Complete factorial designs

Session 6

MATH 80667A: Experimental Design and Statistical Methods
for Quantitative Research in Management
HEC Montréal

Outline

Unbalanced designs

Multifactorial designs

Unbalanced designs

Premise

So far, we have exclusively considered balanced samples

balanced = same number of observational units in each subgroup

Most experiments (even planned) end up with unequal sample sizes.

Noninformative drop-out

Unbalanced samples may be due to many causes, including randomization (need not balance) and loss-to-follow up (dropout)

If dropout is random, not a problem

- Example of Baumannn, Seifert-Kessel, Jones (1992):

> Because of illness and transfer to another school, incomplete data were obtained for one subject each from the TA and DRTA group

Problematic drop-out or exclusion

If loss of units due to treatment or underlying conditions, problematic!

Rosensaal (2021) rebuking a study on the effectiveness of hydrochloriquine as treatment for Covid19 and reviewing allocation:

> Of these 26, six were excluded (and incorrectly labelled as lost to follow-up): three were transferred to the ICU, one died, and two terminated treatment or were discharged

Sick people excluded from the treatment group! then claim it is better.

Worst: "The index [treatment] group and control group were drawn from different centres."

Why seek balance?

Two main reasons

1. Power considerations: with equal variance in each group, balanced samples gives the best allocation
2. Simplicity of interpretation and calculations: the interpretation of the F test in a linear regression is unambiguous

Finding power in balance

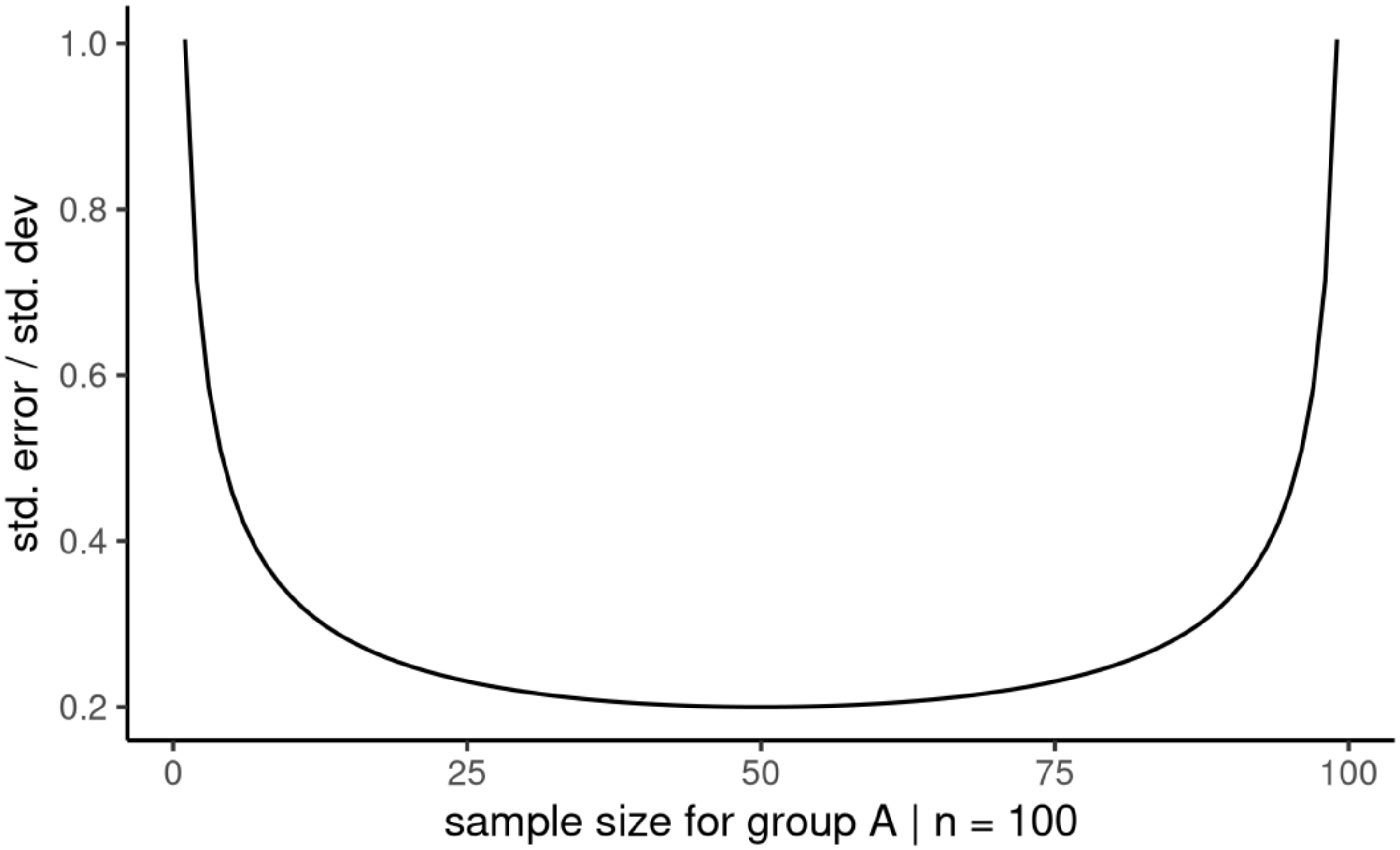
Consider a t-test for assessing the difference between treatments A and B with equal variability

