1 Quick Review of wave equation on $\mathbb{R}$.

1.1 An example and Causality

The wave equation on the entire real line $x \in \mathbb{R}$ corresponds to a string of infinite length[1].

The wave equation then describes the dynamics of the amplitude $u(x,t)$ of the point at position x on the string at time t, has the following form

$$u_{tt} = c^2 u_{xx},$$

or

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.1}$$

The wave equation is hyperbolic. One can then rewrite this equation as

$$u_{\xi\eta} = 0. \tag{1.2}$$

To see this we pass to the *characteristic variables* as follows: let

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases} \tag{1.3}$$

To see that (1.2) is equivalent to (1.1), let us compute the partial derivatives of u with respect to x and t in the new variables using the chain rule.

$$u_t = cu_\xi - cu_\eta,$$

$$u_x = u_\xi + u_\eta.$$

We can differentiate the above first order partial derivatives with respect to t, respectively x using the chain rule again, to get

$$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta},$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

[1] Although physically unrealistic, as we will see later, when considering the dynamics of a portion of the string away from the endpoints, the boundary conditions have no effect for some (nonzero) finite time. Due to this, one can neglect the endpoints, and make the assumption that the string is infinite.

Substituting these expressions into the left hand side of equation (**??**), we see that

$$u_{tt}-c^2u_{xx}=c^2u_{\xi\xi}-2c^2u_{\xi\eta}+c^2u_{\eta\eta}-c^2(u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta})=-4c^2u_{\xi\eta}=0,$$

which is equivalent to (1.2).

Equation (1.2) can be treated as a pair of successive ODEs. Integrating first with respect to the variable η, and then with respect to ξ, we arrive at the solution

$$u(\xi,\eta)=f(\xi)+g(\eta).$$

Recalling the definition of the characteristic variables (1.3), we can switch back to the original variables (x,t), and obtain the general solution

$$u(x,t)=f(x+ct)+g(x-ct). \tag{1.4}$$

Another way to solve equation (1.1) is to realize that the second order linear operator of the wave equation factors into two first order operators

$$\mathcal{L}=\partial_t^2-c^2\partial_x^2=(\partial_t-c\partial_x)(\partial_t+c\partial_x).$$

Hence, the wave equation can be thought of as a pair of transport (advection) equations

$$(\partial_t-c\partial_x)v=0, \tag{1.5}$$

$$(\partial_t+c\partial_x)u=v. \tag{1.6}$$

It is no coincidence, of course, that

$$x+ct=\text{constant}, \tag{1.7}$$

and

$$x-ct=\text{constant}, \tag{1.8}$$

are the characteristic lines for the transport equations (1.5), and (1.6) respectively, hence our choice of the characteristic coordinates (1.3). We also see that for each point in the xt plane there are two distinct characteristic lines, each belonging to one of the two families (1.7) and (1.8), that pass through the point. This is illustrated in Figure 1.1 below.

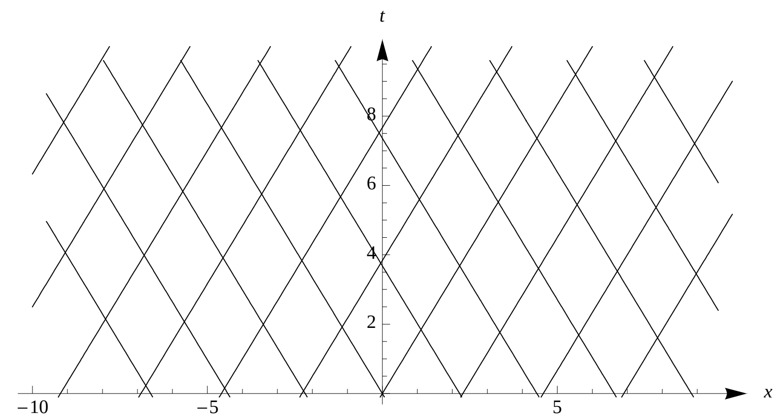

Figure 1.1: Characteristic lines for the wave equation with $c=0.6$.

