1 Calculation of receiver position

1.1 Satellite and receiver clocks are synchronized

$$transittime^{(k)} = rxtime - txtime^{(k)} \tag{1}$$

$$r^{(k)} = (rxtime - txtime^{(k)}) * c \tag{2}$$

$$r^{(k)} = \sqrt{(x–x^{(k)})^2 + (y–y^{(k)})^2 + (z–z^{(k)})^2} \tag{3}$$

$$\sqrt{(x–x^{(k)})^2 + (y–y^{(k)})^2 + (z–z^{(k)})^2} = (rxtime - txtime^{(k)}) * c \tag{4}$$

$txtime^{(k)}$ = time of transmission (measured by $k^{th's}$ satellite clock)
$rxtime$ = time of reception (measured by receiver clock)
$r^{(k)}$ = real (geometrical) range

1.2 Satellite and receiver clocks are not synchronized

$$\rho^{(k)} = ((rxtime + \Delta t) - (txtime^{(k)} + \delta t^{(k)})) * c \tag{5}$$

$$\rho^{(k)} = (rxtime - txtime^{(k)}) * c - \delta t^{(k)} * c + \Delta t * c \tag{6}$$

$$\rho^{(k)} = r^{(k)} - \delta t^{(k)} * c + \Delta t * c \tag{7}$$

$$\begin{aligned} \rho'^{(k)} &= \rho^{(k)} + \delta t^{(k)} * c \\ &= r^{(k)} + \Delta t * c \end{aligned} \tag{8}$$

$txtime^{(k)}$ = time of transmission (relative to gps system time)
$rxtime$ = time of reception (relative to gps system time)
$\rho^{(k)}$ = pseudorange
$\delta t^{(k)}$ = advance of the $k^{th's}$ satellite clock with respect to system time
Δt = advance of the receiver clock with respect to system time

1.3 Time to solve navigation equations

Let's introduce function $f^{(k)} = f^{(k)}(x, y, z, \Delta t)$ of four variables

$$f^{(k)}(x, y, x, \Delta t) = \sqrt{(x–x^{(k)})^2 + (y–y^{(k)})^2 + (z–z^{(k)})^2} + \Delta t * c \tag{9}$$

Above function can be expanded about the point $(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})$ using the Taylor series. It is deliberately truncated after the first-order partial derivatives to eliminate non-linear terms.

$$f^{(k)}(x, y, x, \Delta t) \approx f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t}) + \frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial x}\Delta x + \frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial y}\Delta y + \frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial z}\Delta z + \frac{\partial f(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial(\Delta t)}\Delta(\Delta t) \tag{10}$$

The partial derivatives evaluate as follows:

$$\frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial x} = \frac{\hat{x} - x^{(k)}}{\hat{r}} \tag{11}$$

$$\frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial y} = \frac{\hat{y} - y^{(k)}}{\hat{r}} \tag{12}$$

$$\frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial z} = \frac{\hat{z} - z^{(k)}}{\hat{r}} \tag{13}$$

$$\frac{\partial f^{(k)}(\hat{x}, \hat{y}, \hat{z}, \hat{\Delta t})}{\partial(\Delta t)} = c \tag{14}$$

where

$$\hat{r} = \sqrt{(\hat{x}–x^{(k)})^2 + (\hat{y}–y^{(k)})^2 + (\hat{z}–z^{(k)})^2} \tag{15}$$

We shall have already noticed that our $f^{(k)}$ has the same form as our pseudorange $\rho^{'(k)}$. Thus we can write

$$\rho^{'(k)} = \hat{\rho^{'(k)}} + \frac{\hat{x} - x^{(k)}}{\hat{r}}\Delta x + \frac{\hat{y} - y^{(k)}}{\hat{r}}\Delta y + \frac{\hat{z} - z^{(k)}}{\hat{r}}\Delta z + c * \Delta(\Delta t) \tag{16}$$

If we introduce new variables as follows

$$\Delta\rho^{'(k)} = \rho^{'(k)} - \hat{\rho^{'(k)}} \tag{17}$$

$$a_x^{(k)} = \frac{\hat{x} - x^{(k)}}{\hat{r}} \tag{18}$$

$$a_y^{(k)} = \frac{\hat{y} - y^{(k)}}{\hat{r}} \tag{19}$$

$$a_z^{(k)} = \frac{\hat{z} - z^{(k)}}{\hat{r}} \tag{20}$$

Then we can rewrite equation (16) in a simpler form as

$$\Delta\rho^{'(k)} = a_x^{(k)}\Delta x + a_y^{(k)}\Delta y + a_z^{(k)}\Delta z + c * \Delta(\Delta t) \tag{21}$$

So, we have 4 unknowns: $\Delta x, \Delta y, \Delta z$ and $\Delta(\Delta t)$. Thus we need pseudorange measurements from 4 satellites which will give us set of four linear equations

$$\Delta\rho^{'(1)} = a_x^{(1)}\Delta x + a_y^{(1)}\Delta y + a_z^{(1)}\Delta z + c * \Delta(\Delta t) \tag{22}$$

$$\Delta\rho^{'(2)} = a_x^{(2)}\Delta x + a_y^{(2)}\Delta y + a_z^{(2)}\Delta z + c * \Delta(\Delta t) \tag{23}$$

$$\Delta\rho^{'(3)} = a_x^{(3)}\Delta x + a_y^{(3)}\Delta y + a_z^{(3)}\Delta z + c * \Delta(\Delta t) \tag{24}$$

$$\Delta\rho^{'(4)} = a_x^{(4)}\Delta x + a_y^{(4)}\Delta y + a_z^{(4)}\Delta z + c * \Delta(\Delta t) \tag{25}$$

Those equations can be rewritten using matrix notation

$$\Delta\boldsymbol{\rho}^{\boldsymbol{'}} = \begin{pmatrix} \Delta\rho^{'(1)} \\ \Delta\rho^{'(2)} \\ \Delta\rho^{'(3)} \\ \Delta\rho^{'(4)} \end{pmatrix} \tag{26}$$

$$\boldsymbol{H} = \begin{pmatrix} a_x^{(1)} & a_y^{(1)} & a_z^{(1)} & 1 \\ a_x^{(2)} & a_y^{(2)} & a_z^{(2)} & 1 \\ a_x^{(3)} & a_y^{(3)} & a_z^{(3)} & 1 \\ a_x^{(4)} & a_y^{(4)} & a_z^{(4)} & 1 \end{pmatrix} \tag{27}$$

$$\Delta\boldsymbol{s} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ c * \Delta(\Delta t) \end{pmatrix} \tag{28}$$

And finally

$$\Delta\boldsymbol{\rho}^{\boldsymbol{'}} = \boldsymbol{H}\Delta\boldsymbol{s} \tag{29}$$

Naturally (30) has the following solution

$$\Delta\boldsymbol{s} = \boldsymbol{H}^{\boldsymbol{-1}}\Delta\boldsymbol{\rho}^{\boldsymbol{'}} \tag{30}$$