Inventory Models for Probabilistic Demand: Basic Concepts

MIT Center for Transportation & Logistics

Notation

D = Average Demand (units/time)

c = Variable (Purchase) Cost ($/unit)

h = Carrying or Holding Charge ($/inventory $/time)

c_t = Fixed Ordering Cost ($/order)

$c_e = c*h$ = Excess Holding Cost ($/unit/time)

c_s = Shortage Cost ($/unit/time)

Q = Replenishment Order Quantity (units/order)

L = Replenishment Lead Time (time)

T = Order Cycle Time (time/order)

$N = 1/T$ = Orders per Time (order/time)

IP = Inventory Position (units)

IOH = Inventory on Hand (units)

IOO = Inventory On Order (units)

μ_{DL} = Expected Demand over Lead Time (units/time)

σ_{DL} = Standard Deviation of Demand over Lead Time (units/time)

k = Safety Factor

s = Reorder point (units)

S = Order up to Point (units)

R = Review Period (time)

IFR = Item Fill Rate (%)

CSL = Cycle Service Level (%)

CSOE = Cost of Stock Out Event ($/event)

CSI = Cost per item short

E[US] = Expected Units Short (units)

G(k) = Unit Normal Loss Function

Inventory Replenishment Policies

- Policy: How much to order and when

- Five Methods
  - EOQ Policy – deterministic demand
    - Order Q* every T* time periods
    - Order Q* when IP= μ_{DL}
  - Single Period Models – variable demand
    - Order Q* at start of period where P[x≤Q]=CR
  - Base Stock Policy – one-for-one replenishment
    - Order what was demanded when it was demanded
  - Continuous Review Policy (s,Q) - event based
    - Order Q* when IP≤s
  - Periodic Review Policy (R,S) – time based
    - Order up to S units every R time periods.

Recall:
Inventory Position (IP) =
Inventory on Hand (IOH)
+ Inventory on Order (IOO)
- Backorders

Demand over
Leadtime=D*L= μ_{DL}
(be careful with dimensions)

Quick Aside on Converting Times

- What is the μ and σ of demand during the replenishment lead time?
  - Lead time for replenishment is 7 days (1 week)
  - Annual demand is ~N(450,000, 22,000)

- What is the expected demand over lead time?
  - μ_{DL}= (450,000 units/year)(1 week) / (52 weeks/year)
  - = 8653.8 = 8,654 units

- What is the standard deviation of demand over lead time?
  - Recall that if we assume that each period (week) is identically and independently distributed (iid) then; $\sigma^2_1 + \sigma^2_{2+} \ldots \sigma^2_n = n\sigma^2$
  - So, in our problem, $\sigma^2_{year} = 52(\sigma^2_{week})$
  - Which means $\sigma_{week} = (\sigma_{year})/(\sqrt{52})$
  - σ_{DL}= (22,000 units/year) /(√52) = 3,050 units

- Demand over leadtime ~N(8,654, 3,050)

Quick Aside on Converting Times

- Suppose we have two periods to consider:
  - D_S = Demand over short time period (e.g., week)
  - D_L = Demand over long time period (e.g., year)
  - n = Number of short periods within a long (e.g., 52)

- Converting from Long to Short
  - $E[D_S] = E[D_L]/n$
  - $VAR[D_S] = VAR[D_L]/n$ so that $\sigma_S = \sigma_L/\sqrt{n}$

