Coherent Upper Conditional Previsions Defined by Hausdorff Outer Measures for Unbounded Random Variables.

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Abstract

A model of upper conditional previsions for bounded and unbounded random variables with finite Choquet integral with respect to the Hausdorff outer and inner measures is proposed in a metric space. They are defined by the Choquet integral with respect to Hausdorff outer measures if the conditioning event has positive and finite Hausdorff outer measure in its dimension, otherwise, when the conditioning event has Hausdorff outer measure equal to zero or infinity in its Hausdorff dimension, they are defined by a 0-1 valued finitely, but not countably, additive probability.

Keywords: coherent upper conditional previsions, Hausdorff outer measures, unbounded random variables, Choquet integral, Monotone Convergence Theorem

1. Introduction

The theory of coherent upper conditional previsions has been developed by Walley ([24], [23]) for bounded random variables, but the possibility to consider arbitrary random variables has been investigated in literature (Troffaes and de Cooman [20],[21], Troffaes [22]). There is no requirement of continuity for a coherent prevision P, that is coherence does not imply that given a sequence of random variables X_n converging point-wise to the random variable X then $P(X_n)$ converges to $P(X)$.

One of the motivating issues to propose a new model of coherent prevision which is continuous is that when coherent, real-valued previsions are not continuous from below, that creates the awkward situation where the agent is prevented from assigning equal previsions to each pair of equivalent (unbounded) random variables.

It implies that coherent previsions preclude indifference between equivalent random variables as may occur for random variables with geometric distribution (Seidenfeld et al. [19]).

A way to avoid this problem, when the coherent prevision has an integral representation with respect to a coherent probability μ, is to require that coherent conditional prevision satisfies the Monotone Convergence Theorem, which assures the convergence of $\int X_n d\mu$ to $\int X d\mu$ when the sequence X_n converging point-wise to the random variable X.

In this paper coherent upper conditional previsions defined by Hausdorff outer measures, are introduced, in a metric space, for bounded and unbounded random variables with finite Choquet integral with respect to the Hausdorff outer and inner measures and the following results are proven:

a) coherent upper conditional previsions are continuous from below and they satisfy the Monotone Convergence Theorem when the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension s. Denoted by h^s the s-dimensional Hausdorff outer measure and by S the σ-field of all h^s-measurable sets a consequence of the continuity from below is that the restrictions to the class of all S-measurable random variables on B of the coherent upper conditional previsions coincide with expectation so that positive boost function avoids and the problem of losing indifference between random variables with the same distribution can be solved;

b) if Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension coherent upper prevision $\overline{P}$ satisfies the disintegration property $\overline{P}(X) = \overline{P}(\overline{P}(X|\mathbf{B}))$ on every non- null partition $\mathbf{B}$;

c) all monotone set functions on $\wp(B)$ which are submodular, continuous from below and which represent as Choquet integral a coherent upper conditional prevision defined on a linear lattice $\mathbf{F}$, agree on the set system of weak upper level sets $M = \{\{X \geq x\} | X \in \mathbf{F}, x \in \Re\}$, with the coherent upper conditional probability $\mu^*(A) = \frac{h^s(AB)}{h^s(B)}$ for $A \in \wp(B)$.

The necessity to propose a new tool to define coherent upper conditional previsions arises because they cannot be obtained as extensions of linear expectations defined, by the Radon-Nikodym derivative, in the axiomatic approach [1]; it occurs because one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ-field of the conditioning events, contradicts the necessary condition for the coherence [7, Theorem 1] $P(X|\mathbf{B}) = X$ for every $\mathbf{B}$-measurable random variable.

2. Coherent Upper Conditional Previsions Defined by the Choquet Integral with Respect to Hausdorff Outer Measures

In this section coherent upper and lower conditional previsions are defined on the class of all bounded and unbounded random variables with finite Choquet integral with respect to the Hausdorff outer and inner measures. These upper conditional previsions are defined by the Choquet integral with respect to Hausdorff outer measures if the conditioning event has positive and finite Hausdorff outer measure in its dimension; otherwise they are defined by a 0-1 valued finitely, but not countably additive probability.

2.1. Coherent Upper Conditional Previsions

Let Ω be a non empty set, let $\mathbf{B}$ be a partition of Ω. A random variable is a function $X : \Omega \rightarrow \widehat{\Re} = \Re \cup \{+\infty\}$; for every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by $\sup(X|B)$ the supremum of values that X assumes on B. Coherent upper conditional previsions are functional defined on a linear space by the axioms of coherence ([24]).

Definition 1 *Coherent upper conditional previsions are functionals $P(\cdot|B)$ defined on a linear space K, such that the following axioms of coherence hold for every X and Y in K and every strictly positive constant λ:*

1) $\overline{P}(X|B) \leq \sup(X|B)$;

2) $\overline{P}(\lambda X|B) = \lambda \overline{P}(X|B)$ (positive homogeneity);

3) $\overline{P}(X+Y|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ (subadditivity);

4) $\overline{P}(I_B|B) = 1$.

