First order ODE – Mixing Problems

$S(t)$: the amount of salt in the tank at time t.

$$\frac{dS}{dt} = (\text{rate at which salt flows in at time } t)$$

$$- (\text{rate at which it flows out at time } t)$$

. salt rate = concentration × flow rate

salt inflow rate = inflow concentration × inflow rate

(lb / gallon) (gallon / min)

salt out rate = outflow concentration × outflow rate

$$\frac{S}{V \text{ (volume of tank)}}$$

Q1:

(a)

$$\frac{dS}{dt} = \text{salt inflow rate} - \text{salt outflow rate}$$

$$= (0.10) \times 5 - \frac{S}{250} \times 5$$

$$\begin{cases} S' = 0.5 - 0.02\,S \\ S(0) = 40 \end{cases}$$

(b)

$$s' = -0.02(s - 25)$$

$$\int \frac{1}{s-25}\, ds = -0.02 \int dt$$

$$\ln|s - 25| = -0.02t + c^*$$

$$s - 25 = c e^{-0.02t}$$

$$s = 25 + c e^{-0.02t}$$

$$40 = S(0) = 25 + c \Rightarrow c = 15$$

$$S = 25 + 15 e^{-0.02t}$$

(c)

$$\lim_{t \to \infty} S(t) = 25 = S_L$$

(d)

$$S(T) = 3\% \times S_L + S_L \Rightarrow T = ?$$

$$25 + 15 e^{-0.02T} = 1.03 \times 25$$

$$\Rightarrow 25 + 15 e^{-0.02T} = 25.75$$

$$\Rightarrow 15 e^{-0.02T} = 0.75$$

$$\Rightarrow e^{-0.02T} = \frac{0.75}{15}$$

$$\Rightarrow -0.02T = \ln\left(\frac{0.75}{15}\right)$$

$$\Rightarrow T = \frac{1}{-0.02} \ln\left(\frac{0.75}{15}\right)$$

(e) $\quad$ rate $= r$

$$S' = 0.1r - \frac{r}{250} S$$

$$= -\frac{r}{250}(S - 25)$$

$$\int \frac{1}{S-25}\, dt = -\frac{r}{250} \int dt$$

$$\ln|S-25| = -\frac{r}{250} t + c^*$$

$$S = 25 + c^* e^{-\frac{r}{250} t}$$

$$40 = S(0) = 25 + c^* \Rightarrow c^* = 15$$

$$S = 25 + 15 e^{-\frac{r}{250} t}$$

$$S_L + 3\% \cdot S_L = 25 + 15 e^{-\frac{r}{250} \cdot 45}$$

solve for r.

$S(0)=0$

Q2
$$\begin{cases} S' = 0.5 \times 2 - \frac{S}{100} \cdot 2 & 0 \le t \le 10 \quad ① \\ S' = 0 - \frac{S}{100} \cdot 2 & t > 10 \quad ② \end{cases}$$

S is continuous

$① \Rightarrow \quad S' = 1 - 0.02S = -0.02(S-50)$

$\Rightarrow \quad \int \frac{1}{S-50}\, dS = \int -0.02\, dt$

$\Rightarrow \quad \ln|S-50| = -0.02t + c^* \quad \Rightarrow \quad S-50 = c e^{-0.02t}$

$\Rightarrow \quad S = 50 + Ce^{-0.02t}$

$0 = S(0) = 50 + C \quad \Rightarrow \quad C = -50$

$S = 50 - 50e^{-0.02t} \qquad 0 \le t \le 10$

$S(10) = 50 - 50e^{-0.2}$

$② \quad S' = -0.02S \quad \Rightarrow \quad \int \frac{1}{S}\, dS = -0.02t + C_2^* \quad \Rightarrow \quad \ln|S| = -0.02t + C_2^*$

$\Rightarrow \quad S = C_2 e^{-0.02t}$

$50 - 50e^{-0.2} = S(10) = C_2 e^{-0.2} \quad \Rightarrow \quad 50 = (50 + C_2)e^{-0.2}$

$\Rightarrow \quad 50e^{0.2} = 50 + C_2$

$\Rightarrow \quad C_2 = 50e^{0.2} - 50$

$$\begin{cases} S(t) = 50 - 50e^{-0.02t} & 0 \le t \le 10 \\ S(t) = (50e^{0.2} - 50)e^{-0.02t} & t > 10 \end{cases}$$

$S(20) =$

Q3 (a) salt in : 1×3 gal/min $= 1$ lb/min (lb/gal)

salt out : $\frac{S(t)}{V(t)} \times 2$

where $V(t) = 200 + (3-2)t$

$= 200 + t$

$V(t) \le 500$

$200 + t \le 500$

$t \le 300$

$$\begin{cases} S' = 1 - \frac{S(t)}{200+t} \times 2 & (t \le 300) \\ S(0) = 100 \end{cases}$$

