PROBLEM SET 4

Due Thursday Nov. 8, 2001

Problem 1 (#28 from Folland p.39). *Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$, and $\mu_F((a,b)) = F(b-) - F(a)$.*

Remark. Recall that

$$F(a+) = \lim_{x \searrow a} F(x) = \inf_{x>a} F(x) \qquad F(a-) = \lim_{x \nearrow a} F(x) = \sup_{x<a} F(x).$$

Also F is called right continuous if $F(a) = F(a+)$ and left continuous if $F(a) = F(a-)$.

Problem 2 (#30 from Folland p.40). *If E is Lebesgue measurable (i.e. if $E \in \mathcal{L}$) and $m(E) > 0$, then for any $0 < \alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha\, m(I)$*

Problem 3 (#31 from Folland p.40). *If $E \in \mathcal{L}$ and $m(E) > 0$ the set $E - E := \{x - y \,:\, x, y \in E\}$ contains an interval centered at 0.*

(Hint. Prove that if I is as in the previous exercise withn $\alpha > 3/4$ then $E - E$ containes $(-1/2m(I), 1/2m(I))$.

Problem 4 (#27 from Folland p.39). *Prove that the ternary Cantor set C constructed in class (see Proposition 1.22 in Folland, pg. 38) is <u>compact</u>, <u>nowhere dense</u> and <u>totally disconnected</u> (the latter means that the only conected subsets of C are single points). Prove that moreover, C has no isolated points.*

Recall that a set C is **nowhere dense** *if $\bar{C}$ has empty interior.*

(Hint. Show that if $x, y \in C$ and $x < y$, there exists a $z \notin C$ such that $x < z < y$).

Problem 5. *Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}$, for some $0 < \delta < 1$. Show the resulting set if perfect, has measure $1 - \delta$ and contains no intervals.*

Problem 6. *Let $\{a_k\}$ be a sequence of real numbers, $a_k \in (0,1)$ for all $k \geq 1$. Prove:*

i) $\quad \Pi_{k \geq 1} (1 - a_k)$ *converges iff* $\displaystyle\sum_{k \geq 1} \log(1 - a_k)$ *converges iff* $\displaystyle\sum_{k \geq 1} a_k$ *converges*

ii) *Construct a Cantor-type subset of $[0,1]$ by removing from each interval remaining at the $k-th$ stage a subinterval of relative length $0 < \theta_k < 1$. Show that the remainder set (ie. the infinite intersection of the sets that remained at each stage - as in the Cantor ternary contruction-) has measure zero if and only if $\sum_{k \geq 1} \theta_k = \infty$ (Hint. Use i)).*