Single Period Inventory Model:
Calculating Expected Profitability

MIT Center for
Transportation & Logistics

Agenda

- Expected Profits
- Expected Units Short
- Unit Normal Loss Function
- NFL Jersey Example

Expected Profits

Profit Maximization

$$\boxed{P[x \le Q] = \frac{p-c}{p} = \frac{c_s}{c_s + c_e}}$$

$$\text{Profit}(Q,x) = \begin{cases} px - cQ & \text{if } x \le Q \\ pQ - cQ & \text{if } x \ge Q \end{cases}$$

$$E\left[\text{Profit}(Q)\right] = \int_0^\infty P(Q,x_0) f_x(x_0)\,dx_0$$

$$E\left[\text{Profit}(Q)\right] = \int_0^Q (px - cQ) f_x(x_0)\,dx_0 + \int_Q^\infty (pQ - cQ) f_x(x_0)\,dx_0$$

$$E\left[\text{Profit}(Q)\right] = p\int_0^\infty (x) f_x(x_0)\,dx_0 - cQ - p\int_Q^\infty (x - Q) f_x(x_0)\,dx_0$$

$$\boxed{E\left[\text{Profit}(Q)\right] = pE[x] - cQ - pE\left[\text{UnitsShort}\right]}$$

Expected Profits with Salvage & Penalty

It gets a little more complicated when we use B and g:

g = Salvage value, $/unit

B = Penalty for not satisfying demand, beyond lost profit, $/unit

$$P(Q) = \begin{cases} -cQ + px + g(Q - x) & \text{if } x \le Q \\ -cQ + pQ - B(x - Q) & \text{if } x \ge Q \end{cases}$$

$$E[P(Q)] = (p - g)E[x] - (c - g)Q - (p - g + B)E[\text{US}]$$

Rearranging this:

$$E[P(Q)] = p(E[x] - E[US]) - cQ + g(Q - (E[x] - E[US])) - B(E[\text{US}])$$

Expected Units Short

Expected Values

Continuous | Discrete

E[Units Demanded]

$$\int_{x=0}^{\infty} x f_x\left(x\right)dx = \hat{x}$$

$$\sum_{x=0}^{\infty} xP\left[x\right] = \hat{x}$$

E[Units Sold]

$$\int_{x=0}^{Q} x f_x\left(x\right)dx + Q\int_{x=Q}^{\infty} f_x\left(x\right)dx$$

$$\sum_{x=0}^{Q} xP\left[x\right] + Q\sum_{x=Q+1}^{\infty} P\left[x\right]$$

E[Units Short]