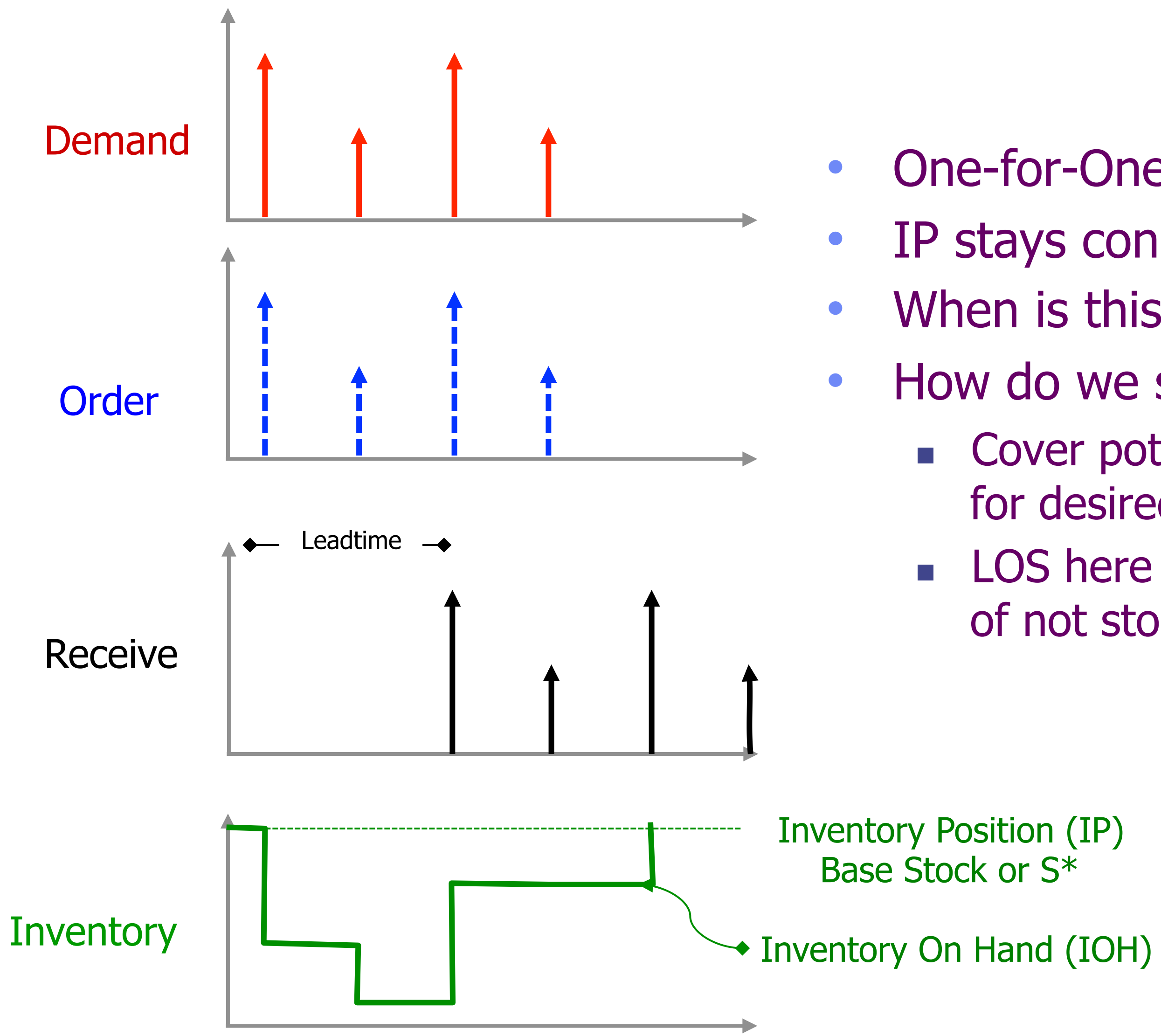
- Converting from Short to Long
  - $E[D_L] = nE[D_S]$
  - $VAR[D_L] = nVAR[D_S]$ so that $\sigma_L = \sqrt{n}\,\sigma_S$

Base Stock Policy

Assumptions: Base Stock Policy

- Demand
  - Constant vs **Variable**
  - Known vs **Random**
  - **Continuous** vs Discrete
- Lead Time
  - Instantaneous
  - **Constant** vs Variable
  - **Deterministic** vs Stochastic
  - Internally Replenished
- Dependence of Items
  - **Independent**
  - Correlated
  - Indentured
- Review Time
  - **Continuous** vs Periodic
- Number of Locations
  - **One** vs Multi vs Multi-Echelon
- Capacity / Resources
  - **Unlimited**
  - Limited / Constrained
- Discounts
  - **None**
  - All Units vs Incremental vs One Time
- Excess Demand
  - None
  - All orders are backordered
  - **Lost orders**
  - Substitution
- Perishability
  - **None**
  - Uniform with time
  - Non-linear with time
- Planning Horizon
  - Single Period
  - Finite Period
  - **Infinite**
- Number of Items
  - **One** vs Many
- Form of Product
  - **Single Stage**
  - Multi-Stage

Base Stock Policy



- One-for-One Order Policy
- IP stays constant at Base Stock
- When is this used?
- How do we set the Base Stock (S*)?
  - Cover potential demand over lead time for desired level of service (LOS)
  - LOS here is defined as the probability of not stocking out = P[$\mu_{DL} \leq S^*$]

Base Stock Policy

- How do I set the Level of Service (LOS)?
  - Management decision
  - Using Critical Ratio

$$LOS^* = P\left[\mu_{DL} \leq S^*\right] = CR = \frac{c_s}{c_s + c_e}$$

- LOS is the probability that no stock outs will occur during the lead time replenishment period

- Base Stock, S*, is set to this amount:

$$S^* = \mu_{DL} + k_{LOS}\,\sigma_{DL}$$

Standard deviation of demand during lead time

Base Stock

Expected Demand during lead time

Safety factor for given LOS

Base Stock Policy Example

- Set the Base Stock Policy for an item:
  - Daily demand ~N(100, 15)
  - Lead time is 2 days
  - Excess cost is $5 per unit per day
  - Shortage cost is $25 per unit per day.