(b) $S' + \frac{2}{200+t} S = 1$ $p(t) = \frac{2}{200+t}$ $r(t) = 1$

$$h = \int p(t)\,dt = \int \frac{2}{t+200}\,dt = 2\ln|t+200| = \ln(t+200)^2$$

$$S = e^{-h}\left(\int e^h r\,dt + c\right) = e^{-\ln(t+200)^2}\left(\int (t+200)^2 \cdot 1\,dt + c\right)$$

$$= (t+200)^{-2}\left(\underbrace{\int (t+200)^2\,dt}_{\frac{1}{3}(t+200)^3} + c\right) = (t+200)^{-2}\left[\frac{1}{3}(t+200)^3 + c\right]$$

$$S = \frac{1}{3}(t+200) + c\,(t+200)^{-2}$$

$$100 = S(0) = \frac{200}{3} + c \cdot 200^{-2} \Rightarrow \frac{100}{3} = c \cdot 200^{-2} \Rightarrow c = \frac{100}{3} \cdot 200^2$$

$$S = \frac{1}{3}(t+200) + \frac{100 \cdot 200^2}{3}(t+200)^{-2}$$

(c) $\text{concentration}(t) = \frac{S(t)}{V(t)} = \frac{1}{3} + \frac{100 \cdot 200^2}{3}(t+200)^{-3}$

$t = 300$ $c(t) = \frac{1}{3} + \frac{100 \cdot 200^2}{3}(300+200)^{-3}$

$\lim_{t \to \infty} c(t) = \frac{1}{3}$

No damping

Q4

$$u'' + \frac{k}{m} u = 0 \qquad \boxed{u'' + 640 u = 0}$$

(A, B constants) $\quad u = A \cos \omega_0 t + B \sin \omega_0 t \qquad \omega_0^2 = k/m$

$u(0) = u_0$ initial position

initial velocity v_0 : $u'(0) = v_0$

$$k = \frac{2 \text{ lb}}{6 \text{ in.}} = \frac{1 \text{ lb}}{3 \text{ in.}} = 4 \text{ lb/ft}$$

$$m = \frac{w}{g} = \frac{2 \text{ lb}}{32 \text{ ft/s}^2}$$

$$\Rightarrow \frac{k}{m} = 4 \times \frac{32}{2} = 64$$

$$\omega_0 = 8$$

$$\begin{cases} u = A \cos(8\,t) + B \sin(\overset{8}{8\sqrt{10}}\,t) \\ u(0) = 3 \times \frac{1}{12} \text{ ft} = \frac{1}{4} \text{ ft} \\ u'(0) = 0 \end{cases}$$

$$\frac{1}{4} = u(0) = A$$

$$u'(t) = -\overset{8}{8\sqrt{10}} A \sin(\overset{8}{8\sqrt{10}}\,t) + \overset{8}{8\sqrt{10}} B \cos(\overset{8}{8\sqrt{10}}\,t)$$

$$0 = u'(0) = \underset{8}{8\sqrt{10}}\,B \Rightarrow B = 0$$

b) $\quad u = \frac{1}{4} \cos(\underset{8}{8\sqrt{10}}\,t)$

c) $\quad \omega_0 = \sqrt{\frac{k}{m}} = \overset{8}{8\sqrt{10}} \qquad T = \frac{2\pi}{\omega_0} = 2\pi \left(\frac{m}{k}\right)^{1/2} = \frac{2\pi}{\underset{8}{8\sqrt{10}}} = \frac{\pi}{4}$