$$t = \frac{\text{estimated difference}}{\text{estimated variability}} = \frac{(\widehat{\mu}_A - \widehat{\mu}_B) - 0}{\mathsf{se}(\widehat{\mu}_A - \widehat{\mu}_B)}.$$

The standard error of the average difference is

$$\sqrt{\frac{\text{variance}_A}{\text{nb of obs. in } A} + \frac{\text{variance}_B}{\text{nb of obs. in } B}} = \sqrt{\frac{\sigma^2}{n_A} + \frac{\sigma^2}{n_B}}$$

Optimal allocation of ressources



The allocation of $n = n_A + n_B$ units that minimizes the std error is $n_A = n_B = n/2$.

Example: tempting fate

We consider data from Multi Lab 2, a replication study that examined Risen and Gilovich (2008) who

> explored the belief that tempting fate increases bad outcomes. They tested whether people judge the likelihood of a negative outcome to be higher when they have imagined themselves [...] tempting fate [...] (by not reading before class) or not [tempting] fate (by coming to class prepared). Participants then estimated how likely it was that [they] would be called on by the professor (scale from 1, not at all likely, to 10, extremely likely).

The replication data gathered in 37 different labs focuses on a 2 by 2 factorial design with gender (male vs female) and condition (prepared vs unprepared) administered to undergraduates.

Load data | Check balance | Marginal means

- We consider a 2 by 2 factorial design.
- The response is `likelihod`
- The experimental factors are `condition` and `gender`
- Two data sets: `RS_unb` for the full data, `RS_bal` for the artificially balanced one.

Load data | **Check balance** | Marginal means

```r
summary_stats <-
  RS_unb |>
  group_by(condition) |>
  summarize(nobs = n(),
            mean = mean(likelihood))
```

Summary statistics

condition	nobs	mean
unprepared	2192	4.606
prepared	2241	4.060

Load data | Check balance | **Marginal means**

```r
# Enforce sum-to-zero parametrization
options(contrasts = rep("contr.sum", 2))
# Anova is a linear model, fit using 'lm
# 'aov' only for *balanced data*
model <- lm(
  likelihood ~ gender * condition,
  data = RS_unb)
library(emmeans)
emm <- emmeans(model,
               specs = "condition")
```

Marginal means for condition

condition	emmean	SE
unprepared	4.504	0.0540
prepared	4.022	0.0535

Note unequal standard errors.

Explaining the discrepancies

Estimated marginal means are based on equiweighted groups:

$$\widehat{\mu} = \frac{1}{4}(\widehat{\mu}_{11} + \widehat{\mu}_{12} + \widehat{\mu}_{21} + \widehat{\mu}_{22})$$

where $\widehat{\mu}_{ij} = n_{ij}^{-1} \sum_{r=1}^{n_{ij}} y_{ijr}$.

The sample mean is the sum of observations divided by the sample size.

The two coincide when $n_{11} = \cdots = n_{22}$.

