**NAME:**

**ID #:**

# TAKE HOME FINAL MATH 534H

Tuesday May 7th, 2013

**Instructions.**

(1) **This exam consists of 4 problems with parts for a total of 100%.**

(2) **It is due no later than Thursday May 9th by noon in LGRT 1338**

(3) **You should work on it alone. You may consult Strauss' book, the class notes and your homework ONLY. No other material is allowed.**

(4) **Show all the work needed to reach your answer for full credit.**

(5) **You cannot discuss the problems with other people, including classmates.**

(6) **You may ask me any questions you have.**
(If via email use nahmod at math dot umass dot edu)

(7) **Write each problem and its solution following each problem and staple them all together with this cover. Insert additional pages if needed.**

**1**) Waves in medium where there is a transverse elastic force (as in a coiled sprig) satisfy the following type of wave equation equation

$$(1) \qquad \begin{cases} u_{tt} - u_{xx} + 5u = 0 & 0 < x < \pi \\ u(0,t) = 0 = u(\pi,t) & \\ u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x) & \end{cases}$$

where $\phi, \psi$ are smooth functions.

Use separation of variables and solve the corresponding eigenvalue problems associated to (1) to write down the sine-cosine series expansion of the solution $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$. Describe the coefficients in terms of the initial data $\phi(x)$ and $\psi(x)$.

**2**) Let $L$, $\alpha > 0$ and $\phi$ be a smooth function on $[0, L]$. Consider the boundary/initial value problem:

$$
(2) \qquad \begin{cases} u_t = ku_{xx} - \alpha u & 0 < x < L \\ u(x, 0) = \phi(x) & \\ u(0, t) = 0 = u(L, t) & \end{cases}
$$

(**a**) Find solutions $u(x)$ which are independent of time ($u_t = 0$) by solving,

$$
\begin{cases} ku_{xx} - \alpha u & = 0 \\ u(0) = u(L) & = 0 \end{cases}
$$

These solutions are the *equilibrium* solutions.

(**b**) Solve (2) using separation of variables. Make sure you analyze all the eigenvalues associated to the problem.

(**c**) Find the $\lim_{t \to 0+} u(t, x)$ where $u$ is the solution found in (b). Justify your answer and interpret the result in terms of your answer in (a).

**3**) Consider the differential operator

$$A = -\frac{d}{dx}(x^2 \frac{d}{dx})$$

on twice differentiable functions $f$ on the interval $[0, L]$ which satisfy Dirichlet boundary conditions $f(0) = f(L) = 0$. That is, for this functions $Af = -(x^2 f'x)'$. Consider the inner product $\langle f, g \rangle := \int_0^L f(x)\,\overline{g(x)}\,dx$

(**a**) Show that $A$ is positive: that is, $\langle Af, f \rangle \geq 0$ for all $f$ as above.

(**b**) Use (a) to prove that all eigenvalues of $A$ are real and positive

(**c**) Show that $A$ is symmetric; that is prove $\langle Af, g \rangle = \langle f, Ag \rangle$ for all function $f$ and $g$ satisfying the boundary conditions above.

(**d**) Use (c) to prove that all eigenvectors associated to different eigenvalues are orthogonal with respect to the inner product above.

**4**) (**a**) Consider $f(x) = x\,(\sin(x))^2$ on $[0, \pi]$. Find its associated Fourier <u>cosine</u> series and state whether such series converges pointwise for each $x \in [0, \pi]$ and what does it converge to. Explain and justify the reasoning that led you to your conclusions.

(**b**) Consider the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -1 \leq x < 0 \\ 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad 0 < x < 1 \end{cases}$$

Find its <u>full</u> Fourier series $\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\big(A_n \cos n\pi x + B_n \sin n\pi x\big)$ on $(-1,\, 1)$.

(**c**) Find a periodic extension of the function $f$ in part (**b**) . Explain.