Classical and Modern Fourier Analysis

Loukas Grafakos

Department of Mathematics, University of Missouri, Columbia, Missouri 65211

E-mail address: loukas@math.missouri.edu

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CHAPTER 1

Solutions to all the exercises

CHAPTER 1

SECTION 1.1

??. (a) For all $\alpha > 0$ let $E_\alpha = \{x \in X : |f(x)| > \alpha\}$. Fix $\alpha_0 \geq 0$ and let t_n be a decreasing sequence of positive numbers that tends to 0. Use the Lebesgue monotone convergence theorem to show that

$$\mu(E_{\alpha_0 + t_n}) \uparrow \mu(E_{\alpha_0}).$$

(b) Let $E = \{x \in X : |f(x)| > \alpha\}$ and $E_n = \{x \in X : |f_n(x)| > \alpha\}$. Since

$$\mu\big(\cap_{n=m}^\infty E_n\big) \leq \liminf_{n\to\infty} \mu(E_n),$$

and $E \subset \cup_{m=1}^\infty \cap_{n=m}^\infty E_n$ use the Lebesgue monotone convergence theorem to deduce the conclusion. (c) follows from the Lebesgue monotone convergence theorem.

??. (a) First assume that $f_j \geq 0$ and that no p_j is infinite. The case where $p \neq 1$ can be derived from the case where $p = 1$ by replacing p_j by p_j/p. The case $k = 2$ and $p = 1$ can be proved using the inequality

$$f_1(x) f_2(x) \leq \frac{f_1(x)^{p_1}}{p_1} + \frac{f_2(x)^{p_2}}{p_2},$$

where $f_1(x)\, f_2(x) \geq 0$. Use induction on k. (b) Show that equality holds above if and only if $f_1(x) = c f_2(x)$ a.e. To prove part (c) write $f = (fg)g^{-1}$ and use part (a).

??. (a) Given an $0 < \varepsilon < \|f\|_{L^\infty}$ find a $E \subset X$ of positive measure such that $|f(x)| \geq \|f\|_{L^\infty} - \varepsilon$ for all $x \in E$. Then $\|f\|_{L^p} \geq (\|f\|_{L^\infty} - \varepsilon)(\mu(E))^{1/p}$ and thus $\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty} - \varepsilon$. The inequality $\limsup_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$ is trivial.

(b) Fix a sequence $0 < p_n < p_0$ such that $p_n \to 0$. Define

$$h_n(x) = (|f_n(x)|^{p_0} - 1)/p_0 - (|f_n(x)|^{p_n} - 1)/p_n.$$

Use that $(t^p - 1)/p \downarrow \log t$ as $p \downarrow 0$ for all $t > 0$. By the Lebesgue monotone convergence theorem we obtain $\int_X h_n \, d\mu \uparrow \int_X h \, d\mu$, which implies that $\int_X (|f_n|^{p_n} - 1)/p_n \, d\mu \to \int_X \log|f| \, d\mu$. (Observe here that the latter could be $-\infty$.) In view of Jensen's inequality and of the fact that $\log t \leq t - 1$ for $t \geq 0$, the sequence of inequalities is true:

$$\int_X \log|f| \, d\mu = \frac{1}{p_n}\int_X \log|f|^{p_n} \, d\mu \leq \frac{1}{p_n} \log\left(\int_X |f|^{p_n} \, d\mu\right) \leq \frac{1}{p_n}\int_X (|f|^{p_n} - 1)\, d\mu.$$

Now let $n \to \infty$ and use the squeeze property of limits.

(c) It is an easy consequence of Jensen's inequality.

??. (a) Use that $(a_k/(\sum_j a_j))^\theta \ge a_k/(\sum_j a_j)$ and sum over k. For part (b) use (a). For part (c) assume that $\sum_{j=1}^N a_j = 1$ and use Langrange multipliers.

??. (a) Reverse the inequality in part (b) discussed next.
(b) We have

$$\|\sum_j f_j\|_{L^r}^r = \sum_j \int_X f_j(\sum_k f_k)^{-1} d\mu \ge \sum_j \|f_j\|_{L^r} \|\sum_k f_k\|_{L^r}^{r-1}$$

by Exercise **??** (c).
(d) Take $\{f_j\}_{j=1}^N$ to be characteristic functions of disjoint sets with the same measure and use Exercise **??** (c).

??. We have

$$\begin{aligned}
&\left\| \int_X F(x,\,\cdot\,)\,d\mu(x) \right\|_{L^p(d\nu)}^p \\
&= \int_T \left(\int_X |F(x,t)|\,d\mu(x) \right) \left(\int_X |F(x,t)|\,d\mu(x) \right)^{p-1} d\nu(t) \\
&= \int_X \left[\int_T |F(x,t)| \left(\int_X |F(x,t)|\,d\mu(x) \right)^{p-1} d\nu(t) \right] d\mu(x) \\
&\le \int_X \|F(x,\,\cdot\,)\|_{L^p(d\nu)} \|(\int_X |F(x,\,\cdot\,)|\,d\mu(x) \Big)^{p-1} \|_{L^{p'}(d\nu)}\,d\mu(x) \\
&= \left\| \int_X |F(x,\,\cdot\,)|\,d\mu(x) \right\|_{L^p(d\nu)}^{p-1} \int_X \|F(x,\,\cdot\,)\|_{L^p(d\nu)}\,d\mu(x),
\end{aligned}$$

and the conclusion follows by dividing both sides by the first factor above. Using Exercise **??** (c) we see that the (only) inequality above can be reversed when $0 < p < 1$ and $F(x,\,\cdot\,)$ is not zero ν-a.e.

??. Use that $\mu(\{|f_1 + \cdots + f_N| > \alpha\}) \le \sum_{j=1}^N \mu(\{|f_j| > \alpha/N\})$ and Exercise **??** (a) and (c).

??. Let f_n be Cauchy in L^p. Pass to a subsequence such that $\|f_{n_{i+1}} - f_{n_i}\|_{L^p} \le 2^{-i}$. Let

$$g_k = \sum_{i=1}^k |f_{n_{i+1}} - f_{n_i}| \qquad \text{and} \qquad g = \sum_{i=1}^\infty |f_{n_{i+1}} - f_{n_i}|.$$

Since $\|g_k\|_{L^p}^p \le \sum_{i=1}^k \|f_{n_{i+1}} - f_{n_i}\|_{L^p}^p \le C$, Fatou's lemma implies that $\|g\|_{L^p} < \infty$, hence g is finite μ-a.e. Hence the series

$$f(x) = f_{n_1}(x) + \sum_{i=1}^\infty (f_{n_{i+1}} - f_{n_i})$$

converges absolutely. Now use Fatou's lemma to show that f_{n_k} converges to f in L^p. Conclude that f_n converges to f in L^p.

??. Observe that for all $\varepsilon > 0$,

$$\{x :\in X :\; f_n(x) \to f(x)\} \subset \cup_{m=1}^\infty \cap_{n=m}^\infty \{x :\in X :\; |f_n(x) - f(x)| < \varepsilon\}.$$

Since the second set has full measure, its complement must have measure zero which implies that

$$\lim_{m\to\infty} \mu(\cup_{n=m}^\infty \{x :\in X :\; |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

The latter implies convergence in measure.

??. Part (a) is easily checked while part (b) follows from a simple integration. To prove part (c) for f_γ observe that the integral

$$q\int_\gamma^\infty \alpha^{q-1}\lambda_f(\alpha)\,d\alpha$$

converges since $\lambda_f(\alpha)\le \alpha^{-p}\|f\|_{L^{p,\infty}}^p$ and $q<p$. Argue similarly for f^γ.

??. Use that

$$\int_E |f(x)|^q\,d\mu(x)\le \int_0^\infty q\alpha^{q-1}\min\left(\mu(E),\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}\right)d\alpha.$$

??. (a) follows directly from the previous exercise. (b) and (c) are easy.

??. (a) Take $\alpha=N/\sigma(j_0)$. Then the set $\{f_\sigma>\alpha\}$ is the union of the intervals $[\frac{j-1}{N},\frac{j}{N}]$ over all j such that $\sigma(j)\le\sigma(j_0)$. It follows that $\lambda_{|f_\sigma|}(N/\sigma(j_0))=N/\sigma(j_0)$, hence $\|f_\sigma\|_{L^{1,\infty}}=1$. (b) Use that $\sum_{\sigma\in S_N}\frac{1}{\sigma(j)}=N!\big(1+\frac12+\cdots+\frac1N\big)\chi_{[0,1]}$. (c) Assume that $L^{1,\infty}$ were normable and let $P(f)$ be a norm comparable to $\|f\|_{L^{1,\infty}}$. Then apply the triangle inequality with respect to P to the functions f_σ and use the fact that $P(f)$ is comparable to $\|f\|_{L^{1,\infty}}$ and parts (a) and (b) to obtain a contradiction. For part (d) argue similarly: observe that $\|f_\sigma\|_{L^{1,\infty}(\mathbf{R}^n)}=1$ while the weak L^1 norm of the sum of all the f_σ's is equal to $N^n!(1+\frac12+\cdots+\frac{1}{N^n}$.

??. (a) Use that $\int |f|\,d\mu\le\int_0^s\lambda_f(\alpha)\,d\alpha\le\int_0^s\alpha^{-p}\|f\|_{L^{p,\infty}}^p\,d\alpha$.

(b) Need that $\alpha^p\lambda_{\max|f_j|}(\alpha)\le\alpha^p\sum_{j=1}^m\lambda_{f_j}(\alpha)$.

(c) Fix $\alpha>0$. We have

$$\begin{aligned}
&\alpha^p\lambda_{f_1+\cdots+f_m}(\alpha)\\
\le&\alpha^p\mu(\{|f_1+\cdots+f_m|>\alpha,\quad \max|f_j|\le\alpha\})+\alpha^p\lambda_{\max|f_j|}(\alpha)\\
\le&\alpha^{p-1}\int\limits_{\substack{|f_1+\cdots+f_m|>\alpha\\ \max|f_j|\le\alpha}}|f_1+\cdots+f_m|\,d\mu+\|\max|f_j|\|_{L^{p,\infty}}^p\\
\le&\alpha^{p-1}\sum_{j=1}^m\int\limits_{|f_j|\le\alpha}|f_j|\,d\mu+\|\max|f_j|\|_{L^{p,\infty}}^p\le\frac{1}{1-p}\sum_{j=1}^m\|f_j\|_{L^{p,\infty}}^p+\sum_{j=1}^m\|f_j\|_{L^{p,\infty}}^p.
\end{aligned}$$

??. Without loss of generality we may assume that $\|f_j\|_{L^{p_j,\infty}}=1$. Write

$$\begin{gathered}
\lambda_{f_1\dots f_k}(\alpha)\le\\
\mu(\{|f_1|>\alpha/s_1\})+\cdots+\mu(\{|f_{k-1}|>s_{k-2}/s_{k-1}\})+\mu(\{|f_k|>s_{k-1}\})\\
\le(s_1/\alpha)^{p_1}+(s_2/s_1)^{p_2}+\cdots+(s_{k-1}/s_{k-2})^{p_{k-1}}+(1/s_{k-1})^{p_k}.
\end{gathered}$$

Set $x_1=s_1/\alpha$, $x_2=s_2/s_1,\dots,x_k=1/s_{k-1}$. Minimize $x_1^{p_1}+\dots x_k^{p_k}$ subject to the constraint $x_1\dots x_k=1/\alpha$. This could be done using Langrange multipliers.

??. The first inequality is an easy consequence of Hölder's inequality. The second inequality follows from the simple observation that

$$\sup_{\alpha>0}\big(\alpha\lambda_f(\alpha)^{\frac1p}\big)\le\sup_{\alpha>0}\big(\alpha^{1-\theta}\lambda_f(\alpha)^{\frac{1-\theta}{p_0}}\big)\sup_{\alpha>0}\big(\alpha^\theta\lambda_f(\alpha)^{\frac{\theta}{p_1}}\big).$$

SECTION 1.2

??. Trivial. Use that if $(f*g)(x) \neq 0$, then $B \cap A^{-1}\{x\} \neq \emptyset$ and thus $x \in AB$.

??. Follows from a suitable change of variables.

??. Let ϕ be a continuous function with compact support on G which approximates g in $L^p(G)$. Let F be the support of ϕ and let W be a symmetric relatively compact neighborhood of the origin. Given $\varepsilon > 0$ find a neighborhood U of the origin such that $U \subset W$ and such that $ab^{-1} \in U$ implies that $|\phi(a) - \phi(b)| \leq \varepsilon \mu(WF)^{-1/p'}$. Then for $st^{-1} \in U$ we have

$$\int_G |\phi(sx) - \phi(tx)|^{p'} d\mu(x) = \int_{WF} |\phi(st^{-1}x) - \phi(x)|^{p'} d\mu(x) \leq \varepsilon.$$

Now approximate g by ϕ in $L^p(G)$ and use density to prove the right continuity of $t \to {}_tg$. Finally observe that

$$|(f*g)(x) - (f*g)(y)| \leq \int_G |f(z)||{}_x\widetilde{g}(z) - {}_y\widetilde{g}(z)|\, d\mu(z).$$

Apply Hölder's inequality with exponents p and p' to deduce the right continuity of $f*g$. The last conclusion follows easily.

??. Trivial.

??. Just check that $\lambda(sA) = \lambda(A)$ for any measurable set A and any $s > 0$ by changing variables in the integral defining $\lambda(A)$.

??. Check the dilation invariance of these measures with respect to left or right change of variables.

??. On the multiplicative group $(\mathbf{R}^+, \frac{dt}{t})$ consider the convolution of the function $|f(x)|x^{1/p}$ with the function $x^{-1/p'}\chi_{[1,\infty)}$.

??. Similarly, on the multiplicative group $(\mathbf{R}^+, \frac{dt}{t})$ consider the convolution of $|f(x)|x^{1-b/p}$ with the function $x^{-b/p}\chi_{[1,\infty)}$ and $|f(x)|x^{1+b/p}$ with the function $x^{b/p}\chi_{(0,1]}$ to deduce this pair of inequalities due to Hardy.

??. It is trivial that $\|T\|_{L^p \to L^p} \leq \|K\|_{L^1}$. To see the converse inequality, fix $0 < \varepsilon < 1$ and let N be a positive integer. Let $\chi_N = \chi_{B(0,N)}$ and for any $R > 0$ let $K_R = K\chi_{B(0,R)}$, where $B(x,R)$ is the ball of radius R centered at x. Observe that for $|x| \leq (1-\varepsilon)N$, we have $B(x,N) \cap B(0,N\varepsilon) = B(0,N\varepsilon)$ thus $\int_{\mathbf{R}^n} \chi_N(x-y)K_{N\varepsilon}(y)\, dy = \int_{\mathbf{R}^n} K_{N\varepsilon}(y)\, dy = \|K_{N\varepsilon}\|_{L^1}$.

$$\frac{\|K*\chi_N\|_{L^p}^p}{\|\chi_N\|_{L^p}^p} \geq \frac{\|K_{N\varepsilon}*\chi_N\|_{L^p(B(0,(1-\varepsilon)N)}^p}{\|\chi_N\|_{L^p}^p} \geq \frac{\|K_{N\varepsilon}\|_{L^1}^p |B(0,(1-\varepsilon)N)|}{|B(0,N)|},$$

Now let $N \to \infty$ first and then $\varepsilon \to 0$ to obtain the required conclusion.

??. For ε, N as in the previous exercise, let $K_{N^\varepsilon}(x) = K(x)\chi_{N^{-\varepsilon} \leq x \leq N^\varepsilon}$ and $f_N = \chi_{[N^{-1},N]}$. Let $B_{N,\varepsilon} = \{x :\ N^{-(1-\varepsilon)} \leq x \leq N^{1-\varepsilon}\}$. For $x \in B_{N,\varepsilon}$ we have that $\{y :\ N^{-1} \leq xy^{-1} \leq N,\ N^{-\varepsilon} \leq y \leq N^\varepsilon\} = \{y :\ N^{-\varepsilon} \leq y \leq N^\varepsilon\}$ and thus $\int_0^\infty f_N(xy^{-1})K_{N^\varepsilon}(y)\, dy = \int_{\mathbf{R}^n} K_{N^\varepsilon}(y)\, dy = \|K_{N^\varepsilon}\|_{L^1}$. Then

$$\frac{\|K*f_N\|_{L^p}^p}{\|f_N\|_{L^p}^p} \geq \frac{\|K_{N^\varepsilon}*f_N\|_{L^p(B_{N,\varepsilon})}^p}{\|\chi_N\|_{L^p}^p} \geq \frac{\|K_{N^\varepsilon}\|_{L^1}^p \int_{N^{-(1-\varepsilon)}}^{N^{1-\varepsilon}} \frac{dt}{t}}{\int_{N^{-1}}^N \frac{dt}{t}},$$

Now let $N \to \infty$ first and then $\varepsilon \to 0$ to obtain the required conclusion.

??. Suppose first that f is C^∞ on the sphere $\mathbf{S}^{n-1}$. Then by uniform continuity, given $\varepsilon > 0$, there exists an $\delta > 0$ such that $|t-s| < \delta$ implies $|f(t) - f(s)| < \varepsilon/\omega_{n-1}^{1/q}$. Observe that for $x \in \mathbf{S}^{n-1}$ we have $|A(x) - B(x)| = |(B^{-1}A)(x) - (x)| = |(B^{-1}A)(e_1) - e_1| = |A(e_1) - B(e_1)|$. Thus if $|A(e_1) - B(e_1)| < \delta$ then $|f(A(x)) -$

$f(B(x))| < \varepsilon/\omega_{n-1}^{1/q}$ and hence the L^q norm of $x \to f(A(x))-f(B(x))$ is smaller than ε. Now if $f \in L^q(\mathbf{S}^{n-1})$, find a $g \in C^\infty(\mathbf{S}^{n-1})$ such that $\|f-g\|_{L^q(\mathbf{S}^{n-1})} < \varepsilon/3$. Pick a δ that works for g for $\varepsilon/3$ and use the triangle inequality to derive the required conclusion.

??. Observe that

$$\int\limits_{\mathbf{S}^{n-1}}\int\limits_{\mathbf{S}^{n-1}} F(\theta)G(\varphi)K(\theta\cdot\varphi)d\varphi d\theta \le \Big(\int\limits_{\mathbf{S}^{n-1}}\Big(\int\limits_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\theta\Big)^p d\varphi\Big)^{\frac{1}{p}}\|G\|_{L^{p'}(\mathbf{S}^{n-1})}.$$

Hölder's inequality applied to the functions F and 1 with respect to the measure $K(\theta\cdot\varphi)\,d\theta$ gives

$$\int_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\theta \le \left(\int_{\mathbf{S}^{n-1}} F(\theta)^p K(\theta\cdot\varphi)\,d\theta\right)^{1/p}\left(\int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\theta\right)^{1/p'}.$$

Using Fubini's theorem bound this last expression by

$$\|F\|_{L^p(\mathbf{S}^{n-1})}\|G\|_{L^{p'}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\varphi.$$

Observe that equality is attained if and only if both F and G are constants.

SECTION 1.3

??. The proof of Theorem **??** requires only minor modifications.

??. Using the hint we come up with the constant:

$$\left[2\left(\frac{\frac{p+1}{2}}{\frac{p+1}{2}-1}+\frac{\frac{p+1}{2}}{r-\frac{p+1}{2}}\right)^{\frac{2}{p+1}}\right]^{\frac{\frac{1}{p}-\frac{1}{r}}{\frac{2}{p+1}-\frac{1}{r}}} A_0^{\frac{\frac{1}{p}-\frac{1}{r}}{1-\frac{1}{r}}} A_1^{\frac{1-\frac{1}{p}}{1-\frac{1}{r}}}.$$

Since $1<p<r$ we have that

$$\frac{\frac{1}{p}-\frac{1}{r}}{\frac{2}{p+1}-\frac{1}{r}} < \frac{p+1}{2p} \qquad \text{and} \qquad \frac{\frac{p+1}{2}}{r-\frac{p+1}{2}} < \frac{\frac{p+1}{2}}{\frac{p+1}{2}-1}.$$

Using these estimates and the fact that $(p+1)^{1/p} \le 2$ when $p>1$ we obtain the required conclusion.

??. We first consider the case of $p_1=\infty$ and $0<p_0<\infty$ arbitrary. Since $|T(f)| \le T(|f|)$, we can assume that $f \ge 0$. For $0<\lambda<1$ and $\alpha>0$, write $f=f_0+f_1$, where f_0 is $f-\lambda\alpha/A_1$ for $f \ge \lambda\alpha/A_1$ and zero otherwise. Then

$$\alpha < |Tf| \le |Tf_0|+|Tf_1| \le |Tf_0| + A_1\frac{\lambda\alpha}{A_1} \le |Tf_0|+\lambda\alpha$$

which implies that $\mu(\{|Tf|>\alpha\}) \le \mu(\{|Tf_0|>(1-\lambda)\alpha\})$. It follows from the hypothesis that

$$\mu(\{|Tf_0|>(1-\lambda)\alpha\}) \le \frac{A_0^{p_0}\|f\|_{L^{p_0}}^{p_0}}{((1-\lambda)\alpha)^{p_0}} \le \frac{A_0^{p_0}}{((1-\lambda)\alpha)^{p_0}}\int\limits_{f>\frac{\lambda\alpha}{A_1}}\left(f-\frac{\lambda\alpha}{A_1}\right)^{p_0}d\mu.$$

Multiply by $p\alpha^{p-1}$, integrate from 0 to ∞ and apply Fubini's theorem to deduce an estimate that depends on λ. Then choose $\lambda=(p-p_0)/p \in (0,1)$ to minimize the outcome. The required conclusion follows.

In the case where $p_1 < \infty$ split $f = f_0 + f_1$ where f_0 is $f - \delta\alpha$ for $f \geq \delta\alpha$ and zero otherwise. Then use that

$$\begin{aligned}\mu(\{|Tf| > \alpha\}) \leq & \mu(\{|Tf_0| > (1-\lambda)\alpha\}) + \mu(\{|Tf_1| > \lambda\alpha\}) \\ \leq & \frac{A_0^{p_0}\|f_0\|_{L^{p_0}}^{p_0}}{((1-\lambda)\alpha)^{p_0}} + \frac{A_1^{p_1}\|f_1\|_{L^{p_1}}^{p_1}}{(\lambda\alpha)^{p_1}}.\end{aligned}$$

Multiply by $p\alpha^{p-1}$, integrate from 0 to ∞ and apply Fubini's theorem to obtain an estimate that depends on δ. Optimize over δ to deduce the required conclusion.

??. Let $w = (b-z)/(b-a)$ and $S_w = T_z$. Then S_w is an analytic family on the usual strip and the hypotheses imply that S_w satisfies (**??**). Also S_{j+iy} maps L^{p_j} to L^{q_j} with bound $M_j(b-(b-a)y)$, $(j=0,1)$, which satisfy (**??**). Apply Theorem **??** to S_w.

??. We are given that the operator given by convolution with K_λ maps $L^2(\mathbf{R}^n)$ into itself with norm one. Use the definition of Bessel functions in Appendix B1 to rewrite $K_\lambda(x) = c_\lambda |x|^{-\lambda-\frac{n}{2}} J_{\lambda+\frac{n}{2}}(2\pi|x|)$ for some constant c_λ bounded by $A\exp(b|\mathrm{Im}\,\lambda|)$ for some $A, b > 0$. Use the asymptotics for the Bessel functions in Appendix B3 to obtain that K_λ is integrable over $\mathbf{R}^n$ when $\lambda > (n-1)/2$. Then apply Theorem **??**.

??. First show that under the same hypothesis, Lemma **??** gives the stronger conclusion that $|F(z)| \leq B(z)$ for $z \in S$. Indeed, fix a y_0 in $\mathbf{R}^1$ and consider the function $F_1(x+iy) = F(x+iy+iy_0)$. (**??**) implies that $\log|F_1(x+iy)| \leq Ae^{|y_0|}e^{a|y|}$. An application to Lemma **??** to the function F_1 gives an estimate for $|F_1(x+i0)|$ which, when translated to $|F(x+iy_0)|$, is nothing else but $|F(x+iy_0)| \leq B(x+iy_0)$. Using this strengthening of Lemma **??**, we obtain from the proof of Theorem **??** the stronger conclusion that $\|T_z(f)\|_{L^q} \leq B(z)\|f\|_{L^p}$.

??. Write $f = \sum_{k=0}^{\infty} f_k$, where $f_k = f\chi_{S_k}$, $S_k = \{2^k \leq |f| < 2^{k+1}\}$, $f_0 = f\chi_{S_0}$, and $S_0 = \{|f| < 2\}$. Use that

$$\int_Y |Tf_0|\,d\nu \leq (\nu(Y))^{1/2}\|Tf_0\|_{L^2} \leq 2A(\mu(X)\nu(Y))^{1/2}.$$

Hölder's inequality with exponents $k+1$ and $(k+1)/k$ gives

$$\int_Y |Tf_k| d\nu \leq 2A(\nu(Y))^{\frac{1}{k+1}} k^\alpha 2^k \mu(S_k)^{\frac{k}{k+1}}.$$

When $\mu(S_k) \geq 3^{-k-1}$ control $\mu(S_k)^{\frac{k}{k+1}}$ by $3\mu(S_k)$ and when $\mu(S_k) \leq 3^{-k-1}$ control $\mu(S_k)^{\frac{k}{k+1}}$ by 3^{-k}. Obtain the required conclusion by using that

$$\sum_{k=1}^{\infty} 2^k k^\alpha \mu(S_k) \leq \int_X |f|(\log^+|f|)^\alpha d\mu\,.$$

??. To do the first integral in the Exercise write $\cosh(\pi y) = \frac{1}{2}(e^{\pi y} + e^{-\pi y})$. Then use the change of variables $z = e^{\pi y}$ to rewrite it as $\frac{1}{\pi}\int_0^\infty \frac{\sin(\pi x)}{(z+\cos(\pi x))^2 + \sin^2(\pi x)} dz$. This last integral is easily seen to be equal to x. Likewise with the second integral.

SECTION 1.4

??. (a) Take $g = \sum_{j=1}^{k} b_j \chi_{E_j}$. Assume without loss of generality that $E_1 \subset \cdots \subset E_k$ and that $\mu(E_k) < \infty$. Check that

$$\int_A g\, d\mu \le \sum_{j=1}^{k} b_j \min(\mu(A), \mu(E_j)) \le \int_0^{\mu(A)} g^*(t)\, dt.$$

(b) Fix g and take $f = \sum_{j=1}^{m} a_j \chi_{F_j}$ where $F_1 \subset \cdots \subset F_m$. Use exercise 1 and that

$$f^*(t) = \sum_{j=1}^{m} a_j \chi_{[0,\mu(F_j))}.$$

??. Since $g^*(t) \le c\min(t^{-1/q_0}, t^{-1/q_1})$, it follows that $g^*(t)t^{1/q} \le c\min(t^{\varepsilon}, t^{-\delta})$ for some $\varepsilon, \delta > 0$. It follows that $g^*(t)t^{1/q} \in L^s(\frac{dt}{t})$ for all $0 < s < \infty$.

??. (a) is trivial. (b) We claim that

$$f^*(t) \le f^{**}(t) \le \left(\frac{1}{t}\int_0^t (f^*(s))^r\, ds\right)^{1/r} \qquad \text{for all } t > 0.$$

The estimate above together with Hardy's inequality (Exercise **??** in section **??**) with $b = q/r - q/p > 0$ imply the assertion in part (b).

The first inequality in the claim above is a consequence of the fact that if $E = \{|f| \ge f^*(t)\}$, then $\mu(E) \ge t$. Therefore

$$(f^*)^r(t) = (|f|^r)^*(t) \le \frac{1}{\mu(E)}\int_E |f|^r\, d\mu \le (f^{**}(t))^r.$$

To prove the second inequality in the claim above we observe that for $\mu(E) \ge t$

$$\frac{1}{\mu(E)}\int_E |f|^r\, d\mu \le \frac{1}{\mu(E)}\int_0^{\mu(E)} (|f|^r)^*(s)\, ds \le \frac{1}{t}\int_0^t (f^*(s))^r\, ds,$$

which follows from Hardy-Littlewood's inequality $\displaystyle\int_X f_1 f_2 d\mu \le \int_0^\infty f_1^*(s) f_2^*(s)\, ds$, the facts that $(\chi_E)^* = \chi_{[0,\mu(E))}$ and $(f^*)^r = (|f|^r)^*$, and the fact that the function $\displaystyle y \to \frac{1}{y}\int_0^y f^*(s)\, ds$ is decreasing for $y > 0$.

(c) Part (b) implies that $L^{p,q}$ is metrizable, in particular $(f,g) \to |||f-g|||_{L^{p,q}}^r$ is a metric which generates the same topology as the quasinorm $\|\cdot\|_{L^{p,q}}$. Normability for $q > 1$ follows by selecting $r = 1$.

??. It is easy to show that countable linear combinations of simple functions are dense in $L^{p,\infty}$ when X is σ-finite. Furthermore the function $f(x) = x^{-1/p}\chi_{x>0}$ cannot be approximated by a sequence of simple functions $L^{p,\infty}$. To see this, partition the interval $(0,\infty)$ into small subintervals of length $\varepsilon > 0$ and let f_ε be the step function $\sum_{-[1/\varepsilon]}^{[1/\varepsilon]} f(k\varepsilon)\chi_{[k\varepsilon,(k+1)\varepsilon]}(x)$. Show that for some $c > 0$ we have $\|f_\varepsilon - f\|_{L^{p,\infty}} \ge c$ for all $\varepsilon > 0$ which implies that no simple function can converge to f in $L^{p,\infty}$.

??. (a) see hint

(b) First show when d is a simple right continuous decreasing function on $[0,\infty)$ there exists a measurable f on X such that $f^* = d$. Then approximate a general continuous d by an increasing sequence of simple functions d_N. Apply a convergence theorem to show that the sequence of the corresponding f_N's converges to some f with $f^* = d$.

