M731–Partial Differential Equations HOMEWORKS – Fall 2019

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SET 1: Due Thursday 10/03/2019

From McOwen's Book: read the introduction & Sections 1.1d) and 1.3

From McOwen's Book Section 1.1 do: 1, 2, 3, 4a)b), 6a) 9.

Additional Problems.

(1) Write down an explicit formula for a function u solving the inhomogeneous initial value problem

$$\begin{cases} u_t + b\cdot\nabla u = f & \text{on} \quad \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $f = f(x,t)$, $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are all given. (Hint. Solve first the homogeneous problem (i.e. $f = 0$) using the method of characteristics (as in class or section 1.1a)). Then use the Fundamental Theorem of Calculus to write $u(x,t) - g(x - bt)$ for (x,t) along a characteristic ($g(x - bt)$ is the homogeneous solution) as an appropriate integral of f. At the end u will be the sum of the homogeneous solution plus an integral term.)

(2) Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b\cdot\nabla u + c\,u = 0 & \text{on} \quad \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given.

(3) Let f be a continuous function on an open set $D \subset \mathbb{R}^n$ such that

$$\int_{D_0} f(x)\,dx = 0 \qquad \text{for all} \quad D_0 \subset D.$$

Prove that then $f \equiv 0$ on D.

SET 2. Due: Thursday 10/17/2019

From McOwen's Book Section 2.1: 1, 2, 7

From McOwen's Book Section 2.2: Read examples in 2.2a); read all of Section 2.2b). Then **do** 1.

Additional Problems.

1) a) Find the characteristics of the PDE

$$y^2u_{xx} - 2yu_{xy} + u_{yy} = u_x + 6y,$$

and determine if elliptic, parabolic or hyperbolic.

b) Then find the canonical form and use it to find the solution u first in the ξ and η variables and then in the x and y variables.

2) a) Find the characteristics of the PDE

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0, \qquad x > 0, y > 0,$$

and determine if elliptic, parabolic or hyperbolic.

b) Then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$$

for ξ and η are suitably chosen canonical coordinates. and use this to obtain the general solution in the ξ and η variables.

Robert McOwen's Book Section 3.1: 1, 4, 5

Additional Problems.

(1) (a)Show that the general solution to the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G

(b) Using a change of variables $\xi = x + t$ and $\eta = x - t$, show that

$$u_{tt} - u_{xx} = 0 \quad \text{if and only if} \quad u_{\xi\eta} = 0$$

(c) Use parts (a) and (b) to rederive D'Alembert's formula.

(2) Let $u \in C^2(\mathbb{R} \times [0,\infty))$ be a solution to the Cauchy initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u(0,x) = g(x) & \text{in} \quad \mathbb{R} \\ u_t(0,x) = h(x) & \text{in} \quad \mathbb{R} \end{cases}$$

Suppose that g and h are smooth and have compact support. The *kinetic energy* is

$$k(t) := \frac{1}{2}\int_{\mathbb{R}} u_t^2(t,x)\,dx$$

and the *potential energy* is

$$p(t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx$$

Prove :

(a) k(t) + p(t) is constant in t.

(b) k(t) = p(t) for all *large enough* times t.

SET 3 Due: Tuesday 10/29/2019

Read first Robert McOwen's Book Section 2.3. Then do problems: 4, 8, 10, 11c).

Additional Problems on Distributions.

(1). Prove the Remark at the bottom of page 10 of the Notes on Distributions. This is a generalization of Pb 8 in section 2.3.

A generalized version of this statement is the following problem: (how would you do it?)

(1') Let $\{f_j\} \in L^1(\mathbb{R}^n)$ be a sequence of nonnegative functions such that

$$\int_{\mathbb{R}^n} f_j(x)\,dx \to 1 \quad \text{as} \quad j\to\infty \quad \text{and for any } a>0,$$
$$\int_{|x|>a} f_j(x)\,dx \to 0 \quad \text{as} \quad j\to\infty.$$

Let F_{f_j} be the distribution defined by f_j. Prove that $F_{f_j} \to \delta$ in $\mathcal{D}'$.

