NOTE: NOT THE ONLY WAY OF DOING IT !!

#28: If $F \in NBV$, Let $G(x) = |\mu_F|((-\infty, x])$

Prove $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$:

(a) $T_F \leq G$:

By Th. 3.28 $F \in NBV \to \exists!$ complex Borel measure μ_F / $F(x) := \mu_F((-\infty, x])$

For $x \in \mathbb{R}$ and any partition $-\infty < x_0 < x_1 < \dots < x_N = x$

$$\sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{N} |\mu_F((-\infty, x_j]) - \mu_F((-\infty, x_{j-1}])|$$

$$= \sum_{j=1}^{N} |\mu_F(-\infty, x_{j-1}] + \mu_j((x_{j-1}, x_j]) - \mu_j(-\infty, x_{j-1}]|$$

$$= \sum_{j=1}^{N} |\mu_F((x_{j-1}, x_j])|$$

$$\leq \sum_{j=1}^{N} |\mu_F|((x_{j-1}, x_j]) \qquad \text{PROP 3.13 a)}$$

$$= |\mu_F|\left(\bigcup_{j=1}^{N} (x_{j-1}, x_j]\right)$$

$$= |\mu_F|((x_0, x])$$

Hence $T_F(x) \leq \sup_{-\infty < x_0 < x} |\mu_F|((x_0, x])$

$$\leq |\mu_F|((-\infty, x]) = G(x).$$

(b) For any Borel E $|\mu_F(E)| \leq \mu_{T_F}(E)$

(i) We prove this when $E = (a, b]$, $b < \infty$ first

$$|\mu_F(E)| = \mu_F((a,b]) = |F(b) - F(a)|$$
$$\leq T_F(b) - T_F(a)$$

(same as in proof of Lemma 3.26)

$$\| $$
$$\mu_{T_F}((a,b])$$

(ii) Suppose $E \in \mathcal{B}_{\mathbb{R}}$ then since μ_{T_F} is a positive Borel measure

(*) $$\mu_{T_F}(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu_{T_F}((a_j, b_j]) : \bigcup_{j=1}^{\infty} (a_j, b_j] \supset E \right\}$$

Case 1 : $F : \mathbb{R} \to \mathbb{R}$ (real valued).

Write $F = \underbrace{\tfrac{1}{2}(T_F + F)}_{=: F^+} - \underbrace{\tfrac{1}{2}(T_F - F)}_{=: F^-}$

Each F^+, F^- is real valued, positive increasing

$\underset{\text{Th. 1.16}}{\Longrightarrow}$ gives rise to μ_{F^+}, μ_{F^-} positive Borel measures / $\mu_F = \mu_{F^+} - \mu_{F^-}$

So consider μ_{F^+} (for example, μ_{F^-} is similar)

By (*) $\forall \varepsilon > 0 \; \exists \{(a_j, b_j]\}_{j=1}^{\infty}$ /

(**) $\mu_{T_{F^+}}(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mu_{T_{F^+}}((a_j, b_j])$

$\geq \sum_{j=1}^{\infty} \mu_{F^+}((a_j, b_j])$ (by (b) i))

$\geq \mu_{F^+}\left(\bigcup (a_j, b_j]\right)$

$\geq \mu_{F^+}(E)$ by monotonicity of μ_{F^+} (positive msr)

$= |\mu_{F^+}(E)|$

$\Longrightarrow \mu_{T_{F^+}}(E) \geq |\mu_{F^+}(E)|$ (†)

Similarly $\mu_{T_{F^-}}(E) \geq \mu_{F^-}(E) = |\mu_{F^-}(E)|$

On the other hand note that

$\mu_{T_F}((a,b]) = T_F(b) - T_F(a) \geq T_{F^+}(b) - T_{F^+}(a) = \mu_{T_{F^+}}((a,b])$

So by (*) and this it essentially follows that

(††) $\mu_{T_F}(E) \geq \mu_{T_{F^+}}(E) \quad \forall E$ Borel.

