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## Abstract

The purpose of this thesis is to present state of the art in radial basis function generated finite difference (RBF-FD) methods for pricing of financial derivatives. This work provides a detailed overview of RBF-FD properties and challenges that arise when the RBF-FD methods are used in financial applications.

Across the financial markets of the world, financial derivatives such as futures, options, and others, are traded in substantial volumes. Knowing the prices of those financial instruments at any given time is of utmost importance. Many of the theoretical pricing models for financial derivatives can be represented using multi-dimensional PDEs, which are in most cases analytically unsolvable.

We present RBF-FD as a recent numerical method with potential to efficiently approximate solutions of PDEs in finance. As its name suggests, the RBF-FD method is of a finite difference (FD) type, from the radial basis function (RBF) group of methods. When used to approximate differential operators, the method is featured with a sparse differentiation matrix, and it is relatively simple to implement — like the standard FD methods. Moreover, the method is mesh-free, meaning that it does not require a structured discretization of the computational domain, and it is of a customizable order of accuracy — which are the features it inherits from the global RBF approximations.

The results in this thesis demonstrate how to successfully apply RBF-FD to different pricing problems by studying the effects of RBF shape parameters for Gaussian RBF-FD approximations, improving the approximation of differential operators in multiple dimensions by using polyharmonic splines augmented with polynomials, constructing suitable node layouts, and smoothing of the initial data to enable high order convergence of the method. Finally, we compare RBF-FD with other available methods on a plethora of pricing problems to form an objective image of the method's performance.

Future development of RBF-FD is expected to result in a solid mesh-free high order method for multi-dimensional PDEs, that can be used together with dimension reduction techniques to efficiently solve problems of high dimensionality that we often encounter in finance.

*"These violent delights have violent ends"*  
*(Romeo and Juliet: Act 2, Scene 6, Line 9)*



# List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I S. Milovanović and L. von Sydow. *Radial Basis Function generated Finite Differences for Option Pricing Problems*. Comp. Math. Appl., 75(4):1462–1481, 2017.
- II Slobodan Milovanović. *Pricing Financial Derivatives using Radial Basis Function generated Finite Differences with Polyharmonic Splines on Smoothly Varying Node Layouts*. arXiv preprint, arXiv:1808.02365[q-fin.CP], 2018.
- III S. Milovanović and L. von Sydow. *A High Order Method for Pricing of Financial Derivatives using Radial Basis Function generated Finite Differences*. arXiv preprint, arXiv:xxxx.yyyy[q-fin.CP], 2018.
- IV S. Milovanović and V. Shcherbakov. *Pricing Derivatives under Multiple Stochastic Factors by Localized Radial Basis Function Methods*. Journ. Comp. Fin., (in review), 2018.
- V L. von Sydow, L. J. Höök, E. Larsson, E. Lindström, S. Milovanović, J. Persson, V. Shcherbakov, Y. Shpolyanskiy, S. Sirén, J. Toivanen, J. Waldén, M. Wiktorsson, J. Levesley, J. Li, C. W. Oosterlee, M. J. Ruijter, A. Toropov, and Y. Zhao. *BENCHOP — The BENCHmarking Project in Option Pricing*. Int. Journ. Comp. Math., 92(12): 2361–2379, 2015.
- VI L. von Sydow, S. Milovanović, E. Larsson, K. in 't Hout, M. Wiktorsson, C. W. Oosterlee, V. Shcherbakov, M. Wyns, A. Leitao, S. Jain, T. Haentjens, and J. Waldén. *BENCHOP: The BENCHmarking Project in Option Pricing — Stochastic and local volatility problems*. Int. Journ. Comp. Math., (in review), 2018.

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# Related Work

The following ongoing project, although not included, is related to the contents of this thesis.

E. Larsson, S. Milovanović, V. Shcherbakov, L. von Sydow, et al.\* *BENCHOP: The BENCHmarking Project in Option Pricing—Basket Options.* manuscript in preparation, 2018.

## Supervised Theses

The following bachelor and master theses were supervised by the author.

T. Sundvall and D. Trång. *Examination of Impact from Different Boundary Conditions on the 2D Black–Scholes Model.* Bachelor’s thesis, Department of Engineering Sciences, Uppsala University, 2014, supervised by: L. von Sydow and S. Milovanović.

A. Abrahamsson and R. Pettersson. *Smoothing of Initial Conditions for High Order Approximations in Option Pricing.* Bachelor’s thesis, Department of Engineering Sciences, Uppsala University, 2016, supervised by: L. von Sydow and S. Milovanović.

Robin Eriksson. *Stencil Study for RBF-FD in Option Pricing.* Bachelor’s thesis, Department of Engineering Sciences, Uppsala University, 2016, supervised by: S. Milovanović and L. von Sydow.

Stephane Dumanois. *Least Squares Radial Basis Function generated Finite Differences for Option Pricing.* Master’s thesis, Department of Mathematics, Uppsala University, 2016, supervised by: S. Milovanović and L. von Sydow.

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# 1. Introduction

The purpose of this thesis is to present state of the art in radial basis function generated finite difference (RBF-FD) methods for pricing of financial derivatives. Based on the six appended papers which are referred to by their Roman numerals, this work provides a detailed overview of RBF-FD properties and challenges that arise when the RBF-FD methods are used in financial applications. Moreover, with this thesis, we aim to motivate further development of RBF-FD for solving multi-dimensional partial differential equations (PDEs) in finance.

Across the financial markets of the world, financial derivatives such as futures, options, and others, are traded in substantial volumes. The value of all assets that underly outstanding derivatives transactions is several times larger than the gross world product (GWP). Financial derivatives are the most commonly used instruments when it comes to hedging risks, speculation based investing, and performing arbitrage. Therefore, knowing the prices of those financial instruments is of utmost importance at any given time. In order to make that possible in practice, it is often required to employ a set of skills incorporating knowledge in financial theory, engineering methods, mathematical tools, and programming practice — which altogether constitute the field known as *financial engineering*.

Many of the theoretical pricing models for financial derivatives can be represented using PDEs. In many cases, those equations are time-dependent, of high spatial dimension, and with challenging boundary conditions — which most often makes them analytically unsolvable. In those cases, we need to utilize numerical approximation as a mean of estimating their solution. The fields of *numerical analysis* and *scientific computing* are concerned with obtaining approximate solutions while maintaining reasonable bounds on errors. Unfortunately, there is no universal numerical method which can be used to solve all problems of this type efficiently. In fact, there are tremendously many numerical methods for solving different types of differential equations, and all those methods

are featured with their own limitations in performance, stability, and accuracy — mostly dependent on details of the problems they aim to solve. Therefore, carefully selecting and developing numerical methods for particular applications has been the only way to build efficient PDE solvers in ongoing practice.

RBF-FD is a recent numerical method with potential to efficiently approximate solutions of PDEs in finance. Over the past years, besides the purely academic development and research of its numerical properties, the method has been mainly applied for simulations of atmospheric phenomena. As its name suggests, the RBF-FD method is of a finite difference type, from the radial basis function family. As a finite difference method, RBF-FD approximates differential equations by linear systems of algebraic equations, known as difference equations. Radial basis functions (RBFs) are used as interpolants that enable local approximations of differential operators that are necessary for constructing the difference equations. Constructed like that, the method is featured with a sparse matrix of the linear system of difference equations, and it is relatively simple to implement — like the standard finite difference methods. Moreover, the method is mesh-free, meaning that it does not require a structured discretization of the computational domain which makes it equally easy to use in spaces of different dimensions, and it is of a customizable order of accuracy — which are the features it inherits from the global radial basis function approximations. It is those properties that led us to recognize RBF-FD as a method with high potential for efficiently approximating the solutions of some analytically unsolvable and computationally challenging pricing problems in finance.

Nevertheless, being a young method, RBF-FD is still under intense development, and we face many challenges when moving from simple theoretical cases toward more complex real-world applications. The core of this thesis deals with finding solutions for overcoming obstacles when financial derivatives are priced using RBF-FD to solve PDEs of multiple spatial dimensions. Thus, it represents a contribution to making the RBF-FD methods more reliable and efficient for use in financial applications.

The rest of this manuscript is organized as follows. We introduce and define financial derivatives in Chapter 2. An overview of some popular financial models and techniques for the pricing of options are presented

in Chapter 3. We present the features and properties of RBF-FD methods for solving PDEs in finance in Chapter 4. Finally, we conclude with some unsolved challenges and suggestions for further development of the RBF-FD method for financial applications in Chapter 5.

## 2. Financial Derivatives

A *financial derivative* is a market instrument whose value depends on the values of some other underlying variables. Most often, those underlying variables are the prices of another traded asset (e.g., a stock underlying stock options), but they may as well be almost any variables of stochastic nature (e.g., air temperatures underlying weather derivatives). There are numerous financial derivatives in existence, available for almost every type of investment asset, ranging from agricultural grains to cryptocurrencies. Futures and options are best known as *exchange-traded* derivatives, standardized to be bought and sold on derivatives exchanges (e.g., Chicago Mercantile Exchange for futures and Chicago Board Options Exchange for options). On the other hand, much larger volumes of financial derivatives are traded bilaterally *over-the-counter* in a highly customizable fashion. That gave birth to many contracts with tailored properties such as forward contracts, swaps, exotic options, and other custom financial instruments.

When it comes to traders, three categories can be readily identified: *hedgers*, *speculators*, and *arbitrageurs* [1]. Hedgers use derivatives to reduce risks from potential future movements in a market variable, speculators use them to bet on the future outcome of a market variable, and arbitrageurs aim at making riskless profit by exploiting discrepancies in values of the same underlying variable traded under different derivatives or across different markets. Thanks to the traders, derivatives markets have been highly liquid over the past decades as many of the traders find trading derivatives more attractive compared to trading their underlying assets.

Financial derivatives are traded in extremely large volumes across the planet. The estimated total notional value of these financial instruments has been above half a quadrillion of USD during the current decade [2]. That is about an order of magnitude larger than GWP [3]. Moreover, derivatives markets have received significant criticism due to their role in the most recent global financial crisis. As a result of the crisis, strict

regulations in trading of derivatives have been introduced in order to increase transparency on the markets, improve market efficiency, and reduce systemic risk. Now, in the post-crisis period, methods for valuation of financial derivatives are still under the spotlight of financial institutions, as they look for the most efficient ways to solve mathematical problems stemming from the regulations. Those problems involve estimation of value adjustments (known as XVA) that keep the counterparty risks under control.

In order to bring financial derivatives closer to the mathematical framework, it is useful for us to define several of their features. We assume that the contract representing a particular financial derivative is signed at time  $t = t_0 \equiv 0$  and expires at  $t = T$ , where  $T$  is also known as the time of *maturity* of the contract. The contract is issued on the underlying stochastic variable  $S(t)$ . At the expiration of the contract, the holder receives payoff  $g(S(T))$ , which is equivalent to the value of the financial derivative at the time of maturity  $T$ , i.e.,  $u(S(T), T) = g(S(T))$ . The value of the contract is represented by a function  $u(t, S(t))$ .

