Folland § 2.6
Th. 2.40 a)

HANDOUT M624 : Continuity of m^n

Let m be the Lebesgue measure on $\mathbb{R}$ and let m^n denote the completion of $\underbrace{m \times m \times \cdots \times m}_{n\text{-times}}$ on $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}$

(Equiv. the completion of $m \times \cdots \times m$ on $\mathcal{L} \otimes \cdots \otimes \mathcal{L} =: \mathcal{L}^n$)

Domain of $m^n = \mathcal{L}^n =$ measurable sets in $\mathbb{R}^n$

<u>LEMMA</u>: ~~Suppose~~ $E \in \mathcal{L}^n$, $m = m^n$ (ABUSE OF NOTATION)

<u>Then</u>: $m(E) = \inf\{m(U) : U \supset E,\ U \text{ open}\}$

$= \sup\{m(K) : K \subset E,\ K \text{ compact}\}$

<u>Proof</u>:
First we will refer by an n-dimensional <u>rectangle</u> to a set $T \subseteq \mathbb{R}^n$ such that

$T = \prod_{\ell=1}^{n} T^{(\ell)}$ where $T^{(\ell)} \subset \mathbb{R}$

are called the <u>SIDES</u> of T.

Now if $E \in \mathcal{L}^n$ then given $\varepsilon > 0$ $\exists$ a countably family $\{T_j\}$ of n-dim. rectangles such that

$$E \subset \bigcup_{j=1}^{\infty} T_j \quad \text{and} \quad \sum_j m(T_j) \leq m(E) + \varepsilon$$

Now for each j apply Theorem 1.18 (Folland) to the sides of T_j. We have that $\exists$ U_j an n-dimensional rectangle whose sides are open sets in $\mathbb{R}$ / $U_j \supset T_j$ and

$$m(U_j) < m(T_j) + \varepsilon/2^j .$$

Let $U = \bigcup_{j=1}^{\infty} U_j$, U is open and

$$m(U) \leq \sum_{j=1}^{\infty} m(U_j) \leq m(E) + 2\varepsilon \quad (*)$$

This proves the first equality in the Lemma.

To establish the second equality: First suppose E is bounded. If in addition E is closed

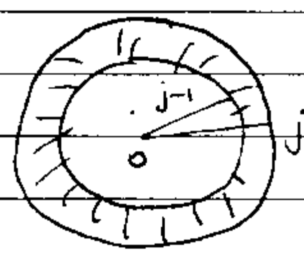
$\Rightarrow E$ is compact and there is nothing to prove.

Suppose then E is not necessarily closed. Then given $\varepsilon > 0$ choose an open set U / $U \supset \bar{E} \setminus E$ and such that $m(U) \le m(\bar{E} \setminus E) + \varepsilon$.

Let $K = \bar{E} \setminus U$. Then K is compact, $K \subset \underline{E}$

and
$$m(K) = m(E) - m(E \cap U)$$
$$= m(E) - [m(U) - m(U \setminus E)]$$
$$\ge m(E) - m(U) + m(\bar{E} \setminus E)$$
$$> m(E) - \varepsilon \quad \text{as desired.}$$

If $\underline{E \text{ is unbounded}}$, let $E_j = E \cap A_j$ where A_j is the annulus in $\mathbb{R}^n$ defined by $j-1 \le |x| < j$, $(j \ge 1)$



Note that $A_j \cap A_{j'} = \emptyset$ $j \ne j'$

Now E_j is bounded so by the preceding argument, given any $\varepsilon > 0$ $\exists$ K_j compact

$$K_j \subset E_j \;/\; m(K_j) \geq m(E_j) - \varepsilon/2^j$$

Let $H_N = \overset{N}{\underset{j=1}{\dot\bigcup}} K_j$ then H_N is compact,

$$H_N \subset E, \quad m(H_N) \geq m\left(\overset{N}{\underset{j=1}{\dot\bigcup}} E_j\right) - \varepsilon$$

