ABSOLUTE CONTINUITY OF GAUSSIAN MEASURES UNDER CERTAIN GAUGE TRANSFORMATIONS

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ABSTRACT. We prove absolute continuity of Gaussian measures associated to complex Brownian bridges under certain gauge transformations. As an application we prove that the invariant measure obtained by Nahmod, Oh, Rey-Bellet and Staffilani in [20], and with respect to which they proved that the periodic derivative nonlinear Schrödinger equation Cauchy initial value problem is almost surely globally well-posed, coincides with the weighted Wiener measure constructed by Thomann and Tzvetkov [24]. Thus, in particular we prove the invariance of the measure constructed in [24].

1. INTRODUCTION

This note is a continuation of the paper [20]. There we construct an invariant weighted Wiener measure associated to the periodic derivative nonlinear Schrödinger equation (DNLS) (2.1) in one dimension and establish global well-posedness for data living in its support. In particular almost surely for data in a certain Fourier-Lebesgue space scaling like $H^{\frac{1}{2}-\epsilon}(\mathbb{T})$, for small $\epsilon > 0$. Roughly speaking, this is achieved by introducing a gauge transformation G (2.12) and considering the *gauged* DNLS equation (GDNLS) (2.13) in order to obtain the necessary estimates. We then constructed an invariant weighted Wiener measure μ, which we proved to be invariant under the flow of the GDNLS, and used it to show the almost surely global well-posedness for the GDNLS initial value problem. To go back to the original DNLS equation we applied the inverse gauge transformation G^{-1} and obtained a new invariant measure $\mu \circ G =: \gamma$ with respect to which almost surely global well-posedness was then proved for the DNLS Cauchy initial value problem[5]. On the other hand, Thomann and Tzvetkov [24] constructed a weighted Wiener measure ν for the DNLS equation directly but its invariance was left unanswered. A natural question then is the absolute continuity of the two measures γ and ν or equivalently, the absolutely continuity of μ and $\nu \circ G^{-1}$, which was left open in [20]. As we will show below this question can be easily answered after one understands the absolute continuity between Gaussian measures naturally associated with complex Brownian motion and their images under certain gauge transformations. This is in fact the heart of the matter of this note. At the end we indeed prove that $\mu = \nu \circ G^{-1}$ (or equivalently that $\gamma = \nu$) thus in particular establishing the invariance of the measure ν constructed in [24]. Furthermore, we prove

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[5] In [20] $\mu \circ G$ is called ν; here we relabel it γ to a priori distinguish it from the name we give to the one constructed in [24].

as a consequence that Theorem 2.1 holds. Our results follow by combining the results on global well-posedness and invariant measure for GDNLS (2.13) obtained by Nahmod, Oh, Rey-Bellet and Staffilani in [20] with the explicit computation of the image of the measure under the gauge transformation. The key to understand the latter is to actually understand how the Gaussian part of the measure changes under the gauge since the transformation of the weight is computed easily (see subsection 2.1 below). This is achieved in Theorem 2.1 in Section 2 of this paper.

Certainly there is a vast literature on the topic of Gaussian measures under nonlinear transformations [5, 22, 18, 3, 7, 15, 4] as well as [1] and other references therein. But as we will show below the nature of the transformations that we want to study does not fit in the context of these works and a different approach needs to be introduced. Since for many nonlinear partial differential equations gauge transformations are an essential tool to convert one kind of nonlinearity into another one, where resonant interactions are more manageable and hence estimates can be proved, we expect the general nature of the central theorem of this note, Theorem 3.1, as well as some of the ideas behind its proof, to be applicable in other situations as well and not just the in the DNLS context.

2. Invariance of weighted Wiener measure for DNLS

As stated in the introduction our motivation arises from the recent paper by Nahmod, Oh, Rey-Bellet, and Staffilani [20] we recall now the set up of that paper and formulate the problem that we want to solve here in that context. We consider the derivative nonlinear Schrödinger equation (DNLS) on the circle $\mathbb{T}$, i.e.,

$$\begin{cases} u_t(x,t) \, - i \, u_{xx}(x,t) \; = \; \lambda \left(|u|^2(x,t) u(x,t) \right)_x \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where $x \in \mathbb{T}$ and $t \in \mathbb{R}$. Our goal is to show that this problem defines a dynamical system, in the sense of ergodic theory. Let us denote by $\Psi(t)$ the flow map associated to our nonlinear equation, i.e., the solution of (2.1), whenever it exists, is given by $u(x,t) = \Psi(t)(u_0(x))$. Let further $(\mathcal{B}, \mathcal{F}, \nu)$ be a probability space where $\mathcal{B}$ is a space (here $\mathcal{B}$ will be a separable Banach space), $\mathcal{F}$ is a σ-algebra (here it will always be the Borel σ-algebra) and ν is a probability measure. The flow map $\Psi(t)$ define a dynamical system on the probability space $(\mathcal{B}, \mathcal{F}, \nu)$ if