- Solution:
  - Find μ_{DL}=100(2) = 200
  - Find σ_{DL}=15(√2)=21.2
  - Find LOS = CR = (25)/(5+25)=0.833
  - Find k_{LOS} from Tables or Spreadsheet
    - k_{LOS}=NORMSINV(0.833) = 0.967
  - Find S*= 200 + (0.967)(21.2) = 220.5 ≅ 221 units

Continuous Review Policies

Assumptions: Continuous Review Policies

- Demand
  - Constant vs **Variable**
  - Known vs **Random**
  - **Continuous** vs Discrete
- Lead Time
  - Instantaneous
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- Number of Items
  - **One** vs Many
- Form of Product
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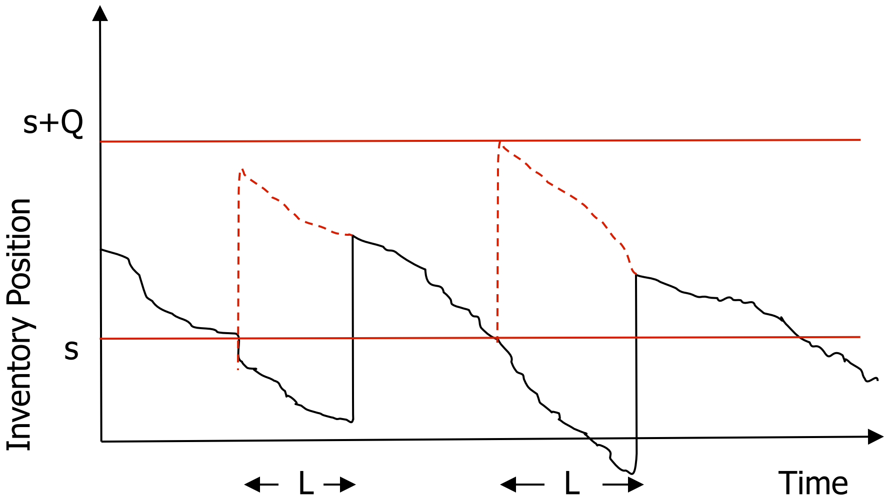
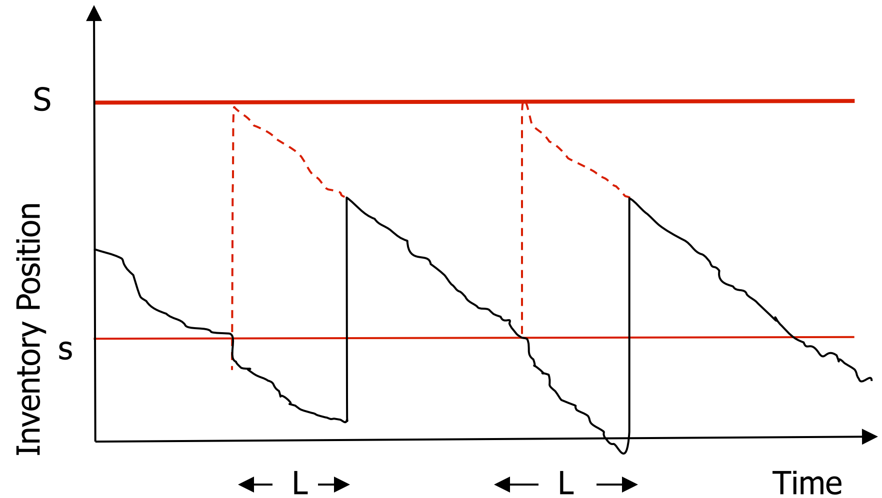
  - Multi-Stage

Continuous Review Policies

- Order-Point, Order-Quantity (s,Q)
  - Policy: **Order Q if IP ≤ s**
  - Two-bin system
- Order-Point, Order-Up-To-Level (s, S)
  - Policy: **Order (S-IP) if IP ≤ s**
  - Min-Max system





Notation

s = Reorder Point	S = Order-up-to Level	L = Replenishment Lead Time
Q = Order Quantity	R = Review Period	IOH= Inventory on Hand

IP = Inventory Position = (IOH) + (Inventory On Order) – (Backorders)

Order Point, Order Quantity Policy (s,Q)

Framework for (s, Q) System

- Finding Q
  - Determines the level of Cycle Stock
  - Usually from EOQ - but other methods maybe?

- Finding s
  - Based on expected demand over lead time (forecasted amount)
  - Added in safety or buffer stock for variability

$$s = \mu_{DL} + k\sigma_{DL}$$

Reorder Point

Forecasted mean demand over the lead time

Safety Stock

k = SS factor

σ_{DL} = RMSE of forecast

What cost and service objectives?