$$\int_{x=Q}^{\infty} \left(x-Q\right) f_x\left(x\right)dx$$

$$\sum_{x=Q+1}^{\infty} \left(x-Q\right) P\left[x\right]$$

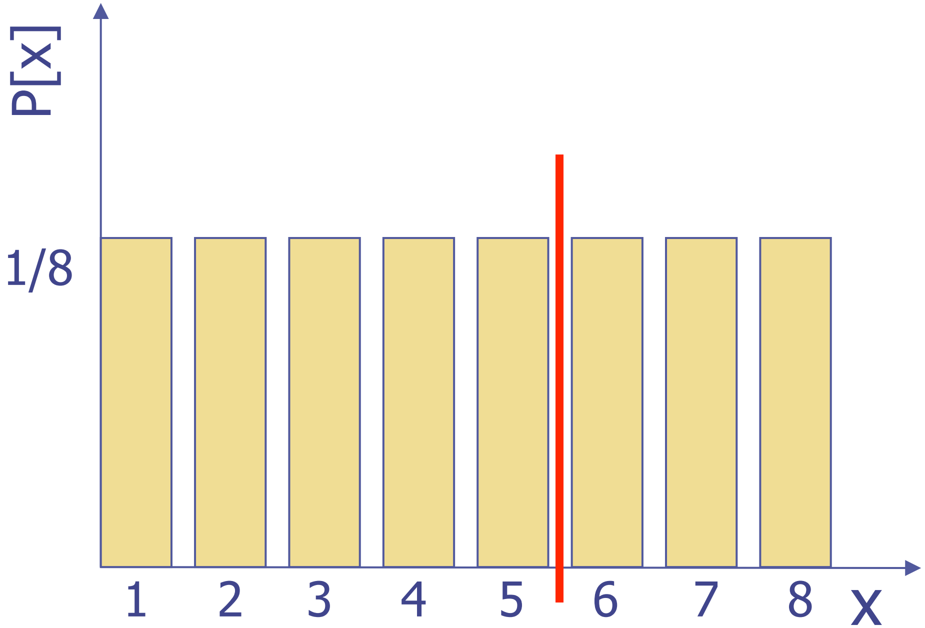
Example: Discrete Case

Each day I can sell between 1 and 8 freshly baked widgets. The demand distribution is shown below. Widgets not sold at end of day are thrown out.

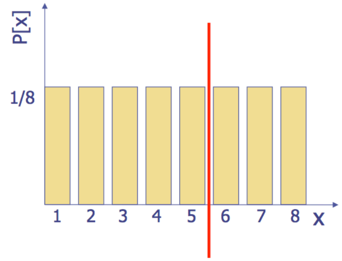
Suppose that I have decided to order 5 each day.

What is my:

- Expected Demand?
- Expected Units Sold?
- Expected Units Short?

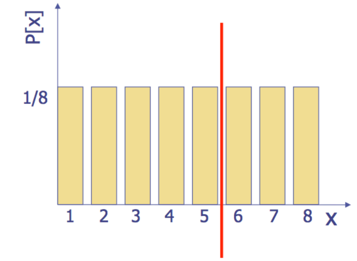


Example:



Demand (x)	P[x]
1	0.125
2	0.125
3	0.125
4	0.125
5	0.125
6	0.125
7	0.125
8	0.125

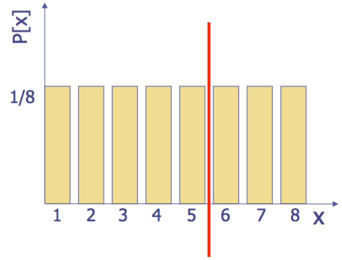
Example: Expected Demand



Demand (x)	P[x]	xP[x]
1	0.125	0.125
2	0.125	0.250
3	0.125	0.375
4	0.125	0.500
5	0.125	0.625
6	0.125	0.750
7	0.125	0.875
8	0.125	1.000
		E[x] = 4.500

$$E[\text{Units Demanded}] = \sum_{x=0}^{\infty} xP\left[x\right] = \hat{x}$$

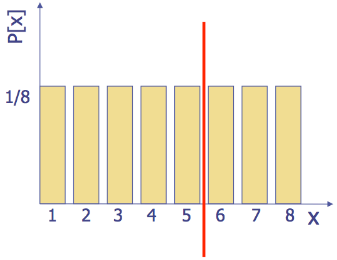
Example: Expected Sales



Demand (x)	P[x]	xP[x]	QP[x]
1	0.125	0.125	0
2	0.125	0.250	0
3	0.125	0.375	0
4	0.125	0.500	0
5	0.125	0.625	0
6	0.125	0	0.625
7	0.125	0	0.625
8	0.125	0	0.625
		E[sales] =	**3.75**

$$\text{E[Units Sold | Q=5]} = \sum_{x=0}^{Q} xP\left[x\right] + Q\sum_{x=Q+1}^{\infty} P\left[x\right]$$

