# Economic Order Quantity (EOQ)

MIT Center for Transportation & Logistics

# Replenishment Model Assumptions

- Demand
  - Constant vs Variable
  - Known vs Random
  - Continuous vs Discrete
- Lead Time
  - Instantaneous
  - Constant vs Variable
  - Deterministic vs Stochastic
  - Internally Replenished
- Dependence of Items
  - Independent
  - Correlated
  - Indentured
- Review Time
  - Continuous vs Periodic
- Number of Locations
  - One vs Multi vs Multi-Echelon
- Capacity / Resources
  - Unlimited
  - Limited / Constrained
- Discounts
  - None
  - All Units vs Incremental vs One Time
- Excess Demand
  - None
  - All orders are backordered
  - Lost orders
  - Substitution
- Perishability
  - None
  - Uniform with time
  - Non-linear with time
- Planning Horizon
  - Single Period
  - Finite Period
  - Infinite
- Number of Items
  - One vs Many
- Form of Product
  - Single Stage
  - Multi-Stage

# Model Assumptions: EOQ

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# Notation

$D$ = Average Demand (units/time)

$c$ = Variable (Purchase) Cost ($/unit)

$c_t$ = Fixed Ordering Cost ($/order)

$h$ = Carrying or Holding Charge ($/inventory $/time)

$c_e = ch$ = Excess Holding Cost ($/unit/time)

$Q$ = Replenishment Order Quantity (units/order)

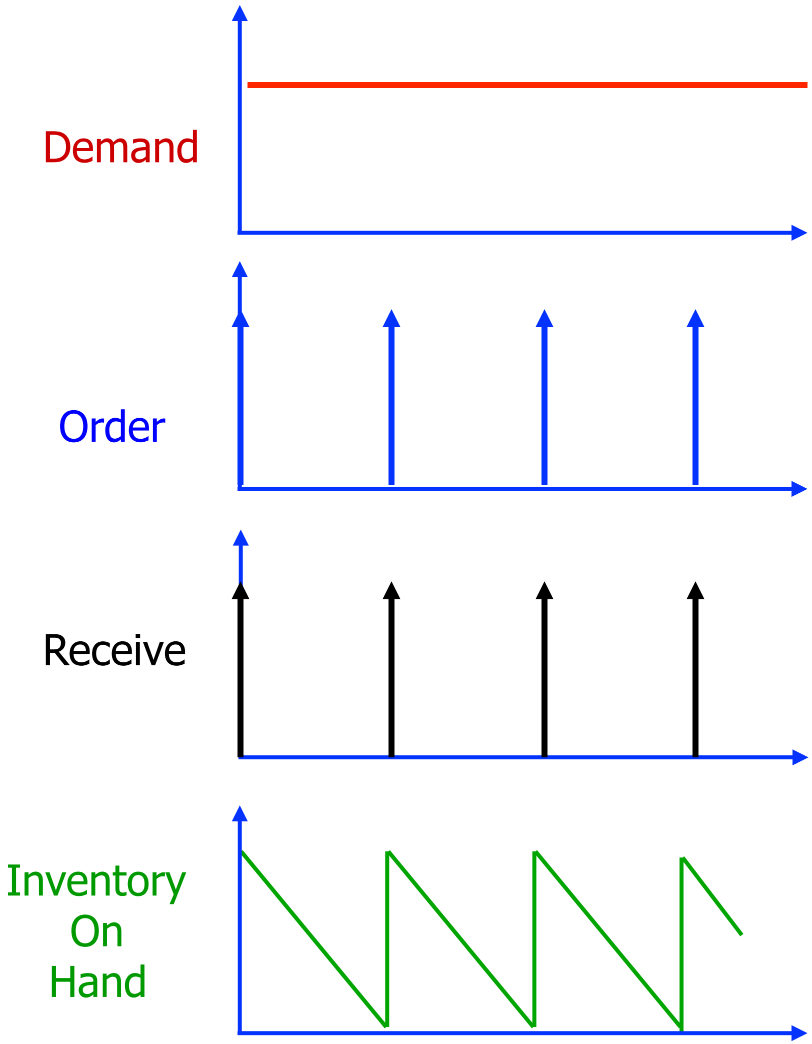
$T$ = Order Cycle Time (time/order)

$N = 1/T$ = Orders per Time (order/time)

$TRC(Q)$ = Total Relevant Cost ($/time)

$TC(Q)$ = Total Cost ($/time)