$$R = \sqrt{A^2 + B^2} = \frac{1}{4}$$

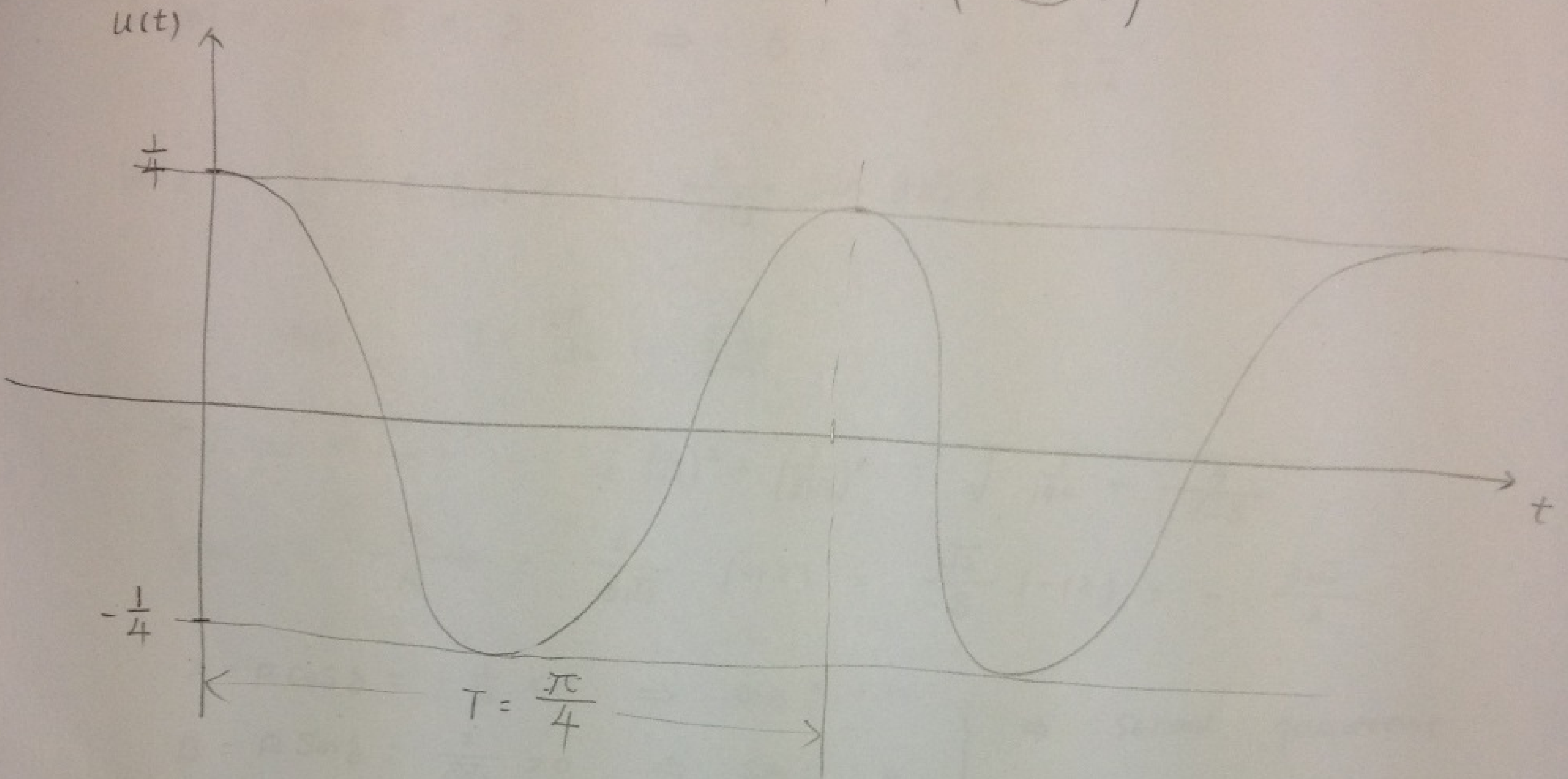
$$\tan\delta = \frac{B}{A} = 0$$

$$\Rightarrow \delta = 0 \text{ or } \pi$$

$$\begin{cases} A = R\overset{=1}{\cos\delta} = \frac{1}{4} > 0 \Rightarrow \cos\delta = 1 \Rightarrow \boxed{\delta = 0} \\ B = R \sin\delta = 0 \end{cases}$$

$$u(t) = R\cos(\omega_0 t - \delta) = \frac{1}{4}\cos(8\sqrt{10}\, t)$$

8



Q5 No damping

$$k = \frac{w \text{ lb}}{3 \text{ in}} = \frac{3 \text{ lb}}{3 \text{ in}} = 1 \text{ lb}/1 \text{ in} = 12 \text{ lb}/\text{ft}$$

$$m = \frac{w}{g} = \frac{3 \text{ lb}}{32 \text{ ft}/s^2}$$

$$\frac{k}{m} = 12 \times \frac{32}{3} = 128$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{128} = 8\sqrt{2}$$

(a)
$$\begin{cases} u'' + 128u = 0 \\ u(0) = -1 \text{ in} \times \frac{1}{12} \text{ ft/in} = -\frac{1}{12} \text{ ft} \\ u'(0) = 2 \text{ ft/s} \end{cases}$$

(b)

$$u = A\cos\omega_0 t + B\sin\omega_0 t$$

$$u(0) = A = -\frac{1}{12}$$

$$u'(t) = -\omega_0 A \sin \omega_0 t + \omega_0 B \cos \omega_0 t$$

$$u'(0) = \omega_0 B = 2 \quad \Rightarrow \quad B = \frac{2}{\omega_0} = \frac{2}{8\sqrt{2}}$$

$$u = -\frac{1}{12} \cos 8\sqrt{2}\, t + \frac{2}{8\sqrt{2}} \sin 8\sqrt{2}\, t$$

(c) $\omega_0 = 8\sqrt{2} \qquad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{8\sqrt{2}}$

