

PROBLEM SET 4

Due November 7th 2002

Problem 1. : Do Problem # 26 in Folland's page 39. Recall that

$$E \triangle A := (E \setminus A) \cup (A \setminus E)$$

Problem 2. : (a) Give the definition of the outer measure m^* that arises in the construction of the Lebesgue measure m on \mathbb{R} .

(b) Use (a), that is the definition of m^* to prove that for any subset A of \mathbb{R} and any $s \in \mathbb{R}$,

$$m^*(A + s) = m^*(A).$$

Problem 3 (#28 from Folland p.39). Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu(\{a\}) = F(a) - F(a-)$, $\mu_F([a, b]) = F(b-) - F(a-)$, $\mu_F([a, b]) = F(b) - F(a-)$, and $\mu_F((a, b)) = F(b-) - F(a)$.

Remark. Recall that

$$F(a+) = \lim_{x \searrow a} F(x) = \inf_{x > a} F(x) \quad F(a-) = \lim_{x \nearrow a} F(x) = \sup_{x < a} F(x).$$

Also F is called right continuous if $F(a) = F(a+)$ and left continuous if $F(a) = F(a-)$.

Problem 4 (#30 from Folland p.40). If E is Lebesgue measurable (i.e. if $E \in \mathcal{L}$) and $m(E) > 0$, then for any $0 < \alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$

Problem 5 (#31 from Folland p.40). If $E \in \mathcal{L}$ and $m(E) > 0$ the set $E - E := \{x - y : x, y \in E\}$ contains an interval centered at 0.

(Hint. Prove that if I is as in the previous exercise with $\alpha > 3/4$ then $E - E$ contains $(-1/2m(I), 1/2m(I))$).

Problem 6 (#27 from Folland p.39). Prove that the ternary Cantor set C constructed in class (see Proposition 1.22 in Folland, pg. 38) is compact, nowhere dense and totally disconnected. Prove that moreover, C has no isolated points.

A set C is said to be **totally disconnected** if the only connected subsets of C are single points. And it is said to be **nowhere dense** if \bar{C} has empty interior.

Hint. Show that if $x, y \in C$ and $x < y$, there exists a $z \notin C$ such that $x < z < y$.

Problem 7. Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the k th stage, each interval removed has length $\delta 3^{-k}$, for some $0 < \delta < 1$. Show the resulting set is perfect, has measure $1 - \delta$ and contains no intervals.

Problem 8. Let $\{a_k\}$ be a sequence of real numbers, $a_k \in (0, 1)$ for all $k \geq 1$. Prove:

$$i) \quad \prod_{k \geq 1} (1 - a_k) > 0 \text{ iff } \sum_{k \geq 1} \log(1 - a_k) \text{ converges iff } \sum_{k \geq 1} a_k \text{ converges}$$

ii) Construct a Cantor-type subset of $[0, 1]$ by removing from each interval remaining at the k - th stage a subinterval of relative length $0 < \theta_k < 1$. Show that the remainder set (ie. the infinite intersection of the sets that remained at each stage - as in the Cantor ternary construction-) has measure zero if and only if $\sum_{k \geq 1} \theta_k = \infty$ (Hint. Use i)).