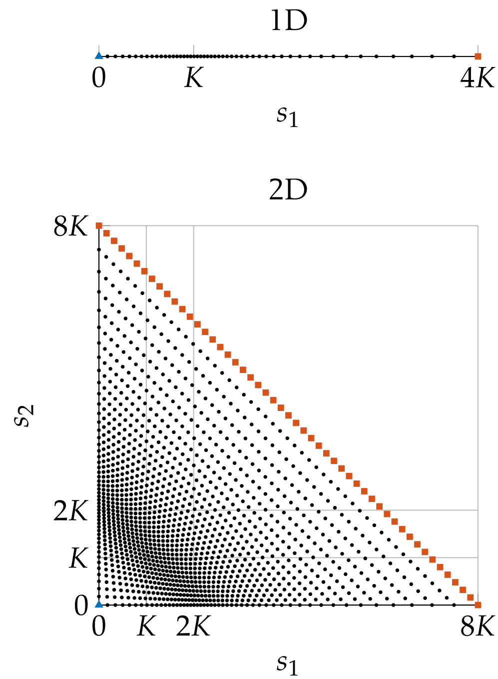

Figure 4.4. Payoff function adapted node layouts for pricing options with different number of underlying assets. The close-field boundary conditions are enforced in the blue triangle node, and in the far-field boundary conditions are enforced in the red square nodes.

By using nonuniform node layouts in our numerical experiments, we have discovered that the RBF-FD stencils become very sensitive to non-smooth variations in density of the node layouts. Namely, if a discretized computational domain contains non-smooth changes in density of the node layout, it is very likely that the stencils constructed across those areas will have very large condition numbers, and therefore make the entire approximation numerically unstable. For a successful implementation of RBF-FD methods, we need to be able to quickly generate node layouts with smoothly varying density. Surprisingly, research on node placing is currently very active, and not many practical results are available. In **Paper II**, we used a two-dimensional node layout suggested by Bengt Fornberg and Natasha Flyer in [62]. An example of layouts with smoothly varying density for pricing two-dimensional arithmetic basket options under the Black–Scholes–Merton model and one-dimensional options under the Heston model are shown in **Figure 4.5**.

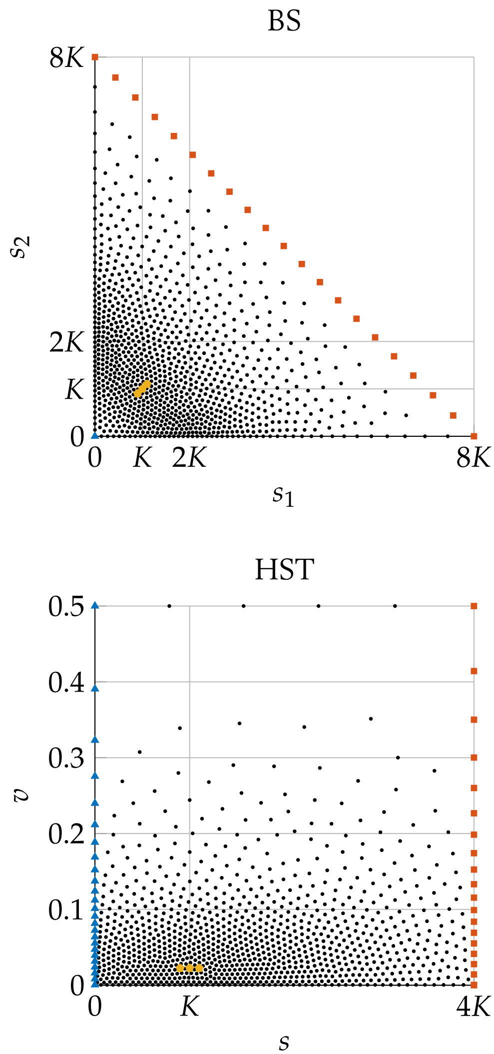

Figure 4.5. Smoothly varying density node layouts for pricing two-dimensional arithmetic basket options under the Black–Scholes–Merton model and one-dimensional options under the Heston model. The close-field boundary conditions are enforced in the blue triangle node, and in the far-field boundary conditions are enforced in the red square nodes. The yellow pentagons show the locations of interest, where the error was measured in the experiments.

The results in **Paper II** show on several examples the advantages of such node layouts over the previously presented ones. The condition numbers of the differentiation matrices in all considered problems are significantly lower, and the accuracy is tremendously improved, as the density can be increased around the areas of interest more easily.

Unfortunately, quick node placing schemes of this sort for higher dimensions are still unavailable. One of the recent works [71], introduces a novel way to quickly generate three-dimensional smoothly varying node layouts, but other than that the resources about this problem are very scarce.