Why equal weight?

- The ANOVA and contrast analyses, in the case of unequal sample sizes, are generally based on marginal means (same weight for each subgroup).
- This choice is justified because research questions generally concern comparisons of means across experimental groups.

Revisiting the F statistic

Statistical tests contrast competing **nested** models:

- an alternative (full) model
- a null model, which imposes restrictions (a simplification of the alternative models)

The numerator of the F-statistic compares the sum of square of a model with (given) main effect, etc. to a model without.

What is explained by condition?

Consider the 2×2 factorial design with factors A: `gender` and B: `condition` (prepared vs unprepared) without interaction.

What is the share of variability (sum of squares) explained by the experimental condition?

Comparing differences in sum of squares (1)

Consider a balanced sample

```
anova(lm(likelihood ~ 1, data = RS_bal),
      lm(likelihood ~ condition, data = RS_bal))
# When gender is present
anova(lm(likelihood ~ gender, data = RS_bal),
      lm(likelihood ~ gender + condition, data = RS_bal))
```

The difference in sum of squares is 141.86 in both cases.

Comparing differences in sum of squares (2)

Consider an unbalanced sample

```
anova(lm(likelihood ~ 1, data = RS_unb),
      lm(likelihood ~ condition,
         data = RS_unb))
# When gender is present
anova(lm(likelihood ~ gender, data = RS_unb),
      lm(likelihood ~ gender + condition,
         data = RS_unb))
```

The differences of sum of squares are respectively 330.95 and 332.34.

Orthogonality

Balanced designs yield orthogonal factors: the improvement in the goodness of fit (characterized by change in sum of squares) is the same regardless of other factors.

So effect of B and $B \mid A$ (read B given A) is the same.

- test for $B \mid A$ compares $\mathsf{SS}(A, B) - \mathsf{SS}(A)$
- for balanced design, $\mathsf{SS}(A, B) = \mathsf{SS}(A) + \mathsf{SS}(B)$ (factorization).

We lose this property with unbalanced samples: there are distinct formulations of ANOVA.

Analysis of variance - Type I (sequential)

The default method in **R** with `anova` is the sequential decomposition: in the order of the variables A, B in the formula

- So F tests are for tests of effect of
  - A, based on $\text{SS}(A)$
  - $B \mid A$, based on $\text{SS}(A, B) - \text{SS}(A)$
  - $AB \mid A, B$ based on $\text{SS}(A, B, AB) - \text{SS}(A, B)$

Ordering matters

Since the order in which we list the variable is **arbitrary**, these F tests are not of interest.

Analysis of variance - Type II

Impact of

- $A \mid B$ based on $\text{SS}(A,B) - \text{SS}(B)$
- $B \mid A$ based on $\text{SS}(A,B) - \text{SS}(A)$
- $AB \mid A,B$ based on $\text{SS}(A,B,AB) - \text{SS}(A,B)$
- tests invalid if there is an interaction.
- In **R**, use `car::Anova(model, type = 2)`

Analysis of variance - Type III

Most commonly used approach

- Improvement due to $A \mid B, AB$, $B \mid A, AB$ and $AB \mid A, B$
- What is improved by adding a factor, interaction, etc. given the rest
- may require imposing equal mean for rows for $A \mid B, AB$, etc.
  - (**requires** sum-to-zero parametrization)
- valid in the presence of interaction
- but F-tests for main effects are not of interest
- In **R**, use `car::Anova(model, type = 3)`

ANOVA for unbalanced data

```r
model <- lm(
  likelihood ~ condition * gender,
  data = RS_unb)
# Three distinct decompositions
anova(model) #type 1
car::Anova(model, type = 2)
car::Anova(model, type = 3)
```

ANOVA (type I)

	Df	Sum Sq	F value
gender	1	164.94	29.1
condition	1	332.34	58.7
gender:condition	1	36.55	6.5
Residuals	4429	25086.33	

ANOVA (type II)

	Df	Sum Sq	F value
gender	1	166.33	29.4
condition	1	332.34	58.7
gender:condition	1	36.55	6.5
Residuals	4429	25086.33	

ANOVA (type III)