(a) (ν-almost sure wellposedness.) There exists a subset $\Omega \subset \mathcal{B}$ with $\nu(\Omega) = 1$ such that the flow map $\Psi(t) : \Omega \to \Omega$ is well defined and continuous in t for all $t \in \mathbb{R}$.

(b) (Invariance of the measure ν.) The measure ν is invariant under the flow $\Phi(t)$, i.e.,

$$\int f \left(\Psi(t)(u) \right) \, d\nu(u) \; = \; \int f(u) d\nu(u).$$

for all $f \in L^1(\mathcal{B}, \mathcal{F}, \nu)$ and all $t \in \mathbb{R}$.

The measure ν here, in a sense, is a substitute for a conserved quantity and in fact ν is constructed by using a certain conserved quantity. Recall that the DNLS equation (2.1) is completely integrable (c.f. [16, 12]) and among the conserved quantities are

$$\text{Mass:} \quad m(u) = \frac{1}{2\pi}\int |u|^2\,dx. \tag{2.2}$$

$$\text{Energy:} \quad E(u) = \int |u_x|^2\,dx + \frac{3}{2}\mathrm{Im}\int u^2\overline{u}\,\overline{u_x}\,dx + \frac{1}{2}\int |u|^6\,dx. \tag{2.3}$$

$$\text{Hamiltonian:} \quad H(u) = \mathrm{Im}\int u\overline{u}_x\,dx + \frac{1}{2}\int |u|^4\,dx. \tag{2.4}$$

We consider a probability measure ν which is based on the conserved quantity $E(u)$ (as well as the mass $m(u)$). Let us decompose $u = a + ib$ into real and imaginary part, and let us consider first the purely formal but very suggestive expression for ν.

$$d\nu = C^{-1}\chi_{\{\|u\|_{L^2}\le B\}}e^{-\frac{\beta}{2}N(u)}e^{-\frac{\beta}{2}\int(|u|^2+|u_x|^2)dx}\prod_{x\in\mathbb{T}}da(x)db(x). \tag{2.5}$$

where

$$N(u) = \frac{3}{2}\mathrm{Im}\int u^2\overline{u}\,\overline{u_x}\,dx + \frac{1}{2}\int |u|^6\,dx\,, \tag{2.6}$$

is the non-quadratic part of the energy $E(u)$. Note that we have added the conserved quantity $\int |u|^2 dx$ to the quadratic part of $E(u)$ such as to make it positive definite. The constant $\beta > 0$ does not play any particular role here and, for simplicity, we choose $\beta = 1$. Note however that all the measures for different β will all be invariant under the flow and they are all mutually singular. The cutoff on the L^2-norm is necessary to make the measure normalizable since $N(u)$ is not bounded below. We will also see that this measure is well-defined only for B under a certain critical value B^*.

The expression in (2.5) at this stage is purely formal since there is no Lebesgue measure in infinite dimensions. In order to give a rigorous definition of the measure ν in (2.5) one needs to:

(a) make sense of the Gaussian part of the measure (2.5); that is make sense of the formal expression

$$d\rho = C'^{-1}e^{-\frac{1}{2}\int(|u|^2+|u_x|^2)dx}\prod_{x\in\mathbb{T}}da(x)db(x). \tag{2.7}$$

(b) construct the measure ν as a measure absolutely continuous with respect to ρ with Radon-Nikodym derivative

$$\frac{d\nu}{d\rho}(u) = \mathcal{Z}^{-1}\chi_{\{\|u\|_{L^2}\le B\}}e^{-\frac{1}{2}N(u)}\,, \tag{2.8}$$

i.e., one needs to show that

$$\mathcal{Z} = \int \chi_{\{\|u\|_{L^2}\le B\}}e^{-\frac{1}{2}N(u)}d\rho < \infty\,. \tag{2.9}$$

Part (b) goes back to the works of Lebowitz, Rose and Speer [19] and of Bourgain [2] for the term $\int |u|^6 dx$ part while the integrability of the term involving $\int u^2\bar{u}\bar{u}_x\,dx$ -and hence the construction of ν- is proved in [24]; see also Section 5 in [20]. Both terms are critical in the sense that integrability requires that B does not exceed a certain critical value B^*.

