Summary of Equations in Aerodynamics Lecture Notes

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1 Introduction

EGH445 is a course that explores fundamental engineering principles, modeling techniques, system dynamics, and control strategies. The course integrates theoretical and practical knowledge, focusing on modern computational tools.

2 System Modelling

System modeling involves representing a physical system mathematically to analyze and predict its behavior. Two major approaches include:

- **Input-State-Output Model:**

$$\dot{x} = f(x, u)$$
$$y = g(x, u)$$

  where:

  - x is the state vector
  - u is the input vector
  - y is the output vector

- **Linear Time-Invariant (LTI) System:**

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

  where A, B, C, and D are system matrices.

2.1 State-Space Representation

The state-space model describes the system using state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

This representation allows a complete description of system dynamics.

2.2 State-Space Model of an Ordinary Differential Equation (ODE)

We consider a second-order differential equation of the form:

$$\ddot{y}(t) + \mathbf{a}\dot{y}(t) + \mathbf{b}y(t) = u(t)$$

Rearranging for $\ddot{y}(t)$:

$$\ddot{y}(t) = -\mathbf{a}\dot{y}(t) - \mathbf{b}y(t) + u(t)$$

2.2.1 Step 1: Define the State Variables

Define the state variables as:

$$x_1(t) = y(t),$$
$$x_2(t) = \dot{y}(t)$$

Substituting into the equation, we obtain the state-space representation:

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -\mathbf{a}x_2(t) - \mathbf{b}x_1(t) + u(t)$$

2.2.2 Step 2: Express in Matrix Form

The state-space equations can be rewritten in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mathbf{b} & -\mathbf{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

2.2.3 Step 3: Define the Output Equation

If we consider the output $y(t)$ as one of the state variables, the output equation is:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2.2.4 Step 4: Final State-Space Representation

Combining the state and output equations, the complete state-space representation is:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ -\mathbf{b} & -\mathbf{a} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

This represents the final state-space form of the given second-order ODE.

2.3 Transfer Function of a State-Space Model

$$\begin{bmatrix} \dot{x} \\ \dot{x} \end{bmatrix}, \quad = \begin{bmatrix} 0 & 1 \\ -\mathbf{b} & -\mathbf{a} \end{bmatrix},$$

3 Control System Analysis

A control system regulates the behavior of dynamic systems. Key elements include:

- **Open-loop Control:** Control action does not depend on output.
- **Closed-loop Control:** Control action adjusts based on feedback.

3.1 Transfer Function Representation

The transfer function relates input and output in the frequency domain:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

where s is the Laplace variable.

4 Stability Analysis

Stability determines if a system returns to equilibrium after disturbances.

- A system is **stable** if all poles of its transfer function have negative real parts.
- A system is **unstable** if any pole has a positive real part.
- **Marginal Stability:** Poles are on the imaginary axis.

5 Feedback Control Systems

Feedback control systems use sensors to adjust control inputs dynamically.

- **Proportional Control (P):** $u(t) = K_p e(t)$
- **Proportional-Integral Control (PI):** $u(t) = K_p e(t) + K_i \int e(t)dt$
- **Proportional-Integral-Derivative (PID):**

$$u(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{de(t)}{dt}$$

where $e(t)$ is the error signal, and K_p, K_i, K_d are control gains.

6 Frequency Response Analysis

The frequency response method analyzes system behavior under sinusoidal inputs.

- **Bode Plot:** Logarithmic magnitude and phase plots.
- **Nyquist Criterion:** Determines stability using contour mapping.
- **Gain and Phase Margins:** Measures robustness.

7 Conclusion

EGH445 covers key principles of system modeling, stability, and control, emphasizing theoretical analysis and computational tools.