	Df	Sum Sq	F value
gender	1	167.71	29.6
condition	1	227.88	40.2
gender:condition	1	36.55	6.5
Residuals	4429	25086.33	

ANOVA for balanced data

```r
model2 <- lm(
  likelihood ~ condition * gender,
  data = RS_bal)
anova(model2) #type 1
car::Anova(model2, type = 2)
car::Anova(model2, type = 3)
# Same answer - orthogonal!
```

ANOVA (type I)

	Df	Sum Sq	F value
condition	1	141.86	24.1
gender	1	121.69	20.6
condition:gender	1	37.88	6.4
Residuals	2500	14733.84	

ANOVA (type II)

	Df	Sum Sq	F value
condition	1	141.86	24.1
gender	1	121.69	20.6
condition:gender	1	37.88	6.4
Residuals	2500	14733.84	

ANOVA (type III)

	Df	Sum Sq	F value
condition	1	141.86	24.1
gender	1	121.69	20.6
condition:gender	1	37.88	6.4
Residuals	2500	14733.84	

Recap

- If each observation has the same variability, a balanced sample maximizes power.
- Balanced designs have interesting properties:
  - estimated marginal means coincide with (sub)samples averages
  - the tests of effects are unambiguous
  - for unbalanced samples, we work with marginal means and type 3 ANOVA
  - if empty cells (no one assigned to a combination of treatment), cannot estimate corresponding coefficients (typically higher order interactions)

Practice

From the OSC psychology replication

> People can be influenced by the prior consideration of a numerical anchor when forming numerical judgments. [...] The anchor provides an initial starting point from which estimates are adjusted, and a large body of research demonstrates that adjustment is usually insufficient, leading estimates to be biased towards the initial anchor.

Replication of Study 4a of Janiszewski & Uy (2008, Psychological Science) by J. Chandler

Multifactorial designs

Beyond two factors

We can consider multiple factors A, B, C, $\ldots$ with respectively n_a, n_b, n_c, $\ldots$ levels and with n_r replications for each.

The total number of treatment combinations is

$$n_a \times n_b \times n_c \times \cdots$$

Curse of dimensionality

Full three-way ANOVA model

Each cell of the cube is allowed to have a different mean

$$\underset{\text{response}}{Y_{ijkr}} = \underset{\text{cell mean}}{\mu_{ijk}} + \underset{\text{error}}{\varepsilon_{ijkr}}$$

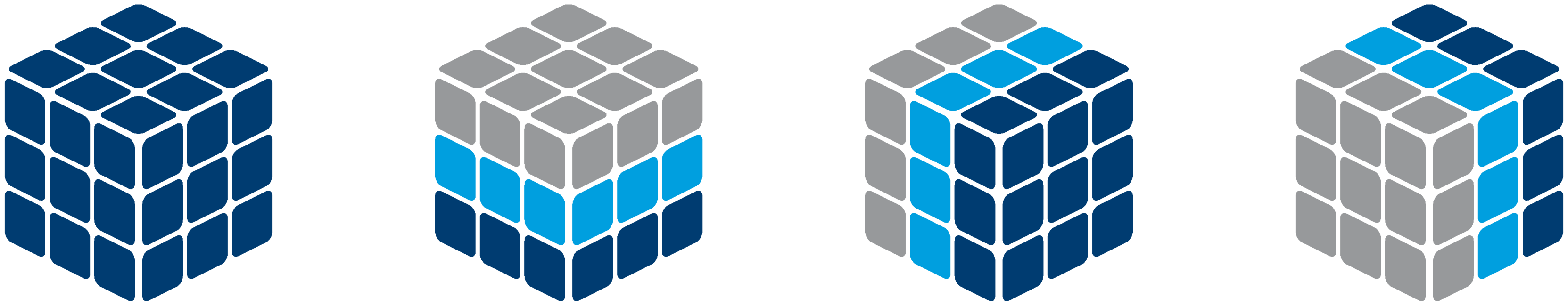
with ε_{ijkt} are independent error term for