# Inventory Charts

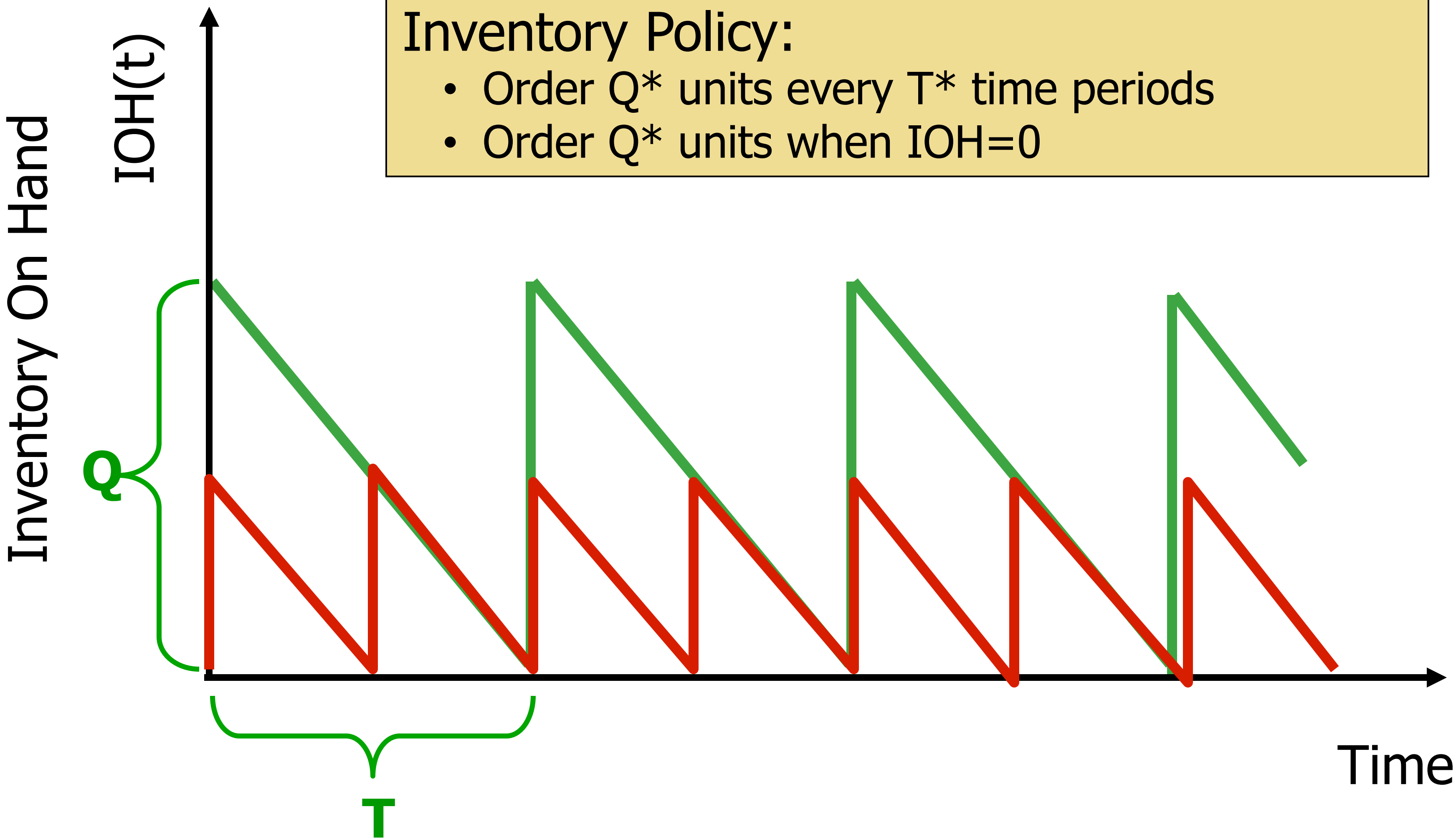



## Model Assumptions (EOQ)

- Demand is uniform and deterministic
- Lead time is instantaneous (0)
- Total amount ordered is received

# Lot Sizing: Finding Optimal Policy

Inventory Policy:
- Order Q* units every T* time periods
- Order Q* units when IOH=0



# What is the total cost?

TC = Purchase + Order + Holding + Shortage

$$\boxed{TC(Q) = cD + c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{Q}{2}\right) + c_s E\left[Units\ Short\right]}$$