1.2 Initial value problem

Along with the wave equation (1.1), we next consider some initial conditions, to single out a particular physical solution from the general solution (1.4). The equation is of second order in time t, so values must be specified both for the initial dispalcement $u(x,0)$, and the initial velocity $u_t(x,0)$. We study the following *initial value problem*, or IVP

$$\begin{cases} u_{tt}-c^2u_{xx}=0 & \text{for} -\infty<x<+\infty, \\ u(x,0)=\phi(x), & u_t(x,0)=\psi(x), \end{cases} \tag{1.9}$$

where ϕ and ψ are arbitrary functions of single variable, and together are called the *initial data* of the IVP. The solution to this problem is easily found from the general solution (1.4). All we need to do is find f and g from the initial conditions of the IVP (1.9). To check the first initial condition, set $t=0$ in (1.4),

$$u(x,0)=\phi(x)=f(x)+g(x). \tag{1.10}$$

To check the second initial condition, we differentiate (1.4) with respect to t, and set $t=0$

$$u_t(x,0)=\psi(x)=cf'(x)-cg'(x). \tag{1.11}$$

Equations (1.10) and (1.11) can be treated as a system of two equations for f and g. To solve this system, we first integrate both sides of (1.11) from 0 to x to get rid of the derivatives on f and g (alternatively we could differentiate (1.10) instead), and rewrite the equations as

$$f(x)+g(x)=\phi(x),$$
$$f(x)-g(x)=\frac{1}{c}\int_0^x\psi(s)ds+f(0)-g(0).$$

We can solve this system by adding the equations to eliminate g, snd subtracting them to eliminate f. This leads to the solution

$$f(x)=\frac{1}{2}\phi(x)+\frac{1}{2c}\int_0^x\psi(s)ds+\frac{1}{2}[f(0)-g(0)],$$
$$g(x)=\frac{1}{2}\phi(x)-\frac{1}{2c}\int_0^x\psi(s)ds-\frac{1}{2}[f(0)-g(0)].$$

Finally, substituting these expressions for f and g back into the solution (1.4), we obtain the solution of the IVP (1.9)

$$u(x,t)=\frac{1}{2}\phi(x+ct)+\frac{1}{2c}\int_0^{x+ct}\psi(s)ds+\frac{1}{2}\phi(x-ct)-\frac{1}{2c}\int_0^{x-ct}\psi(s)ds.$$

Combining the integrals using additivity, and the fact that flipping the integration limits changes the sign of the integral, we arrive at the following form for the solution

$$u(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds. \tag{1.12}$$

This is called the **d'Alambert's formula** for the solution to the IVP (1.9). It implies that as long as ϕ is twice continuously differentiable ($\phi\in C^2$), and ψ is continuously differentiable ($\psi\in C^1$), (1.12) gives a solution to the IVP. We will also consider examples with discontinuous initial data, which after plugging into (1.12) produce *weak solutions*. This notion will be made precise in later lectures.

Example 1.1. *Solve the initial value problem* (1.9) *with the initial data*

$$\phi(x)\equiv 0, \qquad \psi(x)=\sin x.$$

Substituting ϕ and ψ into d'Alambert's formula, we obtain the solution

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}\sin s ds=\frac{1}{2c}(-\cos(x+ct)+\cos(x-ct)).$$

Using the trigonometric identities for the cosine of a sum and difference of two angles, we can simplify the above to get

$$u(x,t)=\frac{1}{c}\sin x\sin(ct).$$

You should verify that this indeed solves the wave equation and satisfies the given initial conditions. □

1.3 An Example

When solving the transport equation, we saw that the initial values simply travel to the right, when the speed is positive (they propagate along the characteristics (1.8)); or to the left (they propagate along the characteristics (1.7)), when the speed is negative. Since the wave equation is made up of two of these type of equations, we expect similar behavior for the solutions of the IVP (1.9) as well. To see this, let us consider the following example with simplified initial data.

Example 1.2. *Find the solution of IVP* (1.9)*, with the initial data*

$$\phi(x) = \begin{cases} h, & |x| \leq a, \\ 0, & |x| > a, \end{cases} \tag{1.13}$$

$$\psi(x) \equiv 0.$$

This data corresponds to an initial disturbance of the string centered at $x=0$ of height h, and zero initial velocity. Notice that $\phi(x)$ is not continuous, let alone twice differentiable, though one can still substitute it into d'Alambert's solution (1.12) *and get a function u, which will be a weak solution of the wave equation.*

