e. The dominated convergence theorem: If $\{f_n\}$ is a sequence in $L^1_{\mathcal{Y}}$ such that $f_n \to f$ a.e., and there exists $g \in L^1$ such that $\|f_n(x)\| \leq g(x)$ for all n and a.e. x, then $\int f_n \to \int f$.
f. If $\mathcal{Z}$ is a separable Banach space, $T \in L(\mathcal{Y}, \mathcal{Z})$, and $f \in L^1_{\mathcal{Y}}$, then $T \circ f \in L^1_{\mathcal{Z}}$ and $\int T \circ f = T(\int f)$.

5.2 LINEAR FUNCTIONALS

Let $\mathcal{X}$ be a vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$. A linear map from $\mathcal{X}$ to K is called a **linear functional** on $\mathcal{X}$. If $\mathcal{X}$ is a normed vector space, the space $L(\mathcal{X}, K)$ of bounded linear functionals on $\mathcal{X}$ is called the **dual space** of $\mathcal{X}$ and is denoted by $\mathcal{X}^*$. According to Proposition 5.4, $\mathcal{X}^*$ is a Banach space with the operator norm.

If $\mathcal{X}$ is a vector space over $\mathbb{C}$, it is also a vector space over $\mathbb{R}$, and we can consider both real and complex linear functionals on $\mathcal{X}$, that is, maps $f : \mathcal{X} \to \mathbb{R}$ that are linear over $\mathbb{R}$ and maps $f : \mathcal{X} \to \mathbb{C}$ that are linear over $\mathbb{C}$. The relationship between the two is as follows:

5.5 Proposition. *Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. If f is a complex linear functional on $\mathcal{X}$ and $u = \operatorname{Re} f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in \mathcal{X}$. Conversely, if u is a real linear functional on $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. In this case, if $\mathcal{X}$ is normed, we have $\|u\| = \|f\|$.*

Proof. If f is complex linear and $u = \operatorname{Re} f$, u is clearly real linear and $\operatorname{Im} f(x) = -\operatorname{Re}[if(x)] = -u(ix)$, so $f(x) = u(x) - iu(ix)$. On the other hand, if u is real linear and $f(x) = u(x) - iu(ix)$, then f is clearly linear over $\mathbb{R}$, and $f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = if(x)$, so f is also linear over $\mathbb{C}$. Finally, if $\mathcal{X}$ is normed, since $|u(x)| = |\operatorname{Re} f(x)| \leq |f(x)|$ we have $\|u\| \leq \|f\|$. On the other hand, if $f(x) \neq 0$, let $\alpha = \overline{\operatorname{sgn} f(x)}$. Then $|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x)$ (since $f(\alpha x)$ is real), so $|f(x)| \leq \|u\| \, \|\alpha x\| = \|u\| \, \|x\|$, whence $\|f\| \leq \|u\|$. ∎

It is not obvious that there are any nonzero bounded linear functionals on an arbitrary normed vector space. The fact that such functionals exist in great abundance is one of the fundamental theorems of functional analysis. We shall now present this result in a more general form that has other important applications.

If $\mathcal{X}$ is a real vector space, a **sublinear functional** on $\mathcal{X}$ is a map $p : \mathcal{X} \to \mathbb{R}$ such that

$$p(x+y) \leq p(x) + p(y) \text{ and } p(\lambda x) = \lambda p(x) \text{ for all } x, y \in \mathcal{X} \text{ and } \lambda \geq 0.$$

For example, every seminorm is a sublinear functional.