Which costs are relevant to the order quantity decision?

$$\boxed{TRC(Q) = c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{Q}{2}\right)}$$

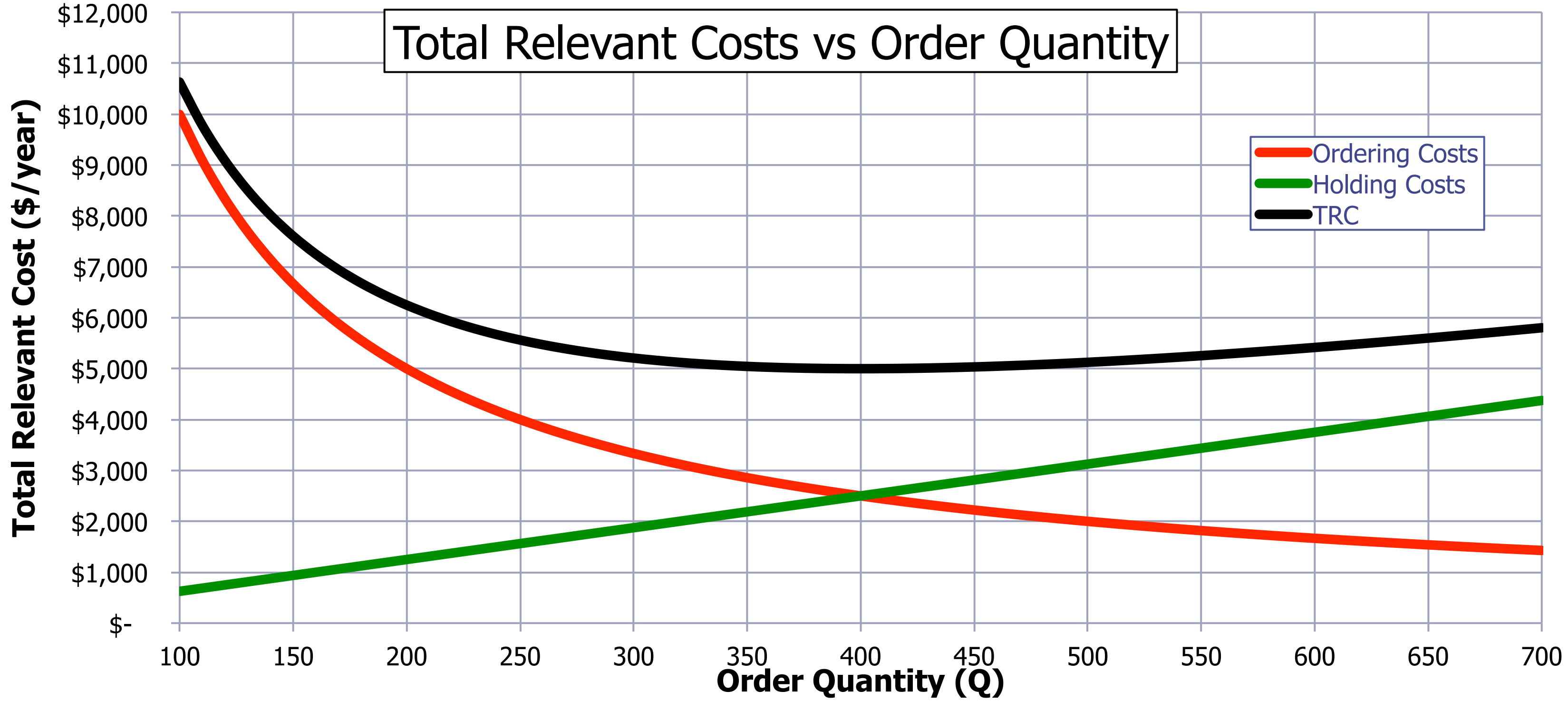
# Building Intuition

$$TRC(Q) = c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{Q}{2}\right)$$

## Example:

What is the optimal order quantity for a product with:

- Demand = 2,000 units/year
- Cost of placing an order = 500 $/order
- Cost of product = 50 $/unit
- Holding cost = 25% of unit cost per year
- Selling price of product = 75 $/unit



