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# Appendix B

# Review of Partial Fractions

In high school algebra, we learn to put fractions over a common denominator. This applies not just to numbers like

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6},$$

but to rational functions, i.e. quotients of polynomials:

$$\frac{1}{x+1} + \frac{1}{x+2} = \frac{2x+3}{(x+1)(x+2)}. \tag{B.1}$$

However, when we want to integrate a complicated rational function, we often want to "go the other way" and separate it into a sum of simpler rational functions. The resulting sum is called a **partial fraction decomposition (PFD)**. For example, if we started with the rational function on the right in (B.1), the expression on the left is its partial fraction decomposition.

In general, suppose we have a rational function

$$R(x) = \frac{p(x)}{q(x)} \quad \text{where } \deg p < \deg q. \tag{B.2}$$

(Recall that if we do not have $\deg p < \deg q$, then we can perform polynomial division to achieve this.) Now we factor $q(x)$ into its linear and irreducible quadratic terms. (Recall that "irreducible quadratic" means it does not contain any real linear factors, so $x^2+1$ is irreducible but $x^2-1$ is not irreducible since it can be factored as $(x-1)(x+1)$.)

- Each factor of $q(x)$ in the form $(ax-b)^k$ contributes the following to the PFD:

$$\frac{A_1}{(ax-b)} + \frac{A_2}{(ax-b)^2} + \cdots + \frac{A_k}{(ax-b)^k}.$$

- Each factor of $q(x)$ in the form $(ax^2+bx+c)^k$ contributes the following to the PFD:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}.$$

Once we have the correct form of the partial fraction decomposition, we simply recombine over the common denominator and compare with the original expression to evaluate the constants.

**Example 1.** Find the PFD for

$$\frac{3x+14}{x^2+x-6}.$$

**Solution.** The denominator factors as $x^2+x-6=(x-2)(x+3)$, so the partial fraction decomposition takes the form

$$\frac{3x+14}{x^2+x-6}=\frac{A}{x-2}+\frac{B}{x+3}.$$

We want to find $A$ and $B$. We recombine over the common denominator and collect terms in the numerator according to the power of $x$:

$$\frac{A}{x-2}+\frac{B}{x+3}=\frac{A(x+3)+B(x-2)}{(x-2)(x+3)}=\frac{(A+B)x+(3A-2B)}{(x-2)(x+3)}.$$

Comparing the numerator of this last expression with that of the original function, we see that $A$ and $B$ must satisfy

$$A+B=3 \qquad \text{and} \qquad 3A-2B=14.$$

We can easily solve these simultaneously to obtain $A=4$ and $B=-1$, so our PFD is

$$\tfrac{3x+14}{x^2+x-6}=\tfrac{4}{x-2}-\tfrac{1}{x+3}. \qquad \square$$

**Example 2.** Find the PFD for

$$\frac{x^2-2x+7}{(x+1)(x^2+4)}.$$

**Solution.** The denominator has already been factored into a linear and an irreducible quadratic factor, so the PFD takes the form

$$\frac{x^2-2x+7}{(x+1)(x^2+4)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}.$$

Now we recombine over the common denominator and collect terms in the numerator according to the power of $x$:

$$\frac{x^2-2x+7}{(x+1)(x^2+4)}=\frac{A(x^2+4)+(x+1)(Bx+C)}{(x+1)(x^2+4)}=\frac{(A+B)x^2+(B+C)x+4A+C}{(x+1)(x^2+4)}.$$

Comparing both sides of this equation, we see that $A, B, C$ must satisfy the following three equations:

$$A+B=1, \quad B+C=-2, \quad 4A+C=7.$$

These can be solved simultaneously (e.g. one can use Gaussian elimination as in Chapter 3) to obtain $A = 2$, $B = -1$, and $C = -1$. In other words, we have the PFD

$$\frac{x^2 - 2x + 7}{(x+1)(x^2+4)} = \frac{2}{x+1} - \frac{x+1}{x^2+4}.$$

**Example 3.** Find the PFD for

$$\frac{2x^3 + 4x^2 + 3}{(x+1)^2(x^2+4)}.$$

**Solution.** The denominator has already been factored into linear and irreducible quadratic factors, so the PFD takes the form

$$\frac{2x^3 + 4x^2 + 3}{(x+1)^2(x^2+4)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{B_1 x + C_1}{x^2+4}.$$

$$= \frac{(A_1 + B_1)x^3 + (A_1 + A_2 + 2B_1 + C_1)x^2 + (4A_1 + B_1 + 2C_1)x + 4A_1 + 4A_1 + C_1}{(x+1)^2(x^2+4)}$$

So we get the following system

$$\begin{aligned} A_1 + B_1 &= 2 \\ A_1 + A_2 + 2B_1 + C_1 &= 4 \\ 4A_1 + B_1 + 2C_1 &= 0 \\ 4A_1 + 4A_2 + C_1 &= 3. \end{aligned}$$

These may be solved simultaneously (for example, using Gaussian elimination) to find $A_1 = 0$, $A_2 = 1$, $B_1 = 2$, $C_1 = -1$. Hence the PFD is

$$\frac{2x^3 + 4x^2 + 3}{(x+1)^2(x^2+4)} = \frac{1}{(x+1)^2} + \frac{2x-1}{x^2+4}.$$

## Trick for Evaluating the Constants

After recombining the terms in our PFD over the common denominator, we want the resultant numerator to equal the numerator of the original rational function. But these are both functions of $x$, so the equality must hold for all $x$. In particular, we can choose convenient values of $x$ which simplify the calculation of the constants; this works especially well with linear factors. Let us redo Examples 1 & 2 using this method.

**Example 1 (revisited).** We wanted to find $A$ and $B$ so that

$$\frac{3x + 14}{x^2 + x - 6} = \frac{A(x+3) + B(x-2)}{(x-2)(x+3)}.$$

Since there are two constants to find, we choose two values of $x$; the obvious choices are $x = 2, -3$. If we plug $x = 2$ into both numerators, the term involving $B$ vanishes and we obtain $3(2) + 14 = A(5)$, i.e. $5A = 20$ or $A = 4$. Similarly, we can plug $x = -3$ into both numerators and obtain $3(-3)+14 = B(-5)$, which means $B = -1$. (These agree with the values found previously.) □

**Example 2 (revisited).** We wanted to find $A$, $B$, and $C$ so that

$$\frac{x^2-2x+7}{(x+1)(x^2+4)} = \frac{A(x^2+4)+(x+1)(Bx+C)}{(x+1)(x^2+4)}.$$

Since there are three constants, we choose three values of $x$. One obvious choice is $x=-1$, and evaluating both numerators yields $10 = 5A$, or $A = 2$. No other choices of $x$ will cause terms to vanish, so let us choose simple values like $x = 0, 1$:

$$\begin{aligned} x=0 \quad &\Rightarrow \quad 7 = 4A + C = 8 + C \quad \Rightarrow \quad C = -1 \\ x=1 \quad &\Rightarrow \quad 6 = 5A + 2(B+C) = 8 + 2B \quad \Rightarrow \quad B = -1, \end{aligned}$$

(These agree with the values found previously.) □