Since $\psi(x) \equiv 0$, only the first term in (1.12) *is nonzero. We compute this term using the particular $\phi(x)$ in* (1.13)*. First notice that*

$$\phi(x+ct) = \begin{cases} h, & |x+ct| \leq a, \\ 0, & |x+ct| > a, \end{cases} \tag{1.14}$$

and

$$\phi(x-ct) = \begin{cases} h, & |x-ct| \leq a, \\ 0, & |x-ct| > a. \end{cases} \tag{1.15}$$

Hence, the solution

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$$

is piecewise defined in 4 different regions in the xt half-plane (we consider only positive time $t \geq 0$), which *correspond to pairings of the intervals in the expressions* (1.14) *and* (1.15)*. These regions are*

$$\begin{aligned}
&I\!: \quad \{|x+ct|\le a, |x-ct|\le a\}, \qquad u(x,t)=h\\
&II\!: \quad \{|x+ct|\le a, |x-ct|> a\}, \qquad u(x,t)=\frac{h}{2}\\
&III\!: \quad \{|x+ct|> a, |x-ct|\le a\}, \qquad u(x,t)=\frac{h}{2}\\
&IV\!: \quad \{|x+ct|> a, |x-ct|> a\}, \qquad u(x,t)=0
\end{aligned} \tag{1.16}$$

The regions are depicted in Figure 1.2. Notice that $|x+ct|\le a$ *is equivalent to*

$$-a\le x+ct\le a, \qquad or \qquad -\frac{1}{c}(x+a)\le t\le -\frac{1}{c}(x-a).$$

Similarly for the other inequalities.

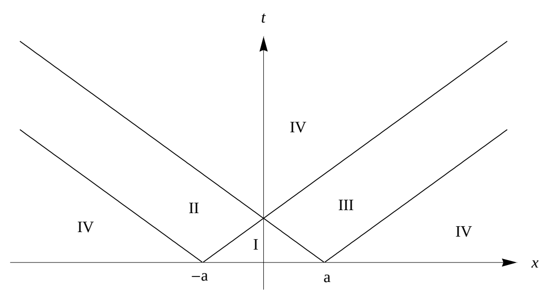

Figure 1.2: Regions where u has different values.

Using the values for the solution in (1.16)*, we can draw the graph of* u *at different times, some of which are depicted in Figures 1.3-1.6.*

We see that the initial box-like disturbance centered at $x=0$ *splits into two disturbances of half the size, which travel in opposite directions with speed* c*.*

The graphs hint that the initial disturbance will not be felt at a point x *on the string (for* $|x|>a$*) before the time* $t=\frac{1}{c}||x|-a|$*. We will shortly see that this is a general property for the wave equation. In this particular case a box-like disturbance appears at the time* $t=\frac{1}{c}||x|-a|$*, and lasts exactly* $t=\frac{2a}{c}$ *units of time, after which it completely moves along. In general, the initial velocity may slow down the speed, and subsequently make the disturbance "last" longer, however, the speed cannot exceed* c*.* □

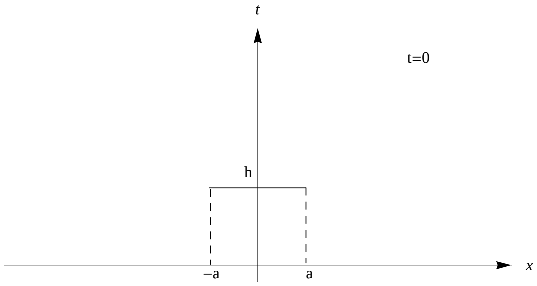

Figure 1.3: The solution at $t\!=\!0$.

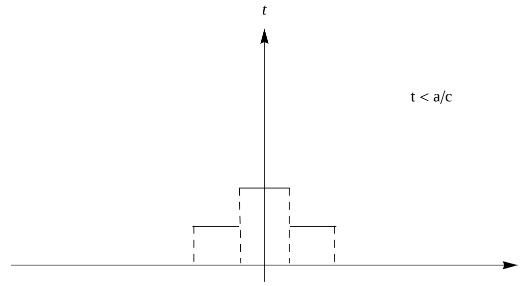

Figure 1.4: The solution for $0\!<\!t\!<\!a/c$.

