Exponential Smoothing: Level & Trend Data

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Treatment of History

- Previous models differed in the *amount* of history considered, but were similar in their *equal* treatment of it.
  - Cumulative & Moving Average – equal weighting to all observations
  - Naïve – all weight to most recent observation

$$\hat{x}_{t,t+1} = \frac{\sum_{i=1}^{t} x_i}{t} \qquad \hat{x}_{t,t+1} = \frac{\sum_{i=t+1-M}^{t} x_i}{M} \qquad \hat{x}_{t,t+1} = x_t$$

- Is there something in between these extremes?
- Should we treat historical data differently?
  - The value of data degrades over time
  - Weight the newer observations more than the older ones
- This is what exponential smoothing does
  - Each observation is weighted
  - Weights decrease exponentially as they age

$$\hat{x}_{t,t+1} = \alpha x_t + \left(1-\alpha\right)\hat{x}_{t-1,t} \qquad 0 \le \alpha \le 1$$

Simple Exponential Smoothing

$$\hat{x}_{t,t+1} = \alpha x_t + (1-\alpha)\hat{x}_{t-1,t} \qquad 0 \le \alpha \le 1$$

Obs.	α=0.2	α=0.4	α=0.6
t	0.2	0.4	0.6
t-1	0.16	0.24	0.24
t-2	0.128	0.144	0.096
t-3	0.1024	0.0864	0.0384
t-4	0.08192	0.05184	0.01536
t-5	0.065536	0.031104	0.006144

Weights attached to observations for different alpha values

Recall that:

$$\hat{x}_{t-1,t} = \alpha x_{t-1} + (1-\alpha)\hat{x}_{t-2,t-1}$$

Expanding and collecting terms:

$$\hat{x}_{t,t+1} = \alpha x_t + (1-\alpha)\left(\alpha x_{t-1} + (1-\alpha)\hat{x}_{t-2,t-1}\right)$$

$$\hat{x}_{t,t+1} = \alpha x_t + \alpha(1-\alpha)x_{t-1} + (1-\alpha)^2\hat{x}_{t-2,t-1}$$

Continuing to substitute:

$$\hat{x}_{t,t+1} = \alpha x_t + \alpha(1-\alpha)x_{t-1} + \alpha(1-\alpha)^2 x_{t-2} + (1-\alpha)^3\hat{x}_{t-3,t-2}$$

Which leads us to the general form:

$$\hat{x}_{t,t+1} = \alpha(1-\alpha)^0 x_t + \alpha(1-\alpha)^1 x_{t-1} + \alpha(1-\alpha)^2 x_{t-2} + \alpha(1-\alpha)^3 x_{t-3}\ldots$$