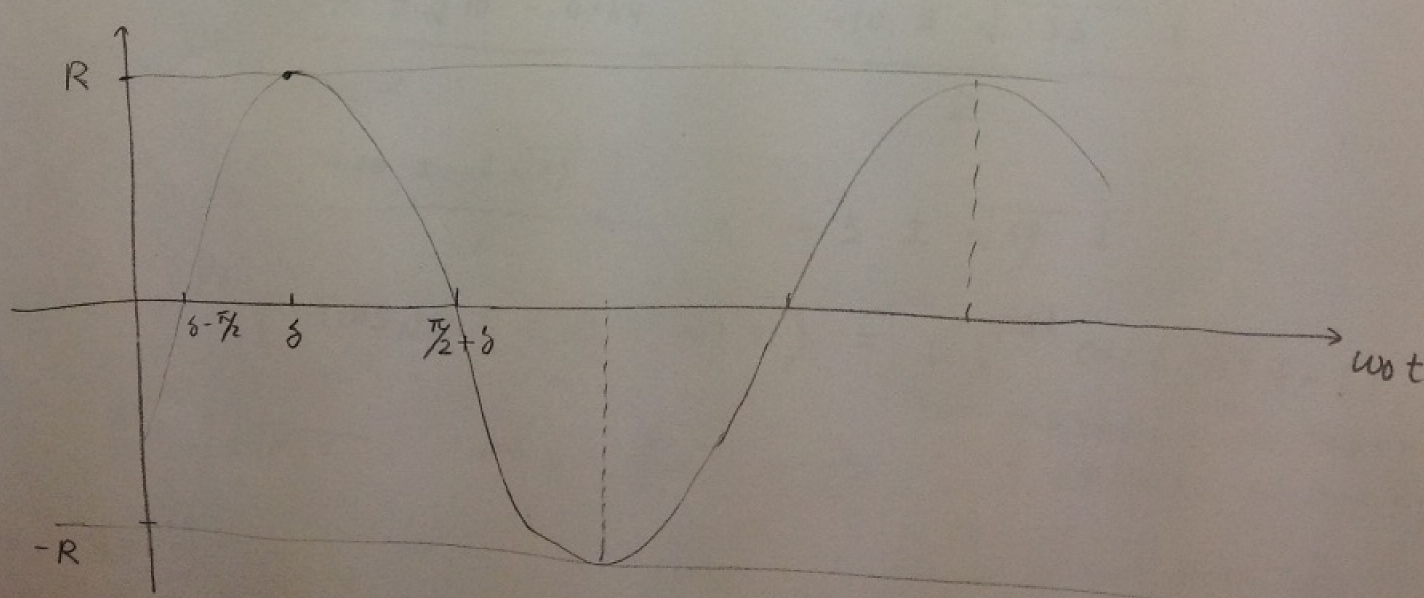
$$R = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{2}{8\sqrt{2}}\right)^2} = \sqrt{\frac{1}{144} + \frac{4}{128}}$$

$$\tan \delta = \frac{B}{A} = \frac{2}{8\sqrt{2}} \cdot (-12) = \frac{\sqrt{2}}{8} \cdot (-12) = -\frac{3\sqrt{2}}{2}$$

$$\left.\begin{array}{lll} A = R\cos\delta = -\frac{1}{12} < 0 & \Rightarrow & \cos\delta < 0 \\ B = R\sin\delta = \frac{2}{8\sqrt{2}} > 0 & \Rightarrow & \sin\delta > 0 \end{array}\right\} \Rightarrow \text{second quadrant}$$

$$\delta = \left(\arctan \frac{3\sqrt{2}}{2}\right) + \frac{\pi}{2}$$

$$u = R \cos(\omega_0 t - \delta)$$



Q6

$$mu'' + \gamma u' + ku = 0$$ → No external force

$$m = \frac{w}{g} = \frac{8\ \text{lb}}{32\ \text{ft/s}^2} = \frac{1}{4}\ \frac{\text{lb}\cdot\text{s}^2}{\text{ft}}$$

damping coefficient $\gamma = \dfrac{5\ \text{lb}}{2\ \text{ft/s}} = 2.5\ \dfrac{\text{lb}\cdot\text{s}}{\text{ft}}$ $\qquad \gamma u' = 5$ when $u' = 2$

spring constant $k = \dfrac{8\ \text{lb}}{6\cdot\frac{1}{12}\ \text{ft}} = 16$

$$\frac{1}{4}u'' + 2.5u' + 16u = 0$$

(a) $\begin{cases} u'' + 10u' + 64u = 0 \\ u(0) = 8\cdot\frac{1}{12} = \frac{2}{3}\ \text{ft} \\ u'(0) = 0 \end{cases}$

(b) $\lambda = \dfrac{-10 \pm \sqrt{10^2 - 4\times 64}}{2} = \dfrac{-10 \pm \sqrt{156}\, i}{2}$

$$= \frac{-10 \pm 2\sqrt{39}\, i}{2} = -5 \pm \sqrt{39}\, i$$

$$u(t) = e^{-5t}\left(A\cos\sqrt{39}\,t + B\sin\sqrt{39}\,t\right) = Ce^{-5t}\cos(\sqrt{39}\,t - \delta)$$