Definition 2 *Given a partition $\mathbf{B}$ and a random variable $X \in L(\Omega)$, a coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is a random variable on Ω equal to $\overline{P}(X|B)$ if $\omega \in B$.*

Definition 3 *A bounded random variable is called $\mathbf{B}$-measurable or measurable with respect to a partition $\mathbf{B}$ if it is constant on the atoms of the partition.*

The following necessary condition for coherence holds [24, p. 292]:

Proposition 4 *If for every B belongs to $\mathbf{B}$ $P(X|B)$ are coherent linear previsions and X is $\mathbf{B}$-measurable then $P(X|\mathbf{B}) = X$*

Suppose that $\overline{P}(X|B)$ is a coherent upper conditional prevision on $L(B)$. Then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$. Let K be a linear space contained in $L(B)$; if for every X belonging to K we have $P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$ then $P(X|B)$ is called a coherent *linear* conditional prevision

[3, 4, 11, 15, 16] and it is a linear, positive and positively homogenous functional on K [24, Corollary 2.8.5].

The unconditional coherent upper prevision $\overline{P} = \overline{P}(\cdot|\Omega)$ is obtained as a particular case when the conditioning event is Ω. Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered.

A coherent upper conditional probability μ_B on $\wp(B)$ is

i) *submodular* or *2-alternating* if $\mu_B(A \cup E) + \mu_B(A \cap E) \leq \mu_B(A) + \mu_B(E)$ for every $A, E \in \wp(B)$;

ii) *continuous from below* if $\lim_{i \to \infty} \mu_B(A_i) = \mu_B(\lim_{i \to \infty} A_i)$ for any increasing sequence of sets $\{A_i\}$, with $A_i \in \wp(B)$.

2.2. Hausdorff Outer Measures

Outer measures are non-negative, monotone set-functions that are sub-additive (Munroe [14]) so they duplicate basic property of upper probability for sets. Hausdorff outer measures are examples of outer measures defined in a metric space.

Let (Ω, d) be a metric space. The diameter of a non empty set U of Ω is defined as $|U| = \sup\{d(x, y) : x, y \in U\}$ and if a subset A of Ω is such that $A \subset \bigcup_i U_i$ and $0 < |U_i| < \delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A.

Let s be a non-negative number. For $\delta > 0$ we define $h_{s,\delta}(A) = \inf \sum_{i=1}^{\infty} |U_i|^s$, where the infimum is over all δ-covers $\{U_i\}$.

The *Hausdorff s-dimensional outer measure* of A ([17], [12]) denoted by $h^s(A)$, is defined as

$$h^s(A) = \lim_{\delta \to 0} h_{s,\delta}(A).$$

This limit exists, but may be infinite, since $h_{s,\delta}(A)$ increases as δ decreases. The *Hausdorff dimension* of a set A, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = \infty \text{ if } 0 \leq s < dim_H(A),$$
$$h^s(A) = 0 \text{ if } dim_H(A) < s < \infty.$$

2.3. Choquet Integral

We recall the definition of the Choquet integral ([2], [5]) with the aim to define upper conditional previsions by Choquet integral with respect to the dimensional Hausdorff outer measures and to prove their properties. The Choquet integral is an integral with respect to a monotone set function. Given a non-empty set Ω and denoted by $\wp(\Omega)$, the family of all subsets of Ω, a monotone set function $\mu : \wp(\Omega) \rightarrow \overline{\Re}_+ = \Re_+ \cup \{+\infty\}$ is such that $\mu(\oslash) = 0$ and if $A, B \in \wp(\Omega)$ with $A \subset B$ then $\mu(A) \leq \mu(B)$. Given a monotone set function μ on $S \subset \wp(\Omega)$ the *outer set function* of μ is the set function μ^* defined on the whole power set $\wp(\Omega)$ by

$$\mu^*(A) = \inf\left\{\mu(B) : B \supset A; B \in S\right\}, A \in \wp(\Omega)$$

The inner set function of μ is the set function μ_* defined on the whole power set $\wp(\Omega)$ by

$$\mu_*(A) = \sup\left\{\mu(B)|B \subset A; B \in S\right\}, A \in \wp(\Omega)$$

Let μ be a monotone set function defined on $\wp(\Omega)$ and $X : \Omega \to \overline{\mathfrak{R}} = \mathfrak{R} \cup \left\{-\infty, +\infty\right\}$ an arbitrary function on Ω. Then the set function

$$G_{\mu,X}(x) = \mu\left\{\omega \in \Omega : X(\omega) > x\right\}$$

is decreasing and it is called *decreasing distribution function* of X with respect to μ. If μ is continuous from below then $G_{\mu,X}(x)$ is right continuous.

