NAME:

ID #:

TAKE HOME FINAL MATH 534H

Saturday May 6th, 2017

Instructions.

(1) **This exam consists of 4 problems with parts for a total of 100%.**

(2) **It is due no later than Tuesday May 9th by 3:30 pm in LGRT 1338**

(3) **You should work on it alone. You may consult Salsa' book, the class notes and your homework ONLY. No other material is allowed.**

(4) **Show <u>all</u> the work needed to reach your answer for full credit.**

(5) **You cannot discuss the problems with other people, including classmates.**

(6) **<u>Type</u> each problem and its solution in an ordered fashion (new page for each problem) and staple them all together with this cover. Insert additional pages if needed.**

Final Problem 1: Find the solution $u(x,t)$ to the following inhomogeneous diffusion boundary/initial value problem with Dirichlet boundary conditions (proceed as in handout example).

$$\begin{cases} u_t \;-\; u_{xx} \;=\; e^{-t}, \qquad 0 < x < 1, \;\; t > 0 \\ u(x,0) \;=\; 1, \quad 0 < x < 1 \\ u(0,t) \;=\; 0 \quad \text{and} \quad u(1,t) \;=\; 0 \end{cases}$$

Hints First find the sine Fourier series for e^{-t} on $(0,1)$. Note that you had to find the sine Fourier series for 1 in Set 2 Problem 4.

At some point you'll encounter an ODE of the form

$$a_n'(t) + c_1 n^2 \pi^2 a_n(t) = c_2 \frac{e^{-t}}{n\pi}$$

for some specific constants c_1, c_2. To find the solution to this ODE, consider $a_n(t) = A_n e^{-t}$ for suitable A_n that you would need to find.

Final Problem 2: a) Let f be a smooth function with compact support (that is the support of f is contained in some fixed ball of radius $R.0$ centered at the origin which means, $f(x) = 0$ for $|x| > R$ for some $R > 0$). Consider the Poisson equation on the whole space $\mathbb{R}^d$

$$(\dagger) \qquad \Delta u = f$$

i) Consider $d = 3$. We know that there exist smooth solutions u that tend to zero at ∞ ($u(x) \to 0$ as $|x| \to \infty$). Use Liouville's theorem to prove that these solutions are unique. Carefully show all steps and justify them.

ii) Consider $d = 2$. We know now that there exist smooth solutions solutions $u(x)$ which could grow like $\ln|x|$ at infinity. Use Liouville's theorem to prove that smooth solutions $u(x)$ whose gradient $|\nabla u(x)| \to 0$ as $|x| \to \infty$ are unique up to a constant; i.e. if u_1 and u_2 are solutions, then $u_1 = u_2 + C$ for some $C > 0$. Carefully show all steps and justify them.

b) Let Ω be a smooth and bounded domain in $\mathbb{R}^d$ and let g be a given smooth function on $\partial\Omega$, the boundary of Ω. Consider the Dirichlet boundary value problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u(\overline{x}) \;=\; g(\overline{x}) & \overline{x} \in \partial\Omega \end{cases}$$

Use the energy method to show that smooth (and continuous up to the boundary) solutions to this problem are unique. To prove this consider $w = u_1 - u_2$ (where u_1, u_2 are two solutions) and write down the boundary value problem that w solves. Next, prove that

$$(1) \qquad E(w) = \int_\Omega |\nabla w(x)|^2 \, dx \;=\; 0$$

and use this in conjunction with the boundary values of w –and the fact that w is continuous on $\overline{\Omega}$ – to conclude uniqueness.

Hint To prove (1), note that

$$0 = \int_\Omega w(x) \, \Delta w(x) \, dx$$

and integrate by parts/divergence theorem.

Final Problem 3

Consider the solution to the initial value problem for the wave equation on $\mathbb{R}$:

$$\begin{cases} u_{tt} - u_{xx} = 0 \qquad x \in \mathbb{R}, \\ u(x,0) = 0 \\ u_t(x,0) = \chi_{[-2,2]}(x) \quad \text{this is 1 on } |x| \leq 2; \text{ and } 0 \text{ otherwise.} \end{cases}$$

(a) Use D' Alembert's formula to write down the solution $u(x,t)$ in terms of its initial data (do not evaluate the integral).

(b) Use differentiation rules for integrals to compute $u_t(x,t)$.
(Hint Recall homework problem 8 in Set 4).

(c) Set $x = 0$ in (b) and prove that for all $|t| > 2$ we have that $u_t(0,t) = 0$.

Final Problem 4:

Consider the following linear wave equation equation on the interval $[0, \pi]$ with zero Dirichlet boundary conditions:

$$
(1) \qquad \begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi \\ u(0,t) = 0 = u(\pi, t) & \\ u(x,0) = \phi(x) & u_t(x,0) = \psi(x) \end{cases}
$$

where ϕ, ψ are smooth functions on $[0, \pi]$

a) Use separation of variables and fully solve the corresponding eigenvalue problems associated to (1) to write down the sine-cosine series expansion of the solution $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$. Describe the coefficients in terms of the initial data $\phi(x)$ and $\psi(x)$.

b) Suppose that $\phi(x) = x$ and $\psi(x) = 0$. Find the sine Fourier series for ϕ in this case and use it to give the explicit solution to (1) by finding the corresponding coefficients in a).