(c) Let $t = \mu(A)$ and define $A_1 = \{x : |g(x)| > g^*(t)\}$ and $A_2 = \{x : |g(x)| \geq g^*(t)\}$. Then $A_1 \subset A_2$ and $\mu(A_1) \leq t \leq \mu(A_2)$ by Proposition **??** (9). Pick $\widetilde{A}$ such that $A_1 \subset \widetilde{A} \subset A_2$ and $\mu(\widetilde{A}) = t$ by part (a). Then

$$\int_{\widetilde{A}} g\, d\mu = \int_X g\chi_{\widetilde{A}}\, d\mu = \int_0^\infty (g\chi_{\widetilde{A}})^* ds = \int_0^{\mu(\widetilde{A})} g^*(s)\, ds.$$

(d) Let $f = \sum_{j=1}^N a_j \chi_{A_j}$ where $a_1 > a_2 > \cdots > a_N > 0$ and the A_j are pairwise disjoint. Write f as $\sum_{j=1}^N b_j \chi_{B_j}$ where $b_j = (a_j - a_{j+1})$ and $B_j = A_1 \cup \cdots \cup A_j$. Pick $\widetilde{B_j}$ as in part (c). Then $\widetilde{B_1} \subset \cdots \subset \widetilde{B_N}$ and the function $f_1 = \sum_{j=1}^N b_j \chi_{\widetilde{B_j}}$ has the same distribution function as f. It follows from part (c) that

$$\int_X f_1 g\, d\mu = \int_0^\infty f^*(s) g^*(s)\, ds.$$

The case of a general f follows from the the case f simple, Exercise **??**, and approximation.

??. A repeated application of the hypothesis $\|x + y\| \leq K(\|x\| + \|y\|)$ gives

$$\|x_1 + \cdots + x_n\| \leq \max_{1\leq j \leq n} [(2K)^j \|x_j\|],$$

for all $x_1, \dots, x_n$ in X. (Here we used $2K \geq 1$.) Now define a function $H : X \to \mathbf{R}^1$ by setting $H(0) = 0$ and $H(x) = 2^{j/\alpha}$ if $2^{j-1} < \|x\|^\alpha \leq 2^j$. Then we clearly have

$$\|x\| \leq H(x) \leq 2^{1/\alpha} \|x\|$$

for all $x \in X$. The result we want to prove will follow from this inequality and the following claim

$$\|x_1 + \cdots + x_n\|^\alpha \leq 2(H(x_1)^\alpha + \cdots + H(x_n)^\alpha).$$

The claim is certainly true when $n = 1$. Suppose it is true when $n = m$. We will show that it is true when $n = m+1$. Without loss of generality assume that $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_{m+1}\|$. Then we must have $H(x_1) \geq H(x_2) \geq \cdots \geq H(x_{m+1})$.

If all the $H(x_j)$'s are distinct, since $H(x_j)^\alpha$ are distinct powers of 2 and therefore they must satisfy $H(x_j)^\alpha \leq 2^{-j+1} H(x_1)^\alpha$. Then

$$\|x_1 + \cdots + x_{m+1}\|^\alpha \leq \big[\max_{1\leq j\leq m+1} (2K)^j \|x_j\|\big]^\alpha \leq \big[\max_{1\leq j\leq m+1} (2K)^j H(x_j)\big]^\alpha \leq$$

$$\big[\max_{1\leq j\leq m+1} (2K)^j 2^{1/\alpha} 2^{-j/\alpha} H(x_1)\big]^\alpha = 2H(x_1)^\alpha \leq 2(H(x_1)^\alpha + \cdots + H(x_{m+1})^\alpha),$$

where the last equality above follows from the choice of α.

We now consider the case where $H(x_j) = H(x_{j+1})$ for some $1 \leq j \leq m$. Then for some integer r we must have $2^{r-1} < \|x_{j+1}\|^\alpha \leq \|x_j\|^\alpha \leq 2^r$ and $H(x_j) = 2^{r/\alpha}$. Next observe that

$$\|x_j + x_{j+1}\|^\alpha \leq K^\alpha (\|x_j\| + \|x_{j+1}\|)^\alpha \leq K^\alpha (2\, 2^r)^\alpha \leq 2^{r+1},$$

which implies that $H(x_j + x_{j+1})^\alpha \leq 2^{r+1} = 2^r + 2^r = H(x_j)^\alpha + H(x_{j+1})^\alpha$. Now apply the inductive hypothesis to $x_1, \dots, x_{j-1}, x_j + x_{j+1}, x_{j+1}, \dots, x_m$ and use the inequality above to obtain the required conclusion.

??. Since a general simple function f can be written as $f_1 - f_2 + if_3 - if_4$ where $f_j \geq 0$, we may restrict our attention to nonnegative f. Let $f = \sum_{j=1}^N a_j \chi_{E_j}$, where $a_1 > a_2 > \cdots > a_N > 0$, $\mu(E_j) < \infty$, and they are pairwise disjoint. Let

$F_j = E_1 \cup \cdots \cup E_j$, $B_0 = 0$, and $B_j = \mu(E_1) + \cdots + \mu(E_j) = \mu(F_j)$ for $j \geq 1$. Then f can be written as $\sum_{j=1}^{N}(a_j - a_{j+1})\chi_{F_j}$, where $a_{N+1} = 0$. Then

$$\|Tf\|_Z = \||Tf|\|_Z \leq \sum_{j=1}^{N}(a_j - a_{j+1})\|T(\chi_{F_j})\|_Z \leq A\sum_{j=1}^{N}(a_j - a_{j+1})(\mu(F_j))^{1/p}$$
$$= C(p)A\sum_{j=0}^{N-1} a_{j+1}(B_{j+1}^{1/p} - B_j^{1/p}) = C(p)A\|f\|_{L^{p,1}},$$

where the penultimate equality follows from a summation by parts.

??. (a) On $\mathbf{R}^1$ consider

$$f(x) = \sum_{k=2}^{\infty} \frac{k}{(\log k)^{1/q_1}}\chi_{[(k+1)^{-p},k^{-p}]}(x).$$

On $\mathbf{R}^n$ consider suitable products of these functions.

(b) Let E be a subset of X of positive measure. Without loss of generality assume that $\mu(E) = 1$. Split E as a union of k disjoint subsets E_j each of which has measure $1/k$. Let $f = \sum_{j=1}^{k} a_j\chi_{E_j}$, where $a_1 > a_2 > \cdots > a_k > 0$. Use the formula in Example **??** to prove that

$$\|f\|_{L^{p,q}} \approx k^{-1/p}\left(\sum_{j=1}^{k} a_j^q\big(j^{q/p} - (j-1)^{q/p}\big)\right)^{1/q}$$

where the constants only depend on p and q. Now take $a_j = j^{-1/p}$. We have $\|f\|_{L^{p,q}} \approx k^{-1/p}(\log k)^{1/q}$ and the conclusion follows if k is large enough.

??.

??. (a) Fix an $\alpha > 0$ and write $f_n = u_n + v_n + w_n$, where $u_n = f_n\chi_{|f_n|\leq\alpha/2}$, $v_n = f_n\chi_{|f_n|>\alpha/2\lambda_n}$, and $w_n = f_n\chi_{\alpha/2<|f_n|\leq\alpha/2\lambda_n}$. Let $u = \sum_n \lambda_n u_n$, $v = \sum_n \lambda_n v_n$, and $w = \sum_n \lambda_n w_n$. Clearly $|u| \leq \alpha/2$. Also $\{v \neq 0\} \subset \cup_n\{|f_n| > \alpha/2\lambda_n\}$, hence $\mu(\{v \neq 0\}) \leq 2/\alpha$. Finally

$$\int_X |w|d\mu \leq \sum_n \lambda_n \int_X |f_n|\chi_{\alpha/2<|f_n|\leq\alpha/2\lambda_n}d\mu$$
$$\leq \sum_n \lambda_n\left[\int_{\alpha/2}^{\alpha/2\lambda_n} \lambda_{f_n}(\beta)d\beta + \int_0^{\alpha/2} \lambda_{f_n}(\alpha/2)d\beta\right]$$
$$\leq \sum_n \lambda_n \log(\frac{1}{\lambda_n}) + 1$$

Now use that

$$\mu(\{|u + v + w| > \alpha\}) \leq \mu(\{|u| > \alpha/2\}) + \mu(\{|v| \neq 0\}) + \mu(\{|w| > \alpha/2\})$$

to deduce the conclusion. (b) follows directly from (a). In part (c), for $\delta > 0$, let $c_\delta = \sum_{n=1}^{\infty} 2^{-\delta n} = (3^\delta - 1)^{-1}$. Then $\alpha\mu\big(\{\sum_{n=1}^{\infty} 2^{-n}f_n > \alpha\}\big) \leq \sum_{n=1}^{\infty} \alpha\mu\big(\{2^{-n}f_n > C_\varepsilon^{-1}2^{-\varepsilon n}\alpha\}\big) \leq C_\varepsilon C_{1-\varepsilon}$.

??. Let $f(x) = |x|^{-n}$ and $f_k(x) = f(x) + k^{-1}\chi_{|x|\leq k^2}$. Then $\|f_k - f\|_{L^\infty} \to 0$ as $k \to \infty$ but $\|f_k\|_{L^{1,\infty}} \geq k^{-1}|\{x \in \mathbf{R}^n : f_k(x) > k^{-1}\}| \geq k^{-1}v_nk^{2n} \to \infty$ as $k \to \infty$.

??. Consider the linear functional $T(\alpha x) = \alpha \|x\|$ defined on the one dimensional subspace generated by x. Use the Hahn-Banach theorem to extend T to the whole space. This statement is false for all non-normable quasi-Banach spaces, since otherwise it would imply that they are normable.

??. Write $f = \sum_{n=-\infty}^{\infty} f\chi_{A_n}$, where A_n is as in the proof of Lemma **??**. Set $c_n = 2^{n/p} f^*(2^n)$ and $f_n = c_n^{-1} f\chi_{A_n}$. Then f_n is supported on A_n which has measure 2^n and it is bounded by $2^{-n/p}$. The equivalence of norms follows easily.

??. We denote below both measures as Lebesgue measures. Fix $1 < p \le 2$. We will show that for any $\delta > 0$ we have the estimate

$$\sup_{0<|A|<\infty} \frac{\|T(\chi_A)\|_{p,\infty}}{|A|^{1/p}} \le 4\delta^{-1+1/p}\|T\| + \frac{2p}{p-1}\delta^{-1+2/p} \sup_{0<|B|<\infty} \frac{\|T^*(\chi_B)\|_{p,\infty}}{|B|^{1/p}}.$$

Once this estimate is established, apply it again for T^* to obtain

$$\sup_{0<|B|<\infty} \frac{\|T^*(\chi_B)\|_{p,\infty}}{|B|^{1/p}} \le 4\varepsilon^{-1+1/p}\|T^*\| + \frac{2p}{p-1}\varepsilon^{-1+2/p} \sup_{0<|A|<\infty} \frac{\|T(\chi_A)\|_{p,\infty}}{|A|^{1/p}}$$

for any $\varepsilon > 0$. Now choose $\delta = \|T^*\|^{-1}\|T\|$ and $\varepsilon = \gamma\|T\|^{-1}\|T^*\|$ for some $\gamma > 0$ and combine the two estimates above to obtain

$$\sup_{0<|A|<\infty} \frac{\|T(\chi_A)\|_{p,\infty}}{|A|^{1/p}} \le C\|T\|^{1/p}\|T^*\|^{1-1/p} + \frac{4p^2\gamma^{-1+2/p}}{(p-1)^2} \sup_{0<|A|<\infty} \frac{\|T(\chi_A)\|_{p,\infty}}{|A|^{1/p}}.$$

Picking $\gamma > 0$ so that $\frac{4p^2}{(p-1)^2}\gamma^{-1+2/p} = \frac{1}{2}$ we obtain the required result for $1 < p \le 2$. The case $2 \le p < \infty$ follows by applying the same argument to T^*.

Now note that for any $\eta, C_0 > 0$ and g integrable we have

$$\sup_{0<|B|\le C_0} |B|^{-\eta} \int_B |g| \le 2 \sup_{0<|B|\le C_0} |B|^{-\eta} \left| \int_B g \right|.$$

Indeed, let $B_1 = B \cap \{g > 0\}$ and $B_2 = B \cap \{g < 0\}$. Then $|B_1|, |B_2| \le |B| \le C_0$ and

$$|B|^{-\eta} \int_B |g| = |B|^{-\eta} \left| \int_{B_1} g - \int_{B_2} g \right| \le |B_1|^{-\eta} \left| \int_{B_1} g \right| + |B_2|^{-\eta} \left| \int_{B_2} g \right|.$$

We now prove the first estimate above. For a fixed set A of finite measure we have

$$\frac{\|T(\chi_A)\|_{p,\infty}}{|A|^{1/p}} \le \sup_{0<|B|<\infty} \frac{|B|^{-2+1/p}\big(\int_B |T(\chi_A)|^{1/2}\big)^2}{|A|^{1/p}}$$

$$\le \sup_{|B|\ge\delta|A|} \frac{|B|^{-2+1/p}\big(\int_B |T(\chi_A)|^{1/2}\big)^2}{|A|^{1/p}} + \sup_{0<|B|\le\delta|A|} \frac{|B|^{-2+1/p}\big(\int_B |T(\chi_A)|^{1/2}\big)^2}{|A|^{1/p}}$$

$$\le |A|^{-1}\delta^{-1+1/p} \sup_{0<|B|<\infty} |B|^{-1}\Big(\int_B |T(\chi_A)|^{1/2}\Big)^2 + \sup_{0<|B|\le\delta|A|} \frac{|B|^{-1+1/p}\int_B |T(\chi_A)|}{|A|^{1/p}}$$

$$\le 4|A|^{-1}\delta^{-1+1/p}\|T(\chi_A)\|_{1,\infty} + 2 \sup_{0<|B|\le\delta|A|} \frac{|B|^{-1+1/p}\big|\int_B T(\chi_A)\big|}{|A|^{1/p}}$$

$$\le 4\delta^{-1+1/p}\|T\| + 2 \sup_{0<|B|\le\delta|A|} \frac{|B|^{-1+2/p}\big|\int_A T^*(\chi_B)\big|}{(|A|\,|B|)^{1/p}}$$

$$\le 4\delta^{-1+1/p}\|T\| + 2\delta^{-1+2/p} \sup_{0<|B|<\infty} \frac{|A|^{-1+1/p}\int_A |T^*(\chi_B)|}{|B|^{1/p}} \qquad \text{(used that } 1<p\le 2)$$

$$\le 4\delta^{-1+1/p}\|T\| + \frac{2p}{p-1}\delta^{-1+2/p} \sup_{0<|B|<\infty} \frac{\|T^*(\chi_B)\|_{p,\infty}}{|B|^{1/p}}.$$

We used throughout that for $0<r<p<\infty$ we have

$$\|g\|_{p,\infty} \le \sup_{0<|B|<\infty} |B|^{-1/r+1/p}\left(\int_B |g|^r\right)^{1/r} \le \Big(\frac{p}{p-r}\Big)^{1/r}\|g\|_{p,\infty},$$

a fact proved in Exercise **??** of section **??**.

??. Imitate Thorin's proof of Riesz's interpolation theorem in the context of multilinear operators.

??. Let T denote the closed triangle with vertices $(1/p_j, 1/q_j, 1/r_j)$ in $\mathbf{R}^3$. First show that for all $(1/p, 1/q, 1/r)$ in T we have an estimate

$$\|T(\chi_A, \chi_B)\|_{L^{r,\infty}} \le M(p,q,r)\mu(A)^{1/p}\nu(B)^{1/q}.$$

Now use Lemma **??** twice to prove an analogous bilinear version. Finally fix a point $(1/p, 1/q, 1/r)$ in the interior of T. Pick four points $(1/a_j, 1/b_j, 1/c_j)$, $1\le j\le 4$, in the interior of T such that the points $(1/a_1, 1/b_1)$, $(1/a_2, 1/b_2)$, $(1/a_3, 1/b_3)$, and $(1/a_4, 1/b_4)$ satisfy $a_1<p$, $b_1<q$, $a_2>p$, $b_2<q$ $a_3<p$, $b_3>q$, $a_4>p$, and $b_4>q$. Now write $f=f_t+f^t$ and $g=g_t+g^t$ as in the proof of Theorem **??** and split $T(f,g)$ into four pieces. For each of the four pieces use an estimate of the form $L^{a_j,m}\times L^{b_j,m}\to L^{c_j,\infty}$ for an appropriate $1\le j\le 4$. The proof follows as in the linear case.

??. For O'Neil's inequality interpolate between the three trivial cases $p=q=1$, $1=p<q=\infty$, and $1=q<p=\infty$. For Hölder's inequality interpolate between the vertices of any triangle whose interior contains the point $(1/p, 1/q, 1/r)$.

CHAPTER 2

SECTION 2.1

??. It is easy to prove a version of Lemma **??** with (**??**) replaced by

$$\mu\Big(\bigcup_{i=1}^{l} B_{j_i}\Big) \geq C^{-2}\mu\Big(\bigcup_{i=1}^{k} B_i\Big).$$

Repeating mutatis-mutandis the proof of Theorem **??**, one obtains that $\widetilde{M}_\mu$ maps $L^1(\mathbf{R}^n,\mu)$ into $L^{1,\infty}(\mathbf{R}^n,\mu)$ with constant at most $D(\mu)$. The L^p result follows from Theorem **??** by interpolation between the cases $p=1$ and $p=\infty$.

The proof of Corollary **??** in this setting follows as in the case of Lebesgue measure.

??. (a) Let $\mathcal{F}$ be the given finite collection of intervals and let $\cup\mathcal{F}$ be the union of all intervals in $\mathcal{F}$. Consider the set $\mathbf{S}$ of all subcollections $\mathcal{G}$ of $\mathcal{F}$ such that $\cup\mathcal{G}=\cup\mathcal{F}$. Pick a subcollection $\mathcal{F}'$ in $\mathbf{S}$ with minimal cardinality. Then the union of all intervals in $\mathcal{F}'$ is equal to $\cup\mathcal{F}$ and no interval in $\mathcal{F}'$ is contained in a union of other intervals in $\mathcal{F}'$. Say that $\mathcal{F}'=\{(\alpha_1,\beta_1),\dots,(\alpha_N,\beta_N)\}$, where $\alpha_1<\alpha_2<\cdots<\alpha_N$. Set $\mathcal{F}_1=\{(\alpha_j,\beta_j):\ 1\le j\le N,\ j \text{ odd}\}$ and $\mathcal{F}_2=\{(\alpha_j,\beta_j):\ 1\le j\le N,\ j \text{ even}\}$. Then $(\cup\mathcal{F}_1)\cup(\cup\mathcal{F}_2)=\cup\mathcal{F}'=\cup\mathcal{F}$ and one can easily see that by of the choice of the subfamily $\mathcal{F}'$, the intervals in each family $\mathcal{F}_1$, $\mathcal{F}_2$ are pairwise disjoint.

We now show that the maximal function $\widetilde{M}_\mu$ of Exercise **??** maps $L^1(\mu)\to L^{1,\infty}(\mu)$ with constant at most 2. For each $x\in E_\alpha=\{x:\ (\widetilde{M}_\mu f)(x)>\alpha\}$ find an open interval I_x containing x such that $\mu(I_x)<\frac{1}{\alpha}\int_{I_x}|f(t)|\,d\mu(t)$. Since for every $y\in I_x$ we have $(\widetilde{M}f)(y)>\alpha$, it follows that $I_x\subset E_\alpha$. Thus the collection $\{I_x\}_{x\in E_\alpha}$ is an open covering of E_α. Let K be a compact subset of E_α and let $\mathcal{F}=\{I_{x_1},\dots,I_{x_m}\}$ be a finite subcover of K. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ as before. Then

$$\mu(K)\le\mu(\cup\mathcal{F})\le\mu(\cup\mathcal{F}_1)+\mu(\cup\mathcal{F}_2)<\frac{1}{\alpha}\int_{\cup\mathcal{F}_1}|f(t)|\,d\mu(t)+\frac{1}{\alpha}\int_{\cup\mathcal{F}_2}|f(t)|\,d\mu(t)$$

and the latter is bounded by $\dfrac{2}{\alpha}\displaystyle\int_{\mathbf{R}}|f(t)|\,d\mu(t)$. Since K was an arbitrary subset of E_α, the same estimate follows for E_α. Observe that proof of Theorem **??** gives the constant 3 instead of 2.

By taking f to be Dirac mass at the origin, or f_ε to be the sequence $(2\varepsilon)^{-1}\chi_{[-\varepsilon,\varepsilon]}$, we see that the best weak type $(1,1)$ norm for $\widetilde{M}$ is 2.

(b) We have

$$\frac{1}{\alpha}\int_A(|f|-\alpha)\,d\mu\le\frac{1}{\alpha}\int_{A\cap\{|f|>\alpha\}}(|f|-\alpha)\,d\mu\le\frac{1}{\alpha}\int_{\{|f|>\alpha\}}(|f|-\alpha)\,d\mu.$$

Now fix $\alpha>0$ and let $E_\alpha=\{\widetilde{M}_\mu f>\alpha\}$. If $\mu(\{f>\alpha\})=\infty$, then there is nothing to prove. Hence we may assume that $\mu(\{f>\alpha\})<\infty$. For every $x\in E_\alpha$ there is an open interval I_x containing x such that $\dfrac{1}{\mu(I_x)}\displaystyle\int_{I_x}|f(t)|\,d\mu(t)>\alpha$. As in part (a), it is easy to see that $I_x\subset E_\alpha$. By Lindelöf's theorem there is a countable subcollection I_j, $j=1,2,\dots$, such that $\cup_{j=1}^{\infty}I_j=E_\alpha$. Let $\mathcal{F}=\mathcal{F}^N=\{I_j:\ j=1,2,...,N\}$. By part (a) we obtain two subcollections $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$, each with pairwise disjoint intervals, a fact that implies

$$\mu(\cup\mathcal{F}_i)=\sum_{I\in\mathcal{F}_i}\mu(I)<\frac{1}{\alpha}\sum_{I\in\mathcal{F}_i}\int_I f(t)|\,d\mu(t)=\frac{1}{\alpha}\int_{\cup\mathcal{F}_i}|f(t)|\,d\mu(t),$$

for $i = 1, 2$. Therefore

$$\begin{aligned}
&\mu(\cup\mathcal{F}^N) = \mu(\cup\mathcal{F}_1) + \mu(\cup\mathcal{F}_2) - \mu((\cup\mathcal{F}_1)\cap(\cup\mathcal{F}_2))\\
<&\frac{1}{\alpha}\int_{\cup\mathcal{F}_1} |f(t)|\,d\mu(t) + \frac{1}{\alpha}\int_{\cup\mathcal{F}_2} |f(t)|\,d\mu(t) - \mu((\cup\mathcal{F}_1)\cap(\cup\mathcal{F}_2))\\
=&\frac{1}{\alpha}\int_{\cup\mathcal{F}^N} |f(t)|\,d\mu(t) + \frac{1}{\alpha}\int_{(\cup\mathcal{F}_1)\cap(\cup\mathcal{F}_2)} |f(t)|\,d\mu(t) - \mu((\cup\mathcal{F}_1)\cap(\cup\mathcal{F}_2))\\
\leq&\frac{1}{\alpha}\int_{\cup\mathcal{F}^N} |f(t)|\,d\mu(t) + \frac{1}{\alpha}\int_{\{|f|>\alpha\}} |f(t)|\,d\mu(t) - \mu(\{|f|>\alpha\}).
\end{aligned}$$

We obtain the final conclusion by letting $N \to \infty$ and by using the fact that $\mathcal{F}^N$ is an increasing sequence of $\mu-$measurable sets whose union is E_α.

(c) Let us take $f \geq 0$. The previous inequality implies

$$\begin{aligned}
&\int_{\mathbf{R}} (\widetilde{M}_\mu f)^p\,d\mu + \int_{\mathbf{R}} f^p\,d\mu\\
=&p\int_0^\infty \alpha^{p-1}\mu(\{\widetilde{M}_\mu f > \alpha\})\,d\alpha + p\int_0^\infty \alpha^{p-1}\mu(\{f>\alpha\})\,d\alpha\\
\leq&p\int_0^\infty \alpha^{p-2}\int_{\{\widetilde{M}_\mu f>\alpha\}} f\,d\mu\,d\alpha + p\int_0^\infty \alpha^{p-2}\int_{\{f>\alpha\}} f\,d\mu\,d\alpha\\
=&\frac{p}{p-1}\int_{\mathbf{R}} (\widetilde{M}_\mu f)^{p-1} f\,d\mu + \frac{p}{p-1}\int_{\mathbf{R}} f^p\,d\mu
\end{aligned}$$

and hence

$$\int_{\mathbf{R}} (\widetilde{M}_\mu f)^p\,d\mu \leq \frac{p}{p-1}\int_{\mathbf{R}} (\widetilde{M}_\mu f)^{p-1} f\,d\mu + \frac{1}{p-1}\int_{\mathbf{R}} f^p\,d\mu.$$

Hölder's inequality gives

$$\int_{\mathbf{R}} (\widetilde{M}_\mu f)^{p-1} f\,d\mu \leq \Big(\int_{\mathbf{R}} (\widetilde{M}_\mu f)^p\,d\mu\Big)^{(p-1)/p}\Big(\int_{\mathbf{R}} f^p\,d\mu\Big)^{1/p}$$

and hence

$$(p-1)\|\widetilde{M}_\mu f\|_{L^p(\mu)}^p \leq p\|\widetilde{M}_\mu f\|_{L^p(\mu)}^{p-1}\|f\|_{L^p(\mu)} + \|f\|_{p,\mu}^p$$

or equivalently

$$(p-1)\Big(\frac{\|\widetilde{M}_\mu f\|_{L^p(\mu)}}{\|f\|_{L^p(\mu)}}\Big)^p - p\Big(\frac{\|\widetilde{M}_\mu f\|_{L^p(\mu)}}{\|f\|_{L^p(\mu)}}\Big)^{p-1} - 1 \leq 0.$$

This shows that the ratio $\|\widetilde{M}_\mu f\|_{L^p(\mu)}/\|f\|_{L^p(\mu)}$ is less than or equal of the unique positive solution A_p of the equation $(p-1)x^p - px^{p-1} - 1 = 0$.

(d) Let $f_0(t) = |t|^{-\frac{1}{p}}$. Homogeneity gives that $\widetilde{M}(f_0)$ is a constant multiple of f_0 and thus it suffices to compute $\widetilde{M}_\mu(f_0)(1)$. An easy calculation gives that

$$\widetilde{M}(f_0)(1) = \frac{p}{p-1}\frac{\gamma^{1/p'}+1}{\gamma+1},$$

where γ is the unique positive solution of the equation

$$\frac{p}{p-1}\frac{\gamma^{1/p'}+1}{\gamma+1} = \gamma^{-1/p}.$$

It is now a matter of simple algebra to show that $\widetilde{M}(f_0)(1) = \gamma^{-1/p}$ is the unique positive root A_p of the equation $(p-1)x^p - px^{p-1} - 1 = 0$.

Let $f_\varepsilon(x) = |x|^{-1/p} g_\varepsilon(x)$, where $g_\varepsilon(x) = \min(|x|^{-\varepsilon}, |x|^\varepsilon)$. It is not difficult to check that for $0 < \varepsilon < 1/p'$ we have

$$(\widetilde{M} f_\varepsilon)(x) \geq \frac{1}{|x| + \gamma|x|} \int_{-\gamma|x|}^{|x|} |t|^{-1/p} g_\varepsilon(t)\, dt \geq \frac{1 + \gamma^{1/p'+\varepsilon}}{(1/p' + \varepsilon)(\gamma + 1)} |x|^{-1/p} g_\varepsilon(x).$$

This implies that

$$\lim_{\varepsilon \to 0} \frac{\|\widetilde{M} f_\varepsilon\|_{L^p}}{\|f_\varepsilon\|_{L^p}} \geq \lim_{\varepsilon \to 0} \frac{1 + \gamma^{1/p'+\varepsilon}}{(1/p' + \varepsilon)(\gamma + 1)} = \gamma^{-1/p} = A_p.$$

The last statement in this exercise is equivalent to the identity

$$2 + 2A_p^p = \int_{|t| \leq 1} |t|^{-1/p} dt + \int_{|t| \leq A_p^p} |t|^{-1/p} dt$$

which is equivalent to the fact that A_p is the unique positive solution of the equation $(p-1)x^p - px^{p-1} - 1 = 0$.

??. We define M_c the centered maximal function with respect to cubes in $\mathbf{R}^n$ and $\widetilde{M}_c$ the uncentered maximal function with respect to cubes in $\mathbf{R}^n$ as follows: For f locally integrable on $\mathbf{R}^n$, $(M_c f)(x)$ is defined as the supremum of the averages of $|f|$ over all cubes in $\mathbf{R}^n$ centered at x, while $(\widetilde{M}_c f)(x)$ is defined as the supremum of the averages of $|f|$ over all cubes containing x. Observe that a ball of radius $R > 0$ is contained in a cube of sidelength $2R$ with the same center, thus $\widetilde{M} f \leq (v_n/2)^n \widetilde{M}_c f$ and $Mf \leq (v_n/2)^n M_c f$. On the other hand any cube of sidelength L is contained in a ball of radius $\sqrt{n}L/2$ with the same center. This implies that $\widetilde{M}_c f \leq (2/v_n\sqrt{n})^n \widetilde{M} f$ and also $M_c f \leq (2/v_n\sqrt{n})^n Mf$. These four estimates imply that M_c and $\widetilde{M}_c$ are of weak type $(1,1)$ and they also map $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ for $1 < p \leq \infty$.

??. (a) Split $f = f\chi_{|f|>\alpha} + f\chi_{|f|\leq\alpha}$. Then $\widetilde{M} f \leq \widetilde{M}(f\chi_{|f|>\alpha}) + \widetilde{M}(f\chi_{|f|\leq\alpha}) \leq \widetilde{M}(f\chi_{|f|>\alpha}) + \alpha$. Then $\{x : (\widetilde{M} f)(x) > 2\alpha\} \subset \{x : (\widetilde{M} f\chi_{|f|>\alpha})(x) > \alpha\}$. The conclusion follows from (**??**).
(b) Apply Proposition **??** to the function $|f|$ and $\alpha > 0$. One obtains open cubes Q_j such that $\alpha < \frac{1}{|Q_j|}\int_{Q_j} f(t)\, dt \leq 2^n\alpha$. Then if $x \in Q_j$ we have $\widetilde{M}_c(f)(x) > \alpha$, hence $\cup_j Q_j \subset \{x : \widetilde{M}_c(f)(x) > \alpha\}$. It follows that

$$|\{x : \widetilde{M}_c(f)(x) > \alpha\}| \geq \sum_j |Q_j| \geq \frac{2^{-n}}{\alpha} \int_{\cup_j Q_j} |f(x)|\, dx \geq \frac{2^{-n}}{\alpha} \int_{\{|f|>\alpha\}} |f(x)|\, dx.$$

The last inequality is a consequence of the fact that for $x \notin \cup_j Q_j$ we have $|f(x)| \leq \alpha$. Use that $\widetilde{M}_c f \leq (2/v_n\sqrt{n})^n \widetilde{M} f$ to obtain the corresponding result about $\widetilde{M}$.