When it comes to hierarchy of financial derivatives, we can see most of them either as a type of a forward/futures contract, or as a type of an option. Therefore, it is common to study forwards and futures as binding contracts ( $-\infty < g(S(T)) < \infty$ ), and options as non-obligatory contracts towards their holders ( $0 \leq g(S(T)) < \infty$ ). In the following sections, we consider them in more detail.

## 2.1 Forwards and Futures

A *forward* contract is an agreement between two parties signed at  $t = t_0$  to buy or sell an underlying  $S(t)$  at a certain future time  $T$  for a certain price  $K(t_0) = K_0$ . The price  $K(t)$  is called the *forward price* of the contract, and it is determined at time  $t_0$  in such a way that the value of the forward contract at the time of signing is equal to zero, i.e.,  $u(t_0, S(t_0)) = u_0 = 0$ . One of the parties in the contract takes a *long* position and agrees to the payoff

$$g_l(S(T)) = S(T) - K_0.$$

The other party assumes a *short* position and agrees to sell  $S(t)$  at the same time  $T$  for the stipulated forward price  $K_0$ , effectively obliging to

the payoff

$$g_s(S(T)) = K_0 - S(T).$$

Forward contracts are traded in over-the-counter markets and may be further customized according to the preferences of the signing parties.

A *futures* contract is an exchange-traded, and thus standardized financial derivative, that is very similar to a forward contract. It is in agreement signed at no cost between two parties at  $t = t_0$  to buy or sell an underlying  $S(t)$  at a certain time  $T$ . The principal difference from the forward contract lies in the way in which the payments are realized. Namely, at every point in time  $t_0 \leq t \leq T$ , there exists a price  $K(t)$ , now called the *futures price* of the contract, that is quoted on the exchange. At time  $T$ , the long position holder of the contract is entitled to the payoff

$$g_l(S(T)) = S(T) - K(T),$$

while the short position holder gets

$$g_s(S(T)) = K(T) - S(T).$$

Moreover, during an arbitrary time interval  $(t_i, t_j]$ , where  $t_0 \leq t_i < t_j < T$ , the long holder of the contract receives the amount  $K(t_j) - K(t_i)$ , and the short holder receives  $K(t_i) - K(t_j)$ . The futures price  $K(t)$  evolves in such way that obtaining the futures contract at any time  $t_0 \leq t \leq T$  incurs a zero cost, i.e.,  $u(t, S(t)) = 0$ .

As far as the pricing of forwards and futures is concerned, it is clear that these contracts are designed in such a way that their prices are equal to zero at the signing. Thus, computational problems of interest here are related to fairly determining the defined forward and futures prices. For more details on forwards and futures, it is wise to turn to [4, 1].

## 2.2 Options

An *option* is a contract that gives its holder the right, but not the obligation to buy or sell an underlying  $S(t)$  by a certain time of maturity  $T$  for a certain price  $K$ . The price  $K$  in the contract is known as the strike price. If the contract gives the buying right to its owner, then it is called a *call* option, and if it gives the selling right, it is called a *put* option. Call

options are characterized with

$$g_c(S(T)) = \max(S(T) - K, 0), \quad (2.1)$$

and put options with

$$g_p(S(T)) = \max(K - S(T), 0), \quad (2.2)$$

as their respective payoff functions. Options that can be exercised at any time  $t_0 < t \leq T$  are called *American* options, and options that can be exercised only at time  $t = T$  are known as *European* options. Since options are traded both on exchanges and in over-the-counter markets, there are many more types of them (e.g., binary options, barrier options, Asian options, Bermudan options, and other exotic options) — as the ways of customizing them are limitless. For instance, *rainbow* options are defined in such a way that their payoffs may depend on more than one underlying asset, consequently requiring a multi-dimensional pricing model in order to estimate their value. An example of such a multi-asset derivative is an arithmetic European call basket option issued on  $D$  underlying assets  $S_1, \dots, S_D$ , whose payoff function is

$$g_{cb}(S_1(T), \dots, S_D(T)) = \max\left(\frac{1}{D} \sum_{d=1}^D S_d(T) - K, 0\right). \quad (2.3)$$

Moreover, for a given underlying  $S$ , there may be a large number of options with different dates of expiration  $T$ , and different strike prices  $K$ .

Due to their versatility, options have been among the most popular financial derivatives on the financial markets, and many examples of their applications can be found in [1].

# 3. Option Pricing

We emphasize that an option gives the right to the holder to do something, and that the holder does not need to use that right. That is the main difference between options and other financial derivatives. Whereas it costs nothing to buy a forward or futures contract, there is always a non-negative price for acquiring an option. That very detail is the cornerstone of one of the most involving fundamental problems in financial markets, known as *option pricing*. Depending on the option characteristics, the pricing problem can be as trivial as deriving an analytical pricing formula — such is the case for the standard European call option with certain market assumptions. Nevertheless, in many other cases of option valuation, we are faced against an eternal struggle of balancing between reasonable market assumptions for deriving delicate mathematical models and developing efficient numerical solvers that can estimate the solutions of the equations posed by those models.

As the option gives stipulated rights, but not the obligations to their holder, it is natural to assume that this contract must have some objective non-negative value at any time. The central task of option pricing is to objectively determine the fair value of an option at any given time  $t \leq T$ . The fundamental mathematical framework for approaching this problem is the *arbitrage theory*. In order to model option prices, the theory heavily relies on carefully argued assumptions about the market and mathematical ingredients such as martingale measures, stochastic differential equations (SDEs), Itô calculus, Feynman–Kac representations, and PDEs. We refer to these topics throughout the manuscript in limited capacity, as the detailed definitions and proofs can be found elegantly presented in [5].

## 3.1 Market Models

In order to be able to price an option, we need a set of assumptions that can be used to build a financial market model. The models range

from the simple ones capturing a rough approximate picture of reality to extremely advanced ones aimed at capturing very fine details of the market. Once the model is defined, we can set up an option pricing problem that needs to be solved in order to estimate the option value. Difficulty of such pricing problems strongly depends on the complexity of the chosen market model as well as on the complexity of the specifics in the option contract that we want to price.

### 3.1.1 Black–Scholes–Merton Model

We start by considering a plain European option on a stock that does not pay dividends, under the famous Black–Scholes–Merton model [6, 7]. Creation of that model in 1973 is considered as one of the most successful quantitative breakthroughs in social sciences, initiating a pricing framework that still keeps occupied thousands of researchers across financial institutions and universities of the world. That was recognized by the Royal Swedish Academy of Sciences, when the *Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel* was awarded to Robert C. Merton and Myron S. Scholes in 1997, while Fischer S. Black was credited with equal contribution since he had passed away two years before the prize was awarded.

The main feature of the Black–Scholes–Merton model is that it allows the prices of European call and put options to be calculated analytically using parameters that are either directly observable on the market or can be easily estimated. The model is still widely used as a benchmark, although more advanced models have been developed over the years to take into account more realistic features of asset price dynamics, such as jumps and stochastic volatility. Being able to calculate prices of some options analytically makes estimation of the model parameters simple — which is useful for calibration of more advanced models and pricing of more complex financial instruments.

The model consists of two assets, a riskless bond  $B(t)$  and a risky stock  $S(t)$ , with dynamics given by the following SDEs

$$dB(t) = rB(t) dt, \quad (3.1)$$

$$dS(t) = \mu S(t) + \sigma S(t) dW(t), \quad (3.2)$$

where  $r$  is the risk-neutral interest rate,  $\mu$  is the drift coefficient, and  $\sigma$  is the volatility of the stock — all three being constant in the model. Moreover,  $W(t)$  is the Wiener process.

The Black–Scholes–Merton model stands on several important assumptions. The main assumption is that the considered financial market is arbitrage free, meaning that it is not possible to make positive earnings on the market without being exposed to risk. The next assumption states that the market is complete and efficient, which means that every contract on the market can be hedged and that the market prices fully reflect all available information. Those assumptions allow us to determine a unique price of the option whose payoff function is  $g(S(T))$ , using the following valuation under the risk-neutral measure  $\mathbb{Q}$ ,

$$u(S(t), t) = \exp(-r(T-t)) \mathbb{E}^{\mathbb{Q}} [g(S(T))]. \quad (3.3)$$

That effectively means that the expected value is calculated on an adapted dynamics by using  $r$  instead of  $\mu$  as the drift constant of the stochastic process  $S(t)$  defined in (3.2).

Moreover, using the Itô's lemma and the Feynman–Kac theorem, we can equivalently express the option price as the solution of the following PDE, known as the Black–Scholes–Merton equation

$$\frac{\partial u}{\partial t} + rs \frac{\partial u}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 u}{\partial s^2} - ru = 0, \\ u(s, T) = g(s), \quad (3.4)$$

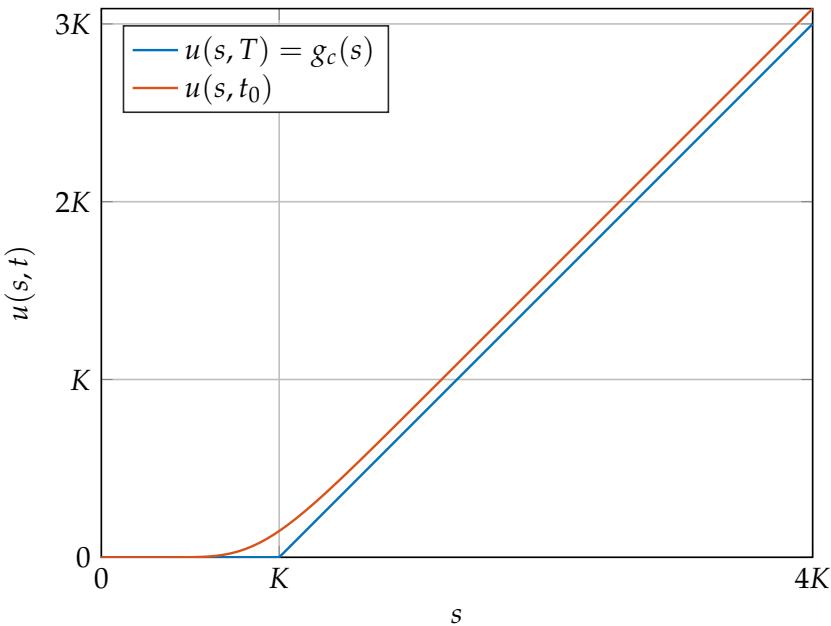
where  $s$  is the deterministic representation of the stochastic asset price  $S$ . Equation (3.4) is a parabolic PDE that has an analytical solution  $u = u(s, t)$  in case of European call and put options.

In order to make better trading decisions, investors often look at the hedging parameters, which are also known as the *greeks*. The most commonly used ones are *delta*  $\Delta = \frac{\partial u}{\partial s}$ , *gamma*  $\Gamma = \frac{\partial^2 u}{\partial s^2}$ , and *vega*  $\nu = \frac{\partial u}{\partial \sigma}$ . As these hedging parameters represent risk sensitivities, being able to compute them is of great importance.

