NAME:

MATH 623 FINAL EXAM

Due: Friday, December 15th, 2017 no later than 11AM

Instructions

1. This exam consists of four (4) problems all counted equally for a total of 100%.

2. You may consult Stein-Shakarchi's Vol. III and class material (class notes, homeworks done, handouts) **only**. No other books or notes are permitted.

3. You should work on the problems alone; do not discuss the problems with other people or classmates. You may ask me any questions you have.

4. **Type** each problem and its solution in an ordered fashion (new page for each problem) and staple them all together with this cover. Insert additional pages if needed.

5. State explicitly all results that you use in your proofs and verify that these results apply.

6. Show all your work and justify the steps in your proofs.

Conventions

1. If a measure is not specified, use Lebesgue measure. This measure is denoted by dm or dx.

1. a) Let $F : [-1, 4] \to \mathbb{R}$ be the function defined by $F(x) = 2|x| - |x - 2|$. Prove that F is of bounded variation and compute the total variation $T_F(-1, 4)$ of F over $[-1, 4]$.

b) Let $F \in BV([a, b])$ and let c be such that $a < c < b$. Show that

$$T_F(a, b) \; = \; T_F(a, c) \; + \; T_F(c, b)$$

c) Show that if L is Lipschitz continuous and F is of bounded variation then the composition $L \circ F$ is of bounded variation. Recall L Lipschitz continuous means that there exists $M > 0$ such that $|L(x) - L(y)| \leq M|x - y|$ for all x, y.

2. For each fixed $N \geq 1$ natural number, let $\mathcal{R}_N := [0, N] \times [0, \infty)$, be a region in $[0, \infty) \times [0, \infty)$ endowed with the Lebesgue measure.

   (a) Consider the continuous function $f(x, t) := (\sin x)e^{-xt}$ defined on $[0, \infty) \times [0, \infty)$. Prove that for each fixed N, f is integrable over $\mathcal{R}_N$; i.e.,

$$\iint_{\mathcal{R}_N} |f(x, t)| \, dxdt \; < \infty.$$

   (Hint. Use Tonelli's Theorem)

   (b) Use part (a) and the fact that for any $x \in \mathbb{R}$, $x \neq 0$

$$\frac{1}{x} = \int_0^\infty e^{-xt} \, dt$$

   to rigorously prove that

$$\lim_{N \to \infty} \int_0^N \frac{\sin\, x}{x} \, dx \; = \; \frac{\pi}{2}.$$

   (Hints. Use Fubini's Theorem in conjunction with the Dominated Convergence Theorem.)

3. Let $f \in L^1(\mathbb{R})$ and let $\widehat{f}$ be its Fourier transform defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)\, e^{-i\xi x}\, dx$$

for all $\xi \in \mathbb{R}$.

a) Prove that $\widehat{f}$ is uniformly continuous on $\mathbb{R}$. Hint. Prove and use the fact that for real numbers a, b and z, $\quad |e^{iaz} - e^{ibz}| \leq |z||a-b|$.

b) Prove that $\widehat{f}$ is also bounded on $\mathbb{R}$. What is the $\sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi)|$ less than or equal to?

c) Assume $f \in L^1(\mathbb{R})$ satisfies

$$\int_{|x| \leq N} |x|\, |f(x)|\, dx \leq N^{1/2}$$

$$\int_{|x| > N} |f(x)|\, dx \leq \frac{1}{N^{1/2}}$$

for all $0 < N < \infty$. Show then that there exists a constant $C > 0$ and $0 < \beta < 1$ such that for all ξ and $\eta \in \mathbb{R}$,

$$|\widehat{f}(\xi) - \widehat{f}(\eta)| \leq C\, |\xi - \eta|^{\beta}$$

(in other words, $\widehat{f}$ is Hölder continuous of order β).

Hint. Obtain an estimate for $\int_{|x| \leq N} (e^{-ix\xi} - e^{-ix\eta})\, f(x)\, dx$ and another estimate for $\int_{|x| > N} (e^{-ix\xi} - e^{-ix\eta})\, f(x)\, dx$. Then optimize your choice of N (in terms of $|\xi - \eta|$) to obtain the desired estimate.

d) Now assume that in addition to having $f \in L^1(\mathbb{R})$, the function $xf(x)$ is also integrable; that is, $\int_{\mathbb{R}} |xf(x)|\, dx < \infty$.

Show then that: $\widehat{f}$ is differentiable and moreover that,

$$\frac{d}{d\xi} \widehat{f}(\xi) = \widehat{(-ixf)}(\xi).$$

Hint. Use the Dominated Convergence Theorem.

4. Let $f \in L^1(\mathbb{R})$ and $r > 0$. Set

$$A_r(f)(x) := \frac{1}{2r} \int_{x-r}^{x+r} f(y)\, dy.$$

(a) Show that the average $A_r(f)(x)$ is a continuous function in **both** x and r.

(b) Show that if in addition f is continuous, then

$$\lim_{r \to 0} A_r(f)(x) \;=\; f(x).$$

(c) Show that A_r is a contraction in $L^1(\mathbb{R})$ in the sense that

$$\|A_r(f)\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}.$$

Hint. Use Tonelli)

(d) Use (b) and (c) to show that if $f \in L^1(\mathbb{R})$ then

$$\lim_{r \to 0} \|A_r(f) - f\|_{L^1(\mathbb{R})} \;=\; 0.$$

Hint. You may use without proof the fact that $L^1(\mathbb{R})$ functions can be approximated in $L^1(\mathbb{R})$ by continuous functions.