Part (a) is a standard problem in Gaussian measures, treated for example by Kuo [17] and Gross [8]; see also in [20] for details. Indeed the measure ρ can be realized as an honest countably additive Gaussian measure on various Hilbert or Banach spaces depending on one's particular needs. For example one can construct ρ as the weak limit of the finite-dimensional Gaussian measures

$$d\rho_N = \mathcal{Z}_{0,N}^{-1} \exp\Big(-\frac{1}{2} \sum_{|n|\leq N} (1+|n|^2)|\widehat{u}_n|^2 \Big) \prod_{|n|\leq N} d\widehat{a}_n d\widehat{b}_n \tag{2.10}$$

where $\widehat{u}_n = \widehat{a}_n + i\widehat{b}_n$ is the Fourier transform of u. For analytical estimates it was convenient in [24] and [20] to consider ρ as measure either on the Hilbert space $H^\sigma(\mathbb{T})$ (Sobolev space) for arbitrary $\sigma < 1/2$ or as a measure on the Banach space $\mathcal{F}L^{s,r}(\mathbb{T})$ (Fourier-Lebesgue space [14, 9, 6]) with norm $\|u\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \| \langle n\rangle^s \, \widehat{u} \, \|_{\ell^r_n(\mathbb{Z})}$ and with the conditions $2 \leq r < \infty$ and $(s-1)r < -1$. Note $\mathcal{F}L^{s,r}(\mathbb{T})$ scales like $H^\sigma(\mathbb{T})$ where $\sigma = s + \frac{1}{r} - \frac{1}{2}$ and the condition $(s-1)r < -1$ is equivalent to $\sigma < 1/2$.

From a probabilistic stand point, however, and to connect the measure ρ with the results in subsequent sections, it is also natural to realize this measure on the space of complex-valued 2π-periodic continuous functions $C(\mathbb{T}, \mathbb{C})$. The measure ρ is closely related to the (complex) Brownian bridges $Z_{u_o}(x)$ where $0 \leq x \leq 2\pi$ and $Z_{u_o}(0) = Z_{u_o}(2\pi) = u_o$. Indeed let $\rho(\cdot|u_o)$ denote the measure ρ conditioned on the event $\{u(0) = u(2\pi) = u_o\}$. If κ denotes the distribution of u_o, then κ is a complex Gaussian probability measure and we have $\rho(\cdot) = \int_{\mathbb{C}} \rho(\cdot|u_o) d\kappa(u_o)$. Then $\rho(\cdot|u_o)$ is absolutely continuous with respect to the probability distribution P_{u_o} of the complex Brownian bridges with

$$\frac{d\rho(\cdot|u_o)}{dP_{u_o}} \;=\; \mathcal{Z}_{u_o}^{-1} e^{-\frac{1}{2}\int_0^{2\pi} |u|^2 \, dx} \,. \tag{2.11}$$

This can be easily seen for example by considering the finite-dimensional distribution of ρ.

By combining the results obtained by Nahmod, Oh, Rey-Bellet, and Stafillani in [20] for the *gauged* equation and the results in the present paper we will prove.

Theorem 2.1. *The DNLS equation* (2.1) *is ν-almost surely well-posed and the measure ν is invariant for the flow map $\Psi(t)$ for* (2.1).

We now explain why in order to prove Theorem 2.1 one needs to introduce a gauge transformation. We go back to the existence of (local) solutions to (2.1). By examining the equation one sees there is a derivative loss arising from the nonlinear term $(|u|^2 u)_x = u^2 \, \overline{u}_x \, + \, 2 \, |u|^2 \, u_x$ and hence for low regularity data one must somehow make up for this loss. Since the worse resonant interactions occur on the second term $|u|^2 \, u_x$ a key idea is to suitably gauge transform the equation to get rid of it, see [11, 12, 23, 13, 10]. In the periodic context a suitable gauge transformation was introduced by Herr [13]. For $f \in L^2(\mathbb{T})$ let us define

$$G(f) = e^{-iJ(f)} f \,, \quad \text{with} \quad J(f)(x) \;=\; \frac{1}{2\pi} \int_0^{2\pi} \int_\theta^x (|f(y)|^2 - m(f)) dy \, d\theta \tag{2.12}$$