- row i
- column j
- depth k
- replication r

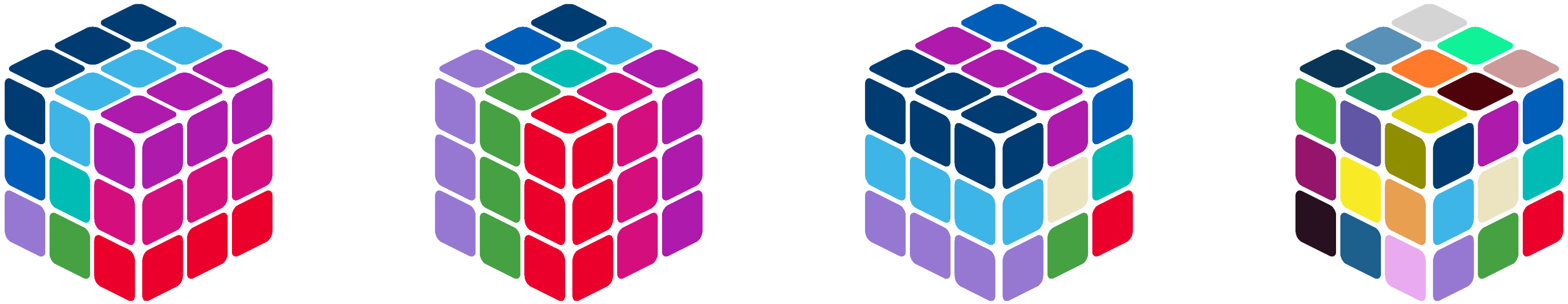
Parametrization of a three-way ANOVA model

With the **sum-to-zero** parametrization with factors A, B and C, write the response as

$$
\begin{aligned}
\underset{\text{theoretical average}}{\mathsf{E}(Y_{ijkr})} &= \underset{\text{global mean}}{\mu} \\
&\quad + \underset{\text{main effects}}{\alpha_i + \beta_j + \gamma_k} \\
&\quad + \underset{\text{two-way interactions}}{(\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}} \\
&\quad + \underset{\text{three-way interaction}}{(\alpha\beta\gamma)_{ijk}}
\end{aligned}
$$



global mean, row, column and depth main effects



row/col, row/depth and col/depth interactions and three-way interaction.

Example of three-way design

Petty, Cacioppo and Heesacker (1981). Effects of rhetorical questions on persuasion: A cognitive response analysis. Journal of Personality and Social Psychology.

