Modelling deadlock with graph theory

Assuming n_f greedy flows with infinite traffic demand are traversing the ingress queues of n_s switches and n_h hosts in a lossless network. Let k be the port number of the switches (assuming all the switches have uniform port number). We logically consider a switch as k ingress queues, and a host as 1 ingress queue. An ingress queue consists of n' ($n' = k * n_s + n_h$) virtual output queues. For any ingress queue q_i, the queue length of its j-th virtual output queue represents the bytes of packets to be sent to ingress queue q_j.

For any flow f_j (j = 1,2,…,n_f), we add a virtual ingress queue $q_{(j+n')}$ as the traffic source. Let q_i be the first ingress queue flow f_j traverses in the network. For $q_{(j+n')}$, the queue length of its i-th virtual output queue is non-zero, and the others are all zero.

Let $\mathbf{G}(\mathbf{V}, \mathbf{E})$ be a directed graph. Let $n = n' + n_f$.

1) Any vertex $v_i \in \mathbf{V}$ (i = 1,2,…,n) represents the i-th ingress queue.
2) $\mathbf{V}_0$ is a subset of $\mathbf{V}$. Any vertex $v_i \in \mathbf{V}_0$ (i = $n'+1, n'+2, \ldots, n$) represents the virtual ingress queue corresponding to the traffic source of $(i - n')$-th flow.
3) Any arc $e_{i,j} \in \mathbf{E}$ (i = 1,2,…,n; j = 1,2,…,n) means there exists a flow traversing ingress queues i and j in sequence.
4) Any arc $e_{i,j}$ in $\mathbf{E}$ has a weight $w_{e_{i,j}}$ (>=0), which represents the queue length of the j-th virtual output queue of ingress queue i.
5) $\forall v_i \in \mathbf{V}$, we have $\sum_{j=1}^{n} w_{e_{i,j}} <= m_\theta$, where m_θ is a constant. Here m_θ represents the PFC threshold. This constraint is used to describe the fact that the queue length of any ingress queue can be no more than the PFC threshold.

$\mathbf{G}(\mathbf{V}, \mathbf{E})$(t) can be viewed as a snapshot of the network state at time t. For any $e_{i,j}$ in $\mathbf{E}$, $w_{e_{i,j}}(t)$ is the weight of $e_{i,j}$ at time t. we define

$$\frac{d(w_{e_{i,j}}(t))}{dt} = \frac{w_{e_{i,j}}(t + \delta t) - w_{e_{i,j}}(t)}{dt}. \tag{1}$$

The behavior of PFC pause can be defined as

$$\frac{d(w_{e_{i,j}}(t))}{dt} >= 0 \text{ when } \sum_{k=1}^{n} w_{e_{j,k}}(t) = m_\theta. \tag{2}$$

The queue length of an ingress queue cannot increase when all the virtual output queues pointing to it are empty, so we have

$$\frac{d(w_{e_{i,j}}(t))}{dt} <= 0 \text{ when } \sum_{k=1}^{n} w_{e_{k,i}}(t) = 0. \tag{3}$$

The queue length of an ingress queue should increase when not all the virtual output queues pointing to it are empty and PFC threshold is not reached, so we have

$$\frac{d(w_{e_{i,j}}(t))}{dt} > 0 \text{ when } \sum_{k=1}^{n} w_{e_{k,i}}(t) > 0 \text{ and } \sum_{k=1}^{n} w_{e_{j,k}}(t) < m_\theta. \tag{4}$$

The queue length of any ingress queue corresponding to the traffic source of any flow should be larger than zero, so we have

$$\sum_{j=1}^{n} w_{e_{i,j}}(t) > 0 \text{ when } v_i \in \mathbf{V}_0. \tag{5}$$

Definition of deadlock: We say $\mathbf{G}(\mathbf{V}, \mathbf{E})$(t) has deadlock when $\exists t_0 < \infty$ and $\exists \mathbf{V}' \subseteq \mathbf{V}$,

1) $\forall t > t_0$ and $\forall v_i \in \mathbf{V}'$, $\frac{d(w_{e_{i,j}}(t))}{dt} = 0$ for any feasible j.
2) $\forall t > t_0$ and $\forall v_i \in \mathbf{V}'$, $\sum_{j=1}^{n} w_{e_{i,j}}(t) > 0$.

Sufficient and necessary condition for deadlock: If $\exists t_1 < \infty$ and $\exists \mathbf{V}' \subseteq \mathbf{V}$,

1) $\forall v_i \in \mathbf{V}'$, $\sum_{j=1}^{n} w_{e_{i,j}}(t_1) = m_\theta$.
2) $\forall v_i \in \mathbf{V}'$ and $\forall v_j \in \mathbf{V} - \mathbf{V}'$, there is either no edge between v_i and v_j, or $w_{e_{i,j}}(t_1) = 0$.

Then $\mathbf{G}(\mathbf{V}, \mathbf{E})$(t) has deadlock.

Proof:

$\Rightarrow$ As $\forall v_i \in \mathbf{V}'$, $\sum_{j=1}^{n} w_{e_{i,j}}(t_1) = m_\theta$, according to equation (2), if $v_i, v_j \in \mathbf{V}'$, $\frac{d(w_{e_{i,j}}(t_1))}{dt} >= 0$. However, as we have the constraint $\sum_{j=1}^{n} w_{e_{i,j}}(t_1) <= m_\theta$, we have $\frac{d(w_{e_{i,j}}(t_1))}{dt} = 0$.

As $\forall v_i \in \mathbf{V}'$ and $\forall v_j \in \mathbf{V} - \mathbf{V}'$, there is either no edge between v_i and v_j, or $w_{e_{i,j}}(t_1) = 0$, according to equation (3), if $v_i \in \mathbf{V}'$, and $v_j \in \mathbf{V} - \mathbf{V}'$, $\frac{d(w_{e_{i,j}}(t_1))}{dt} <= 0$. As $w_{e_{i,j}}(t_1) >= 0$, we have $\frac{d(w_{e_{i,j}}(t_1))}{dt} = 0$

In summary, we have $\forall v_i \in \mathbf{V}'$, $\frac{d(w_{e_{i,j}}(t))}{dt} = 0$ for any feasible j. It is trivial that $\forall v_i \in \mathbf{V}'$, $\sum_{j=1}^{n} w_{e_{i,j}}(t) > 0$. As the two constraints still hold at time $t + \delta t$, we can then prove $\forall t >= t_1$, $\frac{d(w_{e_{i,j}}(t))}{dt} = 0$ is true for any $v_i \in \mathbf{V}'$. So $\mathbf{G}(\mathbf{V}, \mathbf{E})$(t) has deadlock.

$\Leftarrow$ (not finished yet) The key of the proof is to derive a $\mathbf{V}^\star$ from $\mathbf{V}'$ which satisfies the two constraints of the sufficient and necessary condition. I need more time to work out the details.