1. Common Safety Factors Approach
   - Simple, widely used method
   - Apply a common metric to aggregated items
2. Customer Service Approach
   - Establish constraint on customer service
   - Definitions in practice are fuzzy
   - Minimize costs with respect to customer service constraints
3. Cost Minimization Approach
   - Requires costing of shortages
   - Find trade-off between relevant costs

Service & Cost Metrics

$$TC = cD + c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{Q}{2} + k\sigma_{DL}\right) + c_s P[StockOutType]$$

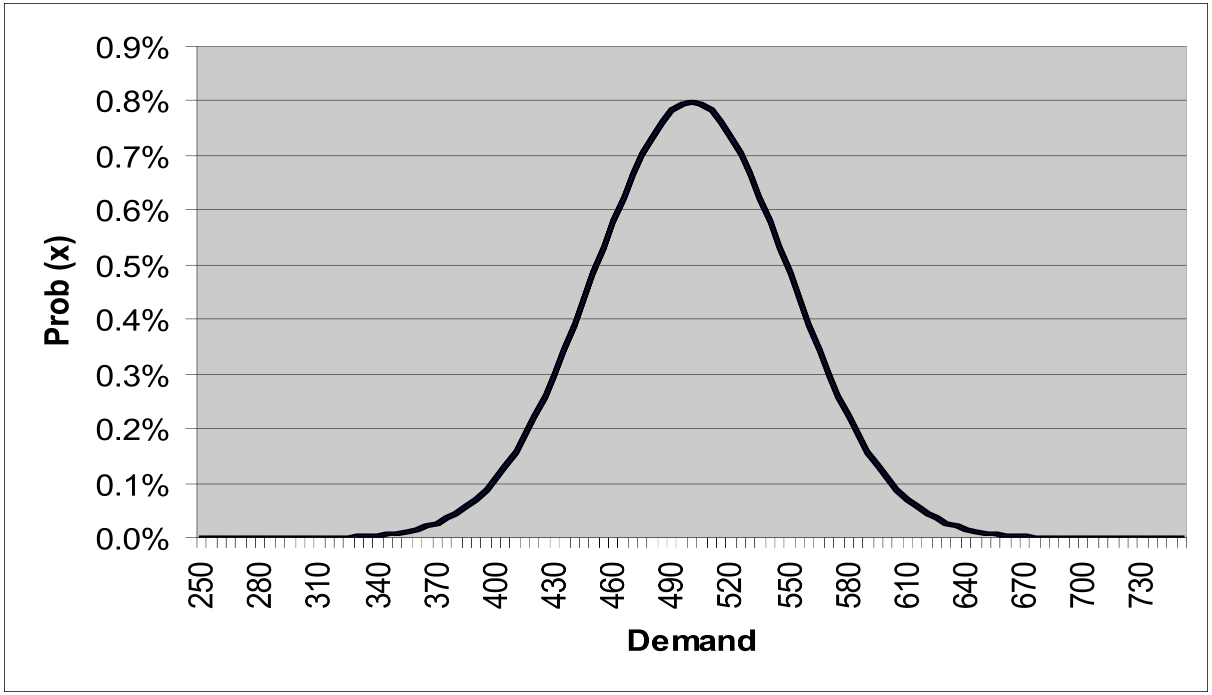
- In this class we will focus on the following:
  - Performance Metrics
    - Cycle Service Level (CSL)
    - Item Fill Rate (IFR)
  - Stockout Cost Metrics
    - Cost per Stockout Event (CSOE)
    - Cost per Item Short (CIS)

- Other forms can be used – these are most common.