Indeed the last statement follows from;

$$m(H_N) = m\left(\overset{N}{\underset{j=1}{\dot\bigcup}} K_j\right) \underset{\text{disjoint union}}{=} \sum_{j=1}^{N} m(K_j) \geq$$

$$\geq \sum_{j=1}^{N} m(E_j) - \sum_{j=1}^{N} \varepsilon/2^j \geq m\left(\bigcup_{j=1}^{N} E_j\right) - \varepsilon$$

On the other hand,

$$m(E) = m\left(\bigcup_{j=1}^{\infty} E_j\right) = m\left(\bigcup_{N=1}^{\infty}\left(\bigcup_{j=1}^{N} E_j\right)\right)$$

$$= \lim_{N\to\infty} m\left(\bigcup_{j=1}^{N} E_j\right)$$

Hence given any $\delta > 0$ $\exists\, N = N(\delta)$ /

$$m(E) = m(E) - m\left(\bigcup_j^N E_j\right) + m\left(\bigcup_j^N E_j\right)$$

$$< \delta/2 + m(H_N) + \delta/2$$

where H_N is the one that exists by the above procedure when $\varepsilon = \delta/2$.

Hence $\exists$ a compact $H_N \subset E$ /

$m(E) < m(H_N) + \delta \implies$

$$m(E) = \sup\{m(K) : K \subset E, K\text{ compact}\}$$

as desired.

<u>COROLLARY:</u> Let $E \in \mathcal{L}^n$ such that $m(E) < \infty$. Then for any $\varepsilon > 0$ $\exists$ a finite collection $\{R_j\}_{j=1}^N$ of disjoint n-dim. rectangles whose sides are intervals such that $m(E \,\Delta\, \bigcup_{j=1}^N R_j) < \varepsilon$.

<u>Proof:</u> If $m(E) < \infty$ then by (*) page 3 of the PROOF OF PREVIOUS lemma, we have

given any $\varepsilon > 0$ the existence of rectangles U_j / that those same U_j's have $m(U_j) < \infty$ for all j. Since the sides of U_j are open (countable union of ~~open~~ intervals) and of finite measure by suitably choosing a subunion of intervals in each side of U_j we can form a rectangle $\tilde{R}_j \subset U_j$ whose sides are finite union of intervals and $m(\tilde{R}_j) \geq m(U_j) - \varepsilon/2^j$

Then given the $\varepsilon > 0$ we can choose N suff. large so that:

$$m\left(E \setminus \bigcup_{j=1}^{N} \tilde{R}_j\right) \leq m\left(\bigcup_{j=1}^{N} (U_j \setminus \tilde{R}_j)\right) + m\left(\bigcup_{j=N+1}^{\infty} U_j\right)$$

$$< \quad \varepsilon + \varepsilon \quad = 2\varepsilon$$

and

$$m\left(\bigcup_{j=1}^{N} \tilde{R}_j \setminus E\right) \leq m\left(\bigcup_{j=1}^{\infty} U_j \setminus E\right) < \varepsilon$$

$$\Rightarrow m\left(E \,\Delta\, \bigcup_{j=1}^{N} \tilde{R}_j\right) < 3\varepsilon .$$

Finally note we can rewrite $\bigcup_{j=1}^{N} \tilde{R}_j = \bigcup_{k=1}^{M} R_k$ (D)

where R_k's are also rectangles whose sides are intervals and $R_k \cap R_{k'} = \phi \quad k \neq k'$

as desired. (CLASS: THINK ABOUT HOW TO DO THIS!)

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- THE following result follows from The COROLLARY ABOVE in conjunction with Theorem 2.26 (1-dim.).

THEOREM: If $f \in L^1(m^n)$ and $\varepsilon > 0$ there exists a simple function $\phi = \sum_{k=1}^{M} a_j \chi_{R_k}$ where each R_k is a product of intervals, such that $\int |f - \phi| \, dm^n < \varepsilon$. There is also a continuous function g, supp g bounded / $\int |f - g| \, dm^n < \varepsilon$