Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability

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Abstract

Choice functions constitute a simple, direct and very general mathematical framework for modelling choice under uncertainty. In particular, they are able to represent the set-valued choices that appear in imprecise-probabilistic decision making. We provide these choice functions with a clear interpretation in terms of desirability, use this interpretation to derive a set of basic coherence axioms, and show that this notion of coherence leads to a representation in terms of sets of strict preference orders. By imposing additional properties such as totality, the mixing property and Archimedeanity, we obtain representation in terms of sets of strict total orders, lexicographic probability systems, coherent lower previsions or linear previsions.

Keywords: choice functions, coherence, desirability, representation, non-binary choice models.

1. Introduction

Choice functions provide an elegant unifying mathematical framework for studying set-valued choice: when presented with a set of options, they generally return a subset of them. If this subset is a singleton, it provides a unique optimal choice or decision. But if the answer contains multiple options, these are incomparable and no decision is made between them. Such set-valued choices are a typical feature of decision criteria based on imprecise-probabilistic uncertainty models, which aim to make reliable decisions in the face of severe uncertainty. Maximality and E-admissibility are well-known examples. When working with a choice function, however, it is immaterial whether it is based on such a decision criterion. The primitive objects on this approach are simply the set-valued choices themselves, and the choice function that represents all these choices serves as an uncertainty model in and by itself.

The seminal work by Seidenfeld et al. [17] has shown that a strong advantage of working with choice functions is that they allow us to impose axioms on choices, aimed at characterising what it means for choices to be rational and internally consistent. This is also what we want to do here, but we believe our angle of approach to be novel and unique: rather than think of choice intuitively, we provide it with a concrete interpretation in terms of desirability [4, 8, 9, 25] or binary preference [15]. Another important feature of our approach is that we consider a very general setting, where the options form an abstract real vector space; horse lotteries and gambles correspond to special cases.

The basic structure of our paper is as follows. We start in Section 2 by introducing choice functions and our interpretation for them. Next, in Section 3, we develop an equivalent way of describing these choice functions: sets of desirable option sets. We use our interpretation to suggest and motivate a number of rationality, or coherence, axioms for such sets of desirable option sets, and show in Section 4 what are the corresponding coherence axioms for choice (or rejection) functions. Section 5 deals with the special case of binary choice, and its relation to the theory of sets of desirable options [4, 8, 9, 25] and binary preference. This is important because our main result in Section 6 shows that any coherent choice model can be represented in terms of sets of such binary choice models. In the remaining Sections 7–9, we consider additional axioms or properties, such as totality, the mixing property, and an Archimedean property, and prove corresponding representation results. This includes representations in terms of sets of strict total orders, sets of lexicographic probability systems, sets of coherent lower previsions and sets of linear previsions.

Proofs have been relegated to the appendix of an extended arXiv version [7].

2. Choice Functions and Their Interpretation

A choice function C is a set-valued operator on sets of options. In particular, for any set of options A, the corresponding value of C is a subset $C(A)$ of A. The options themselves are typically actions amongst which a subject wishes to choose. We here follow a very general approach where these options constitute an abstract real vector space $\mathscr{V}$ provided with a—so-called *background*—vector ordering $\preceq$ and a strict version $\prec$. The elements u of $\mathscr{V}$ are called *options* and $\mathscr{V}$ is therefore called the *option space*. We let $\mathscr{V}_{\succ 0} := \{u \in \mathscr{V} : u \succ 0\}$. The purpose of a choice function is to represent our subject's choices between such options.

Our motivation for adopting this general framework where options are elements of abstract vector spaces, rather than the more familiar one that focuses on choice between, say, horse lotteries [2, 3, 11, 15, 17], is its applicability to various contexts.

A typical set-up that is customary in decision theory, for example, is one where every option has a corresponding reward that depends on the state of a variable X, about which the subject is uncertain. Hence, the reward is uncertain too. As a special case, therefore, we can consider that the variable X takes values x in a set of states $\mathscr{X}$. The reward that corresponds to a given option is then a function u on $\mathscr{X}$. If we assume that this reward can be expressed in terms of a real-valued linear utility scale, this allows us to identify every act with a real-valued map on $\mathscr{X}$. These maps are often taken to be bounded and are then called *gambles* on X. In this context, we can consider the different gambles on X as our options, and let $\mathscr{V}$ be the set of all such gambles. Two popular choices for $\mathscr{V}_{\succ 0}$ are then

$$\{u \in \mathscr{V} \colon u \geq 0 \text{ and } u \neq 0\} \text{ or } \{u \in \mathscr{V} \colon \inf u > 0\},$$

where $\geq$ represents the point-wise ordering of gambles.

A more general framework, which allows us to dispense with the linearity assumption of the utility scale, consists in considering as option space the linear space of all bounded real-valued maps on the set $\mathscr{X} \times \mathscr{R}$, where $\mathscr{R}$ is a (finite) set of rewards. Zaffalon and Miranda [26] have shown that, in a context of binary preference relations, this leads to a theory that is essentially equivalent to the classical horse lottery approach. It tends, however, to be more elegant, because a linear space is typically easier to work with than a convex set of horse lotteries. Van Camp [21] has shown that this idea can be straightforwardly extended from binary preference relations to the more general context of choice functions. We follow his lead in focusing on linear spaces of options here.

In both of the above-mentioned cases, the options are still bounded real-valued maps. In fairly recent work, Van Camp et al. [21, 22] have shown that a notion of indifference can be associated with choice functions quite easily, by moving from the original option space to its quotient space with respect to the linear subspace of all options that are assessed to be equivalent to the zero option. Even when the original options are real-valued maps, the elements of the quotient space will be equivalence classes of such maps—affine subspaces of the original option space—which can no longer be straightforwardly identified with real-valued maps. This provides even more incentives for considering options to be vectors in some abstract linear space $\mathscr{V}$.

Having introduced and motivated our abstract option space $\mathscr{V}$, sets of options can now be identified with subsets of $\mathscr{V}$, which we call *option sets*. We restrict our attention here to *finite* option sets and will use $\mathscr{Q}$ to denote the set of all such finite subsets of $\mathscr{V}$, including the empty set.