In particular the decreasing distribution function of X with respect to the Hausdorff outer measures is right continuous since these outer measures are continuous from below. A function $X : \Omega \to \overline{\mathfrak{R}}$ is called upper μ-measurable if $G_{\mu^*,X}(x) = G_{\mu_*,X}(x)$. Given an upper μ-measurable random variable $X :\Omega \to \overline{R}$ with decreasing distribution function $G_{\mu,X}(x)$, the Choquet integral of X with respect to μ is defined if $\mu(\Omega) < \infty$ through

$$\int X d\mu = \int_{-\infty}^0 (G_{\mu,X}(x) - \mu(\Omega))dx + \int_0^\infty G_{\mu,X}(x)dx$$

The integral is in $\mathfrak{R}$ or can assume the values $-\infty, \infty$ and 'non-existing'.

For upper μ-measurable random variable X we have

$$\int X d\mu = \int X d\mu^* = \int X d\mu_*.$$

For non upper μ-measurable random variables the Choquet integral can be defined by the outer and inner measures generated by μ .

If $X \geq 0$ or $X \leq 0$ the integral always exists. In particular for $X \geq 0$ we obtain

$$\int X d\mu = \int_0^\infty G_{\mu,X}(x)dx$$

If X is bounded and $\mu(\Omega) = 1$ we have that

$$\int X d\mu = \int_{infX}^0 (G_{\mu,X}(x) - 1)dx + \int_0^{supX} G_{\mu,X}(x)dx$$

If μ is a σ-additive measure the Choquet integral coincides with the usual definition except for infinite measures.

Definition 5 *A random variable X is Choquet integrable with respect to a monotone set function μ if the Choquet integral is finite, that is*

$$-\infty < \int^{Cho} X d\mu < +\infty$$

A coherent upper conditional prevision can be represented as Choquet integral with respect to a coherent upper conditional probability μ if and only if μ is submodular ([8][Proposition 1]). A random variable is Choquet integrable with respect to a submoduar coherent upper conditional probability μ if the coherent upper prevision of X with respect to μ if finite.

Definition 6 *A random variable X is Choquet integrable with respect to a submodular coherent upper probability μ if the Choquet integral $\int^{Cho} X d\mu$ is finite that is*

$$-\infty < \overline{P}(X) = \int^{Cho} X d\mu < +\infty$$

Example 1 *Let (Ω, d) be the Euclidean metric space where $\Omega = [0,1]$. Let Q be the set of rational numbers.*
The unbounded random variable X defined by

$$X(\omega) = \left\{ \begin{array}{lll} +\infty & if & x \in Q \cap [0,1] \\ 1 & if & x \in \mathfrak{R} - Q \cap [0,1] \end{array} \right.$$

is Choquet integrable with respect to μ_Ω which is defined by the 1-Hausdorff measure.

By the the conjugacy property, i.e. $\overline{P}(X) = -\underline{P}(-X)$ a dual definition of Choquet integral random variable can be given

Definition 7 *A random variable X is Choquet integrable with respect to a submodular coherent upper probability μ if and only if*

$$-\infty < \int(-X)d\overline{\mu} < +\infty.$$

So a random variable X is Choquet integrable with respect to a submodular coherent upper conditional probability μ if and only if the random variable $-X$ is Choquet integrable with respect to the conjugate lower conditional probability $\overline{\mu}$. If a random variable X is Choquet integrable with respect to a lower conditional probability $\overline{\mu}$ defined on $\wp(\Omega)$ it does not imply that X is Choquet integrable with respect to the conjugate upper conditional probability μ.

Proposition 8 *A random variable X is Choquet integrable with respect to a supermodular coherent lower conditional probability $\overline{\mu}_B$ if and only if the random variable $-X$ is Choquet integrable with respect to the conjugate upper conditional probability μ_B.*

Proof Since X is Choquet integrable with respect to a coherent lower conditional probability $\overline{\mu}_B$, by Definition 6 we obtain

$$-\infty < \int X d\overline{\mu}_B < +\infty \Leftrightarrow -\infty < \int(-X)d\mu_B < +\infty.$$

■

2.4. The Model

A new model of coherent upper conditional previsions for bounded random variables, has been introduced and its properties have been proven in Doria [6], [7], [8], [9], [10].

Theorem 9 *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m_B be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Thus, for each $B \in \mathbf{B}$, the function defined on $\wp(B)$ by*

$$\overline{P}(A|B) = \begin{cases} \frac{h^s(A \cap B)}{h^s(B)} & if \quad 0 < h^s(B) < +\infty \\ m_B(A \cap B) & if \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional probability.

If $B \in \mathbf{B}$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s the fuzzy measure μ_B^* defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^*(AB)}{h^*(B)}$ is a coherent upper conditional probability, which is submodular, continuous from below and such that its restriction to the σ-field of all μ_B^* measurable sets is a Borel regular countably additive probability.

The coherent upper unconditional probability $\overline{P} = \mu_\Omega^*$ defined on $\wp(\Omega)$ is obtained for B equal to Ω.

If $B = \Omega$ we denote μ_B by μ and the Choquet integral by $\int^{Cho} X d\mu$.