# Solving for EOQ Analytically

# Finding Q* - Optimal order quantity

## First Order Conditions

$$TRC[Q] = \frac{c_t D}{Q} + \frac{c_e Q}{2}$$

$$\frac{dTRC[Q]}{dQ} = \frac{-c_t D}{Q^2} + \frac{c_e}{2} = 0$$

$$\frac{c_e}{2} = \frac{c_t D}{Q^2}$$

$$Q^2 = \frac{2c_t D}{c_e}$$

$$Q^* = \sqrt{\frac{2c_t D}{c_e}}$$

## Second Order Conditions

$$\frac{dTRC[Q]}{dQ} = \frac{-c_t D}{Q^2} + \frac{c_e}{2}$$

$$\frac{d^2TRC[Q]}{d^2Q} = \frac{2c_t D}{Q^3}$$

This value is positive for any positive values for $c_t$, D, or Q. Thus, $Q^*$ is a global minimum

$$\boxed{Q^* = \sqrt{\frac{2c_t D}{c_e}}}$$

# Finding T* - Time between replenishments

$$Q^* = \sqrt{\frac{2c_t D}{c_e}} \qquad T^* = \frac{Q^*}{D}$$

Recall that:
- Q= units / order
- D= units / year
- T= years / order
- N= 1/T = orders / year

$$T^* = \left(\frac{1}{D}\right)\sqrt{\frac{2c_t D}{c_e}} = \sqrt{\frac{2c_t D}{D^2 c_e}}$$

Be sure to put T* into units that make sense (days, weeks, months, etc.) don't leave it in years!

$$\boxed{T^* = \sqrt{\frac{2c_t}{Dc_e}}}$$

# Finding TRC* - Optimal Total Relevant costs

$$TRC[Q] = \frac{c_t D}{Q} + \frac{c_e Q}{2} \qquad Q^* = \sqrt{\frac{2c_t D}{c_e}}$$

$$TRC(Q^*) = \frac{c_t D}{\sqrt{\frac{2c_t D}{c_e}}} + \frac{c_e\sqrt{\frac{2c_t D}{c_e}}}{2} = \frac{c_t D\sqrt{c_e}}{\sqrt{2c_t D}} + \frac{c_e\sqrt{2c_t D}}{2\sqrt{c_e}}$$

$$= \frac{\sqrt{c_t c_e D}}{\sqrt{2}} + \frac{\sqrt{c_t c_e D}}{\sqrt{2}} = \frac{2\sqrt{c_t c_e D}}{\sqrt{2}}$$

$$\boxed{\begin{array}{l} TRC(Q^*) = \sqrt{2c_t c_e D} \\ TC(Q^*) = cD + \sqrt{2c_t c_e D} \end{array}}$$

Remember:
Total relevant cost ≠ Total cost.
Need to add purchase cost back in.

# Step by Step Example: EOQ

Find the optimal order quantity, inventory policy, and total relevant costs for a product with the following characteristics:

- Demand = 2,000 units/year
- Cost of placing an order = 500 $/order
- Cost of product = 50 $/unit
- Holding cost = 25% of unit cost per year
- Selling price of product = 75 $/unit

# EOQ Sensitivity to Changes

# Finding Sensitivity of EOQ

Suppose I use a "wrong" order quantity, Q. How much worse will my total relevant cost be using this "wrong" Q versus Q*?

$$\frac{TRC(Q)}{TRC(Q^*)} = \frac{\frac{c_t D}{Q} + \frac{c_e Q}{2}}{\sqrt{2 c_t c_e D}}$$

$$= \frac{c_t D}{Q\sqrt{2 c_t c_e D}} + \frac{c_e Q}{2\sqrt{2 c_t c_e D}}$$

$$= \frac{\sqrt{c_t D}}{Q\sqrt{2 c_e}} + \frac{\sqrt{c_e Q}}{2\sqrt{2 c_t D}}$$

$$= \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\frac{\sqrt{c_t D}}{Q\sqrt{2 c_e}} + \frac{\sqrt{c_e Q}}{2\sqrt{2 c_t D}}$$

$$= \frac{\sqrt{2 c_t D}}{2Q\sqrt{c_e}} + \frac{\sqrt{c_e Q}}{2\sqrt{2 c_t D}}$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2 c_t D}}{Q\sqrt{c_e}} + \frac{\sqrt{c_e Q}}{\sqrt{2 c_t D}}\right)$$

$$\boxed{\frac{TRC(Q)}{TRC(Q^*)} = \left(\frac{1}{2}\right)\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right)}$$