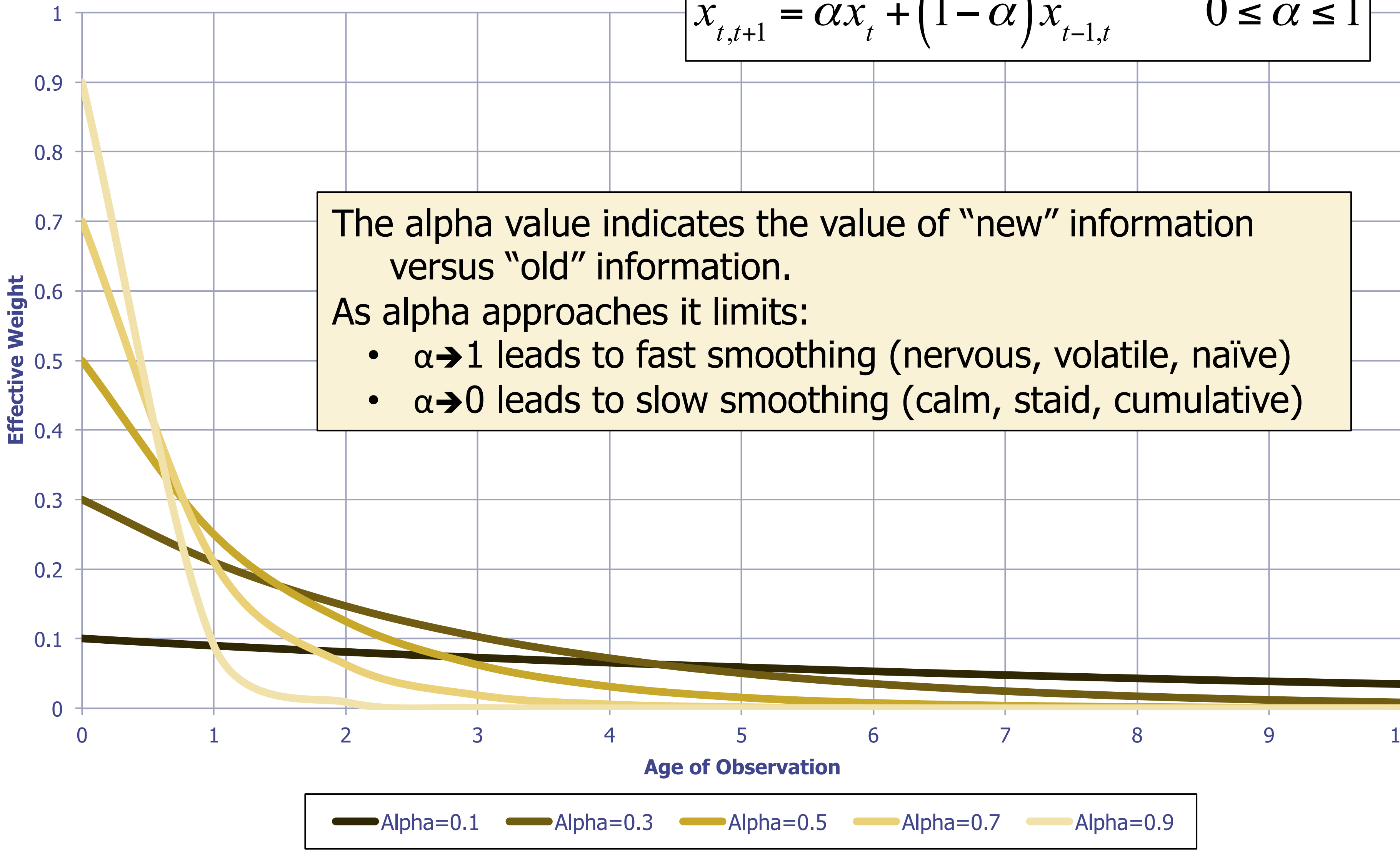
Exponential Smoothing Weights

$$\hat{x}_{t,t+1} = \alpha x_t + \left(1-\alpha\right)\hat{x}_{t-1,t} \qquad 0 \le \alpha \le 1$$

The alpha value indicates the value of “new” information versus “old” information.

As alpha approaches it limits:

- α➔1 leads to fast smoothing (nervous, volatile, naïve)
- α➔0 leads to slow smoothing (calm, staid, cumulative)



Simple Exponential Smoothing

Time Series Analysis

- Simple Exponential Smoothing
  - Stationary demand – no trends or seasonality
  - Value of observations degrade over time
  - Utilizes a smoothing constant (α) where $0 \le \alpha \le 1$
  - In practice $0.1 \le \alpha \le 0.3$

Underlying Model:

$$x_t = a + e_t$$

where:

$$e_t \sim \text{iid}\ (\mu=0,\ \sigma^2=V[e])$$

Forecasting Model:

$$\hat{x}_{t,t+1} = \alpha x_t + (1-\alpha)\hat{x}_{t-1,t} \qquad 0 \le \alpha \le 1$$

We can also think of this as error-correcting.

$$\hat{x}_{t,t+1} = \alpha x_t + (1-\alpha)\hat{x}_{t-1,t}$$

$$\hat{x}_{t,t+1} = \alpha x_t + \hat{x}_{t-1,t} - \alpha\hat{x}_{t-1,t}$$

$$\hat{x}_{t,t+1} = \hat{x}_{t-1,t} + \alpha\left(x_t - \hat{x}_{t-1,t}\right)$$

$$\hat{x}_{t,t+1} = \hat{x}_{t-1,t} + \alpha e_t$$

New estimate is the old estimate
plus some fraction of the most recent error.

