Math 331 Section 3. Solutions to Quiz 2

Problem 1. Given the initial condition below for the differential equation

$$\frac{dy}{dt} = (y-2)\,(y-3)\,y$$

What does the Existence and Uniqueness theorem say about the corresponding solution? **Explain**. Sketch a graph for $y(t)$ indicating its initial condition on it.

Since $y(y-2)(y-3)$ is continuous for all (y, t) the existence theorem guarantees that for the given initial conditions a solution must exist. Moreover, for all (y, t), $y(y-2)(y-3)$ is also differentiable ; hence we have that for each initial condition there must exist a **unique** solution.

By uniqueness, we then know that for different initial conditions the graphs of the solutions cannot intersect.

By inspecting $y(y-2)(y-3)$ we conclude that there are three equilibrium solutions, namely $y = 0$, $y = 2$ and $y = 3$. By checking the signs we have that $\frac{dy}{dt} > 0$ for $y > 3$ and $0 < y < 2$; while $\frac{dy}{dt} < 0$ for $y < 0$ and $2 < y < 3$.

In case **(a)** for the initial condition $y(0) = 1$ we have that the solution is increasing and stays in between the equilibrium solutions $y = 0$ and $y = 2$ by uniqueness (it cannot 'cross' these lines). Moreover the $\lim_{t\to\infty} y(t) = 2$ and $\lim_{t\to-\infty} y(t) = 0$ asymptotically.

In case **(b)** for the initial condition $y(0) = 3$ by uniqueness we must have that the only solution with that initial condition must be the equilibrium solution $y = 3$.

Problem 2. Given the differential equation

$$\frac{dy}{dt} = 3y\,(y-1)$$

(a) Sketch its phase line and identify equilibrium points as sinks, sources or nodes.

(b) For the equation above and initial conditions $y(0) = 1/2$ on the one hand and and $y(0) = 2$on the other; sketch the graphs of the solutions satisfying these initial conditions. Put both graphs on one pair of axes.

Problem 3 Solve the following initial value problem for the given linear differential equation

$$\frac{dy}{dt} = -2t\,y + 4e^{-t^2} \qquad y(0) = 3$$

First rewrite as

$$\frac{dy}{dt} + 2t\,y = 4e^{-t^2} \quad \Longrightarrow \quad \mu(t) = e^{\int 2t\,dt} = e^{t^2}$$

Multiplying both sides by $\mu(t)$ we get

$$e^{t^2}\frac{dy}{dt} + e^{t^2}\,2t\,y = 4 \quad \Longrightarrow \quad (\,e^{t^2}\,y\,)' = 4$$

Integrating both sides with respect to t we obtain,

$$e^{t^2}\,y = 4t + C \quad \Longrightarrow \quad y = 4te^{-t^2} + C\,e^{-t^2}$$

Since $y(0) = 3$, plugging in we have that $C = 3$. Hence

$$y = 4te^{-t^2} + 3\,e^{-t^2}$$