Definition 10 *Let $L^*(B)$ be the class of random variables which are Choquet integrable with respect μ_B^* and with respect $\overline{\mu}_B^*$.*

Theorem 11 *Let μ_B^* the coherent upper conditional probability defined in Theorem 2 and let $\overline{\mu}_B^*$ its dual. Then the class $L^*(B)$ of random variables Choquet integrable with respect μ_B^* and with respect $\overline{\mu}_B^*$, is a linear space.*

Proof We have to prove that for every random variables X and $Y \in L^*(B)$ and $\forall \beta \in \Re$ the random variables $X + Y$ and βX belong to $L^*(B)$. The coherent upper conditional probability μ_B^* defined in Theorem 2 is submodular and continuous from below thus by the Subadditivity Theorem ([5, Theorem 6.3]) we have

$$\int_B^{Cho}(X+Y)d\mu_B^* \le \int_B^{Cho}Xd\mu_B^* + \int_B^{Cho}Yd\mu_B^* < +\infty$$

Since the dual $\underline{\mu}_B^*$ of μ_B^* is defined by the Hausdorff inner measure and it is continuous from above then by the Superadditivity Theorem (Denneberg [5, Corollary 6.4]) we have that

$$-\infty < \int_B^{Cho}Xd\overline{\mu}_B^* + \int_B^{Cho}Yd\overline{\mu}_B^* \le$$
$$\le \int_B^{Cho}(X+Y)d\overline{\mu}_B^* \le \int_B^{Cho}(X+Y)d\mu_B^*$$

so if X and Y are Choquet integrable then also $X + Y$ is Choquet integrable since $-\infty < \int_B^{Cho}(X+Y)d\mu_B < +\infty$

Moreover for any $X \in L^*(B)$, let $\beta \ge 0$, by the positive homogeneity and asymmetry of the Asymmetric Choquet integral we have

$$-\infty < \beta \int_B^{Cho}Xd\overline{\mu}_B^* \le \int_B^{Cho}\beta Xd\overline{\mu}_B^* \le$$
$$\le \int_B^{Cho}\beta Xd\mu_B^* = \beta \int_B^{Cho}Xd\mu_B^* < +\infty.$$

Let $\beta < 0$

$$\int_B^{Cho}(\beta X)d\mu_B^* = -\int_B^{Cho}(-\beta X)d\overline{\mu}_B^* =$$
$$-(-\beta)\int_B^{Cho}Xd\overline{\mu}_B^* < -(-\beta)\int_B^{Cho}Xd\mu_B^* < +\infty.$$

It implies

$$-\infty < \int_B^{Cho}(\beta X)d\mu_B^* < +\infty$$

$\blacksquare$

Theorem 12 *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m_B be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Then for each $B \in \mathbf{B}$ the functional $\overline{P}(X|B)$ defined on the linear space $L^*(B)$ by*

$$\overline{P}(X|B) = \begin{cases} \frac{1}{h^s(B)}\int_B X dh^s & if \quad 0 < h^s(B) < +\infty \\ m_B & if \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional prevision.

Proof Since $L^*(B)$ is a linear space we have to prove that, for every $B \in \mathbf{B}$ the functional $\overline{P}(X|B)$ satisfies conditions 1)-4) of Definition 1. From the definition of $\overline{P}(X|B)$ we have that for every conditioning event B the upper conditional prevision $\overline{P}(\cdot|B)$ satisfies properties 1) and 2) of Definition 1. Moreover property 3) follows from the given definition in the case where B has Hausdorff measure equal to zero or infinity. If B has finite and positive Hausdorff outer measure in its dimension then property 3), follows from the Subadditivity Theorem ([5] Theorem 6.3) since Hausdorff outer measures are monotone, submodular and continuous from below. From the definition we have $\overline{P}(B|B) = 1$. $\blacksquare$

When the conditioning event B has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity, an additive conditional probability is coherent if and only if it takes only $0 - 1$ values. Because linear previsions on $L(B)$ are uniquely determined by their restrictions to events, the class of linear previsions on $L(B)$ whose restrictions to events take only the values 0 and 1 can be identified with the class of $0 - 1$ valued additive probabilities defined on all subsets of B which are in one-to-one correspondence with filter as established in Walley [24, Section 2.9.8].

If the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension the functional $\overline{P}(X|B)$ is monotone, comonotonically additive, submodular and continuous from below.

In Denneberg [Theorem 8.1] [5] the following result has been proven:

Theorem 13 *(Monotone Convergence Theorem) Let μ be a monotone set function on a σ-field $\mathbf{F}$ properly contained in $\wp(\Omega)$, which is continuous from below. For an increasing sequence of non negative, $\mathbf{F}$-measurable random variables X_n in $L^*(B)$ the limit function $X = \lim_{n\to\infty}X_n$ is $\mathbf{F}$-measurable and $\lim_{n\to\infty}\int X_n d\mu = \int X d\mu$.*

Remark 1 *It is not restrictive to consider in the Monotone Convergence Theorem sequence of non-negative random variables since any random variable X can be decomposed in two comonotonic functions which are its positive part X^+ and its negative part X^- given by:*

$$X^+ = 0 \vee X; \quad X^- = (-X)^+$$

where $\vee$ is the maximum so that $X = X^+ - X^-$.