Definition 1 *A* choice function *C is a map from $\mathscr{Q}$ to $\mathscr{Q}$ such that $C(A) \subseteq A$ for every $A \in \mathscr{Q}$.*

Options in A that do not belong to $C(A)$ are said to be *rejected*. This leads to an alternative but equivalent representation in terms of rejection functions: the *rejection function* R_C corresponding to a choice function C is a map from $\mathscr{Q}$ to $\mathscr{Q}$, defined by $R_C(A) := A \setminus C(A)$ for all $A \in \mathscr{Q}$.

Alternatively, a rejection function R can also be independently defined as a map from $\mathscr{Q}$ to $\mathscr{Q}$ such that $R(A) \subseteq A$ for all $A \in \mathscr{Q}$. The corresponding choice function C_R is then clearly defined by $C_R(A) := A \setminus R(A)$ for all $A \in \mathscr{Q}$. Since a choice function is completely determined by its rejection function, any interpretation for rejection functions automatically implies an interpretation for choice functions. This allows us to focus on the former.

Our interpretation for rejection functions—and therefore also for choice functions—now goes as follows. Consider a subject whose uncertainty is represented by a rejection function R, or equivalently, by a choice function C_R. Then for a given option set $A \in \mathscr{Q}$, the statement that an option $u \in A$ is rejected from A—that is, that $u \in R(A)$—is taken to mean that *there is at least one option v in A that our subject strictly prefers over u*.

If we denote the strict preference of one option v over another option u by $v \rhd u$, this can be written succinctly as

$$(\forall A \in \mathscr{Q})(\forall u \in A)\big(u \in R(A) \Leftrightarrow (\exists v \in A) v \rhd u\big). \quad (1)$$

In this paper, such a statement—as well as statements such as those in Equations (2) and (3)—will be interpreted as providing information about a strict preference relation $\rhd$, that may or may not be known or specified. The only requirements that we impose on $\rhd$ is that it should be a strict partial order that extends the background ordering $\succ$ and is compatible with the vector space operations on $\mathscr{V}$:

- $\rhd_0$. $\rhd$ is irreflexive: for all $u \in \mathscr{V}$, $u \not\rhd u$;
- $\rhd_1$. $\rhd$ is transitive: for all $u, v, w \in \mathscr{V}$, $u \rhd v$ and $v \rhd w$ imply that also $u \rhd w$;
- $\rhd_2$. for all $u, v \in \mathscr{V}$, $u \succ v$ implies that $u \rhd v$;
- $\rhd_3$. for all $u, v, w \in \mathscr{V}$, $u \rhd v$ implies that—so is equivalent with—$u + w \rhd v + w$;
- $\rhd_4$. for all $u, v \in \mathscr{V}$ and all $\lambda > 0$, $u \rhd v$ implies that—so is equivalent with—$\lambda u \rhd \lambda v$.

We then call such a preference ordering $\rhd$ *coherent*. Equation (1) now implies that for all $A \in \mathscr{Q}$ and $u \in A$:

$$\begin{aligned} u \in R(A) &\Leftrightarrow (\exists v \in A) v - u \rhd 0 \\ &\Leftrightarrow (\exists v \in A \setminus \{u\}) v - u \rhd 0, \end{aligned} \quad (2)$$

where we use Axiom $\rhd_3$ for the first equivalence, and Axiom $\rhd_0$ for the second. Both equivalences can be conveniently turned into a single one if we no longer require that u should belong to A and consider statements of the form $u \in R(A \cup \{u\})$. Equation (2) then turns into

$$u \in R(A \cup \{u\}) \Leftrightarrow (\exists v \in A) v - u \rhd 0, \tag{3}$$

for all $u \in \mathscr{V}$ and $A \in \mathscr{Q}$. So, according to our interpretation, the statement that u is rejected from $A \cup \{u\}$ is taken to mean that the option set

$$A - u := \{v - u \colon v \in A\} \tag{4}$$

contains at least one option that, according to $\rhd$, is strictly preferred to the zero option 0.

3. Coherent Sets of Desirable Option Sets

A crucial observation at this point is that our interpretation for rejection functions does not require our subject to specify the strict preference $\rhd$. Instead, all that is needed is for her to specify option sets $A \in \mathscr{Q}$ that—to her—contain at least one option that is strictly preferred to the zero option 0. Options that are strictly preferred to zero—so options u for which $u \rhd 0$—are also called *desirable*, which is why we will call such option sets *desirable option sets* and collect them in a *set of desirable option sets* $K \subseteq \mathscr{Q}$. Our interpretation therefore allows a modeller to specify her beliefs by specifying a *set of desirable option sets* $K \subseteq \mathscr{Q}$.

As can be seen from Equations (3) and (4), such a set of desirable option sets K completely determines a rejection function R and its corresponding choice function C_R:

$$(\forall u \in \mathscr{V})(\forall A \in \mathscr{Q})\big(u \in R(A \cup \{u\}) \Leftrightarrow A - u \in K\big). \tag{5}$$

Our interpretation, together with the basic Axioms $\rhd_0$ *and* $\rhd_3$*, therefore allows the study of rejection and choice functions to be reduced to the study of sets of desirable option sets.*

We let $\mathbf{K}$ denote the set of all sets of desirable option sets $K \subseteq \mathscr{Q}$, and consider any such $K \in \mathbf{K}$. The first question to address is when to call K *coherent*: which properties should we impose on a set of desirable option sets in order for it to reflect a rational subject's beliefs? We propose the following axiomatisation, using $(\lambda, \mu) > 0$ as a shorthand notation for '$\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu > 0$'.

