M534H HOMEWORKS– Spring 2013

Prof. Andrea R. Nahmod

• SET 1. Due date: Thursday February 7th

Section 1.1: 2, 3, 4, 11

Section 1.2: 1, 2, 3, 4, 5.

Section 1.3: 7, 9, 10 (in $\mathbb{R}^3$), 11.

Additional Problem:

Let $u = u(\mathbf{x}, t)$ be a solution to the wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^2$. Assuming that $\nabla u \to 0$ fast enough as $|\mathbf{x}| \to \infty$ prove that

$$E(t) = \int_{\mathbb{R}^2} |u_t|^2 + |\nabla u|^2 \, dxdy$$

is constant in t for all time t.

• SET 2. Due date: Thursday February 14th

Section 1.4: 1

Section 1.5: 1, 4 and then do 5, 6 as modified below.

Problem 5: Consider the equation

$$u_x + y u_y = 0$$

with boundary condition $u(x, 0) = \phi(x)$.
(a) For $\phi(x) = x$, for all x, show that no solution exists.
(b) For $\phi(x) = 1$ for all x, show that there are many solutions.

Problem 6: Solve the equation $u_x + 2x\, y^2\, u_y = 0$ and find a solution that satisfies the auxiliary condition $u(0, y) = y$.

Additional Problem:
Find the general solution of $u_x - \sin(x)\, u_y = 0$. Then find the special solution that satisfies $u(0, e^y) = e^y$.

• Set 3. Due date: Thursday February 21st

Section 1.6: 1, 2, 4

Additional Problems.

Problem A.1. Use the method of characteristics to find the solution $u = u(x, y)$ of the semi-linear first order PDE $x^2 u_x + xyu_y = u^2$ passing through the curve $u(y^2, y) = 1$. Determine the set of $(x, y) \in \mathbb{R}^2$ where this solution is defined.

Problem A.2. Use the method of characteristics to find the solution $u = u(x, y)$ of $u_x + x^2 yu_y = -u$ passing through the curve $u(0, y) = y^2$. Determine the set of $(x, y) \in \mathbb{R}^2$ where this solution is defined.

• Set 4. Due date: Thursday February 28th

Section 2.1: 1, 2, 5, 8, 9, 11

Problem 11. Find the general solution of

$$3u_{tt} + 10u_{xt} + 3u_{xx} \;=\; \sin(x + t)$$

Section 2.2: 1, 2, 3. For 1) use the Additional Problem you proved in Set 1.

Additional Problems.

Problem A.1. Find the solution to wave equation with initial conditions $u(x, 0) = \sin x$, $u_t(x, 0) = 0$. Calculate then $u_t(0, t)$.

Problem A.2. Solve the PDE $u_{xx} + u_{xt} - 10u_{tt} \;=\; 0$

• Set 5. Due date: Thursday March 7th

Section 2.2- Additional Problems

Problem A.1. Consider the wave equation in 1D with *damping*

$$u_{tt} = c^2\, u_{xx} - ku - ru_t \qquad k, r > 0$$

show that the *energy functional*

$$E(t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty} |u_t|^2 + c^2|u_x|^2 + k|u|^2\, dx$$

satisfies $dE/dt \leq 0$; that is *energy decreases*. Assume u and its derivatives vanish as $x \to \pm\infty$.

Problem A.2. Prove finite propagation speed for the 1D wave equation. (Hint use Theorem 3 in Appendix A.3 of Strauss's book.) In other words, prove that if the initial conditions $u(0,x) = \phi(x)$ and $u_t(0,x) = \psi(x)$ are zero on the interval $[x_0 - ct_0,\, x_0 + ct_0]$ (for some (t_0, x_0) fixed) then $u(t,x)$ is zero for all (x,t) in $\mathcal{C}$, the domain of dependence of (t_0, x_0).

Hints. Think of $\mathcal{C}$ as the union of segments $I_t := [x_0 - c(t_0 - t),\, x_0 + c(t_0 - t)]$ (at height t) for $0 \le t \le t_0$ and first show that the *local energy* defined by

$$\mathcal{E}(u)(t) := \frac{1}{2}\int_{I_t} |u_t(t,x)|^2 + |u_x(t,x)|^2\, dx$$

is decreasing in t. Then argue as in class to show that then $u(t,x)$ is zero for $(x,t) \in I_t$.

• Set 6. Due date: Thursday March 14th

Section 2.2- More Additional Problems

Problem A.1. Consider the wave equation in 1D with *damping* of the last set, i.e.

$$u_{tt} = c^2\, u_{xx} - ku - ru_t \qquad k, r > 0$$

and show that solutions to the initial value problem with initial conditions $u(0,x) = \phi(x)$ and $u_t(0,x) = \psi(x)$ are unique.

Problem A.2 Consider the Klein-Gordon (KG) equation in 1D

$$u_{tt} - c^2\, u_{xx} + k\, u \,=\, 0 \qquad k > 0$$

and show the following:

(1) The energy associated to KG:

$$E(t) \,=\, \frac{1}{2}\int_{-\infty}^{\infty} |u_t|^2 + c^2 |u_x|^2 + k|u|^2\, dx$$

is *conserved*. (Notice the difference when there is no damping from the ru_t term.)

(2) Use the above to show that solutions to initial value problem for the KG equation with initial conditions $u(0,x) = \phi(x)$ and $u_t(0,x) = \psi(x)$ are unique.

.

Section 2.3: 2, 4, 5, 6, 7.

Section 2.4: 1, 2, 7, 8, 14, 15 16, 18.

Section 2.5: 1

- **SET 7. Due date: Thursday April 4th**

<u>Section 3.1</u>: 1, 2, 3

- **SET 8. NEW Due date: Tuesday April 23rd**

<u>Section 3.2</u>: 1, 3, 4, 5
<u>Section 3.4</u>: 1, 3

<u>Section 4.1</u>: 2, 3

<u>Additional Problems</u> Separate variables to solve the following problems.

A1. $u_{tt} - u_{xx} = 0$ in $0 < x < 3$ with boundary conditions $u(0,t) = u(3,t) = 0$

A2. $u_t = u_{xx} + u$ in $0 < x < \pi$ with boundary conditions $u(0,t) = u(\pi,t) = 0$

- **Set 9. NEW Due date: Tuesday April 23rd**

<u>Section 4.2</u>: 1, 2, 4

<u>Section 4.3</u> 2, 9, 11

<u>Additional Problem</u> Separate variables to solve the following problems.

A1. $u_t = u_{xx}$ in $0 < x < L$ with boundary conditions $u_x(0,t) = u(L,t) = 0$

<u>Section 5.1</u> 3, 5, 6, 7, 9.

- **Set 10. NEW Due date: Tuesday April 30th**

<u>Section 5.2</u> 3, 4, 7, 9, 10, 11, 17.

<u>Section 5.3</u> 1, 2a)b), 3, 4a)b), 5a), 6, 10.

<u>Section 6.1</u> 2, 4, 5 (use $a = 1$), 6 (use $a = 1$ and $b = 2$), 10.

- **Set 11. Do but do not turn in**

<u>Section 5.4</u> 1, 5, 6, 7 , 8a).

<u>Section 6.2</u> 1, 2.