INHOMOGENEOUS TRANSPORT EQUATION

$$(IT) \begin{cases} u_t + b\cdot\nabla u = f(x,t) \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \text{ in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Let's solve the HOMOGENEOUS problem first.

$$(HT) \begin{cases} u_t + b\cdot\nabla u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \quad \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Given $(x,t) \in \mathbb{R}^n \times (0,\infty)$ the Line through (x,t) in the direction of $(b,1)$ can be represented (parametrize) by $(x+\tau b, t+\tau) = X(\tau)$ $(\tau \in \mathbb{R})$

This line intersects $\mathbb{R}^n \times \{t=0\}$ when $\tau = -t$ and does so at $(x-tb, 0)$

Since u must be constant along $X(\tau)$ and $u(x-tb, 0) = g(x-tb)$ by (HT)

$\Rightarrow$ the solution to (HT) must be

$$u(x,t) = \underbrace{g(x-tb)}, \ \forall x \in \mathbb{R}^n \ t>0$$

THIS IS CALLED THE HOMOGENEOUS SOLUTION.

Now for the inhomogeneous equation (IT) we look for solutions along characteristic $X(\tau)$

Then if u is a solution to (IT) we must have that $u(\chi(\tau)) =: z(\tau)$ satisfies

$$\frac{dz}{d\tau} \overset{\text{CHAIN RULE.}}{=} \nabla u(x+\tau b, t+\tau)\cdot b + u_t(x+\tau b, t+\tau)$$

$$= f(x+\tau b, \tau+t) \quad \text{from (IT)}$$

Thus we have that (by FTC)

$$u(x,t) - \underbrace{g(x-tb)}_{\text{HOMOG. SOL}} = z(0) - $$

$$= \int_{-t}^{0} \frac{dz}{d\tau}\, d\tau$$

$$\underset{\text{change variables}}{=} \int_0^t f(x+(\tau-t)b, \tau)\, d\tau$$

All in all then the solution $u(x,t)$ to (IT) is

$$u(x,t) = \underbrace{g(x-tb)}_{\text{HOMOG. SOL.}} + \underbrace{\int_0^t f(x+(\tau-t)b, \tau)\, d\tau}_{\text{"Duhamel integral"}}$$

for all $x \in \mathbb{R}^n$, $t \geq 0$

- CHECK $u(x,t)$ is indeed a solution to (IT)!