and note that the inverse of G is simply given by $G^{-1}(f) = e^{iJ(f)} f$. Under the gauge G, if u is a solution of the DNLS equation (2.1) then $w(x,t) = G(u(x,t))$ is a solution to what we call the GDNLS equation

$$w_t - i w_{xx} - 2m(w) w_x = -w^2 \overline{w}_x + \frac{i}{2} |w|^4 w - i\psi(w) w - i m(w) |w|^2 w \tag{2.13}$$

where

$$\psi(w) := -\frac{1}{\pi} \int_{\mathbb{T}} \mathrm{Im}(w \overline{w}_x) \, dx \;+\; \frac{1}{4\pi} \int_{\mathbb{T}} |w|^4 dx - m(w)^2 \,.$$

The main result of the present paper is to show how the measure ν is transformed under the gauge transformation G. The image of ν under G is denoted by μ and is, by definition, given by

$$\mu(A) \,:=\, \nu(G^{-1}(A)) \,=\, \nu\left(\{x; G(x)\in A\}\right), \tag{2.14}$$

where A is any measurable set. We will use the notation $\mu = \nu\circ G^{-1}$ in the sequel. We have

Theorem 2.2. *For sufficiently small B, the measure $\mu = \nu\circ G^{-1}$ is absolutely continuous with respect to the Gaussian measure ρ and we have*

$$\frac{d\mu}{d\rho}(w) = \tilde{\mathcal{Z}}^{-1}\chi_{\{\|w\|_{L^2}\le B\}}\, e^{-\frac{1}{2}\mathcal{N}(w)}\,. \tag{2.15}$$

where

$$\mathcal{N}(w) \,=\, -\frac{1}{2}\mathrm{Im}\int w^2\overline{w w_x}\,dx + 2m(w)\,\mathrm{Im}\int w\overline{w_x}\,dx - \frac{1}{2}m(w)\int |w|^4\,dx + 2\pi m(w)^3\,.$$

For the measure μ, as given by (2.15), it is proved in [20], see Theorem 6.3, 6.5, 7.1, and 7.2 the following result.

Theorem 2.3. *The GDNLS equation* (2.13) *is μ-almost surely well-posed and the measure μ is invariant for the flow map $\Phi(t)$ for* (2.13).

Remark 2.4. In [13] and [20] one actually performs another supplementary transformation to get rid of the term $2m(w)w_x$ on the left hand side of (2.13). Indeed if we set $v(x,t) = w(x-2tm(w),t)$ then v is a solution of

$$v_t - iv_{xx} = -v^2\overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v \tag{2.16}$$

A simple argument given in section 7 of [20] show that the measure μ is invariant for both the flow maps for (2.13) and (2.16).

To conclude one notes that Theorem 2.1 follows immediately from Theorem 2.2 and 2.3. We are thus left to prove Theorem 2.2.

2.1. **An heuristic introduction of μ.** To understand the form of the measure μ we give here a purely heuristic argument, a rigorous proof is given in the next section. Let us first recall how the invariants for DNLS transform under G. Since $u = e^{iJ(w)}w$ we have $m(u) = m(w)$ and $u_x = e^{iJ(w)}(w_x + iJ(w)_x w)$ with $J(w)_x = |w|^2 - m(w)$ and we obtain after straightforward computations

$$\begin{aligned} H(u) &= \mathrm{Im}\int_{\mathbb{T}} u\overline{u}_x\,dx + \frac{1}{2}\int_{\mathbb{T}}|u|^4\,dx. \\ &= \mathrm{Im}\int_{\mathbb{T}} w\overline{w}_x - \frac{1}{2}\int_{\mathbb{T}}|w|^4\,dx + 2\pi m(w)^2 =: \mathscr{H}(w) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} u_x\overline{u_x} &= (w_x + iJ(w)_x w)\left(\overline{w_x} - iJ(w)_x\overline{w}\right) \\ &= w_x\overline{w_x} - 2\mathrm{Im}\, w^2\overline{w w_x} + 2m(w)\,\mathrm{Im}\, w\overline{w_x} + \left(|w|^6 - 2m|w|^4 + m(w)^2\,|w|^2\right) \end{aligned} \tag{2.18}$$

as well as

$$u^2\overline{u u_x} = w^2\overline{w w_x} - i|w|^6 + im(w)\,|w|^4\,. \tag{2.19}$$