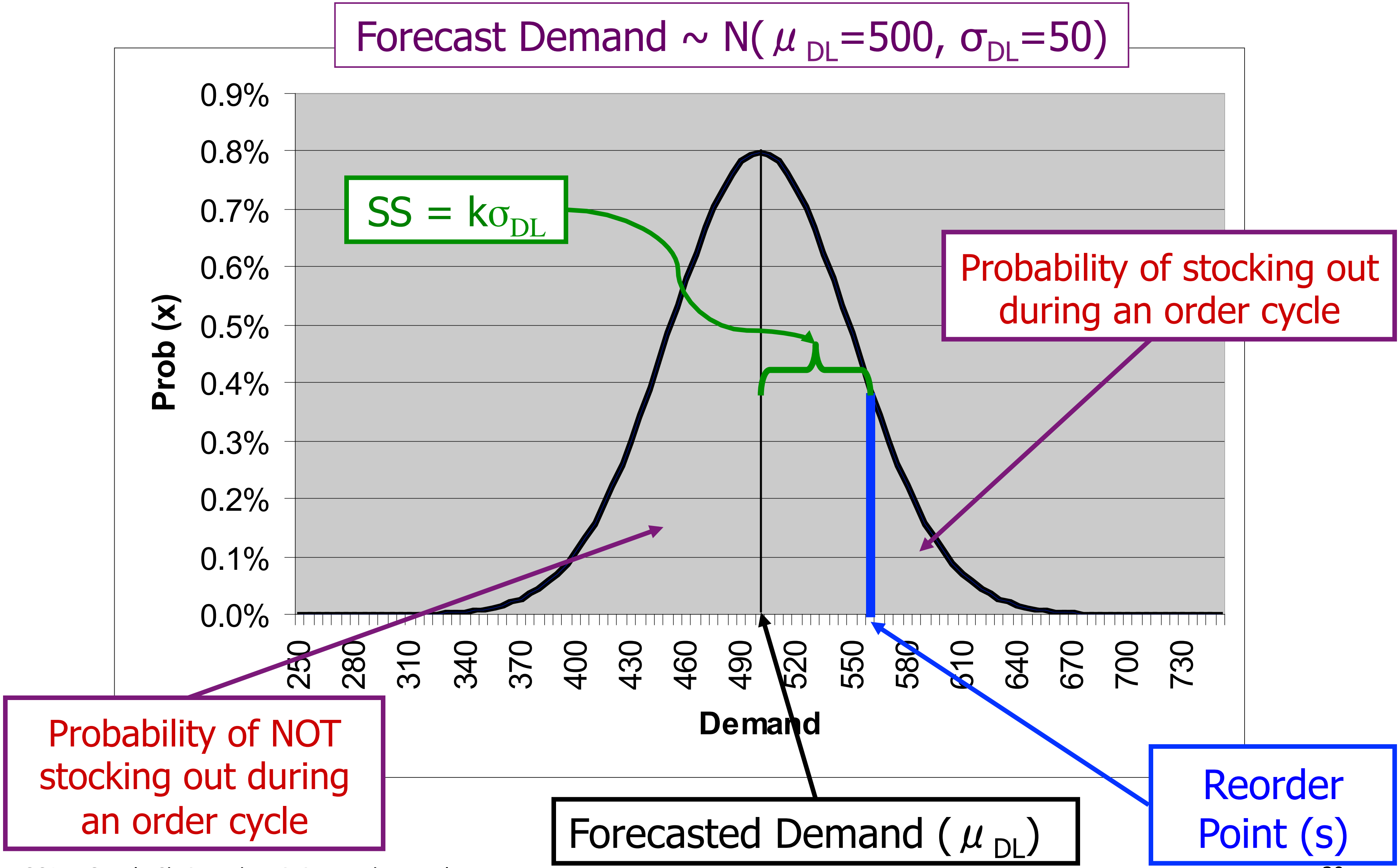
Cycle Service Level (CSL)

Cycle Service Level (CSL)

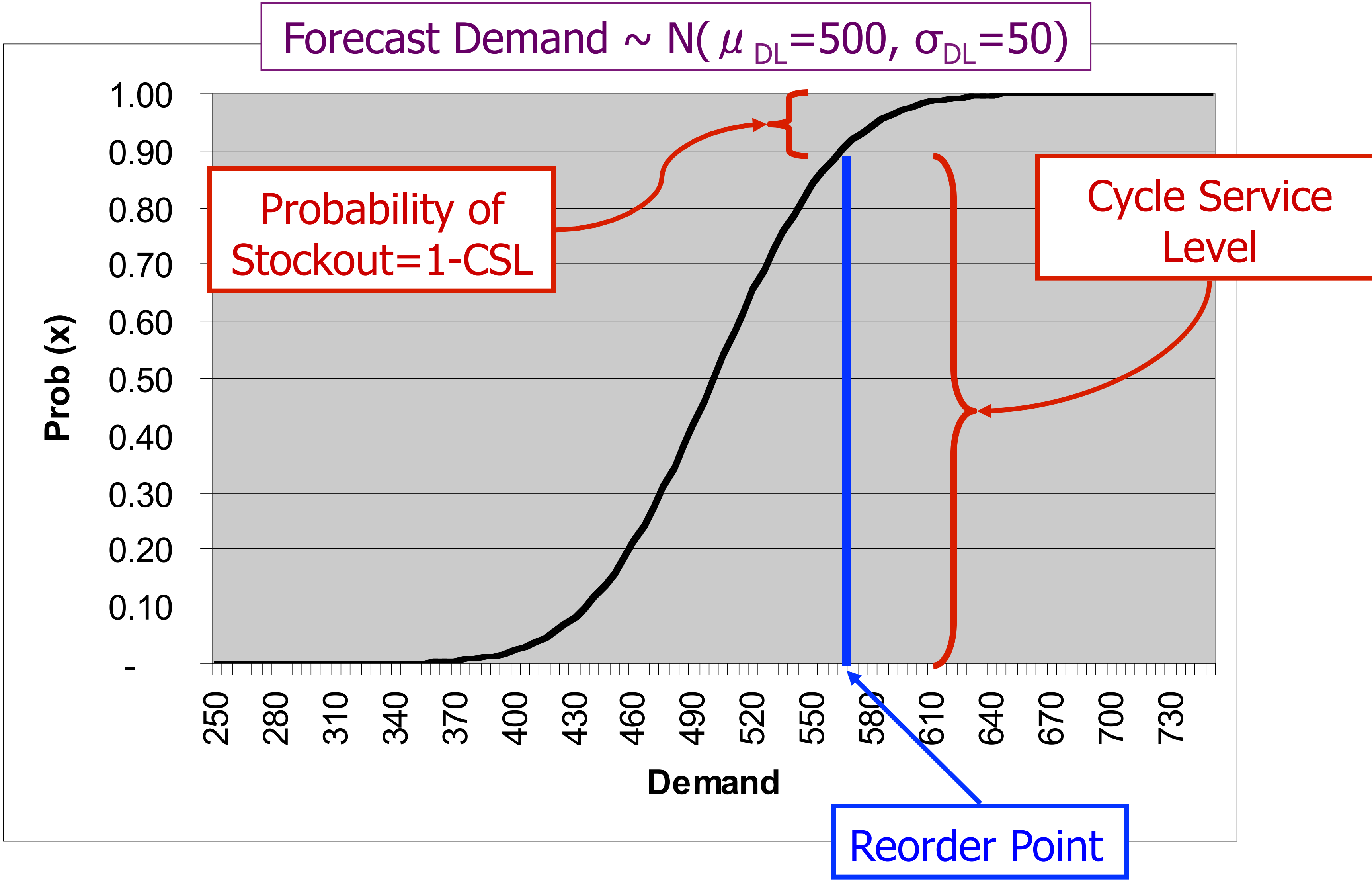
- Probability of no stockouts per replenishment cycle
  - Equal to one minus the probability of stocking out
  - X is the demand during lead time
  - = 1 – P[Stockout] = 1- P[X > s] = P[X ≤ s]