5.6 The Hahn-Banach Theorem. *Let $\mathcal{X}$ be a real vector space, p a sublinear functional on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{X}$ such that $F(x) \leq p(x)$ for all $x \in \mathcal{X}$ and $F|\mathcal{M} = f$.*

Proof. We begin by showing that if $x \in \mathcal{X} \setminus \mathcal{M}$, f can be extended to a linear functional g on $\mathcal{M} + \mathbb{R}x$ satisfying $g(y) \le p(y)$ there. If $y_1, y_2 \in \mathcal{M}$, we have

$$f(y_1) + f(y_2) = f(y_1 + y_2) \le p(y_1 + y_2) \le p(y_1 - x) + p(x + y_2),$$

or

$$f(y_1) - p(y_1 - x) \le p(x + y_2) - f(y_2).$$

Hence

$$\sup\{f(y) - p(y - x) : y \in \mathcal{M}\} \le \inf\{p(x + y) - f(y) : y \in \mathcal{M}\}.$$

Let α be any number satisfying

$$\sup\{f(y) - p(y - x) : y \in \mathcal{M}\} \le \alpha \le \inf\{p(x + y) - f(y) : y \in \mathcal{M}\}$$

and define $g : \mathcal{M} + \mathbb{R}x \to \mathbb{R}$ by $g(y + \lambda x) = f(y) + \lambda\alpha$. Then g is clearly linear, and $g|\mathcal{M} = f$, so that $g(y) \le p(y)$ for $y \in \mathcal{M}$. Moreover, if $\lambda > 0$ and $y \in \mathcal{M}$,

$$g(y + \lambda x) = \lambda[f(y/\lambda) + \alpha] \le \lambda[f(y/\lambda) + p(x + (y/\lambda)) - f(y/\lambda)] = p(y + \lambda x),$$

whereas if $\lambda = -\mu < 0$,

$$g(y + \lambda x) = \mu[f(y/\mu) - \alpha] \le \mu[f(y/\mu) - f(y/\mu) + p((y/\mu) - x)] = p(y + \lambda x).$$

Thus $g(z) \le p(z)$ for all $z \in \mathcal{M} + \mathbb{R}x$.

Evidently the same reasoning can be applied to any linear extension F of f satisfying $F \le p$ on its domain, and it shows that the domain of a maximal linear extension satisfying $F \le p$ must be the whole space $\mathcal{X}$. But the family $\mathcal{F}$ of all linear extensions F of f satisfying $F \le p$ is partially ordered by inclusion (maps from subspaces of $\mathcal{X}$ to $\mathbb{R}$ being regarded as subsets of $\mathcal{X} \times \mathbb{R}$). Since the union of any increasing family of subspaces of $\mathcal{X}$ is again a subspace, one easily sees that the union of a linearly ordered subfamily of $\mathcal{F}$ lies in $\mathcal{F}$. The proof is therefore completed by invoking Zorn's lemma. ∎

If p is a seminorm and $f : \mathcal{X} \to \mathbb{R}$ is linear, the inequality $f \le p$ is equivalent to the inequality $|f| \le p$, because $|f(x)| = \pm f(x) = f(\pm x)$ and $p(-x) = p(x)$. In this situation the Hahn-Banach theorem also applies to complex linear functionals:

5.7 The Complex Hahn-Banach Theorem. *Let $\mathcal{X}$ be a complex vector space, p a seminorm on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a complex linear functional on $\mathcal{M}$ such that $|f(x)| \le p(x)$ for $x \in \mathcal{M}$. Then there exists a complex linear functional F on $\mathcal{X}$ such that $|F(x)| \le p(x)$ for all $x \in \mathcal{X}$ and $F|\mathcal{M} = f$.*

Proof. Let $u = \operatorname{Re} f$. By Theorem 5.6 there is a real linear extension U of u to $\mathcal{X}$ such that $|U(x)| \le p(x)$ for all $x \in \mathcal{X}$. Let $F(x) = U(x) - iU(ix)$ as in Proposition 5.5. Then F is a complex linear extension of f, and as in the proof of Proposition 5.5, if $\alpha = \overline{\operatorname{sgn} F(x)}$ we have $|F(x)| = \alpha F(x) = F(\alpha x) = U(\alpha x) \le p(\alpha x) = p(x)$. ∎

From now on until §5.5, all of our results apply equally to real or complex vector spaces, but for the sake of definiteness we shall assume that the scalar field is $\mathbb{C}$. The principal applications of the Hahn-Banach theorem to normed vector spaces are summarized in the following theorem.

