Lusin's Theorem (Th. 4.5 Ch. 1 of [SS] pg. 34)

Suppose f is measurable and finite-valued on E a msrb. set with $m(E)<\infty$. Then $\forall \varepsilon>0$ $\exists$ a closed set F_ε with $F_\varepsilon \subset E$ and $m(E \setminus F_\varepsilon) \leq \varepsilon$ such that $f\big|_{F_\varepsilon}$ is continuous.

Remark: The statement does not say that f as a function of E is continuous at the points of F_ε. The restriction of f to F_ε, namely $f\big|_{F_\varepsilon}$, is viewed here only as a function on F_ε and so continuity in the statement means that for $x_k, x \in F_\varepsilon$ with $x_k \xrightarrow[k\to\infty]{} x$ we have that $f(x_k) \xrightarrow[k\to\infty]{} f(x)$ (sequential definition of continuity)

Before proving Lusin's theorem. Let us prove the following

- Auxiliary Lemma: Let h be a simple function on E with $m(E)<\infty$. Then $\forall \varepsilon>0$ $\exists$ H_ε closed in E $m(E \setminus H_\varepsilon) \leq \varepsilon$ such that h is continuous on H_ε

Proof of Auxiliary Lemma: Let us write h in canonical form, namely $h = \sum_{k=1}^{K} a_k \chi_{E_k}$ where a_k are nonzero distinct scalars, E_k disjoint $m(E_k) < \infty$, $E = \overset{K}{\underset{k=1}{\biguplus}} E_k$.

For each k, $\exists F_k$ closed $F_k \subset E_k$ $m(E_k \setminus F_k) < \frac{\varepsilon}{k}$.

Define $H_\varepsilon := \bigcup_{k=1}^{K} F_k$, H_ε is closed and since the E_k's are disjoint, F_k are also disjoint. Then we have

$$m(E \setminus H_\varepsilon) = m\left(\biguplus_{k=1}^{K} (E_k \setminus F_k)\right)$$

$$= \sum_{k=1}^{K} m(E_k \setminus F_k) \leq \varepsilon .$$

Note that on H_ε h takes the values a_k on F_k, $1 \leq k \leq K$ (since F_k are disjoint h is well defined on H_ε)

To see that h is continuous on H_ε consider $x_0 \in H_\varepsilon$ then there exists a unique i / $x_0 \in F_i$ and $\exists$ an open set $\ni x_0$ which is disjoint from the closed set $\bigcup_{k \neq i} F_k$ (complement of $\bigcup_{k\neq i} F_k$ is open and contains x_0).

On the intersection of this open set with H_ε, h is constant hence continuous on x_0. #

Proof of Lusin: Let h_n be a sequence of simple functions so that $h_n(x) \to f(x)$ for all x (even a.e x is okay)

Then, by the Auxiliary Lemma,

for each n, we may find sets H_n / $m(E \setminus H_n) < \frac{1}{2^n}$ and h_n is continuous on H_n.

Now, given $\varepsilon > 0$, by Egorov's theorem $\exists\, A_{\varepsilon/3} \subset E$ s.t. $m(E \setminus A_{\varepsilon/3}) \leq \frac{\varepsilon}{3}$ and $h_n \rightrightarrows f$ on $A_{\varepsilon/3}$.

Consider $\mathcal{F}'_\varepsilon := A_{\varepsilon/3} \setminus \bigcup_{n \geq N} (E \setminus H_n)$ for N large enough so that $\sum_{n \geq N} \frac{1}{2^n} \leq \frac{\varepsilon}{3}$.

Now, for every $n \geq N$, h_n is continuous on $\mathcal{F}'_\varepsilon$ ($\mathcal{F}'_\varepsilon$ has the corresponding $E \setminus H_n$ removed). Furthermore $h_n \rightrightarrows f$ on $\mathcal{F}'_\varepsilon \Rightarrow f$ is continuous on $\mathcal{F}'_\varepsilon$.

To finish the proof approximate $\mathcal{F}'_\varepsilon$ by a closed set $F_\varepsilon \subset \mathcal{F}'_\varepsilon$ / $m(\mathcal{F}'_\varepsilon \setminus F_\varepsilon) < \varepsilon/3$.

Then $f/_{F_\varepsilon}$ is continuous and $m(E \setminus F_\varepsilon) < \varepsilon$ #

$\boxed{\text{NOTE}}$: If E is bounded then F_ε is compact.

Remark: There are more general forms of Lusin which rely on Tietze's extension theorem (or some some special version of it).
We'll not cover these but you may research about them