We can use this basic framework to price financial derivatives with different payoffs or extend it in order to be able to valuate options with different underlying assets (e.g., stocks that pay discrete dividends). Also, we can further adapt the model to capture different market features more accurately. (e.g., introduce local volatility instead of the constant one).

Moreover, it is sometimes beneficial to use the Merton model [8] to describe underlying assets with jumps. On the other hand, stochastic volatility models, such as the Heston model presented in Section 3.1.2, are useful when there are prominent volatility smiles in the underlying asset. To push things even further, it is not uncommon to have a stochastic volatility model with jumps — the most known representative is the Bates model [9]. An overview of such extensions of the Black–Scholes–Merton framework can be seen in **Paper V**.

$$r = 0.03, \sigma = 0.15, T = 3$$



**Figure 3.1.** An example of a European call option price  $u(s, t)$  calculated using the Black–Scholes–Merton model, and its payoff function  $g_c(s)$ .

To price multi-asset financial derivatives, such as rainbow options issued on  $D$  underlying assets  $S_1, S_2, \dots, S_D$ , we consider a multi-dimensional analogue to (3.1) and (3.2)

$$\begin{aligned} dB(t) &= rB(t) dt, \\ dS_1(t) &= \mu_1 S_1(t) dt + \sigma_1 S_1(t) dW_1(t), \\ &\vdots \\ dS_D(t) &= \mu_D S_D(t) dt + \sigma_D S_D(t) dW_D(t), \end{aligned} \tag{3.5}$$

where the Wiener processes are correlated such that  $dW_i(t) dW_j(t) = \rho_{ij} dt$ . In this high-dimensional setting, an option with the payoff function  $g(S_1(T), \dots, S_D(T))$ , can be priced by solving the corresponding high-dimensional Black–Scholes–Merton equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}_b u &= 0, \\ u(s_1, \dots, s_D, T) &= g(s_1, \dots, s_D), \end{aligned} \quad (3.6)$$

where

$$\mathcal{L}_b u \equiv r \sum_i^D s_i \frac{\partial u}{\partial s_i} + \frac{1}{2} \sum_{i,j}^D \rho_{i,j} \sigma_i \sigma_j s_i s_j \frac{\partial^2 u}{\partial s_i \partial s_j} - ru. \quad (3.7)$$

We observe (3.6) as a time-dependent PDE with  $D$  spatial dimensions.

When it comes to American options, since these financial derivatives can be exercised at any  $t \leq T$ , as opposed to the European options (that can only be exercised at  $t = T$ ), instead of using a PDE as a model, we formulate the pricing task as a linear complementarity problem (LCP)

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}_b u &\geq 0, \\ u(s_1, \dots, s_D, t) &\geq g(s_1, \dots, s_D), \\ \left( \frac{\partial u}{\partial t} + \mathcal{L}_b u \right) \left( u(s_1, \dots, s_D, t) - g(s_1, \dots, s_D) \right) &= 0, \\ u(s_1, \dots, s_D, T) &= g(s_1, \dots, s_D). \end{aligned} \quad (3.8)$$

This formulation also applies to pricing of a single-asset American option by choosing  $D = 1$ .

### 3.1.2 Multi-Factor Models

Another direction in development of pricing models is to include more stochastic factors. Models with multiple stochastic factors allow for better simulation of market features compared to the standard Black–Scholes–Merton formulation, which is known to fall short in capturing heavy tails of return distributions and volatility smiles. Therefore, various models with local volatilities, local stochastic volatilities, stochastic interest rates, and their combinations have been getting popular. In this section, we present two models with multiple stochastic factors that are used for pricing options.

The attention to local volatility models started with [10]. The first multi-factor model that we introduce is the Heston model [11], featured with a stochastic volatility.

The adapted dynamics for this model is as follows

$$dS(t) = rS(t) dt + \sqrt{V(t)} S(t) dW_s(t), \quad (3.9)$$

$$dV(t) = \kappa(\eta - V(t)) dt + \sigma \sqrt{V(t)} dW_v(t), \quad (3.10)$$

where  $V(t)$  is the stochastic volatility,  $\sigma$  is the constant volatility of volatility,  $\kappa$  is the speed of mean reversion of the volatility process,  $\eta$  is the mean reversion level,  $r$  is the risk-free interest rate,  $W_s(t)$  and  $W_v(t)$  are correlated Wiener processes with constant correlation  $\rho$ , i.e.,  $dW_s(t) dW_v(t) = \rho dt$ . After using the Itô's lemma and the Feynman–Kac theorem, the PDE for the Heston model reads as

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}_h u &= 0, \\ u(s, v, T) &= g(s), \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \mathcal{L}_h u &\equiv \frac{1}{2} vs^2 \frac{\partial^2 u}{\partial s^2} + \rho \sigma vs \frac{\partial^2 u}{\partial s \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 u}{\partial v^2} \\ &\quad + rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} - ru, \end{aligned} \quad (3.12)$$

$s$  and  $v$  are deterministic representations of the stochastic asset price and volatility processes, respectively.

When it comes to path dependent options, the Heston model has a great advantage over the Black–Scholes–Merton model and models with deterministic local volatility. There is clear evidence that in practice the volatility of asset prices is in itself random, and cannot be simply described as a function of time and underlying strike price [12].

The Heston–Hull–White model [13, 14], is an enhancement of the Heston stochastic volatility model. The improvement consists of adding a stochastic interest rate that follows the Hull–White process [15], as the interest rates on the market are not constant. The model is useful when pricing long-term derivatives in which we observe an implied volatility smile in the underlying asset. Another notable property of the

Hull–White model is that the interest rates can be negative, as nowadays happens in some economies.

The adapted dynamics for this model is as follows

$$dS(t) = R(t)S(t) dt + \sqrt{V(t)}S(t) dW_s(t), \quad (3.13)$$

$$dV(t) = \kappa(\eta - V(t)) dt + \sigma_v \sqrt{V(t)} dW_v(t), \quad (3.14)$$

$$dR(t) = a(b - R(t)) dt + \sigma_r dW_r(t), \quad (3.15)$$

where  $R(t)$  is the stochastic interest rate,  $a$  is the speed of mean reversion of the interest rate process,  $b$  is its mean reversion level,  $\sigma_r$  is its volatility,  $W_s(t)$ ,  $W_v(t)$ , and  $W_r(t)$  are correlated Wiener processes.

We can apply the Itô's lemma and the Feynman–Kac theorem to derive the pricing PDE

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}_w u &= 0, \\ u(s, v, r, T) &= g(s), \end{aligned} \quad (3.16)$$

where

$$\begin{aligned} \mathcal{L}_w u \equiv & \frac{1}{2}vs^2 \frac{\partial^2 u}{\partial s^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 u}{\partial r^2} + \\ & \rho_{sv}\sigma_v vs \frac{\partial^2 u}{\partial s \partial v} + \rho_{sr}\sigma_r \sqrt{vs} \frac{\partial^2 u}{\partial s \partial r} + \rho_{vr}\sigma_v \sigma_r \sqrt{v} \frac{\partial^2 u}{\partial v \partial r} + \\ & rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b - r) \frac{\partial u}{\partial r} - ru, \end{aligned} \quad (3.17)$$

Here, it becomes clear how advanced models easily grow in complexity, which in turn makes it difficult to calibrate and value them in practice. Several other multi-factor models are discussed in more detail in **Papers IV and VI**.

## 3.2 Pricing Methods

For a small number of cases, such as plain European call or put options under the Black–Scholes–Merton model, calculating the option price can be done by closed form solutions, derived using analytical methods. In some other cases, it is possible to approximate the solutions using semi-analytical schemes. Commonly used methods for pricing of financial

derivatives in the absence of analytical or semi-analytical solutions can be split into three main groups: stochastic methods, methods based on the Fourier transform, and deterministic methods. Performance of these methods when pricing several option types across different market models is presented in **Papers V** and **VI**.

### 3.2.1 Stochastic Methods

Stochastic methods, such as Monte Carlo (MC), aim at approximating option prices using the form showed in (3.3). The idea of estimating expectations by repeated random sampling was used in different forms for centuries, but it was officially defined in [16]. The first application of an MC method in option pricing was reported in 1977 for European options [17]. A least square MC method for pricing American options was introduced in 2001 [18], and more recently, a new regression based MC method, named stochastic grid bundling method (SGBM), has been developed for efficient pricing of early-exercise options and their hedging parameters [19]. Furthermore, quasi-MC [20] — methods that use deterministic sequences of numbers to boost convergence — became successful at efficiently tackling problems in hundreds of dimensions [21]. More recently, many advanced versions of MC methods have been developed, of which some of the most notable are multilevel MC methods [22], which are inspired by the multigrid ideas for the iterative solution of PDEs. Interestingly, in the time of publishing of this thesis, some pioneering approaches in development of quantum computing MC algorithms for pricing of financial derivatives have been made [23].

Discrete models like binomial trees that appeared in 1979 [24, 25], also fall in the group of stochastic methods. These models work by simulating stochastic trajectories of the underlying dynamics on predefined discrete lattices and are among the simplest nontrivial models of financial markets.

Stochastic methods are most suitable for multi-asset derivatives and multi-factor models — both of which, as well as their combinations, result in problems of high dimensionality. The classical versions of these methods are arguably easy to implement and use. MC methods are significantly less efficient than other methods when used for problems in

smaller dimensions, as their convergence rate is much slower in comparison.

### 3.2.2 Fourier Methods

This group consists of methods based on the Fourier transform such as Carr–Madan fast Fourier transform method [26]. More recently, Fourier-cosine series expansions (COS) for European options [27] and early-exercise options [28], showed to be extremely efficient in pricing. In 2012, the COS method was extended to higher dimensions [29]. The methods from this category are very fast and accurate, but they typically require existence of the characteristic function for the price process of the underlying asset in closed form, or at least its approximation — which is available for a fairly large class of the models, but not all.

### 3.2.3 Deterministic Methods

Deterministic methods are used to solve pricing problems in PDE form such as (3.4), by discretizing its differential operators.

The main methods in this category are the finite differences (FD). The first time an FD method was used for pricing of a contract was in 1976 [30], to solve a one-dimensional Black–Scholes–Merton equation. A few years later, FD schemes, together with MC methods, were established as a standard numerical approach for pricing financial derivatives when analytical solutions are not available [31]. Moreover, a notable operator splitting scheme was introduced in [32], enabling FD methods to price American options efficiently. Over the years, FD methods have been used to solve mostly one-dimensional and two-dimensional pricing problems. More recently, hierarchical approximation using sparse grids [33] and asymptotic expansions [34] of high-dimensional option pricing problems have been developed — enabling state of the art FD [35, 36] to be used for pricing high-dimensional options by solving a sequence of lower dimensional problems. Apparently, many high-dimensional pricing problems have such a configuration of volatilities and correlations that their effective dimensionality is low, and as such can be represented by a small number of lower dimensional components [37].

Although they are used less often, the finite element methods can excel in certain cases [38, 39, 40], and the same applies for finite volumes [41], which finds its use in convection dominated or degenerate cases.