5.8 Theorem. *Let $\mathcal{X}$ be a normed vector space.*

a. *If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{M}$, there exists $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $f|\mathcal{M} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.*
b. *If $x \neq 0 \in \mathcal{X}$, there exists $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*
c. *The bounded linear functionals on $\mathcal{X}$ separate points.*
d. *If $x \in \mathcal{X}$, define $\widehat{x} : \mathcal{X}^* \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$. Then the map $x \mapsto \widehat{x}$ is a linear isometry from $\mathcal{X}$ into $\mathcal{X}^{**}$ (the dual of $\mathcal{X}^*$).*

Proof. To prove (a), define f on $\mathcal{M} + \mathbb{C}x$ by $f(y + \lambda x) = \lambda\delta$ ($y \in \mathcal{M}$, $\lambda \in \mathbb{C}$). Then $f(x) = \delta$, $f|\mathcal{M} = 0$, and for $\lambda \neq 0$, $|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \, \|\lambda^{-1}y + x\| = \|y + \lambda x\|$. Thus the Hahn-Banach theorem can be applied, with $p(x) = \|x\|$ and $\mathcal{M}$ replaced by $\mathcal{M} + \mathbb{C}x$. (b) is the special case of (a) with $\mathcal{M} = \{0\}$, and (c) follows immediately: if $x \neq y$, there exists $f \in \mathcal{X}^*$ with $f(x - y) \neq 0$, i.e., $f(x) \neq f(y)$. As for (d), obviously $\widehat{x}$ is a linear functional on $\mathcal{X}^*$ and the map $x \mapsto \widehat{x}$ is linear. Moroever, $|\widehat{x}(f)| = |f(x)| \leq \|f\| \, \|x\|$, so $\|\widehat{x}\| \leq \|x\|$. On the other hand, (b) implies that $\|\widehat{x}\| \geq \|x\|$. ∎

With notation as in Theorem 5.8d, let $\widehat{\mathcal{X}} = \{\widehat{x} : x \in \mathcal{X}\}$. Since $\mathcal{X}^{**}$ is always complete, the closure $\overline{\widehat{\mathcal{X}}}$ of $\widehat{\mathcal{X}}$ in $\mathcal{X}^{**}$ is a Banach space, and the map $x \mapsto \widehat{x}$ embeds $\mathcal{X}$ into $\overline{\widehat{\mathcal{X}}}$ as a dense subspace. $\overline{\widehat{\mathcal{X}}}$ is called the **completion** of $\mathcal{X}$. In particular, if $\mathcal{X}$ is itself a Banach space then $\overline{\widehat{\mathcal{X}}} = \widehat{\mathcal{X}}$.

If $\mathcal{X}$ is finite-dimensional, then of course $\widehat{\mathcal{X}} = \mathcal{X}^{**}$, since these spaces have the same dimension. For infinite-dimensional Banach spaces it may or may not happen that $\widehat{\mathcal{X}} = \mathcal{X}^{**}$; if it does, $\mathcal{X}$ is called **reflexive**. The examples of Banach spaces we have examined so far are not reflexive except in trivial cases where they turn out to be finite-dimensional. We shall prove some cases of this assertion and present examples of reflexive Banach spaces in later sections.

Usually we shall identify $\widehat{x}$ with x and thus regard $\mathcal{X}^{**}$ as a superspace of $\mathcal{X}$; reflexivity then means that $\mathcal{X}^{**} = \mathcal{X}$.

Exercises

17. A linear functional f on a normed vector space $\mathcal{X}$ is bounded iff $f^{-1}(\{0\})$ is closed. (Use Exercise 12b.)