(c) Suppose first that $|f(x)|\log(1+|f(x)|)$ is integrable over the ball B. Then

$$\int_B \widetilde{M}f\,dx \le \int_B 1\,dx + \int_{\widetilde{M}f>1} \widetilde{M}f\,dx$$
$$\le |B| + \lambda_{\widetilde{M}f}(1) + \int_1^\infty \lambda_{\widetilde{M}f}(\alpha)\,d\alpha$$
$$\le |B| + 3^n\|f\|_{L^1} + 2\cdot 3^n \int_1^\infty \frac{1}{\alpha}\int_{\{|f|>\alpha/2\}} |f(y)|\,dy\,d\alpha$$
$$\le |B| + 3^n\|f\|_{L^1} + 2\cdot 3^n \int_B |f(y)|\log(2|f(y)|)\,dy$$
$$\le |B| + 3^n(1+2\log 2)\|f\|_{L^1} + 2\cdot 3^n \int_B |f(y)|\log(1+|f(y)|)dy$$

Conversely, suppose that $\widetilde{M}$ is integrable over B. Then it is easy to see that $\widetilde{M}$ is integrable over any compact set and that it decays like $\|f\|_{L^1}\mathrm{dist}(x,B)^{-n}$ as $|x|\to\infty$. Let $c_n = (\sqrt{n}v_n/2)^n$ and fix $\alpha \ge 1$. Then

$$\{x:\ (\widetilde{M}f)(x) > c_n\alpha\} \subset \{x:\ (\widetilde{M}f)(x)\chi_{\widetilde{M}f>c_n}(x) > c_n\alpha\},$$

hence the result in (b) gives

$$|\{x:\ (\widetilde{M}f)(x)\chi_{\widetilde{M}f>c_n}(x) > c_n\alpha\}| \ge \frac{2^{-n}}{\alpha}\int\limits_{\{|f|>\alpha\}} |f(x)|dx \ge \frac{2^{-n}}{\alpha}\int\limits_{\{1+|f|>2\alpha\}} |f(x)|dx,$$

where the last inequality is due to the fact that for $\alpha \ge 1$ we have $\{1+|f|>2\alpha\}\subset\{|f|>\alpha\}$. Integrate the last inequality above from $\alpha=1$ to ∞ and apply Fubini's theorem to obtain

$$\infty > \|\widetilde{M}f\chi_{\widetilde{M}f>c_n}\|_{L^1} \ge 2^{-n}\int |f(y)|\,\log\frac{(1+|f(y)|)}{2}\,dy.$$

Since f is integrable, this estimate implies the required conclusion.

??. Set $B = 3^n\|f\|_{L^1}|A|^{-1}$. We have

$$\int_A (\widetilde{M}f)^q\,d\mu = q\int_0^\infty |\{x\in A:\ (\widetilde{M}f)(x)>\alpha\}|\alpha^{q-1}\,d\alpha$$
$$\le q\int_0^\infty \min\big(|A|, \frac{3^n}{\alpha}\|f\|_{L^1}\big)\alpha^{q-1}d\alpha$$
$$\le q\int_0^B |A|\alpha^{q-1}\,d\alpha + q\int_B^\infty \alpha^{q-2}3^n\|f\|_{L^1}\,d\alpha$$
$$\le |A|B^q + \frac{q}{q-1}3^n\|f\|_{L^1}^q|A|^{1-q} = \frac{3^{nq}}{1-q}|A|^{1-q}\|f\|_{L^1}^q.$$

??. (a) Let $\widetilde{M}^{(j)}$ denote the one-dimensional uncentered Hardy-Littlewood maximal function acting on the j^{th} variable in $\mathbf{R}^n$. Observe that $\widetilde{M}_s = \widetilde{M}^{(1)}\circ\cdots\circ\widetilde{M}^{(n)}$ which implies that $\widetilde{M}_s$ maps $L^p(\mathbf{R}^n)$ into itself with norm at most A_p^n.

(b) To obtain that A_p^n is indeed the norm of $\widetilde{M}_s$ on $L^p(\mathbf{R}^n)$ consider the family of functions $f_\varepsilon(x) = \prod_{j=1}^n |x_j|^{-1/p}\min(|x_j|^\varepsilon, |x_j|^{-\varepsilon})$. Repeating the argument in Exercise 2(d) we obtain that $\|\widetilde{M}_s(f_\varepsilon)\|_{L^p}/\|f_\varepsilon\|_{L^p}\to A_p^n$ as $\varepsilon\to\infty$.

(c) Let $f_0 = \chi_{[0,1]\times[0,1]}$ in $\mathbf{R}^2$. Then for $x=(x_1,x_2)$ with $x_1,x_2>10$ we have $\widetilde{M}_s(f_0)\ge x_1^{-1}x_2^{-1}$. But the function $(x_1x_2)^{-1}$ is not in $L^{1,\infty}(R)$ where $R =$

$(10,\infty)\times(10,\infty)$, since the measure of the set of all (x_1,x_2) in R with $(x_1x_2)^{-1}>1$ is equal to infinity!

??. Work with each variable separately. Take for example $n=2$. Then

$$\begin{aligned}
&\sup_{t_1,t_2>0}\int\int |f(x_1-y_1,x_2-y_2)|\,|\phi_{t_1,t_2}(y_1,y_2)|\,dt_1dt_2\\
\le &\sup_{t_1,t_2>0}\int\int |f(x_1-y_1,x_2-y_2)|\,t_1^{-1}(1+|\tfrac{x_1}{t_1}|)^{-1-\varepsilon}t_2^{-1}(1+|\tfrac{x_2}{t_2}|)^{-1-\varepsilon}\,dt_1dt_2\\
\le &c_\varepsilon\sup_{t_2>0}\int |(M^{(1)}f)(x_1,x_2-y_2)|\,t_2^{-1}(1+|\tfrac{x_2}{t_2}|)^{-1-\varepsilon}\,dt_2\\
\le &c_\varepsilon^2\,(M^{(1)}M^{(2)}f)(x_1,x_2)\le c_\varepsilon^2\,(\widetilde{M}^{(1)}\widetilde{M}^{(2)}f)(x_1,x_2)=c_\varepsilon^2\,(\widetilde{M}_sf)(x_1,x_2),
\end{aligned}$$

where we applied Theorem **??** twice above and we set $c_\varepsilon=\int_{\mathbf{R}}(1+|x|)^{-\varepsilon}\,dx$. We denoted by $M^{(j)}$ above the one-dimensional centered Hardy-Littlewood maximal function acting on the j^{th} variable in $\mathbf{R}^n$.

??. Let $f_0=\chi_{B(0,1)}$ and $B_x=B\big(\frac12(|x|-|x|^{-1})\frac{x}{|x|},\frac12(|x|+|x|^{-1})\big)$. Then $B(0,1)\cap B_x$ has measure at least $v_n/2$, where v_n is the volume of $B(0,1)$. It follows that $(\widetilde{M}f_0)(x)\ge 2^{n-1}(|x|+|x|^{-1})^{-n}$ for $|x|>1$. Using polar coordinates we obtain

$$\frac{\|\widetilde{M}f_0\|_{L^p}^p}{\|f_0\|_{L^p}^p}\ge\frac{1}{v_n}\left(v_n+\omega_{n-1}2^{(n-1)p}\int_{r=1}^\infty\frac{r^{n-1}}{(r+1/r)^{np}}\,dr\right),$$

where $\omega_{n-1}=nv_n$ is the surface area of the unit sphere $\mathbf{S}^{n-1}$. It suffices to show that the expression $W_n=n2^{(n-1)p}\displaystyle\int_{r=1}^\infty\frac{r^{n-1}}{(r+1/r)^{np}}\,dr$ tends to infinity as $n\to\infty$. First change variables $r=\tan\phi$ to write

$$W_n=n2^{(n-1)p}\int_{\pi/4}^{\pi/2}(\sin\phi)^{np+n-1}(\cos\phi)^{np-n-1}d\phi$$

and then set $u=\cos^2\phi$ to ontain

$$W_n=2^{-p-1}2^{np}n\int_0^{1/2}(1-u)^{\frac{np}{2}+\frac{n}{2}-1}u^{\frac{np}{2}-\frac{n}{2}-1}du.$$

For any $0<t\le 1/2$ we estimate W_n from below by

$$2^{-p-1}2^{np}n\int_0^t(1-u)^{\frac{np}{2}+\frac{n}{2}-1}u^{\frac{np}{2}-\frac{n}{2}-1}du=2^{-p}(p-1)^{-1}2^{np}(1-t)^{\frac{np}{2}+\frac{n}{2}-1}t^{\frac{np}{2}-\frac{n}{2}}$$

and select $t=(\frac{np}{2}-\frac{n}{2})(np-1)^{-1}$ to maximize the last expression above. One obtains that

$$W_n\ge c_p\big\{(p+1)^{\frac{p}{2}+\frac12}(p-1)^{\frac{p}{2}-\frac12}p^{-p}\big\}^n$$

as $n\to\infty$ and since the expression inside the curly brackets above is strictly bigger than 1 when $1<p<\infty$, it follows that the operator norm of $\widetilde{M}$ on $L^p(\mathbf{R}^n)$ grows exponentially with the dimension.

??. (a) For simplicity we work with closed rectangles. Let f_0 be the characteristic function of the unit ball $B(0,1)$ in $\mathbf{R}^2$. Fix $|x|\ge 10$ and let R_x be the rectangle of sidelengths $|x|$ and 2 having 0 and x as the midpoint of its shortest sides. Then $(M_0f_0)(x)\ge|R_x|^{-1}\int_{R_x}f_0(t)\,dt=\pi|x|^{-1}$. Since the $|x|^{-p}$ is not integrable near infinity in $\mathbf{R}^2$ when $p<2$, we conclude that M_0 cannot map L^p to L^p when $p<2$.

(b) We prove the claim about M_{00}. Suppose that two rectangles of the same eccentricity intersect. Convince yourselves that the worst possible case is when the larger one is tangent to the corner of the smaller one at an angle of 90^o with respect to the diagonal of the smaller one. Then each side of the larger rectangle needs to be increased $2\sqrt{1+b^2}+1$ times so that it covers the smaller rectangle. Thus if two rectangles of the same eccentricity intersect, then the smaller one is contained in $4b$-times the bigger one. Adjust the proof of Lemma **??** to obtain that given any finite family of rectangles in $\mathbf{R}^2$ with the same eccentricity, then there exists a finite subfamily of pairwise disjoint rectangles whose measures add up to at least $1/(4b)^2$ of the measure of the original family. Now adjust the proof of Theorem **??** to obtain that M_{00} maps $L^1(\mathbf{R}^2)$ into $L^{1,\infty}(\mathbf{R}^2)$ with constant at most $16b^2$.
(c) Here one needs to make the observation that if two rectangles $I_1\times\cdots\times I_n$ and $J_1\times\cdots\times J_n$ in $\mathbf{R}^n$ with sidelengths $(|I_1|, a_2|I_1|,\dots,a_n|I_n|)$ and $(|J_1|, a_2|J_1|,\dots,a_n|J_n|)$ intersect, then the smaller one is contained in the triple of the bigger one. Then Lemma **??** applies in this situation and one obtains that the corresponding maximal function is of weak type $(1,1)$ with bound 3^n.

??. The first part of this exercise can be proved in a similar way as the proof of Theorem **??**.

To deduce the improvement of the Lebesgue differentiation theorem, we argue as follows. Let us first work with $f\in L^p(B(0,N))$, for some fixed $N>0$. Define $T_\varepsilon(f)(x)=\sup_{B(z,\varepsilon)\ni x}\big(\frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)}|f(y)-f(x)|^p\,dy\big)^{1/p}$ and $T_*f=\sup_{\varepsilon>0}T_\varepsilon f$ for all $f\in L^p(B(0,N))$. Use Minkowski's inequality to obtain that $T_*f\le|f|+(\widetilde{M}|f|^p)^{1/p}$. Since the operator $f\to(\widetilde{M}|f|^p)^{1/p}$ maps $L^p(B(0,N))$ into $L^{p,\infty}(\mathbf{R}^n)$, we obtain that T_* is bounded from $L^p(B(0,N))$ into $L^{p,\infty}(\mathbf{R}^n)$. It is easy to see that for f continuous function with compact support we have $\lim_{\varepsilon\to0}T_\varepsilon f=0$. We conclude that $\lim_{\varepsilon\to0}(T_\varepsilon f)(x)=0$ for almost all $x\in B(0,N)$. Now given $f\in L^p_{\text{loc}}(\mathbf{R}^n)$ we have $f\in L^p(B(0,N))$ for all $N>0$. Therefore $\lim_{\varepsilon\to0}(T_\varepsilon f)(x)=0$ for almost all $x\in B(0,N)$. Since $\mathbf{R}^n$ is a countable union of balls of the type $B(0,N)$, it follows that $\lim_{\varepsilon\to0}(T_\varepsilon f)(x)=0$ for almost all $x\in\mathbf{R}^n$.

??. (a) Write $E_\alpha=\{t\in\mathbf{R}:\ |(M_Rf)(t)|>\alpha\}$ as a union of disjoint open intervals (β_k,γ_k). For each $x\in(\beta_k,\gamma_k)$ let $W_x=\{y\in(x,\gamma_k):\ \int_x^y|f(t)|\,dt>\alpha(y-x)\}$. We will show that for all k we have

$$\int_{\beta_k}^{\gamma_k}|f(t)|\,dt=\alpha(\gamma_k-\beta_k).$$

Then the required conclusion will follow by summing on k. If $\gamma_k=\infty$ then the above identity clearly holds. Therefore we can assume that $\gamma_k<\infty$. If W_x were empty, then there must exist a $w>\gamma_k$ such that $\int_x^w|f(t)|\,dt>\alpha(w-x)$. Then

$$\int_{\gamma_k}^w|f(t)|\,dt=\int_x^w|f(t)|\,dt-\int_x^{\gamma_k}|f(t)|\,dt>\alpha(w-x)-\alpha(\gamma_k-x)=\alpha(w-\gamma_k).$$

Therefore $\gamma_k\in E_\alpha$ which is a contradiction. Now let $s_x=\sup W_x$. We claim that $s_x=\gamma_k$. If $s_x<\gamma_k$, then a continuity argument gives that $\int_x^{\gamma_k}|f(t)|\,dt=\alpha(s_x-x)$. Since the set N_{s_x} is nonempty, there is a real number $y\in(s_x,\gamma_k)$ such that $\int_{s_x}^{y} |f(t)|\,dt > \alpha(y-s_x)$. Thus $\int_{x}^{y} |f(t)|\,dt > \alpha(y-x)$ which is a contradiction since $y > s_x$. Therefore for all $x \in (\beta_k,\gamma_k)$ we have $s_x = \gamma_k$ and hence the inequality $\int_{x}^{\gamma_k} |f(t)|\,dt \ge \alpha(\gamma_k - x)$ holds. Letting $x \to \beta_k$ we obtain $\int_{\beta_k}^{\gamma_k} |f(t)|\,dt \ge \alpha(\gamma_k - \beta_k)$. Since $\beta_k \notin E_\alpha$, it follows that the converse inequality is also valid. Therefore we have established $\int_{\beta_k}^{\gamma_k} |f(t)|\,dt = \alpha(\gamma_k - \beta_k)$ for all k

(b) Multiply the identities in (a) by $p\alpha^{p-1}$ and integrate from $\alpha = 0$ to ∞ to obtain

$$\|M_L f\|_{L^p}^p = \frac{p}{p-1}\int |f(t)|(M_L f)(t)^{p-1}\,dt$$

and similarly for $M_R f$. The conclusion follows by applying Hölder's inequality with exponents p and p' to the integral above.

(c) Let $f_R(x) = |x|^{-1/p}\chi_{x>0}$ and $f_L(x) = |x|^{-1/p}\chi_{x<0}$. it is easy to see that $M_R(f_R) = (p/(p-1))f_R$ and $M_L(f_L) = (p/(p-1))f_L$. Therefore the functions f_R and f_L suggest that the operator norms of M_R and M_L on $L^p(\mathbf{R})$ are indeed equal to $p/(p-1)$. Since f_R is not actually an L^p function, introduce $f_{R,\varepsilon}(x) = f_R(x)\min(|x|^{-\varepsilon}, |x|^{\varepsilon})$ for $\varepsilon > 0$ and prove that $\|M_R(f_{R,\varepsilon})\|_{L^p}/\|f_{R,\varepsilon}\|_{L^p} \to p/(p-1)$ as $\varepsilon \to 0$ in a way similar to that in Exercise 2 (d).

(d) Clearly $\widetilde{M}f \ge \max(M_R f, M_L f)$. Conversely, if $a < x < b$ then

$$\frac{1}{b-a}\int_a^b |f(t)|\,dt \le \frac{x-a}{b-a}(M_L f)(x) + \frac{b-x}{b-a}(M_R f)(x) \le \max(M_R f, M_L f).$$

Taking the supremum over all $a < x$ and $b > x$ we obtain $\widetilde{M}f \le \max(M_R f, M_L f)$.

(e) It follows from part (b) that

$$\begin{aligned}\int (M_L f)^p + (M_R f)^p\,dx =& \frac{p}{p-1}\int |f|\left((M_R f)^{p-1} + (M_L f)^{p-1}\right)dx\\ =& \frac{p}{p-1}\int |f|\left((\widetilde{M}f)^{p-1} + (Nf)^{p-1}\right)dx.\end{aligned}$$

Use the pointwise bound $|f|(Nf)^{p-1} \le \frac{|f|^p}{p} + \frac{(Nf)^p}{p'}$ and estimate $\int_{\mathbf{R}} |f|(\widetilde{M}f)^{p-1}dx$ by $\|f\|_{L^p}\|\widetilde{M}f\|_{L^p}^{p-1}$ to obtain the claimed conclusion.

??. (a) let us work with open dyadic cubes for simplicity Let $E_\alpha = \{x \in \mathbf{R}^n : (\widetilde{M}_d f)(x) > \alpha\}$. For every $x \in E_\alpha$ select a dyadic cube Q_x containing x such that $|Q_x|^{-1}\int_{Q_x} |f(t)|\,dt > \alpha$. Now each Q_x selected is contained in E_α since for $y \in Q_x$ we have $(\widetilde{M}_d f)(y) \ge |Q_x|^{-1}\int_{Q_x} |f(t)|\,dt > \alpha$. Thus the family of cubes Q_x forms a covering of E_α. Pass to a countable subfamily of these cubes that still covers E_α. Since all the cubes in the subfamily are dyadic, we can use an inductive argument to dispose of the cubes that are contained in some other cubes in the subcollection. After this procedure is terminated, E_α is almost everywhere equal to a countable union of disjoint dyadic cubes Q_j which satisfy $|Q_j|^{-1}\int_{Q_j} |f(t)|\,dt > \alpha$. (We have a.e. equality of sets since some part of the boundaries of the cubes Q_j may also be contained in E_α). The required conclusion follows by summing over j.

To prove (b) multiply the inequality in (a) by $p\alpha^{p-1}$ and integrate from $\alpha = 0$ to ∞ just as in the proof of Exercise **??** (b).

SECTION 2.2

??. (a) Define h on $\mathbf{R}$ by setting $h(x) = e^{-1/x}$ for $x > 0$ and 0 otherwise. Observe that $h(x)$ is differentiable of all orders at $x = 0$ and $h^{(m)}(0) = 0$ for all $m \geq 0$. Set

$$g(x) = \begin{cases} 0 & \text{when } x \leq 0, \\ e^{1-\frac{1}{x}}\big(1 - e^{-\frac{1}{1-x}}\big) & \text{when } 0 < x < 1, \\ 1 & \text{when } x \geq 1. \end{cases}$$

Then $g(x)$ is a C^∞ function on the line which vanishes when $x \leq 0$ and is identically equal to one when $x \geq 1$.
(b) The function $g(\frac{x-1}{2})g(\frac{4-x}{2})$ is therefore C^∞ and vanishes on $(-\infty, 1/2] \cup [4, +\infty)$ and it is identicaly equal to 1 on $[1, 2]$. On $\mathbf{R}^n$ the function $G(x) = g(|x|)$ is equal equal to 1 on the annulus $1 \leq |x| \leq 2$ and vanishes off the annulus $1/2 \leq |x| \leq 4$.
(a) A simple calculation using the results in Proposition **??** (4) and (5) gives the identity $\widehat{f} = (\widehat{\phi}\,\widetilde{\widehat{\phi}}) * \overline{(\widehat{\phi}\,\widetilde{\widehat{\phi}})} = |\widehat{\phi}|^2 * |\widehat{\phi}|^2$ since ϕ is odd. Thus $\widehat{f}$ has compact support.

??. Using Propositions **??** and **??** we obtain

$$\begin{aligned}
\rho_{\alpha,\beta}(\widehat{f}) &= \sup_{x \in \mathbf{R}^n} |x^\alpha (\partial^\beta \widehat{f})(x)| \\
&= \sup_{x \in \mathbf{R}^n} |x^\alpha \big((-2\pi i \xi)^\beta f(\xi)\big)\widehat{\ }(x)| \\
&= (2\pi)^{-|\alpha|} \sup_{x \in \mathbf{R}^n} |\big(\partial^\alpha((-2\pi i \xi)^\beta f(\xi))\big)\widehat{\ }(x)| \\
&\leq (2\pi)^{-|\alpha|} \|\partial^\alpha\big((-2\pi i (\,\cdot\,))^\beta f(\,\cdot\,)\big)\|_{L^1} \\
&\leq (2\pi)^{-|\alpha|} C_n \sum_{|\gamma| = n+2} \rho_{\gamma,\alpha}((-2\pi i(\,\cdot\,))^\beta f(\,\cdot\,)) \\
&\leq C_{\alpha,\beta,n} \sum_{\substack{|\gamma| \leq n+2+|\beta| \\ |\delta| \leq |\alpha|}} \rho_{\gamma,\delta}(f)
\end{aligned}$$

Replacing f by $f_k - f$ and letting $k \to \infty$ we obtain that every seminorm of $\widehat{f_k} - \widehat{f}$ tends to zero, i.e. $\widehat{f_k}$ converges to $\widehat{f}$ in $\mathcal{S}$ as $k \to \infty$. This argument also shows that the Fourier transform of a Schwartz function is also a Schwartz function.

??. Leibnitz's rule gives that any derivative of fg is controlled by a finite sum of produtcs of derivatives of f times derivatives of g. Since the derivatives of g have tempered growth at infinity and the derivatives of f have decay faster than any polynomial at infinity, it follows that any derivative of fg has arbitrarily fast decay at inifinity and the conclusion follows from the observation in Remark **??**.

??. As in the proof of Proposition **??** write

$$|(f * g)(x)| \leq AB \int_{\mathbf{R}^n} (1 + |x - y|)^{-M} (1 + |y|)^{-N} dy$$

and estimate the piece of the integral over the set $\{y : \ |x|/2 \leq |y - x|\}$ by

$$AB \int_{|y-x| \geq |x|/2} (1 + |x|/2)^{-M} (1 + |y|)^{-N} dy \leq AB 2^M (1 + |x|)^{-M} C_N$$

and the piece of the integral over the set $\{y: \ |y-x| \le |x|/2\}$ by

$$AB\int_{|y-x|\le|x|/2}(1+|x-y|)^{-M}(1+|x|/2)^{-N}dy \le AB2^N(1+|x|)^{-N}C_M.$$

The maximum of these estimates gives the claimed decay for $f*g$.

??. Given $f\in L^2(\mathbf{R}^n)$ let $f_N = f\chi_{B(0,N)}$. Then $\|f_N - f\|_{L^2}\to 0$ as $N\to\infty$ which implies that $\{f_N\}_N$ is an L^2-Cauchy sequence and by Plancherel's theorem so is $\{\widehat{f_N}\}_N$. Thus $\widehat{f_N}$ converge to some $K(f)$ in L^2 as $N\to\infty$. It also follows from Plancherel's theorem that the operator K maps L^2 to L^2 with norm 1. We observe that $K(h)=h$ when $h\in\mathcal{S}(\mathbf{R}^n)$. The reason is that for $h\in\mathcal{S}$, h_N converge to h pointwise and also in L^2, hence the two limits must agree. Now a standard $\varepsilon/3$ argument (using that K is L^2 bounded) gives that $K(f)=\widehat{f}$ for all $f\in L^2(\mathbf{R}^n)$. To show the last statement of this exercise, given $f\in L^1\cap L^2$ find a sequence of Schwartz functions f_j which converges to f a.e. and satisfies $|f_j|\le 2|f|$. Then the Lebesgue dominated convergence theorem gives that

$$\widehat{f_j}(\xi)=\int_{\mathbf{R}^n} f_j(x)e^{-2\pi i x\cdot\xi}dx \to \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx,$$

for all $\xi\in\mathbf{R}^n$. But we know that $\widehat{f_j}\to\widehat{f}$ in L^2 and therefore the last integral above coincides with $\widehat{f}(\xi)$.

??. (a) Simply observe that

$$\begin{aligned}|\widehat{f}(\xi)-\widehat{f}(\eta)| \le &\int_{B(0,R)} |f(x)|\,|e^{-2\pi i x\cdot\xi}-e^{-2\pi i x\cdot\eta}|\,dx + 2\|f\|_{L^1(B(0,R)^c)}\\ \le& 2\pi R|\xi-\eta|\|f\|_{L^1} + 2\|f\|_{L^1(B(0,R)^c)}.\end{aligned}$$

Now given $\varepsilon>0$ find $R>0$ such that $2\|f\|_{L^1(B(0,R)^c)}<\varepsilon/2$. Then find $\delta>0$ such that $2\pi R\delta\|f\|_{L^1}<\varepsilon/2$. Then for $|\xi-\eta|<\delta$ we have $|\widehat{f}(\xi)-\widehat{f}(\eta)|\le\varepsilon$.
(b) Follows immediately from Fubini's theorem.
(c) As in (**??**) we obtain

$$\int_{\mathbf{R}^n} f(x)\varepsilon^{-n}e^{-\pi\varepsilon^{-2}|x-t|^2}\,dx = \int_{\mathbf{R}^n}\widehat{f}(x)e^{2\pi i x\cdot t}e^{-\pi|\varepsilon x|^2}\,dx.$$

Now the right hand side converges (everywhere) to $(\widehat{f}\,)^\vee(t)$ as $\varepsilon\to 0$ by the Lebesgue dominated convergence theorem. The left hand side converges to $f(t)$ a.e. in view of Corollary **??**. We conclude that $(\widehat{f}\,)^\vee = f$ a.e. on $\mathbf{R}^n$.
(d) Take $t=0$ in part (c). Since f is continuous at 0, the left hand side of the identity above converges to $f(0)$ as $\varepsilon\to 0$ in view of Theorem **??** (2).
(e) The sequence of positive functions $\widehat{f}(x)e^{-\pi|\varepsilon x|^2}$ increases to $\widehat{f}(x)$ a.e. as $\varepsilon\to 0$. The Lebesgue monotone convergence theorem gives that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n}\widehat{f}(x)e^{-\pi|\varepsilon x|^2}dx = \int_{\mathbf{R}^n}\widehat{f}(x)\,dx.$$

Part (d) gives that the limit above is equal to $f(0)$, hence $\widehat{f}$ is integrable. Fourier inversion follows from part (c).

??. Calculation gives that $(\chi_{[a,b]})\widehat{\ }(\xi) = \dfrac{e^{-2\pi i\xi b} - e^{-2\pi i\xi a}}{-2\pi i\xi}$ which is seen to tend to zero as $|\xi| \to \infty$. In $\mathbf{R}^n$ we have

$$(\chi_{[a_1,b_1]\times\cdots\times[a_n,b_n]})\widehat{\ }(\xi_1,\dots,\xi_n) = \frac{e^{-2\pi i\xi_1 b_1} - e^{-2\pi i\xi_1 a_1}}{-2\pi i\xi_1}\cdots\frac{e^{-2\pi i\xi_n b_n} - e^{-2\pi i\xi_n a_n}}{-2\pi i\xi_n}$$

which also tends to zero as $|\xi| \to \infty$. The same result is valid for a finite linear combination of characteristic functions of rectangles on $\mathbf{R}^n$. Given $f \in L^1(\mathbf{R}^n)$ and $\varepsilon > 0$ find a function g given by a finite linear combination of characteristic functions of rectangles with $\|f-g\|_{L^1} < \varepsilon$. Pick an N_0 such that for $|\xi| > N_0$ we have $|\widehat{g}(\xi)| < \varepsilon$. Then for $|\xi| > N_0$ we have that $|\widehat{f}(\xi)| < 2\varepsilon$ in view of Proposition **??** (1).

??. Given $f \in L^p$ for $1 < p < 2$, find $f_1 \in L^1$ and $f_2 \in L^2$ such that $f = f_1 + f_2$. (You may take $f_1 = f\chi_{|f|>1}$ and $f_2 = f\chi_{|f|\le 1}$.) Then define $\widehat{f} = \widehat{f_1} + \widehat{f_2}$. To see that this is well defined, observe that if $f = h_1 + h_2 = f_1 + f_2$ with $f_1, h_1 \in L^1$ and $f_2, h_2 \in L^2$, then $f_1 - h_1 = h_2 - f_2 = f_0 \in L^1 \cap L^2$. In view of Exercise **??**, the Fourier transform of f_0 is equal to $\displaystyle\int_{\mathbf{R}^n} f_0(\xi)e^{-2\pi i x\cdot\xi}dx$, and thus the functions $f_1 - h_1$ and $f_2 - h_2$ have the same Fourier transform. Thus $\widehat{f_1} + \widehat{f_2} = \widehat{h_1} + \widehat{h_2}$ and $\widehat{f}$ is well-defined. Finally if $f \in \mathcal{S} \subset L^p$, then write $f = f + 0$, where $f \in L^1$ and $0 \in L^2$. Then $\widehat{f}$ (where f is thought as an element of L^p) is equal to the Fourier transform of f (where f is thought as an element of L^1) and is therefore given as the known convergent integral.

??. (a) If $t \le 1$, then $\displaystyle\Big|\int_0^t \frac{\sin\xi}{\xi}\,d\xi\Big| \le \int_0^t \Big|\frac{\sin\xi}{\xi}\Big|\,d\xi \le t \le 1$. Also for $t > 1$

$$\Big|\int_0^t \frac{\sin\xi}{\xi}\,d\xi\Big| \le 1 + \Big|\int_1^t \frac{(\cos\xi)'}{\xi}\,d\xi\Big| \le 1 + \Big|\frac{\cos t}{t}\Big| + \Big|\frac{\cos 1}{1}\Big| + \Big|\int_1^t \frac{\cos\xi}{\xi^2}\,d\xi\Big| \le 4.$$

(b) Since f is odd, we have that $\displaystyle\widehat{f}(\xi) = -i\int_{\mathbf{R}} f(x)\sin(2\pi x\xi)\,d\xi$. Divide by ξ, and integrate from $\xi = 0$ to ∞. We obtain

$$\begin{aligned}\Big|\int_0^t \frac{\widehat{f}(\xi)}{\xi}\,d\xi\Big| &= \Big|\int_{\mathbf{R}} f(x)\int_0^t \frac{\sin(2\pi x\xi)}{\xi}\,d\xi\,dx\Big| \\ &\le \int_{\mathbf{R}} |f(x)|\Big|\int_0^{2\pi t x} \frac{\sin(\xi)}{\xi}\,d\xi\Big|\,dx \le 4\|f\|_{L^1},\end{aligned}$$

where the first equality above follows from Fubini's theorem.