# EOQ Sensitivity wrt Order Size

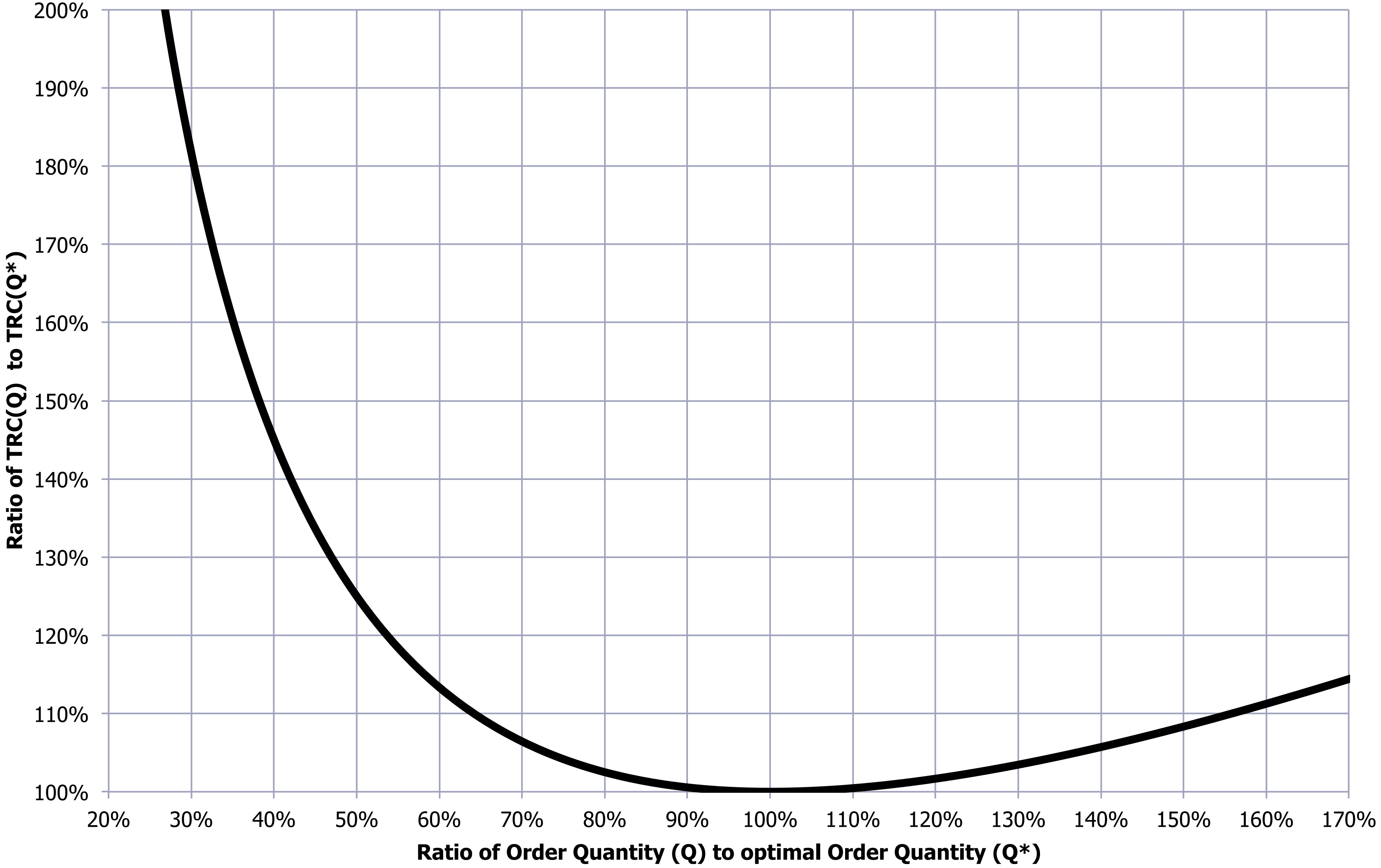
Previous example where:

$D = 2000$ $c=50$ $c_e = 12.5$ $c_t = 500$

| Q | Ordering Costs ($c_tD/Q$) | Holding Costs ($c_eQ/2$) | Total Relevant Costs | Q/Q* | TRC/TRC* |
|---|---|---|---|---|---|
| 800 | $1,250 | $5,000 | $6,250 | 200% | 125% |
| 600 | $1,667 | $3,750 | $5,417 | 150% | 108.3% |
| Q*=400 | $2,500 | $2,500 | $5,000 | -- | -- |
| 200 | $5,000 | $1,250 | $6,250 | 50% | 125% |
| 20 | $50,000 | $125 | $50,125 | 5% | 1002.5% |

Would you rather order Q>Q* or Q<Q*?