4.3 Choosing Shape Parameters

The properties of the RBF-FD methods presented so far are quite attractive, but when it comes to implementing the methods in practice, with infinitely smooth RBFs we need to deal with the selection of the shape parameter ε. As it can be seen in the plot to the left of Figure 4.1, the shape parameter tunes the support of the RBF. The larger the shape parameter is, the smaller the support — and vice versa. Moreover, the greater the support of an RBF, the approximation becomes more accurate. Nevertheless, if an RBF becomes too flat, the system of equations (4.6) becomes nearly singular, and the computations of the differentiation weights become ill-conditioned. There have been several approaches to stabilize the RBF-FD stencils and make them independent of the choice of the shape parameter [63], but all those treatments came with significant increases in computational costs.

Therefore, in **Paper I**, we suggested choosing the shape parameter such that it has the smallest value before the problem becomes ill-conditioned, in order to maintain high accuracy of the RBF-FD approximation. In that article, we showed by spatial error analysis and verified by numerical experiments, that an efficient RBF-FD method can be constructed if the shape parameter for approximating the Black–Scholes–Merton differential operator is chosen as

$$\varepsilon = \varepsilon^*(h) \equiv \frac{\alpha}{h}, \tag{4.11}$$

where α is a real positive constant, and h is the characteristic distance between the nodes. The constant α is obtained by linear regression between two points in the h-ε plane with the minimal error for the given h, obtained experimentally. The suggestion has been verified on one-dimensional and two-dimensional European and American option pricing problems, on equidistant Cartesian and problem-adapted nonuniform layouts. The result is illustrated in **Figure 4.6**.

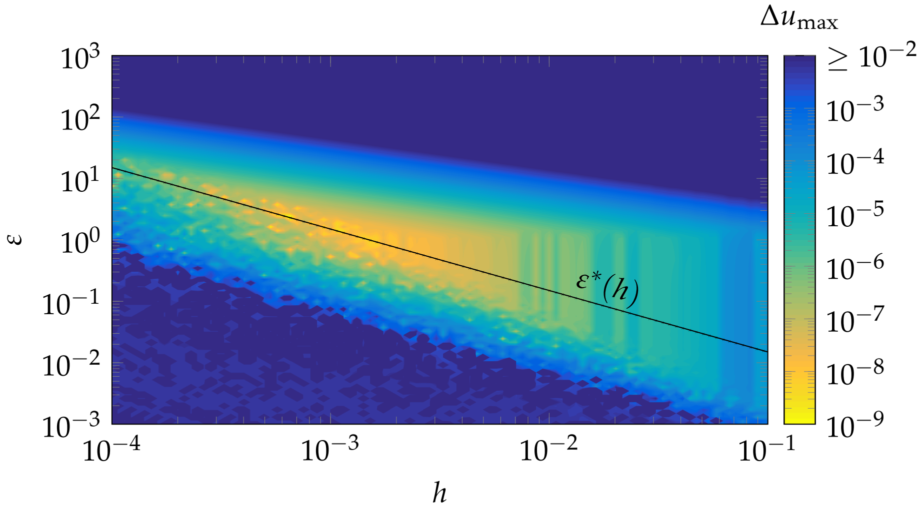

Figure 4.6. The maximum absolute error $\Delta u_{\max}$ measured in the subdomain $\hat{\Omega} = [\frac{1}{3}K, \frac{5}{3}K]$ around the strike price K, as a function of h and ε, for a one-dimensional European call option priced on an equidistant Cartesian grid with RBF-FD stencil size $n = 3$. The black line shows the appropriate choice for the shape parameter. The RBF-FD approximation is performed using GS basis functions.

If we take a closer look at **Figure 4.6**, we can see that as h decreases, the high error areas are joining together from top and bottom — leaving no space for high accuracy if h is sufficiently small. The truncation error analysis results presented in **Paper I** explain the presented behavior, and we consider the solutions for this issue in the following section.

4.4 Role of Polynomials

Many infinitely smooth RBFs have been successfully used for approximating differential operators of PDEs by RBF-FD. Nevertheless, the linear systems of equations that needed to be solved in order to obtain the weights $\mathbf{w}_j$ have been often ill-conditioned, especially as h was becoming smaller. Several works [64, 65, 66, 67, 63, 68], addressed this problem by adding low-order polynomials together with RBFs into the presented interpolation. The linear system that we need to solve to obtain the differentiation weights for each node in our problem then becomes

$$\begin{bmatrix} \mathbf{A}_j & \mathbf{P}_j^T \\ \mathbf{P}_j & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}_j \\ \mathbf{v}_j \end{bmatrix} = \begin{bmatrix} \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^1\|) \\ \vdots \\ \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^{n_j}\|) \\ \mathcal{L}p_1(\underline{x}_j) \\ \vdots \\ \mathcal{L}p_{m_j}(\underline{x}_j) \end{bmatrix}, \tag{4.12}$$