Finally, RBF methods are a more recent group of deterministic methods to be used for option pricing — first time applied in 1999 for one-dimensional European options [42]. Ever since, these methods have been becoming popular, as they possess potential to cope with PDEs of moderately high dimensions.

### 3.2.4 Method Selection

Based on the presented details and the results reported in **Papers V** and **VI**, a basic guide for selecting an appropriate option pricing method is first to check if it is possible to calculate the solution to the pricing problem analytically. In case that is not possible, the next best option is a Fourier transform based method. Deterministic methods come into play as robust numerical schemes when Fourier methods are not applicable. Nevertheless, they often suffer from the curse of dimensionality as the degrees of freedom in the resulting approximations grow exponentially with the dimensionality of the problem. Therefore, if the pricing problem is of a higher dimensionality that cannot be reduced, Monte Carlo methods are the most common alternative.

Typically, deterministic methods are used to solve pricing problems of up to no more than three dimensions. In the following chapter, we present a localized RBF method that might become an alternative to Monte Carlo methods for moderately high-dimensional problems, i.e., of dimensionality three to five.

## 4. Radial Basis Function generated Finite Differences

Using the RBF methods for approximating solutions of PDEs dates back to the beginning of the nineties in the previous century [43, 44]. Ever since, these methods have been used in different fields, including financial engineering [42, 45, 46].

In order to apply an RBF method, we observe option pricing problems on the truncated computational domain  $\Omega \subset \mathbb{R}^d$  in the following PDE form

$$\frac{\partial}{\partial t} u(\underline{x}, t) + \mathcal{L}u(\underline{x}, t) = 0, \quad \underline{x} \in \Omega \quad (4.1)$$

$$\mathcal{B}u(\underline{x}, t) = f(\underline{x}, t), \quad \underline{x} \in \partial\Omega, \quad (4.2)$$

$$u(\underline{x}, T) = g(\underline{x}), \quad \underline{x} \in \Omega, \quad (4.3)$$

where  $u(\underline{x}, t)$  is the option price,  $\underline{x}$  is the spatial variable representing underlying assets and/or stochastic factors, with  $\mathcal{L}$  as the differential operator of the pricing model;  $\mathcal{B}$  is the boundary operator which together with the function  $f(\underline{x}, t)$  models the boundary conditions; initial data are defined by the terminal condition  $g(\underline{x})$ .

To construct a global RBF approximation in space, we scatter  $N$  nodes  $\underline{x}_j$ , where  $j = 1, \dots, N$ , across the computational domain  $\Omega$ . Then, we consider an interpolant

$$\tilde{u}(\underline{x}, t) = \sum_{j=1}^N \lambda_j(t) \phi(\|\underline{x} - \underline{x}_j\|), \quad (4.4)$$

where  $\phi$  is the RBF, and  $\lambda_j(t)$  are the time-dependent interpolation coefficients. At any time  $t$ , the value of the interpolant in every point  $\underline{x}$  only depends on the distance to the nodes and this expression is valid for any number of dimensions.

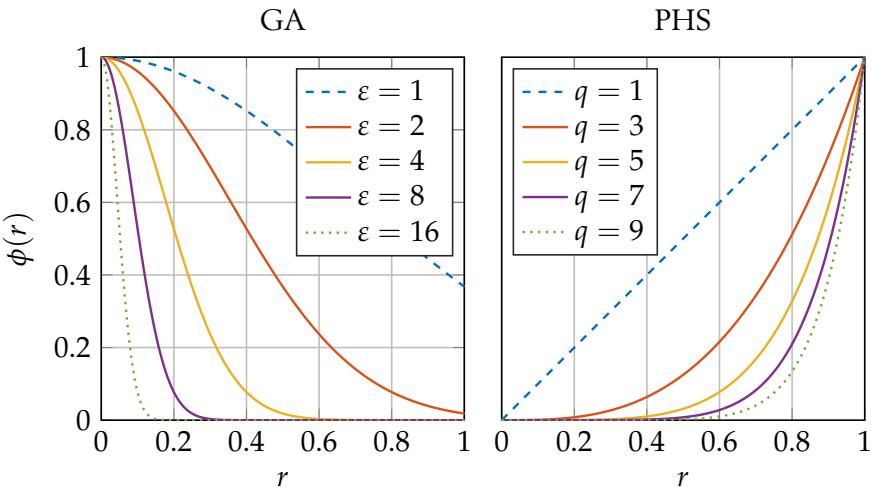
Some examples of commonly used RBFs are listed in **Table 4.1**, split into two groups. The first group in the table consists of infinitely smooth

RBFs that can provide spectral accuracy for interpolation and are featured with a shape parameter  $\varepsilon$ . The second kind is a piecewise smooth RBF that can give algebraic convergence for interpolation.

**Table 4.1.** Commonly used RBFs, where  $\varepsilon \in \mathbb{R}^+$  is the shape parameter for the infinitely smooth RBFs, and  $q \in \{2m - 1, m \in \mathbb{N}\}$  is the degree of the polyharmonic spline as a piecewise smooth RBF.

RBF	$\phi(r)$
Gaussian (GA)	$\exp(-\varepsilon^2 r^2)$
Multiquadric (MQ)	$\sqrt{1 + \varepsilon^2 r^2}$
Inverse Multiquadric (IMQ)	$1/\sqrt{1 + \varepsilon^2 r^2}$
Inverse Quadratic (IQ)	$1/(1 + \varepsilon^2 r^2)$
Polyharmonic Spline (PHS)	$r^q$

In this thesis, we consider GA and PHS basis functions for approximating solutions of the pricing equations. Those two RBFs are shown plotted on a unit domain in **Figure 4.1**.



**Figure 4.1.** Examples of GAs with different shape parameter values and PHSs of different degrees.

We can apply the global RBF method by collocating at the same  $x_j$  points through substituting (4.4) into (4.1). Thus, we obtain a dense linear system of ordinary differential equations (ODEs) of size  $N$ , where  $\lambda_j(t)$  are the unknowns. Starting from the terminal condition (4.3), we

can use a backward time integration method of our choice to compute the coefficients  $\lambda_j(t)$ , and therefore evaluate the interpolant  $\tilde{u}$  which approximates the option price.

Even though the global RBF methods possess desirable properties such as spectral convergence and mesh-free domain discretization, they are featured with dense system matrices which makes the method very computationally demanding. To overcome that weakness, several localized RBF approaches were introduced, among which radial basis function partition of unity (RBF-PU) methods [47] and RBF-FD [48, 49], are the most popular, and still actively developed. These localized RBF methods are featured with sparser system matrices while still maintaining great properties from the global RBF methods, such as being mesh-free and of high order.

The RBF-PU method has been used in finance for pricing multi-asset derivatives [50, 51, 52], and its performance when pricing one-dimensional options and their hedging parameters is also documented in **Paper V**. Moreover, RBF-PU is extensively compared with the RBF-FD method at solving multiple stochastic factors problems, which is reported in **Paper IV**. While in that paper both methods performed similarly, on a more comprehensive study with stochastic and local volatility problems, presented in **Paper VI** — RBF-FD showed as a robust method that performs more efficiently in most of the considered cases. As RBF-FD and RBF-PU are still in development, it is hard to say which method has a better future potential. Therefore, it is important for the field of computational finance that both of these methods continue developing.

## 4.1 Method Definition

In this thesis, we focus on development of the RBF-FD methods. RBF-FD can be seen as a kind of an FD method that belongs to the family of RBF methods. To construct an RBF-FD approximation, as firstly introduced by Andrei I. Tolstykh in 2000 [48], we can reuse the same  $N$  scattered nodes across the computational domain  $\Omega$ . For each node  $\underline{x}_j$ , we define an array of nodes  $\mathbf{x}_j$  consisting of  $n_j - 1$  neighboring nodes and  $\underline{x}_j$  itself, and consider it as a stencil of size  $n_j$  centered at  $\underline{x}_j$ . The differential operator  $\mathcal{L}$  defined in (4.1) is approximated in every node  $\underline{x}_j$  as

$$\mathcal{L}u(\underline{x}_j, t) \approx \sum_{i=1}^{n_j} w_j^i u_j^i \equiv \mathbf{w}_j u(\mathbf{x}_j, t), \quad j = 1, \dots, N, \quad (4.5)$$

where  $u_j^i = u(\underline{x}_j^i, t)$  and  $\underline{x}_j^i$  is a locally indexed node in  $\mathbf{x}_j$ , while  $\mathbf{w}_j$  is the array of differentiation weights for the stencil centered at  $\underline{x}_j$ . In the standard RBF-FD methods, the weights  $w_j^i$  are calculated by enforcing (4.5) to be exact for RBFs centered at each of the nodes in  $\mathbf{x}_j$ , yielding

$$\underbrace{\begin{bmatrix} \phi(\|\underline{x}_j^1 - \underline{x}_j^1\|) & \dots & \phi(\|\underline{x}_j^1 - \underline{x}_j^{n_j}\|) \\ \vdots & \ddots & \vdots \\ \phi(\|\underline{x}_j^{n_j} - \underline{x}_j^1\|) & \dots & \phi(\|\underline{x}_j^{n_j} - \underline{x}_j^{n_j}\|) \end{bmatrix}}_{\mathbf{A}_j} \underbrace{\begin{bmatrix} w_j^1 \\ \vdots \\ w_j^{n_j} \end{bmatrix}}_{\mathbf{w}_j} = \underbrace{\begin{bmatrix} \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^1\|) \\ \vdots \\ \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^{n_j}\|) \end{bmatrix}}_{\mathbf{l}_j}. \quad (4.6)$$

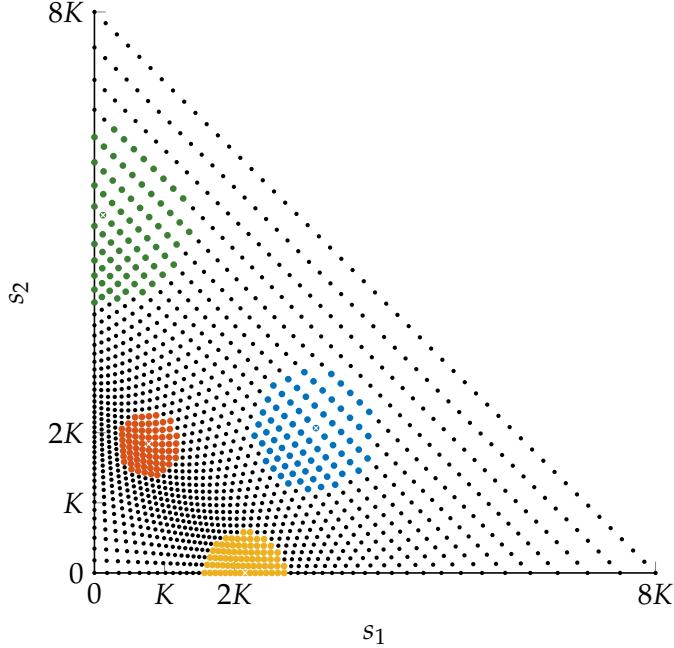
In theory on RBF interpolation, it is known that (4.6) forms a nonsingular system of equations. Therefore, a unique set of weights can be computed for each node  $\underline{x}_j$  by solving  $N$  linear systems of size  $n_j \times n_j$ . We arrange those weights in a differentiation matrix  $\mathbf{L}$ , which now represents a discrete approximation of the spatial operator  $\mathcal{L}$  on the chosen set of nodes  $\{\underline{x}_j\}_{j=1}^N$ . Since  $n_j \ll N$ , the resulting differentiation matrix is sparse, having  $n_j$  non-zero elements per row. It is important to note that the linear systems (4.6) can be solved in parallel, which significantly reduces the weights computation overhead that RBF-FD has compared to FD. Moreover, when it comes to the boundary nodes and the nodes that are close to the boundary, the nearest neighbor based stencils automatically form according to the shape of the boundary and require no special treatment for computing the differentiation weights — which can be seen in **Figure 4.2**. The only data that is required for approximation of differential operators are Euclidian distances between the nodes. This means that (4.6) represents a way to approximate a differential operator in any number of dimensions.