Definition 2 *A set of desirable option sets* $K \subseteq \mathscr{Q}$ *is called* coherent *if it satisfies the following axioms:*

K_0. *if* $A \in K$ *then also* $A \setminus \{0\} \in K$*, for all* $A \in \mathscr{Q}$*;*

K_1. $\{0\} \notin K$*;*

K_2. $\{u\} \in K$*, for all* $u \in \mathscr{V}_{\succ 0}$*;*

K_3. *if* $A_1, A_2 \in K$ *and if, for all* $u \in A_1$ *and* $v \in A_2$*,* $(\lambda_{u,v}, \mu_{u,v}) > 0$*, then also*[1]

$$\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \in K;$$

K_4. *if* $A_1 \in K$ *and* $A_1 \subseteq A_2 \in \mathscr{Q}$*, then also* $A_2 \in K$*.*

We denote the set of all coherent sets of desirable option sets by $\overline{\mathbf{K}}$*.*

This axiomatisation is entirely based on our interpretation and the following three axioms for desirability:

d_1. 0 is not desirable;

d_2. all $u \in \mathscr{V}_{\succ 0}$ are desirable;

d_3. if u, v are desirable and $(\lambda, \mu) > 0$, then $\lambda u + \mu v$ is desirable.

Each of these three axioms follows trivially from our assumptions on the preference relation $\rhd$: Axiom d_1 follows from $\rhd_0$, Axiom d_2 follows from $\rhd_2$ and Axiom d_3 follows from $\rhd_1$ and $\rhd_4$.[2]

That the coherence Axioms K_0–K_4 are implied by our rationality requirements d_1–d_3 for the concept of desirability, can now be seen as follows. Since a desirable option set is by definition a set of options that contains at least one desirable option, Axiom K_4 is immediate. Axioms K_0 and K_1 follow naturally from d_1, and Axiom K_2 is an immediate consequence of d_2. The argument for Axiom K_3 is more subtle. Since A_1 and A_2 are two desirable option sets, there must be at least one desirable option $u \in A_1$ and one desirable option $v \in A_2$. Since for these two options, the positive linear combination $\lambda_{u,v} u + \mu_{u,v} v$ is again desirable by d_3, at least one of the elements of the option set $\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\}$ must be a desirable option. Hence, it must be a desirable option set.

4. Coherent Rejection Functions

Now that we have formulated our basic rationality requirements K_0–K_4 for a set of desirable option sets K, we are in a position to use their link (5) with rejection functions to derive equivalent rationality requirements for the latter.

Equation (5) already allows us to derive a first and very basic axiom for rejection functions—and a very similar one for choice functions, left implicit here—without imposing *any* requirements on the set of desirable option sets K:

R_0. for all $A \in \mathscr{Q}$ and $u \in A$: $u \in R(A) \Leftrightarrow 0 \in R(A - u)$.

1. The following simple example might help the reader understand what this axiom allows for. Consider any two $a, b \in \mathscr{V}$, let $A_1 = A_2 = A := \{a, b\}$ and choose $(\lambda_{a,a}, \mu_{a,a}) = (1, 0)$, $(\lambda_{a,b}, \mu_{a,b}) = (1, 1)$, $(\lambda_{b,a}, \mu_{b,a}) = (1, 1)$ and $(\lambda_{b,b}, \mu_{b,b}) = (1, 1)$. Then if $A \in K$, it follows from Axiom K_3 that also $\{a, a+b, 2b\} \in K$.
2. Conversely, under Axiom $\rhd_3$ for the preference relation $\rhd$, these three Axioms d_1–d_3 imply the remaining Axioms $\rhd_0$–$\rhd_2$ and $\rhd_4$.

When we do impose requirements on sets of desirable option sets K, Equation (5) allows us to turn them into requirements for rejection (and hence also choice) functions. In particular, we will see in Proposition 4 below that our Axioms K_0–K_4 imply that

R_1. $R(\emptyset) = \emptyset$, and $R(A) \neq A$ for all $A \in \mathscr{Q} \setminus \{\emptyset\}$;

R_2. $0 \in R(\{0, u\})$, for all $u \in \mathscr{V}_{\succ 0}$;

R_3. if $A_1, A_2 \in \mathscr{Q}$, $0 \in R(A_1 \cup \{0\})$ and $0 \in R(A_2 \cup \{0\})$ and if $(\lambda_{u,v}, \mu_{u,v}) > 0$ for all $u \in A_1$ and $v \in A_2$, then

$$0 \in R(\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \cup \{0\});$$

R_4. if $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$, for all $A_1, A_2 \in \mathscr{Q}$.

Axiom R_4 is Sen's condition α [18, 19].

Definition 3 *A rejection function R is called* coherent *if it satisfies the Axioms R_0–R_4. A choice function C is called* coherent *if the associated rejection function R_C is.*

Our next result establishes that these notions of coherence are perfectly compatible with the coherence for sets of desirable option sets that we introduced in Section 3.

Proposition 4 *Consider any set of desirable option sets $K \in \mathbf{K}$ and any rejection function R that are connected by Equation (5). Then K is coherent if and only if R is.*

We will from now on work directly with (coherent) sets of desirable option sets and will use the collective term *(coherent) choice models* for (coherent) choice functions, rejection functions, and sets of desirable option sets. Of course, our primary motivation for studying coherent sets of desirable option sets is their connection with the other two choice models. This being said, it should however also be clear that our results do not depend on this connection. The theory of sets of desirable option sets that we are about to develop can therefore be used independently as well.

5. The Special Case of Binary Choice

According to our interpretation, the statement that A belongs to a set of desirable option sets K is taken to mean that A contains at least one desirable option. This implies that singletons play a special role: for any $u \in \mathscr{V}$, stating that $\{u\} \in K$ is equivalent to stating that u is desirable. For any set of desirable option sets K, these singleton assessments are captured completely by the set of options

$$D_K := \{u \in \mathscr{V} \colon \{u\} \in K\}$$

that, according to K, are definitely desirable—preferred to 0. A set of desirable option sets $K \in \mathbf{K}$ that is completely determined by such singleton assessments is called *binary*.

Definition 5 *A set of desirable option sets K is* binary *if*

$$A \in K \Leftrightarrow (\exists u \in A)\{u\} \in K, \text{ for all } A \in \mathscr{Q}.$$

In order to explain how any binary set of desirable option sets K is indeed completely determined by D_K, we need a way to associate a rejection function with sets of options such as D_K. To that end, we consider the notion of a *set of desirable options*: a subset D of $\mathscr{V}$ whose interpretation will be that it consists of the options $u \in \mathscr{V}$ that our subject considers desirable. We denote the set of all such sets of desirable options $D \subseteq \mathscr{V}$ by $\mathbf{D}$.