Theorem 14 *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the s-dimensional Hausdorff outer measure. Let $\mathbf{F} \subset \wp(\Omega)$ be a σ-field and let $\mathbf{K}$ be the class of all $\mathbf{F}$-measurable random variables. If B has positive and finite Hausdorff outer measure in its dimension then the functional defined as in Theorem 12 is continuous from below, that is given an increasing sequence of non negative random variables X_n of $\mathbf{K}$ converging point-wise to the random variable X we have that $\lim_{n\to\infty}\overline{P}(Xn|B) = \overline{P}(X|B)$.*

Proof If B has positive and finite Hausdorff outer measure in its Hausdorff dimension s we have that $\overline{P}(X|B) = \frac{1}{h^s(B)}\int_B X dh^s$. Since each s-dimensional Hausdorff outer measure is continuous from below then by the Monotone Convergence Theorem it follows that the given upper conditional prevision is continuous from below, that is

$$\lim_{n\to\infty}\overline{P}(Xn|B) = \lim_{n\to\infty}\frac{1}{h^s(B)}\int_B X_n dh^s =$$
$$\frac{1}{h^s(B)}\int_B X dh^s = \overline{P}(X|B).$$

$\blacksquare$

Example 2 *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the s-dimensional Hausdorff outer measure. Let $\mathbf{F}$ be the σ-field of h^s-measurable sets and let $\mathbf{K}$ be the class of all $\mathbf{F}$-measurable random variables. If B has positive and finite Hausdorff outer measure in its dimension then the functional defined as in Theorem 12 is continuous from below, that is given an increasing sequence of non negative random variables X_n of $\mathbf{K}$ converging point-wise to the random variable X we have that $\lim_{n\to\infty}\overline{P}(Xn|B) = \overline{P}(X|B)$.*

Corollary 15 *When the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension the coherent upper conditional prevision defined in Theorem 12 preserves equivalence between unbounded random variables with the same distribution.*

3. Different Convergences of Coherent Upper Previsions

It is important to note that, if the convergence of the sequence X_n of random variables to X is not point-wise as required in the Monotone Convergence Theorem, we cannot avoid positive boost function. If for example X_n converges in probability to X ([1] p. 274) and coherent (conditional) previsions are defined to be continuous with respect to convergence in probability then for unbounded random variables, prevision cannot be equal to the expectation and $b(X) = P(X) - E(X)$ is positive. It occurs because convergence in probability does not assure the convergence of the integral (see Example 3) so that, for unbounded measurable random variables, prevision may be not equal to the expectation and $b(X) = P(X) - E(X)$ is positive.

The convergence in probability can be extended to an upper probability μ (it is called μ-stochastic convergence in Denneberg [5] p. 97)

Definition 16 *Given a coherent upper probability μ, a sequence X_n of random variables converges in μ upper probability to the random variable X if $\forall \varepsilon > 0$*

$$\lim_{n\to+\infty}\mu(\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon) = 0.$$

In the paper of Troffaes and de Cooman ([20]) the problem to extend any given coherent lower prevision to a set including some unbounded random variables is investigated. Moreover a theory of conditional coherent lower prevision for random quantities, including unbounded ones is introduced in Troffaes ([22]). In ([20]) the authors construct a limit procedure, approximating unbounded random variables by bounded ones. They extend the notion of convergence in probability to an upper probability and call it convergence hazy since it is similar to the limit procedure of the Dunford integration. The lower prevision can be defined for an arbitrary random variable X that is the limit, according to the hazy convergence, of a sequence of bounded random variables X_n. The lower prevision of X is defined as the limit of the lower prevision of X_n and can be written as the Dunford integral that is an integral with respect to a finitely additive probability. For this reason this integral does not satisfy the Monotone Convergence Theorem. Moreover the limit procedure proposed in ([20]) does not assure the continuity of the (lower) prevision and so it cannot be used to solve the problem of losing equivalence between unbounded random variables with the same distribution. It occurs because the proposed method is based on the convergence in probability; in fact even if we define the prevision by the integral with respect to a countably additive probability instead of a merely finitely additive probability as occurs with the Dunford integral, we have that the convergence in probability does not assure the convergence of the integral and so the continuity of the prevision as shown by the following example.