(c) Assume that such an integrable f exists. Let $\widetilde{f}(x) = f(-x)$. The L^1 functions $\widetilde{f}$ and $-f$ have the same Fourier transforms (since $\widehat{f} = g$ is odd) and thus they coincide. Hence f is odd. But any odd integrable function on the line must satisfy the conclusion of part (b) which clearly $\widehat{f} = g$ fails to satisfy.

??. Observe that the map $x \to x - x^{-1}$ is a bijection from $(0,\infty)$ onto $(-\infty,\infty)$ and its inverse is $y \to \frac{1}{2}(y + \sqrt{y^2+4})$. Similarly the map $x \to x - x^{-1}$ is a bijection from $(-\infty,0)$ onto $(-\infty,\infty)$ and its inverse is $y\to\frac{1}{2}(y-\sqrt{y^2+4})$. Therefore

$$\begin{aligned}\int_0^\infty f(x-1/x)\,dx =&\frac{1}{2}\int_{-\infty}^{+\infty} f(y)\left(dy+d\sqrt{y^2+4}\right)\\ \int_{-\infty}^0 f(x-1/x)\,dx =&\frac{1}{2}\int_{-\infty}^{+\infty} f(y)\left(dy-d\sqrt{y^2+4}\right).\end{aligned}$$

Summing the above two identities we obtain the required conclusion.

??. (a) Apply the previous exercise with $f(x)=e^{-tx^2}$ to obtain

$$e^{-2t}\int_{-\infty}^{+\infty} e^{-tx^2}dx=\int_{-\infty}^{+\infty} e^{-tx^2}e^{-t/x^2}dx.$$

Change variables $z=\pi^{-1/2}t^{1/2}x$ in the first integral above and $y=tx^2$ in the second integral above. Use that $\displaystyle\int_{-\infty}^{+\infty} e^{-\pi z^2}dz=1$ to get the conclusion.

(b) Set $t=\pi|x|$ in the identity of part (a). Integrate with respect to the measure $e^{-2\pi i\xi\cdot x}dx$ and apply Fubini's theorem to obtain

$$\begin{aligned}\left(e^{-2\pi|x|}\right)\widehat{\ }(\xi) =&\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-y}\int_{\mathbf{R}^n} e^{-\pi^2|x|^2/y}e^{-2\pi i\xi\cdot x}dx\frac{dy}{\sqrt{y}}\\ =&\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-y}y^{n/2}\pi^{-n/2}e^{-y|x|^2}\frac{dy}{\sqrt{y}}\\ =&\frac{1}{\pi^{\frac{n+1}{2}}}\int_0^\infty e^{-y(|x|^2+1)}y^{\frac{n+1}{2}-1}dy\\ =&\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.\end{aligned}$$

??. For part (a) observe that

$$|f(x)^2|=\left|\int_{-\infty}^x [f(t)^2]'\,dt\right|=\left|2\int_{-\infty}^x f(t)f'(t)\,dt\right|\le 2\|f\|_{L^p}\|f\|_{L^{p'}}\,.$$

For part (b) write

$$f(x)^2=\int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_n}\partial_1\dots\partial_n[f(t)^2]dt_n\dots dt_1$$

and compute the multiple derivative above to obtain the conclusion.

??. Suppose that for all $f\in L^p(\mathbf{R})$ we had $\widehat{f}\in L^1_{\text{loc}}$. Then the linear functional $T(f)=\int_{B(0,1)}\widehat{f}(\xi)\,d\xi$ would satisfy $|T(f)|\le C_f<\infty$ for all $f\in L^p(\mathbf{R}^n)$ and thus it would have to be bounded by the closed graph theorem, that is there would exist a $B>0$ such that $\|T(f)\|\le B\|f\|_{L^p}$ for all $f\in L^p(\mathbf{R}^n)$. For $b>0$ take $f_b(x)=(1+ib)^{-n/2}e^{-\pi(1+ib)^{-1}|x|^2}$ and observe that $\widehat{f_b}(\xi)=e^{-\pi(1+ib)|\xi|^2}$. This is true by analytic continuation from the corresponing result where $1+ib$ is replaced by $t>0$. Now observe that $\int_{B(0,1)}|\widehat{f_b}(\xi)|\,d\xi=\int_{B(0,1)}e^{-\pi|\xi|^2}\,d\xi=c$ while $\|f_b\|_{L^p}=c_p(1+b^2)^{\frac{n}{2p}-\frac{n}{4}}$. Thus $\|f_b\|_{L^p}\to 0$ as $b\to\infty$ which provides a contradiction.

??. Fix a y and z in $\mathbf{R}^n$. Then $\widehat{\partial_j f}(\xi-z)=-2\pi i(\xi-z)\widehat{f}(\xi)e^{2\pi iz\cdot x}$ and

$$\|f\|_{L^2}^2=\frac{1}{n}\int_{\mathbf{R}^n} f(x)\overline{f(x)}\sum_{j=1}^n\partial_j(x_j-y_j)\,dx$$

$$
\begin{aligned}
&= -\frac{1}{n}\int_{\mathbf{R}^n}\sum_{j=1}^{n}\big(f(x)\partial_j\overline{f(x)}+\partial_j f(x)\overline{f(x)}\big)(x_j-y_j)\,dx\\
&\le \frac{2}{n}\int_{\mathbf{R}^n}\Big[\sum_{j=1}^{n}|\partial_j f(x)|^2\Big]^{\frac12}|f(x)|\Big[\sum_{j=1}^{n}|x_j-y_j|^2\Big]^{\frac12}\,dx\\
&\le \frac{2}{n}\bigg[\int_{\mathbf{R}^n}\sum_{j=1}^{n}|\partial_j f(x)|^2dx\bigg]^{\frac12}\bigg[\int_{\mathbf{R}^n}|x-y|^2|f(x)|^2dx\bigg]^{\frac12}\\
&= \frac{2}{n}\bigg[\int_{\mathbf{R}^n}\sum_{j=1}^{n}|\widehat{\partial_j f}(\xi-z)|^2dx\bigg]^{\frac12}\bigg[\int_{\mathbf{R}^n}|x-y|^2|f(x)|^2dx\bigg]^{\frac12}\\
&= \frac{2}{n}\bigg[\int_{\mathbf{R}^n}(2\pi|\xi-z|)^2|\widehat{f}(\xi-z)|^2dx\bigg]^{\frac12}\bigg[\int_{\mathbf{R}^n}|x-y|^2|f(x)|^2dx\bigg]^{\frac12}
\end{aligned}
$$

which proves the required estimate.

SECTION 2.3

??. Any such measure defines a linear functional on $\mathcal{S}$ via integration. This linear functional is continuous since if $f_k \to f$ in $\mathcal{S}$, then

$$
\Big|\int_{\mathbf{R}^n}(f_k(x)-f(x))\,d\mu(x)\Big| \le \sup_{x\in\mathbf{R}^n}\big((1+|x|)^k|f_k(x)-f(x)|\big)\int_{\mathbf{R}^n}\frac{d\mu(x)}{(1+|x|)^k}
$$

and the last expression above is controlled by a constant multiple of finite sum of $\rho_{\alpha,\beta}$ seminorms of $f_k - f$, thus it converges to zero as $k\to\infty$. It follows that μ defines a continuous linear functional on $\mathcal{S}$ and it can be identified with an eleent of $\mathcal{S}'$. The actions of Lebesgue measure and of the function 1 (both thought as tempered distributions) coincide.

??. Without loss of generality take $b = 1$. We need to show that for all multi-indices α and β we have $x^\alpha(\partial^\beta(\phi_\varepsilon * g))(x)$ converges uniformly to $x^\alpha(\partial^\beta g)(x)$ as $\varepsilon\to 0$. Setting $\partial^\beta g = h$, we see that it suffices to show that $x^\alpha(\phi_\varepsilon * h)(x)$ converges uniformly to $x^\alpha h(x)$. This will follow from the fact that $\partial^\alpha(\phi_\varepsilon*h)\widehat{\ }$ converges to $\partial^\alpha\widehat{h}$ in L^1. Using Leibnitz's rule we write $\partial^\alpha(\widehat{\phi_\varepsilon}\widehat{h}) = \widehat{\phi_\varepsilon}\partial^\alpha\widehat{h}+\sum_{0<|\gamma|\le|\alpha|}c_\gamma(\partial^\gamma\widehat{\phi_\varepsilon})\partial^{\alpha-\gamma}\widehat{h}$. The sum above converges to zero in L^1 as $\varepsilon\to 0$ since $\widehat{\phi_\varepsilon}(\xi) = \widehat{\phi}(\varepsilon\xi)$ and every derivative of $\widehat{\phi_\varepsilon}$ contains a factor of ε.

??. We have

$$
\begin{aligned}
&((\delta^a f)*(\delta^a u))(x) = (\delta^a u)(\tau^x\widetilde{\delta^a f}) = (\delta^a u)(\tau^x\delta^a\widetilde{f}) = a^{-n}u(\delta^{1/a}\tau^x\delta^a\widetilde{f})\\
=&a^{-n}u(\tau^{ax}\widetilde{f}) = a^{-n}u(\widetilde{\tau^{ax}f}) = a^{-n}(f*u)(ax) = a^{-n}(\delta^a(f*u))(x),
\end{aligned}
$$

since $\delta^{1/a}\tau^x\delta^a = \tau^{\delta a}$ for all $x\in\mathbf{R}^n$ and $a>0$.

??. (a) Observe that $\chi'_{[a,b]}(f) = -\int_a^b f'(t)\,dt = -f(b)+f(a) = (\delta_a-\delta_b)(f)$.

(b) We have

$$
\begin{aligned}
(\partial_1\chi_{B(0,1)})(f) &= -\int_{B(0,1)} \partial_1 f\, dx_1 dx_2 = \int_{-1}^{+1}\int_{-\sqrt{1-x_2^2}}^{+\sqrt{1-x_2^2}} (\partial_1 f)(x_1,x_2)\, dx_1 dx_2 \\
&= \int_{-1}^{+1} f(\sqrt{1-t^2},t)\, dt - \int_{-1}^{+1} f(-\sqrt{1-t^2},t)\, dt.
\end{aligned}
$$

The latter can be identified with a measure that lives on the boundary of the circle. Similarly one obtains

$$
(\partial_2\chi_{B(0,1)})(f) = \int_{-1}^{+1} f(t,-\sqrt{1-t^2})\, dt - \int_{-1}^{+1} f(t,\sqrt{1-t^2})\, dt.
$$

(c) Since the Fourier transform of $x \to e^{2\pi i x y}$ is δ_y and $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, it follows that the Fourier transform of $\cos x$ is $\frac{1}{2}(\delta_{1/2\pi} + \delta_{-1/2\pi})$. Similarly the the Fourier transform of $\sin(2\pi x)$ is $\frac{1}{2i}(\delta 1/2\pi - \delta_{-1/2\pi})$.
(d) Fix $f \in \mathcal{S}(\mathbf{R})$. We have

$$
\begin{aligned}
(\log|x|)'(f) &= -\int_{\mathbf{R}} \log|x| f'(x)\, dx = -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \log|x|\, f'(x)\, dx \\
&= -\lim_{\varepsilon\to 0}\Big[-f(\varepsilon)\log\varepsilon + f(-\varepsilon)\log\varepsilon - \int_{|x|\geq\varepsilon} \frac{1}{x}\, f(x)\, dx\Big] \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{1}{x}\, f(x)\, dx
\end{aligned}
$$

since for C^∞ functions f the expression $(f(\varepsilon) - f(-\varepsilon))\log\varepsilon$ tends to zero as $\varepsilon \to 0$.

??. (a) We need to show that every $\rho_{\alpha,\beta}$ norm of the expression in part (a) tends to zero as $h \to 0$. Use the mean value theorem twice to write

$$
\begin{aligned}
&\sup_{x\in\mathbf{R}^n}\left| x^\alpha \frac{(\partial^\beta f)(x+he_j) - (\partial^\beta f)(x)}{h} - x^\alpha(\partial^\beta\partial_j f)(x)\right| \\
=& \sup_{x\in\mathbf{R}^n}\left| x^\alpha\big((\partial_j\partial^\beta f)(x+\xi_{x,h}e_j) - (\partial^\beta\partial_j f)(x)\big)\right| \\
\leq& |h| \sup_{x\in\mathbf{R}^n}\left| x^\alpha(\partial_j^2\partial^\beta f)(x+\xi'_{x,h}e_j)\right| = A(h),
\end{aligned}
$$

since $|\xi'_{x,h}| \leq |\xi_{x,h}| \leq |h| \leq 1/2$. We have

$$
|(\partial_j^2\partial^\beta f)(x+\xi'_{x,h}e_j)| \leq C_N(1+|x+\xi'_{x,h}e_j|)^{-N} \leq C'_N(\tfrac{1}{2}+|x|)^{-N},
$$

and by picking $N > |\alpha|$, we estimate the supremum above by a constant, and we conclude that $A(h)$ tends to zero as $h \to 0$.
(b) Say that ϕ is equal to one on $B(0,1)$ and vanishes off $B(0,2)$. Then

$$
\begin{aligned}
&\rho_{\alpha,\beta}(\phi_k f - f) = \sup_{x\in\mathbf{R}^n} |x^\alpha\partial^\beta(\phi_k f - f)(x)| \\
=& \sup_{x\in\mathbf{R}^n}\Big| x^\alpha\big[\phi_k(x)(\partial^\beta f)(x) - (\partial^\beta f)(x)\big] + \sum_{0<|\gamma|\leq|\beta|} c_\gamma(\partial^\gamma\phi_k)(x)(\partial^{\beta-\gamma} f)(x)\Big| \\
\leq& \sup_{|x|\geq k} |x|^{|\alpha|}\,|(\partial^\beta f)(x)| + \frac{1}{k}\sum_{0<|\gamma|\leq|\beta|} c'_\gamma \sup_{x\in\mathbf{R}^n} |x|^{|\alpha|}|(\partial^{\beta-\gamma} f)(x)|,
\end{aligned}
$$

which clearly tends to zero as $k \to \infty$.

(c) We need to show that

$$\rho_{\alpha,\beta}\big(\phi_k(\phi_k\widehat{f})^\vee - f^\vee\big) = \sup_{x\in\mathbf{R}^n}\left|\int_{\mathbf{R}^n} f(\xi)x^\alpha\big[\partial_x^\beta\big(\phi_k(x)e^{2\pi i x\cdot\xi}\big)\phi_k(\xi) - (\partial_x^\beta e^{2\pi i x\cdot\xi})\big]\,d\xi\right|$$

tends to zero as $k\to\infty$ for all multi-indices α and β. Control the expression above by the sum of the following three expressions

$$I_k^1 = \sup_{x\in\mathbf{R}^n}\left|x^\alpha\Big(\sum_{0<|\gamma|\le|\beta|} c_\gamma(\partial^\gamma\phi_k)(x)(2\pi i x)^\beta\Big)\int_{\mathbf{R}^n} f(\xi)\phi_k(\xi)e^{2\pi i x\cdot\xi}\,d\xi\right|$$

$$I_k^2 = \sup_{x\in\mathbf{R}^n}\left|x^\alpha(2\pi i x)^\beta\phi_k(x)\int_{\mathbf{R}^n} f(\xi)(\phi_k(\xi)-1)e^{2\pi i x\cdot\xi}\,d\xi\right|$$

$$I_k^3 = \sup_{x\in\mathbf{R}^n}\left|x^\alpha(2\pi i x)^\beta(\phi_k(x)-1)\int_{\mathbf{R}^n} f(\xi)e^{2\pi i x\cdot\xi}\,d\xi\right|$$

Now observe that I_k^3 is bounded by $C\sup_{|x|\ge k}(1+|x|)^{|\alpha|+|\beta|}|f^\vee(x)|$ which tends to zero as $k\to\infty$. In I_k^2 integrate by parts N times with respect to the differential operator $(I-\Delta_\xi)^N$ to obtain the estimate

$$I_k^2 \le \sup_{x\in\mathbf{R}^n}\frac{|x|^{|\alpha|+|\beta|}}{(1+4\pi^2|x|^2)^N}\int_{\mathbf{R}^n}|(I-\Delta_\xi)^N(f(1-\phi_k))(\xi)|\,d\xi$$

which also tends to zero as $k\to\infty$ by the Lebesgue dominated convergence theorem. Finally a similar integration by parts yields the bound

$$I_k^1 \le \frac{C_\beta}{k}\sup_{x\in\mathbf{R}^n}\frac{|x|^{|\alpha|+|\beta|-1}}{(1+4\pi^2|x|^2)^N}\int_{\mathbf{R}^n}|(I-\Delta_\xi)^N(f\phi_k)(\xi)|\,d\xi$$

and this also tends to zero as $k\to\infty$.

??. The Fourier transform of a C_0^∞ function has to be real analytic and hence it cannot vanish on an open set, unless it it is identically equal to zero.

??. Fix $g\in\mathcal{S}(\mathbf{R}^n)$ and test against this g. It suffices to show that

$$\int_{\mathbf{R}^n} g(\xi)\int_{B(0,N)} f(x)e^{-2\pi i x\cdot\xi}dx\,d\xi \to \widehat{f}(g)$$

as $N\to\infty$. By Fubini's theorem it suffices to show that $\displaystyle\int_{B(0,N)} f(x)\widehat{g}(x)\,dx$ tends to $\displaystyle\int f(x)\widehat{g}(x)\,dx$ as $N\to\infty$. But this easily follows from Hölder's inequality since $\widehat{g}\in L^{p'}$ and $\|f\chi^c_{B(0,N)}\|_{L^p}\to 0$ as $N\to\infty$.

??. Define a linear functional u on $\mathcal{S}$ by setting $u(f)=\sum_{k\in\mathbf{Z}^n}c_kf(k)$. To see that $u\in\mathcal{S}'$ simply observe that

$$|u(f)| \le \sup_{y\in\mathbf{R}^n}(1+|y|)^{M+n+1}|f(y)|\sum_{k\in\mathbf{Z}^n}\frac{c_k}{(1+|k|)^{M+n+1}}$$

and that the expression inside the supremum above is bounded by a finite sum of $\rho\alpha,\beta$ seminorms of f, while the sum converges since $|c_k|\le(1+|k|)^M$. By testing against Schwartz functions it is easy to see that the sequence of tempered distributions $\sum_{|k|\le N}c_k\delta_k$ converges to u in $\mathcal{S}'(\mathbf{R}^n)$ as $N\to\infty$. We also have

$$\widehat{u}(f) = \sum_{k\in\mathbf{Z}^n}c_k\int_{\mathbf{R}^n}e^{-2\pi i x\cdot k}f(x)\,dx = \lim_{N\to\infty}\int_{\mathbf{R}^n}h_N(x)f(x)\,dx$$

and this implies the second statement in the exercise.

??. (a) If u were a homogeneous function of degree α, then

$$u(\delta^\lambda f) = \int_{\mathbf{R}^n} u(x)f(\lambda x)\,dx = \lambda^{-n}\int_{\mathbf{R}^n} u(\lambda^{-1}x)f(x)\,dx = \lambda^{-n-\gamma}\int_{\mathbf{R}^n} u(x)f(x)\,dx$$

and the latter is $\lambda^{-n-\gamma}u(f)$. Hence this definition agrees with the usual definition of homogeneous functions. To see (b) simply observe that $\delta_0(\delta^\lambda f) = \delta_0(f) = \lambda^{-n-(-n)}\delta_0(f)$ and thus δ_0 is a homogeneous distribution of degree $-n$. Parts (c) and (d) follow by first observing that they hold for functions and then by the applying the identity $u(\delta^\lambda f) = \lambda^{-n-\gamma}u(f)$ to $\partial^\alpha f$ and $\widehat{f}$ respectively.

??. The sequence of functions e^{inx} tends to zero in $\mathcal{S}'$ in view of the Riemann-Lebesgue Lemma (Exercise **??** in section **??**). However the product $e^{inx}e^{-inx} = 1$ does not tend to zero in $\mathcal{S}'$. Observe that for $f \in C_0^\infty(\mathbf{R})$ we have

$$\int_{\mathbf{R}} f(x)\frac{\sqrt{n}}{1+n|x|^2}\,dx = \int_{\mathbf{R}} f(y/\sqrt{n})\frac{1}{1+|y|^2}\,dy \to 0$$

as $n \to \infty$.

??. Fix a Schwartz function h in $\mathbf{R}^n$ whose Fourier transform is equal to one on the ball $B(0,1)$ and vanishes off the ball $B(0,2)$. Then $\widehat{f}(\xi) = \widehat{f}(\xi)\widehat{h}(\xi/R)$. Taking inverse Fourier transforms we obtain $f = f * h_{1/R}$ and thus $\partial^\alpha f = R^{|\alpha|} f * (\partial^\alpha h)_{1/R}$. The conclusion follows by taking L^∞ norms.

SECTION 2.4

??. We have $\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$. Use the Lebesgue differentiation theorem to show that $(\widehat{f}(\xi+he_j)-\widehat{f}(\xi))/h$ converges to $\int_{\mathbf{R}^n}(-2\pi i x_j)f(x)e^{-2\pi i x\cdot\xi}dx$ and that the latter is continuous if $1 < N-n$. Use induction for the general case.

??. Corollary **??** gives that every harmonic function on $\mathbf{R}^n$ is a polynomial. Therefore a bounded harmonic function on $\mathbf{R}^n$ has to be a constant. Let us now derive the fundamental theorem of algebra from this. If the nonconstant polynomial $P(z)$ with complex coefficients had no complex roots, then $1/P(z)$ would be a well defined entire and (thus harmonic function) on $\mathbf{C}$. If the degree of P were greater than or equal to 1, then $1/P(z) \to 0$ as $|z| \to \infty$ and thus $1/P(z)$ would have to be bounded on the whole plane. It follows that $1/P(z)$ would have to be a constant and so would P which contradicts our assumption that P is nonconstant. Therefore P has to have a complex root.

??. Pick a C^∞ function ϕ equal to zero in a neighborhood of $-\infty$ and equal to one in a neighborhood of $+\infty$. The function e^x is not in $\mathcal{S}'$ since its integral against the Schwartz function $\phi(x)e^{-x}$ does not converge. Now for $f \in \mathcal{S}$ we have

$$\int_{-\infty}^{+\infty} f(x)e^x e^{ie^x}\,dx = \int_{-\infty}^{0} f(x)e^x e^{ie^x}\,dx + \int_0^1 f(\log y)e^{iy}\,dy.$$

The first integral above converges absolutely and is poitwise controlled by a multiple of $\|f\|_{L^\infty}$. Integrate by parts twice to write the second integral above as

$-\displaystyle\int_1^\infty (f''(\log y) - f'(\log y))e^{iy}\frac{dy}{y^2}$ and observe that this integral also converges absolutely and is pointwise controlled by a constant multiple of $\|f'\|_{L^\infty} + \|f''\|_{L^\infty}$. This makes $e^x e^{ie^x}$ a continuous linear functional on $\mathcal{S}$.

??. Since the function f is even, we have

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} \frac{\cos(2\pi x\xi)}{e^{x\pi} + e^{-x\pi}}\, dx = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos(2\xi x)}{e^x + e^{-x}}\, dx.$$

Consider the meromorphic function $G(z) = e^{2\xi z}(e^z + e^{-z})^{-1}$ and integrate it over the rectangular contour with corners $(-R, 0)$, $(R, 0)$, $(R, i\pi)$, and $(-R, i\pi)$. The function G has only one pole inside this contour at the point $i\pi/2$ and the residue at this pole is $-ie^{-\pi\xi}/2$. The exponential decay of the integrant gives that the parts of integral of $G(z)$ over the two vertical sides of the contour tend to zero as $R \to \infty$. The part of the integral of G over the top horizontal line of the contour tends to $e^{-2\pi\xi}\pi\widehat{f}(\xi)$, while the integral of G over the bottom horizontal line of the contour tends to $\pi\widehat{f}(\xi)$. Cauchy's residue theorem now gives

$$\pi\widehat{f}(\xi) + e^{-2\pi\xi}\pi\widehat{f}(\xi) + 0 + 0 = 2\pi i(-ie^{-\pi\xi}/2),$$

which says that $\widehat{f}(\xi) = \operatorname{sech}(\pi\xi) = f(\xi)$.

??. Since ϕ is periodic with period 1 we can expand $e^{i\phi(x-a)}$ in Fourier series as

$$e^{i\phi(x-a)} = \sum_{k\in\mathbf{Z}} c_k e^{2\pi ik(x-a)}.$$

Then $\widehat{f_a}(\xi) = \sum_{k\in\mathbf{Z}} c_k e^{-2\pi ika)}\widehat{w}(\xi - k)$ and the absolute value of the latter is equal to $\sum_{k\in\mathbf{Z}} |c_k||\widehat{w}(\xi - k)|$ since w is supported in the interval $[-1/2, 1/2]$. If ϕ is chosen so that for any $a_1, \dots, a_m \in \mathbf{R}$ there exist $x_1, \dots, x_m \in \mathbf{R}$ such that $\det(e^{i\phi(x_j - a_k)})_{1\le j,k\le n} \neq 0$, then every finite subfamily of the family $\{f_a : a \in \mathbf{R}\}$ is linearly independent.

??. First observe that since $P_y \in L^{p'}$, it follows that the convolution $f * P_y$ is well defined for $f \in L^p(\mathbf{R}^n)$. Moreover, every second derivative of the function $(x, y) \to P_y(x)$ is also in $L^{p'}$ and thus the integral

$$\int_{\mathbf{R}^n} \left(\frac{d^2}{dy^2}P_y(x-t) + \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}P_y(x-t)\right) f(t)\, dt$$

converges absolutely and is equal to the the Laplacian of the function $(x, y) \to (P_y * f)(x)$ on $\mathbf{R}^{n+1}_+$. Now an explicit calculation gives that the Laplacian of the function $(x, y) \to P_y(x)$ is identically equal to zero and this proves the conclusion. Exercise **??** in section **??** gives that the Fourier transform (in x) of $P_y(x)$ is $e^{-2\pi|\xi|}$. Therefore the identity $(P_{y_1} * P_{y_2})(x) = P_{y_1+y_2}(x)$ follows easily by taking Fourier transforms.

??. Part (a) follows by an explicit calculation. To prove part (b) simply observe that when $|\theta - x_0| \ge \delta$, then $|\theta - rx_0| \ge \delta/2$ uniformly in x_0 when $r > 1 - \delta/2$. For part (c) we will need the additional observation that when $r < 1$, the function $\theta \to v(\theta, rx_0)$ has integral equal to 1 over the sphere $\mathbf{S}^{n-1}$. This can be proved directly but can also be proved using the fact that $v(\theta, rx_0)$ is harmonic and therefore its value at the origin is equal to its average over the unit sphere. To show that u is harmonic just interchange the Laplacian and the integral that defines u. This interchange is justified for x in any open ball strictly smaller than the unit ball, hence for all x in the unit ball. Clearly u is continuous on the interior of $B(0,1)$ and it remains to show that u is continuous on its boundary $\mathbf{S}^{n-1}$. Let $x_0 \in \mathbf{S}^{n-1}$, $0 < r < 1$ and let $B_\delta(x_0) = \{\theta \in \mathbf{S}^{n-1} : |\theta - x_0| > \delta\}$. Using that the function $\theta \to v(\theta, rx_0)$ has integral equal to 1 over the sphere $\mathbf{S}^{n-1}$ we obtain

$$|u(rx_0) - u(x_0)| = \left| \int_{\mathbf{S}^{n-1}} \big(f(\theta) - f(x_0)\big) v(\theta; rx_0)\, d\theta \right|$$
$$\leq \int_{B_\delta(x_0)} |f(\theta) - f(x_0)| v(\theta; rx_0)\, d\theta + \int_{\mathbf{S}^{n-1} - B_\delta(x_0)} |f(\theta) - f(x_0)| v(\theta; rx_0)\, d\theta.$$

Since f is uniformly continuous on $\mathbf{S}^{n-1}$, given $\varepsilon > 0$ find a $\delta > 0$ such that $\sup_{\theta \notin B_\delta(x_0)} |f(\theta) - f(x_0)| < \varepsilon/2$ uniformly in $x_0 \in \mathbf{S}^{n-1}$. For this $\delta > 0$ pick an r_0 such that for $r > r_0$ we have $\displaystyle\int_{B_\delta} v(\theta; rx_0)\, d\theta < \varepsilon/2\|f\|_{L^\infty}$. Using the inequalities above we see that for $r > r_0$ we have that $|u(rx_0) - u(x_0)| < \varepsilon$ uniformly in x_0. Now if $x \in B(0,1)$ let $x' = x/|x|$ and observe that

$$|u(x) - u(x_0)| \leq |u(x) - u(x')| + |f(x') - f(x_0)| < 2\varepsilon$$

if $|x| > r_0$ and x' is sufficiently close to x_0. This proves the continuity of u on the boundary of the sphere.

??. (a) Let Π be the stereographic projection of Appendix D4. Rotational invariance implies that the integral $I = \displaystyle\int_{\mathbf{S}^n} |\xi - \eta|^{-\lambda} d\xi$ is independent of $\eta \in \mathbf{S}^{n-1}$. On the other hand the stereographic projection gives

$$\begin{aligned} I &= \int_{\mathbf{R}^n} |\Pi(x) - \Pi(y)|^{-\lambda} 2^n (1+|x|^2)^{-n}\, dx \\ &= \int_{\mathbf{R}^n} 2^{n-\lambda} |x-y|^{-\lambda} (1+|x|^2)^{\frac{\lambda}{2}-n} (1+|y|^2)^{\frac{\lambda}{2}}\, dx \\ &= (1+|y|^2)^{\frac{\lambda}{2}} \int_{\mathbf{R}^n} 2^{n-\lambda} |x|^{-\lambda} (1+|x|^2)^{\frac{\lambda}{2}-n}\, dx = J \end{aligned}$$

since $y = \Pi(\eta)$ is an arbitrary element of $\mathbf{R}^n$, (choose $y = \Pi(-e_{n+1}) = 0$. The integral J above converges because $0 < \lambda < n$ can it be computed by switching to polar coordinates $x = r\theta$ and using the change of variables $r = \tan(\phi)$. We obtain

$$J = 2^{n-\lambda} |\mathbf{S}^{n-1}| \int_0^{\pi/2} (\sin\phi)^{n-\alpha-1} (\cos\phi)^{n-1} d\phi = 2^{n-\lambda} \frac{\pi^{\frac{\pi}{2}} \Gamma(\frac{n-\lambda}{2})}{\Gamma(n - \frac{\lambda}{2})},$$

by the formulas in Appendices A3 and A4. The conclusion in part (b) follows from the fact that the function I above is constant in y.