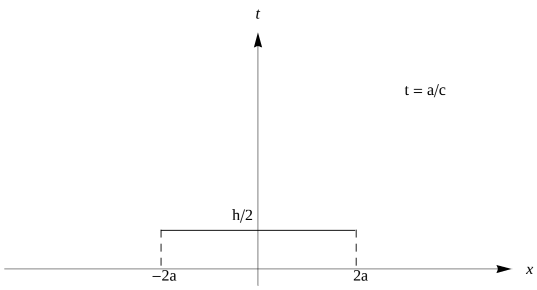

Figure 1.5: The solution at $t\!=\!a/c$.

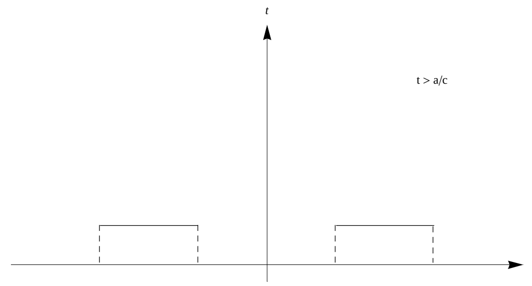

Figure 1.6: The solution for $t\!>\!a/c$.

1.4 Causality

The value of the solution to the IVP (1.9) at a point (x_0,t_0) can be found from d'Alambert's formula (1.12)

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0)+\phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)ds. \tag{1.17}$$

We can see that this value depends on the values of ϕ at only two points on the x axis, x_0+ct_0, and x_0-ct_0, and the values of ψ only on the interval $[x_0-ct_0,x_0+ct_0]$. For this reason the interval $[x_0-ct_0,x_0+ct_0]$ is called *interval of dependence* for the point (x_0,t_0). The triangular region with vertices at x_0-ct_0 and x_0+ct_0 on the x axis and the vertex (x_0,t_0) is called the *domain of dependence*, or the *past history* of the point (x_0,t_0), as depicted in Figure 1.7. The sides of this triangle are segments of characteristic lines passing through the point (x_0,t_0). Thus, we see that the initial data travels along the characteristics to give the values at later times. In the previous example of the box wave, we saw that the jump discontinuities travel along the characteristic lines as well.

An inverse notion to the domain of dependence is the notion of *domain of influence* of the point x_0 on the x axis. This is the region in the xt plane consisting of all the points, whose domain of dependence contains the

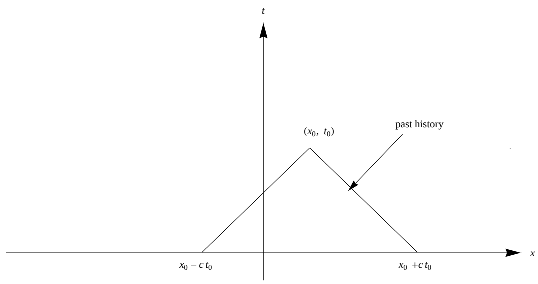

Figure 1.7: Domain of dependence for (x_0,t_0).

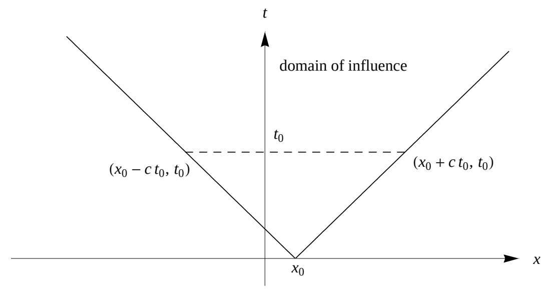

Figure 1.8: Domain of influence of x_0.

point x_0. The region has an upside-down triangular shape, with the sides being the characteristic lines emanating from the point x_0, as shown in Figure 1.8. This also means that the value of the initial data at the point x_0 effects the values of the solution u at all the points in the domain of influence. Notice that at a fixed time t_0, only the points satisfying $x_0 - ct_0 \le x \le x_0 + ct_0$ are influenced by the point x_0 on the x axis.