Finding P[Stockout]



Cumulative Normal Distribution



Example: Finding (s,Q) Policy with CSL

- Problem:
  - You are managing the inventory for a production part with annual demand ~N(62,000, 8,000). The cost of the item, c, is $100 and the holding charge is 15% per year. You have determined that the economic order quantity, Q*, is 5,200 units. Lead time is 2 weeks.
  - Assuming a CSL of 95%, find the appropriate (s, Q) policy.

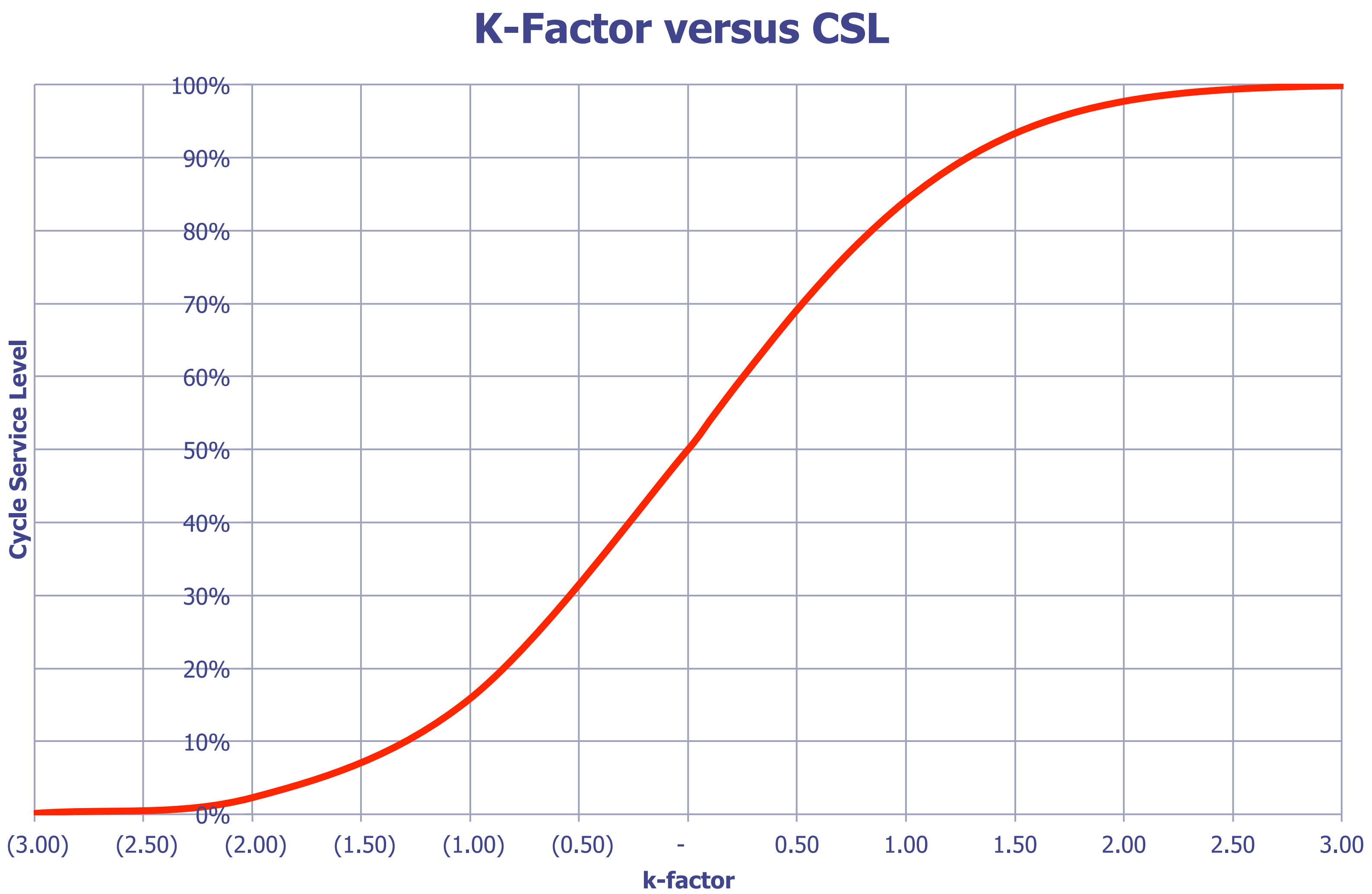
- Solution
  - Find μ_{DL} = (62,000)/(26) = 2,384.6 = 2,385 units
  - Find σ_{DL} = (8,000)/(√26) = 1,568.9 = 1,569 units
  - Find k where CSL = 0.95 or P[x≤k] = 0.95, k=1.644 = 1.64
  - Find s = $\mu_{DL} + k\sigma_{DL}$ = 2,385 + (1.64)(1,569) = 4,958 units