Example

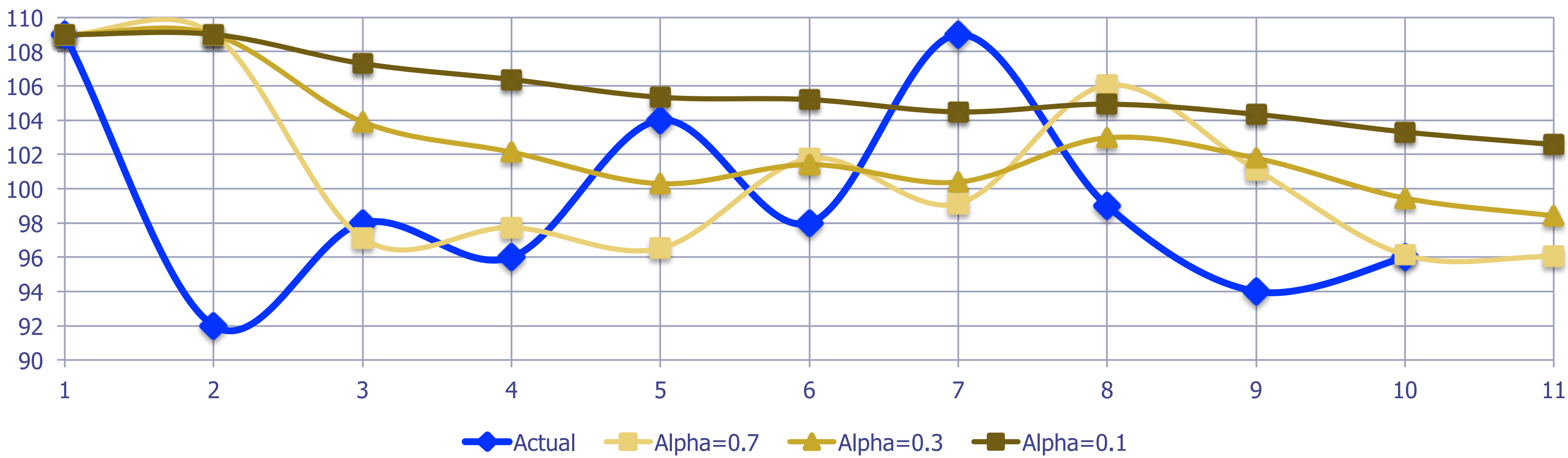
Calculate the forecast for period 6 from period 5 with alpha = 0.3:

$$\hat{x}_{5,6} = (\alpha)x_5 + (1-\alpha)\hat{x}_{4,5}$$
$$= (.3)(104) + (0.7)(100.3) = 101.4$$

What is the forecast for period 12 from period 10 with alpha = 0.3?

(hint: it is the same as the forecast for period 13, 14, . . .)

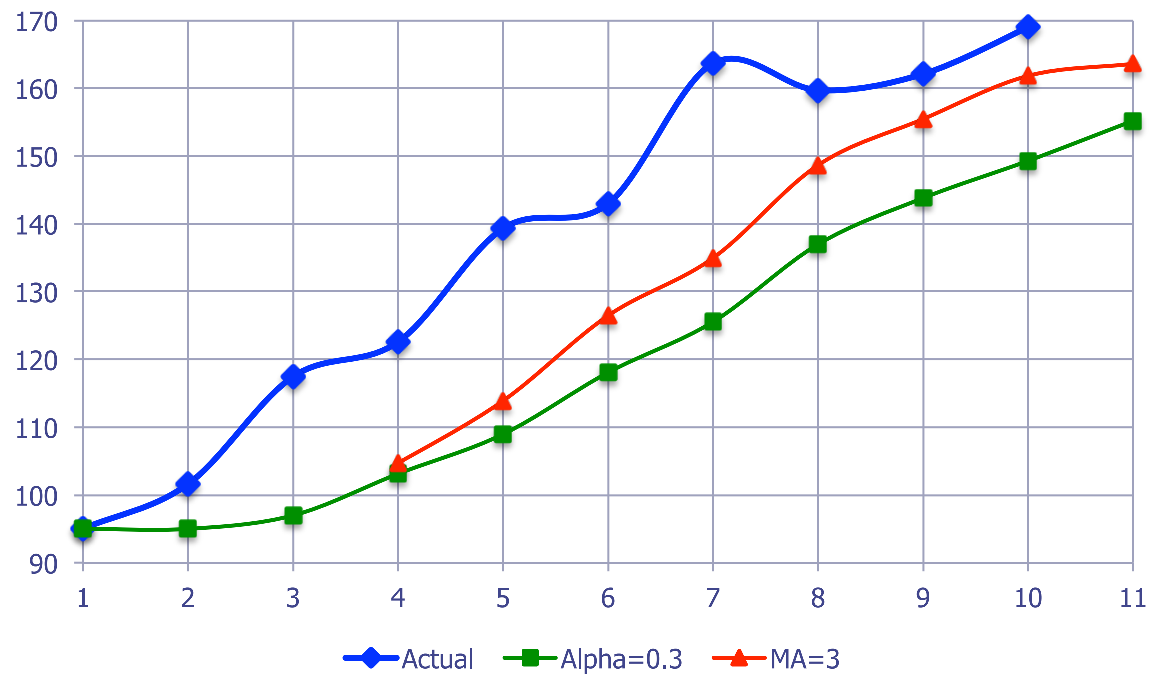
		$\hat{x}_{t,t+1}$ Exp. Smoothing		
t	x_t	Alpha =0.7	Alpha =0.3	Alpha =0.1
1	109	*109.0*	*109.0*	*109.0*
2	92	97.1	103.9	107.3
3	98	97.7	102.1	106.4
4	96	96.5	100.3	105.3
5	104	101.8	101.4	105.2
6	98	99.1	100.4	104.5
7	109	106.0	103.0	104.9
8	99	101.1	101.8	104.3
9	94	96.1	99.4	103.3
10	96	96.0	98.4	102.6



