

M534H Special Assignment– Spring 2009

Due Date: Thursday March 26th, 2009

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Theorem (Finite Propagation Speed Theorem).

Consider the *initial value problem for the wave equation* on \mathbb{R} :

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

where $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$ are two smooth given functions (data). Let $x_0 \in \mathbb{R}$ $t_0 > 0$ be fixed and suppose that $\phi(x)$ and $\psi(x)$ vanish for all x in the interval $[x_0 - t_0, x_0 + t_0]$.

Prove that $u(x, t)$ vanishes for all (x, t) within \mathcal{C} , the domain of dependence of (x_0, t_0) .

Recall $\mathcal{C} := \{(x, t) : 0 \leq t \leq t_0 \text{ and } x_0 - (t_0 - t) \leq x \leq x_0 + (t_0 - t)\}$

Notes The Theorem is also valid in higher dimensions but for simplicity prove it only in one (space) dimension. In one dimension, one can trivially prove the above theorem directly using the representation formulas for the solution $u(x, t)$ in terms of the initial data which are available in one dimension.

Or, one could prove it without using this explicit representation of u , but by using the energy method instead –as we have seen in class-. This is a harder proof but the advantage of the method is that it also works in higher dimensions.

This assignment is then to prove the Finite Propagation Speed Theorem using the *energy method*.

To do so, for each $0 \leq t \leq t_0$, let $I_t := [x_0 - (t_0 - t), x_0 + (t_0 - t)]$. Note I_t is contained in the interval $(x_0 - t_0, x_0 + t_0)$. Define the modified energy:

$$\tilde{E}(t) = \frac{1}{2} \int_{I_t} |u_t|^2 + |u_x|^2 dx$$

Note $\tilde{E}(t) \geq 0$ for any t and that $\mathcal{C} = \bigcup_{0 \leq t \leq t_0} I_t$. The goal is to show that for each $0 \leq t \leq t_0$, $u(x, t) = 0$ for all $x \in I_t$. Do so by proving the following:

(1) Prove that $\tilde{E}(t)$ is a decreasing function of t by showing that $\frac{d\tilde{E}}{dt} \leq 0$

To compute the derivative in time use: (see A.3 Theorem 3 in Strauss's book p.421).

$$\frac{d}{dt} \int_{a(t)}^{b(t)} F(x, t) dx = \int_{a(t)}^{b(t)} \frac{d}{dt} F(x, t) dx + [F(b(t), t)b'(t) - F(a(t), t)a'(t)]$$

(2) Show that $\tilde{E}(0) = 0$

(3) By (1) you then have that $\tilde{E}(t) \leq \tilde{E}(0)$ for any $0 \leq t \leq t_0$ and by (1) you can conclude that $\tilde{E}(t) = 0$ for any $0 \leq t \leq t_0$. Prove then that this implies that $u(x, t) = 0$ for any $x \in I_t$ and any $0 \leq t \leq t_0$.