where $\mathbf{A}_j$ is the RBF matrix and $\mathbf{w}_j$ is the array of differentiation weights; $\mathbf{P}_j$ is the matrix of size $m_j \times n_j$ that contains all monomials up to order p (corresponding to m_j monomial terms) that are evaluated in each node $\underline{x}_j^i$ of the stencil $\mathbf{x}_j$ and $\mathbf{0}$ is a zero square matrix of size $m_j \times m_j$; $\mathbf{v}_j$ is the array of dummy weights that are discarded, and $\{p_1, p_2, \ldots, p_{m_j}\}$ is the array of monomial functions indexed by their position relative to the total number of monomial terms m_j, such that it contains all the combinations of monomial terms up to degree p.

We used monomials of $p = 0$ augmented to GS basis functions in **Paper I**, and the result is shown in **Figure** 4.7.

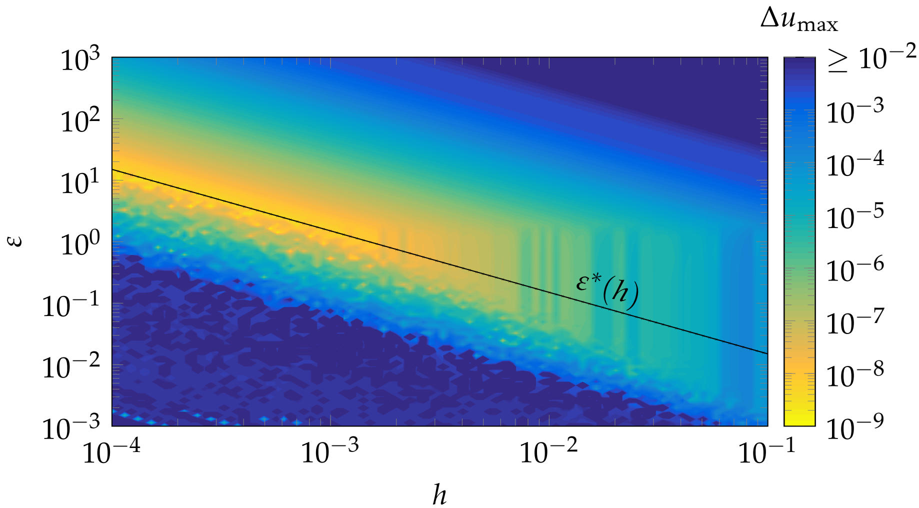

Figure 4.7. The maximum absolute error $\Delta u_{\max}$ measured in the subdomain $\hat{\Omega} = [\frac{1}{3}K, \frac{5}{3}K]$ around the strike price K, as a function of h and ε, for a one-dimensional European call option priced on an equidistant Cartesian grid with RBF-FD stencil size $n = 3$. The black line shows the appropriate choice for the shape parameter. The RBF-FD approximation is performed using GS augmented with monomials of degree $p = 0$.

The figure no longer shows joining high error fields, which ensures well-conditioned convergence of the method as h is decreased.

Even though the problem of choosing the shape parameter for GA based RBF-FD schemes is thoroughly examined for option pricing problems in **Paper I**, it still remains unsolved for general applications. Nevertheless, recent developments [69, 70], have demonstrated that the RBF-FD approximation can be greatly improved by using high order polynomials together with PHSs as piecewise smooth RBFs in the interpolation, shown in the plot to the right of **Figure 4.1**. With that approach, it seems as if the polynomial degree takes the role of controlling the rate of convergence. This allows us to use piecewise smooth PHSs as RBFs without a shape parameter, since the approximation accuracy is no longer controlled by the smoothness of the RBFs. Still, the RBFs do contribute to reduction of approximation errors, and therefore are necessary in order to have both stable and accurate approximation.

In **Paper II**, we successfully apply the PHS based RBF-FD method to pricing two-dimensional European call and American put arithmetic basket options under the Black–Scholes–Merton model, and a one-dimensional European call option under the Heston model. The PHS based method, free of any hassle of picking the shape parameters, outperforms the standard FD method despite the computational overhead from the differentiation weights.

4.5 Smoothing Payoff Functions

Paper VI

5. Outlook and Further Development

It works. It is not the best. It is still promisingly developing. Write about the node placement in high-D!

Although the used smoothly varying density node placing algorithm works only in two-dimensional domains, some recent work has been done to come up with more robust and efficient ways to construct adaptable smooth node layouts in higher dimensions [71]. Research on efficient generation of high-dimensional node layouts is expected to give a significant improvement in performance of the higher-dimensional RBF-FD methods and improve the competitiveness of these methods in different financial applications.

References

[1] J. C. Hull. *Options, Futures and Other Derivatives*. Pearson Education, 2017. ISBN 9780134631493.