Example: Expected Units Short



Demand (x)	P[x]	Loss L(Q)=x-Q	L(Q)P[x]
1	0.125	0	0
2	0.125	0	0
3	0.125	0	0
4	0.125	0	0
5	0.125	0	0
6	0.125	1	0.125
7	0.125	2	0.250
8	0.125	3	0.375
			E[US] = 0.75

$$E[\text{Units Short} \mid Q=5] = \sum_{x=Q+1}^{\infty} (x-Q)P[x]$$

Unit Normal Loss Function

Finding Expected Units Short

Loss Function L(Q) = expected amount that random variable X exceeds a given threshold value, Q. Expected units short per replenishment cycle

$$E[US] = \sum_{x=Q+1}^{\infty} (x - Q)\, p[x] \qquad E[US] = \int_{x=Q}^{\infty} (x - Q)\, f_x(x)\, dx$$

For a Normal Distribution we can use G(k), the Unit Normal Loss Function:

$$E\left[US\right] = \int_{x=Q}^{\infty} \left(x - Q\right) f_x\left(x\right) dx \quad = \sigma G\left(\frac{Q-\mu}{\sigma}\right) = \sigma G\left(k\right)$$

Recall that k is the Standard Normal Distribution where (μ =0, σ=1)
Transform: k=(Q - μ)/σ In Spreadsheets: k=NORMSINV(CR)

Unit Normal Loss function is:

$$G(k) = f_x(x_O) - k * Prob[x_O \geq k])$$

In Spreadsheets:

=NORMDIST(**k**,0,1,0) – **k***(1 - NORMSDIST(**k**))

Standard Normal Table

Suppose:
~N(160, 45)
and Q = 190

What is my E[US]?

k=(190-160)/45
= 0.67

G(k) = 0.1503

E[US] = (45)(0.1503)
= 6.76 units

k	P[x≤k]	G(k)	k	P[x≤k]	G(k)	k	P[x≤k]	G(k)
0.00	0.5000	0.3989	0.50	0.6915	0.1978	1.00	0.8413	0.0833
0.01	0.5040	0.3940	0.51	0.6950	0.1947	1.01	0.8438	0.0817
0.02	0.5080	0.3890	0.52	0.6985	0.1917	1.02	0.8461	0.0802
0.03	0.5120	0.3841	0.53	0.7019	0.1887	1.03	0.8485	0.0787
0.04	0.5160	0.3793	0.54	0.7054	0.1857	1.04	0.8508	0.0772
0.05	0.5199	0.3744	0.55	0.7088	0.1828	1.05	0.8531	0.0757
0.06	0.5239	0.3697	0.56	0.7123	0.1799	1.06	0.8554	0.0742
0.07	0.5279	0.3649	0.57	0.7157	0.1771	1.07	0.8577	0.0728
0.08	0.5319	0.3602	0.58	0.7190	0.1742	1.08	0.8599	0.0714
0.09	0.5359	0.3556	0.59	0.7224	0.1714	1.09	0.8621	0.0700
0.10	0.5398	0.3509	0.60	0.7257	0.1687	1.10	0.8643	0.0686
0.11	0.5438	0.3464	0.61	0.7291	0.1659	1.11	0.8665	0.0673
0.12	0.5478	0.3418	0.62	0.7324	0.1633	1.12	0.8686	0.0659
0.13	0.5517	0.3373	0.63	0.7357	0.1606	1.13	0.8708	0.0646
0.14	0.5557	0.3328	0.64	0.7389	0.1580	1.14	0.8729	0.0634
0.15	0.5596	0.3284	0.65	0.7422	0.1554	1.15	0.8749	0.0621
0.16	0.5636	0.3240	0.66	0.7454	0.1528	1.16	0.8770	0.0609
0.17	0.5675	0.3197	0.67	0.7486	0.1503	1.17	0.8790	0.0596
0.18	0.5714	0.3154	0.68	0.7517	0.1478	1.18	0.8810	0.0584
0.19	0.5753	0.3111	0.69	0.7549	0.1453	1.19	0.8830	0.0573
0.20	0.5793	0.3069	0.70	0.7580	0.1429	1.20	0.8849	0.0561
0.21	0.5832	0.3027	0.71	0.7611	0.1405	1.21	0.8869	0.0550