??. By a translation, a dilation, and a rotation we may assume that $x = e_1 = (1, 0, \ldots, 0)$ and $y = 0$. Then

$$\int_{\mathbf{R}^n} |e_1 - t|^{-\alpha_1} |t|^{-\alpha_2} dt = \Big(c_1 |\xi|^{-n+\alpha_1} c_2 |\xi|^{-n+\alpha_2} \Big)^{\vee} (e_1) = c_1 c_2 c_3$$

where Theorem **??** was repeatedly used and these constants can be explicitly computed.

??. Part (a) is subsumed in the proof of part (b) below. Write $x = (x_1, x') \in \mathbf{R}^n$ with $x' \in \mathbf{R}^{n-1}$. Suppose that h is constant on the sphere S given by the equations

$x_1 = a_1$ and $(x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = R^2$. Let (a_1, x') and (a_1, y') be two points in S. Pick a $B \in O(n-1)$ such that $y = (a_1, y') = (a_1, Bx')$. Then

$$\begin{aligned}
\widehat{h}(y) = &\int_{\mathbf{R}^n} e^{-2\pi i t_1 a_1} \int_{\mathbf{R}^{n-1}} h(t_1, t') e^{-2\pi i t' \cdot y'} \, dt' \, dt_1 \\
&\int_{\mathbf{R}^n} e^{-2\pi i t_1 a_1} \int_{\mathbf{R}^{n-1}} h(t_1, t') e^{-2\pi i t' \cdot Bx'} \, dt' \, dt_1 \\
&\int_{\mathbf{R}^n} e^{-2\pi i t_1 a_1} \int_{\mathbf{R}^{n-1}} h(t_1, t') e^{-2\pi i B^{-1} t' \cdot x'} \, dt' \, dt_1 \\
&\int_{\mathbf{R}^n} e^{-2\pi i t_1 a_1} \int_{\mathbf{R}^{n-1}} h(t_1, Bt') e^{-2\pi i t' \cdot x'} \, dt' \, dt_1 \\
&\int_{\mathbf{R}^n} e^{-2\pi i t_1 a_1} \int_{\mathbf{R}^{n-1}} h(t_1, t') e^{-2\pi i t' \cdot x'} \, dt' \, dt_1 \\
=&\widehat{a_1, x'} = \widehat{h}(x).
\end{aligned}$$

??. By a translation, a dilation, and a rotation we may assume that $y = e_1 = (1, 0, \dots, 0)$ and $z = 0$. Let $g_1(x) = |x - e_1|^{d_1 - n} |x|^{d_3 - n}$ and $g_2(x) = \pi^{n/2} \prod_{j=1}^{3} \frac{\Gamma\left(n - \frac{d_j}{2}\right)}{\Gamma\left(\frac{d_j}{2}\right)} |x - e_1|^{d_1 - n} |x|^{d_3 - n}$. We will show that $\widehat{g_1} = \widehat{g_2}$. Theorem **??** gives

$$(|x|^{d-n})^\wedge(\xi) = \pi^{n/2 - d} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{-d+n}{2}\right)} \, |\xi|^{-d} := c(d) |\xi|^{-d}.$$

We now caluclate

$$\begin{aligned}
\widehat{g_1}(\xi) =&c(d_3) c(d_1) |\xi|^{-d_3} * \left(|\xi|^{-d_1} e^{-2\pi i \xi \cdot e_1} \right) \\
=&c(d_3) c(d_1) \int_{\mathbf{R}^n} |\xi - \eta|^{-d_3} |\eta|^{-d_1} e^{-2\pi i \eta \cdot e_1} d\eta \\
=&c(d_3) c(d_1) |\xi|^{n - d_1 - d_3} \int_{\mathbf{R}^n} |\xi' - t|^{-d_3} |t|^{-d_1} e^{-2\pi i |\xi| t \cdot e_1} dt,
\end{aligned}$$

where $\xi' = \xi / |\xi|$. Now, for given ξ find a rotation A_ξ so that $A_\xi e_1 = \xi'$. Clearly $|\xi' - t| = |e_1 - A_\xi^{-1} t|$, $|t| = |A_\xi^{-1} t|$, and $t \cdot e_1 = t \cdot A_\xi^{-1} \xi' = A_\xi^{-1} t \cdot A_\xi^{-2} \xi'$. Hence, with $s = A_\xi^{-1} t$ and the last integral above is equal to

$$\begin{aligned}
c(d_3) c(d_1) |\xi|^{n - d_1 - d_3} &\int_{\mathbf{R}^n} |e_1 - s|^{-d_3} |s|^{-d_1} e^{-2\pi i s \cdot A_\xi^{-2} \xi} ds \\
&= c(d_3) c(d_1) |\xi|^{n - d_1 - d_3} \widehat{h}(A_\xi^{-2} \xi),
\end{aligned}$$

where $h(t) = |t - e_1|^{-d_3} |t|^{-d_1}$. We have that $\widehat{g}(\xi) = \left(h * |t|^{-d_2}\right)^\wedge(\xi) = c(n - d_2) \widehat{h}(\xi) |\xi|^{d_2 - n}$. Using that $d_1 + d_2 + d_3 = 2n$ and that $c(n-d)^{-1} = c(d)$ we deduce $\widehat{g_1}(\xi) = \widehat{g_2}(\xi)$ if and only if

$$\widehat{h}(\xi) = \widehat{h}(A_\xi^{-2} \xi) \qquad \text{for almost all } \xi \in \mathbf{R}^n.$$

Now use that the function h is constant along spheres orthogonal to e_1 and the previous exercise to obtain the above identity.

??. (a) Integrating e^{iz^2} over the described contour we obtain

$$\int_0^R e^{ix^2} (x + i0)' \, dx + \int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} (Re^{i\theta})' \, d\theta + \int_{R\frac{\sqrt{2}}{2}}^0 e^{i(x + ix)^2} (x + ix)' \, dx = 0$$

But the middle integral above is bounded in absolute value by

$$R\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}\,d\theta \le R\int_0^{\frac{\pi}{4}} e^{-R^2\frac{4\theta}{\pi}}\,d\theta \le \frac{\pi}{4R}\int_0^\infty e^{-x}dx$$

which tends to zero as $R\to\infty$. Using this fact and letting $R\to\infty$ we obtain

$$\int_0^\infty e^{ix^2}\,dx = (1+i)\int_0^\infty e^{-2x^2}\,dx = (1+i)\frac{\sqrt{2\pi}}{4}$$

by using the result in Appendix A1. The Fresnel integral identity follows.
(b) The n-dimensional case follows from the 1-dimensional case below:

$$\int_{-\infty}^\infty e^{i\pi x^2}e^{-2\pi i x\xi}dx = \int_{-\infty}^\infty e^{i\pi(x^2-2x\xi)}dx = e^{-i\pi\xi^2}\int_{-\infty}^\infty e^{i\pi(x-\xi)^2}dx = e^{-i\pi\xi^2}e^{i\frac{\pi}{4}}$$

by using the result in part (a).

SECTION 2.5

??. If $\phi\in C_0^\infty$, then for h large $\tau^h\phi$ has disjoint support from ϕ and therefore $\|\tau^h\phi+\phi\|_{L^q}^q = \|\phi\|_{L^q}^q + \|\tau^h\phi\|_{L^q}^q = 2\|\phi\|_{L^q}^q$. Given $f\in L^q$ and $\varepsilon>0$ find a $\phi\in C_0^\infty$ such that $\|f-\phi\|_{L^q}<\varepsilon$. Now find a $h_0>0$ such that for $|h|>h_0$ we have $\|\tau^h\phi+\phi\|_{L^q} = 2^{1/q}\|\phi\|_{L^q}$. Then for $|h|>h_0$ we have

$$\|\tau^h f+f\|_{L^q}\le 2\varepsilon + \|\tau^h\phi+\phi\|_{L^q}\le (2+2^{1/q})\varepsilon + 2^{1/q}\|f\|_{L^q}$$

and similarly $\|\tau^h f+f\|_{L^q}\ge 2^{1/q}\|f\|_{L^q}-(2+2^{1/q})\varepsilon$. The assertion follows.

??. Translation of a multiplier corresponds to conjugation of the operator by a character, an operation that clearly does not affect its norm. The multiplier $\delta^h m$ corresponds to the kernel K_h, where $K=m^\vee$ and $K_h = h^{-n}\delta^{h^{-1}}K$. By Exercise **??** in section **??** we have $f*K_h = (f_{h^{-1}}*K)_h$ for all $f\in\mathcal{S}$. Thus

$$\frac{\|f*K_h\|_{L^p}}{\|f\|_{L^p}} = \frac{\|(f_{h^{-1}}*K)_h\|_{L^p}}{\|f\|_{L^p}} = \frac{h^{-n/p}\|f_{h^{-1}}*K\|_{L^p}}{\|f\|_{L^p}} = \frac{\|f_{h^{-1}}*K\|_{L^p}}{\|f_{h^{-1}}\|_{L^p}}.$$

Taking the supremum over $f\neq 0$ we obtain that the operators $f\to f*K$ and $f\to f*K_h$ have the same norm from L^p to L^p. The statement with n-parameter family of dilations follows similarly. Reflection of multiplier corresponds to reflection of the kernel which does not affect the norm of the operator. Modulation of the multiplier corresponds to translation of the operator which also does not affect the norm of the operator. Finally rotation of the multiplier corresponds to rotation of the kernel and this operation also does not change the norm of the operator.

??. In case (a) we have for $f\in\mathcal{S}$

$$\|(\widehat{f}\psi m)^\vee\|_{L^p} = \|f*\psi^\vee * m^\vee\|_{L^p}\le \|m\|_{\mathcal{M}_p}\|f*\psi^\vee\|_{L^p}\le \|m\|_{\mathcal{M}_p}\|\psi^\vee\|_{L^1}\|f\|_{L^p}.$$

This proves the first assertion in the exercise. Since $(\psi*m)(\xi) = \int_{\mathbf{R}^n}\psi(z)m(\xi-z)\,dz$ we have that

$$\big(\widehat{f}(\psi*m)\big)^\vee = \int_{\mathbf{R}^n}\psi(z)\big(\widehat{f}\tau^z m)\big)^\vee\,dz$$

for all $f\in\mathcal{S}$. Taking L^p norms and using that the multipliers $\tau^{-z}m$ and m have the same $\mathcal{M}_p$ norm proves assertion (b).

??. Since $\rho_{\alpha,\beta}(\partial^\gamma f) = \rho_{\alpha,\beta+\gamma}(f)$, for all multi-indices α and β, it follows that every finite sum of seminorms of $\partial^\gamma f$ is controlled by a finite sum of seminorms of

f. This proves part (a). To see part (b) assume that $\|\partial^\gamma f\|_{L^q} \le C\|f\|_{L^p}$ for some $0 < C < \infty$ and all $f \in \mathcal{S}(\mathbf{R}^n)$. Let $f_0(x) = e^{-\pi|x|^2}$ and apply this inequality to the functions $(\delta^\lambda f_0)(x) = f_0(\lambda x)$ with $\lambda > 0$. Since $\partial^\gamma \delta^\lambda f_0 = \lambda^{|\gamma|}\delta^\lambda \partial^\gamma f_0$ one obtains

$$\lambda^{|\gamma|}\lambda^{-n/q}\|\partial^\gamma f_0\|_{L^q} \le C\lambda^{-n/p}\|f_0\|_{L^p}.$$

If $1/q-1/p \neq |\gamma|/n$, then by sending λ to zero or infinity we obtain $\infty \le C\|f_0\|_{L^p} < \infty$ or $0 < \|\partial^\gamma f_0\|_{L^q} \le 0$ which are both impossible.

??. Observe that the function $K_\gamma(x) = |x|^{-n+\gamma}$ is in $L^{r,\infty}$ with $r = (n-\gamma)/\gamma$. Theorem **??** gives that the operator $f \to f * K_\gamma$ maps $L^p \to L^q$ when $1 < p < q < \infty$ and $1/p + 1/r = 1/q + 1$. This is the same as $1/p - 1/q = \gamma/n$.

??. Part (a) was essentially proved in Proposition **??**. Part (b) is a corollary of part (a) by setting $m_j(\xi) = \displaystyle\int_{1/j}^{j} m_t(\xi)\,\frac{dt}{t}$. Part (c) is also a consequence of part (a) and of the fact that dilations do not affect the norm of a multiplier. Finally part (d) follows from the sequence of inequalities

$$\|m\|_{\mathcal{M}_p} \ge \|m_0\|_{\mathcal{M}_p} \ge \|m_0\|_{L^\infty} \ge \max(|m(0+)|, |m(0-)|).$$

??. Simply observe that

$$\|f\|_{L^p} = \|(\widehat{f} m \tfrac{1}{m})^\vee\|_{L^p} \le \|m\|_{\mathcal{M}_p}\|(\widehat{f}\tfrac{1}{m})^\vee\|_{L^p} = \|m\|_{\mathcal{M}_p}\|Tf\|_{L^p}.$$

??. Part (a) follows directly from Exercise **??** in section **??**. To prove part (b) observe that Exercise **??** gives that $\|m_2\|_{\mathcal{M}_p} \ge \|m_1\|_{\mathcal{M}_p}$. Now $m_1 m_2 = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}$ and $(m_1 m_2)^\vee = [(\chi_{[-1/2,1/2]})^\vee]^2 \ge 0$. By part (a), $\|m_1 m_2\|_{\mathcal{M}_p} = \displaystyle\int [(\chi_{[-1/2,1/2]})^\vee]^2(t)\,dt = 1$, in view of Plancherel's Theorem. Thus

$$\|m_2\|_{\mathcal{M}_p} = \|m_2 m_1 m_1\|_{\mathcal{M}_p} \le \|m_2 m_1\|_{\mathcal{M}_p}\|m_1\|_{\mathcal{M}_p} = \|m_1\|_{\mathcal{M}_p}.$$

??. (1) and (2) are clearly equivalent. Let A be the norm of the operator in (1) and B the norm of the operator in (3). We can easily obtain that $B \le A$ by applying the operator in (1) to step functions. For the converse write

$$\begin{aligned}\int_{\mathbf{R}^n}\left|\sum_{m\in\mathbf{Z}^n} a_m f(x-m)\right|^p dx &= \int_{[0,1]^n}\sum_{k\in\mathbf{Z}^n}\left|\sum_{m\in\mathbf{Z}^n} a_m f(x+k-m)\right|^p dx\\ &\le B^p\int_{[0,1]^n}\sum_{k\in\mathbf{Z}^n}|f(x-k)|^p dx = B^p\|f\|^p_{L^p(\mathbf{R}^n)},\end{aligned}$$

which implies that $A \le B$.

??. Write $m(x) = \sum_{j\in\mathbf{Z}^n} a_j e^{2\pi i j\cdot x}$ in Fourier series for $x \in [0,1]^n$. Then $M(x) = \sum_{j\in\mathbf{Z}^n} a_j e^{2\pi i j\cdot x}$ for $x \in \mathbf{R}^n$. Let $h \in C_0^\infty$ to be determined later and let $Q = [-1/2,1/2]^n$ be the unit cube centered at the origin. Define operators

$$\begin{aligned}E((c_j)) &= \sum_{k\in\mathbf{Z}^n} c_k \chi_Q(x+k)\\ R(f)(j) &= \int_{-j+Q} f(x)\,dx\end{aligned}$$

Let T_{mh} be the operator with multiplier mh. An easy calculation shows that $(RT_{mh}E)((c_k))(j) = \sum_{k\in\mathbf{Z}^n} c_k q_{j-k}$, where

$$q_j = \int_Q (\widehat{\chi_Q} mh)^\vee (x-j)dx = \sum_{k\in\mathbf{Z}^n} a_k \int_Q (\widehat{\chi_Q})^2(\xi)h(\xi)e^{2\pi i(k-j)\cdot\xi}d\xi = a_j,$$

if we pick a $C_0^\infty(\mathbf{R}^n)$ function h equal to $(\widehat{\chi_Q})^{-2}(\xi) = \prod_{j=1}^n \pi^2\xi^2\sin^{-2}(\pi\xi)$ on Q. Therefore the operator $RT_{mh}E$ is given by convolution with the fixed sequence (a_j). Since both R and E are contractions, we have $\|RT_{mh}E\|_{l^p\to l^p} \le \|T_{mh}\|_{L^p\to L^p} \le \|h^\vee\|_{L^1}\|m\|_{\mathcal{M}_p}$ by Exercise **??** above. The conlcusion follows from the previous Exercise.

??. In view of Exercise **??** in section **??**, the distributional Fourier transform $\widehat{u}$ of u is a homogeneous distribution of degree $-i\tau$. But Proposition **??** gives that $\widehat{u}$ is a C^∞ function on $\mathbf{R}^n - \{0\}$, and in particular bounded on the sphere $\mathbf{S}^{n-1}$. Since $\widehat{u}(\xi) = |\xi|^{-i\tau}\widehat{u}(\frac{\xi}{|\xi|})$ it follows that $\widehat{u}$ is a bounded function on $\mathbf{R}^n - \{0\}$. Thus $\|\widehat{u}\|_{L^\infty} < \infty$ and the operator given by convolution with u is L^2 bounded.

??. The trilinear operator $(m_1, m_2, f) \to \big((m_1 * m_2)\widehat{f}\big)^\vee$ maps $L^\infty \times L^1 \times L^2 \to L^2$ and also $L^2 \times L^2 \times L^1 \to L^1$. The multilinear Riesz-Thorin interpolation theorem gives the desired conclusion.

CHAPTER 3

SECTION 3.1

??. Both parts of the exercise easily follow from Proposition **??** and the observation that the Fourier coefficients of the functions on each side are equal.

??. Given a trigonometric polynomial $P(x) = \sum_{j=-N}^N a_j e^{2\pi i j x}$ consider the meromorphic function $Q(z) = \sum_{j=-N}^N a_j z^j$ on $\mathbf{C}$. Then $Q(z) = z^{-N}R(z)$ where $R(z) = \sum_{j=0}^{2N} a_{j-N}z^j$ is a polynomial with at most $2N$ zeros on the plane. Therefore $Q(z)$ has at most $2N$ zeros on the plane and so does $P(x)$ on the circle. The trigonometric polynomial $e^{2\pi i N x} + e^{-2\pi i N x} = 2\cos(2\pi x)$ has exactly $2N$ zeros on the circle, namely $x_k = (2k+1)/4N$, where $k \in \{0, 1, \dots, 2N-1\}$.

??. Since $\sin^2(\pi t) = \frac{1}{2}(1-\cos(2\pi t)) = \frac{1}{2} - \frac{1}{4}e^{2\pi i t} - \frac{1}{4}e^{-2\pi i t}$, a simple calculation gives

$$\left(\frac{1}{2} - \frac{1}{4}e^{2\pi i x} - \frac{1}{4}e^{-2\pi i x}\right)\sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right)e^{2\pi i x}$$
$$= \frac{1}{N+1}\left(\frac{1}{2} - \frac{1}{4}e^{2\pi(N+1)ix} - \frac{1}{4}e^{-2\pi(N+1)ix}\right),$$

and this proves the required identity. We now prove that F_N is an approximate identity. We have that $F_N \ge 0$ and using the first part of the identity in (**??**), we easily obtain that $\int_{\mathbf{T}^1} F_N(x)\,dx = 1$ which verifies parts (i) and (ii) of the definition of approximate identities. To verify part (iii) we use the second part of the identity in (**??**). Given $|x| \le 1/2$ we have $|\sin(\pi x)| \ge 2|x|$ by Appendix E. Therefore if

$1/2 \geq |x| \geq \delta$ we have

$$\int_{\delta \leq |x| \leq 1/2} \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)} \right)^2 dx \leq \frac{1}{N+1} \frac{1-2\delta}{4\delta^2} \to 0$$

as $N \to \infty$. This proves property (iii) of approximate identities. Because of the product structure of the multiple Fejér kernel, the fact that $F(n, N)$ is an approximate identity on $\mathbf{T}^n$ is a direct consquence of the one-dimensional result.

??. Properties (i) and (iii) of approximate identities follow from the corresponding properties for the Fejér kernel $F_N = F(1, N)$. Since $\displaystyle\int_{\mathbf{T}^1} F_N(x)\, dx = 1$ for all N, it follows that this is also true for V_N, hence property (ii) of approximate identities follows. Using (**??**) we obtain that $\widehat{F_N}(j) = 1 - \frac{|j|}{N+1}$ when $|j| \leq N+1$ and zero otherwise. Therefore

$$\widehat{V_N}(j) = \begin{cases} 2\big(1 - \frac{|j|}{2N+2}\big) - \big(1 - \frac{|j|}{N+1}\big) = 1 & \text{when } |j| \leq N+1, \\ 2\big(1 - \frac{|j|}{2N+2}\big) & \text{when } 2N+1 \geq |j| > N+1, \\ 0 & \text{when } |j| > 2N+1. \end{cases}$$

??. This is a consequence of the Riesz-Thorin interpolation theorem. Interpolate between the estimates $\|\{\widehat{f}\}\|_{l^\infty} \leq \|f\|_{L^1}$ and $\|\{\widehat{f}\}\|_{l^2} \leq \|f\|_{L^2}$ to obtain for $1 < p < 2$ the estimate $\|\{\widehat{f}\}\|_{l^{p'}} \leq \|f\|_{L^p}$ for all f simple functions. Use density to extend this to all functions. To see that the norm of the operator $f \to \{\widehat{f}\}$ is 1, take $f(x) = 1$. To obtain the converse inequality, let $\{a_m\}_{m \in \mathbf{Z}^n} \in l^{p'}$ for some $1 \leq p \leq 2$. Then $\{a_m\}_{m \in \mathbf{Z}^n} \in l^2$ and the function $h(x) = \sum_{m \in \mathbf{Z}^n} a_m e^{2\pi i x \cdot m}$ is in $L^2(\mathbf{T}^n)$ and hence in $L^p(\mathbf{T}^n)$. Clearly we have $\widehat{h}(m) = a_m$ for all $m \in \mathbf{Z}^n$.

??. Let $g_N(x) = \displaystyle\sum_{k=2}^{N} \frac{e^{ik \log k}}{k^{1/2} (\log k)^2} e^{2\pi i k x}$ for $x \in [0, 1]$. Set $b_k = k^{-1/2} (\log k)^{-2}$ for $k \geq 2$ and $b_1 = 0$. Also set $a_k = e^{ik(\log k + 2\pi x)}$. for $k \geq 1$. Summation by parts gives

$$\sum_{k=1}^{N} a_k b_k = A_N b_N - \sum_{k=1}^{N-1} A_k (b_{k+1} - b_k),$$

where $A_k = \sum_{j=1}^{k} a_j$. Since $|b_k - b_{k+1}| \leq c k^{-3/2} (\log k)^{-2}$ and $|A_k| \leq C k^{1/2}$, the Weirstrass M-test gives that the sequence of functions g_N converges uniformly to a continuous function g on $[0, 1]$. Because of the uniform convergence, one can interchange the integration with the limit below

$$\lim_{N \to \infty} \int_{\mathbf{T}^1} g_N(x) e^{-2\pi i x \cdot \xi}\, dx = \int_{\mathbf{T}^1} \lim_{N \to \infty} g_N(x) e^{-2\pi i x \cdot \xi}\, dx,$$

to obtain that the Fourier coefficients $\widehat{g}(k)$ of g are $\dfrac{e^{ik \log k}}{k^{1/2} (\log k)^2}$ when $k \geq 2$ and zero otherwise. For this sequence we certainly have $\sum_{k \in \mathbf{Z}} |\widehat{g}(k)|^q = \infty$ for all $q < 2$.

??. For $r_1, \ldots, r_n < 1$ the geometric series that defines $P_{r_1, \ldots, r_n}$ converges absolutely and uniformly for $x \in \mathbf{T}^n$. Because of the multiplicative structure of the kernel $P_{r_1, \ldots, r_n}$, it suffices to show the required identity for $n = 1$. First observe that

$$\frac{1 + re^{2\pi i x}}{1 - re^{2\pi i x}} = \frac{(1 + re^{2\pi i x})(1 - re^{-2\pi i x})}{|1 - re^{2\pi i x}|^2} = \frac{1 - r^2 + 2i \sin(2\pi x)}{1 - 2r \cos(2\pi x) + r^2},$$

which implies the second identity by taking real parts. Now

$$\frac{1+re^{2\pi ix}}{1-re^{2\pi ix}} = (1+re^{2\pi ix})\sum_{j=0}^{\infty} r^j e^{2\pi ijx} = 1+2\sum_{j=1}^{\infty} r^j e^{2\pi ijx}$$

and the first identity also follows by taking real parts. Let us now prove that the Poisson kernel P_r gives an approximate identity as $r \uparrow 1$. The last identity above gives that P_r is integrable and that its integral is equal to 1. Since P_r is positive, this proves properties (i) and (ii) of the definition of approximate identities. To prove property (iii) of approximate identities, let us fix a $\delta > 0$. If $|x| \le 1/2$, then $|1-e^{2\pi ix}| \ge 2\cdot 2\pi|x|/\pi$ by Appendix E. Hence $|1-e^{2\pi ix}| \ge 4\delta$ if $1/2 \ge |x| \ge \delta$. It follows that $|1-re^{2\pi ix}| \ge |r-re^{2\pi ix}| - (1-r) \ge 4\delta r - (1-r)$ and thus

$$\int_{1/2\ge|x|\ge\delta} P_r(x)\,dx \le (1-2\delta)\frac{1-r^2}{(4\delta r-(1-r))^2}.$$

Now this last expression tends to zero as $r \uparrow 1$ (take $r > (4\delta+1)^{-1}$). We now extend this result to $\mathbf{T}^n$ which we identify with $[-1/2,1/2)^n$. Since $P_{r,\dots,r}(x_1,\dots,x_n) = P_r(x_1)\dots P_r(x_n)$, properties (i) and (ii) of approximate identities follow by multiplying each coordinate separately. To prove property (iii), let $V_\delta = \{x \in \mathbf{T}^n : |x_j| < \delta, \quad 1 \le j \le n\}$ be the cubic neighborhood of 0. For $x = (x_1,\dots,x_n) \notin V_\delta$ we have that $|x_j| \ge \delta$ for some $1 \le j \le n$. Therefore

$$\int_{V_\delta^c} P_{r,\dots r}(x_1,\dots,x_n)\,dx \le \sum_{j=1}^{n}\Big(\prod_{k\ne j}\int_{\mathbf{T}^1} P_r(x_k)\,dx_k\Big)\int_{1/2\ge|x_j|\ge\delta} P_r(x_j)\,dx_j$$
$$\le n\,(1-2\delta)\,\frac{1-r^2}{(4\delta r-(1-r))^2}$$

by the previous argument, and the last expression above tends to 0 as $r \uparrow 1$.

??. Write $\|D_N\|_{L^1} = \displaystyle\int_{-1/2}^{1/2}\left|\frac{\sin((2N+1)\pi x)}{\sin(\pi x)}\right|dx = 2\int_0^{1/2}\frac{|\sin((2N+1)\pi x)|}{\sin(\pi x)}\,dx$ as the sum

$$2\int_0^{1/2}\frac{|\sin((2N+1)\pi x)|}{\pi x}\,dx + 2\int_0^{1/2}|\sin((2N+1)\pi x)|\left(\frac{1}{\sin(\pi x)}-\frac{1}{\pi x}\right)dx$$

Now the second term above is bounded by $\pi/4$ and change variables $t = (2N+1)\pi x$ in the first integral above. Since $|\sin(t)| = |\sin(t-\pi)|$ for all t, the first term above can be written as

$$\frac{2}{\pi}\left[\int_0^{\pi}\frac{\sin t}{t}\,dt + \int_0^{\pi}\frac{\sin t}{t+\pi}\,dt + \cdots + \int_0^{\pi}\frac{\sin t}{t+(N-1)\pi}\,dt + \int_0^{\pi/2}\frac{\sin t}{t+N\pi}\,dt\right].$$

This is easily estimated from above by

$$\frac{2}{\pi}\left[\int_0^{\pi} dt + \Big(\int_0^{\pi}\sin t\,dt\Big)\Big(\frac{1}{\pi}+\frac{2}{\pi}+\cdots+\frac{1}{N\pi}\Big)\right] = 2 + \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}$$

and from below by

$$\frac{2}{\pi}\Big(\int_0^{\pi}\sin t\,dt\Big)\left[\frac{1}{\pi}+\frac{2}{\pi}+\cdots+\frac{1}{N\pi}\right] = \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}$$

??. One can easily see that $D_N(x)$ has N roots in the interval $(0,1/2)$, the smallest one being $x = (2N+1)^{-1}$. The maximum of D_N attained at zero is $2N+1$

and we therefore have $|D_N(x)| \le 2N+1$ while the estimate $|D_N(x)| \le \pi/(2|x|)$ is always true. We have

$$\|D_N\|_{L^p}^p \le 2\int_0^{\frac{1}{2N+1}} (2N+1)^p\,dx + 2\int_{\frac{1}{2N+1}}^{\frac{1}{2}} \frac{\pi^p}{(2|x|)^p}\,dx \le C_p(2N+1)^{p-1}.$$

The estimate from below follows by observing that when $|x| \le 1/2(2N+1)$ we have $|D_N(x)| \ge \dfrac{2(2N+1)|x|}{\pi|x|}$ (by Appendix E) and thus

$$\|D_N\|_{L^p}^p \ge 2\int_0^{\frac{1}{2(2N+1)}} (2(2N+1)/\pi)^p\,dx \ge c_p(2N+1)^{p-1}.$$

??. We check this identity by comparing Fourier coefficients. The m^{th} Fourier coefficient of the left hand side is $\dfrac{m}{N}\,\widehat{P}(m)$ while the Fourier coefficient of the right hand side is

$$\widehat{P}(m)\bigg(1-\frac{|m-N|}{N}\bigg)_+ - \widehat{P}(m)\bigg(1-\frac{|m+N|}{N}\bigg)_+ = \frac{m}{N}\,\widehat{P}(m).$$

Now take L^∞ norms of both terms of the identity in the hint. Using that $\|F_{N-1}\|_{L^1} = 1$ we obtain the requird conclusion.