[2] Bank for International Settlements. *Derivatives Statistics*. https://www.bis.org/statistics, 2018.

[3] World Bank. *World Development Indicators Database*. http://databank.worldbank.org/data/download/GDP.pdf, 2018.

[4] D. Duffie. *Futures Markets*. Prentice-Hall, 1989. ISBN 9780133456202.

[5] T. Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, 2009. ISBN 9780199574742.

[6] F. Black and M. Scholes. *The pricing of options and corporate liabilities*. Journal of Political Economy, 81: 637–654, 1973.

[7] R. C. Merton. *Theory of rational option pricing*. The Bell Journal of Economics and Management Science, 4: 141–183, 1973.

[8] R. C. Merton. *Option pricing when underlying stock returns are discontinuous*. Journal of financial economics, 3(1-2): 125–144, 1976.

[9] D. S. Bates. *Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options*. The Review of Financial Studies, 9(1): 69–107, 1996.

[10] B. Dupire et al. *Pricing with a smile*. Risk, 7(1): 18–20, 1994.

[11] S. L. Heston. *A closed-form solution for options with stochastic volatility with applications to bond and currency options*. The review of financial studies, 6(2): 327–343, 1993.

[12] J. Hull and A. White. *Pricing interest-rate derivative securities*. Rev. Financial Stud., 3: 573–592, 1990.

[13] N. Metropolis and S. Ulam. *The Monte Carlo method*. Journal of the American statistical association, 44(247): 335–341, 1949.

[14] P. P. Boyle. *Options: A Monte Carlo approach*. Journal of financial economics, 4(3): 323–338, 1977.

[15] F. A. Longstaff and E. S. Schwartz. *Valuing American options by simulation: a simple least-squares approach*. The review of financial studies, 14(1): 113–147, 2001.

[16] S. Jain and C. W. Oosterlee. *The stochastic grid bundling method: Efficient pricing of bermudan options and their greeks*. Applied Mathematics and Computation, 269: 412–431, 2015.

[17] S. H. Paskov, J. F. Traub, et al. *Faster valuation of financial derivatives*. Journal of portfolio management, 22(1): 113, 1995.

[18] J. Dick, F. Y. Kuo, and I. H. Sloan. *High-dimensional integration: the quasi-Monte Carlo way*. Acta Numerica, 22: 133–288, 2013.

[19] M. B. Giles. *Multilevel monte carlo path simulation*. Operations Research, 56(3): 607–617, 2008.

[20] P. Rebentrost, B. Gupt, and T. R. Bromley. *Quantum computational finance: Monte Carlo pricing of financial derivatives*. arXiv preprint, arXiv:1805.00109[quant-ph], 2018.

[21] J. C. Cox, S. A. Ross, M. Rubinstein, et al. *Option pricing: A simplified approach*. Journal of financial Economics, 7(3): 229–263, 1979.

[22] R. J. Rendleman Jr. and B. J. Bartter. *Two-state option pricing*. The Journal of Finance, 34(5): 1093–1110, 1979.

[23] P. Carr and D. Madan. *Option valuation using the fast Fourier transform*. Journal of computational finance, 2(4): 61–73, 1999.

[24] F. Fang and C. W. Oosterlee. *A novel pricing method for European options based on Fourier-cosine series expansions*. SIAM Journal on Scientific Computing, 31(2): 826–848, 2008.

[25] F. Fang and C. W. Oosterlee. *Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions*. Numerische Mathematik, 114(1): 27, 2009.

[26] M. J. Ruijter and C. W. Oosterlee. *Two-dimensional fourier cosine series expansion method for pricing financial options*. SIAM Journal on Scientific Computing, 34(5): B642–B671, 2012.

[27] M. J. Brennan and E. S. Schwartz. *The pricing of equity-linked life insurance policies with an asset value guarantee*. Journal of Financial Economics, 3(3): 195–213, 1976.

[28] M. J. Brennan and E. S. Schwartz. *Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis*. Journal of Financial and Quantitative Analysis, 13(3): 461–474,

1978.

[29] S. Ikonen and J. Toivanen. *Operator splitting methods for American option pricing*. Applied mathematics letters, 17(7): 809–814, 2004.

[30] C. Reisinger and G. Wittum. *Efficient hierarchical approximation of high-dimensional option pricing problems*. SIAM Journal on Scientific Computing, 29(1): 440–458, 2007.

[31] C. Reisinger and R. Wissmann. *Numerical valuation of derivatives in high-dimensional settings via partial differential equation expansions*. Journal of Computational Finance, 18(4): 95–127, 2015.

[32] K. J. in 'T Hout and S. Foulon. *ADI finite difference schemes for option pricing in the heston model with correlation*. International Journal of Numerical Analysis and Modeling, 7(2): 303–320, 2010.