Example 3 *Let (Ω, F, P) be the probability space with $\Omega =$ [0,1], F the Borel σ-field of [0,1] and P equal to the Lebesgue measure on F. Denote by $I_{(0,\frac{1}{n})}$ the indicator function of the interval $(0, \frac{1}{n})$ and consider the sequence of random variables $X_n = n^2 I_{(0,\frac{1}{n})}$. We have that X_n converges in probability to $X = 0$, but $\int X dP = 0$ while the $\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} n = \infty$.*

4. Disintegration Property of a Coherent Upper Conditional Prevision Defined with Respect to Its Associated Hausdorff Outer Measure

In this section the coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$, defined in Theorem 10, is proven to satisfy the disintegration property with respect to every non-null partition $\mathbf{B}$ if Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension. In Doria [9] the following theorem has been proven. Disintegration property for coherent linear previsions has been investigated in Seidenfeld et al. [18] and for coherent lower and upper conditional previsions in Miranda et al. [13] and Doria [9].

Definition 17 *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t and let $\mathbf{B}^*$ be a countable subclass of disjoint subsets of Ω. The chain $M(\mathbf{B}^*)$ generated by the subclass $\mathbf{B}^*$ is defined as the class containing $\oslash$, Ω and the following sets*

$$C_1 = B_1, ... C_n = C_{n-1} \cup B_n, ..., C = \cup_{n=1}^{+\infty} B_n$$

In Doria [9] the following result has been proven.

Theorem 18 *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t and and let μ_Ω^* be the coherent upper conditional probability defined on $\wp(\Omega)$ by $\mu_\Omega^* = \frac{h^t(A)}{h^t(\Omega)}$ for $A \in \wp(\Omega)$. Given a countable subclass $\mathbf{B}^*$ of Ω containing disjoint sets, denote by A the σ-field generated by the chain $M(\mathbf{B}^*)$. Then each $B \in \mathbf{B}$ is μ_Ω^*-measurable.*

Definition 19 *Let Ω be a set with positive and finite Hausdorff measure in its Hausdorff dimension t. Let $\mathbf{B}$ be a partition of Ω and let $\mathbf{B}^*$ be the subclass of $\mathbf{B}$ of sets B with positive upper coherent probability μ_Ω^*, thus $\mathbf{B}$ is a non-null partition if $\mu_\Omega^*(\Omega - \cup_{B \in \mathbf{B}^*} B) = 0$.*

Theorem 20 *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t. Thus the coherent conditional prevision $\overline{P}(X|\mathbf{B})$ defined in Theorem 10 satisfies the disintegration property on every non-null partition $\mathbf{B}$ of Ω.*

Proof Ω is a set with with positive and finite Hausdorff outer measure in its Hausdorff dimension t so that the restriction $\mu_\Omega(\cdot) = \frac{h^t(\cdot)}{h^t(\Omega)}$ to the σ-field of the h^t-measurable

sets, of the upper unconditional probability defined in Theorem 12, is a countably additive probability. Moreover, since $\mathbf{B}$ is a non-null partition, there is at most a non-empty countable subclass $\mathbf{B}^*$ of $\mathbf{B}$ of sets B with positive upper coherent probability μ_Ω^*. By Theorem 18 these sets are μ_Ω^*-measurable.

Since every random variable X and every constant c in $L^*(\Omega)$ are comonotonic, we consider the two comonotonic classes $\mathbf{C} = \{\overline{P}(X|\mathbf{B}), c\}$ and $\mathbf{C}_1 = \{X, c\}$ so that by Proposition 10.1 of [5] there exist two additive set functions α, and α' on $\wp(\Omega)$, which agree with h^t on the σ-field of h^t-measurable sets, such that

$$\int_\Omega \overline{P}(X|\mathbf{B}) dh^t = \int_\Omega \overline{P}(X|\mathbf{B}) d\alpha$$

and

$$\int_B X dh^t = \int_\Omega I_B X dh^t = \int_\Omega I_B X d\alpha = \int_B X d\alpha'.$$

Then for every random variable $X \in L^*(\Omega)$ the disintegration property is satisfied for every non-null partition $\mathbf{B}$ since the following equalities hold:

$$\overline{P}(\overline{P}(X|\mathbf{B})) = \frac{1}{h^t(\Omega)} \int_\Omega \overline{P}(X|\mathbf{B}) dh^t =$$

$$= \frac{1}{h^t(\Omega)} \int_\Omega \overline{P}(X|\mathbf{B}) d\alpha =$$

$$= \sum_{B \in \mathbf{B}^*} \left(\frac{1}{h^t(B)} \int_B X d\alpha' \right) \frac{h^t(B)}{h^t(\Omega)} =$$

$$= \frac{1}{h^t(\Omega)} \sum_{B \in \mathbf{B}^*} \int_B X d\alpha = \frac{1}{h^t(\Omega)} \int_\Omega X dh^t = \overline{P}(X).$$

■

5. Uniqueness of the Representing Coherent Upper Conditional Probability

Given a metric space (Ω, d) we can determine conditions under which the upper conditional probability $\mu^* = \frac{h^s(AB)}{h^s(B)}$ where h^s is the s-dimensional Hausdorff outer measure, is the unique fuzzy measure that represents a coherent upper conditional prevision $\overline{P}(\cdot|B)$ as Choquet integral.