After the weights are computed and stored in the differentiation matrix  $\mathbf{L}$ , an approximation of (4.1) can be presented in the form of the following semi-discrete equation

$$\frac{d}{dt} \mathbf{u}(t) = \mathbf{L} \mathbf{u}(t), \quad (4.7)$$

$$\mathbf{u}(T) = \mathbf{g}, \quad (4.8)$$

where  $\mathbf{u}(t) \equiv u(\mathbf{x}, t)$  is the discrete numerical solution of the pricing equation,  $\mathbf{g} \equiv g(\mathbf{x})$ , while  $\mathbf{x}$  is the array of all nodes in the computational domain. To compute the option price  $\mathbf{u}$ , we need to integrate (4.7) in time.



**Figure 4.2.** An example of nearest neighbor based stencils, used for approximating the differential operator  $\mathcal{L}$  on a nonuniform node layout adapted for pricing of two-dimensional basket options with the underlying assets  $s_1$  and  $s_2$ , and strike price  $K$ . The central node of each displayed stencil is denoted by a white cross mark. All stencils are of the size  $n_j = n = 75$ .

For the time discretization, in all of our reported research, we use the second order backward differentiation method (BDF2). The BDF2 scheme requires two known previous states in order to compute the current one. To initiate the method, the Euler backward method (BDF1) is often used for the first time step. In order to avoid factoring two different matrices, we use BDF2 with BDF1 as described in [53], so that we get a single differentiation matrix with nonuniform time steps.

We split the time interval  $[0, T]$  into  $M$  non-uniform steps of length  $\tau^k = t^{M-k} - t^{M-k+1}$ ,  $k = 1, \dots, M$  and define the BDF2 weights as

$$\beta_0^k = \tau^k \frac{1 + \omega_k}{1 + 2\omega_k}, \quad \beta_1^k = \frac{(1 + \omega_k)^2}{1 + 2\omega_k}, \quad \beta_2^k = \frac{\omega_k^2}{1 + 2\omega_k}, \quad (4.9)$$

where  $\omega_k = \tau^k / \tau^{k-1}$ ,  $k = 2, \dots, M$ . In [53], it is shown how the time steps can be chosen in such a way that  $\beta_0^k \equiv \beta_0$ . Therefore, the coefficient matrix is the same in all time steps, and only one matrix factorization is needed. Applying the BDF2 scheme to (4.7) we obtain a fully discretized system of equation

$$\underbrace{(\mathbf{E} - \beta_0 \mathbf{L})}_{\mathbf{C}} \mathbf{u}^k = \beta_1^k \mathbf{u}^{k-1} - \beta_2^k \mathbf{u}^{k-2} \quad (4.10)$$

where  $\mathbf{E}$  is the identity matrix of the appropriate size. To solve this system, we employ the iterative GMRES method with an incomplete LU factorization as the preconditioner.

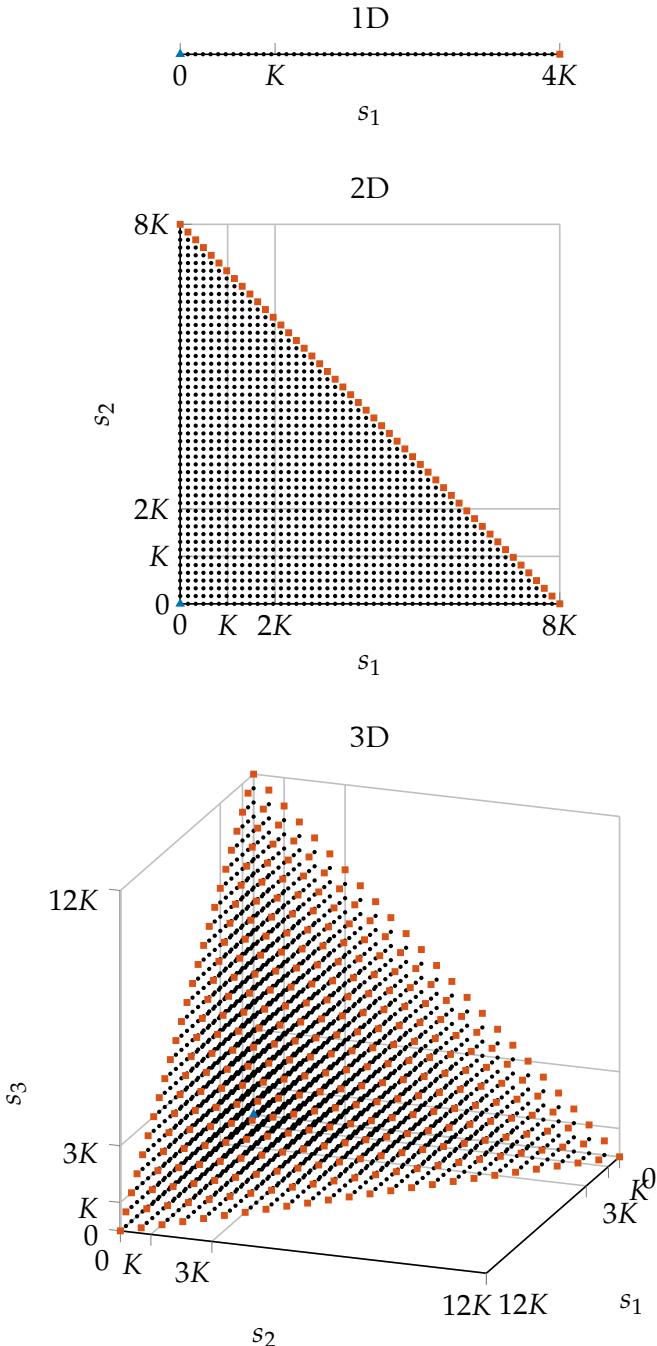
RBF-FD may be seen as a generalization of classical FD methods — where a polynomial interpolant is used instead of an RBF. Ever since its introduction, the RBF-FD methods have been successfully applied for solving convection-diffusion equations [54, 55], incompressible Navier–Stokes [56, 57, 58], and elliptic equations [59, 49]. To the best of our knowledge, RBF-FD methods were introduced to finance with a master thesis [60] at Uppsala University in 2013, and the results reported in this thesis represent the continuation of that work. In parallel with our research, classical versions of RBF-FD using infinitely smooth RBFs with constant shape parameters have been applied on equidistant Cartesian grids for pricing of different contracts [61, 62, 63, 64]. Although those articles noted the importance of RBF shape parameters for the RBF-FD approximation stability and accuracy, no special attention was paid to choosing them appropriately. Moreover, the RBF-FD examples in those articles were not exploiting the great RBF advantage of being mesh-free, as the method was applied to pricing problems using node layouts that correspond to standard equidistant FD grids. The first results of option pricing with the RBF-FD method using nonuniform node layouts and recommendations for choosing the shape parameter for GA RBFs were reported in **Paper I**.

## 4.2 Scattering Nodes

In general, pricing PDEs are defined on infinite real domains. In many cases, the domain may be limited from one side, e.g., because a stock as an underlying cannot be negative, but in most cases the domain remains open towards  $+\infty$ . Since we want to use a numerical discretization scheme, it is first required to truncate the domain and assign appropriate boundary conditions at the boundaries. When it comes to pricing multi-asset options under single-factor models, we truncate the far positive side of each dimension at  $s_{\max} = aKD$ ,  $a \in \mathbb{N}$ . In practice, most of the time, using  $a = 4$  keeps the approximation in the area around the strike price  $K$  safe from the artificial boundary effects. In that case, the close-field boundary is usually set at  $s_{\min} = 0$ . The details about the domain truncation for the multi-factor models, as well as the specifics about the boundary conditions, can be found in the appended papers next to each problem used in the numerical experiments.

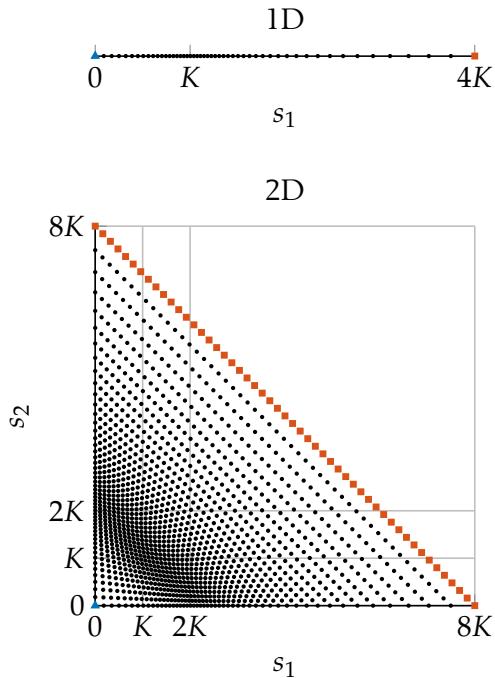
Once the computational domain boundaries are defined, we discretize the domain by scattering nodes across it. In order to study RBF-FD approximation and to be able to compare it with the standard FD methods, we started by using equidistant-Cartesian-grid-based node layouts. **Figure 4.3** shows equidistant-Cartesian-grid-based node layouts for arithmetic basket option pricing problems of up to three dimensions in space. Although not fully exploiting the mesh-free feature of the RBF-FD methods, the presented node layouts for  $D \geq 2$  are truncated diagonally via a hyperplane that is parallel to the hyperplane of discontinuity in the first derivative of the payoff function for call and put basket options — as it is not necessary to have computations in the parts of the domain that are far away from the strike value.

For many contracts, it is common to have discontinuities in the first derivative of the payoff function, and for some instruments even in the payoff function itself. Those discontinuities pose an obstacle for accurate numerical approximations and can often limit the order of convergence of the numerical methods. Knowing the location of a discontinuity allows us to scatter the nodes such that their density is higher around the discontinuity, and therefore the accuracy improved in that area. One way to do that is to use a one-dimensional nonuniform node scattering scheme from [35].