18. Let $\mathcal{X}$ be a normed vector space.

a. If $\mathcal{M}$ is a closed subspace and $x \in \mathcal{X} \setminus \mathcal{M}$ then $\mathcal{M} + \mathbb{C}x$ is closed. (Use Theorem 5.8a.)

b. Every finite-dimensional subspace of $\mathcal{X}$ is closed.

19. Let $\mathcal{X}$ be an infinite-dimensional normed vector space.

a. There is a sequence $\{x_j\}$ in $\mathcal{X}$ such that $\|x_j\| = 1$ for all j and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$. (Construct x_j inductively, using Exercises 12b and 18.)

b. $\mathcal{X}$ is not locally compact.

20. If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$.

21. If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, define $\alpha : \mathcal{X}^* \times \mathcal{Y}^* \to (\mathcal{X} \times \mathcal{Y})^*$ by $\alpha(f,g)(x,y) = f(x) + g(y)$. Then α is an isomorphism which is isometric if we use the norm $\|(x,y)\| = \max(\|x\|, \|y\|)$ on $\mathcal{X} \times \mathcal{Y}$, the corresponding operator norm on $(\mathcal{X} \times \mathcal{Y})^*$, and the norm $\|(f,g)\| = \|f\| + \|g\|$ on $\mathcal{X}^* \times \mathcal{Y}^*$.

22. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a. Define $T^\dagger : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the **adjoint** or **transpose** of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}|\mathcal{X} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

23. Suppose that $\mathcal{X}$ is a Banach space. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{X}^*$, let $\mathcal{M}^0 = \{f \in \mathcal{X}^* : f|\mathcal{M} = 0\}$ and $\mathcal{N}^\perp = \{x \in \mathcal{X} : f(x) = 0$ for all $f \in \mathcal{N}\}$. (Thus, if we identify $\mathcal{X}$ with its image in $\mathcal{X}^{**}$, $\mathcal{N}^\perp = \mathcal{N}^0 \cap \mathcal{X}$.)

a. $\mathcal{M}^0$ and $\mathcal{N}^\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively.

b. $(\mathcal{M}^0)^\perp = \mathcal{M}$ and $(\mathcal{N}^\perp)^0 \supset \mathcal{N}$. If $\mathcal{X}$ is reflexive, $(\mathcal{N}^\perp)^0 = \mathcal{N}$.

c. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural projection, and define $\alpha : (\mathcal{X}/\mathcal{M})^* \to \mathcal{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathcal{X}/\mathcal{M})^*$ onto $\mathcal{M}^0$, where $\mathcal{X}/\mathcal{M}$ has the quotient norm.

d. Define $\beta : \mathcal{X}^* \to \mathcal{M}^*$ by $\beta(f) = f|\mathcal{M}$; then β induces a map $\overline{\beta} : \mathcal{X}^*/\mathcal{M}^0 \to \mathcal{M}^*$ as in Exercise 15, and $\overline{\beta}$ is an isometric isomorphism.

24. Suppose that $\mathcal{X}$ is a Banach space.

a. Let $\widehat{\mathcal{X}}$, $(\mathcal{X}^*)\widehat{}$ be the natural images of $\mathcal{X}$, $\mathcal{X}^*$ in $\mathcal{X}^{**}$, $\mathcal{X}^{***}$, and let $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{\mathcal{X}} = 0\}$. Then $(\mathcal{X}^*)\widehat{} \cap \widehat{\mathcal{X}}^0 = \{0\}$ and $(\mathcal{X}^*)\widehat{} + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

b. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.

25. If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.) *Note:* Separability of $\mathcal{X}$ does not imply separability of $\mathcal{X}^*$.

26. Let $\mathcal{X}$ be a real vector space and let P be a subset of $\mathcal{X}$ such that (i) if $x, y \in P$, then $x + y \in P$, (ii) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$, (iii) if $x \in P$ and $-x \in P$,