In Denneberg ([5] Chapter 13), representation theorems for functionals with minimal requirements on the domain are examined. Let $\mathbf{F}$ be a class of random variables such that

a') $X \geq 0$ for all $X \in \mathbf{F}$ (non negativity)

b') $aX, X \wedge a, X - X \wedge a \in \mathbf{F}$ if $X \in \mathbf{F}, a \in \Re^+$

c') $X \wedge Y, X \vee Y$ if $X, Y \in \mathbf{F}$ (lattice property).

In Proposition 13.5 of ([5]) it is proven that if a functional Γ, defined on the domain $\mathbf{F}$, is monotone, comonotonically additive, submodular and continuous from below then Γ is representable as Choquet integral with respect to a monotone, submodular set function which is continuous from below. Furthermore all set functions on $\wp(\Omega)$ with these properties agree on the set system of weak upper level sets $M = \{\{X \geq x\} | X \in F, x \in \Re\}$.

The uniqueness of the representing set function is due to the fact that the function $\Gamma(X \wedge x)$ determines the distribution function $G_{\mu,X}$ of an upper μ-measurable and positive random variable X with respect to any set function μ representing Γ since $G_{\mu,X} = \frac{d}{dx}\Gamma(X \wedge x)$ for $X \in \mathbf{F}$ and for all $x \in \Re^+$. If $G_{\mu,X}$ is right continuous then $G_{\mu,X}$ is the derivative from right of $\int(X \wedge x)d\mu$ ([5] Lemma 13.1). If the domain $\mathbf{F}$ is a linear lattice containing all constants this result can be extended to every bounded random variable. In fact if X is ≤ 0 then, since X is bounded, there exists a constant k such that $Y = X - k \in \mathbf{F}$ and $Y = X - k \geq 0$ so that $G_{\mu,Y} = \frac{d}{dx}\Gamma(Y \wedge x)$. (Doria [7])

In the next theorem it is proven that if B has positive and finite Hausdorff outer measure in its dimension s and the coherent upper conditional prevision $\overline{P}(\cdot|B)$ is monotone, comonotonically additive, submodular and continuous from below then the upper conditional probability defined by the s-dimensional Hausdorff outer measure h^s is the unique monotone set function on the set system of weak upper level sets $M = \{\{X \geq x\} | X \in \mathbf{F}, x \in \Re\}$, which is submodular, continuous from below and representing $\overline{P}(\cdot|B)$ as Choquet integral.

Theorem 21 *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let $F \subset L^*(B)$ be a linear lattice of non-negative random variables on B. If B has positive and finite Hausdorff outer measure in its dimension and the coherent upper conditional prevision $\overline{P}(\cdot|B)$ on $\mathbf{F}$ is monotone, comonotonically additive, submodular and continuous from below then $\overline{P}(\cdot|B)$ is representable as Choquet integral with respect to a monotone, submodular set function which is continuous from below. Furthermore all monotone set functions on $\wp(B)$ with these properties agree on the set system of weak upper level sets $M = \{\{X \geq x\} | X \in \mathbf{F}, x \in \Re\}$, with the coherent upper conditional probability $\mu^*(A) = \frac{h^s(AB)}{h^s(B)}$ for $A \in \wp(B)$.*

Proof $\mathbf{F}$ is a linear lattice containing all constants then conditions a'), b') and c') are satisfied. So from Proposition 13.5 of [5] we obtain that the functional $\overline{P}(\cdot|B)$ is representable by a monotone, submodular, continuous from below set function. Moreover all set functions with these properties agree on the set system of weak upper level sets $M = \{\{X \geq x\} | X \in \mathbf{F}, x \in \Re\}$. ∎

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Every t-dimensional Hausdorff outer measure is monotone, submodular and continuous from below but the functional $\overline{P}(\cdot|B)$ is representable only by the s-dimensional Hausdorff measure, whre s is the Hausdorff dimension of the conditioning event, for this reasoning it is called Hausdorff outer measure *associated* with $\overline{P}(\cdot|B)$. In fact the following cases can be considered:

$t < s$; the functional $\overline{P}(\cdot|B)$ cannot be represented by the Choquet integral with respect to h^t because in this case we would have $\overline{P}(B|B) = \int_B dh^t = h^t(B) = \infty$ and it is a contradiction since $\overline{P}(\cdot|B)$ is a coherent upper conditional prevision.

$t > s$ then $h^t(B) = 0$; so the functional $\overline{P}(\cdot|B)$ cannot be represented by the Choquet integral with respect to the t-dimensional Hausdorff outer measure because in this case $\overline{P}(\cdot|B)$ would be equal to 0 on $\mathbf{L}*(B)$ and so it would not be a coherent conditional prevision since it does not satisfy the condition $\overline{P}(B|B) = 1$, necessary for the coherence.

So all monotone set functions on $\wp(\Omega)$ which are submodular, continuous from below and represent the functional $\overline{P}(\cdot|B)$ agree on the set system of weak upper level sets with the coherent upper conditional probability $\mu^*(A) = \frac{h^s(AB)}{h^s(B)}$.