**Figure 4.3.** Equidistant-Cartesian-grid-based node layouts for pricing options with a different number of underlying assets. The close-field boundary conditions are enforced in the blue triangle node, and the far-field boundary conditions are enforced in the red square nodes.

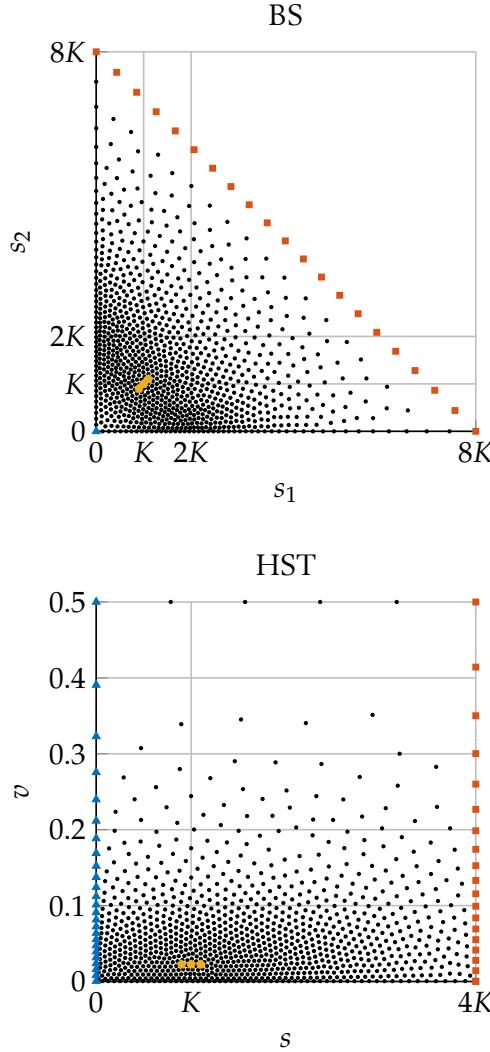
That scheme in the case of arithmetic basket options can be expanded to higher dimensions by generating one-dimensional layouts along the axes of the domain, and connecting them (via equidistant node scattering) diagonally across the domain, as seen in **Figure 4.4**. In **Papers I, II, III**, we show the advantages of this payoff-function-adapted node layout over the equidistant-Cartesian-grid-based node layouts.



**Figure 4.4.** Payoff-function-adapted node layouts for pricing options with a different number of underlying assets. The close-field boundary conditions are enforced in the blue triangle node, and the far-field boundary conditions are enforced in the red square nodes.

By using nonuniform node layouts in our numerical experiments, we have discovered that the RBF-FD stencils become very sensitive to non-smooth variations in their density, making the approximation numerically unstable. For a successful implementation of RBF-FD methods, we need to be able to quickly generate node layouts with smoothly varying density. Unfortunately, not many practical results on node scattering are readily available, but research on this topic has become quite active lately. In **Paper II**, we used a two-dimensional node layout suggested by

Bengt Fornberg and Natasha Flyer in [65]. An example of layouts with smoothly varying density for pricing two-dimensional arithmetic basket options under the Black–Scholes–Merton model and one-dimensional options under the Heston model are shown in Figure 4.5.



**Figure 4.5.** Smoothly-varying-density node layouts for pricing two-dimensional arithmetic basket options under the Black–Scholes–Merton model and one-dimensional options under the Heston model. The close-field boundary conditions are enforced in the blue triangle node, and the far-field boundary conditions are enforced in the red square nodes. The yellow pentagons show the locations of interest.

The results in **Paper II** show on several examples the advantages of such node layouts over the previously presented ones. The condition numbers of the differentiation matrices in all considered problems are significantly lower, and the accuracy is tremendously improved, as the density can be increased around the areas of interest more easily.

Unfortunately, quick node placing schemes of this sort for higher dimensions are still unavailable. One of the recent works [66], introduces a novel way to quickly generate three-dimensional smoothly varying node layouts, but other than that the resources about this problem are very scarce.

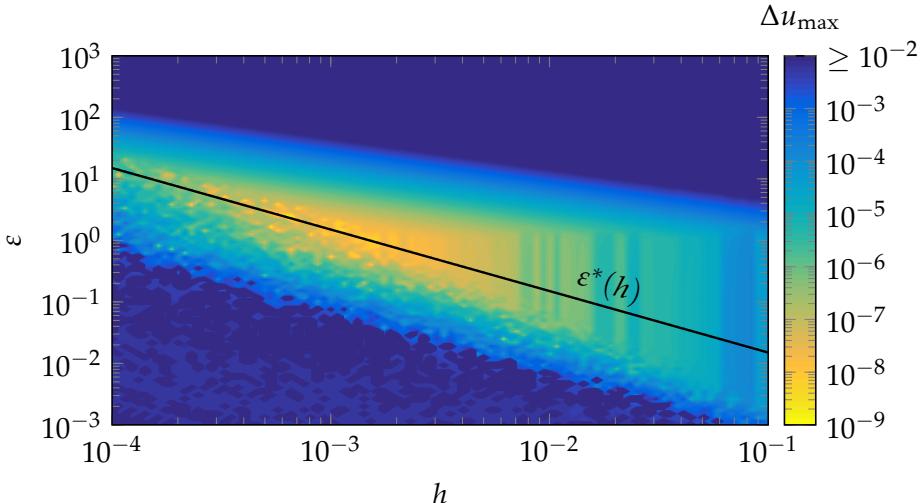
### 4.3 Choosing Shape Parameters

The properties of the RBF-FD methods presented so far are quite attractive, but when it comes to implementing the methods in practice, with infinitely smooth RBFs we need to deal with the selection of the shape parameter  $\varepsilon$ . As it can be seen in the plot to the left of Figure 4.1, the shape parameter tunes the support of the RBF. The larger the shape parameter is, the smaller the support — and vice versa. Moreover, the greater the support of an RBF, the approximation becomes more accurate. Nevertheless, if an RBF becomes too flat, the system of equations (4.6) becomes nearly singular, and the computations of the differentiation weights become ill-conditioned. There have been several approaches to stabilize the RBF-FD stencils and make them independent of the choice of the shape parameter [67], but all those treatments came with significant increases in computational costs.

Therefore, in **Paper I**, we suggested choosing the shape parameter such that it has the smallest value before the problem becomes ill-conditioned, in order to maintain high accuracy of the RBF-FD approximation. In that article, we showed by spatial error analysis and verified by numerical experiments, that an efficient RBF-FD method can be constructed if the shape parameter for approximating the Black–Scholes–Merton differential operator is chosen as

$$\varepsilon = \varepsilon^*(h) \equiv \frac{\alpha}{h}, \quad (4.11)$$

where  $\alpha$  is a real positive constant, and  $h$  is the characteristic distance between the nodes. The constant  $\alpha$  is obtained by linear regression between two points in the  $h$ - $\varepsilon$  plane with the minimal error for the given  $h$ , obtained experimentally. The suggestion has been verified on one-dimensional and two-dimensional European and American option pricing problems, on equidistant Cartesian and problem-adapted nonuniform layouts. The result is illustrated in **Figure 4.6**.



**Figure 4.6.** The maximum absolute error  $\Delta u_{\max}$  measured in the subdomain  $\hat{\Omega} = [\frac{1}{3}K, \frac{5}{3}K]$  around the strike price  $K$ , as a function of  $h$  and  $\varepsilon$ , for a one-dimensional European call option priced on an equidistant Cartesian grid with RBF-FD stencil size  $n = 3$ . The black line shows the appropriate choice for the shape parameter. The RBF-FD approximation is performed using GA basis functions.

If we take a closer look at **Figure 4.6**, we can see that as  $h$  decreases, the high error areas are joining together from top and bottom — leaving no space for high accuracy if  $h$  is sufficiently small. The truncation error analysis results presented in **Paper I** explain the presented behavior, and we consider the solutions for this issue in the following section.

## 4.4 Role of Polynomials

Many infinitely smooth RBFs have been successfully used for approximating differential operators of PDEs by RBF-FD. Nevertheless, the linear systems of equations that needed to be solved in order to obtain the weights  $\mathbf{w}_j$  have been often ill-conditioned, especially as  $h$  was becoming smaller. Several works [68, 69, 70, 71, 67, 72], addressed this problem by adding low-order polynomials together with RBFs into the presented interpolation. The linear system that we need to solve to obtain the differentiation weights for each node in our problem then becomes

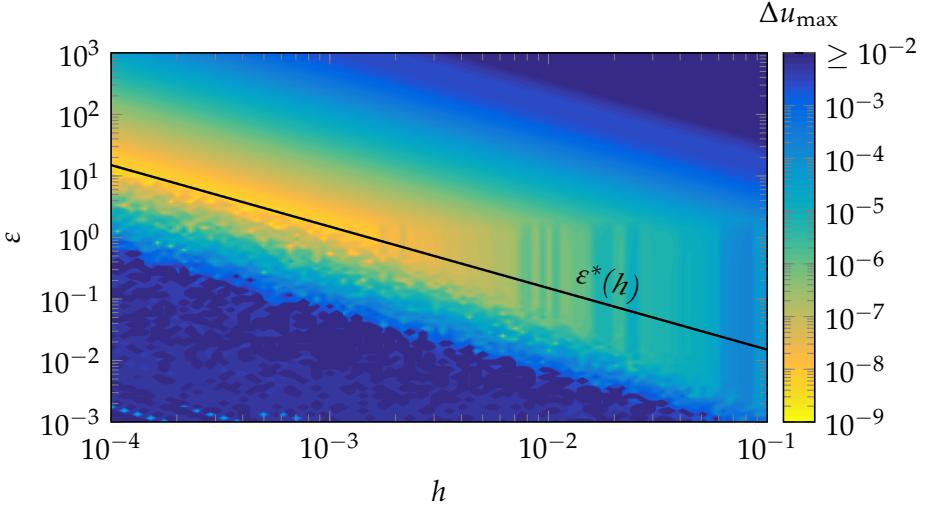
$$\begin{bmatrix} \mathbf{A}_j & \mathbf{P}_j^T \\ \mathbf{P}_j & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}_j \\ \mathbf{v}_j \end{bmatrix} = \begin{bmatrix} \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^1\|) \\ \vdots \\ \mathcal{L}\phi(\|\underline{x}_j - \underline{x}_j^{n_j}\|) \\ \mathcal{L}p_1(\underline{x}_j) \\ \vdots \\ \mathcal{L}p_{m_j}(\underline{x}_j) \end{bmatrix}, \quad (4.12)$$

where  $\mathbf{A}_j$  is the RBF matrix and  $\mathbf{w}_j$  is the array of differentiation weights;  $\mathbf{P}_j$  is the matrix of size  $m_j \times n_j$  that contains all monomials up to degree  $l$  (corresponding to  $m_j$  monomial terms) that are evaluated in each node  $\underline{x}_j^i$  of the stencil  $\underline{x}_j$  and  $\mathbf{0}$  is a zero square matrix of size  $m_j \times m_j$ ;  $\mathbf{v}_j$  is the array of dummy weights that are discarded, and  $\{p_1, p_2, \dots, p_{m_j}\}$  is the array of monomial functions indexed by their position relative to the total number of monomial terms  $m_j$ , such that it contains all the combinations of monomial terms up to degree  $l$ .