Hence by using (2.3), (2.18), (2.19) we find

$$\begin{aligned} E(u) &= \int w_x \overline{w_x}\, dx - \frac{1}{2}\mathrm{Im} \int w^2 \overline{w w_x}\, dx + 2m(w)\, \mathrm{Im} \int w \overline{w_x}\, dx \qquad (2.20) \\ &\quad -\frac{1}{2} m(w) \int |w|^4\, dx + 2\pi m(w)^3 \\ &=: \mathcal{E}(w)\,. \qquad (2.21) \end{aligned}$$

Remark 2.5. Notice that $\mathcal{E}(w)$ involves the other conserved quantities $\mathscr{H}(w)$ and $m(w)$ and if we define

$$\mathscr{E}(w) := \int_{\mathbb{T}} |w_x|^2\, dx - \frac{1}{2}\mathrm{Im} \int_{\mathbb{T}} w^2 \overline{w}\, \overline{w_x}\, dx + \frac{1}{4\pi} \left(\int_{\mathbb{T}} |w(t)|^2\, dx \right) \left(\int_{\mathbb{T}} |w(t)|^4\, dx \right), \qquad (2.22)$$

we then have

$$E(u) = \mathscr{E}(w) + 2m\, \mathscr{H}(w) - 2\pi\, m^3 = \mathcal{E}(w), \qquad (2.23)$$

and so $\mathscr{E}(w)$ is also a conserved quantity. One could build invariant measures using $\mathscr{E}(w)$ rather than $\mathcal{E}(w)$ but they would turn out to be equivalent measures.

Let us pretend that the measure ν is the measure with density

$$\chi_{\{\|u\|_{L^2} \le B\}} e^{-\frac{1}{2} N(u)} e^{-\frac{1}{2} \int (|u|^2 + |u_x|^2) dx} = \chi_{\{\|u\|_{L^2} \le B\}} e^{-\frac{1}{2} E(u) - \frac{1}{2} \int |u|^2 dx} \qquad (2.24)$$

with respect to the (nonexistent!) infinite dimensional Lebesgue measure $\prod_{x \in \mathbb{T}} da(x) db(x)$. Let us assume furthermore that this nonexistent Lebesgue measure is left invariant under G. Then we would simply obtain from (2.20) that

$$\begin{aligned} d\mu &= \tilde{C}^{-1} \chi_{\{\|w\|_{L^2} \le B\}}\, e^{-\frac{1}{2}\mathcal{E}(w) - \frac{1}{2} \int |w|^2 dx} \prod_{x \in \mathbb{T}} da(x) db(x) \qquad (2.25) \\ &= \mathscr{Z}^{-1} \chi_{\{\|w\|_{L^2} \le B\}}\, e^{-\frac{1}{2}\mathcal{N}(w)} d\rho \end{aligned}$$

where

$$\mathcal{N}(w) = -\frac{1}{2}\mathrm{Im} \int w^2 \overline{w w_x}\, dx + 2m(w)\, \mathrm{Im} \int w \overline{w_x}\, dx - \frac{1}{2} m(w) \int |w|^4\, dx + 2\pi m(w)^3$$

is the nonquadratic part of the energy $\mathcal{E}(w)$.

The crucial problem to understand rigorously the transformation of μ under G is actually to understand the transformation of the Gaussian part ρ of μ under G since the transformation of the weight is computed easily as in the formal computation above. This is achieved in Theorem 2.1 below where in order to analyze the transformation of ρ the main ingredients will be:

(i) The relation between ρ and Brownian bridges, see eq. (2.11).

(ii) The well-known fact that a Brownian bridge can be obtained by conditioning a Brownian motion to return at its starting point.

(iii) The conformal invariance of Brownian motions. Note that since $w = e^{-iJ(u)} u$, $J(u) = J(w)$ and $J(u) = J(|u|)$, it is more convenient to consider this transformation in terms of the variables

$$|u| = |w|\,, \qquad \arg(w) = \arg(u) - iJ(|u|). \qquad (2.26)$$

By conformal invariance of Brownian motion and (ii), $|u|$ and $\arg(u)$ have a Gaussian distribution after a suitable reparametrization. The transformation (2.26) is easy to understand. In particular if we condition on $|u|$, the transformation of $\arg(u)$ is a simple translation by a fixed vector which leads to the next item (iv).

(iv) The Cameron-Martin formula for the transformation of Gaussian measure under a translation by a fixed vector, see e.g. [5, 1].