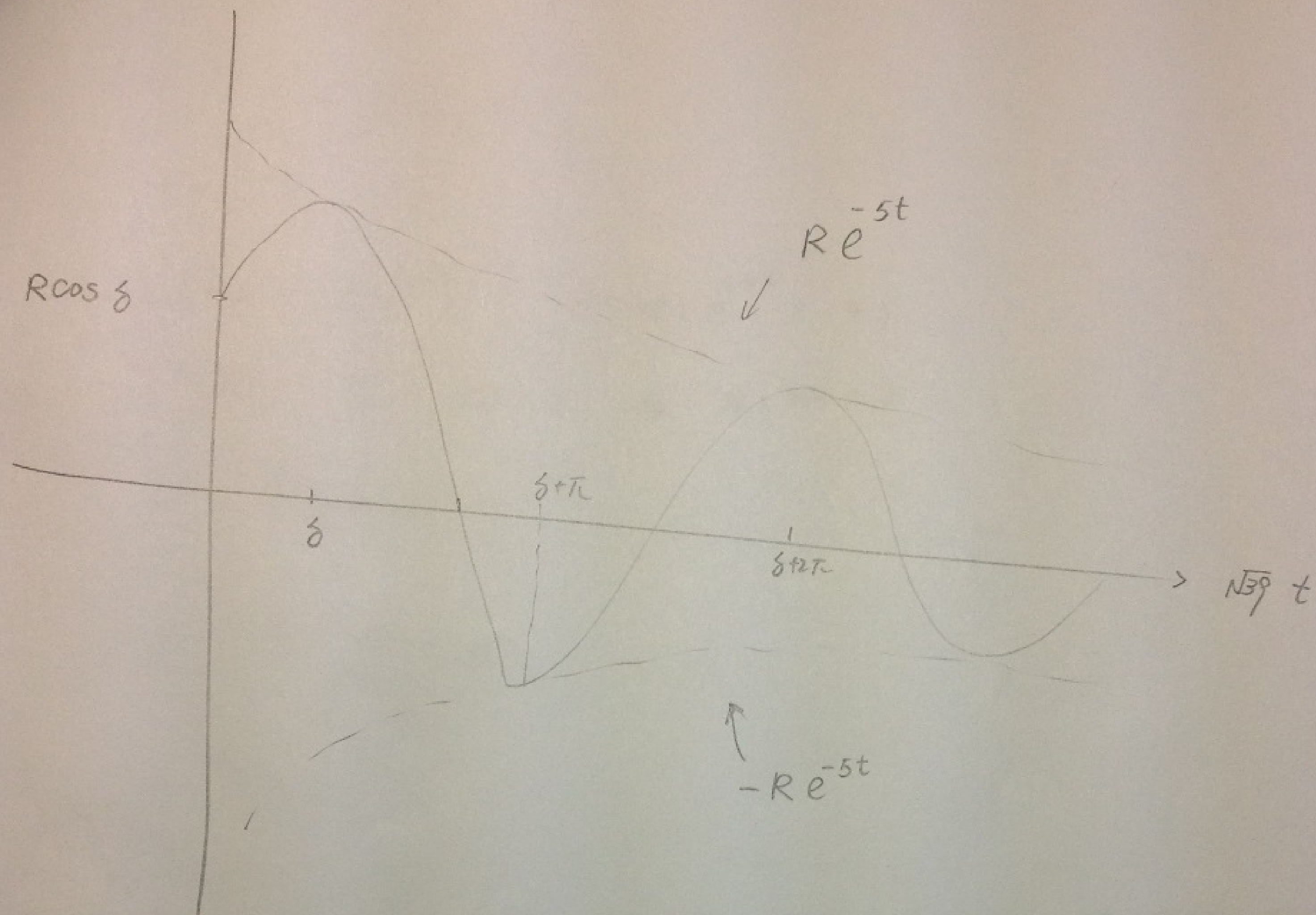
$C = \sqrt{A^2+B^2}$ $\qquad \tan\delta = \dfrac{B}{A} = -\dfrac{5}{\sqrt{39}}$

$B = R\sin\delta < 0 \Rightarrow \sin\delta < 0$
$A = R\cos\delta > 0 \Rightarrow \cos\delta > 0$ $\Rightarrow$ 4-th quadrant

$\delta = -\arctan\dfrac{5}{\sqrt{39}}$

$\begin{cases} u(0) = A = \frac{3}{2} \\ u'(t) = 5e^{-5t}(A\cos\sqrt{39}\,t + B\sin\sqrt{39}\,t) + e^{-5t}(-\sqrt{39}A\sin\sqrt{39}\,t + \sqrt{39}B\cos\sqrt{39}\,t) \\ u'(0) = 5A + \sqrt{39}B = 0 \Rightarrow B = \frac{-5A}{\sqrt{39}} = -\frac{15}{2\sqrt{39}} \end{cases}$

(c).