??. Define $R(z) = \sum_{k=-N}^{N} a_k z^{k+N}$ where $z \in \mathbf{C}$. Since P is positive we have $a_k = \overline{a_{-k}}$ and hence $R(z) = z^{2N}\overline{R(1/\overline{z})}$. It is easy to see that the zeros of R are away from the origin and away from the unit circle. Therefore there exist $0 < |z_k| < 1$ and $r_k \ge 1$ with $\sum_{k=1}^{s} r_k = N$ such that

$$R(z) = a_N \prod_{k=1}^{s} (z-z_k)^{r_k}(z-1/\overline{z_k})^{r_k},$$

Since $z - 1/\overline{w} = -z(1/z-\overline{w})/\overline{w}$, for $z = e^{2\pi i\xi}$ on the unit circle we have

$$\begin{aligned}
P(\xi) = e^{-2\pi iN\xi}R(e^{2\pi i\xi}) &= a_N e^{-2\pi i\xi}\prod_{k=1}^{s}(e^{2\pi i\xi}-z_k)^{r_k}(e^{2\pi i\xi}-1/\overline{z_k})^{r_k}\\
&= a_N e^{-2\pi iN\xi}\prod_{k=1}^{s}(e^{2\pi i\xi}-z_k)^{r_k}(e^{-2\pi i\xi}-\overline{z_k})^{r_k}(-1)^{r_k}e^{2\pi i r_k\xi}\ \overline{z_k}^{-r_k}\\
&= a_N(-1)^N B\prod_{k=1}^{s}\prod_{k=1}^{s}|e^{2\pi i\xi}-z_k|^{2r_k} = \bigg|B_0\prod_{k=1}^{s}(e^{2\pi i\xi}-z_k)^{r_k}\bigg|^2,
\end{aligned}$$

where $B = \prod_{k=1}^{s}\overline{z_k}^{-r_k}$ and B_0 is a complex square root of $a_N(-1)^N B$. Finally set $Q(\xi)$ to be the polynomial (of degree N) inside the last absolute value above.

??. Follow the hints to obtain the solution. Use the estimate

$$\begin{aligned}
&\sum_{m\in\mathbf{Z}^n\setminus\{0\}} R^n(1+R|m|)^{-\frac{n+1}{2}}(1+\varepsilon R|m|)^{-N}\\
\le &\sum_{1\le|m|\le(R\varepsilon)^{-1}} R^{n-\frac{n+1}{2}}|m|^{-\frac{n+1}{2}} + \sum_{|m|\ge(R\varepsilon)^{-1}} R^{n-\frac{n+1}{2}-N}\varepsilon^{-N}|m|^{-\frac{n+1}{2}-N}\\
&\le C_nR^{n-\frac{n+1}{2}}(R\varepsilon)^{-(n-\frac{n+1}{2})} + C_nR^{n-\frac{n+1}{2}-N}\varepsilon^{-N}((R\varepsilon)^{-1})^{n-\frac{n+1}{2}-N}\\
&\qquad 2C_n\varepsilon^{-\frac{n-1}{2}}\,.
\end{aligned}$$

Now pick ε so that $\varepsilon R^n = \varepsilon^{-\frac{n-1}{2}}$ and for this ε we obtain the claimed error. The estimate from above is identical.

??. Assume that S contains no lattice point other than 0 and set $f = \chi_{\frac{1}{2}S} * \chi_{\frac{1}{2}S}$. Then $f(m) = 0$ when $m \neq 0$. The function f is supported in S and since S is symmetric we have $\chi_{\frac{1}{2}S}(x) = \chi_{\frac{1}{2}S}(-x)$. Then $\widehat{f}(m) = |\widehat{\chi_{\frac{1}{2}S}}|^2 \geq 0$. The Poisson summation formula gives

$$|\tfrac{1}{2}S| = \int \chi_{\frac{1}{2}S}(x)\chi_{\frac{1}{2}S}(-x)dx = f(0) = \sum_{m\in\mathbf{Z}^n} f(m) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m) \geq \widehat{f}(0) = |\tfrac{1}{2}S|^2.$$

It follows that $|\frac{1}{2}S| \leq 1$ and thus $|S| \leq 2^n$. The Poisson summation formula can be applied since f is integrable and $\widehat{f}$ has decay of the order $|\xi|^{-(n+1)}$ as $|\xi| \to \infty$, hence it is also integrable.

SECTION 3.2

??. We will show that the function

$$h = e\sum_{k=0}^{\infty}(\sup_{j\geq k} a_j - \sup_{j\geq k+1} a_j)(k+2)\chi_{[\frac{k+1}{k+2},1]}$$

works. Clearly $\|h\|_{L^1} = e\sum_{k=0}^{\infty}(a_k - a_{k+1}) = e\,a_0 < \infty$. For every $m \geq 0$ and $k \geq m$ we have

$$e\,\frac{k+2}{m+1}\left(1-\left(\frac{k+1}{k+2}\right)^{m+1}\right) = e\,(k+2)\frac{\xi^m}{k+2} \geq e\left(\frac{m+1}{m+2}\right)^m \geq 1,$$

by the mean-value theorem, where $\xi \in \left[\frac{k+1}{k+2},1\right] \subset \left[\frac{m+1}{m+2},1\right]$. Therefore

$$\int_0^1 t^m h(t)\,dt \geq \sum_{k=m}^{\infty}(a_k - a_{k+1})e\,\frac{k+2}{m+1}\left(1-\left(\frac{k+1}{k+2}\right)^{m+1}\right) \geq \sum_{k=m}^{\infty}(a_k - a_{k+1}) = a_m.$$

??. For $j \in \mathbf{Z}$ define $a_j = \left(\sup_{|k|\geq j} d_{(k_1,\dots,k_n)}\right)^{1/n}$ for $j \geq 0$ and $a_j = a_{-j}$ for $j < 0$. Then a_j is decreasing for $j > 0$ and tends to zero as $|j| \to \infty$. Moreover,

$$a_{j_1}\dots a_{j_n} \geq \prod_{r=1}^{n}\big(\sup_{|k|\geq j_r} d_{(k_1,\dots,k_n)}\big)^{1/n} \geq \sup_{|k|\geq \max(j_1,\dots,j_n)} d_{(k_1,\dots,k_n)} \geq d_{(j_1,\dots,j_n)}.$$

??. (a) The function f is decreasing and bounded below by zero, thus it has a limit as $x \to \infty$. Then $(f(x)-f(x/2)) \to 0$ as $x \to \infty$. We have $-(f(x)-f(x/2)) = \displaystyle\int_{x/2}^{x} -f'(t)\,dt \geq -f'(x)x/2 \geq 0$, since $-f'(t)$ is decreasing and nonnegative. Now let $x \to \infty$ and use the squeeze law to deduce that $\lim_{x\to\infty} f'(x)x = 0$.

(b) Set $f(x) = \displaystyle\int_1^x g(t)\,dt$. Then $f \geq 0$, $f' \geq 0$, and $f'' = g' \leq 0$. Also f has a limit at infinity (the number $\int_1^\infty g(t)\,dt$. Thus part (a) gives the required conclusion. Here is an alternative solution. Given $\varepsilon > 0$ find x_1 such that $\displaystyle\int_{x_1}^{\infty} g(t)\,dt < \frac{\varepsilon}{2}$. If for all $t > x_1$ we had $g(t) \geq \dfrac{\varepsilon}{2t}$, then integrating from x_1 to ∞ we would obtain

$\displaystyle\int_{x_1}^{\infty} g(x)\,dx = +\infty$, which is a contradiction. Therefore for some $x_0 > x_1$ we have $g(x_0) \le \dfrac{\varepsilon}{2x_0}$. Now for $x > x_0$ we have

$$xf(x) = x_0 g(x_0) + \int_{x_0}^{x} \big(g(t) + tg'(t)\big)\,dt \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since the second term in the integral above is negative. This proves part (b).

??. In the definition $\|f\|_{\dot{\Lambda}^\gamma} = \sup\limits_{x,h\in\mathbf{T}^n} \dfrac{|f(x+h)-f(x)|}{|h|^\gamma}$ we may identify $\mathbf{T}^n$ with $[-1/2,1/2]^n$ and take $h \in [-1/2,1/2]^n$. Then $|h| \le \sqrt{n}/2$ and therefore

$$\frac{|f(x+h)-f(x)|}{|h|^\gamma} = |h|^{\delta-\gamma}\frac{|f(x+h)-f(x)|}{|h|^\delta} \le \big(\frac{\sqrt{n}}{2}\big)^{\delta-\gamma}\frac{|f(x+h)-f(x)|}{|h|^\delta}$$

Taking the supremum over all x, h we obtain the conclusion.

??. (a) Use the mean value theorem to deduce that the $Var(f)$ is controlled by the supremum of Riemann sums of the integral of $|f'(t)|$.

(b) Observe that $|f(x)-f(0)| \le Var(f,[0,x]) \le Var(f)$ for any $x \in [0,1)$.

??. Since f is differentiable at every point and its derivative is integrable, then f is absolutely continuous. Since the product of two absolutely continuous functions is absolutely continuous, we have that the integral of the function $x \to f(x)e^{-2\pi i x m}$ over the torus is zero which implies that $\widehat{f'}(m) = 2\pi i m \widehat{f}(m)$. Then

$$\sum_{m\in\mathbf{Z}} |\widehat{f}(m)| = |\widehat{f}(0)| + \frac{1}{2\pi}\sum_{m\neq 0}\frac{|\widehat{f}(m)2\pi i m|}{|m|} \le \|f\|_{L^1} + \frac{1}{2\pi}\big(\sum_{j\neq 0} j^{-2}\big)^{1/2}\|f'\|_{L^2},$$

where we used Planchrel's theorem in the last step above.

??. (a) The Fourier coefficients of the product of two functions are obtained by the l^1 convolution of the Fourier coefficients of each function. Since convolution maps $l^1 \times l^1 \to l^1$ the first conclusion follows.

(b) A Fourier coefficient of the convolution of two L^2 functions is the product of the corresponding Fourier coefficients of these functions. Apply the Cauchy-Schwartz inequality to control the sum of the absolute values of the Fourier coefficients of the convolution of two functions by the product of their L^2 norms.

??. We have $f(x+h) - f(x) = \sum\limits_{k=0}^{\infty} 2^{-\alpha k} e^{2\pi i 2^k x}\big(e^{2\pi i 2^k h} - 1\big)$. The part of the sum where $2^k \le |h|^{-1}$ is controlled by

$$\sum_{k=0}^{\log_2 |h|^{-1}} 2^{-\alpha k}|e^{2\pi i 2^k h} - 1| \le \sum_{k=0}^{\log_2 |h|^{-1}} 2^{-\alpha k} 2\pi 2^k |h| = C|h|^\alpha,$$

while the part of the sum where $2^k > |h|^{-1}$ is controlled by

$$\sum_{k > \log_2 |h|^{-1}} 2^{-\alpha k} \le C|h|^\alpha.$$

??. Let $g_N(x) = \sum_{k=2}^{N} \dfrac{e^{ik\log k}}{k} e^{2\pi ikx}$ for $x \in [0,1]$. Set $a_k = e^{ik\log k}e^{2\pi ikx}$ for $k \geq 2$ and $a_1 = 0$. Also set $b_k = 1/k$. Summation by parts gives

$$\sum_{k=1}^{N} a_k b_k = A_N b_N - \sum_{k=1}^{N-1} A_k (b_{k+1} - b_k),$$

where $A_k = \sum_{j=1}^{k} a_j$. Since $|b_k - b_{k+1}| \leq k^{-2}$ and $|A_k| \leq Ck^{1/2}$, letting $N \to \infty$ we obtain

$$g(x) = \sum_{k=2}^{\infty} \bigg(\sum_{j=2}^{k} e^{ij\log j} e^{2\pi ijx} \bigg) \frac{1}{k(k+1)}.$$

Writting g as above we can estimate $|g(x+h) - g(x)| \leq S_1 + S_2$, where

$$S_1 = \sum_{k>|h|^{-1}} \bigg(\sum_{j=2}^{k} e^{ij\log j} e^{2\pi ijx} \big(e^{2\pi ijx} - 1\big) \bigg) \frac{1}{k(k+1)} \leq 2C \sum_{k>|h|^{-1}} \frac{\sqrt{k}}{k^2} = C'|h|^{1/2},$$

$$S_2 = \sum_{k=2}^{|h|^{-1}} \bigg(\sum_{j=2}^{k} e^{ij\log j} e^{2\pi ijx} \big(e^{2\pi ijh} - 1\big) \bigg) \frac{1}{k(k+1)} \leq C'' \sum_{k=2}^{|h|^{-1}} k^{3/2}|h| \frac{1}{k^2} = C'''|h|^{1/2},$$

provided we justify that

$$\sup_{x\in\mathbf{R}} \bigg| \sum_{j=2}^{k} e^{ij\log j} e^{2\pi ijx} \big(e^{2\pi ijh} - 1\big) \bigg| \leq C'' k^{3/2}|h|.$$

To see this, write $e^{2\pi ijh} - 1 = 2\pi ijh(\cos(2\pi ij\xi_1) + i\sin(2\pi ij\xi_2))$ for some ξ_1, ξ_2 between 0 and h, by applying the mean value theorem to the real and imaginary parts of $e^{2\pi ijh} - 1$. Expressing the sine and the cosine above in terms of exponentials we can reduce the last inequality above to the inequality

$$\sup_{y\in\mathbf{R}} \Big| \sum_{j=2}^{k} j e^{ij\log j} e^{2\pi ijy} \Big| \leq C'' k^{3/2}$$

which can be easily proved by a summation by parts. This proves that $g \in \dot{\Lambda}^{1/2}(\mathbf{T}^1)$. Finally it is trivial that $g \notin A(\mathbf{T}^1)$.

??. Let c_0 be the space of all sequences on $\mathbf{Z}^n$ that tend to zero at infinity equipped with the L^∞ norm. The Fourier transform defines an injective continuous map from $L^1(\mathbf{T}^n)$ into c_0. If this map were onto, then there would exist a positive constant A such that for all $f \in L^1$ the inequality $\|f\|_{L^1} \leq A\|\widehat{f}\|_{L^\infty}$ would be true. We will show that this inequality is false. Fix a smooth nonzero function h supported in the set $[-1/4, 1/4]^n$. Define $g_b(x) = h(x)e^{-\pi(1+ib)|x|^2}$ and extend g_b to a periodic function on $\mathbf{R}^n$ with period one in each variable. The Fourier coefficients of g_b are obtaining by convolving the Fourier coefficients of h with the sequence $\{(1+ib)^{-n/2}e^{-\frac{\pi}{1+ib}|m|^2}\}_{m\in\mathbf{Z}^n}$. Hence $\|\widehat{g_b}\|_{l^\infty} \leq \|\widehat{h}\|_{l^1}(1+b^2)^{-n/4}$ which tends to zero as $b \to \infty$. Since $\|g_b\|_{L^1} = \|h\|_{L^1} > 0$, this contradicts the inequality $\|f\|_{L^1} \leq A\|\widehat{f}\|_{L^\infty}$.

SECTION 3.3

??. All of the results in this exercise follow by a few simple calculations.

??. Since the Fourier series of the function $|x|$ converges and the function $|x|$ is continuous at zero, Proposition **??** gives that

$$0 = \frac{1}{4} + \sum_{m\neq 0} \Big(-\frac{1}{2m^2\pi^2} + \frac{(-1)^m}{2m^2\pi^2}\Big) e^{2\pi i 0} = \frac{1}{4} + \frac{1}{\pi^2} \sum_{m \text{ odd}} \frac{1}{m^2}.$$

It follows that the sum of the squares of the reciprocals of all odd integers, call it S_o, is equal to $\pi^2/4$. Since $S_e + S_o = \frac{1}{4}(S_o + S_e) + S_o$, it follows that $S_e = \pi^2/12$ and thus $S_o + S_e = \pi^2/3$. The third identity in this exercise follows by considering the Fourier series of the function x^2 at the origin. The fourth identity follows by considering the Fourier series of the function $\cosh(2\pi x)$ at the origin.

??. Part (a) is proved by comparing Fourier coefficients. If $|j| \le N$ then the j^{th} Fourier coefficient on the right is

$$\frac{M+1}{M-N}\left(1 - \frac{|j|}{M+1}\right) - \frac{N+1}{M-N}\left(1 - \frac{|j|}{N+1}\right) = 1,$$

while for $N < |j| \le M$ the j^{th} Fourier coefficient on the right is

$$\frac{M+1}{M-N}\left(1 - \frac{|j|}{M+1}\right) - \frac{M+1}{M-N}\left(1 - \frac{|j|}{M+1}\right) = 0,$$

and likewise if $|j| > M$.

(b) Given $\varepsilon > 0$ find an $a > 0$ and k_0 as in the condition. Choose k_1 so that $k \ge k_1$ implies $|(F_k * f)(x) - A(x)| \le \varepsilon$. Apply the identity of part (a) with M=[a N]. Then $M - N \ge M/2$ for $N \ge k_2$. For $N \ge \max(k_0, k_1, k_2, 10)$ we obtain

$$\left| \frac{[aN]+1}{[aN]-N} \sum_{N<|j|\le[aN]} \left(1 - \frac{|j|}{[aN]+1}\right) \widehat{f}(j) e^{2\pi i j x} \right| \le \frac{2[aN]\varepsilon}{[aN]-N} \le 4\varepsilon,$$

$$\left| \frac{[aN]+1}{[aN]-N} (F_{[aN]} * f)(x) - \frac{N+1}{[aN]-N} (F_N * f)(x) - A(x) \right| \le \varepsilon \frac{2[aN]+2}{[aN]-N} \le 8\varepsilon$$

which proves the conclusion.

??. Functions of bounded variation are given by differences of increasing functions and therefore they have side limits everywhere. Proposition **??** gives that the Fourier coefficients of a function f of bounded variation decay like $(1+|m|)^{-1}$ as $|m| \to \infty$ and thus they satisfy the condition in part (b) of the previous exercise. Theorem **??** gives that $(F_N * f)(x_0)$ converges to the average of the left and right limits of f at x_0. Hence, so does $(D_N * f)(x_0)$ by the previous exercise. Now the function $g_b = \chi_{[-b,b]}$ is a function of bounded variation on $\mathbf{T}^1$. Thus $(D_N * g_b)(b)$ must converge to $1/2$ as $N \to \infty$. Using Exercise **??** (b) we obtain the required identity.

??. (a) If K is compact in $L^1(\mathbf{T}^n)$, given $\varepsilon > 0$ there exist finitely many trigonometric polynomials $P_1, \dots, P_k$ such that given any $f \in K$, there exists a P_j with $\|f - P_j\|_{L^1} \le \varepsilon$. Let N_0 be the highest degree of all these polynomials. For $|m| \ge N_0$ we have $\|\widehat{f}(m)| \le |(f - P_j)\widehat{\ }(m)| \le \|f - P_j\|_{L^1} \le \varepsilon$.

(b) Let K be the set of all functions $\phi_w(x) = f(w-x)\prod_{j=1}^n \dfrac{\cos(\pi x_j)}{\sin(\pi x_j)}$, $w \in B_0$, where B_0 is a compact subset of B. Then $K \cup \{f\}$ is a compact subset of $L^1(\mathbf{T}^n)$

which implies that the integrals

$$\int_{\mathbf{T}^n} f(w-x) \prod_{j=1}^{n} \frac{\sin(2N\pi x_j)\cos(\pi x_j)}{\sin(\pi x_j)}\, dx + \int_{\mathbf{T}^n} f(w-x) \prod_{j=1}^{n} \cos(2N\pi x_j)\, dx$$

converge to zero uniformly in $w \in B_0$ as $N \to \infty$.

??. Part (a) is immediate but also follows from Exercise **??**. The identity in part (b) follows by looking at Fourier coefficients. Exercise **??** gives that the left hand side in part (b) is equal to

$$\sum_{1\le |m|\le M} \widehat{g}(m-N)\widehat{D_N}(m-N)e^{2\pi mx} = \sum_{1\le |r+N|\le M} \frac{1}{r}\widehat{D_N}(r)e^{2\pi mx}.$$

Observe that only for those r's for which $1 \le |r| \le N$ the terms of the second sum are not zero. Consider the cases $M \ge 2N$ and $M < 2N$ separately. The second statement in part (b) follows by a simple summation by parts, by using that $|\sum_{r=1}^{N} e^{2\pi irx}| \le 2|e^{2\pi ix}-1|^{-1} \le \frac{1}{2}|x|^{-1}$ since $|x| \le \frac{1}{2}$ by Appendix E.
(c) If $x = 0$, there is nothing to prove. If $x \neq 0$, we have that

$$\left|\sum_{1\le |r|\le N} \frac{1}{r}e^{2\pi irx}\right| = \left|\sum_{1/|x|\le |r|\le N} \frac{1}{r}e^{2\pi irx}\right| + \left|\sum_{1\le r\le 1/|x|} \frac{1}{r}\big(e^{2\pi irx}-1\big)\right|,$$

where one of the two terms above does not appear if $1/|x| > N$. The first term above is bounded by

$$C\sum_{1/|x|\le |r|} \frac{1}{r^2}\frac{2}{|e^{2\pi ix}-1|} \le C\sum_{1/|x|\le |r|} \frac{1}{r^2}\frac{1}{2|x|} \le C_1$$

by a summation by parts, while the second term is trivially bounded by

$$\sum_{1\le r\le 1/|x|} \frac{1}{r}\, 2\pi r|x| = C_1.$$

(d) The continuity of f follows from the uniform convergence of the series, a consequence of part (c). The second conclusion in part (b) gives that the Fourier series of f converges absolutely at any $x \neq 0$. It remains to proves that $|(f * D_M)(0)|$ diverges to infinity as $M \to \infty$. Take $M = e^{e^m}$, $m \in \mathbf{Z}^+$. Part (b) gives

$$\begin{aligned}\big|(f*D_M)(0)\big| &= \left|\sum_{\lambda_k\le M/2}\frac{1}{k^2}\sum_{1\le |r|\le \lambda_k}\frac{1}{r} + \sum_{\lambda_k> M/2}\frac{1}{k^2}\sum_{\substack{-\lambda_k\le r\le M-\lambda_k\\ r\neq 0}}\frac{1}{r}\right| \\ &= 0 + \sum_{\lambda_k> M/2}\frac{1}{k^2}\sum_{|M-\lambda_k|\le r\le \lambda_k}\frac{1}{r} \ge \frac{1}{m^2}\sum_{1\le r\le \lambda_m}\frac{1}{r} \ge \frac{\log \lambda_m}{m^2} \ge \frac{e^m}{m^2},\end{aligned}$$

where we picked $\lambda_m = M+1 = e^{e^m}+1$. Thus $\limsup\limits_{M\to\infty} |(f*D_M)(0)| = \infty$.

SECTION 3.4

??. Observe that $(D(n,N) * F)(x_1,\dots,x_n) = \prod_{j=1}^{n}(D_N * f)(x_j)$. Now if $\limsup_{N\to\infty} |(D_N*f)(x)| = \infty$ for all $x \in A$ (where $|A| = 1$), then $\limsup_{N\to\infty} |(D(n,N)* F)(x)| = \infty$ for all $x \in A^n$. and thus

??. First observe that statements (a) and (b) are equivalent if $f = \chi_Q$ and Q is a square in $\mathbf{T}^n$. Thus if (a) holds, then (b) holds for all step functions. Approximate a general C^∞ function with step functions in the uniform norm and use an $\varepsilon/3$ argument to obtain (b) for all f smooth. The statement (b) $\implies$ (c) is trivial. To prove (c) $\implies$ (b) we write

$$\frac{1}{N}\sum_{k=0}^{N-1} f(a_k) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m\in\mathbf{Z}^n}\widehat{f}(m)e^{2\pi i m\cdot x} = \widehat{f}(0) + \sum_{m\in\mathbf{Z}^n\setminus\{0\}}\widehat{f}(m)\Big(\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i m\cdot x}\Big).$$

Because of the rapid decay of the Fourier coefficients of f we can pass the limit as $N\to\infty$ inside the sum in m. It follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} f(a_k) = \widehat{f}(0) = \int_{\mathbf{T}^n} f(x)\,dx\,.$$

To prove that (b) $\implies$ (a) given a square Q in $\mathbf{T}^n$ pick two smooth functions g and h such that

$$0 \le h \le \chi_Q \le g$$

and such that g is equal to 1 on Q and vanishes off $(1+\varepsilon)Q$ while h is equal to 1 on $(1-\varepsilon)Q$ and vanishes off Q. Observe that

$$|Q| - c_n\,\varepsilon \le \int_{\mathbf{T}^n} h(x)\,dx \le |Q| \le \int_{\mathbf{T}^n} g(x)\,dx \le |Q| + c_n\,\varepsilon\,.$$

for some $c_n > 0$. Since

$$\frac{1}{N}\sum_{k=0}^{N-1} h(a_k) \le \frac{1}{N}\sum_{k=0}^{N-1}\chi_Q(a_k) \le \frac{1}{N}\sum_{k=0}^{N-1} g(a_k)\,,$$

the sandwich theorem implies that

$$|Q| - c_n\,\varepsilon \le \liminf_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\chi_Q(a_k) \le \limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\chi_Q(a_k) \le |Q| + c_n\,\varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary the conclusion follows.

??. It is easy to check that condition (c) in the previous exercise holds. Indeed if $m \in \mathbf{Z}^n\setminus\{0\}$ then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i m\cdot kx} = \lim_{N\to\infty}\frac{1}{N}\frac{e^{2\pi i N(m\cdot x)}-1}{e^{2\pi i(m\cdot x)}-1} = 0$$

since $m\cdot x$ is never a rational and thus the denominator never vanishes.

??. Following the hint we obtain the identity

$$(1-\tfrac{|m|^2}{R^2})^\alpha = \frac{2\Gamma(\alpha+1)}{\Gamma(\frac{n+1}{2})\Gamma(\beta)}\frac{1}{R^{2\alpha}}\int_{|m|}^{R} K_r(x)(R^2-r^2)^{\beta-1}(1-\tfrac{|m|^2}{r^2})^\delta\, r^n\,dr\,,$$

where $\beta = \alpha - \frac{n-1}{2}$. Multiplying by $e^{2\pi i m\cdot x}$, summing over $|m|\le R$ and changing the summation and integration yields the required conclusion.

??. This easily follows from the previous exercise and the fact that

$$R^{-2\alpha}\int_0^R (R^2-r^2)^{\beta-1} r^n\,dr \le C_{n,\alpha,\beta} < \infty$$

whenever $\beta \ge \alpha \ge \frac{n-1}{2}$ and $R \ge 1$. The case where $R < 1$ is trivial.

??. It is obvious that N is right continuous and constant on intervals of the form $[r_j, r_{j+1})$. For $t \ge 0$ set $\delta(t) = \#\{m \in \mathbf{Z}^n : \ |m| = t\}$. Observe that $N(r_0) = \delta(r_0)$ and that $\delta(r_{j+1}) = N(r_{j+1}) - N(r_j)$ for $j \ge 0$. For a Schwartz function ϕ on the line we have

$$\begin{aligned}\langle N', \phi \rangle = - \langle N, \phi' \rangle = &- \sum_{j=0}^{\infty} \int_{\mathbf{R}} N(t)\phi'(t)\, dt \\ =&- \sum_{j=0}^{\infty} \int_{[r_j, r_{j+1})} N(t)\phi'(t)\, dt \\ =&- \sum_{j=0}^{\infty} N(r_j) \int_{[r_j, r_{j+1})} \phi'(t)\, dt \\ =&- \sum_{j=0}^{\infty} N(r_j)\big(\phi(r_{j+1}) - \phi(r_j)\big) \\ =&N(r_0)\phi(r_0) + \sum_{j=0}^{\infty} \big[N(r_{j+1}) - N(r_j)\big]\phi(r_{j+1}) \\ =&\sum_{j=0}^{\infty} \delta(r_j)\phi(r_j) = \int_{\mathbf{R}} \phi(t) d\delta(t) = \langle \delta, \phi \rangle\, .\end{aligned}$$

We conlcude $N' = \delta$.

??. We have

$$\begin{aligned}\sum_{a \le |m| \le b} f(|m|) = &\sum_{j:\ a \le r_j \le b} f(r_j)\delta(r_j) = \int_a^b f(t) d\delta = \int_a^b f(x)N'(x)\, dx \\ =&f(b)N(b) - f(a)N(a) - \int_a^b f'(x)N(x)\, dx\, .\end{aligned}$$

??. We pick a transcendental number in γ in $(0, 1)$ and we write

$$\sum_{\substack{m \in \mathbf{Z}^n \setminus \{0\} \\ |m| \le R}} \frac{e^{i|m|}}{|m|^\lambda} = \sum_{k=0}^{R} \sum_{\substack{m \in \mathbf{Z}^n \setminus \{0\} \\ k+\gamma \le |m| \le k+1+\gamma}} \frac{e^{i|m|}}{|m|^\lambda}$$

But the previous exercise the last sum above is equal to

$$\begin{aligned}&\frac{e^{i(k+1+\gamma)}}{(k+1+\gamma)^\lambda} N(k+1+\gamma) - \frac{e^{i(k+\gamma)}}{(k+\gamma)^\lambda} N(k+\gamma) - \int_{k+\gamma}^{k+1+\gamma} \left(\frac{e^{it}}{t^\lambda}\right)' N(t) dt \\ &= \frac{v_n e^{i(k+1+\gamma)}}{(k+1+\gamma)^{\lambda-n}} - \frac{v_n e^{i(k+\gamma)}}{(k+\gamma)^{\lambda-n}} - \int_{k+\gamma}^{k+1+\gamma} \left(\frac{e^{it}}{t^\lambda}\right)' v_n t^n dt + O\big(k^{-\lambda+n\frac{n-1}{n+1}}\big) \\ &= \int_{k+\gamma}^{k+1+\gamma} \frac{e^{it}}{t^\lambda}\big(v_n t^n\big)' dt + O\big(k^{-\lambda+n\frac{n-1}{n+1}}\big) = \int_{k+\gamma}^{k+1+\gamma} \frac{n v_n e^{it}}{t^{\lambda-n+1}} dt + O\big(k^{-\lambda+n\frac{n-1}{n+1}}\big) \\ &= \frac{-i\omega_{n-1} e^{i(k+1+\gamma)}}{(k+1+\gamma)^{\lambda-(n-1)}} - \frac{-i\omega_{n-1} e^{i(k+\gamma)}}{(k+\gamma)^{\lambda-(n-1)}} + \int_{k+\gamma}^{k+1+\gamma} \frac{-i\omega_{n-1} e^{it} dt}{t^{\lambda-n+2}/(\lambda-n+1)} + O\big(k^{-\lambda+n\frac{n-1}{n+1}}\big) \\ &= \frac{-i\omega_{n-1} e^{i(k+1+\gamma)}}{(k+1+\gamma)^{\lambda-(n-1)}} - \frac{-i\omega_{n-1} e^{i(k+\gamma)}}{(k+\gamma)^{\lambda-(n-1)}} + O\big(k^{-\lambda+n\frac{n-1}{n+1}}\big) = \beta_k + O(k^{-\lambda+(n-1)-\frac{n-1}{n+1}})\end{aligned}$$

For part (b) suppose that $\lambda \le n-1$ and that the series $\sum_{m\in\mathbf{Z}^n\setminus\{0\}} \frac{e^{i|m|}}{|m|^\lambda}$ converges. Then $\beta_k \to 0$ as $k \to \infty$. But then $k^{\lambda-(n-1)}\beta_k$ would also tend to zero since the error $k^{\lambda-(n-1)}O(k^{-\lambda+(n-1)-\frac{n-1}{n+1}})$ does tend to zero. But the term $k^{\lambda-(n-1)}\beta_k$ oscillates, hence it cannot tend to zero, a contradiction. For part (c) observe that if $\lambda > n - \frac{n-1}{n+1}$, then the error term above gives a converges series in k.

