M623 HOMEWORK Part II – Fall 2014

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SET 8 - DUE 12/11/14 (WITH EXAM)

Chapter 3 (pp 145-155): 11, 12, 15, 16, 22. Do also: $1, 4$ (from last time), 7.

Note: Please start working on the above. I will add a couple more on BV and absolute continuity soon.

SET 7 - DUE 11/25/14

Do the three problems I assigned in class last week plus:
From Chapter 2 (pp 88-97): 16, 21d)e), 24, 25.
From Chapter 3 (pp 146-Section 5): 4, 5.
From Chapter 3 (pp 152- Section 6): 1^*. This problem is based on a good understanding of Lemma 1.2 p.102 and of Lemma 3.9 p.128-definition of Vitali covering is just before the statement of Lemma 3.9.
$*$: do not turn in.

SET 6 - DUE 11/13/14

From Chapter 2 (pp 88-97): Read Cor. 3.7- 3.8, then do 7. Recall Invariance (p. 73). Read Prop 3.9 then do 21.

Additional Problems

I. Let s be a fixed positive number. Prove that

$$\int_0^\infty e^{-sx}\,\frac{\sin^2 x}{x}\,dx = \frac{1}{4}\log(1+4s^{-2})$$

by integrating $e^{-sx}\sin(2xy)$ with respect to $x \in (0,\infty)$, $y \in (0,1)$ and with respect to $y \in (0,1)$, $x \in (0,\infty)$. Justify all your steps. (**Hints.** $\cos(2\theta) = 1 - 2\sin^2\theta$. In order to do one of the integrations, either integrate by parts twice or use the definition of the appropriate trigonometric function in terms of complex exponentials.)

II. Consider the function $f(x,y) := e^{-xy} - 2e^{-2xy}$ where $x \in [0,\infty)$ and $y \in [0,1]$.
i) Prove that for a.e. $y \in [0,1]$ f^y is integrable on $[0,\infty)$ with respect to $m_{\mathbb{R}}$.
ii) Prove that for a.e. $x \in [0,\infty)$ f^x is integrable on $[0,1]$ with respect to $m_{\mathbb{R}}$.
iii) Use Fubini to prove that $f(x,y)$ is not integrable on $[0,\infty)\times[0,1]$ with respect to $m_{\mathbb{R}^2}$.

SET 5 – Due Thursday 11/13/14 (from before Midterm. Please turn in)

From S-K's Book, do problems: 8 (p. 39), 13a) (p 41), 17 (p 42), 19 (p 42), 20 (p42-43), 30 (p 44).

Additional Problems:

I. Consider the sequence of functions $f_n(x) := \frac{n}{1+(nx)^2}$. For $a \in \mathbb{R}$ be a fixed number consider the Lebesgue integral $I_a(f_n)(x) := \int_a^\infty f_n(x)\, dm$. Compute $\lim_{n\to\infty} I_a(f_n)(x)$ in each case: i) $a = 0$ ii) $a > 0$ and iii) $a < 0$. Carefully justify your calculations (recall the transformation of integrals under dilations).

II. Use the DCT to prove the following: let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions on $\mathbb{R}^d$ such that $\sum_{n=1}^\infty \int |f_n(x)|\, dm < \infty$. Show that $\sum_{n=1}^\infty f_n(x)$ *converges* a.e. $x \in \mathbb{R}^d$ to an *integrable function* and that $\sum_{n=1}^\infty \int f_n(x)\, dm = \int \sum_{n=1}^\infty f_n(x) dm$.

III. Let $0 < c < d$ be fixed real numbers and for $n \geq 1$ consider the sequence function $f_n(x) = ce^{-cnx} - de^{-dnx}$, where $x \geq 0$. Prove that:

(1) $\sum_{n=1}^\infty \int |f_n(x)|\, dm$ diverges
(2) $\sum_{n=1}^\infty \int f_n(x)\, dm = 0$
(3) $\sum_{n=1}^\infty f_n(x)$ is an integrable function on $[0,\infty)$ and that

$$\int_{[0,\infty)} \sum_{n=1}^\infty f_n(x)\, dm = \ln(\frac{d}{c}).$$

IV. Consider the function on $\mathbb{R} \times \mathbb{R}$ given by

$$f(x,y) = ye^{-(x^2+1)y^2} \quad \text{if } x \geq 0, y \geq 0 \quad \text{and} \quad 0 \quad \text{otherwise .}$$

a) Integrate $f(x,y)$ over $\mathbb{R} \times \mathbb{R}$ (justify your steps carefully)
b) Use a) to prove that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.
c) Use b) and dilation to prove that $\int_{-\infty}^\infty e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$.

V. Let $f(x)$ be a measurable function over $\mathbb{R}^{d_1}$ and $g(y)$ be a measurable function over $\mathbb{R}^{d_2}$. Prove that $F(x,y) = f(x)g(y)$ is a measurable function over $\mathbb{R}^{d_1+d_2}$.