3. WIENER MEASURES UNDER GAUGE TRANSFORMATIONS

Let $Z_{u_o}(x)$ be a standard complex Brownian bridge on the interval $0 \le x \le 2\pi$ and with $Z_{u_o}(0) = Z_{u_o}(2\pi) = u_o$. The law of Z_{u_o} is denoted by P_{u_o} and is a Gaussian probability on $\{Z \in C(\mathbb{T}\,;\mathbb{C}), Z(0) = u_o\}$. Since no confusion arises we will omit the index u_o in the sequel and denote the Brownian bridge simply by Z and its probability distribution by P. We consider first the transformation of a complex Brownian bridge under a class of transformations which contains in particular the gauge transformation G given in (2.12).

We assume that the map G satisfies the following condition

(C) The map $G : C(\mathbb{T},\mathbb{C}) \to C(\mathbb{T},\mathbb{C})$ has the form

$$G(Z)(x) \;=\; e^{-iJ(Z)(x)} Z(x) \tag{3.1}$$

where $J : C(\mathbb{T},\mathbb{C}) \to C(\mathbb{T},\mathbb{R})$ depends only on $|Z|$ and is such that

$$\frac{d}{dx} J(Z)(x) = h(|Z|)(x)\,, \tag{3.2}$$

and $h(|Z|)(x)$ is continuous in x.

The gauge transformation (2.12) in Section 1 satisfies condition **(C)** since we have

$$J(Z)(x) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x \left(|Z(y)|^2 - \frac{1}{2\pi}\int_0^{2\pi}|Z(t)|^2\,dt\right) dy\,d\theta\,, \tag{3.3}$$

$$h(|Z|)(x) \;=\; |Z(x)|^2 - \frac{1}{2\pi}\int_0^{2\pi}|Z(t)|^2\,dt\,. \tag{3.4}$$

Theorem 3.1. *Let Z be a standard complex Brownian bridge with probability distribution P. Let G be a map which satisfies the condition* **(C)**. *If we have*

$$\mathbb{E}_P\left[\exp\left(\mathrm{Im}\int_0^{2\pi} h(|Z|)(t)Z(t)\,d\overline{Z}(t) - \frac{1}{2}\int_0^{2\pi}|h(|Z|)(t)|^2|Z(t)|^2\,dt\right)\right] = 1 \tag{3.5}$$

Then $P\circ G^{-1}$ is absolutely continuous with respect to the Brownian bridge P with Radon-Nikodym derivative

$$\exp\left(\mathrm{Im}\int_0^{2\pi} h(|Z|)(t)Z(t)\,d\overline{Z}(t) - \frac{1}{2}\int_0^{2\pi}|h(|Z|)(t)|^2|Z(t)|^2\,dt\right)\,. \tag{3.6}$$

Remark 3.2. Assume for a moment that we are not in a periodic setting, that Z is a standard Brownian motion instead of a Brownian bridge, and that

$$\frac{d}{dx} J(Z)(x) = f(|Z|(x)) \tag{3.7}$$

for some real-valued continuous function f. The expression in the right hand side of (3.7) looks very similar to the corresponding one in (3.2), but it is actually easier to handle since it is non anticipative, in the sense that it depends on the Brownian motion up to "time" x and not later. Thanks to this fact the Radon-Nikodym derivative for the transformation G

can be computed as a consequence of Girsanov formula. Indeed if we set $R = J(Z)$ then we have

$$\tilde{Z} \equiv G(Z) = e^{-iR}Z\,.$$

By Ito's formula we have $dR = f(|Z|)dx$ and

$$\begin{aligned} d\tilde{Z} &= -ie^{-iR}ZdR + e^{-iR}dZ - \frac{1}{2}e^{-iR}ZdR^2 + ie^{-iR}dRdZ + 0\frac{1}{2}dZ^2 \\ &= e^{-iJ(Z)}(-iZ)f(|Z|)dx + e^{-iJ(Z)}dZ \\ &= -i\tilde{Z}f(|\tilde{Z}|)dx + e^{-iJ(Z)}dZ\,. \end{aligned}$$

where we have used that $|\tilde{Z}| = |Z|$. Since $J(Z)(x)$ is a nonanticipating functional $e^{-iJ(Z)}dZ$ is a Brownian motion (see [21]) and therefore $\tilde{Z}$ has the same law as the solution of the SDE

$$d\tilde{Z} = -i\tilde{Z}f(\tilde{Z})dx + dZ\,.$$

An application of Girsanov Theorem gives now the form of the Radon-Nikodym derivative as in (3.6).

Proof of Theorem 3.1 The remark above explains why the kind of gauge transformations we consider cannot be studied directly by the Girsanov Theorem and some manipulation needs to be performed.