??. The Fourier series in question is

$$(2\pi)^{-\frac{n}{2}} \sum_{m\in\mathbf{Z}^n} \frac{J_{\frac{n}{2}}(|m|)}{|m|^{\frac{n}{2}}} e^{2\pi i m\cdot x}\,.$$

We take $x = 0$ and we apply the calculation of the previous exercise with the function $J_{\frac{n}{2}}(t)t^{-\frac{n}{2}}$ replacing the function $e^{it}t^{-\lambda}$. We obtain

$$\sum_{\substack{m\in\mathbf{Z}^n\setminus\{0\}\\ k+\gamma\le|m|\le k+1+\gamma}} \frac{J_{\frac{n}{2}}(|m|)}{|m|^{\frac{n}{2}}}$$

$$\frac{J_{\frac{n}{2}}(k+1+\gamma)}{(k+1+\gamma)^{\frac{n}{2}}}N(k+1+\gamma) - \frac{J_{\frac{n}{2}}(k+\gamma)}{(k+\gamma)^{\frac{n}{2}}}N(k+\gamma) - \int_{k+\gamma}^{k+1+\gamma} \left(\frac{J_{\frac{n}{2}}(t)}{t^{\frac{n}{2}}}\right)' N(t)\,dt$$

$$\frac{v_n J_{\frac{n}{2}}(k+1+\gamma)}{(k+1+\gamma)^{-\frac{n}{2}}} - \frac{v_n J_{\frac{n}{2}}(k+\gamma)}{(k+\gamma)^{-\frac{n}{2}}} - \int_{k+\gamma}^{k+1+\gamma} \left(\frac{J_{\frac{n}{2}}(t)}{t^{\frac{n}{2}}}\right)' v_n t^n\,dt + O\big(k^{-\frac{n+1}{2}+n\frac{n-1}{n+1}}\big)$$

$$= \int_{k+\gamma}^{k+1+\gamma} \frac{J_{\frac{n}{2}}(t)}{t^{\frac{n}{2}}}\big(v_n t^n\big)'\,dt + O\big(k^{\frac{n-1}{2}-\frac{2n}{n+1}}\big)$$

$$= \int_{k+\gamma}^{k+1+\gamma} c_n \cos(t - \tfrac{\pi n+\pi}{4})t^{\frac{n-1}{2}}\,dt + O\big(k^{\frac{n-1}{2}-1}\big) = \beta_k + O\big(k^{\frac{n-1}{2}-1}\big)$$

by an integration by parts, where we set

$$\beta_k = c_n \sin(k+\gamma+1-\tfrac{\pi n+\pi}{4})(k+\gamma+1)^{\frac{n-1}{2}} - c_n \sin(k+\gamma-\tfrac{\pi n+\pi}{4})(k+\gamma)^{\frac{n-1}{2}}\,.$$

If the above Fourier series converged, then $k^{-\frac{n-1}{2}}\beta_k$ would tend to zero. Now now express $k^{-\frac{n-1}{2}}\beta_k$ in terms of exponentials. Using that the expression $e^{ik}k^{-\frac{n-1}{2}}\beta_k$ would also tend to zero as $k \to \infty$ we obtain a contradiction.

SECTION 3.5

??. (a) For $0 < r < 1$ sum the geometric series to obtain

$$-i\sum_{m=-\infty}^{+\infty} \operatorname{sgn}(m)\, r^{|m|} e^{2\pi i m t} = \frac{-i}{1-re^{2\pi i m t}} - \frac{-i}{1-re^{-2\pi i m t}} = \frac{2r\sin(2\pi t)}{1-2r\cos(2\pi t)+r^2}\,.$$

(b) If f is smooth, then $|\widehat{f}(m)| \le C(1+|m|)^{-M}$ for all $M > 0$. Therefore the series $(Q_r * f)(t) = -i\sum_{m=-\infty}^{+\infty} \operatorname{sgn}(m)\, r^{|m|} e^{2\pi i m t}$ converges to $-i\sum_{m=-\infty}^{+\infty} \operatorname{sgn}(m)\, e^{2\pi i m t}$ $= \widetilde{f}(t)$ as $r \uparrow 1$. Since Q_r is real-valued, the last conclusion follows.
(c) The functions $(r,t) = z \to r^{|m|}e^{2\pi i m t}$ are harmonic in the unit disc $|z| < 1$. It is easy to see that $(P_r * f)(t) = \sum_{m=-\infty}^{+\infty} r^{|m|}\widehat{f}(m)e^{2\pi i m t}$ and $(Q_r * f)(t) = -i\sum_{m=-\infty}^{+\infty} \operatorname{sgn}(m)\, r^{|m|}\widehat{f}(m)e^{2\pi i m t}$. Since $|\widehat{f}(m)| \le \|f\|_{L^1}$ and $r < 1$, the series defining $(P_r * f)(t)$ and $(Q_r * f)(t)$ converge uniformly on compact subsets of the

unit disc and it follows that $(P_r * f)(t)$ and $(Q_r * f)(t)$ are harmonic functions on the unit disc.
(d) An easy calculation shows that $P_r(t) + iQ_r(t) = 2\sum_{m=0}^{\infty} \widehat{f}(m) r^m e^{2\pi imt}$ which which is nothing else but the analytic function $2\sum_{m=0}^{\infty} \widehat{f}(m) z^m$ of $z = re^{2\pi it}$. It follows that $(P_r * f)(t)$ and $(Q_r * f)(t)$ are conjugate harmonic functions.

??. Since Q_r has integral zero over the circle, we can write

$$(f * Q_r)(x) = \int_{|t|\le 1/2} \big(f(x-t) - f(x)\big)\, Q_r(t)\, dt.$$

Now observe that for $1/2 \le r < 1$ we have

$$1 + r^2 - 2r\cos(2\pi t) = (1-r)^2 + 2r(1 - \cos(2\pi t)) \ge 1 - \cos(2\pi t) \ge \frac{2|2\pi t|^2}{\pi^2} = 8t^2.$$

Therefore $|Q_r(t)| \le \frac{\pi}{2}|t|^{-1}$ and clearly $Q_r(t) \to \cot(\pi t)$ as $r \uparrow 1$. Since f is smooth, we have that $|f(x-t) - f(x)| \le C|t|$ and the Lebesgue dominated convergence theorem implies that

$$\lim_{r\to 1} (f * Q_r)(x) = \int_{|t|\le 1/2} \big(f(x-t) - f(x)\big) \cot(\pi t)\, dt.$$

Finally, it quite simple to see that the latter is equal to

$$\lim_{\varepsilon\to 0} \int_{\varepsilon \le |t| \le 1/2} f(x-t)\cot(\pi t)\, dt.$$

SECTION 3.6

??.
??.
??. **??**. **??**. **??**. **??**. **??**.

SECTION 3.7

??. Suppose there exist two such N-tuples $(\varepsilon_1, \dots, \varepsilon_N)$ and $(\varepsilon_1', \dots, \varepsilon_N')$. Pick the largest $k \le N$ such that the coefficients of λ_k are different. Then setting $\varepsilon_j - \varepsilon_j' = \delta_j$ for $1 \le j \le k$ we obtain

$$-\delta_k \lambda_k = \delta_1 \lambda_1 + \cdots + \delta_{k-1}\lambda_{k-1}.$$

Then each $|\delta_j|$ is at most 2 and $|\delta_k| \ge 1$. Since $\lambda_{k-1} \le A^{-1}\lambda_k$, $\lambda_{k-2} \le A^{-2}\lambda_k$, $\dots$, $\lambda_1 \le A^{-(k-1)}\lambda_k$. It follows that

$$\lambda_k \le 2(A^{-1} + A^{-2} + \cdots + A^{-(k-1)})\lambda_k < 2(A-1)^{-1}\lambda_k\,.$$

But this would imply that $A \le 3$ which contradicts our assumption on A.

??. Observe that the Fourier coefficients $\widehat{f}(m)$ of this function f decay like $|m|^{-1}$ as $|m| \to \infty$.

??. (a) Let $2N = [\min(A-1, 1-\frac{1}{A})\lambda_k]$ and let K_N be as in the proof of Proposition **??**. We write

$$\widehat{f}(\lambda_k) = \int_{|x|\le\frac{1}{2}} f(x)e^{-2\pi i\lambda_k x}\,dx = \int_{|x|\le\frac{1}{2}} (f(x)-f(0))e^{-2\pi i\lambda_k x}dx$$

$$= \int_{|x|\le\frac{1}{N}} (f(x)-f(0))e^{-2\pi i\lambda_k x}K_N(x)\,dx + \int_{\frac{1}{N}\le|x|\le\frac{1}{2}} (f(x)-f(0))e^{-2\pi i\lambda_k x}K_N(x)dx.$$

But the first integral above is bounded by $K\int_{|x|\le\frac{1}{N}} |x|^\alpha\,dx \le cN^{-\alpha}$. Using (**??**) we obtain that the second integral above is bounded by a multiple of $K\int_{\frac{1}{N}\le|x|\le\frac{1}{2}} |x|^\alpha N^{-3}|x|^{-4}\,dx$, which is bounded by a multiple of $N^{-\alpha}$. Keeping in mind that $N \approx \lambda_k$, we obtain the required conclusion.

(b). Then f is in $A(\mathbf{T}^1)$ and thus f can be represented as a sum of its Fourier series. Given x and y in $[-\frac{1}{2}, \frac{1}{2}]$, using the result in part (a) we obtain

$$|f(x)-f(y)| \le \sum_{k=1}^\infty C\lambda_k^{-\alpha}|e^{2\pi i\lambda_k x} - e^{2\pi i\lambda_k x}| \le C'\sum_{k=1}^\infty \lambda_k^{-\alpha}\min(1, (\lambda_k|x-y|)^\alpha)\,.$$

It is fairly easy to see that the series above converges and is bounded above by a multiple of $|x-y|^\alpha$.

??. Using that f vanishes in $|x|\le\delta$, the idenity

$$\widehat{f}(\lambda_k) = \int_{\delta\le|x|\le\frac{1}{2}} f(x)e^{-2\pi i\lambda_k x}K_N(x)\,dx\,,$$

and the estimate in (**??**), we obtain $|\widehat{f}(\lambda_k)| \le C\lambda_k^{-3}$. This implies that f is in C^2 since the series

$$f(x) = \sum_{k=1}^\infty \widehat{f}(\lambda_k)e^{2\pi i\lambda_k x}$$

can be differentiated twice. Next we have

$$\begin{aligned}\widehat{f}(\lambda_k) =& \frac{1}{(-2\pi i\lambda_k)^2}\int_{|x|\le\frac{1}{2}} f''(x)e^{-2\pi i\lambda_k x}dx\\ =& \frac{1}{(-2\pi i\lambda_k)^2}\int_{|x|\le\frac{1}{2}} f''(x)e^{-2\pi i\lambda_k x}K_N(x)\,dx\end{aligned}$$

and applying the same argument to f'' we obtain that f is in C^4. Continuing this way we conclude that f is in C^∞.

??. Statement (b) follows from (a) using Proposition **??**. Statement (c) follows easily from (b). We have

$$\frac{f(x)-f(x_0)}{x-x_0} = \sum_{k=0}^\infty a^{-k}\frac{e^{2\pi i b^k x} - e^{2\pi i b^k x_0}}{x-x_0}$$

and since the k^{th} term of the last series above is bounded by $a^{-k}b^k$, the series above converges absolutely and uniformly. Then we can pass the limit $asx \to x_0$ inside the sum above to obtain that f is everywhere differentiable and

$$f'(x) = 2\pi i\sum_{k=0}^\infty (b/a)^{-k}e^{2\pi i b^k x}\,.$$

Finally (c) trivially implies (a).

CHAPTER 4

SECTION 4.1

??. (a) Differentiation gives $I'(a) = -\int_0^\infty e^{-ax}\sin x\,dx$. By a double integration by parts the latter is equal to $-(1+a^2)^{-1}$. Now integration gives $I(a) = C - \arctan(a)$, letting $a\to\infty$ gives $C=\pi/2$.

(b) Differentiation gives that the second derivative in a of the integral in (b) is equal to $a^{-1} - a(1+a^2)^{-1}$. Integrate twice to obtain that the the integral in (b) is equal to

$$-\arctan(a) + a\log\frac{a}{\sqrt{1+a^2}} + Ca + C'.$$

Letting $a\to\infty$ we obtain that $C=0$ and $C'=\pi/2$.

??. (a) Suppose thet ϕ is supported in $[-R,R]$. Then for $|x|\ge 2R$ we have $|\text{p.v.}\int\frac{\phi(y)}{x-y}dy| \le 2\|\phi\|_{L^1}/|x|$ since $|x-y|\ge|x|-|y|\ge|x|/2$ since $|y|\le R$. On the other hand for $|x|\le 2R$ we have $|\text{p.v.}\int\frac{\phi(y)}{x-y}dy| = |\text{p.v.}\int\frac{\phi(y)-\phi(x)}{x-y}dy| \le \int_{|x-y|\le|x|+R}\frac{|\phi(y)-\phi(x)|}{|x-y|}dy \le 6R\|\phi'\|_{L^\infty}$.

(b) For $|x|\ge 2R$ an integration by parts m times gives $|\text{p.v.}\int\frac{\phi^{(m)}(y)}{x-y}dy| = m!|\text{p.v.}\int\frac{\phi(y)}{(x-y)^{m+1}}dy| \le 2^{m+1}m!\|\phi^{(m)}\|_{L^1}/|x|^{m+1}$ since $|x-y|\ge|x|-|y|\ge|x|/2$ as before. For $|x|\le 2R$ we obtain as before that $|\text{p.v.}\int\frac{\phi^{(m)}(y)}{x-y}dy| \le 6R\|\phi^{(m+1)}\|_{L^\infty}$.

(c) Follows similarly.

SECTION 4.2

??.

SECTION 4.3

??. Select a cube Q if $\left(\frac{1}{|Q|}\int_Q|f(x)|^q\,dx\right)^{1/q} > \alpha$. Observe that if Q was selected, then $\left(\frac{1}{|Q|}\int_Q|f(x)|^q\,dx\right)^{1/q} \le 2^{\frac{n}{q}}\alpha$. Let Q_j denote all the selected cubes. Define

$$b_j = \left(f - \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx\right)\chi_{Q_j}$$

and observe that $\|b_j\|_{L^q} \le 2^{\frac{n}{q}+1}\alpha|Q_j|^{\frac{1}{q}}$. Let F be the complement of the union of the Q_j. We define g to be equal to f on F and equal to $\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx$ on Q_j. Then $f=g+b$ and we observe that g is at most $2^{\frac{n}{q}}\alpha$ on Q_j and at most α on F by the Lebesgue differentiation theorem. It is easy to see that $\|g\|_{L^q}\le\|f\|_{L^q}$ and that $\|b\|_{L^q}\le 2^{\frac{n}{q}+1}\|f\|_{L^q}$.

SECTION 4.4

??. (a) Set $z = x/|y|$ and $\xi = y/|y|$ to reduce the required estimate to

$$\left| \frac{z-\xi}{|z-\xi|} - \frac{z}{|z|} \right| \le \frac{2}{|z|}\,.$$

whenever $|z| \ge 2$. But the above can be written as

$$\left| \frac{|z|z - |z|\xi - |z-\xi|z}{|z-\xi|} \right| \le 2$$

or equivalently to

$$\left| \frac{-|z-\xi|\xi + (|z| - |z-\xi|)(z-\xi)}{|z-\xi|} \right| \le 2\,.$$

But the above is clearly true since $|\xi| = 1$ and $\big||z| - |z-\xi|\big| \le |\xi| = 1$.
(b) For a point $x \in \mathbf{R}^n$, let us set x' be the project of x on the unit sphere $\mathbf{S}^{n-1}$, i.e. $x' = x/|x|$.We have

$$\big|K(x-y) - K(x)\big| \le \left| \frac{\Omega((x-y)') - \Omega(x')}{|x|^n} \right| + |\Omega((x-y)')| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right|$$

Using that Ω is Lipschitz and the mean-value theorem, the above is at most

$$B_0 \frac{|x' - (x-y)'|^\alpha}{|x|^n} + \|\Omega\|_{L^\infty} \frac{n|y|}{|x-\theta y|^{n+1}}$$

for some $0 < \theta < 1$. But if $|x| \ge 2|y|$ we have $|x - \theta y| \ge \frac{1}{2}|x|$ and by part (a) the above expression is at most

$$B_0 \frac{(2|y|)^\alpha}{|x|^\alpha} \frac{1}{|x|^n} + \|\Omega\|_{L^\infty} \frac{n2^{n+1}|h|}{|x|^{n+1}} \le B_0 \frac{(2|y|)^\alpha}{|x|^{n+\alpha}} + \|\Omega\|_{L^\infty} \frac{n2^{n+1}|y|^\alpha}{|x|^{n+\alpha}}\,.$$

This estimate gives the required conclusion

$$\int_{|x|\ge 2|y|} \big|K(x-y) - K(x)\big|\, dx \le C_n(B_0 + \|\Omega\|_{L^\infty}).$$

??. We only prove part (b) which is more general than part (a) since it is easy to see that $\omega_\infty(t) \le \omega_1(t)$ for all $t > 0$. Let $|x| \ge 2|y$. We use the estimate

$$\big|K(x-y) - K(x)\big| \le \left| \frac{\Omega((x-y)') - \Omega(x')}{|x-y|^n} \right| + |\Omega(x')| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right|\,.$$

Using the mean-value theorem and switching to polar coordinates we obtain

$$\int_{|x|\ge 2|y|} |\Omega(x')| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right| dx \le C_n \|\Omega\|_{L^1}\,.$$

It remains to control the integral

$$\int_{|x|\ge 2|y|} \frac{\big|\Omega((x-y)') - \Omega(x')\big|}{|x-y|^n}\, dx\,.$$

Pick a matrix $A_{x,y} \in O(n)$ such that $A_{x,y}(x') = (x-y)'$. Then $\Omega((x-y)') = \Omega(A_{x,y}(x'))$. Then for $|x| \ge 2|y|$ we have

$$\|A_{x,y}\| \le \sup_{\theta \in \mathbf{S}^{n-1}} |A_{x,y}(\theta) - \theta| \le \frac{2|y|}{|x|}\,.$$

Therefore

$$\int_{\mathbf{S}^{n-1}} |A_{x,y}(\theta)-\theta|\,d\theta \le \sup_{\substack{A\in O(n)\\ \|A\|\le \frac{2|y|}{|x|}}} \int_{\mathbf{S}^{n-1}} |\Omega(A\theta)-\Omega(\theta)|\,d\theta \le \omega_1\big(\frac{2|y|}{|x|}\big)$$

Switching into polar coordinates and using the easy estimate $|x-y|\ge \frac{1}{2}|x|$ when $|x|\ge 2|y|$ we obtain

$$\int_{|x|\ge 2|y|} \frac{\big|\Omega((x-y)')-\Omega(x')\big|}{|x-y|^n}\,dx \le C_n\int_{r=2|y|}^{\infty}\int_{\mathbf{S}^{n-1}} |A_{x,y}(\theta)-\theta|\,d\theta\frac{dr}{r}$$

$$C_n\int_{r=2|y|}^{\infty}\omega_1\big(\frac{2|y|}{|x|}\big)\frac{dr}{r} = C_n\int_{r=0}^{1}\omega_1(r)\frac{dr}{r}<\infty\,.$$

(c)

??. It is easy to see that $K(\varepsilon,\cdot\,)$ satisfies (**??**), (**??**), and (**??**) uniformly in $\varepsilon>0$. Also observe that

$$K_{(\varepsilon)} - K*\phi_\varepsilon = (K(\varepsilon,\cdot\,)_{(1)} - K(\varepsilon,\cdot\,)*\phi)_\varepsilon$$

Therefore

$$f*K_{(\varepsilon)} = f*K*\phi_\varepsilon + f*(K(\varepsilon,\cdot\,)_{(1)} - K(\varepsilon,\cdot\,)*\phi)_\varepsilon$$

for all $f\in\mathcal{S}(\mathbf{R}^n)$. We will show that

$$|K(\varepsilon,\cdot\,)_{(1)}(x) - (K(\varepsilon,\cdot\,)*\phi)(x)| \le C\frac{A_1+A_2+A_3}{(1+|x|)^{n+\delta}}$$

uniformly in $\varepsilon>0$. For $|x|\ge 2$ we have

$$|K(\varepsilon,\cdot\,)_{(1)}(x) - (K(\varepsilon,\cdot\,)*\phi)(x)|$$

$$=\left|\int_{\mathbf{R}^n}(K(\varepsilon,\cdot\,)(x)-K(\varepsilon,\cdot\,)(y))\phi(x-y)\,dy\right| \le A_2\frac{\displaystyle\int |z|^\delta|\phi(z)|\,dz}{|x|^{n+\delta}},$$

while for $|x|\le 2$ we have $|y|\le 3$ in the integrals below and thus

$$\begin{aligned}
&|K(\varepsilon,\cdot\,)_{(1)}(x) - (K(\varepsilon,\cdot\,)*\phi)(x)| \le A_1 + |(K(\varepsilon,\cdot\,)*\phi)(x)|\\
\le&A_1+\left|\int_{|y|\le 3}K(\varepsilon,\cdot\,)(y)(\phi(x-y)-\phi(x))\,dy\right| + \left|\phi(x)\lim_{\eta\to 0}\int_{\eta\le|y|\le 3}K(\varepsilon,\cdot\,)(y)\,dy\right|\\
\le&A_1+\|\nabla\phi\|_{L^\infty}\int_{|y|\le 3}A_1|y|^{-n}|y|\,dy + \|\phi\|_{L^\infty}A_3\\
\le&C_\phi(A_1+A_3).
\end{aligned}$$

Corollary **??** now gives that

$$\sup_{\eta>0}\big|f*(K(\varepsilon,\cdot\,)_{(1)} - K(\varepsilon,\cdot\,)*\phi)_\eta\big| \le c\,(A_1+A_2+A_3)Mf$$

uniformly in ε. Taking $\eta=\varepsilon$ we obtain Cotlar's inequality

$$(T_*f)(x)\le M(f*K) + c\,(A_1+A_2+A_3)M(f).$$

SECTION 4.5

??. (a) Observe that $\sum_{j=1}^{\infty}|f_j|^r = 1$ and thus $\|\{f_j\}_j\|_{L^\infty(\mathbf{R},l^r)} = 1$. However $(Mf_j)(x) \geq 2^{j-1}/2(2^j - x) \geq 1/4$ for $0 \leq x \leq 2^{j-1}$ which implies that $\sum_{j=1}^{\infty}|Mf_j|^r = \infty$ on $(0,\infty)$.

(b) We have that $\sum_{j=1}^{N}(Mf_j)(x) \geq c\sum_{k=1}^{N}|Nx-k|^{-1}$ for x irrational in $[0,1]$ which implies that $\|\sum_{j=1}^{N}|Mf_j|\|_{L^p} \geq \log N$. On the other hand $\|\sum_{j=1}^{N}|f_j|\|_{L^p} \approx 1$.

CHAPTER 5

SECTION 5.1

??. Choose a function η supported in $\frac{8}{9} \leq |x| \leq \frac{9}{4}$ which is equal to one on the annulus $\frac{8}{9} + 10^{-10} \leq |x| \leq \frac{9}{4} - 10^{-10}$. Then the sum $\sum_{k\in\mathbf{Z}}\eta(2^{-k}\xi)$ is never zero. Define $\widehat{\Psi}(\xi) = \eta(\xi)\big(\sum_{k\in\mathbf{Z}}\eta(2^{-k}\xi)\big)^{-1}$ and observe that the Schwartz function Ψ thus constructed satifies (**??**).

??. For fixed $\xi \neq 0$, split the sum into the pieces $2^j \leq |\xi|^{-1}$ and $2^j \geq |\xi|^{-1}$. In each case use the appropriate bound to obtain the required conclusion.

??. (a) Using that Ψ has mean value zero we write

$$\widehat{\Psi}(\xi) = \int_{|x|\leq 1}(e^{-2\pi i x\cdot\xi} - 1)\Psi(x)\,dx + \int_{|x|\geq 1}(e^{-2\pi i x\cdot\xi} - 1)\Psi(x)\,dx$$

from which we conclude

$$\big|\widehat{\Psi}(\xi)\big| \leq |\xi|\|\Psi\|_{L^1} + (2\pi|\xi|)^{\frac{\varepsilon}{2}}\int_{|x|\geq 1}|x|^{\frac{\varepsilon}{2}}|\Psi(x)|\,dx \leq C_n B|\xi|^{\frac{\varepsilon}{2}}$$

when $|\xi| \leq 1$. For $|\xi| \geq 1$ we use the hint to obtain

$$2\widehat{\Psi}(\xi) \leq \int_{\mathbf{R}^n}|\Psi(x-y) - \Psi(x)|\,dx \leq B|y|^{\varepsilon} = B(2|\xi|)^{-\varepsilon}.$$

Using the previous exercise we conclude that (**??**) holds for $p = 2$.
(b) We have that

$$\sum_{2^j\leq|y|^{-1}}\int_{|x|\geq 2|y|}\big|\Psi_{2^{-j}}(x-y) - \Psi_{2^{-j}}(x)\big|\,dx = \sum_{2^j\leq|y|^{-1}}\int_{|x|\geq 2(2^j|y|)}\big|\Psi(x-y) - \Psi(x)\big|\,dx$$

and using that the last integral above is at most $B(2^j|y|)^{\varepsilon}$ we obtain that this term is at most a dimensional multiple of B. We also have

$$\sum_{2^j>|y|^{-1}}\int_{|x|\geq 2|y|}\big|\Psi_{2^{-j}}(x-y) - \Psi_{2^{-j}}(x)\big|\,dx$$

$$\leq \sum_{2^j>|y|^{-1}}\int_{|x|\geq 2|y|}2^{jn}B\big((2^j|x-y|)^{-n-\varepsilon} + (2^j|x|)^{-n-\varepsilon}\big)\,dx$$

$$2B\sum_{2^j>|y|^{-1}}\int_{|x|\geq|y|}2^{-j\varepsilon}|x|^{-n-\varepsilon}\,dx$$

$$C_n B\sum_{2^j>|y|^{-1}}2^{-j\varepsilon}|y|^{-\varepsilon} = C_n' B\,.$$

This proves (**??**).

??. (a) Theorem **??** implies that for any subset S of $\mathbf{Z}$ and any Schwartz function f we have

$$\Big\|\Big(\sum_{j\in S}|\Delta_j f|^2\Big)^{1/2}\Big\|_{L^p} \le C_{n,p}\|f\|_{L^p}.$$

By duality this is saying that for every sequence $\{g_j\}_j$ of functions we have

$$\Big\|\sum_{j\in S}\Delta_j g_j\Big\|_{L^p} \le C_{n,p'}\Big\|\Big(\sum_{j\in S}|g_j|^2\Big)^{1/2}\Big\|_{L^p}.$$

Applying the above inequality to the sequence $\Delta_j f$ we obtain

$$\Big\|\sum_{j\in S}\Delta_j^2 f\Big\|_{L^p} \le C_{n,p'}\Big\|\Big(\sum_{j\in S}|\Delta_j f|^2\Big)^{1/2}\Big\|_{L^p}$$

(b) Take $S=\{j:\ |j|>N\}$. For f in Schwartz we have $f=\sum_{j\in\mathbf{Z}}\Delta_j^2 f$ hence

$$\Big\|\sum_{|j|\le N}\Delta_j^2 f - f\Big\|_{L^p} = \Big\|\sum_{|j|>N}\Delta_j^2 f\Big\|_{L^p} \le C_{n,p'}\Big\|\Big(\sum_{|j|>N}|\Delta_j f|^2\Big)^{1/2}\Big\|_{L^p}.$$

The sequence $\big(\sum_{|j|>N}|\Delta_j f|^2\big)^{1/2}$ tends to zero as $N\to\infty$ and is bounded above by the L^p function $\big(\sum_{j\in\mathbf{Z}}|\Delta_j f|^2\big)^{1/2}$. By the Lebesgue dominated convergence theorem we obtain that $\big\|\big(\sum_{|j|>N}|\Delta_j f|^2\big)^{1/2}\big\|_{L^p}\to 0$ as $N\to\infty$ from which it follows that the sequence $\sum_{|j|\le N}\Delta_j^2 f$ converges to f in $L^p(\mathbf{R}^n)$ for $1<p<\infty$.

SECTION 5.2

??

SECTION 5.3

??

SECTION 5.4

??

SECTION 5.5

??

CHAPTER 6

SECTION 6.1

??

SECTION 6.2

??

??

??. The required set of p's concides with the set of all p for which the function $G_s(x) = \big((1+|\xi|^2)^{-s/2}\big)^\vee(x)$ is in $L^p(\mathbf{R}^n)$. Using Proposition **??** we see that this is the case when $(-n+\alpha)p > -n$, i.e. when $p < n/(n-\alpha)$.

??.

SECTION 6.3

??. Observe that

$$|f(x+t)+f(x-t)-2f(x)| = 2\Big|\sum_{k=1}^{\infty} 2^{-k}(\cos(2\pi 2^k t)-1)e^{2\pi i 2^k x}\Big|$$

$$\leq 2\sum_{k=1}^{\infty} 2^{-k}\min(2, \tfrac{1}{2}(2\pi 2^k|t|)^2) = C|t|.$$

Suppose we now had $|f(x+t)-f(x)| \leq A|t|$ for all x and t. Since the function f is 2π-periodic, Bessel's inequality gives

$$(A|t|)^2 \geq \int_0^1 |f(x+t)-f(x)|^2\,dx = \sum_{k=1}^{\infty} 2^{-2k}|2^{2\pi i 2^k t}-1|^2.$$

But when $2^k|t| \leq 1/2$ we have $|e^{2\pi i 2^k t}-1| \geq 2^{k+2}|t|$ which implies that

$$\sum_{k=1}^{\infty} 2^{-2k}|2^{2\pi i 2^k t}-1|^2 \geq \sum_{2^k|t|\leq 1/2} 2^{-2k}2^{2k+4}|t|^2 \geq C|t|^2\log\frac{1}{2|t|}$$

which provides a contradiction when $|t|$ is near zero.