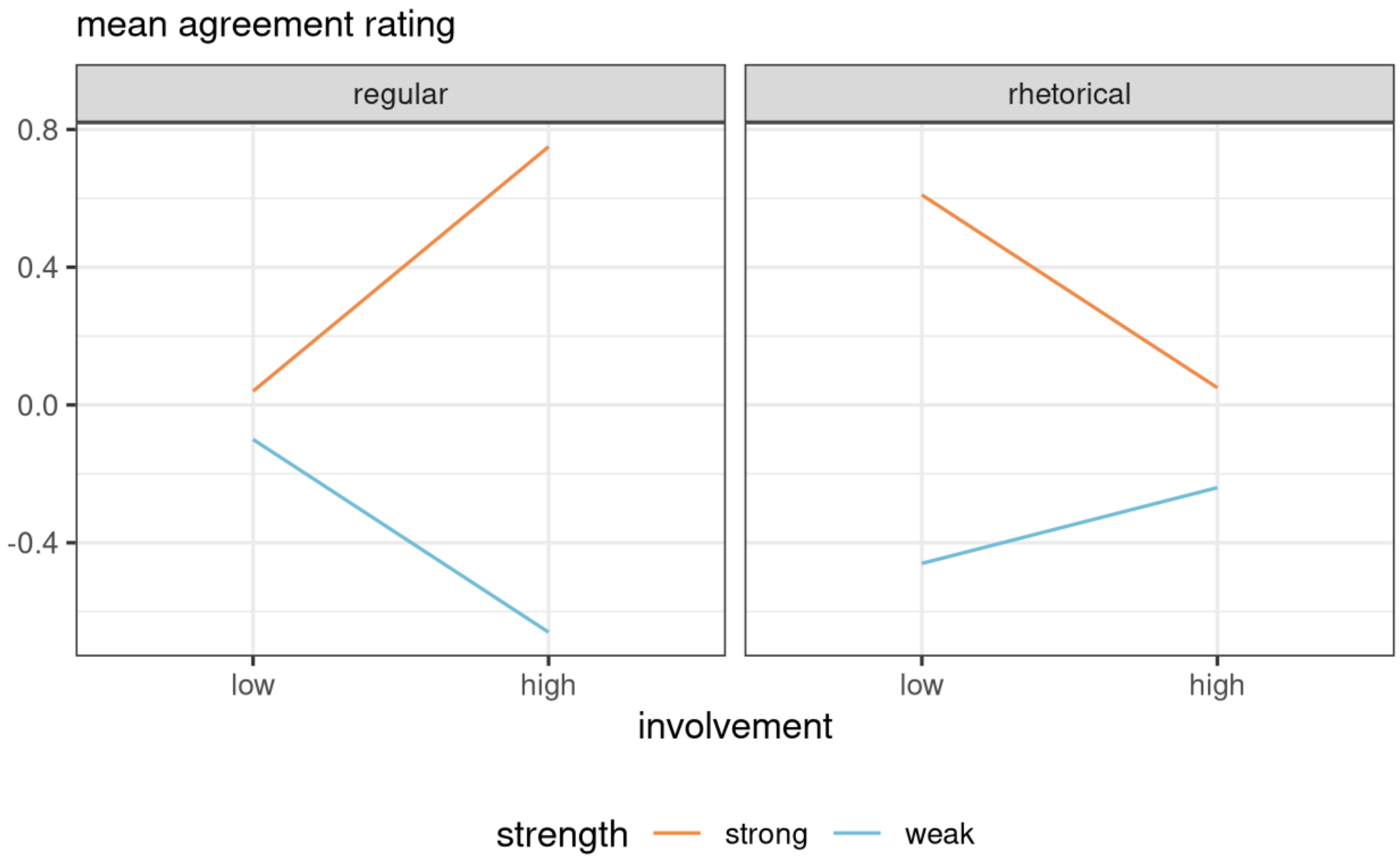
A $2 \times 2 \times 2$ factorial design with 8 treatments groups and $n = 160$ undergraduates.

Setup: should a comprehensive exam be administered to bachelor students in their final year?

- **Response** Likert scale on -5 (do not agree at all) to 5 (completely agree)
- **Factors**
- A: strength of the argument (`strong` or `weak`)
- B: involvement of students `low` (far away, in a long time) or `high` (next year, at their university)
- C: style of argument, either `regular` form or `rhetorical` (Don't you think?, ...)

Interaction plot

Interaction plot for a $2 \times 2 \times 2$ factorial design from Petty, Cacioppo and Heesacker (1981)



The microwave popcorn experiment

What is the best brand of microwave popcorn?

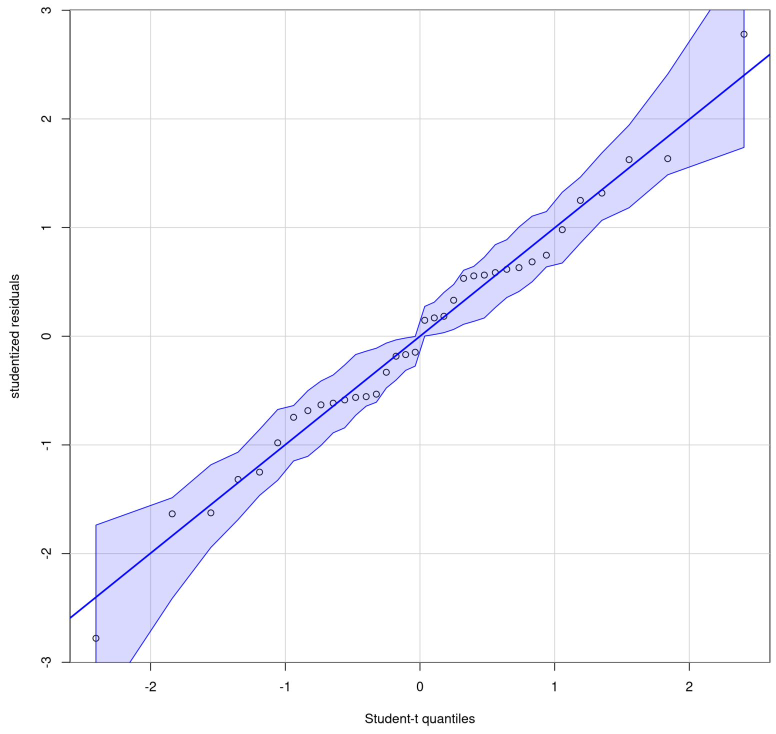
- **Factors**
- brand (two national, one local)
- power: 500W and 600W
- time: 4, 4.5 and 5 minutes
- **Response**: ~~weight~~, ~~volume~~, ~~number~~, percentage of popped kernels.
- Pilot study showed average of 70% overall popped kernels (10% standard dev), timing values reasonable
- Power calculation suggested at least $r = 4$ replicates, but researchers proceeded with $r = 2$...