NFL Jersey Example Solution

Example: NFL Replica Jerseys

- Data:
  - Total cost = c = 10.90 $/jersey
  - Selling price = p = 24 $/jersey
  - Forecast demand ~N(32000, 11000)
- Solutions:
  - Case 1: No salvage value (g=0); Q* = 33,267
  - Case 2: Salvage value (g= 7 $/jersey); Q* = 40,149
- What is Expected Profit for each case?

$$\boxed{E\left[P(Q)\right]=(p-g)E\left[x\right]-(c-g)Q-(p-g+B)E\left[\text{UnitsShort}\right]}$$

Case adapted from Parsons, J. (2004) "Using A Newsvendor Model for Demand Planning of NFL Replica Jerseys," MIT Supply Chain Management Program Thesis.
Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg

Example: NFL Replica Jerseys

$$\boxed{E\left[P\left(Q\right)\right]=\left(p-g\right)E\left[x\right]-\left(c-g\right)Q-\left(p-g+B\right)E\left[\text{UnitsShort}\right]}$$

- First term; (p-g)E[x]
  - Case 1: (24-0)(32000) = $768,000
  - Case 2: (24-7)(32000) = $544,000
- Second term; (c-g)Q
  - Case 1: = (10.9 - 0)(33267) = $362,610
  - Case 2: = (10.9 - 7)(40149) = $156,581
- Third term; (p-g+B)E[US]
  - Using Unit Normal Loss Function; E[US] = σ G(k)

Example: NFL Replica Jerseys

- Third term; (p-g+B)E[US]
  1. Find $k = (Q-\mu)/\sigma$
  2. Look up or calculate Unit Normal Loss function, G(k)
  3. Find $E[US] = \sigma G(k)$
  4. Multiply E[US] by (p-g+B)
- Case 1: No salvage value, Q* = 33,267
  1. k = (33267 – 32000)/11000 = 0.115
  2. G(k) = 0.3441
     - =NORMDIST(**0.115**,0,1,0) – **0.115***(1 - NORMSDIST(**0.115**))
     - Use Standard Normal Table (interpolation)

Standard Normal Table

k= 0.115
What is G(0.115)?

Interpolating:
=(½)(0.3464 – 0.3418)
= 0.0023

G(0.115) = 0.3418+0.0023
= 0.3441

k	P[x≤k]	G(k)	k	P[x≤k]	G(k)	k	P[x≤k]	G
0.00	0.5000	0.3989	0.50	0.6915	0.1978	1.00	0.8413	0.
0.01	0.5040	0.3940	0.51	0.6950	0.1947	1.01	0.8438	0.
0.02	0.5080	0.3890	0.52	0.6985	0.1917	1.02	0.8461	0.
0.03	0.5120	0.3841	0.53	0.7019	0.1887	1.03	0.8485	0.
0.04	0.5160	0.3793	0.54	0.7054	0.1857	1.04	0.8508	0.
0.05	0.5199	0.3744	0.55	0.7088	0.1828	1.05	0.8531	0.
0.06	0.5239	0.3697	0.56	0.7123	0.1799	1.06	0.8554	0.
0.07	0.5279	0.3649	0.57	0.7157	0.1771	1.07	0.8577	0.
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0.09	0.5359	0.3556	0.59	0.7224	0.1714	1.09	0.8621	0.
0.10	0.5398	0.3509	0.60	0.7257	0.1687	1.10	0.8643	0.
0.11	0.5438	0.3464	0.61	0.7291	0.1659	1.11	0.8665	0.
0.12	0.5478	0.3418	0.62	0.7324	0.1633	1.12	0.8686	0.
0.13	0.5517	0.3373	0.63	0.7357	0.1606	1.13	0.8708	0.
0.14	0.5557	0.3328	0.64	0.7389	0.1580	1.14	0.8729	0.
0.15	0.5596	0.3284	0.65	0.7422	0.1554	1.15	0.8749	0.
0.16	0.5636	0.3240	0.66	0.7454	0.1528	1.16	0.8770	0.
0.17	0.5675	0.3197	0.67	0.7486	0.1503	1.17	0.8790	0.
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0.20	0.5793	0.3069	0.70	0.7580	0.1429	1.20	0.8849	0.
0.21	0.5832	0.3027	0.71	0.7611	0.1405	1.21	0.8869	0.