With any $D \in \mathbf{D}$, our interpretation for rejection functions in Section 2 inspires us to associate a set of desirable option sets K_D, defined by

$$K_D := \{A \in \mathscr{Q} \colon A \cap D \neq \emptyset\}.$$

It turns out that a set of desirable options sets K is binary if and only if it has the form K_D, and the unique *representing* D is then given by D_K.

Proposition 6 *A set of desirable options sets $K \in \mathbf{K}$ is binary if and only if there is some $D \in \mathbf{D}$ such that $K = K_D$. This D is then necessarily unique, and equal to D_K.*

Just like we did for sets of desirable option sets in Section 3, we can use the basic rationality principles d_1–d_3 for the notion of desirability—or binary preference—to infer basic rationality criteria for sets of desirable options. When they do, we call them coherent.

Definition 7 *A set of desirable options $D \in \mathbf{D}$ is called* coherent *if it satisfies the following axioms:*[3]

D_1. *$0 \notin D$;*

D_2. *$\mathscr{V}_{\succ 0} \subseteq D$;*

D_3. *if $u, v \in D$ and $(\lambda, \mu) > 0$, then $\lambda u + \mu v \in D$.*

$\overline{\mathbf{D}}$ denotes the set of all coherent sets of desirable options.

So a coherent set of desirable options is a convex cone [Axiom D_3] in $\mathscr{V}$ that does not contain 0 [Axiom D_1] and includes $\mathscr{V}_{\succ 0}$ [Axiom D_2]. Sets of desirable options are an abstract version of the sets of desirable gambles that have an important part in the literature on imprecise probability models [4, 9, 12, 25]. This abstraction was first introduced and studied in great detail in [8, 14].

Our next result shows that the coherence of a binary set of desirable option sets is completely determined by the coherence of its corresponding set of desirable options.

3. The Axioms D_1–D_3 for sets of desirable options should not be confused with the rationality criteria d_1–d_3 for our primitive notion of desirability—or binary preference. Like the Axioms K_0–K_4, they are only derived from these primitive assumptions on the basis of their interpretation.

Proposition 8 *Consider any binary set of desirable option sets $K \in \mathbf{K}$ and let $D_K \in \mathbf{D}$ be its corresponding set of desirable options. Then K is coherent if and only if D_K is. Conversely, consider any set of desirable options $D \in \mathbf{D}$ and let K_D be its corresponding binary set of desirable option sets, then K_D is coherent if and only if D is.*

So the binary coherent sets of desirable option sets are given by $\{K_D : D \in \overline{\mathbf{D}}\}$, allowing us to call any coherent set of desirable option sets in $\overline{\mathbf{K}} \setminus \{K_D : D \in \overline{\mathbf{D}}\}$ *non-binary*.

What makes coherent sets of desirable options $D \in \overline{\mathbf{D}}$—and hence also coherent binary sets of desirable option sets—particularly interesting is that they induce a binary preference order $\rhd_D$—a strict vector ordering—on $\mathscr{V}$, defined by $u \rhd_D v \Leftrightarrow u - v \in D$, for all $u, v \in \mathscr{V}$. The preference order $\rhd_D$ is coherent—satisfies Axioms $\rhd_0$–$\rhd_4$—and furthermore fully characterises D: one can easily see that $u \in D$ if and only if $u \rhd_D 0$. Hence, coherent sets of desirable options and coherent binary sets of desirable option sets are completely determined by a single binary strict preference order between options. This is of course the reason why we reserve the moniker *binary* for choice models that are essentially based on *singleton* assessments.

6. Representation in Terms of Sets of Desirable Options

It should be clear—and it should be stressed—at this point that making a direct desirability assessment for an option u typically requires more of a subject than assessing the desirability of an option set A: the former requires that our subject should state that u is desirable, while the latter only requires the subject to state that some option in A is desirable, but not to specify which. It is this difference—this greater latitude in making assessments—that guarantees that our account of choice is much richer than one that is purely based on binary preference. In the framework of sets of desirable option sets, it is for instance possible to express the belief that *at least* one of two options u or v is desirable, while remaining undecided about which of them actually is; in order to express this belief, it suffices to state that $\{u, v\} \in K$. This is not possible in the framework of sets of desirable options. Sets of desirable option sets therefore constitute a much more general uncertainty framework than sets of desirable options.

So while it is nice that there are sets of desirable option sets K_D that are completely determined by a set of desirable options D, such binary choice models are typically *not* what we are interested in here. No, it is the *non-binary* coherent choice models that we have in our sights. If we replace such a non-binary coherent set of desirable option sets K by its corresponding set of desirable options D_K, we lose information, because then necessarily $K_{D_K} \subset K$. Choice models are therefore more expressive than sets of desirable options. But it turns out that our coherence axioms lead to a representation result that allows us to still use sets of desirable options, or rather, sets of them, to completely characterise *any* coherent choice model.

Theorem 9 *A set of desirable option sets $K \in \mathbf{K}$ is coherent if and only if there is a non-empty set $\mathscr{D} \subseteq \overline{\mathbf{D}}$ of coherent sets of desirable options such that $K = \bigcap\{K_D : D \in \mathscr{D}\}$. The largest such set $\mathscr{D}$ is then $\overline{\mathbf{D}}(K) := \{D \in \overline{\mathbf{D}} : K \subseteq K_D\}$.*

Due to the one-to-one correspondence between coherent sets of desirable options D and coherent preference orders $\rhd_D$, this representation result tells us that working with a coherent set of desirable option sets K is equivalent to working with the set of those coherent preference orders $\rhd_D$ for which $K \subseteq K_D$. For the rejection function R that corresponds to K through Equation (5), $u \in R(A)$ means that u is dominated in A for all these representing coherent preference orders $\rhd_D$. Similarly, $u \in C_R(A)$ means that u is undominated according to at least one of these representing coherent preference orders $\rhd_D$. This effectively tells us that our coherence axioms K_0–K_4 for choice models characterise a generalised type of choice under Levi's notion of E-admissibility [10, 20, 22], but with representing preference orders $\rhd_D$ that need not be total orders based on comparing expectations.

