Math 331 Section 1. Solutions to Quiz 2

Problem 1. Find the general solution $y = y(x)$ to following second order differential equations by *reduction of order*.

$$(a) \qquad y'' + (y')^2 = 0$$

Solution. This is a nonlinear second order equation. So we have no method to solve it other than try if there is a substitution that might work. Indeed, note that if we let $v(x) = y'(x)$ then the equation becomes the first order ODE:

$$v' + v^2 = 0$$

which we can solve by separation of variables as follows:

$$v' + v^2 = 0 \quad \Longrightarrow \quad v' = -v^2 \quad \Longrightarrow \quad -\frac{v'}{v^2} = 1.$$

By integrating both sides with respect to x we get that

$$\int -\frac{v'}{v^2}\, dx = \int 1\, dx \quad \Longrightarrow \quad \frac{1}{v} = x + C \quad \Longrightarrow \quad v(x) = \frac{1}{x+C}$$

for any constant C.

Now, since $v(x) = y'(x)$, we have integrating once again with respect to x that

$$\int y'(x)\, dx = \int \frac{1}{x+C}\, dx \quad \Longrightarrow \quad y(x) = \ln|x+C| + \tilde{C}$$

where $C, \tilde{C}$ are arbitrary constants.

$$(b) \qquad x^2 y'' + 2xy' - 2y = 0, \qquad y_1(x) = x \qquad (x > 0)$$

Recall that $a \ln x = \ln(x^a)$ for any real number a.

Solution. This is a second order equation with variable coefficients. We are given one solution; namely $y_1(x) = x$ (check is a solution by plugging in the equation!).

By the method of reduction of order we can find a second solution $y_2(x)$ (which will be linearly independent to y_1) by setting $y_2(x) := u(x)y_1(x)$ and plugging into the equation to find $u(x)$.

We thus obtain:

$$x^2u''y_1 + 2x^2u'y_1' + x^2u\, y_1'' + 2xu'y_1 + 2xu\, y' - 2u\, y_1 = 0$$

Grouping all the terms that have u we find that

$$x^2u''y_1 + 2x^2u'y_1' + 2xu'y_1 + u\big(x^2\, y_1'' + 2x\, y' - 2\, y_1\big) = 0.$$

Since the term within parenthesis is zero because we know that y_1 is a solution we have that

$$x^2u''y_1 + 2x^2u'y_1' + 2xu'y_1 = 0$$

Since $y_1(x) = x$ we have then that to find u we need to solve

$$x^3 u'' + 2x^2 u' + 2x^2 u' = 0 \quad \Longrightarrow \quad u'' + \frac{4}{x} u' = 0$$

Once again, to solve we let $v(x) = u'(x)$ and reduce the last equation to the first order ODE $v' + \frac{4}{x} v = 0$ which by separation of variables (or integrating factor since is linear in this case) gives that

$$\int \frac{v'}{v} dx = \int -\frac{4}{x}\, dx \quad \Longrightarrow \quad \ln|v| = -4 \ln|x| + C \quad \Longrightarrow \quad \ln|v| = \ln(x^{-4}) + C_1.$$

Thus

$$v(x) = C_2\, x^{-4} \qquad \text{for any constants} \quad C_1, C_2.$$

Therefore $u(x) = \int v(x) dx = C_2\, x^{-3} + C_3$. Since we are interested in finding one y_2 to use as fundamental solution together with y_1 we can choose $C_2 = 1$ and $C_3 = 0$. Hence

$$y_2(x) = u(x) y_1(x) = x^{-2}$$

(you should check that indeed x^{-2} is a solution); and the general solution to our equation is

$$y(x) \;=\; k_1 y_1(x) \;+\; k_2 y_2(x) \;=\; k_1 x \;+\; k_2 x^{-2}$$

for any constants k_1, k_2.

Problem 2 Find the solution $y = y(x)$ to the following initial value problem:

$$(a) \qquad y'' + 16y = 0 \qquad\qquad y(0) \;=\; 10, \quad y'(0) = 12$$

This an undamped second order linear homogeneous equation with constant coefficients (undamped free vibration). Its solution should be a *harmonic oscillation.* Indeed to find the two fundamental solutions solve

$$\lambda^2 + 16 = 0 \quad \Longrightarrow \quad \lambda = \sqrt{-16} \quad \Longrightarrow \quad \lambda_1 = 4i, \quad \lambda_2 = -4i$$

Hence the general solution is

$$y(x) = A\cos(4x) \;+\; B\sin(4x)$$

Since $y(0) \;=\; 10, \quad y'(0) = 12$ we must then have

$$\begin{aligned} 10 &= A\cos(0) \;+\; B\sin(0) \quad \Longrightarrow \quad A = 10 \\ 12 &= -4A\sin(0) + 4B\cos(0) \quad \Longrightarrow \quad B = 3 \end{aligned}$$

All in all the solution is

$$y(x) = 10\cos(4x) \;+\; 3\sin(4x).$$

$$(b) \qquad 2y'' + 4y' + 2y = 0 \qquad\qquad y(0) = 1, \quad y'(0) = 5$$

This a damped second order linear homogeneous equation with constant coefficients. To determine whether there is *overdamping, critical damping* or *underdamping* we need first to find the fundamental solutions first. We solve

$$2\lambda^2 + 4\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = \frac{-4 \pm \sqrt{16 - 16}}{4} \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = -1$$

We then have a double real root which means there is *critical damping* and that the general solution has the form

$$y(x) = C_1 e^{-x} + C_2 x\, e^{-x}$$

Since $y(0) = 1, \quad y'(0) = 5$ we must then have

$$\begin{aligned} 1 &= C_1 \\ 5 &= -4C_1 + C_2 \qquad \Longrightarrow \qquad C_2 = 9 \end{aligned}$$

All in all the solution is

$$y(x) = e^{-x} + 9\, x\, e^{-x}$$

(Again check the answer is indeed a solution).