[33] H. T. and K. in 't Hout. *Alternating direction implicit finite difference schemes for the Heston–Hull–White partial differential equation*. The Journal of Computational Finance, 16(1): 83–110, 2012.

[34] X. Wang and I. H. Sloan. *Why are high-dimensional finance problems often of low effective dimension?* SIAM Journal on Scientific Computing, 27(1): 159–183, 2005.

[35] R. Zvan, P. Forsyth, and K. Vetzal. *A general finite element approach for pde option pricing models*. methods, 19: 3, 1998.

[36] P. Forsyth, K. Vetzal, and R. Zvan. *A finite element approach to the pricing of discrete lookbacks with stochastic volatility*. Applied Mathematical Finance, 6(2): 87–106, 1999.

[37] A. Heinecke, S. Schraufstetter, and H.-J. Bungartz. *A highly parallel black–scholes solver based on adaptive sparse grids*. International Journal of Computer Mathematics, 89(9): 1212–1238, 2012.

[38] R. Zvan, P. A. Forsyth, and K. Vetzal. *A finite volume approach for contingent claims valuation*. IMA Journal of Numerical Analysis, 21(3): 703–731, 2001.

[39] Y.-C. Hon and X.-Z. Mao. *A radial basis function method for solving options pricing model*. Financial Engineering, 8(1): 31–49, 1999.

[40] E. J. Kansa. *Multiquadrics — A scattered data approximation scheme with applications to computational fluid-dynamics — I surface approximations and partial derivative estimates*. Computers & Mathematics with applications, 19(8-9): 127–145, 1990.

[41] E. J. Kansa. *Multiquadrics — A scattered data approximation scheme with applications to computational fluid-dynamics — II solutions to parabolic, hyperbolic and elliptic partial differential equations*. Computers & mathematics with applications, 19(8-9): 147–161, 1990.

[42] G. E. Fasshauer, A. Q. M. Khaliq, and D. A. Voss. *Using meshfree approximation for multi-asset american options*. Journal of the Chinese Institute of Engineers, 27(4): 563–571, 2004.

[43] U. Pettersson, E. Larsson, G. Marcusson, and J. Persson. *Improved radial basis function methods for multi-dimensional option pricing*. Journal of Computational and Applied Mathematics, 222(1): 82–93, 2008.

[44] H. Wendland. *Fast evaluation of radial basis functions: Methods based on partition of unity*. In *Approximation Theory X: Wavelets, Splines, and Applications*. Citeseer, 2002.

[45] A. I. Tolstykh. *On using RBF-based differencing formulas for unstructured and mixed structured–unstructured grid calculations*. In *Proceedings of the 16th IMACS World Congress on Scientific Computation, Applied Mathematics and Simulation, Lausanne, Switzerland,* 6. 2000.

[46] G. B. Wright and B. Fornberg. *Scattered node compact finite difference-type formulas generated from radial basis functions*. Journal of Computational Physics, 212(1): 99–123, 2006.

[47] A. Safdari-Vaighani, A. Heryudono, and E. Larsson. *A radial basis function partition of unity collocation method for convection–diffusion equations arising in financial applications*. Journal of Scientific Computing, 1–27, 2015.

[48] V. Shcherbakov and E. Larsson. *Radial basis function partition of unity methods for pricing vanilla basket options*. Computers & Mathematics with Applications, 71(1): 185–200, 2016.

[49] V. Shcherbakov. *Radial basis function partition of unity operator splitting method for pricing multi-asset american options*. BIT Numerical Mathematics, 56(4): 1401–1423, 2016.

[50] E. Larsson, K. Åhlander, and A. Hall. *Multi-dimensional option pricing using radial basis functions and the generalized Fourier transform*. Journal of Computational and Applied Mathematics, 222(1):

175–192, 2008.

[51] G. Chandhini and Y. S. Sanyasiraju. *Local RBF-FD solutions for steady convection-diffusion problems*. International Journal for Numerical Methods in Engineering, 72(3): 352–378, 2007.

[52] D. Stevens, H. Power, M. Lees, and H. Morvan. *The use of PDE centres in the local RBF Hermitian method for 3D convective-diffusion problems*. Journal of Computational Physics, 228(12): 4606–4624, 2009.

[53] C. Shu, H. Ding, and K. S. Yeo. *Local radial basis function-based differential quadrature method and its application to solve two-dimensional incompressible Navier–Stokes equations*. Computer Methods in Applied Mechanics and Engineering, 192(7-8): 941–954, 2003.

[54] Y. Y. Shan, C. Shu, and Z. L. Lu. *Application of local MQ-DQ method to solve 3D incompressible viscous flows with curved boundary*. Computer modeling in engineering and sciences, 25(2): 99, 2008.

