PROBLEM SET 1

Due September 19th, 2002

Problem 1. *Let $\{E_n\}_{n\geq 1}$ be a family of sets. Prove the following facts about the* $\liminf E_n$ *and* $\limsup E_n$ *respectively.*

$$i) \qquad \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n = \{x : x \in E_n \textit{ for all but finitely many } n\} \text{ (this is by def. the } \liminf E_n)$$

$$ii) \qquad \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n = \{x : x \in E_n \textit{ for infinitely many } n\} \text{ (this is by def. the} \limsup E_n)$$

Problem 2. *Let $\{E_n\}_{n\geq 1}$ be a family of sets. For each $n \geq 1$ let $\chi_{E_n}(x)$ denote the characteristic function of the set E_n (i.e. the function that equals 1 when $x \in E_n$ and zero otherwise). Prove the following :*

$$i) \qquad \liminf \chi_{E_n}(x) = \chi_{\liminf E_n}(x)$$

$$ii) \qquad \limsup \chi_{E_n}(x) = \chi_{\limsup E_n}(x)$$

Problem 3. *Let (X, ρ) be a metric space.*

a) Prove that the finite intersection of open sets is open.

b) Prove that the finite union of closed sets is closed

Problem 4.

a) Prove that a <u>closed</u> subset of a complete metric space is also complete. Show with an example that if the subset of a complete metric space is not closed then it may fail to be complete.

b) Prove that a complete subset of an arbitrary metric space is <u>closed</u>. Find an example of an arbitrary metric space and a complete subset of it and check is indeed closed.

Problem 5. *Let (X, ρ) be a metric space. Prove that if X is totally bounded then it is bounded.*