Interestingly, any potential property of sets of desirable option sets that is preserved under taking arbitrary intersections, and that the binary choice models satisfy, is inherited from the binary models through the representation result of Theorem 9. It is easy to see that this applies in particular to Aizermann's condition [1].

7. Imposing Totality

We have just shown that every coherent choice model K can be represented by a collection of coherent sets of desirable options D. This leads us to wonder whether it is possible to achieve representation using only particular types of coherent D, and, if yes, for which types of coherent sets of desirable option sets K—and hence for which types of rejection functions R and choice functions C—this is possible. In this section, we clear the air by starting with a rather simple case, where we restrict attention to *total* sets of desirable options D, corresponding to total preference orders $\rhd_D$.

Definition 10 *We call a set of desirable options $D \in \mathbf{D}$* total *if it is coherent and*

$\mathrm{D_T}$. *for all $u \in \mathscr{V} \setminus \{0\}$, either $u \in D$ or $-u \in D$.*

$\overline{\mathbf{D}}_{\mathrm{T}}$ *denotes the set of all total sets of desirable options.*

That the binary preference order $\rhd_D$ corresponding to a total set of desirable options D is indeed a total order can be seen as follows. For all $u, v \in \mathscr{V}$ such that $u \neq v$, the property $\mathrm{D_T}$ implies that either $u - v \in D$ or $v - u \in D$.

Hence, for all $u,v \in \mathscr{V}$, we have that either $u=v$, $u \rhd_D v$ or $v \rhd_D u$, which indeed makes $\rhd_D$ a total order.

It was shown in [4, 9] that what we call *total* sets of desirable options here, are precisely the *maximal* or undominated coherent ones, i.e. those coherent $D \in \overline{\mathbf{D}}$ that are not included in any other coherent set of desirable option sets: $(\forall D' \in \overline{\mathbf{D}})(D \subseteq D' \Rightarrow D = D')$. The question of which types of binary sets of desirable option sets K_D the total D correspond to, is answered by the following definition and proposition.

Definition 11 *We call a set of desirable option sets $K \in \mathbf{K}$* total *if it is coherent and*

$\mathrm{K_T}$. $\{u,-u\} \in K$ *for all* $u \in \mathscr{V} \setminus \{0\}$.

$\overline{\mathbf{K}}_{\mathrm{T}}$ *denotes the set of all total sets of desirable options.* .

Proposition 12 *For any set of desirable options $D \in \mathbf{D}$, D is total if and only if K_D is, so $K_D \in \overline{\mathbf{K}}_{\mathrm{T}} \Leftrightarrow D \in \overline{\mathbf{D}}_{\mathrm{T}}$.*

So a binary K is total if and only if its corresponding D_K is. For general total sets of desirable option sets $K \in \overline{\mathbf{K}}_{\mathrm{T}}$, which are not necessarily binary, we nevertheless still have representation in terms of total binary ones.

Theorem 13 *A set of desirable option sets $K \in \mathbf{K}$ is total if and only if there is a non-empty set $\mathscr{D} \subseteq \overline{\mathbf{D}}_{\mathrm{T}}$ of total sets of desirable options such that $K = \bigcap\{K_D : D \in \mathscr{D}\}$. The largest such set $\mathscr{D}$ is then $\overline{\mathbf{D}}_{\mathrm{T}}(K) := \{D \in \overline{\mathbf{D}}_{\mathrm{T}} : K \subseteq K_D\}$.*

This representation result shows that our total choice models correspond to a generalised type of choice under Levi's notion of E-admissibility [10, 20], but with representing preference orders $\rhd_D$ that are now maximal, or undominated. They correspond to what Van Camp et al. [22, Section 4] have called *M-admissible* choice models. Our discussion above provides an axiomatic characterisation for such choice models.

We conclude our study of totality by characterising what it means for a rejection function to be total.

Proposition 14 *Consider a set of desirable option sets $K \in \mathbf{K}$ and a rejection function R that are connected by Equation (5). Then K is total if and only if R is coherent and satisfies*

$\mathrm{R_T}$. $0 \in R(\{0,u,-u\})$, *for all* $u \in \mathscr{V} \setminus \{0\}$.

8. Imposing the Mixing Property

Totality is, of course, a very strong requirement, and it leads to a very special and restrictive type of representation. We therefore now turn to weaker requirements, and their consequences for representation. One such additional property, which sometimes pops up in the literature about choice and rejection functions, is the following *mixing property* [17, 21], which asserts that an option that is rejected continues to be rejected if one removes mixed options—convex combinations of other options in the option set:[4]

$\mathrm{R_M}$. if $A \subseteq B \subseteq \mathrm{conv}(A)$ then also $R(B) \cap A \subseteq R(A)$, for all $A, B \in \mathscr{Q}$,

where $\mathrm{conv}(\cdot)$ is the *convex hull operator*, defined by

$$\mathrm{conv}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, \sum_{k=1}^{n} \lambda_k = 1, u_k \in V \right\}$$

for all $V \subseteq \mathscr{V}$. $\mathbb{N}$ is the set of natural numbers, excluding 0, and $\mathbb{R}_{>0}$ is the set of all (strictly) positive reals. A rejection function that satisfies this mixing property is called *mixing*.

The following result characterises the mixing property in terms of the corresponding set of desirable option sets. We provide two equivalent conditions: one in terms of the convex hull operator, and one in terms of the $\mathrm{posi}(\cdot)$ operator, which, for any subset V of $\mathscr{V}$, returns the set of all positive linear combinations of its elements:

$$\mathrm{posi}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in V \right\}.$$

Proposition 15 *Consider any set of desirable option sets $K \in \mathbf{K}$ and any rejection function R that are connected by Equation (5). Then R is coherent and mixing if and only if K is coherent and satisfies any—and hence both—of the following conditions:*

$\mathrm{K_M}$. *if $B \in K$ and $A \subseteq B \subseteq \mathrm{posi}(A)$, then also $A \in K$, for all $A, B \in \mathscr{Q}$;*

$\mathrm{K'_M}$. *if $B \in K$ and $A \subseteq B \subseteq \mathrm{conv}(A)$, then also $A \in K$, for all $A, B \in \mathscr{Q}$.*

In the context of sets of desirable options in linear spaces, we prefer to use the $\mathrm{posi}(\cdot)$ operator because it fits more naturally with $\mathrm{d_3}$. We therefore adopt $\mathrm{K_M}$ in our definition for mixingness.