[55] P. P. Chinchapatnam, K. Djidjeli, P. B. Nair, and M. Tan. *A compact RBF-FD based meshless method for the incompressible Navier–Stokes equations*. Proceedings of the Institution of Mechanical Engineers, Part M: Journal of Engineering for the Maritime Environment, 223(3): 275–290, 2009.

[56] A. I. Tolstykh and D. A. Shirobokov. *On using radial basis functions in a "finite difference mode" with applications to elasticity problems*. Computational Mechanics, 33(1): 68–79, 2003.

[57] X. Wang. *Finite differences based on radial basis functions to price options*. Master's thesis, Uppsala University, 2014.

[58] M. K. Kadalbajoo, A. Kumar, and L. P. Tripathi. *Application of radial basis function with L-stable Padé time marching scheme for pricing exotic option*. Computers & Mathematics with Applications, 66(4): 500–511, 2013.

[59] M. K. Kadalbajoo, A. Kumar, and L. P. Tripathi. *Application of the local radial basis function-based finite difference method for pricing American options*. International Journal of Computer Mathematics, 92(8): 1608–1624, 2015.

[60] A. Kumar, L. P. Tripathi, and M. K. Kadalbajoo. *A numerical study of Asian option with radial basis functions based finite differences*

method. Engineering Analysis with Boundary Elements, 50: 1–7, 2015.

[61] M. K. Kadalbajoo, A. Kumar, and L. P. Tripathi. *Radial-basis-function-based finite difference operator splitting method for pricing american options*. International Journal of Computer Mathematics, 1–17, 2017.

[62] B. Fornberg and N. Flyer. *Fast generation of 2-D node distributions for mesh-free PDE discretizations*. Computers & Mathematics with Applications, 69(7): 531–544, 2015.

[63] B. Fornberg, E. Lehto, and C. Powell. *Stable calculation of Gaussian-based RBF–FD stencils*. Comput. Math. Appl., 65(4): 627–637, 2013.

[64] O. Davydov and D. T. Oanh. *Adaptive meshless centres and RBF stencils for Poisson equation*. Journal of Computational Physics, 230(2): 287–304, 2011.

[65] B. Fornberg and E. Lehto. *Stabilization of RBF-generated finite difference methods for convective PDEs*. Journal of Computational Physics, 230(6): 2270–2285, 2011.

[66] N. Flyer, E. Lehto, S. Blaise, G. B. Wright, and A. St-Cyr. *A guide to RBF-generated finite differences for nonlinear transport: Shallow water simulations on a sphere*. Journal of Computational Physics, 231(11): 4078–4095, 2012.

[67] E. Larsson, E. Lehto, A. Heryudono, and B. Fornberg. *Stable computation of differentiation matrices and scattered node stencils based on Gaussian radial basis functions*. SIAM Journal on Scientific Computing, 35(4): A2096–A2119, 2013.

[68] N. Flyer, G. A. Barnett, and L. J. Wicker. *Enhancing finite differences with radial basis functions: experiments on the Navier–Stokes equations*. Journal of Computational Physics, 316: 39–62, 2016.

[69] N. Flyer, B. Fornberg, V. Bayona, and G. A. Barnett. *On the role of polynomials in RBF–FD approximations: I. Interpolation and accuracy*. Journal of Computational Physics, 321: 21–38, 2016.

[70] V. Bayona, N. Flyer, B. Fornberg, and G. A. Barnett. *On the role of polynomials in RBF–FD approximations: II. Numerical solution of elliptic PDEs*. Journal of Computational Physics, 332: 257–273, 2017.

[71] O. Vlasiuk, T. Michaels, N. Flyer, and B. Fornberg. *Fast high-dimensional node generation with variable density*. arXiv preprint arXiv:1710.05011, 2017.

Contributions

Paper I

Paper II

Paper III

Paper IV

Paper V

Paper VI

Acknowledgments

Woohoo!

Summary

The purpose of this thesis is to report on state of the art in Radial Basis Function generated Finite Difference (RBF-FD) methods for pricing of financial derivatives. Based on the six appended papers which are referred to by their Roman numerals, this work provides a detailed overview of RBF-FD properties and challenges that arise when the RBF-FD methods are used in financial applications. Moreover, with this manuscript, we aim to motivate further development of RBF-FD for finance.

Across the financial markets of the world, financial derivatives such as futures, options, and others, are traded in substantial volumes. The value of all assets that underly outstanding derivatives transactions is several times larger than the gross world product (GWP). Financial derivatives are the most commonly used instruments when it comes to hedging risks, speculation based investing, and performing arbitrage. Therefore, knowing the prices of those financial instruments is of utmost importance at any given time. In order to make that possible in practice, it is often required to employ a set of skills incorporating knowledge in financial theory, engineering methods, mathematical tools, and programming practice — which altogether constitute the field known as *financial engineering*.