We used monomials of degree  $l = 0$  augmented to GA basis functions in **Paper I**, and the result is shown in **Figure 4.7**. The figure no longer shows joining high error fields, which ensures well-conditioned convergence of the method as  $h$  is decreased.

Even though the problem of choosing the shape parameter for GA based RBF-FD schemes is thoroughly examined for option pricing problems in **Paper I**, it remains unsolved for general applications. Nevertheless, recent developments [73, 74], have demonstrated that the RBF-FD approximation can be greatly improved by using high order polynomials together with PHSs as piecewise smooth RBFs in the interpolation, shown in the plot to the right of **Figure 4.1**. With that approach, it seems as if the polynomial degree takes the role of controlling the rate of con-

vergence. This allows us to use piecewise smooth PHSs as RBFs without a shape parameter, since the approximation accuracy is no longer controlled by the smoothness of the RBFs. Still, the RBFs do contribute to reduction of approximation errors, and therefore are necessary in order to have both stable and accurate approximation.

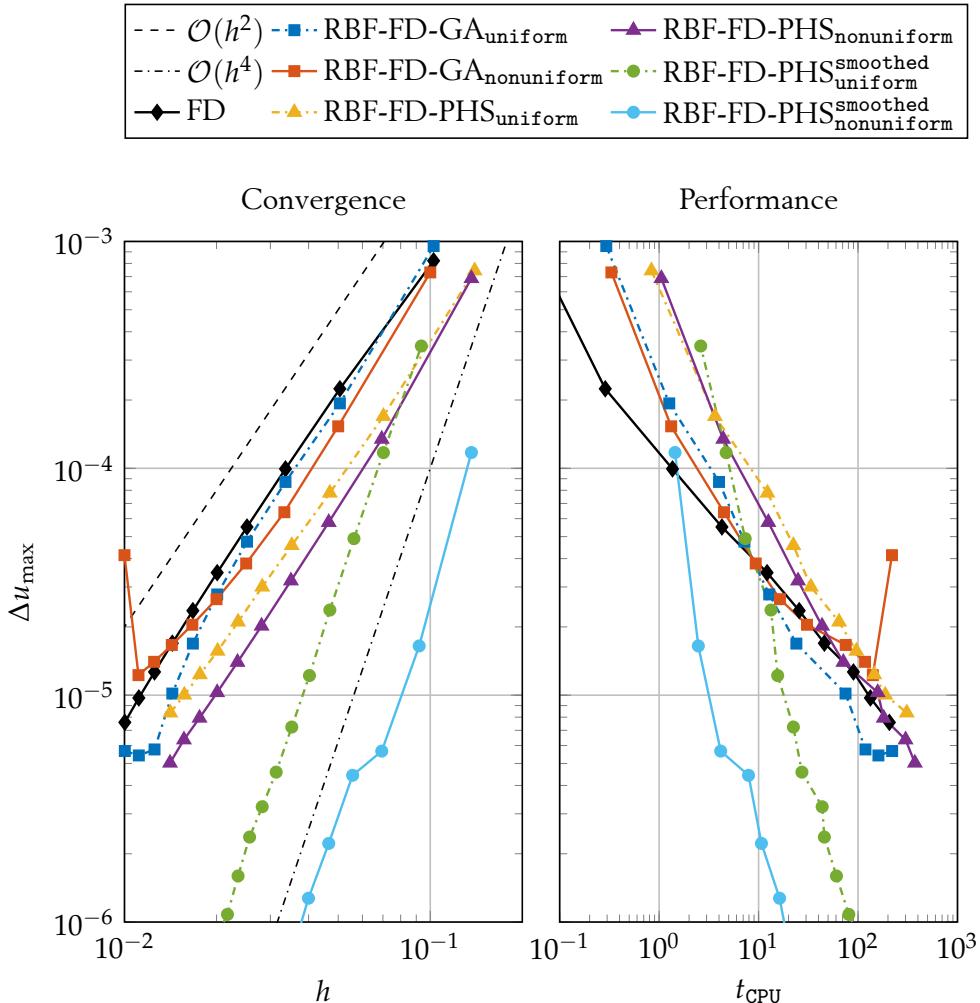


**Figure 4.7.** The maximum absolute error  $\Delta u_{\max}$  measured in the subdomain  $\hat{\Omega} = [\frac{1}{3}K, \frac{5}{3}K]$  around the strike price  $K$ , as a function of  $h$  and  $\varepsilon$ , for a one-dimensional European call option priced on an equidistant Cartesian grid with RBF-FD stencil size  $n = 3$ . The black line shows the appropriate choice for the shape parameter. The RBF-FD approximation is performed using GA augmented with monomials of degree  $l = 0$ .

In Paper II, we successfully apply the PHS based RBF-FD method to pricing two-dimensional European call and American put arithmetic basket options under the Black–Scholes–Merton model, and a one-dimensional European call option under the Heston model. The PHS based method, free of any hassle of picking the shape parameters, outperforms the standard FD method despite the computational overhead from the differentiation weights.

## 4.5 Smoothing of Payoff Functions

For many option pricing problems, the payoff has a discontinuity in the function itself or its derivatives. Such discontinuities limit the order of convergence obtained in numerical simulations. For that reason, before applying the numerical method, we suggest smoothing the payoff function according to [75]. This approach has been successfully used for option pricing problems in, e.g., [76, 77]. The smoothing of the payoff function enables the numerical method to converge with the expected order of the discretization used.



**Figure 4.8.** Absolute error  $\Delta u_{\max}$  as a function of  $h$  and CPU-time in seconds for a two-dimensional European call basket option.

In **Paper III**, we apply the smoothing on the equidistant-Cartesian-grid-based node layouts as the theory in [75] suggests, to price two-dimensional European basket options. Moreover, we find a way to use the smoothing on the payoff-function-adapted nonuniform node layouts to enable RBF-FD spatial order of convergence of four. The results of the numerical experiments show that replacing the payoff function with the smoothed one gives convergence of the appropriate order, at an acceptable increase in the computational cost. **Figure 4.8** shows the performance of the PHS based RBF-FD methods with smoothing against other RBF-FD and FD methods on uniform and nonuniform node layouts.

Nevertheless, the smoothing on the presented payoff-function-adapted nonuniform node layouts relies on the parallel placement of the nodes with the hyperplane of discontinuity. Smoothing of the initial data on the node layouts with smoothly varying density still remains an open question.

## 5. Outlook and Further Development

In this thesis, we present recent developments of RBF-FD methods and their potential to efficiently approximate solutions of PDEs for pricing of financial derivatives. We show how to apply RBF-FD to a number of option pricing problems in a way that utilizes the best features of the method such as mesh-free node placement, high order approximation, and sparsity of the differentiation matrices. By performing numerical experiments, we confirm the robustness of the method and compare its performance with other currently used and newly developed methods in financial practice. Moreover, we identify some challenges and limitations for which we suggest appropriate treatments, such as dealing with RBF shape parameters and smoothing of initial data, but there are still some problems left open.

As far as future research is concerned, we note that the smoothly-varying-density node placing algorithm can be efficiently employed only in two-dimensional domains. On the other hand, the payoff-function-adapted node layout is not as easily customizable for different cases as the smoothly-varying-density one. Some recent work has been done to come up with more robust and efficient ways of constructing customizable node layouts with smoothly varying density in higher dimensions [66], but the results are yet to be seen in practice. Research on efficient generation of high-dimensional node layouts is expected to give a significant improvement in performance of the higher-dimensional RBF-FD methods and improve the competitiveness of these methods in different financial applications. Moreover, the problem of smoothing of payoff functions on smoothly-varying-density node layouts still remains open, as the available theory covers only equidistant Cartesian grids.

Therefore, future development of RBF-FD is expected to result in a solid mesh-free high order method for multi-dimensional PDEs, that can be used together with dimension reduction techniques to efficiently solve problems of high dimensionality that we often encounter in finance.

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# Contributions

## Paper I

The author of the thesis contributed to this paper by developing all the codes and performing all the numerical experiments reported in the paper. The manuscript was written in close collaboration with the coauthor.

## Paper II

The author of the thesis is the sole author of the paper.

## Paper III

The author of the thesis contributed to this paper by developing all the codes and performing all the numerical experiments reported in the paper. The manuscript was written in close collaboration with the coauthor.

## Paper IV

The author of the thesis contributed to this paper by developing the RBF-FD codes. The manuscript was written and the numerical experiments were performed in close collaboration with the coauthor.

## Paper V

The author of the thesis contributed to this paper by developing the RBF-FD codes and writing the code necessary for performing numerical experiments on a supercomputer for all methods presented in the paper. Moreover, the author performed all the numerical experiments and produced all the figures for the manuscript. The manuscript was written in collaboration with the coauthors.

## Paper VI

The author of the thesis contributed to this paper by developing the RBF-FD codes and writing the code necessary for performing numerical experiments on a supercomputer for all methods presented in the paper. Moreover, the author performed all the numerical experiments and produced all the figures for the manuscript. The manuscript was written in collaboration with the coauthors. The author was a member of the project management group.

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## ***Summary***

The purpose of this thesis is to present state of the art in radial basis function generated finite difference (RBF-FD) methods for pricing of financial derivatives. This work provides a detailed overview of RBF-FD properties and challenges that arise when the RBF-FD methods are used in financial applications. Moreover, with this manuscript, we aim to motivate further development of RBF-FD for solving multi-dimensional partial differential equations (PDEs) in finance.

Across the financial markets of the world, financial derivatives such as futures, options, and others, are traded in substantial volumes. The value of all assets that underly outstanding derivatives transactions is several times larger than the gross world product (GWP). Financial derivatives are the most commonly used instruments when it comes to hedging risks, speculation based investing, and performing arbitrage. Therefore, knowing the prices of those financial instruments is of utmost importance at any given time. In order to make that possible in practice, it is often required to employ a set of skills incorporating knowledge in financial theory, engineering methods, mathematical tools, and programming practice — which altogether constitute the field known as *financial engineering*.

Many of the theoretical pricing models for financial derivatives can be represented using PDEs. In many cases, those equations are time-dependent, of high spatial dimension, and with challenging boundary conditions — which most often makes them analytically unsolvable. In those cases, we need to utilize numerical approximation as a mean of estimating their solution. The fields of *numerical analysis* and *scientific computing* are concerned with obtaining approximate solutions while maintaining reasonable bounds on errors. Unfortunately, there is no universal numerical method which can be used to solve all problems of this type efficiently. In fact, there are tremendously many numerical methods for solving different types of differential equations, and all those methods are featured with their own limitations in performance, stability, and accuracy — mostly dependent on details of the problems they aim to solve. Therefore, carefully selecting and developing numerical methods for particular applications has been the only way to build efficient PDE solvers in ongoing practice.

