M331 Section 1 Additional Word Problems – Spring 2013

Prof. Andrea R. Nahmod

First Order ODE - Mixing Problems.

In the following problems we assume that salt is neither created nor destroyed in the tank. Hence variations in the amount of salt are due solely to the flows in and out of the tank. In other words, if $S(t)$ denotes the amount of salt in the tank at time t then

$$\frac{dS}{dt} = (\text{rate at which salt flows in at time } t) - (\text{rate at which it flows out at time } t).$$

Problem 1. At time $t = 0$ a tank contains 40 pounds of salt dissolved in 250 gallons of water. Assume that water containing 0.10 pound of salt per gallon is entering the tank at a rate $r = 5$ gallon per minute and that the well-stirred mixture is draining from the tank at the same rate.

(a) Set up the initial value problem that describes this flow process (ie. set up the ODE and initial condition).
(b) Find the amount of salt $S(t)$ at any time t.
(c) Find the limiting amount S_L of salt that is present in the tank after a very long time ($t \to \infty$).
(d) Find the time T after which the salt level is within 3% of S_L.
(e) Find the flow rate that is required if the value of T is not to exceed 45 minutes.

Problem 2. A tank originally contains 100 gallons of fresh water. Then water containing 0.5 pound of salt per gallon is poured into the tank at a rate of 2 gallons per minute, and the mixture is allowed to leave at the same rate. After 10 minutes the process is stopped and fresh water is poured into the tank at a rate of 2 gallons per minute, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 minutes.

Problem 3.(Variable volume). A tank with capacity 500 gallons originally contains 200 gallon of water with 100 pound of salt in solution. Water containing 1 pound of salt per gallon is entering at a rate of 3 gallons per minute, and the mixture is allowed to flow out the tank at a rate of 2 gallons a minute.

(a) Set up the initial value problem that describes this flow process (ie. set up the ODE and initial condition).
(b) Find the amount of salt $S(t)$ in the tank at any time t prior to the instant when the solution begins to overflow the tank.

(c) Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing.
(d) In the theoretical scenario that the tank had infinite capacity, what would the limiting concentration be?

Second Order ODE - Free Vibrations.

In Problems 4, 5, 6, 7 and 9 use pounds, feet and seconds as units. In Problem 8 use kilograms, meters and second as units (recall $1N = kg\frac{m}{s^2}$).

Problem 4. A body weighing 2 pounds stretches a spring 6 inches. Suppose the body is displaced an additional 3 inches in the positive direction (down) and then released (ie. zero initial velocity). Assume there is no damping.
(a) Formulate the initial value problem that governs the motion of the body (an ODE and initial conditions. Measure the displacement $y(t)$ of the body in feet.)
(b) Find the displacement $y(t)$ at any time t.
(c) Find the frequency ω_0, period T, phase δ and amplitude R of the motion. Re-express the solution $y(t)$ as $R\cos(\omega_0 t - \delta)$ and sketch its graph in the $(\omega t,\, y)$ axes, indicating the amplitude lines $y = \pm R$.

Problem 5. A body weighing 3 pounds stretches a spring 3 inches. Suppose the body is pushed upwards (negative direction) contracting the spring a distance of 1 inch and then set into motion with a downward (positive) velocity of 2 feet per second. Assume there is no damping.
(a) Formulate the initial value problem that governs the motion of the body (an ODE and initial conditions. Measure the displacement $y(t)$ of the body in feet.)
(b) Find the displacement $y(t)$ at any time t.
(c) Find the frequency ω_0, period T, phase δ and amplitude R of the motion. Re-express the solution $y(t)$ as $R\cos(\omega_0 t - \delta)$ and sketch its graph in the $(\omega_0 t,\, y)$ axes, indicating the amplitude lines $y = \pm R$.

Problem 6. A body weighing 8 pounds stretches a spring 6 inches. Suppose the body is displaced from the rest position an additional 8 inches in the positive direction (down) and then released (ie. zero initial velocity). The body is in a medium that exerts a viscous resistance of 5 pounds when the body has velocity 2 feet per second.
(a) Formulate the initial value problem that governs the motion of the body (an ODE and initial conditions. Measure the displacement $y(t)$ of the body in feet.)
(b) Find the displacement $y(t)$ at any time t
(c) Re-express the solution $y(t)$ as $Re^{-\alpha t}\cos(\theta t - \delta)$ and sketch its graph in the $(\theta t,\, y)$ axes, indicating the amplitude curves $y = \pm Re^{-\alpha t}$.

(d) Classify as either undamped or damped (free) vibration. If damped determined if there is overdamping, critical damping or underdamping,

Problem 7. A body weighing 32 pounds stretches a spring 2 feet. Suppose the body is displaced from the rest position an additional 6 inches in the positive direction (down) and then released (ie. zero initial velocity). The body is in a medium that exerts a viscous resistance of 24 pounds when the body has velocity 3 feet per second.
(a) Formulate the initial value problem that governs the motion of the body (an ODE and initial conditions. Measure the displacement $y(t)$ of the body in feet.)
(b) Find the displacement $y(t)$ at any time t
(c) Classify as either undamped or damped (free) vibration. If damped determined if there is underdamping, critical damping or overdamping.

Second Order ODE - Forced Vibrations.

Problem 8. A body of mass $m = 5$ kg stretches a spring 20 cm. Suppose the body is acted on by an external force of $4\sin(\frac{t}{2})N$ and moves in a medium that imparts a viscous force of $2N$ when the speed of the body is 4 cm per second. The body is then set in motion from its equilibrium position with an initial velocity of 3 cm per second.
(a) Formulate the initial value problem that governs the motion of the body (an ODE and initial conditions.)
(b) Find the displacement $y(t)$ at any time t (solution to the problem above).
(c) Identify the transient and steady-state parts of the solution.
(d) Sketch the graph of the steady state solution.
(e) If the given external force is replaced by another force equal to $2\cos(\omega t)$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum. Recall the amplitude of the **forced response** is given by $R = \frac{F_0}{\Delta}$ where $F_0 = 2$ the amplitude of the new external force and $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ (γ = damping coefficient). Thus R is maximum when Δ is minimum.

Problem 9. An undamped spring-mass system with a body that weighs 12 pounds and a spring constant 1.5 pounds per inch is suddenly set in motion at $t = 0$ by an external force $4\cos(7t)$ pounds.
(a) Determine the position $y(t)$ of the mass at any time t
(b) Find $R = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}$ where ω_0 is the natural frequency, m is the mass and F_0 and ω are respectively, the amplitude and frequency of the external force.
(c) Show that the solution found in (a) can be re-express as $y(t) = 2R\,\sin(\frac{(\omega_0-\omega)t}{2})\sin(\frac{(\omega_0+\omega)t}{2})$
(d) Sketch a graph of the displacement $y(t)$.