Exponential Smoothing with Trend

Time Series: Non-Stationary Models

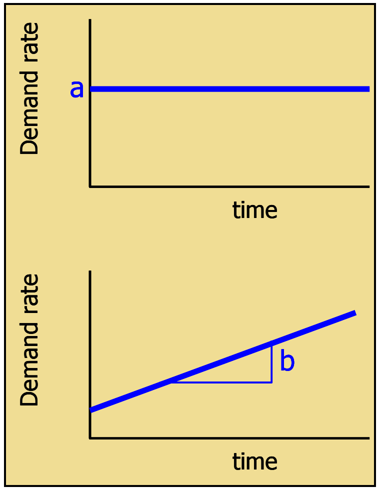
		$\hat{x}_{t,t+1}$ Forecasts	
t	x_t	Alpha = 0.3	MA = 3
1	95	*95.0*	
2	102	97.0	
3	117	103.1	104.7
4	123	109.0	113.9
5	139	118.1	126.5
6	143	125.5	135.0
7	164	137.0	148.6
8	160	143.8	155.4
9	162	149.3	161.8
10	169	155.2	163.6



Challenges:

- Moving average & simple exponential smoothing models will always lag a trend
- They only look at history to find the stationary level
- Need to capture the 'trend' or 'seasonality' factors

Time Series Analysis



- Exponential Smoothing for Level & Trend
  - Expand exponential smoothing to include trend
  - Often referred to as Holt's Method
  - Uses smoothing constants (α,β) where $0 \leq \alpha, \beta \leq 1$

Underlying Model:

$$x_t = a + bt + e_t$$

where: $e_t \sim$ iid $(\mu=0\ ,\ \sigma^2=V[e])$

Forecasting Model:

$$\hat{x}_{t,t+\tau} = \hat{a}_t + \tau\hat{b}_t$$

Updating Procedure:

$$\boxed{\hat{a}_t} = \alpha x_t + (1-\alpha)\boxed{\left(\hat{a}_{t-1} + \hat{b}_{t-1}\right)}$$

$$\boxed{\hat{b}_t} = \beta\boxed{\left(\hat{a}_t - \hat{a}_{t-1}\right)} + (1-\beta)\boxed{\hat{b}_{t-1}}$$

This is just $\hat{x}_{t-1,t}$

These are estimates of the level and trend components at end of time period t.

The "old" trend - estimated trend from last period

The "new" trend - difference between this period and last period's estimated level.