Many of theoretical pricing models for financial derivatives can be represented using partial differential equations (PDEs). In many cases, those equations are time-dependent, of high spatial dimension, and with challenging boundary conditions — which most often makes them analytically unsolvable. In those cases, we need to utilize numerical approximation as a mean of estimating their solution. The fields of *numerical analysis* and *scientific computing* are concerned with obtaining approximate solutions while maintaining reasonable bounds on errors. Unfortunately, there is no universal numerical method which can be used to solve all problems of this type efficiently. In fact, there are tremendously many numerical methods for solving different types of differential equations, and all those methods are featured with their own limitations in performance, stability, and accuracy — mostly dependent on details of the problems they aim to solve. Therefore, carefully selecting and developing numerical methods for particular

applications has been the only way to build efficient PDE solvers in ongoing practice.

In this thesis, we present RBF-FD as a recent numerical method with potential to efficiently approximate solutions of PDEs in finance. Over the past years, besides the purely academic development and research of its numerical properties, the method has been mainly applied for simulations of atmospheric phenomena. As its name suggests, the RBF-FD method is of a finite difference type, from the radial basis function family. As a finite difference method, RBF-FD approximates differential equations by linear systems of algebraic equations, known as difference equations. Radial basis functions (RBFs) are used as interpolants that enable local approximations of differential operators that are necessary for constructing the difference equations. Constructed like that, the method is featured with a sparse matrix of the linear system of difference equations, and it is relatively simple to implement like the standard finite difference methods. Moreover, the method is mesh-free, meaning that it does not require a structured discretization of the computational domain which makes it equally easy to use in spaces of different dimensions, and it is of a customizable order of accuracy — which are the features it inherits from the global radial basis function approximations. It is those properties that led us to recognize RBF-FD as a method with high potential for efficiently solving some analytically unfeasible and computationally challenging pricing problems in finance.

Nevertheless, being a young method, RBF-FD is still under intense development and we face many challenges when moving from simple theoretical cases toward more complex real-world applications. The core of this thesis deals with finding solutions for overcoming obstacles when financial derivatives are priced using RBF-FD to solve PDEs with several spatial dimensions. Thus, it represents a contribution to making the RBF-FD methods more reliable and efficient for use in financial applications.

The results in this thesis demonstrate how to successfully apply RBF-FD to different problems in finance by studying the effects of RBF shape parameters for Gaussian RBF-FD approximations, improving the approximation of differential operators by using polyharmonic splines with polynomials, constructing suitable node layouts, and smoothing of the initial data.

Additionally, we compare the RBF-FD method against other available methods on a plethora of pricing problems to give an objective image of the method's performance.

Sammanfattning

Syftet med denna avhandling är att presentera forskningsfronten för finita differenser genererade via radiella basfunktioner (RBF-FD) för prissättning av finansiella derivat. Baserat på de sex bifogade artiklarna presenterar denna avhandling en detaljerad överblick av egenskaper hos RBF-FD metoder samt de svårigheter som uppstår då dessa används inom finansiella tillämpningar. Vidare ämnar denna avhandling motivera fortsatt utveckling av RBF-FD inom finans.

Inom värdlens finansiella markader handlas finansiella derivat, så som terminer, optioner med mera, i stora volymer. Värdet hos alla tillgångar underliggande utstående finansiella transaktioner är flera gånger högre än bruttovärldsprodukten. Finansiella derivat är det styrmedel som används mest inom risk hedging, spekulationsbaserad investering, samt utnyttjande av arbitrage möjligheter. Av denna anledning är det synnerligen viktigt att, vid varje given tid, veta priset på dessa finansiella derivat. För att göra detta möjligt är det nödvändigt att använda färdigheter så som kunskaper från finansiell teori, ingenjörsmässiga metoder, matematiska verktyg, och programmeringserfarenhet. Dessa formar tillsammans området *finansiell ingenjörskonst*.

Många av de teoretiska modeller som finns för att prissätta finansiella derivat kan beskrivas som partiella differentialekvationer. I många fall är dessa ekvationer tidsberoende, högdimensionalla och har svårhanterliga randvillkår – vilket oftast gör de analytiskt olösbara. När detta är fallet behöver numeriska metoder användas för att approximera lösningen. Områderna numerisk analys och beräkningsvetenskap handlar om att få fram approximativa lösningar och samtidig garantera att felet i lösningen hålls inom rimliga begränsningar. Dessväre finns ingen universell numerisk metod som kan användas för att lösa alla problem effektivt. Dett finns faktiskt en uppsjö av olika numeriska metoder för att lösa olika typer av differentialekvationer och alla dessa metoder har sina egna begräsningar i effektivitet, stabilitet och noggrannhnet — oftast beroende på egenskaper hos problemet som ska lösas. Att noggrant välja och utvecka numeriska metoder för specifika problem är därför det enda sättet att konstruera effektiva PDE-lösare.