Definition 16 *We call a set of desirable option sets $K \in \mathbf{K}$* mixing *if it is coherent and satisfies $\mathrm{K_M}$. The set of all mixing sets of desirable option sets is denoted by $\overline{\mathbf{K}}_{\mathrm{M}}$.*

We now proceed to show that these mixing sets of desirable option sets allow for a representation in terms of sets of desirable options that are themselves mixing, in the following sense.

Definition 17 *We call a set of desirable options $D \in \mathbf{D}$* mixing *if it is coherent and satisfies*

4. Van Camp [21] refers to this property as 'convexity', but we prefer to stick to the original name suggested by Seidenfeld et al. [17] for the sake of avoiding confusion. We nevertheless want to point out that in a context that focuses on rejection rather than choice, the term '*unmixing*' would be preferable, because the rejection is preserved under *removing* mixed options—whereas the choice is preserved under *adding* mixed options.

$\mathrm{D_M}$. *for all* $A \in \mathscr{Q}$, *if* $\operatorname{posi}(A) \cap D \neq \emptyset$, *then also* $A \cap D \neq \emptyset$.

$\overline{\mathbf{D}}_{\mathrm{M}}$ *denotes the set of all mixing sets of desirable options.*

The *binary* elements of $\overline{\mathbf{K}}_{\mathrm{M}}$ are precisely the ones based on such a mixing set of desirable options; they can be represented by a single element of $\overline{\mathbf{D}}_{\mathrm{M}}$.

Proposition 18 *For any set of desirable options* $D \in \mathbf{D}$, K_D *is mixing if and only if* D *is, so* $K_D \in \overline{\mathbf{K}}_{\mathrm{M}} \Leftrightarrow D \in \overline{\mathbf{D}}_{\mathrm{M}}$.

For general mixing sets of desirable option sets that are not necessarily binary, we nevertheless still obtain a representation theorem analogous to Theorems 9 and 13, where the representing sets of desirable options are now mixing.

Theorem 19 *A set of desirable option sets* $K \in \mathbf{K}$ *is mixing if and only if there is a non-empty set* $\mathscr{D} \subseteq \overline{\mathbf{D}}_{\mathrm{M}}$ *of mixing sets of desirable options such that* $K = \bigcap\{K_D : D \in \mathscr{D}\}$. *The largest such set* $\mathscr{D}$ *is then* $\overline{\mathbf{D}}_{\mathrm{M}}(K) := \{D \in \overline{\mathbf{D}}_{\mathrm{M}} : K \subseteq K_D\}$.

This representation result is akin to the one proved by Seidenfeld et al [17], but without the additional condition of weak Archimedeanity they impose. In order to better explain this, and to provide this result with some extra intuition, we take a closer look at the mixing sets of desirable options that make up our representation. The following result is an equivalent characterisation of such sets.

Proposition 20 *Consider any coherent set of desirable options* $D \in \overline{\mathbf{D}}$ *and let* $D^{\mathrm{c}} := \mathscr{V} \setminus D$. *Then* D *is mixing if and only if* $\operatorname{posi}(D^{\mathrm{c}}) = D^{\mathrm{c}}$.

So we see that the coherent sets of desirable options that are also mixing are precisely those whose complement is again a convex cone.[5] They are therefore identical to the *lexicographic* sets of desirable options sets introduced by Van Camp et al. [21, 23]. What makes this particularly relevant and interesting is that these authors have shown that when $\mathscr{V}$ is the set of all gambles on some finite set $\mathscr{X}$ and $\mathscr{V}_{\succ 0} = \{u \in \mathscr{V} : u \geq 0 \text{ and } u \neq 0\}$, then the sets of desirable options in $\overline{\mathbf{D}}$ that are lexicographic—and therefore mixing—are exactly the ones that are representable by some lexicographic probability system that has no non-trivial Savage-null events. This is, of course, the reason why they decided to call such coherent sets of desirable options *lexicographic*. Because of this connection, it follows that in their setting, Theorem 19 implies that every mixing choice model can be represented by a set of lexicographic probability systems.

Due to the equivalence between coherent lexicographic sets of desirable options and mixing ones on the one hand, and between total sets of desirable options and maximal coherent ones on the other, the following proposition is an immediate consequence of a similar result by Van Camp et al. [21, 23]. It shows that the total sets of desirable options constitute a subclass of the mixing ones: mixingness is a weaker requirement than totality.

Proposition 21 *Every total set of desirable options is mixing:* $\overline{\mathbf{D}}_{\mathrm{T}} \subseteq \overline{\mathbf{D}}_{\mathrm{M}}$.

By combining this result with Theorems 13 and 19, it follows that every total set of desirable options sets is mixing, and similarly for rejection and choice functions. So mixingness is implied by totality for non-binary choice models as well. Since totality is arguably the more intuitive of the two, one might therefore be inclined to discard the mixing property in favour of totality. We have nevertheless studied the mixing property in some detail, because it can be combined with other properties, such as the notions of Archimedeanity studied in the next section. As we will see, this combination leads to very intuitive representation, where the role of lexicographic probability systems is taken over by expectation operators—called linear previsions.

9. Imposing Archimedeanity

There are a number of ways a notion of Archimedeanity can be introduced for preference relations and choice models [2, 3, 15, 17, 11]. Its aim is always to guarantee that the real number system is expressive enough, or more precisely, that the preferences expressed by the models can be represented by (sets of) *real-valued* probabilities and utilities, rather than, say, probabilities and utilities expressed using hyperreals. Here, we consider a notion of Archimedeanity that is close in spirit to an idea explored by Walley [24, 25] in his discussion of so-called *strict desirability*.