SECTION 6.4

??.

SECTION 6.5

??.

CHAPTER 7

SECTION 7.1

??.

??. Given f in BMO, $0 < \alpha \le 1$, and a ball B, pick $C_B = |f_B|^\alpha$. Then

$$\frac{1}{|B|}\int_B \big||f(x)|^\alpha - C_B\big|\,dx \le \frac{1}{|B|}\int_B |f(x) - f_B|^\alpha\,dx$$
$$\frac{1}{|B|}\left(\int_B 1\,dx\right)^{1-\alpha}\left(\int_B |f(x) - f_B|\,dx\right)^\alpha \le \|f\|_{BMO}^\alpha.$$

Thus $\||f|^\alpha\|_{BMO} \le 2\|f\|_{BMO}^\alpha$.

SECTION 7.2

??.

SECTION 7.3

??.
??. Let $B(x,r)$ be a ball in $\mathbf{R}^n$. Then

$$\begin{aligned}\mu(B(x,r)\times(0,r)) &= \sum_j b_j |B(x,r)\times(0,r)\cap\{(x,a_j):\ x\in\mathbf{R}^n\}| \\ &= \sum_{j:\,a_j<r} b_j|\{(x,a_j):\ x\in B(x,r)\}| = |B(x,r)|\sum_{j:\,a_j<r} b_j.\end{aligned}$$

It follows that $\mu(B(x,r)/|B(x,r)| = \sum_{j:\,a_j<r} b_j$ and the supremum of these expressions over all $r>0$ is equal to $\sum_j b_j$. Thus $\|\mu\|_{\mathcal{C}} = \sum_j b_j$.

??. Every $x\in\Omega$ belongs to some cube Q_j but by Proposition **??** (d) there are at most 12^n Q_k that touch Q_j. Thus there are at most 12^n Q_k^* that intersect with Q_j and these are all the possible Q_s^*'s that x can belong to.

SECTION 7.4

??.

SECTION 7.5

??. We follow the hint. Initially we observe that

$$\int_{\mathbf{R}^n}\frac{dx}{(1+|x|)^M} \le \int_{|x|\le 1} dx + \int_{|x|\ge 1}\frac{dx}{|x|^M} = v_n + nv_n\int_1^\infty \frac{r^{n-1}}{r^M}dr = \frac{v_n M}{M-n}.$$

In the case $2^\nu|a-b|\le 1$ we have the estimate

$$\begin{aligned} I(a,\mu,M;b,\nu,N) &\le \left(\int_{\mathbf{R}^n}\frac{dx}{(1+|x|)^M}\right)2^{\nu n} \\ &\le \left(\int_{\mathbf{R}^n}\frac{dx}{(1+|x|)^M}\right)2^{\nu n}\,\frac{2^{\min(M,N)}}{(1+2^\nu|a-b|)^{\min(M,N)}} \\ &\le \frac{v_n M 4^N}{M-n}\,\frac{2^{\nu n}}{(1+2^\nu|a-b|)^{\min(M,N)}}.\end{aligned}$$

In the case $2^\nu|a-b| \geq 1$ first observe that when $x \in H_a$ we have $|x-b| \geq \frac{1}{2}|a-b|$ and thus

$$\int_{H_a} \frac{2^{\mu n}}{(1+2^\mu|x-a|)^M} \frac{2^{\nu n}}{(1+2^\nu|x-b|)^N}\, dx \leq \frac{v_n M 2^N}{M-n} \frac{2^{\nu n}}{(1+2^\nu|a-b|)^N}.$$

When $x \in H_b$ we have $|x-a| \geq \frac{1}{2}|a-b|$ and thus

$$\begin{aligned}
&\int_{H_b} \frac{2^{\mu n}}{(1+2^\mu|x-a|)^M} \frac{2^{\nu n}}{(1+2^\nu|x-b|)^N}\, dx \\
\leq &\frac{v_n N}{N-n} \frac{2^{\mu n}}{(1+2^\mu \frac{1}{2}|a-b|)^M} \\
\leq &\frac{v_n N}{N-n} \frac{2^M 2^{\mu n}}{2^{\mu M}|a-b|^M} \\
\leq &\frac{v_n N}{N-n} \frac{2^M 2^{(\nu-\mu)M} 2^{\mu n}}{(2^\nu|a-b|)^M} \\
\leq &\frac{v_n N}{N-n} \frac{2^M 2^{(\nu-\mu)(M-n)} 2^{\nu n} 2^M}{(1+2^\nu|a-b|)^M} \\
\leq &\frac{v_n N 4^M}{N-n} \frac{2^{\nu n}}{(1+2^\nu|a-b|)^M}.
\end{aligned}$$

Summing the last two estimates for the integral over H_a and the integral over H_b we obtain the required conclusion for $I(a,\mu,M;b,\nu,N)$.

??. We have

$$\begin{aligned}
I = &\left| \int_{\mathbf{R}^n} \phi_a(x)\phi_b(x)\, dx \right| \\
= &\left| \int_{\mathbf{R}^n} \big(\phi_a(x) - \sum_{|\alpha|\leq L-1} \frac{(\partial_x^\alpha \phi_a)(b)}{\alpha!}(x-b)^\alpha\big)\phi_b(x)\, dx \right| \\
\leq &B \sum_{|\alpha|=L} \frac{A_\alpha}{\alpha!} \int_{\mathbf{R}^n} \frac{|x-b|^L 2^{\mu n} 2^{\mu L}}{(1+2^\mu|\xi_x - a|)^M} \frac{2^{\nu n}}{(1+2^\nu|x-b|)^N}\, dx,
\end{aligned}$$

for some ξ_x on the segment $[b,x]$. Now observe that

$$\begin{aligned}
1+2^\mu|a-b| \leq &1+2^\mu|a-\xi_x| + 2^\mu|\xi_x - b| \\
\leq &1+2^\mu|a-\xi_x| + 2^\nu|x-b| \\
\leq &(1+2^\nu|b-x|)(1+2^\mu|a-\xi_x|).
\end{aligned}$$

We use this estimate in the last integral above to obtain

$$I \leq B \sum_{|\alpha|=L} \frac{A_\alpha}{\alpha!} \frac{2^{\mu n}}{(1+2^\mu|a-b|)^M} \int_{\mathbf{R}^n} \frac{2^{-(\nu-\mu)L} 2^{\nu n}}{(1+2^\nu|b-x|)^{N-L-M}}\, dx$$

which clearly implies the required conclusion.

CHAPTER 8

SECTION 7.1

??.

SECTION 8.2

??.

SECTION 8.3

??.

SECTION 8.4

??.

??. Split up the integral in t over the regions $|t| \le 1$, $1 \le |t| \le 2^{-r}$ and $2^{-r} \le |t|$. We obtain

$$\begin{aligned}&\int_{\mathbf{R}^n} \frac{\min(1, 2^r|t|)}{(1+|t|)^{n+\delta}}\, dt\\ \le&\int_{|t|\le 1} 2^r\, dt + \int_{1\le |t|\le 2^{-r}} \frac{2^r\, dt}{|t|^{n+\delta-1}} + \int_{2^{-r}\le |t|} \frac{dt}{|t|^{n+\delta}}\\ =&v_n 2^r + \frac{n v_n}{1-\delta}(2^{r\delta} - 2^r) + \frac{n v_n}{\delta} 2^{r\delta},\end{aligned}$$

whenever $\delta \neq 1$ from which the corresponding conclusion follows. The second to last integral above needs to be modified when $\delta = 1$. In this case it gives $n v_n 2^r \log 2^{-r}$ which is easily seen to be controlled by $2 n v_n 2^r(|r|+1)$. The conclusion for $\delta = 1$ follows similarly.

The last statement of the exercise follows by changing variables $t' = 2^j t$, setting $k - j = r$ and using the first part.

SECTION 8.5

??.

CHAPTER 9

SECTION 9.1

??. It is easy to see that $[kw]_{A_p} \le \|k\|_{L^\infty} \|k^{-1}\|_{L^\infty} [w]_{A_p}$.

??. For all cubes Q in $\mathbf{R}^n$ we have

$$\left(\frac{1}{|Q|}\int_Q w_1 w_2^{1-p}\, dx\right)\left(\frac{1}{|Q|}\int_Q w_1^{-\frac{1}{p-1}} w_2\, dx\right)^{p-1}$$

is controlled by

$$\left(\frac{1}{|Q|}\int_Q w_1\, dx\right)(\operatorname*{ess\,inf}_Q w_2)^{1-p}\left(\frac{1}{|Q|}\int_Q w_2\, dx\right)^{p-1}(\operatorname*{ess\,inf}_Q w_1)^{-1}$$

which is at most $[w_1]_{A_1}[w_2]_{A_1}^{p-1}$. Taking supremum over all Q we obtain the required estimate.

??. For all cubes Q in $\mathbf{R}^n$ we have that

$$\left(\frac{1}{|Q|}\int_Q w^\delta\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{\delta}{q-1}}\,dx\right)^{q-1}$$

is controlled by

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)^\delta\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\,dx\right)^{\delta(p-1)} \le [w]_{A_p}^\delta\,.$$

Taking supremum over all Q we obtain the required estimate.

??. The measure $w\,dx$ is doubling with constant $3^n[w]_{A_1}$. Using Exercise **??** in section **??** we obtain the required conclusion.

??. We have that $w = w_0^{(1-\theta)\frac{p}{p_0}} w_1^{\theta\frac{p}{p_1}}$. We apply Hölder's inequality with exponents $\frac{p_0}{p(1-\theta)}$ and $\frac{p_1}{p\theta}$ in the integral $\frac{1}{|Q|}\int_Q w\,dx$ and Hölder's inequality with exponents $\frac{p_0'}{p'(1-\theta)}$ and $\frac{p_1'}{p'\theta}$ in the integral $\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\,dx\right)^{p-1}$. Observe that both pairs of exponents are dual to each other. The required conclusion follows. This argument can be modified to include the endpoint case $p_0 = 1$.

??. (a) Use that for $x \in Q$, $(Mf)(x) \ge c_n \frac{1}{|Q|}\int_Q f(y)\,dy$ which implies that $(Mf)^{-\frac{1}{p-1}} \le c_n^{-\frac{1}{p-1}}\big(\frac{1}{|Q|}\int_Q f(y)\,dy\big)^{-\frac{1}{p-1}}$ on Q.

(b) The pair $\big((Mg)^{1-p}, |g|^{1-p}\big)$ is of class (A_p, A_p) for all $1 < p < \infty$ because of part (a) since the pair $(|g|, Mg)$ is of class $(A_{p'}, A_{p'})$. If we had the estimate

$$\int_{\mathbf{R}^n}(Mf)^p(Mg)^{1-p}\,dx \le \int_{\mathbf{R}^n}|f|^p g^{1-p}\,dx$$

then $f = g$ would provide a contradiction if $0 < |g| < \infty$.

SECTION 9.2

??. $\|S(f)\|_{L^{p'q}(w)} = \left(\int_{\mathbf{R}^n} M(|f|^q w)^{p'} w^{-\frac{1}{p-1}}\,dx\right)^{\frac{1}{p'q}} \le C_w\|f\|_{L^{p'q}(w)}$, by using the fact that the Hardy-Littlewood maximal operator maps $L^{p'}(w^{-\frac{1}{p-1}})$ into itself.

??. It is easy to see that the product of the first quantity times the second quantity to the power $p-1$ controls the A_p constant of e^v, thus $[e^v]_{A_p} \le C^p$. Conversely, we have

$$\frac{1}{|Q|}\int_Q e^{v(t)-v_Q}\,dt = (e^{-\frac{v_Q}{p-1}})^{p-1}\,(e^v)_Q \le (e^{-\frac{v}{p-1}})_Q^{p-1}\,(e^v)_Q \le [e^v]_{A_p}$$

by Jensen's inequality. Similarly one obtains

$$\frac{1}{|Q|}\int_Q e^{-\frac{v(t)-v_Q}{p-1}}\,dt = (e^{v_Q})^{\frac{1}{p-1}}\,(e^{-\frac{v}{p-1}})_Q \le (e^v)_Q^{\frac{1}{p-1}}\big(e^{-\frac{v}{p-1}}\big)_Q \le [e^v]_{A_p}^{\frac{1}{p-1}}\,.$$

??. (a) The two conditions of the previous exercise imply that

$$\sup_{Q \text{ cubes}}\frac{1}{|Q|}\int_Q e^{|v(t)-v_Q|}\,dt \le [e^v]_{A_2}\,.$$

Thus if $\phi = e^v \in A_2$, then $\log\phi = v$ satisfies

$$\sup_{Q \text{ cubes}} \frac{1}{|Q|}\int_Q |v(t) - v_Q|\,dt \le \sup_{Q \text{ cubes}} \frac{1}{|Q|}\int_Q e^{|v(t)-v_Q|}\,dt \le [e^v]_{A_2} = [\phi]_{A_2}\,.$$

(b) This is easy.

(c) We have that $\phi^{-\frac{1}{p-1}} \in A_{p'}$ and thus in A_2. Therefore $\log\phi^{-\frac{1}{p-1}}$ is in BMO by part (a). Hence so is $\log\phi$.

??. For parts (a) and (b) one just needs to use that the constant C in Theorem **??** increases as $[w]_{A_p}$ does while γ decreases as $[w]_{A_p}$ increases. For part (c) we use Proposition **??** and the Marcinkiewicz interpolation theorem (estimate (**??**) to obtain the estimate

$$\|M\|_{L^p(w)\to L^p(w)} \le C_n\big(\tfrac{p}{p-q}\big)^{\frac{1}{p}}\big([w]_{A_q}^{\frac{2}{q}}\big)^{\frac{q}{p}}\,.$$

The above is easily shown to be bounded by

$$C_n(p')^{\frac{1}{p}}\big(1+\tfrac{1}{\gamma}\big)^{\frac{1}{p}}\big(C_1^p C_2^{p'}\big)^2[w]_{A_p}^{\frac{2}{p}}\,,$$

where γ, C_1, and C_2 are as in Theorem **??** and Corollary **??**. The conclusion follows by noting that the last constant above increases as $[w]_{A_p}$ increases.

SECTION 9.3

??. **??**. **??**.

??. In proving the estimate

$$\frac{1}{|Q|}\int_Q |P(x)|\,dx \ge C_{n,k}\bigg(\frac{1}{|Q|}\int_Q |P(x)|^2\,dx\bigg)^{\frac{1}{2}}$$

for all polynomials of degree k in $\mathbf{R}^n$, one may assume that Q is the unit cube $[0,1]^n$. Indeed, by a translation we can assume that the leftmost corner of Q is 0. Then a dilation allows us to take $Q = [0,1]^n$. To prove the estimate above for $Q = [0,1]^n$ we consider the vector space V_k of all polynomials of degree at most k in $\mathbf{R}^n$. Then all norms on V_k are equivalent since V_k is finite-dimensional and the above inequality follows. Therefore $|P(x)|$ satisfies the reverse Hölder inequality with constant only depending on n and k. Thus $|P(x)|$ is an A_∞ weight and hence in A_p for some p. It follows from Exercise **??** in section **??** that $\log|P(x)|$ is in BMO. Keeping track of the constants, we obtain that the BMO of $\log|P(x)|$ only depends on n and k.

SECTION 9.4

??.

SECTION 9.5

??.

??. Indeed, if $v \in A_1$ we have

$$\begin{aligned}
&\bigg(\int_{\mathbf{R}^n} |(Tf)(x)|^q v(x)\,dx\bigg)^{\frac{1}{q}}\\
=&\bigg(\int_{\mathbf{R}^n} \big\{|(Tf)(x)|^r (Mf)(x)^{-(r-q)}\big\}^{\frac{q}{r}} \big\{(Mf)(x)^q\big\}^{\frac{r-q}{r}} v(x)\,dx\bigg)^{\frac{1}{q}}\\
\le&\bigg(\int_{\mathbf{R}^n} |(Tf)(x)|^r (Mf)(x)^{-(r-q)}\, v(x)dx\bigg)^{\frac{1}{r}}\bigg(\int_{\mathbf{R}^n} (Mf)(x)^q v(x)\,dx\bigg)^{\frac{r-q}{rq}}.
\end{aligned}$$

Using Theorem **??** with $\varepsilon = \frac{r-q}{r-1}$ and $k(x) = 1$, we obtain that $(Mf)^{\frac{r-q}{r-1}}$ is an A_1 weight with constant depending only on q, r, and n. Since $v \in A_1$, Exercise **??** in section **??** gives

$$\big[(Mf)^{-(r-q)}\, v\big]_{A_r} = \Big[\big((Mf)^{\frac{r-q}{r-1}}\big)^{1-r}\, v\Big]_{A_r} \le \big[(Mf)^{\frac{r-q}{r-1}}\big]_{A_1}^{r-1}[v]_{A_1} \le C_{nqr}[v]_{A_1}\,,$$

where C_{nqr} depends on the specified parameters. Thus $(Mf)^{-(r-q)}\, v$ is an A_r weight with constant at most a multiple of $[v]_{A_1}$. Using our hypothesis and the result of Exercise **??** in section **??** we obtain

$$\begin{aligned}
&\bigg(\int_{\mathbf{R}^n} |(Tf)(x)|^r (Mf)(x)^{-(r-q)}\, v(x)\,dx\bigg)^{\frac{1}{r}}\bigg(\int_{\mathbf{R}^n} (Mf)(x)^q v(x)\,dx\bigg)^{\frac{r-q}{rq}}\\
\le& N_r(C_{nqr}[v]_{A_1})\bigg(\int_{\mathbf{R}^n} |f(x)|^r (Mf)(x)^{-(r-q)} v(x)dx\bigg)^{\frac{1}{r}}\bigg(C_{nq}[v]_{A_1}\int_{\mathbf{R}^n} |f(x)|^q v(x)dx\bigg)^{\frac{r-q}{rq}}\\
\le& C'_{nqr} N_r(C_{nqr}[v]_{A_1})\,[v]_{A_1}^{\frac{r-q}{rq}}\bigg(\int_{\mathbf{R}^n} |f(x)|^q\, v(x)dx\bigg)^{\frac{1}{r}}\bigg(\int_{\mathbf{R}^n} |f(x)|^q v(x)\,dx\bigg)^{\frac{r-q}{rq}},
\end{aligned}$$

since $Mf \ge |f|$. Combining this estimate with the one obtained earlier, we obtain that for all $v \in A_1$, T maps $L^q(v)$ into $L^q(v)$ with norm at most a constant depending only on q, r, and $[v]_{A_1}$.

??. Fix $B > 1$, w in A_p with $[w]_{A_p} \le B$ and $f \in L^p(w)$. Given $1 < p < \infty$ and $B > 1$, we use the result of Exercise **??** (b) in section **??** to find positive numbers $\delta = \delta(n, p, B)$ and $B' = B'(n, p, B)$ such that

$$[\omega]_{A_p} \le B \implies [\omega]_{A_{p-\delta}} \le B'\,. \tag{1.0.1}$$

We choose δ small enough so that $1 < \frac{p}{p-\delta} < r$. Similarly we can find positive numbers $B'' = B''(n, (p-\delta)', B')$, $\theta = \theta(n, (p-\delta)', B')$, $\theta < (p-\delta)' - 1$ such that

$$[\omega]_{A_{(p-\delta)'}} \le B' \implies [\omega]_{A_{(p-\delta)'-\theta}} \le B''\,. \tag{1.0.2}$$

By (1.0.1), $w \in A_{p-\delta}$, thus Proposition **??** (4) gives that $w^{-\frac{1}{p-\delta-1}}$ is in $A_{(p-\delta)'}$ with constant $[w^{-\frac{1}{p-\delta-1}}]_{A_{(p-\delta)'}} = [w]_{A_{p-\delta}}^{\frac{1}{p-\delta-1}}$. It follows from (1.0.2) that

$$[w^{-\frac{1}{p-\delta-1}}]_{A_{(p-\delta)'-\theta}} \le (B'')^{\frac{1}{p-\delta-1}}\,. \tag{1.0.3}$$

Now set $(p-\delta)' - \theta = \frac{(p-\delta)'}{1+\eta}$ and we observe that η also depends on n, p, and B. Clearly $|Tf|^{\frac{p}{(p-\delta)'}} w \le M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)^{\frac{1}{1+\eta}}$, and if the latter is finite almost everywhere (otherwise see the end of the proof), it is an A_1 weight with constant

$$\Big[M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)^{\frac{1}{1+\eta}}\Big]_{A_1} \le C(n)\,\eta^{-1} = F(n,p,B)\,.$$

Since $\frac{p}{p-\delta} < r$ we apply the previous exercise and Hölder's inequality to get

$$\|Tf\|_{L^p(w)} = \left(\int_{\mathbf{R}^n} |Tf(x)|^{\frac{p}{p-\delta}} |Tf(x)|^{\frac{p}{(p-\delta)'}} \, w(x)\, dx \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbf{R}^n} |Tf(x)|^{\frac{p}{p-\delta}} M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)(x)^{\frac{1}{1+\eta}} \, dx \right)^{\frac{1}{p}}$$

$$\leq C(n,p,r,B) \left(\int_{\mathbf{R}^n} |f(x)|^{\frac{p}{p-\delta}} M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)(x)^{\frac{1}{1+\eta}} \, dx \right)^{\frac{1}{p}}$$

$$= C(n,p,r,B) \left(\int_{\mathbf{R}^n} |f(x)|^{\frac{p}{p-\delta}} w(x)^{\frac{1}{p-\delta}} M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)(x)^{\frac{1}{1+\eta}} w(x)^{-\frac{1}{p-\delta}} \, dx \right)^{\frac{1}{p}}$$

$$\leq C(n,p,r,B) \|f\|_{L^p(w)}^{\frac{1}{p-\delta}} \left(\int_{\mathbf{R}^n} M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)(x)^{\frac{(p-\delta)'}{1+\eta}} w(x)^{-\frac{1}{p-\delta-1}} \, dx \right)^{\frac{1}{p(p-\delta)'}},$$

where we set $C(n,p,r,B) = \big\{ C'_{n(p-\delta)r} N_r\big(C_{n(p-\delta)r} F(n,p,B)\big) F(n,p,B)^{\frac{r-q}{rq}} \big\}^{\frac{1}{p-\delta}}$ in accordance with the notation of Lemma **??**. In view of (1.0.3) and Exercise **??** (c) in section **??**, the Hardy-Littlewood maximal operator M maps $L^{\frac{(p-\delta)'}{1+\eta}}(w^{-\frac{1}{p-\delta-1}})$ into itself with norm at most a constant depending on n, p, and B'' (hence B). Thus, the last displayed expression above is bounded by

$$C'(n,p,r,B) \|f\|_{L^p(w)}^{\frac{1}{p-\delta}} \left(\int_{\mathbf{R}^n} |Tf(x)|^p w(x)^{(p-\delta)'} w(x)^{-\frac{1}{p-\delta-1}} \, dx \right)^{\frac{1}{p(p-\delta)'}}$$

$$= C'(n,p,r,B) \|f\|_{L^p(w)}^{\frac{1}{p-\delta}} \|Tf\|_{L^p(w)}^{\frac{1}{(p-\delta)'}} .$$

Dividing through by $\|Tf\|_{L^p(w)}^{\frac{1}{(p-\delta)'}}$ (assuming it is finite), we obtain the required estimate (**??**) with constant $N_p(B) = C'(n,p,r,B)^{p-\delta}$.

We finally need to address the technical issue concerning the situations in which $M\big(|Tf|^{\frac{p(1+\eta)}{(p-\delta)'}} w^{1+\eta}\big)$ or $\|Tf\|_{L^p(w)}$ may not be finite a.e. In this case we simply rerun the argument above replacing $|Tf|$ by $|Tf|_L = \chi_{B(0,L)} |Tf| \chi_{|Tf| \leq L}$ to obtain the estimate $\| |Tf|_L \|_{L^p(w)} \leq N_p(B) \leq \|f\|_{L^p(w)}$, since $M\big(\chi_{B(0,L)} w^{1+\eta}\big) < \infty$ a.e. and $|Tf|_L$ is in $L^p(w)$. The desired conclusion follows by letting $L \to \infty$ and applying the Lebesgue monotone convergence theorem.

CHAPTER 10

SECTION 10.1

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SECTION 10.2

??.

SECTION 10.3

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SECTION 10.4

??.

SECTION 10.5

??.

??. Observe that

$$\widehat{\phi_s}(y) = 2^{\frac{m}{2}}\widehat{\phi}(2^m y - \tfrac{l}{2})e^{2\pi i k(\frac{l}{2}-2^m y)}\,.$$

Then we have

$$\begin{aligned}&\sum_{s\in\mathbf{D}_m}\phi_s(x)\overline{\widehat{\phi_s}(y)}\\ =&\sum_{l\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}\phi(2^{-m}x-k)e^{2\pi i 2^{-m}x\frac{l}{2}}\overline{\widehat{\phi}(2^m y-\tfrac{l}{2})}e^{-2\pi i k(\frac{l}{2}-2^m y)}\\ =&\sum_{l\in\mathbf{Z}}\left\{\sum_{k\in\mathbf{Z}}\phi(2^{-m}x-k)e^{-2\pi i k(\frac{l}{2}-2^m y)}\right\}e^{2\pi i 2^{-m}x\frac{l}{2}}\overline{\widehat{\phi}(2^m y-\tfrac{l}{2})}\,.\end{aligned}$$

But using the Poisson summation formula $\sum_k g(k)e^{-2\pi i k b} = \sum_k \widehat{g}(b+k)$ to the sum inside the curly brackets we obtain that the expression above is equal to

$$\sum_{l\in\mathbf{Z}}\left\{e^{2\pi i x y}\sum_{k\in\mathbf{Z}}\widehat{\phi}(2^m y-\tfrac{l}{2}-k)e^{-2\pi i 2^{-m}x(\frac{l}{2}+k)}\right\}e^{2\pi i 2^{-m}x\frac{l}{2}}\overline{\widehat{\phi}(2^m y-\tfrac{l}{2})}$$

But since $\widehat{\phi}$ is supported inside an interval of length at most $3/4$, it follows that $\widehat{\phi}(2^m y-\frac{l}{2}-k)\overline{\widehat{\phi}(2^m y-\frac{l}{2})}=0$ unless possibly when $k=0$. Thus the last expression above reduces to

$$\begin{aligned}&\sum_{l\in\mathbf{Z}}\left\{e^{2\pi i x y}\widehat{\phi}(2^m y-\tfrac{l}{2})e^{-2\pi i 2^{-m}x\frac{l}{2}}\right\}e^{2\pi i 2^{-m}x\frac{l}{2}}\overline{\widehat{\phi}(2^m y-\tfrac{l}{2})}\\ =&e^{2\pi i x y}\sum_{l\in\mathbf{Z}}|\widehat{\phi}(2^m y-\tfrac{l}{2})|^2 = c_0 e^{2\pi i x y}\,.\end{aligned}$$

Next we have

$$\begin{aligned}&\sum_{s\in\mathbf{D}_m}\langle\!\langle f,\phi_s\rangle\!\rangle\phi_s(x)\\ =&\sum_{s\in\mathbf{D}_m}\langle\!\langle\widehat{f},\widehat{\phi_s}\rangle\!\rangle\phi_s\\ =&\int_{\mathbf{R}}\widehat{f}(y)\sum_{s\in\mathbf{D}_m}\overline{\widehat{\phi_s}(y)}\phi_s(x)\,dy\\ =&\int_{\mathbf{R}}\widehat{f}(y)c_0 e^{2\pi i x y}\,dy = c_0 f(x)\,.\end{aligned}$$

??. First observe that an easy calculation gives

$$\widehat{\phi_{y,\xi}}(z) = e^{2\pi i(\xi - z)\cdot y}\widehat{\phi}(z-\xi)\,.$$

Then

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \phi_{y,\xi}(x)\overline{\widehat{\phi_{y,\xi}}(z)}\, dyd\xi \\
=&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{2\pi i\xi\cdot x}e^{2\pi i(z-\xi)\cdot y}\phi(x-y)\overline{\widehat{\phi}(z-\xi)}\, dyd\xi \\
=&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{2\pi i(z-\xi')\cdot x}e^{2\pi i\xi'\cdot y}\phi(x-y)\overline{\widehat{\phi}(\xi')}\, dyd\xi' \\
=&e^{2\pi iz\cdot x}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot(x-y)}\phi(x-y)\overline{\widehat{\phi}(\xi)}\, dyd\xi \\
=&e^{2\pi iz\cdot x}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot y}\phi(y)\overline{\widehat{\phi}(\xi)}\, dyd\xi \\
=&e^{2\pi iz\cdot x}\int_{\mathbf{R}^n} \widehat{\phi}(\xi)\overline{\widehat{\phi}(\xi)}\, d\xi \\
=&e^{2\pi iz\cdot x}\|\phi\|_{L^2}^2\,.
\end{aligned}$$

To prove the required identity, we write

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \phi_{y,\xi}(x)\langle\!\langle f,\phi_{y,\xi}\rangle\!\rangle\, dyd\xi \\
=&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \phi_{y,\xi}(x)\langle\!\langle \widehat{f},\widehat{\phi_{y,\xi}}\rangle\!\rangle\, dyd\xi \\
=&\int_{\mathbf{R}^n} \widehat{f}(z)\bigg(\int_{\mathbf{R}^n} \phi_{y,\xi}(x)\overline{\widehat{\phi_{y,\xi}}(z)}\, dyd\xi\bigg)dz \\
=&\|\phi\|_{L^2}^2\int_{\mathbf{R}^n} \widehat{f}(z)e^{2\pi iz\cdot x}dz \\
=&\|\phi\|_{L^2}^2 f(x)\,.
\end{aligned}$$

The convergence of the integrals above is justified by the rapid decay of the functions at infinity.

SECTION 10.6

??. We prove the result for $\widetilde{M}$ since this implies the corrsponding result for M. Exercise **??** (a) in section **??** gives the estimate

$$|\{\widetilde{M}f>\lambda\}|\lambda \le 3^n\int_{\{\widetilde{M}f>\lambda\}}|f(z)|\, dz\,.$$

It follows that

$$|\{\widetilde{M}f>\lambda\}|^{\frac{1}{p}}\lambda \le 3^n|\{\widetilde{M}f>\lambda\}|^{-1+\frac{1}{p}}\int_{\{\widetilde{M}f>\lambda\}}|f(z)|\, dz \le C_p\|f\|_{L^{p,\infty}}\,.$$

Taking the supremum over all $\lambda>0$ we obtain the required conclusion.