Example: NFL Replica Jerseys

- Expected Cost of Units Short = (p-g+B)E[US]
  1. Find $k = (Q-\mu)/\sigma$
  2. Look up or calculate Unit Normal Loss function, G(k)
  3. Find $E[US] = \sigma G(k)$
  4. Multiply E[US] by (p-g+B)
- Case 1: No salvage value g=0; Q* = 33,267
  1. k = (33267 – 32000)/11000 = 0.115
  2. G(k) = 0.3441
  3. E[US] = (11000)(0.3441) = 3785 jerseys
  4. (p-g+B)E[US] = (24-0-0)(3785) = $90,840
- Case 2: Salvage value g=7; Q* = 40,149
  1. k = (40149 – 32000)/11000 = 0.741
  2. G(k) = 0.1332
  3. E[US] = (11000)(0.1332) = 1465 jerseys
  4. (p-g+B)E[US] = (24-7-0)(1465) = $24,905

Example: NFL Replica Jerseys

$$E\left[P(Q)\right] = (p-g)E\left[x\right] - (c-g)Q - (p-g+B)E\left[\text{US}\right]$$

	(p-g)E[x]	(c-g)Q	(p-g+B) E[US]	E[Profit]
Case 1 (g=0)	$768,000	$362,610	$90,840	**$314,550**
Case 2 (g=7)	$544,000	$156,581	$24,905	**$362,514**

Example: NFL Replica Jerseys

$$E\left[P(Q)\right] = (p-g)E\left[x\right] - (c-g)Q - (p-g+B)E\left[\mathrm{US}\right]$$

	(p-g)E[x]	(c-g)Q	(p-g+B) E[US]	E[Profit]
Case 1 (g=0)	\$768,000	\$362,610	\$90,840	**\$314,550**
Case 2 (g=7)	\$544,000	\$156,581	\$24,905	**\$362,514**

$$E\left[P(Q)\right] = p\left(E\left[x\right] - E\left[US\right]\right) - cQ + g\left(Q - \left(E\left[x\right] - E\left[US\right]\right)\right) - B\left(E\left[\mathrm{US}\right]\right)$$

Units	E[Demand]	E[US]	Ordered Q	E[Sold at Full Price]	E[Sold at Discount]
Case 1 (g=0)	32,000	3,785	33,267	28,215	0
Case 2 (g=7)	32,000	1,465	40,149	30,535	9,614

Key Points from Lesson

Key Points from Lesson

- Expected Profitability

$$E\left[P\left(Q\right)\right] = p\left(E\left[x\right] - E\left[US\right]\right) - cQ + g\left(Q - \left(E\left[x\right] - E\left[US\right]\right)\right) - B\left(E\left[\mathrm{US}\right]\right)$$