Example

Suppose we are in time 101 and we use alpha=0.3 and beta=0.1.

a) Forecast demand for t=102

b) Forecast demand for t=110

Forecasting Model

$$\hat{x}_{t,t+\tau} = \hat{a}_t + \tau \hat{b}_t$$

Updating Procedure

$$\hat{a}_t = \alpha x_t + (1-\alpha)(\hat{a}_{t-1} + \hat{b}_{t-1})$$

$$\hat{b}_t = \beta(\hat{a}_t - \hat{a}_{t-1}) + (1-\beta)\hat{b}_{t-1}$$

Data

t	x_t	$\hat{a}_t$	$\hat{b}_t$	$\hat{x}_{t,t+1}$
100	92	90	8.5	98.5
101	95	97.5	8.4	105.9

Part a) Estimating demand for t=102

1. Find $\hat{a}_{101}$

$$\hat{a}_{101} = (0.3)x_{101} + (0.7)(\hat{a}_{100} + \hat{b}_{100})$$

$$= (0.3)(95.0) + (0.7)(90.0 + 8.5) \approx 97.5$$

2. Find $\hat{b}_{101}$

$$\hat{b}_{101} = (0.1)(\hat{a}_{101} - \hat{a}_{100}) + (0.9)\hat{b}_{100}$$

$$= (0.1)(97.5 - 90.0) + (0.9)(8.5) \approx 8.4$$

3. Find $\hat{x}_{101,102}$ =97.5+8.4 = 105.9

Part b) Estimating demand for t=110

This means τ=9, so

$\hat{x}_{101,110}$ =97.5+(9)8.4 = 173.1

Example

Example: #VMI1984

Spreadsheets are in resources link for this video

You need to develop monthly forecasts (in pallets) for item #VMI1984 that seems to have an upward trend. Looking at past year's data, you have determined that α=0.25 and β= 0.10. Your estimated level ($\hat{a}_0$) in January (t=0) is 28 pallets/month and the estimate of trend ($\hat{b}_0$) is 1.35.

a) Using exponential smoothing, estimate a forecast for February

This is easy – just plug in the numbers.

$\hat{x}_{J,F}=\hat{a}_J + (1)(\hat{b}_J) = 28 + (1)(1.35) = 29.35$ pallets

b) It is now the end of February and demand was 27 pallets. What is your forecast for March?

	A	B	C	D	E	F	G	H
1	Alpha	0.250						
2	Beta	0.100						
3								
4		t	x(t)	a^(t)	b^(t)	x^(t,t+1)	e(t)	e(t)^2
5	January	0		28.00	1.35	29.35		
6	February	1	27	28.76	1.29	30.05	-2.35	5.52

=D5+E5

=C6–F5

=$B$1*C6+(1–$B$1)*F5

=$B$2*(D6–D5)+(1–$B$2)*E5

Example: #VMI1984

Spreadsheets are in resources link for this video

c) Build a spreadsheet for "next month" estimates for the next 8 months.

Actual Demand

"Next Month" Forecasts

	A	B	C	D	E	F	G	H	I
1	Alpha	0.250					Omega =	0.05	
2	Beta	0.100							
3									
4		t	x(t)	a^(t)	b^(t)	x^(t,t+1)	e(t)	e(t)^2	MSE
5	January	0	28	28.00	1.35	29.35			4.20
6	February	1	27	28.76	1.29	30.05	-2.35	5.52	4.27
7	March	2	30	30.04	1.29	31.33	-0.05	0.00	4.05
8	April	3	34	32.00	1.36	33.35	2.67	7.13	4.21
9	May	4	32	33.02	1.32	34.34	-1.35	1.83	4.09
10	June	5	33	34.00	1.29	35.29	-1.34	1.79	3.97
11	July	6	32	34.47	1.21	35.68	-3.29	10.85	4.32
12	August	7	36	35.76	1.22	36.97	0.32	0.10	4.11
13	September	8	33	35.98	1.12	37.10	-3.97	15.78	4.69
14	October	9	36	36.82	1.09	37.91	-1.10	1.20	4.52

How good are the forecasts? Look at MSE=(1/n)Σe^2, but which one?

We will need an estimate of the the forecast error for finding safety stock.