(d) damped (free) vibration

underdamping $\gamma^2 - 4km < 0$

$\boxed{Q7}$

$$m = \frac{w}{g} = \frac{32\ lb}{32\ ft/s^2} = 1\ \frac{lb \cdot s^2}{ft}$$

$$\gamma = \frac{24\ lb}{3\ ft/s} = 8\ \frac{lb \cdot s}{ft}$$

$$k = \frac{32\ lb}{2\ ft} = 16\ lb/ft$$

$$u'' + 8u' + 16u = 0 \qquad u(0) = 6 \times \frac{1}{12} = \frac{1}{2} ft \qquad u'(0) = 0$$

$$\gamma^2 - 4km = 8^2 - 4 \cdot 16 \cdot 1 = 0 \qquad u = (A+Bt)e^{-\gamma t/2m}$$

$$u = (A + Bt)e^{-4t}$$

$$u(0) = A = \frac{1}{2}$$

$$u'(t) = Be^{-4t} + (A+Bt)e^{-4t} \cdot (-4)$$

$$u'(0) = B - 4A = 0 \Rightarrow B = 4A = 2$$

$$u = \left(\frac{1}{2} + 2t\right)e^{-4t}$$

critical damping

$\boxed{Q8}$

(i) $y(t)$: position of the body of mass at time t

$$my'' + \gamma y' + ky = F(t)$$

$m = 5$ kg

$F(t) = 4\sin(t/2)$ N $= 4\sin(t/2)$ $kg\,m/s^2$

$(g = 9.8\ m/s^2)$

$L = 20$ cm $= 0.2$ m

$k = mg/L = 5 \cdot 9.8/0.2 = 245$

$F_d = -\gamma v$ where $F_d = -2N$ when $v = 4$ cm/s $= 0.04$ m/s

$\gamma = \frac{2}{v} = \frac{2}{0.04} = 50$

ODE

$$\begin{cases} 5y'' + 50y' + 245y = 4\sin\left(\frac{t}{2}\right) \\ y(0) = 0 \\ y'(0) = 3\ \text{cm/s} = 0.03\ \text{m/s} \end{cases}$$

(a)

homo : $y'' + 10y' + 49 = 0$

$\lambda^2 + 10\lambda + 49 = 0$ $\qquad \lambda = \dfrac{-10 \pm \sqrt{10^2 - 4\cdot 49}}{2} = \dfrac{-10 \pm \sqrt{96}\, i}{2}$

$$= \frac{-10 \pm 4\sqrt{6}\, i}{2} = -5 \pm 2\sqrt{6}\, i$$

Free response :

$$y_h = e^{-5t}\left(A\cos(2\sqrt{6}t) + B\sin(2\sqrt{6}t)\right)$$

Try $\quad y_p = K\cos(t/2) + M\sin(t/2) \qquad$ (K, M constants)

$$y_p' = -\tfrac{1}{2}K\sin(t/2) + \tfrac{1}{2}M\cos(t/2)$$

$$y_p'' = -\tfrac{1}{4}K\cos(t/2) - \tfrac{1}{4}M\sin(t/2)$$

cos - term :

$$5\cdot\left(-\tfrac{1}{4}K\right) + 50\left(\tfrac{1}{2}M\right) + 245(K) = 0$$

sin- term

$$5\left(-\tfrac{1}{4}M\right) + 50\left(-\tfrac{1}{2}K\right) + 245(M) = 4$$

$$\Rightarrow \begin{cases} 975K + 100M = 0 \\ 975M - 100K = 16 \end{cases}$$

$$M = \frac{15600}{960625} \qquad K = \frac{-100}{975}M = \frac{-100}{975}\cdot\frac{15600}{960625} = \frac{-1600}{960625}$$

$$y(t) = y_h + y_p = \underbrace{e^{-5t}\left(A\cos(2\sqrt{6}t) + B\sin(2\sqrt{6}t)\right)}_{\text{transient solution}} + \underbrace{K\cos(t/2) + M\sin(t/2)}_{\text{Steady state solution}}$$

d)

Steady state solution :