(argue as (*)(*))

Similarly for $\mu_{T_F}(E) \geq \mu_{T_{F^-}}(E)$

Let P, N be the Hahn decomposition for μ_F then

$$|\mu_F(E)| = |\mu_{F^+}(E\cap P) - \mu_{F^-}(E\cap N)|$$

$$\leq \mu_{F^+}(E\cap P) + \mu_{F^-}(E\cap N)$$

by previous parts $$\leq \mu_{T_{F^+}}(E\cap P) + \mu_{T_{F^-}}(E\cap N)$$

$$\leq \mu_{T_F}(E\cap P) + \mu_{T_F}(E\cap N)$$

$$= \mu_{T_F}(E).$$

$\therefore$ When F is real valued we have

$$|\mu_F(E)| \leq \mu_{T_F}(E).$$

<u>Case 2</u> : $F: \mathbb{R} \to \mathbb{C}$ $\quad F = F_{re} + i F_{Im}$

we have $\mu_{T_F}(E) \geq \mu_{T_{F_j}}(E) \quad \forall E \in \mathcal{B}_{\mathbb{R}} \quad j = 1,2$ real, Im

(first for intervals, then open, then Borel).

$\mu_F = \mu_F^{re} + i\mu_F^{Im} \qquad \mu_F^{re}, \mu_F^{Im} << \mu_{T_F}$

$|\mu_F^{re}(E)| \leq \mu_{T_{F_{Real}}}(E) \leq \mu_{T_F}(E)$

$|\mu_F^{Im}(E)| \leq \mu_{T_{F_{Im}}}(E) \leq \mu_{T_F}(E)$

Since μ_F^{re}, μ_F^{Im} are finite by Theorem 3.5

$\forall \varepsilon > 0 \ \exists \ \delta = \delta(\varepsilon) > 0 \ /$

$$\mu_{T_F}(E) < \delta \implies \begin{cases} |\mu_F^{re}(E)| < \varepsilon \\ |\mu_F^{Im}(E)| < \varepsilon \end{cases}$$

Given $\delta' < \delta$ $\exists \{(a_j, b_j]\} / E \subset \bigcup_{j=1}^{\infty} (a_j, b_j]$

and
$$\mu_{T_F}(E) + \delta' \geq \sum_j \mu_{T_F}((a_j, b_j])$$
$$\geq \sum_j |\mu_F((a_j, b_j])|$$
$$\geq |\mu_F(\bigcup_j (a_j, b_j])|$$

If we let $A = E \setminus \bigcup_j (a_j, b_j]$ then

$\mu_{T_F}(A) \leq \delta'$ and so $|\mu_F^{re}(A)|, |\mu_F^{Im}(A)| < \varepsilon$

$\implies |\mu_F(A)| \leq \sqrt{2}\,\varepsilon$

$\implies |\mu_F(E)| \leq \mu_{T_F}(E) + \delta' + \sqrt{2}\,\varepsilon$

$\left(E = \bigcup (a_j, b_j] \cup A\right)$ Choose $\delta' \leq \min(\delta, \varepsilon)$ and let $\varepsilon \to 0$

to get $|\mu_F(E)| \leq \mu_{T_F}(E)$.

(c) By Exercise 21

$$|\mu_F|(E) = \sup \left\{ \sum_{j=1}^{m} |\mu_F(E_j)| \;\; E_1 \dots E_m \text{ disjoint} \;\; E = \dot\bigcup E_j \right\}$$

By part (b)

$$|\mu_F(E_j)| \le \mu_{T_F}(E_j).$$

$$\Rightarrow \sum_{j=1}^{m} |\mu_F(E_j)| \le \mu_{T_F}(E)$$

by taking sup get $\quad |\mu_F| \le \mu_{T_F}$

$\therefore \; G \le T_F$