ANOVA | QQ-plot | R code | Interaction plot

```
data(popcorn, package = 'hecedsm')
# Fit model with three-way interaction
model <- aov(percentage ~ brand*power*time,
             data = popcorn)
# ANOVA table - 'anova' is ONLY for balanced designs
anova_table <- anova(model)
# Quantile-quantile plot
car::qqPlot(model)
```

Model assumptions: plots and tests are meaningless with `(n_r=2)` replications per group...

ANOVA | QQ-plot | R code | Interaction plot

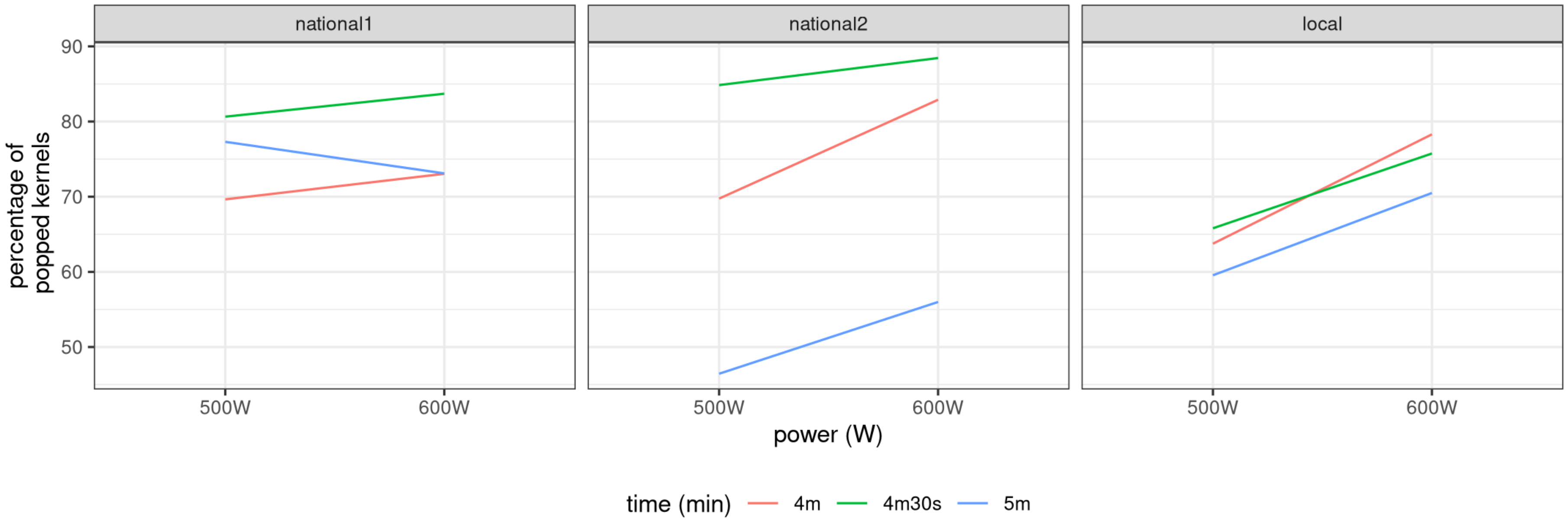


All points fall roughly on a straight line.

