

# The Cauchy problem for the hyperbolic–elliptic Ishimori system and Schrödinger maps\*

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## Abstract

We show an improved local in time existence and uniqueness result for Schrödinger maps and for the hyperbolic–elliptic nonlinear system proposed by Ishimori in analogy with the two-dimensional classical continuous isotropic Heisenberg spin (2d-CCIS) chain. The proof uses fairly standard gauge geometric tools and energy estimates in combination with Kenig’s version of the Koch–Tzvetkov method, to obtain *a priori*  $L_t^q L_x^\infty$  estimates for classical solutions to certain dispersive equations.

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## 1. Introduction

The purpose of this note is to prove a new local in time existence and uniqueness result for the two-dimensional hyperbolic–elliptic Ishimori system as well as for two-dimensional Schrödinger maps.

Schrödinger maps are solutions to

$$\partial_t s = J(s) \sum_k \nabla_k \partial_k s, \quad (1)$$

where  $s : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathcal{M}$  a Kähler manifold with complex structure  $J$ ,  $J^2 = -I$  and metric  $g$ . By  $\nabla_k$  we denote the covariant derivative on the  $s^{-1}\mathcal{T}(\mathcal{M})$  bundle in the direction of  $x^k$ . These

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are evolution equations that arise from the same geometric considerations as wave maps. They are examples of gauge theories which are abelian in the case of Riemann surfaces—Kähler manifolds of dimension one—such as the 2-sphere  $\mathbb{S}^2$  or the 2-hyperbolic space  $\mathbb{H}^2$  [44]. In this note we focus on  $\mathbb{S}^2$ ; then  $J = -\mathbf{i}$  the  $2 \times 2$  matrix with 1,  $-1$  in the off-diagonal. Physically, just as wave maps are studied as examples of relativistic nonlinear field theories [22], Schrödinger maps arise naturally from the Landau–Lifshitz equations (a  $U(1)$ -gauge theory) governing the static as well as dynamical properties of magnetization [27, 35]. To be more precise, for maps  $s : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$  equation (1) takes the form

$$\partial_t s = s \times \Delta s, \quad |s|^2 = 1, \quad (2)$$

which is the Landau–Lifshitz equation at zero dissipation when only the exchange field is retained [25, 35]. In two spatial dimensions equation (2) is also known as the two-dimensional classical continuous isotropic Heisenberg spin model (2d-CCIHS); i.e. the long wave length limit of the isotropic Heisenberg ferromagnet [25, 35, 41].

The hyperbolic–elliptic Ishimori system,

$$\begin{aligned} \partial_t s &= s \times \square_{xy} s + \kappa(\zeta_x s_y - \zeta_y s_x) \\ \Delta \zeta &= 2s \cdot (s_x \times s_y) \end{aligned} \quad (3)$$

with  $s : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ ,  $\lim_{|x|, |y| \rightarrow \infty} s(x, y, t) = (0, 0, -1)$  and  $\kappa$  a real constant, was proposed in 1984 by Ishimori. In [12], Ishimori—seeking to show that the dynamics of topological vortices need not be generally non-integrable—introduced the system (3) in analogy with the 2d CCIHS chain, as a model having the same topological properties as the latter yet permitting topological vortices whose dynamics are integrable; the system (3) is completely integrable when  $\kappa = 1$  (cf [2, 12, 26, 29, 43]). The linearized equation of (3) in this case is the same as that of the hyperbolic–elliptic Davey–Stewartson system and has been tackled by inverse scattering methods [2, 26, 36]. On the other hand, system (3) with  $\Delta_{xy}$  in lieu of  $\square_{xy}$  and  $\kappa = 0$  reduces to (2). The Ishimori system (3) describes the time evolution of a system of static spin vortices in the plane. The right-hand side of the equation for the scalar potential function  $\zeta(x, y, t)$  is the topological charge density of the system. The integer values of the topological charge  $Q$ ,

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} s \cdot (s_x \times s_y) dx dy$$

classify the static spin vortices [12]. Geometrically  $\zeta(x, y, t)$  is a multiple of the curvature tensor and the second equation can be viewed as describing the pull back of a piece of the volume in the target  $\mathbb{S}^2$ .

Our main result here is that both the 2d Schrödinger map equation (2) into  $\mathbb{S}^2$  and the Ishimori system (3) admit a local in time solution for the Cauchy initial value problem with large data in the Sobolev<sup>4</sup> class  $H^\gamma(\mathbb{R}^2)$ ,  $\gamma > 3/2$  suitably avoiding the north pole<sup>5</sup>. Uniqueness holds in  $H^2(\mathbb{R}^2)$ .

In the context of Schrödinger maps it was shown in [31] that one can find an appropriate frame on  $s^{-1}\mathcal{T}(\mathbb{S}^2)$  so that the derivatives of the solution of (2) satisfy a certain nonlinear

<sup>4</sup> Data  $s_0$  in the inhomogeneous Sobolev space should be thought of as  $s_0 \in \dot{W}^{1,2}$  and  $\partial s_0$ , the tangent vector to  $\mathbb{S}^2 \subset \mathbb{R}^3$  at  $s_0$ , being in  $H^{\gamma-1}(\mathbb{R}^2)$ . Note the unit normal to  $\mathbb{S}^2$  at a point  $u \in \mathbb{R}^3$  is  $u$  itself.

