Problem 33.

Let us call denote by

$$I = \int f dx \qquad a_n = \int f_n dx \qquad \text{and} \qquad A = \liminf \int f_n dx$$

We wish to prove that

$$I \le A$$

By assumption we know

(1) $f_n \ge 0$ (in particular $a_n \ge 0$ also)
(2) $f_n \to f$ in measure.

Now since A is the smallest of all limit points, there must exist a subsequence of a_{n_k} of a_n that converges to A ; i.e.

$$\lim_{k\to\infty} a_{n_k} = A.$$

In particular note that

$$A = \lim_{k\to\infty} \int f_{n_k} dx \qquad \text{and that } f_{n_k} \to f \text{ in measure as well}$$

Then for every subsequence $a_{n_{k_j}}$ of a_{n_k} we also have that

$$A = \lim_{j\to\infty} a_{n_{k_j}} = \lim_{j\to\infty} \int f_{n_{k_j}} dx$$

and $f_{n_{k_j}} \to f$ in measure as well .

In particular then, there exists one such sub(sub)sequence for which we have a.e. convergence to f. By abuse of notation let us refer to *this particular* sub(sub)sequence as $f_{n_{k_j}}$ once again. Now, by Fatou's lemma we have that

$$I = \int f = \int \lim f_{n_{k_j}} = \int \liminf f_{n_{k_j}} \le \liminf a_{n_{k_j}} = \lim_{j\to\infty} a_{n_{k_j}} = A$$

as desired. $\square$

Problem 38 Part b).

First note that :

$$f_n g_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f)$$

Then **prove** that if a function is finite a.e. and $\mu(X) < \infty$ then the function is *almost* bounded : i.e.

$$\forall\varepsilon > 0 \text{there exists} M = M(\varepsilon) > 0 \ \mu(\{x \in X \, : \, f(x) > M\}) < \varepsilon$$

To see this look at the sequence of sets $E_n = \{x \in X \, : \, f(x) > n\}$ and use the continuity from *above* of the measure μ. Note $\mu(E_1) \leq \mu(X) < \infty$.

To find a counterexample in the case $\mu(X) = \infty$ consider $X = \mathbb{R}$ and μ Lebesgue measure. Define a sequence of functions $f_n(x) = a_n$ for all x in $\mathbb{R}$ where $\{a_n\}$ is any sequence of positive real numbers you care to **choose** such that $a_n \to 0$ as $n \to \infty$. And define $g_n(x) = g(x)$ for all $n \geq 1$ where $g(x)$ is any function you care to **choose** over $\mathbb{R}$ such that $g(x) \to \pm\infty$ as $x \to \pm\infty$. Show f_n converges in measure to zero, g_n converges in measure to g but $f_n \, g_n$ does not converge in measure to zero.

Problem 41.

The argument below might need some fine tuning: check it carefully.

Since X is σ-finite we can write

$$X = \bigcup_{i=1}^{\infty} X_i \qquad \mu(X_i) < \infty \qquad X_i \cap X_{i'} = \emptyset, \qquad i \neq i'$$

Since $f_m \to f \quad$ a.e. in X then $f_m \to f \quad$ a.e. in X_i, $\forall i$.

Let $\varepsilon > 0$ be given and fixed. For each $i \geq 1$ we will apply Egoroff's theorem with $\frac{\varepsilon}{2^i}$.

Then there exists $F_i \subset X_i$, $\mu(F_i) < \frac{\varepsilon}{2^i}$ such that

$$f_m \to f \quad \text{uniformly on } \ X_i \setminus F_i$$

Note that $F_i \cap F_{i'} = \emptyset \quad i \neq i'$.

Let us denote by $G_i = X_i \setminus F_i$ then since

$$G_i^c = X \setminus G_i = \bigcup_{j \neq i} X_j \, \cup \, F_i$$

where all unions are disjoints one can <u>*prove*</u> (homework: prove it !) that

$$\bigcap_{i=1}^{\infty} G_i^c \subseteq \bigcup_{i=1}^{\infty} F_i$$

(again last union is disjoint union). Hence

$$\mu((\bigcup_{i=1}^{\infty} G_i)^c) \, = \, \mu(\bigcap_{i=1}^{\infty} G_i^c) \leq \sum_{i=1}^{\infty} \mu(F_i) \leq \varepsilon$$

For later use we now denote by

$$H_1 = \big(\bigcup_{i=1}^{\infty} G_i\big)^c = X \setminus \big(\bigcup_{i=1}^{\infty} G_i\big) \text{ and } \mathcal{E}_1 = \big(\bigcup_{i=1}^{\infty} G_i\big)$$

Now consider the sequence $\varepsilon_n = 2^{-n}$, $n \geq 1$.

Let $n = 1$ and run the argument above with $\varepsilon = \varepsilon_1 = 1/2$. We get $\mathcal{E}_1$ and H_1 such that

(1) $X = \mathcal{E}_1 \cup H_1$ where the union is disjoint and $\mu(X \setminus \mathcal{E}_1) = \mu(H_1) < 1/2$

(2) $f_m \to f$ uniformly on each set G_i in $\mathcal{E}_1$

(3) $f_m \to f$ a.e in X; hence in particular, $f_m \to f$ a.e in H_1

For $n = 2$ we now consider as full space $X = H_1$ and apply Egoroff's theorem with $\varepsilon_2 = 1/4$. Then there exists a set $\mathcal{E}_2$ such that

(1) $X = \mathcal{E}_2 \cup H_2$, where the union is disjoint, $H_2 = H_1 \setminus \mathcal{E}_2$ and $\mu(H_1 \setminus \mathcal{E}_2) = \mu(H_2) < 1/4$

(2) $f_m \to f$ uniformly in $\mathcal{E}_2$

(3) $f_m \to f$ a.e in H_1; hence in particular, $f_m \to f$ a.e in H_2

Repeat the step $n = 2$ above inductively for all $n \geq 3$ applying Egoroff to $X = H_{n-1}$ and $\varepsilon = \varepsilon_n = 2^{-n}$ to get sets $\mathcal{E}_n$ and H_n such that

(1) $X = \mathcal{E}_n \cup H_n$, where the union is disjoint, $H_n = H_{n-1} \setminus \mathcal{E}_n$ and $\mu(H_{n-1} \setminus \mathcal{E}_n) = \mu(H_n) < 2^{-n}$

(2) $f_m \to f$ uniformly in $\mathcal{E}_n$

(3) $f_m \to f$ a.e in H_{n-1}; hence in particular, $f_m \to f$ a.e in H_n

Note that $\mu(H_1) < \infty$ and $\ldots \cdots \subseteq H_n \subseteq H_{n-1} \ldots \cdots \subseteq H_2 \subseteq H_1$. Then

$$X = \bigcup_{n=1}^{\infty} \mathcal{E}_n \cup H \qquad \text{where} \quad H = \Big(\bigcup_{n=1}^{\infty} \mathcal{E}_n\Big)^c \text{ and}$$

$$\mu(H) = \mu(X \setminus \bigcup_{n=1}^{\infty} \mathcal{E}_n) = \mu(\bigcap_{n=1}^{\infty} H_n) = \lim_{n\to\infty} \mu(H_n) = 0$$

On the other hand by relabeling all the G_i in $\mathcal{E}_1$ and all the $\mathcal{E}_n$, $n \geq 2$ as E_m, $m \geq 1$ (note that the union of two countable families of sets is countable) for example by sending G_i , $i \geq 1$ to E_{2k}, $k \geq 1$ and $\mathcal{E}_n$, $n \geq 2$ to E_{2k+1}, $k \geq 0$ we obtained the desired conslusion. $\square$