In this thesis, we present RBF-FD as a recent numerical method with potential to efficiently approximate solutions of PDEs in finance. Over the past years, besides the purely academic development and research of its numerical properties, the method has been mainly applied for simulations of atmospheric phenomena. As its name suggests, the RBF-FD method is of a finite difference type, from the radial basis function family. As a finite difference method, RBF-FD approximates differential equations by linear systems of algebraic equations, known as difference equations. Radial basis functions (RBFs) are used as interpolants that enable local approximations of differential operators that are necessary for constructing the difference equations. Constructed like that, the method is featured with a sparse matrix of the linear system of difference equations, and it is relatively simple to implement — like the standard finite difference methods. Moreover, the method is mesh-free, meaning that it does not require a structured discretization of the computational domain which makes it equally easy to use in spaces of different dimensions, and it is of a customizable order of accuracy — which are the features it inherits from the global radial basis function approximations. It is those properties that led us to recognize RBF-FD as a method with high potential for efficiently solving some analytically unfeasible and computationally challenging pricing problems in finance.

Nevertheless, being a young method, RBF-FD is still under intense development, and we face many challenges when moving from simple theoretical cases toward more complex real-world applications. The core of this thesis deals with finding solutions for overcoming obstacles when financial derivatives are priced using RBF-FD to solve PDEs with multiple spatial dimensions. Thus, it represents a contribution to making the RBF-FD methods more reliable and efficient for use in financial applications.

The results in this thesis demonstrate how to successfully apply RBF-FD to different problems in finance by studying the effects of RBF shape parameters for Gaussian RBF-FD approximations, improving the approximation of differential operators in multiple dimensions by using polyharmonic splines augmented with polynomials, constructing suitable node layouts, and smoothing of the initial data in order to enable high order convergence of the method. Finally, we compare the RBF-FD method with other available methods on a plethora of pricing problems to give an objective image of the method's performance.

# ***Sammanfattning***

Syftet med denna avhandling är att presentera forskningsfronten för finita differenser genererade via radiella basfunktioner (RBF-FD) för prissättning av finansiella derivat. Baserat på de sex bifogade artiklarna presenterar denna avhandling en detaljerad överblick av egenskaper hos RBF-FD metoder samt de svårigheter som uppstår då dessa används inom finansiella tillämpningar. Vidare ämnar denna avhandling motivera fortsatt utveckling av RBF-FD metoder för lösning av partiella differentialekvationer (PDEer) för finansiella tillämpningar.

Inom världens finansiella markader handlas finansiella derivat, såsom terminer, optioner med mera, i stora volymer. Värdet av alla underliggande tillgångar hos utställda finansiella derivat är flera gånger större än bruttovärldsprodukten. Finansiella derivat är det instrument som används mest inom risk hedging, spekulationsbaserad investering, samt utnyttjande av möjligheter till arbitrage. Det är därför synnerligen viktigt att veta priset på dessa finansiella derivat vid varje given tid. För att möjliggöra detta behövs en uppsättning färdigheter såsom kunskaper inom finansiell teori, ingenjörsmässiga metoder, matematiska verktyg, och programmeringserfarenhet. Dessa bildar tillsammans området *finansiell ingenjörskonst*.

Många av de teoretiska modeller som finns för att prissätta finansiella derivat kan beskrivas med PDEer. I många fall är dessa ekvationer tidsberoende, hög-dimensionella och med utmanande randvillkor – vilket oftast gör dem analytiskt olösbara. När så är fallet behöver numeriska metoder användas för att approximera lösningen. Områdena numerisk analys och beräkningsvetenskap handlar om att beräkna approximativa lösningar och samtidigt garantera att felet i lösningen hålls inom rimliga gränser. Dessvärre finns ingen universell numerisk metod som kan användas för att lösa alla problem effektivt. Det finns i själva verket en stor mängd numeriska metoder för att lösa olika typer av differentialekvationer vilka alla har sina egna begränsningar med avseende på effektivitet, stabilitet och noggrannhet — oftast beroende på egenskaper hos problemet som ska lösas. Att noggrant välja och utveckla numeriska metoder för specifika problem har därför i praktiken varit det enda sättet att konstruera effektiva PDE-lösare.

I denna avhandling presenteras RBF-FD som en nyutvecklad metod med potential att effektivt beräkna approximativa lösningar till PDEer inom finans. Under de senaste åren, förutom rent akademisk utveckling och forskning kring metoden numeriska egenskaper, har den huvudsakligen används för simuleringar av atmosfäriska fenomen. Som dess namn antyder, är RBF-FD en finit differensmetod, baserad på klassen av från radiella basfunktioner. Som vanliga finita differenser, approximerar RBF-FD differential ekvationer som linjära system av algebraiska ekvationer, också känt som differensekvationer. Radiella basfunktioner (RBFer) används som interpolanter för att möjliggöra lokala approximationer av differentialoperatorer som behövs för att konstruera differensekvationerna. Genom att konstruera metoden på detta vis får metoden ett glest linjärt ekvationssystem och är liksom vanliga finita differensmetoder relativt enkel att implementera. Vidare är metoden nät-fri, vilket innebär att den inte kräver en strukturerad diskretisering av beräkningsområdet. Detta gör den lika enkel att använda oberoende av antal rumsliga dimensioner och den kan erhålla den noggrannhetsordning man önskar. De senare egenskaperna ärver metoden från globala approximationer med RBFer. Sammantaget gör detta att vi kan identifiera RBF-FD som en metod med hög potential att effektivt lösa analytiskt olösbara och beräkningsmässigt utmanande prissättningsproblem inom finans.

Icke desto mindre är RBF-FD en ung metod som fortfarande befinner sig under intensiv utveckling och vi möter många utmaningar när vi går från enkla teoretiska fall mot mer komplexa problem från riktiga tillämpningar. Kärnan i denna avhandling är att hitta sätt att överbrygga de hinder som uppstår då finansiella derivat prissätts genom lösning av PDEer med hjälp av RBF-FD i flera rumsliga dimensioner. På så sätt representerar denna avhandling ett bidrag till att göra RBF-FD metoder mer tillförlitliga och effektiva för användning inom finansiella tillämpningar.

Resultaten i denna avhandling visar hur man framgångsrikt tillämpar RBF-FD på olika problem inom finans genom att studera effekten av formparameter för Gaussiska RBF-FD approximationer, visa hur man förbättrar approximationen av flerdimensionella differentialoperatorer genom att använda polyharmoniska splines förstärkta med polynom, hur man distribuerar nodpunkter på ett lämpligt sätt, samt hur man glättar initialdata för att uppnå hög konvergensordning. Slutligen jämför vi RBF-FD mot andra tillgängliga metoder på en stor mängd prissättningsproblem. Detta bidrar till att skapa en objektiv bild av metodens effektivitet.

# Преглед

Циљ ове докторске дисертације јесте да извести о тренутном стању развоја метода коначних разлика изведенних из радијалних базних функција (КР-РБФ) за одређивање цене финансијских деривата. Овај докторски рад представља детаљан преглед својства КР-РБФ и изазова који настају када се КР-РБФ методе примене у финансијској пракси. Такође, резултатима ове докторске тезе желимо да мотивишемо даљи развој КР-РБФ као поуздане методе за решавање високодимензионих парцијалних диференцијалних једначина (ПДЈ) које проистичу из финансијских модела.

Финансијским дериватима као што су форвард уговори, опције и други, широм светских финансијских тржишта тргује се у огромним количинама. Укупна вредност имовине која у овом тренутку покрива активне финансијске деривате је неколико пута већа од вредности бруто светског производа. Финансијски деривати су најкоришћенији инструменти за управљање ризиком, инвестирање засновано на шпекулисању, као и коришћење арбитражних прилика. Стога, изузетно је важно знати њихове вредности у сваком тренутку. Како би то било могуће у пракси, неретко је потребно применити скуп вештина заснованих на познавању финансијске теорије, математичких алата, метода природних наука и техника програмирања — које заједно чине арсенал знања струке познате као *финансијско инжењерство*.

Многи теоријски модели за одређивање цена финансијских деривата могу се представити помоћу ПДЈ. У великом броју случаја, те једначине су временски зависне, високодимензионе, са проблематичним граничним условима — што их најчешће чини аналитички нерешивима. У тим ситуацијама, принуђени смо да користимо нумериčко апроксимирање као средство за одређивање њихових приближних решења. Научно-техничке дисциплине као што су *нумериčка анализа* и *рачунарство*, баве се развојем метода за приближно решавање аналитички нерешивих проблема и изучавањем својстава тих метода. Нажалост, не постоји једна универзална нумериčка метода коју бисмо могли да користимо за ефикасно решавање свих проблема овог типа. Штавише, постоји огроман број нумеричких метода за приближно решавање различитих врста диференцијалних једначина, а свака од њих

долази са себи својственим скупом ограничења у прецизности, ефикасности и стабилности — која најчешће зависе од особина самог проблема који покушавамо да решимо. Стoga, једини начин за ефикасно апроксимирање решења ПДЈ јесте пажљиво бирање, развијање и прилагођавање нумеричких метода за сваки проблем, понаособ.

У овој дисертацији представљамо КР-РБФ као младу нумеричку методу која има изванредан потенцијал за ефикасно апроксимирање решења ПДЈ у финансијама. Претходних година, поред чистог академског развоја и истраживања везаних за њена нумеричка својства, ова метода је успешно коришћена у пракси за симулирање атмосферских феномена. Као што њено само име наговештава, КР-РБФ је врста методе коначних разлика (КР) која потиче из породице метода радијалних базних функција (РБФ). Тако конструисану, ову методу карактерише ретка матрица система и једноставно имплементирање — слично као код класичних КР метода. Такође, ова метода не захтева мрежу за дискретизовање домена, већ дискретизовање можемо да обављамо слободним рас простирањем тачака. То чини дискретизовање вишедимензионих домена врло једноставним, а поред тога метода је прилагодљивог реда прецизности — што су особине које наслеђује од глобалних РБФ метода. Представљена својства служе као основ за препознавање КР-РБФ као прикладне методе за решавање одређених аналитички нерешивих и рачунарски изазовних проблема у финансијама.

Ипак, то што је КР-РБФ млада метода у развоју носи са собом хргу изазвана са којима се суочавамо када се са идеалних академских примера пређе на проблеме стварног света. Језгро ове дисертације се бави управо проналажењем решења за препреке на које наилазимо када се цене финансијских деривата одређују решавањем вишедимензионих ПДЈ помоћу КР-РБФ.

Проучавајући утицај параметара облика Гаусових РБФ на КР-РБФ апроксимирање, користећи полихармоничне сплајнове са полиномима за апроксимирање вишедимензионих диференцијалних оператора, конструишући прикладне распореде тачака током дискретизовања домена и глачањем почетних услова како би метод могао да конвергира високим редом, резултати ове дисертације показују како успешно можемо применити КР-РБФ методу на разне проблеме у финансијама. Такође, ова теза садржи резултате поређења КР-РБФ са другим методама на плетори финансијских проблема, како бисмо имали што објективнију слику о перформансама представљене методе.