6. Conclusions

The aim of this paper is to extend the model of coherent upper conditional previsions defined by Hausdorff outer measures to domains containing unbounded random variables. Since the class of the bounded and unbounded random variables which admit Choquet integral is not a linear space, first it is proven that the class $L^*(B)$ of all random variables which have finite Choquet integral with respect to the coherent upper conditional probability μ_B^* and with respect to its conjugate lower conditional probability is a linear space. Then, if the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension, a coherent upper conditional prevision $\overline{P}(X|B)$, defined as Choquet integral with respect to its associated Hausdorff outer measure on these domains is proven to satisfy the Monotone Convergence Theorem. Given a non-null partition $\mathbf{B}$ the coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is proven to satisfy the disintegration property. The coherent upper conditional probability μ_B defined by Hausdorff outer measure and all monotone set functions which are submodular and continuous from below and represent a coherent upper conditional $\overline{P}(X|B)$, defined on a linear lattice of random variables contained in $L^*(B)$, are proven to coincide on the class of all weak upper level sets. This result represents a strong connection between coherent upper conditional probability defined by Hausdorff outer measure and other monotone set functions which represent the same coherent upper conditional prevision.

References

[1] Billingsley, P. *Probability and measure*. Wiley, 1986

[2] Choquet, G. *Theory of capacity*. Ann. Inst. Fourier, 5, 131-295, 1953

[3] de Finetti, B. *Probability Induction and Statistics*. Wiley, New York, 1970

[4] de Finetti, B. *Theory of Probability*. Wiley, London, 1974

[5] Denneberg, D., *Non-additive measure and integral*. Kluwer Academic Publishers, 1994

[6] Doria, S.Probabilistic independence with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures. *International Journal of Approximate Reasoning*, 46, 617-635, 2007

[7] Doria, S. Characterization of a coherent upper conditional prevision as the Choquet integral with respect to its associated Hausdorff outer measure. *Annals of Operations Research*, 33-48, 2012

[8] Doria, S. Symmetric coherent upper conditional prevision by the Choquet integral with respect to Hausdorff outer measure. *Annals of Operations Research*, 377-396, 2014

[9] Doria, S. On the disintegration property of a coherent upper conditional prevision by the Choquet integral with respect to its associated Hausdorff outer measure. *Annals of Operations Research*, Volume 216, Issue 2, 253-269, 2017

[10] Doria, S., Dutta, B., Mesiar, R. Integral representation of coherent upper conditional prevision with respect to its associated Hausdorff outer measure: a comparison among the Choquet integral, the pan-integral and the concave integral. *International Journal of General Systems*, Volume 216, Issue 2, 569-592, 2018

[11] Dubins, L.E. Finitely additive conditional probabilities, conglomerability and disintegrations. *The Annals of Probability*, Vol. 3, 89-99, 1975

[12] Falconer, K.J. *The geometry of fractal sets*. Cambridge University Press, 1986

[13] Miranda E., Zaffalon M., de Cooman G., Conglomerable natural extensions, *International Journal of Approximate Reasoning*, Vol 53, Issue 8, 1200-1227, 2012

[14] Munroe, M. E., *Introduction to Measure and Integration*, Addison-Wesley, Massachussetts, 1953 *International Journal of Approximate Reasoning*, Vol 53, Issue 8, 1200-1227, 2012

[15] Regazzini, E. *Finitely additive conditional probabilities*. Rend. Sem. Mat. Fis., Milano, 5 5,69-89, 1985

[16] Regazzini, E. De Finetti's coherence and statistical inference. *The Annals of Statistics*, Vol. 15, No. 2, 845-864, 1987

[17] Rogers, C. A *Hausdorff measures*. Cambridge University Press, 1970

[18] Seidenfeld T., Schervish M. J., and Kadane J. K. , Non-conglomerability for finite-valued, finitely additive probability. *The Indian Journal of Statistics*, Special issue on Bayesian Analysis, Vol.60, Series A, 476-491, 1998

[19] Seidenfeld T., Schervish M.,Kadane J.B.. Preferences for equivalent random variables: A price for unbounded utilities. *Journal of Mathematical Economics* 45, 329-340, 2009

[20] Troffaes M. C. M., de Cooman G. Extension of coherent lower previsions to unbounded random variables. *Proceedings of the 9th International Conference IPMU 2002*, 735-42, 2002

[21] Troffaes M. C. M., de Cooman G. Lower previsions for unbounded random variables *Soft Methods in Probability, statistics and Data Analysis*, 146-155, 2002

[22] Troffaes M. C. M. Conditional lower previsions for unbounded random quantities. *Soft Methods for Integrated Uncertainty Modelling* 6, 201-209, 2006.

[23] Walley, P. *Coherent lower (and upper) probabilities*, Statistics Research Report, University of Warwick, 1981

[24] Walley, P. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991