For the sake of simplicity, we will restrict ourselves to a particular case of our abstract framework,[6] where $\mathscr{V} = \mathscr{L}(\mathscr{X})$ is the set of all gambles on a set of states $\mathscr{X}$ and $\mathscr{V}_{\succ 0} = \{u \in \mathscr{L}(\mathscr{X}) : \inf u > 0\}$. We identify every real number $\mu \in \mathbb{R}$ with the constant gamble that takes the value μ, and then define Archimedeanity as follows.

Definition 22 *We call a set of desirable options* $D \in \mathbf{D}$ Archimedean *if it is coherent and satisfies the following openness condition:*

$\mathrm{D_A}$. *for all* $u \in D$, *there is an* $\varepsilon \in \mathbb{R}_{>0}$ *such that* $u - \varepsilon \in D$.

We denote the set of all Archimedean sets of desirable options by $\overline{\mathbf{D}}_{\mathrm{A}}$, *and let* $\overline{\mathbf{D}}_{\mathrm{M,A}}$ *be the set of all Archimedean sets of desirable options that are also mixing.*

What makes Archimedean and mixing Archimedean sets of desirable options particularly interesting, is that they are in a one-to-one correspondence with coherent lower previsions and linear previsions [24], respectively.

5. Recall that coherent sets of desirable options are convex cones because of Axiom D_3.

6. It is possible to introduce a version of our notion of Archimedeanity in our general framework as well, but explaining this would take up much more space than we are allowed in this conference paper.

Definition 23 *A* coherent lower prevision $\underline{P}$ *on* $\mathscr{L}(\mathscr{X})$ *is a real-valued map on* $\mathscr{L}(\mathscr{X})$ *that satisfies*

LP$_1$. $\underline{P}(u) \geq \inf u$ *for all* $u \in \mathscr{L}(\mathscr{X})$*;*

LP$_2$. $\underline{P}(\lambda u) = \lambda \underline{P}(u)$ *for all* $u \in \mathscr{L}(\mathscr{X})$ *and* $\lambda \in \mathbb{R}_{>0}$*;*

LP$_3$. $\underline{P}(u+v) \geq \underline{P}(u) + \underline{P}(v)$ *for all* $u, v \in \mathscr{L}(\mathscr{X})$*;*

A linear prevision P *on* $\mathscr{L}(\mathscr{X})$ *is a coherent lower prevision that additionally satisfies*

P$_3$. $P(u+v) = P(u) + P(v)$ *for all* $u, v \in \mathscr{L}(\mathscr{X})$*;*

We denote the set of all coherent lower previsions on $\mathscr{L}(\mathscr{X})$ *by* $\underline{\mathbf{P}}$ *and let* $\mathbf{P}$ *be the set of all linear previsions.*

In order to make the aforementioned one-to-one correspondences explicit, we introduce the following maps. With any set of desirable options D in $\mathbf{D}$, we associate a (possibly extended) real functional $\underline{P}_D$ on $\mathscr{L}(\mathscr{X})$, defined by

$$\underline{P}_D(u) := \sup\{\mu \in \mathbb{R} \colon u - \mu \in D\}, \text{ for all } u \in \mathscr{L}(\mathscr{X}).$$

Conversely, with any (possibly extended) real functional $\underline{P}$ on $\mathscr{L}(\mathscr{X})$, we associate a set of desirable options

$$D_{\underline{P}} := \{u \in \mathscr{L}(\mathscr{X}) \colon \underline{P}(u) > 0\}.$$

Our next result shows that these maps lead to an isomorphism between $\overline{\mathbf{D}}_{\mathrm{A}}$ and $\underline{\mathbf{P}}$, and similarly for $\overline{\mathbf{D}}_{\mathrm{M,A}}$ and $\mathbf{P}$.

Proposition 24 *For any Archimedean set of desirable options D, $\underline{P}_D$ is a coherent lower prevision on $\mathscr{L}(\mathscr{X})$ and $D_{\underline{P}_D} = D$. If D is moreover mixing, then $\underline{P}_D$ is a linear prevision. Conversely, for any coherent lower prevision $\underline{P}$ on $\mathscr{L}(\mathscr{X})$, $D_{\underline{P}}$ is an Archimedean set of desirable options and $\underline{P}_{D_{\underline{P}}} = \underline{P}$. If $\underline{P}$ is furthermore a linear prevision, then $D_{\underline{P}}$ is mixing.*

The importance of these correspondences is that any representation in terms of sets of Archimedean (mixing) sets of desirable options is effectively a representation in terms of sets of coherent lower (or linear) previsions. As we will see, these kinds of representations can be obtained for sets of desirable option sets—and hence also rejection and choice functions—that are themselves Archimedean in the following sense.

Definition 25 *We call a set of desirable option sets $K \in \mathbf{K}$* Archimedean *if it is coherent and satisfies*

K$_\mathrm{A}$. *for all $A \in K$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $A - \varepsilon \in K$.*

We denote the set of all Archimedean sets of desirable option sets by $\overline{\mathbf{K}}_{\mathrm{A}}$, and let $\overline{\mathbf{K}}_{\mathrm{M,A}}$ be the set of all Archimedean sets of desirable options that are also mixing.

This notion easily translates from sets of desirable option sets to rejection functions.

Proposition 26 *Consider any set of desirable option sets $K \in \mathbf{K}$ and any rejection function R that are connected by Equation (5). Then K is Archimedean if and only if R is coherent and satisfies*

R$_\mathrm{A}$. *for all $A \in \mathscr{Q}$ and $u \in \mathscr{V}$ such that $u \in R(A \cup \{u\})$, there is some $\varepsilon \in \mathbb{R}_{>0}$ such that $u \in R((A-\varepsilon) \cup \{u\})$.*

A first and basic result is that our notion of Archimedeanity for sets of desirable option sets is compatible with that for sets of desirable options.