# EOQ Sensitivity wrt Order Size



# EOQ Sensitivity wrt Demand

- How sensitive is the TRC to changes in actual demand?
- Notation:
  - $D_F$ = Forecasted Demand
  - $Q_F$* = EOQ using Forecasted Demand
  - $D_A$ = Actual Demand (what really occurred)
  - $Q_A$* = EOQ calculated with Actual Demand

$$Q_F^* = \sqrt{\frac{2D_F c_t}{c_e}}$$

$$Q_A^* = \sqrt{\frac{2D_A c_t}{c_e}}$$

$$\frac{TRC(Q)}{TRC(Q^*)} = \left(\frac{1}{2}\right)\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) \qquad \frac{TRC(Q_F^*)}{TRC(Q_A^*)} = \left(\frac{1}{2}\right)\left(\frac{Q_A^*}{Q_F^*} + \frac{Q_F^*}{Q_A^*}\right)$$

$$\frac{Q_A^*}{Q_F^*} = \frac{\sqrt{\frac{2D_A c_t}{c_e}}}{\sqrt{\frac{2D_F c_t}{c_e}}} = \sqrt{\frac{D_A}{D_F}}$$

$$\boxed{\frac{TRC(Q_F^*)}{TRC(Q_A^*)} = \frac{1}{2}\left(\sqrt{\frac{D_A}{D_F}} + \sqrt{\frac{D_F}{D_A}}\right)}$$

# EOQ Sensitivity wrt Demand

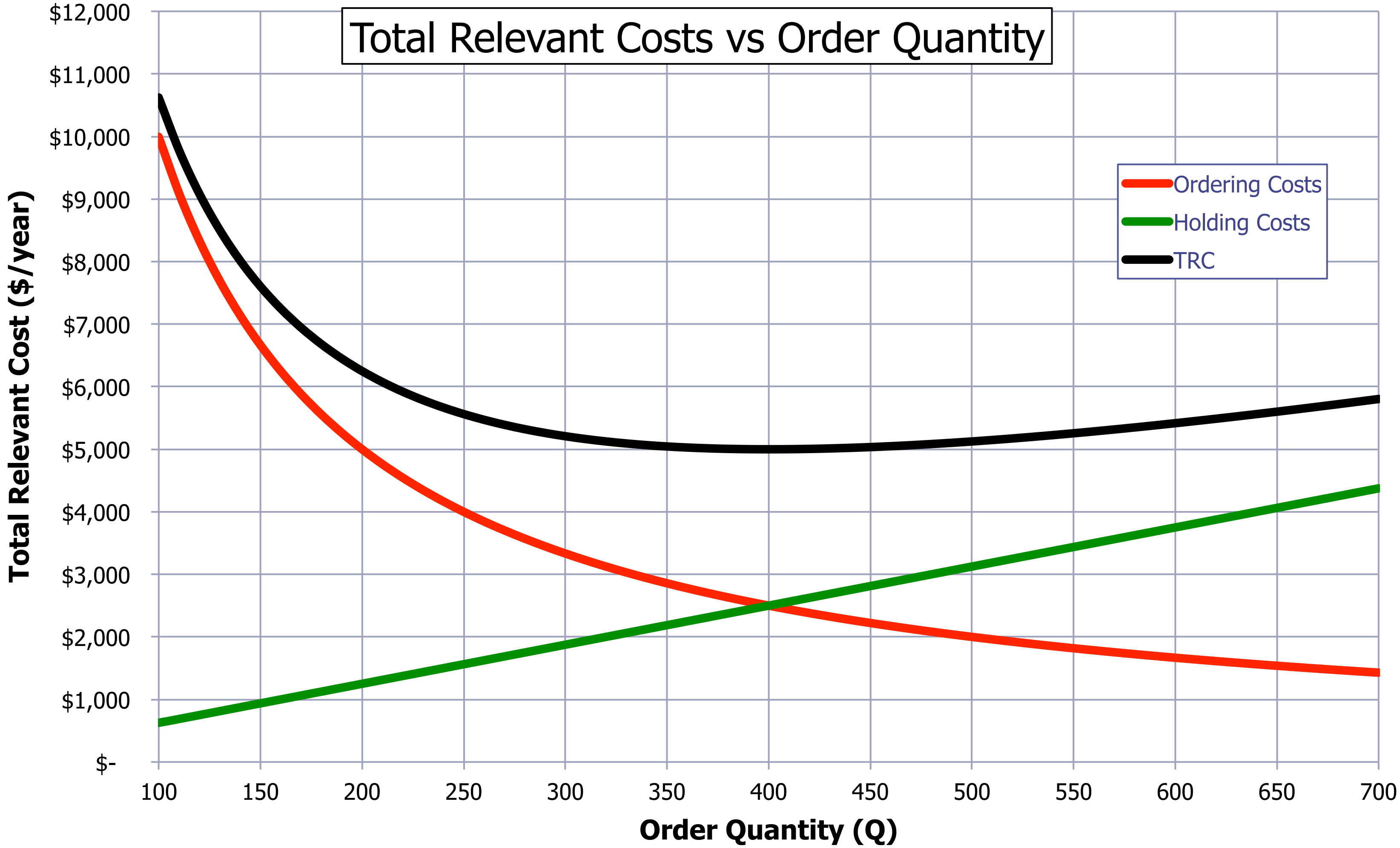
Previous example where:

$D = 2000$ $c=50$ $c_e = 12.5$ $c_t = 500$

| $D_{ACTUAL}$ | $D_A/D_F$ | $Q^*_A/Q^*_F$ | $TRC^*_A/TRC^*_F$ |
|---|---|---|---|
| 200 | 0.10 | 0.32 | 1.74 |
| 1,000 | 0.50 | 0.71 | 1.06 |
| 1,500 | 0.75 | 0.87 | 1.01 |
| 1,800 | 0.90 | 0.95 | 1.00 |
| 2,000 | 1.00 | 1.00 | 1.00 |
| 3,000 | 1.50 | 1.22 | 1.02 |
| 4,000 | 2.00 | 1.41 | 1.06 |
| 20,000 | 10.00 | 3.16 | 1.74 |

How much will TRC change if I change $c_t$, $c$, or $c_e$?

# EOQ – Very Robust Solution



# EOQ Sensitivity to Order Cycle Time

# EOQ Sensitivity wrt Order Cycle Time

- How sensitive is TRC to T*?
  - Why do we care and why is "time" different?
  - How do I find the "best" T that is also practical?

- Previous Example:
  - Q* = 400 units for TRC* = $5,000 per year
  - T* = 0.2 years or 73 days or 10.4 weeks
  - What if I order weekly? monthly? Other?
    - TRC(1)/TRC(10.4) = .5(1/10.4 + 10.4/1) = 5.24 or 524% higher costs
    - TRC(4)/TRC(10.4) = 1.49 or 50% higher costs

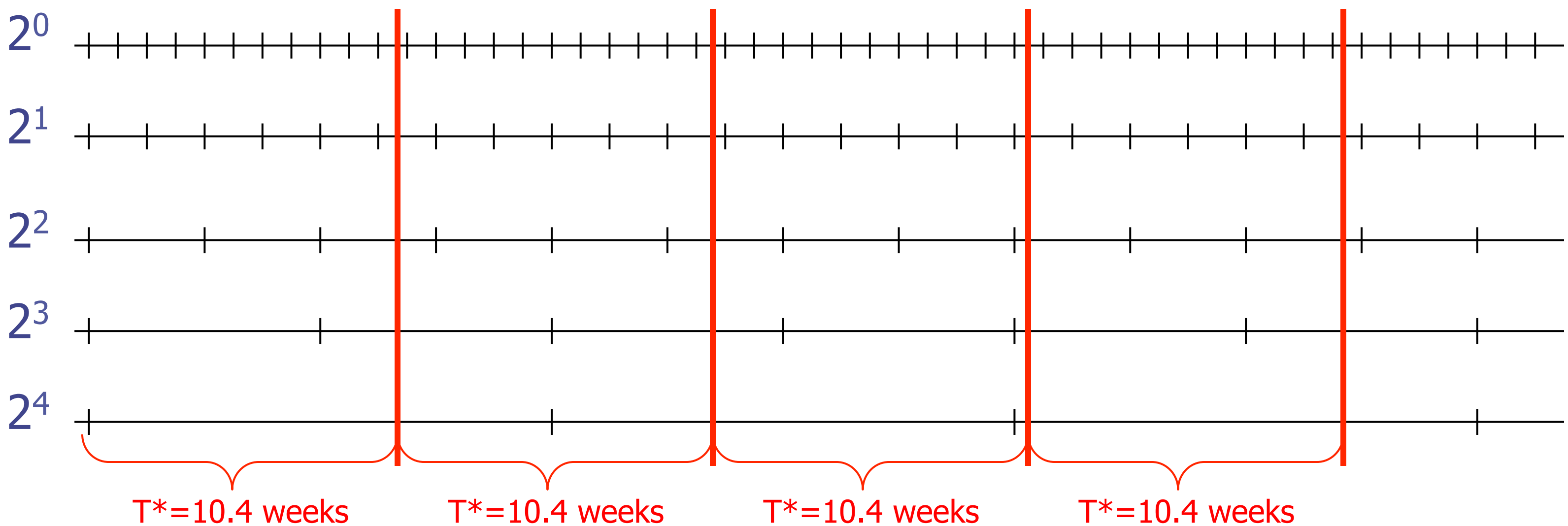
$$\boxed{\frac{TRC(T)}{TRC(T^*)} = \frac{1}{2}\left(\frac{T}{T^*} + \frac{T^*}{T}\right)}$$