In Spreadsheets:
- k=NORMSINV(CSL)
- CSL=NORMSDIST(k)

In Standard Normal Tables:
- CSL=P[x≤k]

Policy: Order 5,200 when Inventory Position ≤ 4,958 units

k Factor versus Cycle Service Level



Cost Per Stockout Event (CSOE)

What if I know CSOE?

- Consider total costs
  - Purchase Price - no change
  - Order Costs – no change from EOQ
  - Holding Costs – add in Safety Stock
  - StockOut Costs product of:
    - Cost per stockout event (CSOE), B_1
    - Number of replenishment cycles
    - Probability of a stockout per cycle

$$TC = PurchaseCosts + OrderCosts + HoldingCosts + StockOutCosts$$

$$TC = cD + c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{Q}{2} + k\sigma_{DL}\right) + (B_1)\left(\frac{D}{Q}\right)P\left[x \geq k\right]$$

What costs are relevant?

How do I solve for k?

Finding k that minimizes TRC

Take First Order Conditions wrt k . . .

$$TRC\left(k\right) = c_e k\sigma_{DL} + (B_1)\left(\frac{D}{Q}\right)P\left[x \geq k\right]$$

Recall that for Normal Distribution:

$$\frac{d(P[x \geq k])}{dk} = -f_x\left(k\right) = \frac{-e^{\frac{-k^2}{2}}}{\sqrt{2\pi}}$$

Which gives us:

$$\frac{dTRC\left(k\right)}{dk} = c_e\sigma_{DL} + (B_1)\left(\frac{D}{Q}\right)\left(\frac{-e^{\frac{-k^2}{2}}}{\sqrt{2\pi}}\right) = 0$$

Solving for k . . .

$$\boxed{k = \sqrt{2\ln\left(\frac{B_1 D}{c_e\sigma_{DL}Q\sqrt{2\pi}}\right)}}$$

Cost per Stockout Event (B_1)

- Decision Rule for B_1 Costs
  - If $\dfrac{B_1 D}{c_e Q \sigma_{DL} \sqrt{2\pi}} > 1$ then $k = \sqrt{2 \ln\left(\dfrac{B_1 D}{c_e Q \sigma_{DL} \sqrt{2\pi}}\right)}$
  - Otherwise, set k as low as management allows

- Questions
  - Why is the first condition there?
  - What k would management allow?

Example: Finding (s,Q) Policy with CSOE

- Problem:
  - You are managing the inventory for a production part with annual demand ~N(62,000, 8,000). The cost of the item, c, is $100 and the holding charge is 15% per year. You have determined that the economic order quantity, Q*, is 5,200 units. Lead time is 2 weeks.
  - Assuming CSOE=B_1=$50,000 per event since it shuts the production line down, find the appropriate (s, Q) policy.
- Solution
  - Find μ_{DL} = (62,000)/(26) = 2,384.6 = 2,385 units
  - Find σ_{DL} = (8,000)/(√26) = 1,568.9 = 1,569 units
  - Check that the first condition is met $$\frac{B_1 D}{c_e Q \sigma_{DL}\sqrt{2\pi}} = \frac{(50,000)(62,000)}{(15)(5,200)(1,569)\sqrt{2\pi}} = 10.1 > 1$$
  - Solve for k $$k = \sqrt{2\ln(10.1)} = 2.15$$
  - Find s = μ_{DL} + kσ_{DL} = 2,385 + (2.15)(1,569) = 5,758 units

Policy: Order 5,200 when Inventory Position ≤ 5,758 units

Item Fill Rate (IFR)