In the course of the proof we will use some properties of complex Brownian motions which we denote by $B(x)$ with probability distribution Q (again we omit from the notation the choice of $B(0)$.) We recall first the well-known fact, see e.g. [21], that a Brownian bridge $Z(x)$ is obtained from a Brownian motion by conditioning B on the event $\{B(2\pi) = B(0)\}$.

Furthermore we will use the conformal invariance of Brownian motion, that is if $A = A_1 + iA_2$ is a complex Brownian motion, and ϕ is analytic function then $B = \phi(A)$ is, after a suitable time change, again a complex Brownian motion (see e.g. [21], Example 8.22). For $B(x) = \exp(A(s))$ the time change is given by

$$x = x(s) = \int_0^s \left|e^{A(r)}\right|^2 dr = \int_0^s e^{2A_1(r)}dr \qquad \frac{dx}{ds} = \left|e^{A(s)}\right|^2 = |B(x(s))|^2, \tag{3.8}$$

or equivalently

$$s(x) = \int_0^x \frac{dr}{|B(r)|^2}, \qquad \frac{ds}{dx} = \frac{1}{|B(x)|^2}\,. \tag{3.9}$$

If we write $B(x)$ in polar coordinate

$$B(x) = |B(x)|e^{i\arg(B)(x)} \tag{3.10}$$

we have

$$A(s) = A_1(s) + iA_2(s) = \log|B(x(s))| + i\arg(B)(x(s)) \tag{3.11}$$

and A_1 and A_2 are real independent Brownian motions.

In view of conditioning we are interested in $B(x)$ for $0 \le x \le 2\pi$ and thus we introduce the stopping time

$$S = \inf\left\{s\,;\ \int_0^s \left|e^{A(r)}\right|^2 dr = 2\pi\right\}. \tag{3.12}$$

and remark, for future use, the important fact that the stopping time $\mathcal{S}$ depends only on the real part $A_1(s)$ of $A(s)$, or equivalently only $|B|(x)$.

For the Brownian bridge $Z(x)$ let us set

$$W_1(s) = \log |Z(x(s))| \,, \quad W_2(s) = \arg(Z)(x(s)) \tag{3.13}$$

where $x(s) = \int_0^s e^{2W_1(r)} dr$ and $0 \le x \le 2\pi$ is equivalent to $0 \le s \le S$.

Since the stopping time depends only $|Z|$ then once we condition on the process $W_1(s) = \log |Z(x(s))|$, the conditional law of the process $\arg(Z)(t(s))$ is now a Brownian bridge on the interval $0 \le s \le S$.

If we define $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L}(W_1, W_2)(s) &:= W_1(s) + i \left[W_2(s) - J(|Z|)(x(s))\right]. \\ &= W_1(s) + i \left[W_2(s) - J(e^{W_1})(s)\right] \end{aligned} \tag{3.14}$$

then we have

$$e^{\mathcal{L}(W_1,W_2)(s)} = G(Z)(x(s)),$$

where g is as in (3.1). In terms of the variables W_1, W_2 the transformation $\mathcal{L}$ is easy to analyze. Let us denote by P the law of Z, Q the law of (W_1, W_2), $\tilde{P} = P \circ G^{-1}$ the law of $G(Z)$ and $\tilde{Q} = Q \circ \mathcal{L}^{-1}$ the law of $\mathcal{L}(W_1, W_2)$.

By conditioning on W_1 we write Q as

$$dQ(W_1, W_2) \;=\; dQ_{W_1}(W_2) dQ(W_1), \tag{3.15}$$

where $Q(W_1)$ is the marginal law of W_1 under Q and $Q_{W_1}(W_2)$ is the conditional law of W_2 given W_1. Similarly we decompose the distribution $\tilde{Q}$ as

$$d\tilde{Q}(W_1, W_2) \;=\; d\tilde{Q}_{W_1}(W_2) d\tilde{Q}(W_1). \tag{3.16}$$

By (3.14) we have

$$dQ(W_1) = d\tilde{Q}(W_1)$$

since the real part is left unchanged. Furthermore $d\tilde{Q}_{W_1}(W_2)$ can be computed by Cameron-Martin formula. In fact conditioned on W_1, W_2 is a *real* brownian bridge with $W_2(0) = W_2(S)$ and the imaginary part of $\mathcal{L}(W_1, W_2)$ is obtained by translating $W_2(s)$ by a function which depends only on W_1.