$$K = \frac{-1600}{960625}$$
$$M = \frac{15600}{960625}$$

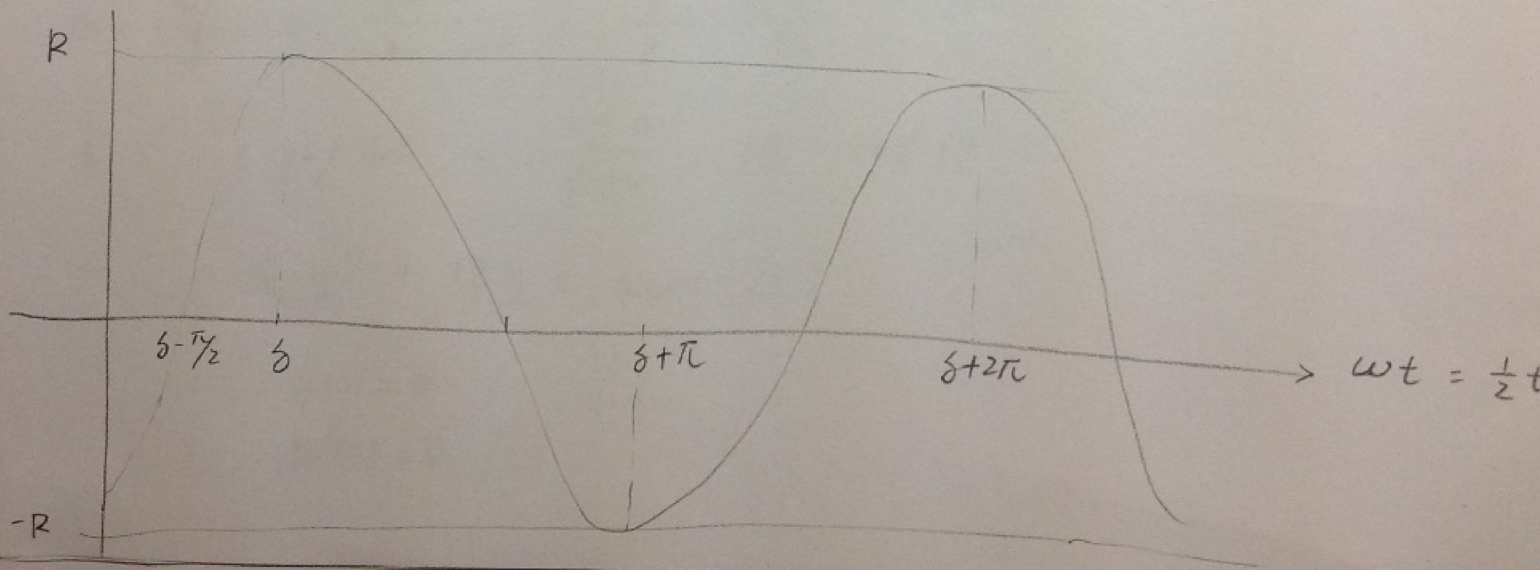
$$K \cos(t/2) + M \sin(t/2) = R \cos(\omega t - \delta)$$

$$R = \sqrt{K^2 + M^2} \qquad \omega = \frac{1}{2}$$

$$\tan \delta = \frac{M}{K} = -\frac{975}{100}$$

$K = R\cos\delta < 0 \Rightarrow \cos\delta < 0$
$M = R\sin\delta > 0 \Rightarrow \sin\delta > 0$
$\Big\} \Rightarrow$ Second quadrant

$$\delta = \arctan\left(\frac{975}{100}\right) + \frac{\pi}{2}$$



(e)

find w when $\Delta = \sqrt{m^2(w_0^2 - w^2)^2 + \gamma^2 w^2}$ is mini

$w_0^2 = \frac{k}{m} = \frac{245}{5} = 49$

$D = \Delta^2 = m^2(w_0^2 - w^2)^2 + \gamma^2 w^2$

$\frac{\partial D}{\partial w} = m^2 \cdot 2(w_0^2 - w^2) \cdot (-2w) + 2\gamma^2 w = 0$

$\Rightarrow \quad -4m^2 w_0^2 w + 4m^2 w^3 + 2\gamma^2 w = 0$

$\Rightarrow \quad 4m^2 w^3 + (2\gamma^2 - 4m^2 w_0^2) w = 0$

$4 \cdot 5^2 = 100 \qquad 2 \cdot 50^2 - 4 \cdot 5^2 \cdot 49^2 = 5000 - (49^2) \times 100$

$= 100(50 - 49^2)$

$\Rightarrow \quad 100w \left[w^2 + (50 - 49^2) \right] = 0$

$\Rightarrow \quad w^2 = 49^2 - 50 = 2351$

$\boxed{Q9}$

$my'' + ky = F(t)$

$m = \frac{w}{g} = \frac{12 \text{ lb}}{32 \text{ ft/s}^2} = \frac{3}{8} \cdot \frac{\text{lb} \cdot \text{s}^2}{\text{ft}}$

$k = 1.5 \text{ lb/inch} = \frac{1.5 \text{ lb}}{\frac{1}{12}\text{ft}} = 18 \text{ lb/ft}$

$$\begin{cases} \frac{3}{8} y'' + 18y = 4\cos(7t) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

For $m'u'' + ku = F_0 \cos \omega t$

$$u = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

(C_1, C_2 constants)

If $u(0) = 0$ $u'(0) = 0$

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

$F_0 = 4$ $\omega = 7$ $m = \frac{3}{8}$ $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\cancel{18}^{6} \cdot \frac{8}{\cancel{3}}}$

$= \sqrt{3 \cdot 16} = 4\sqrt{3}$

$$u = \frac{4}{\frac{3}{8}(48 - 49)} (\cos 7t - \cos 4\sqrt{3} t)$$ $\omega_0^2 = 48$

$$= -\frac{32}{3} (\cos 7t - \cos 4\sqrt{3} t) = \frac{32}{3} (\cos 4\sqrt{3} t - \cos 7t)$$

$$R = \frac{32}{3}$$

(c)

$$u = R(\cos \omega_0 t - \cos \omega t)$$

$$= 2R \sin\left[\frac{\omega_0 + \omega}{2} t\right] \cdot \sin\left[\frac{\omega - \omega_0}{2} t\right]$$

($\cos \alpha - \cos \beta = -2 \sin[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$
$= 2 \sin[(\beta + \alpha)/2] \sin[(\beta - \alpha)/2]$)