Item Fill Rate (IFR)

- Item Fill Rate
  - Fraction of customer demand met routinely from IOH
  - This is equal to one minus the fraction we expect to be short

- Logic for Rule
  - We order Q each cycle
  - The fraction we are short = E[US]/Q
  - Therefore, item fill rate = 1 - E[US]/Q
  - Assuming ~Normal, E[US]= σ_{DL}G(k)
  - Calculate the desired G(k)
  - Find appropriate k

$$IFR = 1 - \frac{E[US]}{Q}$$

$$IFR = 1 - \frac{\sigma_{DL} G[k]}{Q}$$

$$G[k] = \frac{Q}{\sigma_{DL}}\left(1 - IFR\right)$$

Example: Finding (s,Q) Policy with IFR

- Problem:
  - You are managing the inventory for a production part with annual demand ~N(62,000, 8,000). The cost of the item, c, is $100 and the holding charge is 15% per year. You have determined that the economic order quantity, Q*, is 5,200 units. Lead time is 2 weeks.
  - Assuming IFR = 99%, find the appropriate (s, Q) policy.
- Solution
  - Find μ_{DL} = (62,000)/(26) = 2,384.6 = 2,385 units
  - Find σ_{DL} = (8,000)/(√26) = 1,568.9 = 1,569 units
  - Solve for G(k)

$$G[k] = \frac{Q}{\sigma_{DL}}\left(1 - IFR\right) = \frac{5{,}200}{1{,}569}\left(1 - .99\right) = 0.0331$$

  - Solve for k=1.45 from tables,
  - Find s = $\mu_{DL} + k\sigma_{DL}$ = 2,385 + (1.45)(1,569) = 4,660 units

Policy: Order 5,200 when Inventory Position ≤ 4,660 units

Cost per Item Short (CIS)

What if I know cost per item short?

- Consider total costs
  - Purchase Price - no change
  - Order Costs – no change from EOQ
  - Holding Costs – add in Safety Stock
  - StockOut Costs product of:
    - Cost per item stocked out (c_s)
    - Estimated number of units short
    - Number of replenishment cycles

$$TC = PurchaseCosts + OrderCosts + HoldingCosts + StockOutCosts$$

$$TC = cD + c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{Q}{2} + k\sigma_{DL}\right) + c_s\sigma_{DL}G_u(k)\left(\frac{D}{Q}\right)$$

- What costs are relevant?
- How do I solve for k?

$$P\left[StockOut\right] = P\left[x \geq k\right] = \frac{Qc_e}{Dc_s}$$

Cost per Item Short (c_s)

- Decision Rule for c_s Costs
  - If $\frac{Qc_e}{Dc_s} \le 1$ Then $P\left[StockOut\right] = P\left[x \ge k\right] = \frac{Qc_e}{Dc_s}$
  - Otherwise, set k as low as management allows
- Questions
  - Why is the first condition there?
  - What k would management allow?

Example: Finding (s,Q) Policy with CIS

- Problem:
  - You are managing the inventory for a production part with annual demand ~N(62,000, 8,000). The cost of the item, c, is $100 and the holding charge is 15% per year. You have determined that the economic order quantity, Q*, is 5,200 units. Lead time is 2 weeks.
  - Assuming CIS = c_s = 45 $/unit-year, find the appropriate (s, Q) policy.
- Solution
  - Find μ_{DL} = (62,000)/(26) = 2,384.6 = 2,385 units
  - Find σ_{DL} = (8,000)/(√26) = 1,568.9 = 1,569 units
  - Check decision rule: $$\frac{Qc_e}{Dc_s} = \frac{(5{,}200)(15)}{(62{,}000)(45)} = 0.02795 \leq 1$$
  - Find k where: $$P\left[x \geq k\right] = 1 - P\left[x \leq k\right] = 0.02795 \qquad P\left[x \leq k\right] = 0.97205 \qquad k = 1.91$$
  - Find s = μ_{DL} + kσ_{DL} = 2,385 + (1.91)(1,569) = 5,382 units

Policy: Order 5,200 when Inventory Position ≤ 5,382 units

Key Points from Lesson

Key Points

- Base Stock Policy
  - Order one for one with initial quantity of S*

$$S^* = \mu_{DL} + k_{LOS}\sigma_{DL}$$

- Continuous Review Policy (s,Q)
  - Order Q when IP ≤ s

$$s = \mu_{DL} + k\sigma_{DL}$$

- Safety Stock set by service or cost metrics
  - Cycle Service Level
  - Item Fill Rate
  - Cost per Stockout Event
  - Cost per Item Short

Metric	Value	k	SS
CSL	95%	1.64	$2,573
IFR	99%	1.45	$2,275
CSOE	$50,000	2.15	$3,373
CIS	$45	1.91	$2,997

- Safety stock only buffers for demand over lead time

Questions, Comments, Suggestions?
Use the Discussion!

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