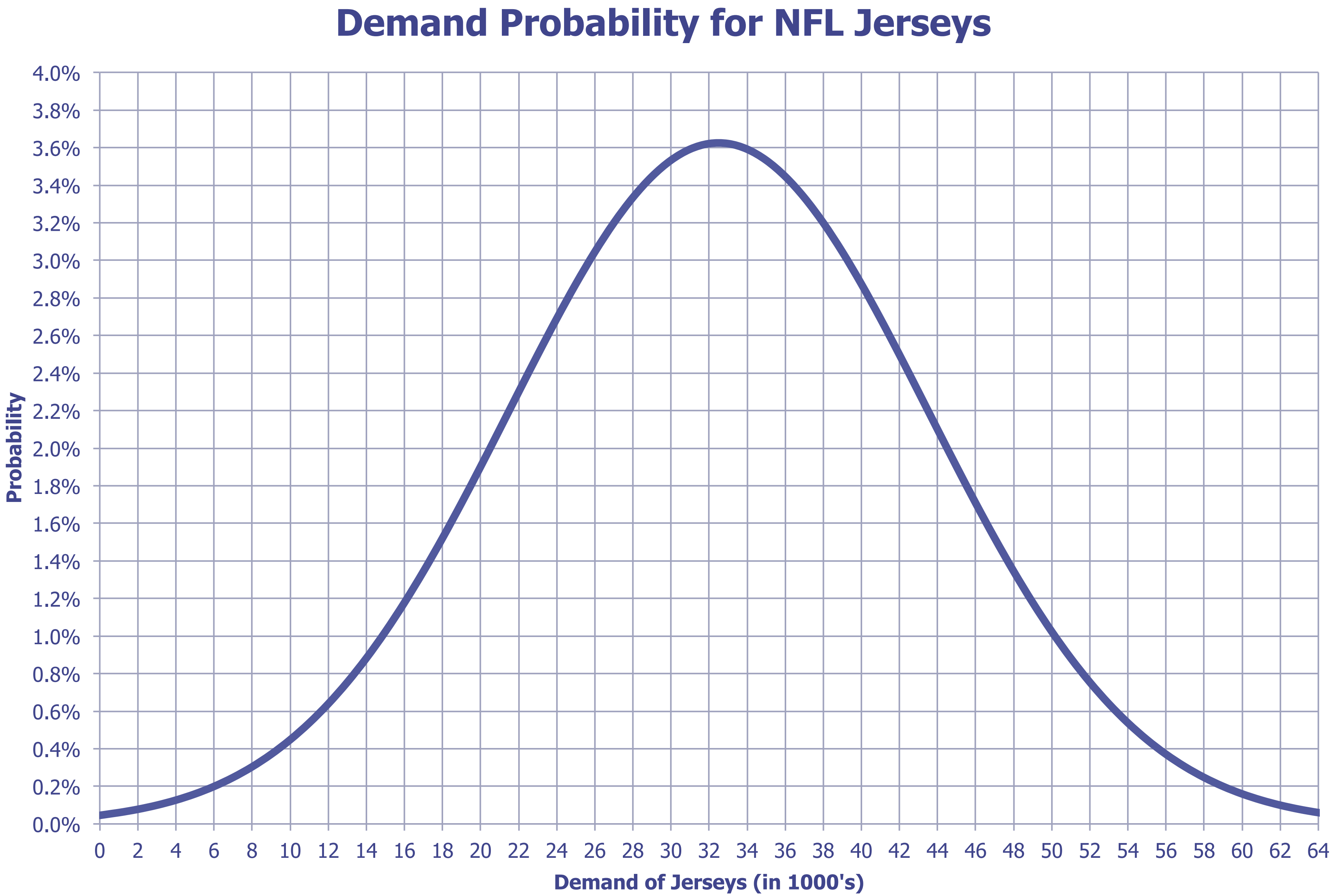
- Expected Units Short
  - Loss Function

$$\int_{x=Q}^{\infty} \left(x - Q\right) f_x\left(x\right) dx \qquad \sum_{x=Q+1}^{\infty} \left(x - Q\right) P\left[x\right]$$

  - Tabulate the Loss Function for any distribution
  - Unit Normal Loss Function, G(k)

=NORMDIST(**k**,0,1,0) – **k***(1 - NORMSDIST(**k**))

Demand ~Normal (32000, 11000)



Questions, Comments, Suggestions?
Use the Discussion!

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