## EGOROV'S THEOREM: (Th. 4.4 Ch. 1 of [SS] pg. 33).

Suppose $\{f_k\}_{k\geq 1}$ is a sequence of measurable fcs. defined on a measurable set $E$, $m(E)<\infty$ and assume that $f_k(x) \to f(x)$ a.e. in $E$. THEN: Given $\varepsilon>0$, we can find a set $A_\varepsilon \subset E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ and

$$f_k \rightrightarrows f \text{ on } A_\varepsilon$$

(ie. $f_k$ converges uniformly to $f$ on $A_\varepsilon$).

Proof: First note that by replacing $E$ by a subset $E'$ / $(E \setminus E')$ has measure zero one may (if necessary). assume $f_k(x) \to f(x)$ $\forall x \in E'$.

So WLOG let's assume from the beginning that $f_k(x) \to f(x)$ $\forall x \in E$.

Now for each $n, k \geq 1$ let us define

$$E_k^n := \left\{ x\in E : \left| f_j(x) - f(x) \right| < \frac{1}{n} \ \forall j > k \right\}$$

NOTE that for fixed $n$ $E_k^n \subset E_{k+1}^n$ and $E_k^n \nearrow E$ as $k \to \infty$.

Hence using Corollary 3.3(i) ("regularity" of Lebesgue measure) we find that there exists $k_n$ such that

$$m(E \setminus E^n_{k_n}) < \frac{1}{2^n} . \quad (\dagger)$$

By construction we then have

$$|f_j(x) - f(x)| < \frac{1}{n} \text{ if } j > k_n \text{ and } x \in E^n_{k_n}$$

Choose $N$ so that $\sum_{n=N}^{\infty} 2^{-n} < \frac{\varepsilon}{2}$ $(\dagger\dagger)$

and let

$$\tilde{A}_\varepsilon = \bigcap_{n \geq N} E^n_{k_n} .$$

Note $m(E \setminus \tilde{A}_\varepsilon) \leq \sum_{n=N}^{\infty} m(E \setminus E^n_{k_n})$

$$\leq \frac{\varepsilon}{2} \text{ by } (\dagger) \,\&\, (\dagger\dagger)$$

Next let $\delta > 0$ be given, and choose $n_0 \geq N$ and $\frac{1}{n_0} < \delta$. Then $\forall\ j > k_{n_0}$ and $\forall\ x \in \tilde{A}_\varepsilon$, $|f_j(x) - f(x)| < \delta$

Here we used that any $x \in \tilde{A}_\varepsilon$ is in $E^{n_0}_{k_{n_0}}$ as well by definition of $\tilde{A}_\varepsilon$ and choice of $n_0$.

Hence $f_k$ converges uniformly to $f$ on $\tilde{A}_\varepsilon$.

To conclude we need to choose a <u>closed</u> subset $A_\varepsilon$ of $\tilde{A}_\varepsilon$ / $m(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \frac{\varepsilon}{2}$

This is possible thanks to Theorem 3.4 Ch 1.

As a result $m(E \setminus A_\varepsilon) < \varepsilon$ and of course $f_k$ converges uniformly to $f$ on $A_\varepsilon$ as well

#