# EOQ Sensitivity wrt Order Cycle Time

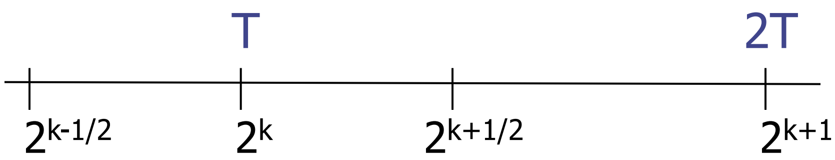
## Power of Two Policies

- Order in intervals of powers of two
- Select a realistic base period, $T_{Base}$ (day, week, month)
- Guarantees that TRC will be within 6% of optimal!

- We want to find the smallest k where $2^k \leq T^* \leq 2^{k+1}$
  - $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, . . .
  - . . . So for T*=10.4 weeks . . . k=3 so we order every 8 weeks



# EOQ Sensitivity wrt Order Cycle Time




$$\frac{TRC(T)}{TRC(T^*)} = \frac{1}{2}\left(\frac{T}{T^*} + \frac{T^*}{T}\right) = \frac{1}{2}\left(\frac{2T}{T^*} + \frac{T^*}{2T}\right)$$

$$\frac{2T^* - T^*}{2T} = \frac{2T - T}{T^*}$$

$$\frac{T^*}{2T} = \frac{T}{T^*}$$

$$\frac{T}{T^*} = \sqrt{\frac{1}{2}} \quad and \quad \frac{T^*}{T} = \sqrt{2}$$

$$\frac{TRC(2^k)}{TRC(T^*)} \le \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{1}\right) \approx 1.06$$

To find a practical T:

1. Calculate T*
2. Pick a base time period (day, week, etc.)
3. Find the lowest value of k that satisfies:

$$\boxed{\frac{T^*}{\sqrt{2}} \le 2^k \le \sqrt{2}\,T^*}$$

A Power of Two time interval is guaranteed to be within 6% of the TRC* with the optimal time interval.

# Step by Step Example: Power of Two

Find the optimal order quantity, optimal order cycle time, and TRC for a product with the following characteristics:

- Demand = 25,000 units/year
- Cost of placing an order = 750 \$/order
- Cost of product = 25.50 \$/unit
- Holding cost = 15% of unit cost per year

Q* =3,131 units/order
T*=0.125 years/order
TRC*=\$11,976 \$/year

Assume that we want to order on a power of two weekly schedule, what is my new T and Q and what is my percent error from TRC*?

# Step by Step Example: Power of Two

# Key Points from Lesson

# Key Points – The EOQ model . . .

- has restrictive assumptions, but is still widely used
- trades fixed (order) and variable (holding) costs
- is relatively insensitive to changes in . . .
  - rounding of order quantities (Q)
  - rounding of order cycle time (T)
  - errors in forecasting (D)
  - errors in cost parameters ($c$, $c_t$, $h$, $c_e$)
- helps focus management on process improvements
  - How do I lower $c_t$?
  - How do I lower $c$?
  - How do I lower $h$?
- is a good starting point, but not always a good end point.

# Questions, Comments, Suggestions?
# Use the Discussion!