Since

$$\frac{dJ(u(t(s)))}{ds} = (J(u))'(t(s)) \frac{dt(s)}{ds} = h(|u|)(t(s)) \frac{dt(s)}{ds} \,, \tag{3.17}$$

Cameron-Martin formula implies that the Radon-Nikodym derivative of the law of imaginary part of $\mathcal{L}(W_1, W_2)$ with respect to the law of a real brownian bridge on the interval $0 \le s \le S$ is given by

$$\exp \left(\int_0^S h(|Z|)(x(s)) \frac{dt}{ds} dW_2(s) - \frac{1}{2} \int_0^S h(|Z|)^2(x(s)) \left(\frac{dt}{ds}\right)^2 ds \right) .$$

To conclude we finally re-express the Radon-Nikodym derivative in terms of Z and t. We have

$$dW(s) = \frac{1}{Z(x(s))} \frac{dx}{ds} dZ(t(s)) = \overline{Z}(x(s)) dZ(x(s)) \tag{3.18}$$

and thus

$$\int_0^S h(|Z|)(x(s)) \frac{dx}{ds} dW_2(s) \;=\; \mathrm{Im} \int_0^{2\pi} h(|Z|)(x) \overline{Z} \, dZ(x) \,, \tag{3.19}$$

and

$$\int_0^S h(|Z|)^2(x(s))\left(\frac{dx}{ds}\right)^2 ds = \int_0^{2\pi} |h(|Z|)(x)|^2|Z(x)|^2\,dx. \tag{3.20}$$

This concludes the proof of Theorem 3.1.

3.1. **Application to the periodic derivative NLS.** Let us consider the measure μ given in the introduction, see (2.7) and (2.8). In this section we prove Theorem 2.2 using Theorem 3.1.

Proof of Theorem 2.2 We note that, using (2.19), the Radon-Nykodym derivative $\frac{d\nu}{d\rho}$ transforms under the gauge G as

$$N(G^{-1}(w)) = \frac{3}{2}\mathrm{Im}\int w^2\overline{ww_x}\,dx - \int |w|^6\,dx + \frac{3}{2}m\int |w|^4\,dx\,. \tag{3.21}$$

Furthermore by the results of [2, 24, 20] $\chi_{\{\|w\|_{L^2}\leq B\}}e^{-\frac{1}{2}N(G^{-1}(w))} \in L^1(\rho)$ for sufficiently small B.

Therefore it is enough to consider how ρ transforms under the gauge transformation, i.e., we consider the measure $\tilde{\rho} = \rho\circ G^{-1}$. Without cutoff on $\|u\|_{L^2}$ one cannot expect $\tilde{\rho}$ to be absolutely continuous with respect to ρ, but all we really need is that the restriction of $\tilde{\rho}$ on $\{m(w) \leq \frac{B^2}{2\pi}\}$ be absolutely continuous with respect to ρ. Alternatively we can incorporate the cutoff in J by redefining J to be

$$J(w)(x) := \begin{cases} \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x(|w(y)|^2 - m(w))dy\,d\theta & \text{if } m(w) \leq \frac{B^2}{2\pi} \\ 0 & \text{otherwise.}\end{cases} \tag{3.22}$$

so that we have

$$\frac{d}{dx}J(w)(x) = h(|w|)(x) = \begin{cases} |w(x)|^2 - m(w) & \text{if } m(w) \leq \frac{B^2}{2\pi} \\ 0 & \text{otherwise.}\end{cases} \tag{3.23}$$

By the results in [2, 24, 20]

$$\begin{aligned} &\exp\left(\mathrm{Im}\int h(|w|)w\overline{w}_x\,dx - \frac{1}{2}\int h(|w|)^2|w|^2\,dx\right) \\ &\quad = \chi_{\{m(w)\leq\frac{B^2}{2\pi}\}}\exp\left(\mathrm{Im}\int(|w|^2 - m(w))w\overline{w}_x\,dx - \frac{1}{2}\int(|w(x)|^2 - m(w))^2|w(x)|^2\,dt\right)\end{aligned} \tag{3.24}$$

belongs to $L^1(\rho)$ for sufficiently small B. By conditioning on $\{u(0) = u(2\pi) = u_o\}$ and using equation (2.11) we conclude that the Novikov condition (3.5) is satisfied for almost every u_o. Therefore using Theorem 3.1 the Radon-Nikodym derivative $\frac{d\tilde{\rho}}{d\rho}$ is given by (3.24). Finally combining the equations (3.24) and (3.21) we obtain equation (2.15). □

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