Example 1.3 (The hammer blow)**.** *Analyze the solution to the IVP* (1.9) *with the following initial data*

$$\begin{aligned} &\phi(x) \equiv 0, \\ &\psi(x) = \begin{cases} h, & |x| \le a, \\ 0, & |x| > a. \end{cases} \end{aligned} \tag{1.18}$$

From d'Alambert's formula (1.12)*, we obtain the following solution*

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{1.19}$$

Similar to the previous example of the box wave, there are several regions in the xt plane, in which u has different values. Indeed, since the initial velocity $\psi(x)$ is nonzero only in the interval $[-a,a]$, the integral in (1.19) *must be computed differently according to how the intervals $[-a,a]$ and $[x-ct,x+ct]$ intersect. This corresponds to the cases when ψ is zero on the entire integration interval in* (1.19)*, on a part of it, or is nonzero on the entire* *integration interval. These different cases are:*

$$
\begin{aligned}
&I: && \{x-ct<x+ct<-a<a\}, && u(x,t)=0\\
&II: && \{x-ct<-a<x+ct<a\}, && u(t,x)=\frac{1}{2c}\int_{-a}^{x+ct} h ds = h\frac{x+ct+a}{2c}\\
&III: && \{x-ct<-a<a<x+ct\}, && u(t,x)=\frac{1}{2c}\int_{-a}^{a} h ds = h\frac{a}{c}\\
&IV: && \{-a<x-ct<x+ct<a\}, && u(t,x)=\frac{1}{2c}\int_{x-ct}^{x+ct} h ds = ht\\
&V: && \{-a<x-ct<a<x+ct\}, && u(t,x)=\frac{1}{2c}\int_{x-ct}^{a} h ds = h\frac{a-(x-ct)}{2c}\\
&VI: && \{-a<a<x-ct<x+ct\}, && u(x,t)=0
\end{aligned}
\tag{1.20}
$$

The regions are depicted in Figure 1.9 below. Notice that to find the value of the solution at a point (x_0,t_0)*, one simply needs to trace the point back to the* x *axis along the characteristic lines, and determine how the interval of dependence intersects the segment* $[-a,a]$.

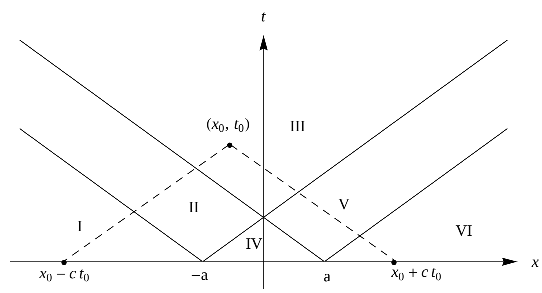

Figure 1.9: Regions where u has different values.

□

1.5 Conclusion

The wave equation, being a constant coefficient second order PDE of hyperbolic type, posseses two families of characteristic lines, which correspond to constant values of respective characteristic variables. Using these variables the equation can be treated as a pair of successive ODEs, integrating which leads to the general solution. This general solution was used to arrive at d'Alambert's solution (1.12) for the IVP on the whole line. Unfortunately this simple derivation relies on having two families of characteristics and does not work for the heat and Laplace's equations.

Exploring a few examples of initial data, we established causality between the initial data and the values of the solution at later times. In particular, we saw that the initial values travel with speeds bounded by the wave speed c, and that the discontinuities of the initial data travel along the characteristic lines.

2 The energy method

2.1 Energy for the wave equation

Let us consider an infinite string with constant linear density ρ and tension magnitude T. The wave equation describing the vibrations of the string is then

$$\rho u_{tt} = T u_{xx}, \qquad -\infty < x < \infty. \tag{2.1}$$

Since this equation describes the mechanical motion of a vibrating string, we can compute the kinetic energy associated with the motion of the string. Recall that the kinetic energy is $\frac{1}{2}mv^2$. In this case the string is infinite, and the speed differs for different points on the string. However, we can still compute the energy of small pieces of the string, add them together, and pass to a limit in which the lengths of the pieces go to zero. This will result in the following integral

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 dx.$$

We will assume that the initial data vanishes outside of a large interval $|x| \leq R$, so that the above integral is convergent due to the finite speed of propagation. We would like to see if the kinetic energy KE is conserved in time. For this, we differentiate the above integral with respect to time to see whether it is zero, as is expected for a constant function, or whether it is different from zero.

$$\frac{d}{dt}KE = \frac{1}{2}\rho \int_{-\infty}^{\infty} 2u_t u_{tt} dx = \int_{-\infty}^{\infty} \rho u_t u_{tt} dx.$$

Using the wave equation (2.1), we can replace the ρu_{tt} by $T u_{xx}$, obtaining

$$\frac{d}{dt}KE = T\int_{-\infty}^{\infty} u_t u_{xx} dx.$$

The last quantity does not seem to be zero in general, thus the next best thing we can hope for, is to convert the last integral into a full derivative in time. In that case the difference of the kinetic energy and some other quantity will be conserved. To see this, we perform an integration by parts in the last integral

$$\frac{d}{dt}KE = Tu_t u_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} Tu_{xt}u_x dx.$$