I denna avhandling presenteras RBF-FD som en nyutvecklad metod med potential att effektivt lösa PDE:er approximativt inom finans. Under de senaste åren, förutom rent akademisk utveckling och forskning kring dess numeriska egenskaper, har metoden huvudsakligen används för simuleringar av atmosfäriska fenomen. Som dess namn antyder, är RBF-FD baserat på finita differenser, från radiella basfunktions klassen. Som vanliga finita differenser, approximerar RBF-FD differential ekvationer som linjära system av algebraiska ekvationer, också känt som differensekvationer. Radiella basfuntioner (RBFs) används för att bilda interpolanter vilket möjliggör lokala approximaitoner av differential operatorer som behövs för att konstruera differensekvationer. Genom att konstruera metoden på detta vis får det linjära systemet en gles matris och är relativt enkel att implementera, som vanliga finita differenser metoder. Vidare är metoden nät-fri, vilket betyder att den inte kräver ett strukturerad diskretisering av beräkningsdomänen, vilket gör den lika enkel att använda oberoende av den rumsliga dimensionen, och den har skräddarsydd noggranhetsordning — vilka alla är egenskaper som metoden ärver från globala radiella basfunktions-approximationer. Det är dessa egenskaper som gör att vi kan identifiera RBF-FD som en metod med hög potential för att effektivt kunna lösa analytiskt olösbara och beräkningsmässigt utmanande prissättningsproblem inom finans.

Likväl är RBF-FD en ung metod som fortfarande befinner sig under intensiv utveckling och vi möter många utmaningar när vi går från enkla teoretiska fall mot mer komplexa problem från riktiga tillämpningar. Kärnarn i denna avhandling är att hitta sätt att överbrygga de hinder som uppstår då finansiella deriva prissätts genom att lösa PDE:er med hjälp av RBF-FD i flera rumsliga dimensioner. Därför representerar denna avhandling ett bidrag till att göra RBF-FD metoder mer tillförlitliga och effektiva för användning inom finansiella tillämpningar.

Resultaten i denna avhandling demonstrerar för det första hur man framgångsrikt tillämpar RBF-FD på olika problem i finans genom att studera effektet av RBFs formparameter för Gaussiska RBF-FD approximationer, för det andra hur man förbättrar approximationen av differential operatorer genom att använda polyharmoniska splines med polynom, för det tredje hur man distribuerar nodpunkter på ett lämpligt sätt, och slutligen hur man slätar ut initialvillkåren.

Vidare jämför vi RBF-FD mot andra metoder i den uppsjö av metoder som finns tillgängliga för prissättningsproblem. Detta bidrar till att skapa en objektiv bild av metodens effektivitet.

Преглед

Већим делом свога тока река Дрина протиче кроз тесне гудуре између стрмих планина или кроз дубоке кањоне окомито одсечених обала. Само на неколико места речног тока њене се обале проширују у отворене долине и стварају, било на једној било на обе стране реке, жупне, делимично равне, делимично таласасте пределе, подесне за обрађивање и насеља. Такво једно проширење настаје и овде, код Вишеграда, на месту где Дрина избија у наглом завоју из дубоког и уског теснаца који стварају Бутковe Стијене и Узавничке планине. Заокрет који ту прави Дрина необично је оштар а планине са обе стране тако су стрме и толико ублизу да изгледају као затворен масив из којег река извире право, као из мрког зида. Али ту се планине одједном размичу у неправилан амфитеатар чији промер на најширем месту није већи од петнаестак километара ваздушне линије.

На том месту где Дрина избија целом тежином своје водене масе, зелене и запењене, из привидно затвореног склопа црних и стрмих планина, стоји велики и складно срезани мост од камена, са једанаест лукова широког распона. Од тог моста, као од основице, шири се лепезасто цела таласаста долина, са вишеградском касабом и њеном околином, са засеоцима полеглим у превоје брежуљака, прекривена њивама, испашама и шљивицима, изукрштана међама и плотовима и пошкропљена шумарцима и ретким скуповима црногорице. Тако, посматрано са дна видика, изгледа као да из широких лукова белог моста тече и разлива се не само зелена Дрина него и цео тај жупни и питоми простор, са свим што је на њему и јужним небом над њим.

На десној обали реке, почињући од самог моста, налази се главнина касабе, са чаршијом, делом у равници, а делом на обронцима брегова. На другој страни моста, дуж леве обале, протеже се Малухино поље, раштркано предграђе око друма који води пут Сарајева. Тако мост, састављајући два краја сарајевског друма, веже касабу са њеним предграђем.

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