Keep a current running update of the MSE – using exponential smoothing. Select an omega where $0.01 \leq \omega \leq 0.1$

$$MSE_t = \omega\left(x_t - \hat{x}_{t-1,t}\right)^2 + \left(1-\omega\right)MSE_{t-1}$$

=\$H\$1*(C14-F13)^2+(1-\$H\$1)*I13

Damped Trends

Damped Trends

- Problems with trend terms
  - Trends do not continue unchanging indefinitely
  - Constant linear trends can lead to over-forecasting
  - This is especially true for longer forecast horizons
- Damped trend model
  - Slight modification to exponential smoothing model
  - Select dampening parameter phi, where $0 \leq \phi \leq 1$
  - If $\phi = 1$, this is just a linear trend

Forecasting Model $$\hat{x}_{t,t+\tau} = \hat{a}_t + \sum_{i=1}^{\tau} \phi^i \hat{b}_t$$

Updating Procedure
$$\hat{a}_t = \alpha x_t + (1-\alpha)\left(\hat{a}_{t-1} + \phi \hat{b}_{t-1}\right)$$
$$\hat{b}_t = \beta\left(\hat{a}_t - \hat{a}_{t-1}\right) + (1-\beta)\phi \hat{b}_{t-1}$$

Same Example

Spreadsheets are in resources link for this video

Build a spreadsheet for "next month" estimates for the next 8 months using a damped trend.

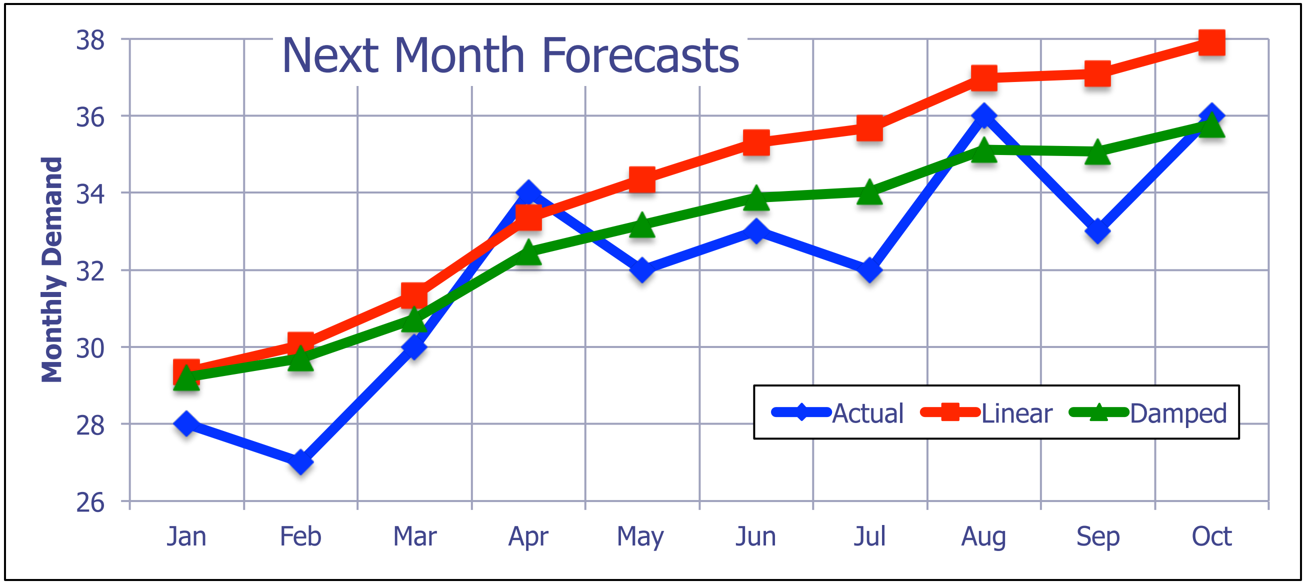
	A	B	C	D	E	F	G	H	I
1	Alpha	0.250					Omega =	0.05	
2	Beta	0.100							
3	Phi	0.900							
4									
5		t	x(t)	a^(t)	b^(t)	x^(t,t+1)	e(t)	e(t)^2	MSE
6	January	0	28	28.00	1.35	29.22			4.20
7	February	1	27	28.66	1.16	29.70	-2.22	4.91	4.24
8	March	2	30	29.78	1.05	30.72	0.30	0.09	4.03
9	April	3	34	31.54	1.03	32.47	3.28	10.73	4.36
10	May	4	32	32.35	0.91	33.17	-0.47	0.22	4.16
11	June	5	33	33.13	0.82	33.87	-0.17	0.03	3.95
12	July	6	32	33.40	0.69	34.02	-1.87	3.48	3.93
13	August	7	36	34.51	0.67	35.12	1.98	3.92	3.93
14	September	8	33	34.59	0.55	35.08	-2.12	4.48	3.95
15	October	9	36	35.31	0.52	35.78	0.92	0.84	3.80

