Hints for Some Problems in 6.1 & 6.2

6.1

Pb 4: Use that $\cos^2(\omega t) = \frac{1}{2} + \frac{\cos(2\omega t)}{2}$

Pb 7: Here ω and θ are given constants

Use then that

$$\sin(\omega t + \theta) = \sin(\omega t)\overbrace{\cos\theta}^{\text{constant}} + \overbrace{\sin\theta}^{\text{constant}}\cos(\omega t)$$

So $\mathcal{L}(\sin(\omega t+\theta))(s) =$

$$= \overbrace{\cos\theta}^{\text{constant}}\ \mathcal{L}(\sin(\omega t))(s) + \overbrace{\sin\theta}^{\text{constant}}\ \mathcal{L}(\cos(\omega t))(s)$$

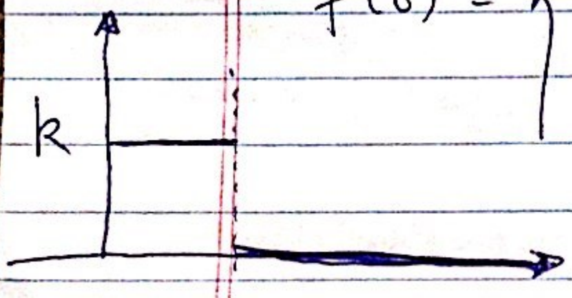
Linearity

Use Table next.

Pb 10: We are given the function

$$f(t) = \begin{cases} k & 0 \leqslant t < c \\ 0 & t \geqslant c \end{cases}$$

where k, c are given constants



To compute $\mathcal{L}(f(t))$ we need to rewrite $f(t)$ because it is <u>not</u> the same constant for all t.

Suppose we call $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$.

Then $u(t-c) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$

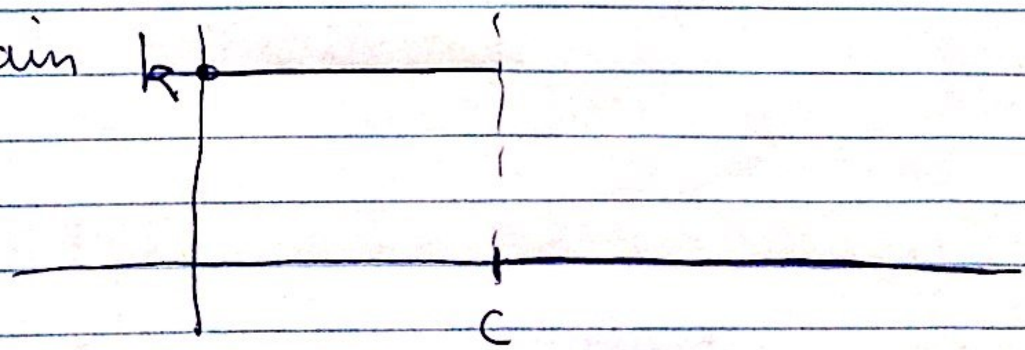
c = given constant.

So if we rewrite.

given constant   given constant

$$f(t) = k\,u(t) - k\,u(t-c)$$

we obtain



Then $\mathcal{L}(f)(s) = k\,\mathcal{L}(u(t))(s) - k\,\mathcal{L}(u(t-c))(s)$

Linearity.

$$= k \cdot \underbrace{\frac{1}{s}}_{\mathcal{L}(u(t))} - k \underbrace{\frac{e^{-cs}}{s}}_{\mathcal{L}(u(t-c))} = \frac{k}{s}\left(1 - e^{-cs}\right)$$

(k, c constants)

NOTE: We used that $\mathcal{L}(u(t-c)) = \dfrac{e^{-cs}}{s}$ which one can compute:

$$\mathcal{L}(u(t-c))(s) = \int_0^\infty e^{-st}\, u(t-c)\, dt$$

(by definition of $u(t-c)$) $$= \int_c^\infty e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_c^\infty = \frac{e^{-cs}}{s}$$

Pb 18 : Use Pb 10 with $k=1$, $c=2$

Pb 36 : Recall $\sinh t = \frac{1}{2}\left(e^{t} - e^{-t}\right)$

so $\sinh t \cos t = \frac{1}{2}\left(e^{t}\cos t - e^{-t}\cos t\right)$

Then → Use Linearity and Shift properties of $\mathcal{L}$.

Pb 38 (37 is similar)

Recall $\mathcal{L}^{-1}\left(F(s-a)\right) = e^{at} f(t)$

where $F = \mathcal{L}(f)$ or $f(t) = \mathcal{L}^{-1}(F)$

(*) Hence $\frac{6}{(s+1)^3} = \frac{6}{(s-(-1))^3} = F(s-(-1))$

where $F(s) = \frac{6}{s^3} \Rightarrow f(t) = 3t^2$

SHORT TABLE 6.1

(here we use that $\mathcal{L}(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$)

Hence from (*) we get

$$\mathcal{L}^{-1}\left(\frac{6}{(s+1)^3}\right) = e^{-t} \cdot 3t^2$$

#6.2 : Pb 4 : Use partial fractions (after taking Laplace transform) to rewrite

$$\frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} - \frac{s-1}{s^2+9}$$

Then take $\mathcal{L}^{-1}$ of $\rightarrow$ $= \frac{1}{s+1} - \frac{s}{s^2+9} + \frac{1}{s^2+9}$

(use linearity, shift and table)

Pb 8 : After taking Laplace transform of the equation you get that the solution of the algebraic equation is

$$Y(s) = \frac{8.1s - 28.5}{(s-2)^2}$$

$$= \frac{8.1(s-2) + 16.2 - 28.5}{(s-2)^2}$$

by algebra (or partial fractions)

$$= \frac{8.1}{(s-2)} - \frac{12.3}{(s-2)^2}$$

Now compute $\mathcal{L}^{-1}$ of this (using linearity, shift, table)