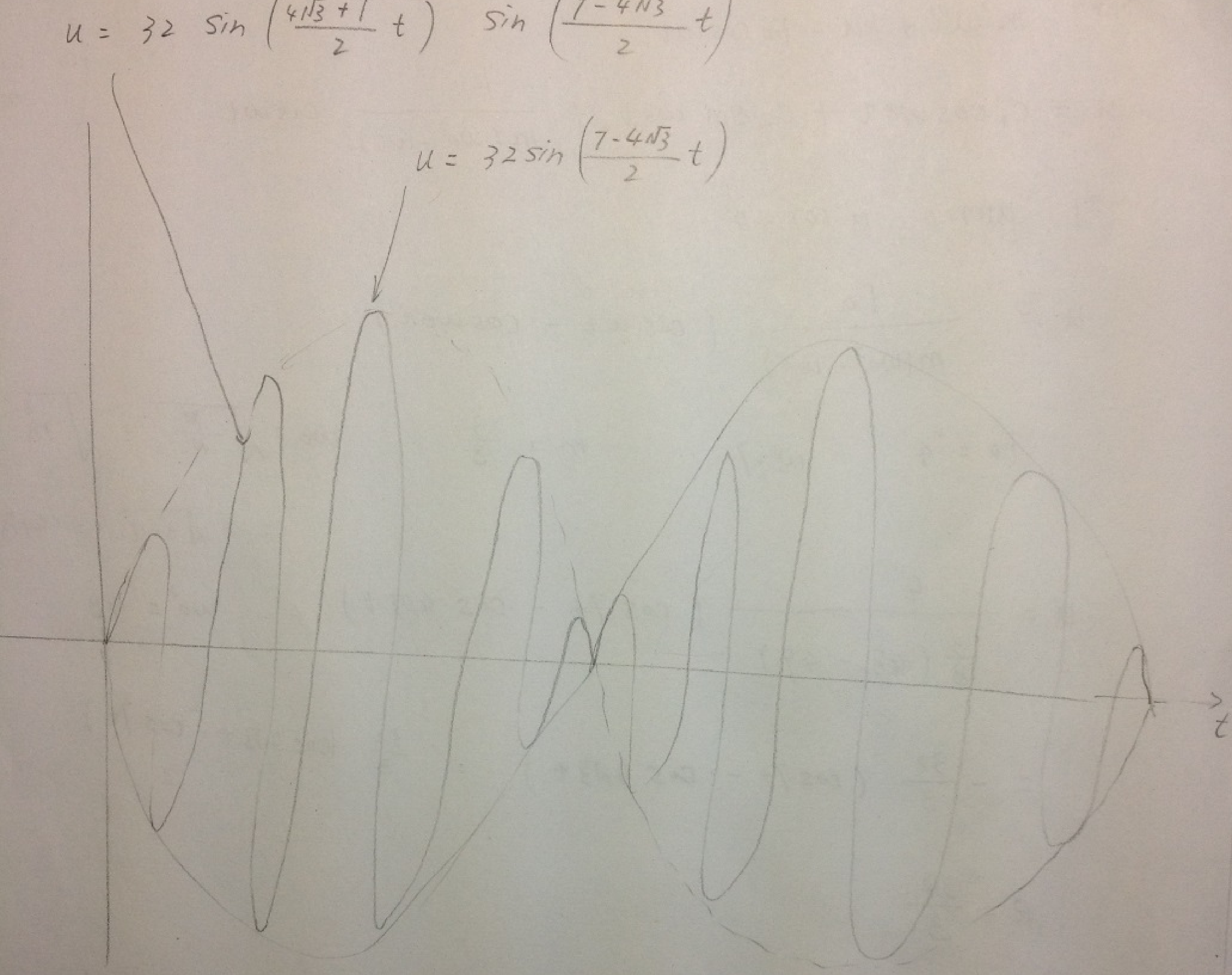
(d)

$$= 32 \sin\left(\frac{4\sqrt{3} + 7}{2} t\right) \sin\left(\frac{7 - 4\sqrt{3}}{2} t\right)$$

$|\omega_0 - \omega| = |4\sqrt{3} - 7| = |\sqrt{48} - \sqrt{49}|$ is small $\omega_0 + \omega >> |\omega_0 - \omega|$

$\Rightarrow$ $\sin(\omega_0 + \omega)t/2$ rapidly oscillating function compared to $\sin\left(\frac{\omega_0 - \omega}{2} t\right)$

Thus motion is a rapid oscillation with freq $\frac{\omega_0 + \omega}{2}$ but with a slowly varying sinusoidal amplitude $\frac{2F_0}{m|\omega_0^2 - \omega^2|} \left|\sin \frac{\omega_0 - \omega}{2} t\right|$


$u = 32 \sin\left(\frac{4\sqrt{3}+7}{2} t\right) \sin\left(\frac{7-4\sqrt{3}}{2} t\right)$
$u = 32 \sin\left(\frac{7-4\sqrt{3}}{2} t\right)$
t