=D6+$B$3*E6

=$B$1*C15+(1-$B$1)*(D14+$B$3*E14)

=$B$2*(D15-D14)+(1-$B$2)*$B$3*E14

Comparing Linear versus Damped

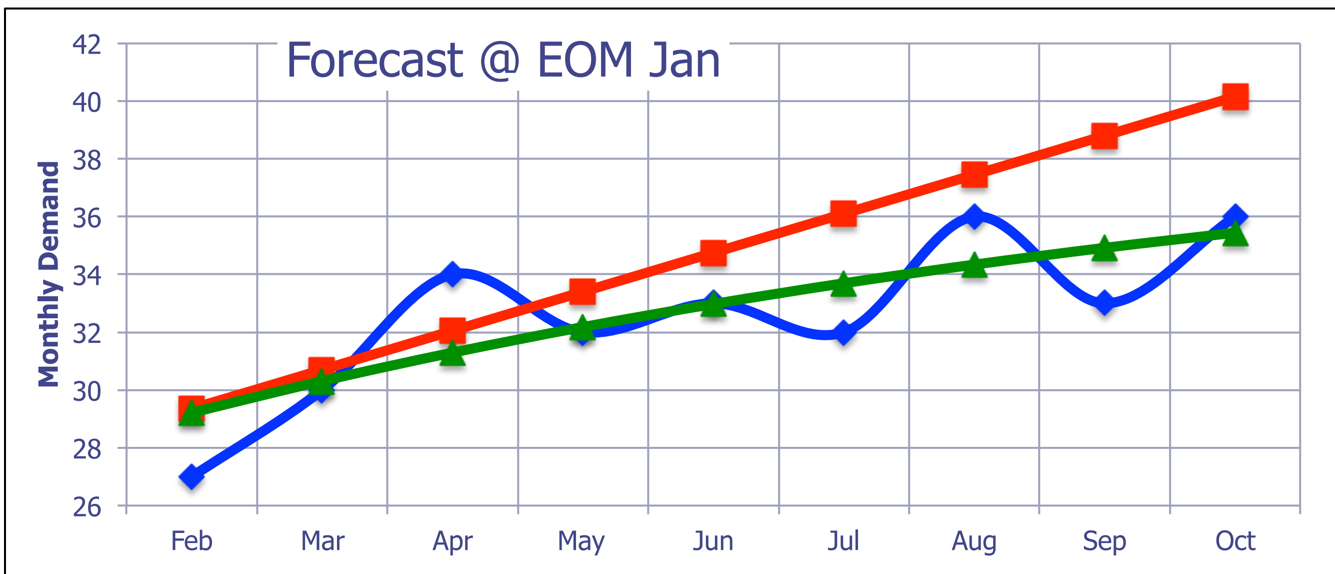


Comparing MSE:
- Linear: 4.52
- Damped: 3.80

Results are data specific, obviously

Nine month forecast made at EOM Jan.

Note the tapering effect of the damped model.



Key Points from Lesson

Key Points

- Exponential Smoothing Models
  - Simple Model (level)

$$\hat{x}_{t,t+1} = \alpha x_t + (1-\alpha)\hat{x}_{t-1,t}$$

  - Holt Model (level & trend)

$$\hat{x}_{t,t+\tau} = \hat{a}_t + \tau \hat{b}_t$$

$$\hat{a}_t = \alpha x_t + (1-\alpha)(\hat{a}_{t-1} + \hat{b}_{t-1})$$

$$\hat{b}_t = \beta(\hat{a}_t - \hat{a}_{t-1}) + (1-\beta)\hat{b}_{t-1}$$

- Other Smoothing Models
  - MSE Trending – for use in inventory models (ω)
  - Damped Trends – tapering effect (ϕ)
- Core Concepts:
  - Value of information degrades over time
  - Balance of using both old & new information

Questions, Comments, Suggestions? Use the Discussion!

"Lexi"
Photo courtesy Yankee Golden Retriever Rescue (www.ygrr.org)

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