(2) For δ the Dirac delta distribution, find the k-th (weak) derivative $\delta^{(k)}$ for any $k \geq 1$.

(3) Find the derivative in the sense of distributions of the sgn$x$, the function on $\mathbb{R}$ defined to be equal to 1 for $x>0$, equal to 0 for $x=0$ and equal to -1 for $x<0$.

(4) Compute $\frac{d}{dx}(\log|x|)$ on $\mathbb{R}$ in the sense of distributions.
Recall that the principal value of $\frac{1}{x}$ (pv $\frac{1}{x}$) is defined as

$$\langle \mathrm{pv}\frac{1}{x}, \phi\rangle = \mathrm{pv}\int_{\mathbb{R}} \frac{\phi(x)}{x}\,dx := \lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx$$

(5) Prove that the Dirac delta distribution δ_{x_0} with point mass at $x_0 \in \mathbb{R}^n$ (fixed) is not given by a locally integrable function. In other words prove that there does not exist any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\langle \delta_{x_0}, v\rangle = \langle f, v\rangle \qquad \text{for all} \quad v \in C_0^\infty(\mathbb{R}^n)$$

SET 4 Due: Thursday 11/07/2019

Robert McOwen's Book Section 3.2: 1, 2, 2*, 3, 5, 6a)

Hint for Problem 2 First show that if $y = (y_1, y_2, y_3)$ is a point in the unit sphere $\mathbb{S}^2$ then

$$\int_{\mathbb{S}^2} y_j \, d\sigma(y) \;=\; 0 \qquad\qquad j = 1, 2, 3$$

where as always $d\sigma$ is the area surface element. This can be proved by an explicit calculation using spherical coordinates or also by symmetry, splitting for each j, the integral over the sphere $\mathbb{S}^2$ into the two integrals for the half spheres $y_j \geq 0$ and $y_j \leq 0$ and showing the two integrals cancel out).

Problem 2* Same as problem 2 but with initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = x_2$.

Hint for Problem 5 Follow the hint in the back of the book and define $u(x_1, x_2, x_3) = \cos(\frac{m}{c}x_3)v(x_1, x_2, t)$. Then prove by a direct calculation u satisfies the $3d$ (linear homogeneous) wave equation. Hence u can be represented in terms of g and h using Kirchhoff's formula. Assume first for simplicity that $g = 0$ and write the explicit formula for u in this case. Then set $x_3 = 0$ and proceed as in the 'method of descent' (parametrize the two halves of $\mathbb{S}^2$ corresponding by graphs $y_3 = \pm\sqrt{1 - (y_1^2 + y_2^2)}$) to obtain a formula for v which should look like:

$$v(x_1, x_2, t) = C\,t \int_D \frac{\cos\left(mt\sqrt{1 - (y_1^2 + y_2^2)}\right) h(x_1 + cty_1, x_2 + cty_2)}{\sqrt{1 - (y_1^2 + y_2^2)}} \, dy_1 dy_2$$

where $D := \{(y_1, y_2) : \, y_1^2 + y_2^2 \leq 1\,\}$.

Finally drop the assumption that $g = 0$ to get the general formula.

Hint for Problem 6a) The decay in t in $3d$ is due to waves spreading out in space on expanding spheres $\partial B(x, ct)$ as $t \to \infty$. This bound reflects the *dispersion* of waves. Assume first $g = 0$ and use Kirchhoff's formula in conjunction with the fact that h is bounded (i.e. $|h(x)| \leq M$ for some $M > 0$) and compactly supported (hence $h(x) = 0$ if $|x| \geq R$ for some large $R > 0$). From these (plus change of variables) one can prove that

$$|u(x, t)| \leq \frac{M}{Ct}\text{area}(\partial B(x, ct) \cap B(0, R))$$

where $C > 0$ depends on the area of the unit sphere and the speed c. Next show that the area of the *spherical cap* $\partial B(x, ct) \cap B(0, R)$ can be bounded by some absolute constant times R^2- and hence by a quantity that is independent of x and t. Next note that if $h = 0$ instead then both terms in g in Kirchhoff's formula can be treated similarly as the previous case (do it!). Finally, for $g, h \in C_0^\infty(\mathbb{R}^3)$ we have the sum of 3 terms all of which we know how to treat.