Due to the finite speed of propagation, the endpoint terms vanish. The last integral is a full derivative, thus we have

$$\frac{d}{dt}KE = -\int_{-\infty}^{\infty} Tu_{xt}u_x dx = -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} Tu_x^2 dx\right).$$

Defining

$$PE = \frac{1}{2}T\int_{-\infty}^{\infty} u_x^2 dx,$$

we see that

$$\frac{d}{dt}KE = -\frac{d}{dt}PE, \qquad \text{or} \qquad \frac{d}{dt}(KE+PE) = 0.$$

The quantity $E = KE + PE$ is then conserved, which is the total energy of the string undergoing vibrations. Notice that PE plays the role of the potential energy of a stretched string, and the conservation of energy implies conversion of the kinetic energy into the potential energy and back without a loss.

Another way to see that the energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2) dx \tag{2.2}$$

is conserved, is to multiply equation (2.1) by u_t and integrate with respect to x over the real line.

$$0 = \int_{-\infty}^{\infty} \rho u_{tt}u_t dx - \int_{-\infty}^{\infty} Tu_{xx}u_t dx.$$

The first integral above is a full derivative in time. Integrating by parts in the second term, and realizing that the subsequent integral is a full derivative as well, while the boundary terms vanish, we obtain the identity

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 + Tu_x^2 dx\right) = 0,$$

which is exactly the conservation of total energy.

The conservation of energy provides a straightforward way of showing that the solution to an IVP associated with the linear equation is unique. We demonstrate this for the wave equation next, while a similar procedure will be applied to establish uniqueness of solutions for the heat IVP in the next section.

Example 2.1. *Show that the initial value problem*

$$\begin{cases} u_{tt}-c^2u_{xx}=f(x,t) \qquad for -\infty<x<+\infty, \\ u(x,0)=\phi(x), \qquad u_t(x,0)=\psi(x), \end{cases} \tag{2.3}$$

has a unique solution.

Arguing from the inverse, let as assume that the IVP (2.3) *has two distinct solutions, u and v. But then their difference $w=u-v$ will solve the homogeneous wave equation, and will have the initial data*

$$w(x,0)=u(x,0)-v(x,0)=\phi(x)-\phi(x)\equiv 0,$$

$$w_t(x,0)=u_t(x,0)-v_t(x,0)=\psi(x)-\psi(x)\equiv 0.$$

Hence the energy associated with the solution w at time $t=0$ is

$$E[w](0)=\frac{1}{2}\int_{-\infty}^{\infty}[(w_t(x,0))^2+c^2(w_x(x,0))^2]dx=0$$

This differs from the energy defined above by a constant factor of $1/\rho$ (recall that $T/\rho=c^2$), and is thus still a conserved quantity. It will subsequently be zero at any later time as well. Thus,

$$E[w](t)=\frac{1}{2}\int_{-\infty}^{\infty}[(w_t(x,t))^2+c^2(w_x(x,t))^2]dx=0, \qquad \forall t.$$

But since the integrand in the expression of the energy is nonnegative, the only way the integral can be zero, is if the integrand is uniformly zero. That is,

$$\nabla w(t,x)=(w_t(x,t),w_x(x,t))=0, \qquad \forall x,t.$$

This implies that w is constant for all values of x and t, but since $w(x,0)\equiv 0$, the constant value must be zero. Thus,

$$u(x,t)-v(x,t)=w(x,t)\equiv 0,$$

which is in contradiction with our initial assumption that u and v are different. This implies that the solution to the IVP (2.3) *is unique.* □

The procedure used in the last example, called the *energy method*, is quite general, and works for other linear evolution equations possessing a conserved (or decaying) positive definite energy. The heat equation, considered next, is one such case.

2.2 Conclusion

Using the energy motivated by the vibrating string model behind the wave equation, we derived a conserved quantity, which corresponds to the total energy of motion for the infinite string. This positive definite quantity was then used to prove uniqueness of solution to the wave IVP via the energy method, which essentially asserts that zero initial total energy precludes any (nonzero) dynamics. A similar approach was used to prove uniqueness for the heat IBVP, concluding that zero initial heat implies steady zero temperatures at later times.