ANOVA QQ-plot **R code** Interaction plot

```
popcorn |>
   group_by(brand, time, power) |>
   summarize(meanp = mean(percentage)) |>
ggplot(mapping = aes(x = power,
                     y = meanp,
                     col = time,
                     group = time)) +
  geom_line() +
  facet_wrap(~brand)
```

ANOVA | QQ-plot | R code | **Interaction plot**



No evidence of three-way interaction (hard to tell with $r = 2$ replications).

Analysis of variance table for balanced designs

terms	degrees of freedom
A	$n_a - 1$
B	$n_b - 1$
C	$n_c - 1$
AB	$(n_a - 1)(n_b - 1)$
AC	$(n_a - 1)(n_c - 1)$
BC	$(n_b - 1)(n_c - 1)$
ABC	$(n_a - 1)(n_b - 1)(n_c - 1)$
residual	$n_a n_b n_c (R - 1)$
total	$n_a n_b n_c n_r - 1$

Analysis of variance table for microwave-popcorn

	Degrees of freedom	Sum of squares	Mean square	F statistic	p-value
brand	2	331.10	165.55	1.89	0.180
power	1	455.11	455.11	5.19	0.035
time	2	1554.58	777.29	8.87	0.002
brand:power	2	196.04	98.02	1.12	0.349
brand:time	4	1433.86	358.46	4.09	0.016
power:time	2	47.71	23.85	0.27	0.765
brand:power:time	4	47.33	11.83	0.13	0.967
Residuals	18	1577.87	87.66		

Omitting terms in a factorial design

The more levels and factors, the more parameters to estimate (and replications needed)

- Costly to get enough observations / power
- The assumption of normality becomes more critical when $r = 2$!

It may be useful not to consider some interactions if they are known or (strongly) suspected not to be present

- If important interactions are omitted from the model, biased estimates/output!

Guidelines for the interpretation of effects

Start with the most complicated term (top down)

- If the three-way interaction ABC is significative:
  - don't interpret main effects or two-way interactions!
  - comparison is done cell by cell within each level
- If the ABC term isn't significative:
  - can marginalize and interpret lower order terms
  - back to a series of two-way ANOVAs

What contrasts are of interest?

- Can view a three-way ANOVA as a series of one-way ANOVA or two-way ANOVAs...

Depending on the goal, could compare for variable A

- marginal contrast ψ_A (averaging over B and C)
- marginal conditional contrast for particular subgroup: ψ_A within c_1
- contrast involving two variables: ψ_{AB}
- contrast differences between treatment at $\psi_A \times B$, averaging over C.
- etc.

See helper code and chapter 22 of Keppel & Wickens (2004) for a detailed example.

Effects and contrasts for microwave-popcorn

Following preplanned comparisons

- Which combo (brand, power, time) gives highest popping rate? (pairwise comparisons of all combos)
- Best brand overall (marginal means marginalizing over power and time, assuming no interaction)
- Effect of time and power on percentage of popped kernels
- pairwise comparison of time $\times$ power
- main effect of power
- main effect of time

Preplanned comparisons using `emmeans`

Let A=brand, B=power, C=time

Compare difference between percentage of popped kernels for 4.5 versus 5 minutes, for brands 1 and 2

$$\mathscr{H}_0 : (\mu_{1.2} - \mu_{1.3}) - (\mu_{2.2} - \mu_{2.3}) = 0$$

```r
library(emmeans)
# marginal means
emm_popcorn_AC <- emmeans(model,
                          specs = c("brand","time"))
contrast_list <-
  list(
    brand12with4.5vs5min = c(0, 0, 0, 1, -1, 0, -1, 1,0))
contrast(emm_popcorn_AC,  # marginal mean (no time)
         method = contrast_list) # list of contrasts
```

Preplanned comparisons

Compare all three times (4, 4.5 and 5 minutes)

At level 99% with Tukey's HSD method

- Careful! Potentially misleading because there is a `brand * time` interaction present.

```
# List of variables to keep go in `specs`: keep only time
emm_popcorn_C <- emmeans(model, specs = "time")
pairs(emm_popcorn_C,
      adjust = "tukey",
      level = 0.99,
      infer = TRUE)
```