Note related to part 6b). In $2d$ there is 'one less direction' than in $3d$ for waves to spread out so intuitively we expect the amplitude of the waves to decay slower as time increases.

And indeed, in $2d$ there is decay bound of the form $|u(x,t)| \leq \frac{C}{\sqrt{t}}$ but this is harder to prove. In $1d$, however, as we can clearly see from D'Alembert's formula there is no decay at all as $t \to \infty$.

Robert McOwen's Book Section 3.3: 1, 2, 4, 5.

Additional Problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $x \in \mathbb{R}^n$ be fixed. For $r > 0$ let

$$B_r(x) := \{\, y \in \mathbb{R}^n \,:\, |x-y| \,\leq\, r \,\}, \quad \text{and} \quad \partial B_r(x) := \{\, y \in \mathbb{R}^n \,:\, |x-y| \,=\, r \,\}.$$

a) Prove that

$$\frac{d}{dr} \int_{B_r(x)} f(y)\, dy \;=\; \int_{\partial B_r(x)} f(y)\, d\sigma(y)$$

where $d\sigma$ is the surface measure.
Hint. Use polar coordinates to write

$$\int_{B_r(x)} f(y)\, dy \;=\; \int_0^r \int_{S^{n-1}} f(x+\rho z)\, d\sigma(z) \rho^{n-1}\, d\rho$$

where $\mathbb{S}^{n-1}$ is the unite sphere in $\mathbb{R}^n$

b) Suppose now $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f = f(r,x)$, $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$ (fixed). Let

$$\phi(r) := \int_{B_r(x)} f(r,y)\, dy$$

and assume that f and $\partial_r f$ are continuous. Show then that

$$\frac{d}{dr}\phi(r) \;=\; \int_{B_r(x)} \partial_r f(r,y)\, dy \;+\; \int_{\partial B_r(x)} f(r,y)\, d\sigma(y).$$

Hint. Write

$$\frac{\phi(r+h) - \phi(r)}{h}$$

as

$$\int_{B_{r+h}(x)} \frac{f(r+h,y) - f(r,y)}{h}\, dy + \frac{1}{h}\left\{ \int_{B_{r+h}(x)} f(r,y) dy - \int_{B_r(x)} f(r,y)\, dy \right\}.$$

Then use the Dominated Convergence Theorem on the first term and part **a)** on the second term.

SET 5 Due: Thursday November 21, 2019

Robert Owen's Book Section 4.1: 1, 2, 3, 5, 6

Robert Owen's Book Section 4.2: 3, 4, 5, 6, 8, 10a)

SET 6 Due: Thursday December 5

Robert Owen's Book Section 4.1: 7, 9

Robert Owen's Book Section 4.2: 7, 11

Additional Problem: Let $\Omega \subset \mathbb{R}^n$, bounded open set and $g \in C^1(\partial\Omega)$. Find the PDE for $u \in C^2(\Omega)$ such that u is the minimizer over $\mathcal{A}$ of

$$E(v) = \int_\Omega \sqrt{1 + |\nabla v|^2}\, dx,$$

where

$$\mathcal{A} := \{v \in C^2(\Omega) \cap C^1(\partial\Omega) \ : \ v \equiv g \ \text{ on } \partial\Omega\}$$

SET 7 Due: Tuesday December 10th

Robert Owen's Book Section 5.1: 2, 6, 7

Robert Owen's Book Section 5.2: 1, 2, 3, 4, 11

Additional Problems. Compute (in the sense of distributions) the following Fourier transforms on $\mathbb{R}^n$:

(i) $\widehat{\delta_0}$ (ii) $\widehat{D^\alpha \delta_0}$ (iii) $\widehat{x^\alpha}$ (iv) $\widehat{H}$ where $H(x)$ is the Heaviside function.

EXTRA $-$Do these but do not turn in:

Robert Owen's Book Section 6.1: 5a), 6, 15, 16

Robert Owen's Book Section 6.2: 1, 2, 4, 6