Proposition 27 *For any set of desirable options $D \in \mathbf{D}$, K_D is Archimedean (and mixing) if and only if D is, so $K_D \in \overline{\mathbf{K}}_{\mathrm{A}} \Leftrightarrow D \in \overline{\mathbf{D}}_{\mathrm{A}}$ and $K_D \in \overline{\mathbf{K}}_{\mathrm{M,A}} \Leftrightarrow D \in \overline{\mathbf{D}}_{\mathrm{M,A}}$.*

In order to state our representation results for Archimedean choice models that are not necessarily binary, we require a final piece of machinery: a topology on $\overline{\mathbf{D}}_{\mathrm{A}}$ and $\overline{\mathbf{D}}_{\mathrm{M,A}}$, or equivalently, a notion of closedness. We do this by defining the convergence of a sequence of Archimedean sets of desirable options $\{D_n\}_{n\in\mathbb{N}}$ in terms of the point-wise convergence of the corresponding sequence of coherent lower previsions:

$$\lim_{n\to+\infty} D_n = D \Leftrightarrow \big(\forall u \in \mathscr{L}(\mathscr{X})\big) \lim_{n\to+\infty} \underline{P}_{D_n}(u) = \underline{P}_D(u).$$

A set $\mathscr{D} \subseteq \overline{\mathbf{D}}_{\mathrm{A}}$ of Archimedean sets of desirable options is then called *closed* if it contains all of its limit points, or equivalently, if the corresponding set of coherent lower previsions—or linear previsions when $\mathscr{D} \subseteq \overline{\mathbf{D}}_{\mathrm{M,A}}$—is closed with respect to point-wise convergence.

Our final representation results state that a set of desirable option sets K is Archimedean if and only if it can be represented by such a closed set, and if K is moreover mixing, the elements of the representing closed set are as well.

Theorem 28 *A set of desirable option sets $K \in \mathbf{K}$ is Archimedean if and only if there is some non-empty closed set $\mathscr{D} \subseteq \overline{\mathbf{D}}_{\mathrm{A}}$ of Archimedean sets of desirable options such that $K = \bigcap\{K_D \colon D \in \mathscr{D}\}$. The largest such set $\mathscr{D}$ is then $\overline{\mathbf{D}}_{\mathrm{A}}(K) := \{D \in \overline{\mathbf{D}}_{\mathrm{A}} \colon K \subseteq K_D\}$.*

Theorem 29 *A set of desirable option sets $K \in \mathbf{K}$ is mixing and Archimedean if and only if there is some non-empty closed set $\mathscr{D} \subseteq \overline{\mathbf{D}}_{\mathrm{M,A}}$ of mixing and Archimedean sets of desirable options such that $K = \bigcap\{K_D \colon D \in \mathscr{D}\}$. The largest such set $\mathscr{D}$ is then $\overline{\mathbf{D}}_{\mathrm{M,A}}(K) := \{D \in \overline{\mathbf{D}}_{\mathrm{M,A}} \colon K \subseteq K_D\}$.*

If we combine Theorem 29 with the correspondence result of Proposition 24, we see that Axioms K$_0$–K$_4$ together with K$_\mathrm{M}$ and K$_\mathrm{A}$ characterise exactly those choice models that are based on E-admissibility with respect to a closed—but not necessarily convex—set of linear previsions. In much the same way, Theorem 28 can be seen to characterise a generalised notion of E-admissibility, where the representing objects are coherent lower previsions. Walley–Sen maximality [20, 24] can be regarded as a special case of this generalised notion, where only a single representing coherent lower prevision is needed.

10. Conclusion

The main conclusion of this paper is that the language of desirability *is* capable of representing non-binary choice models, provided we extend it with a notion of disjunction, allowing statements such as 'at least one of these two options is desirable'. When we do so, the resulting framework of sets of desirable options turns out to be a very flexible and elegant tool for representing set-valued choice. Not only does it include E-admissibility and maximality, it also opens up a range of other types of choice functions that have so far received little to no attention. All of these can be represented in terms of sets of strict preference orders or—if additional properties are imposed—in terms of sets of strict total orders, sets of lexicographic probability systems, sets of coherent lower previsions or sets of linear previsions.

Another important conclusion is that our axiomatisation for general (possibly non-binary) choice models allows for representations in terms of 'atomic' models, which in themselves represent binary choice. However, this should of course not be taken to mean that our choice models are essentially binary. Indeed, it follows readily from our representation theorems that the binary aspects D_K of a non-binary choice model K are captured by the intersections of the representing sets of desirable options, but the representation is much more powerful than that, because it also extends to the non-binary aspects of choice.

This distinction between the binary level and the non-binary one also leads us to the following important words of caution, which are akin to an earlier observation made by Quaeghebeur [13]. At the binary level, choice is represented by a set of desirable options, which can—under certain assumptions such as Archimedeanity—be identified with a convex closed set of linear previsions. We have also seen in Theorem 29 that the (binary *and* non-binary) aspects of mixing and Archimedean choice can be fully represented by a closed set of mixing and Archimedean sets of desirable options, each of which is, by Proposition 24, equivalent to a linear prevision. So, in this case there is a representation in terms of a set of linear previsions both at the binary level and at the general (binary *and* non-binary) level, but these two sets of linear previsions will typically be different, and they play a very different role. To put it bluntly: sets of linear previsions à la Walley [24] should not be confused with sets of linear previsions—credal sets—à la Levi [10].

To conclude, what we have done here, in a very specific sense, is to introduce a way of dealing with statements of the type 'there is some option in the option set A that is strictly preferred to 0'. Axioms such as K_0–K_4 can then be seen as the logical axioms—for deriving new statements from old—that govern this language. Our representation theorems provide a *semantics* for this language in terms of desirability, and they show that the corresponding logical system is sound and complete.

In our future work on this topic, we intend to investigate how we can let go of the closedness condition in Theorems 28 and 29. We expect to have to turn to other types of Archimedeanity; variations on Seidenfeld et al.'s weak Archimedeanity [17, 5] come to mind. We also intend to show in more detail how the existing work on choice models for horse lotteries [17] fits nicely within our more general and abstract framework of choice on linear option spaces. And finally, we intend to further develop conservative inference methods for coherent choice functions, by extending our earlier natural extension results [6] to the more general setting that we have considered here.

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Author Contributions

As with most of our joint work, there is no telling, after a while, which of us two had what idea, or did what, exactly. We have both contributed equally to this paper. But since a paper must have a first author, we decided it should be the one who took the first significant steps: Jasper, in this case.

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