<sup>5</sup> More precisely we will request the range of the data to omit a small neighbourhood of the north pole. This is a stronger than needed technicality that simplifies the translation back and forth between the Ishimori or Schrödinger map system and the modified Ishimori or Schrödinger map equation. In principle, the weaker assumption of the map having degree zero should suffice; but it alone makes the translation back to the map more involved. Note that for nearly parallel spins—i.e. sufficiently small data—initially close to the south pole none of these requirements is needed since for a short time the map will stay close to the south pole.

Schrödinger system, referred to as the modified Schrödinger map system (MSM). Instead of repeating the argument we will describe below how the same ideas transform very similarly the Ishimori system into a nonlinear hyperbolic Schrödinger equation. In analogy with Schrödinger maps, we will refer to it as the modified Ishimori system (MIS).

To prove our main result we rely on the latter transformation in combination with energy estimates as in [32], Strichartz estimates and a formulation devised by Kenig [16, 18] of the ideas in Koch–Tzvetkov’s work [24], to obtain *a priori* estimates for classical smooth maps (see also Burq *et al* [5], Bahouri and Chemin [1] and Vega [45] for earlier uses of these ideas).

Standard approximation methods and compactness then give local in time existence in  $H^s(\mathbb{R}^2)$ ,  $s > 1/2$  for the MSM and the MIS and uniqueness in  $H^{s'}(\mathbb{R}^2)$ ,  $s' > 1$ . Hence as in section 2 of [31], one obtains local existence in  $H^\gamma(\mathbb{R}^2)$ ,  $\gamma > 3/2$  and uniqueness in  $H^{\gamma'}(\mathbb{R}^2)$ ,  $\gamma' > 2$  for the Ishimori system and the Schrödinger map equation.

Independently and simultaneously to our work here, Kato proved for the MSM the same local in time existence result using the same method. In addition, he was able to show uniqueness in  $H^1(\mathbb{R}^2)$  as opposed to  $H^{s'}(\mathbb{R}^2)$ ,  $s' > 1$  [13]. We have adapted his uniqueness argument here also for the MSI.

Various local and global existence results for smooth Schrödinger maps have been obtained by several authors and can be found in [6, 9, 21, 28, 31, 32, 37, 41].

For the hyperbolic–elliptic Ishimori system there is Soyeur’s [40] work who showed for small  $H^3$  smooth data local and global existence. He also showed uniqueness of large data solutions in  $H^4(\mathbb{R}^2)$ . Global existence for small smooth data followed from Strichartz estimates (dispersive inequality) [43] plus local existence as in Shatah [38], Klainerman and Ponce [23], Sulem *et al* [41]. Local existence for large smooth data does not follow from the results in [21] because  $\nabla\phi$  is not necessarily in  $L^1$ . Thus in this paper we also need to do a parabolic regularization in the covariant derivative equation and prove energy estimates in  $H^k$ ,  $k$  large. This will be done in the appendix and for clarity in the exposition of ideas we momentarily assume in the earlier sections existence of large data local in time smooth solutions.

Finally we note that our results in sections 3–5 actually hold in any  $n \geq 2$  with  $H^{(n-1)/2+}$  in lieu of  $H^{1/2+}$ . We restrict ourselves to the more physical two-dimensional case. The interested reader could easily adapt the statements accordingly, to the higher dimensional setup.

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## 2. The modified Ishimori system

Consider the system (3) with  $\kappa \neq 0$  any real constant. The precise value of  $\kappa$  will not enter but in the constants bounding the estimates. Without any loss of generality we can re-normalize  $\zeta =: 1/\kappa\phi$  and write

$$\begin{aligned}\partial_t s &= s \times \square_{xy} s + (\phi_x s_y + \phi_y s_x), \\ \Delta \phi &= 2\kappa s \cdot (s_x \times s_y).\end{aligned}\tag{4}$$

In what follows relabel for notational convenience  $(t, x, y) = (x_0, x_1, x_2)$ . Following [31] we start with a description in terms of the stereographic projection of  $\mathbb{S}^2 \setminus \{N\} \rightarrow \mathbb{C}$  where  $N$  is the north pole. Note that for  $s : \mathbb{R}^2 \rightarrow \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$ , the energy of  $s$  is

$$E(s) = \frac{1}{2} \int_{\mathbb{R}^2} \sum_{j=1}^2 \frac{|\partial s / \partial x_j|^2}{(1 + |s|^2)^2} dx dy.$$

We rewrite system (4) using covariant derivatives,

$$\nabla_j := \frac{\partial}{\partial x_j} - 2 \frac{(\partial s / \partial x_j) \cdot \bar{s}}{1 + |s|^2}$$

as

$$i\partial_t s = \left\{ \left( \nabla_1 - i \frac{\partial \phi}{\partial x_2} \right) \frac{\partial s}{\partial x_1} - \left( \nabla_2 + i \frac{\partial \phi}{\partial x_1} \right) \frac{\partial s}{\partial x_2} \right\}. \quad (5)$$

Just as with Schrödinger maps we have two sets of consistency conditions, the first one follows from the Levi–Civita connection having no torsion; the second one follows from the curvature of  $\{\nabla_j\}$  being the pull back of constant curvature on  $\mathbb{S}^2$  by the map  $s$  [31]:

$$\begin{aligned} \nabla_j \frac{\partial s}{\partial x_k} &= \nabla_k \frac{\partial s}{\partial x_j}, & j = 0, 1, 2, \quad k = 1, 2, \\ [\nabla_j, \nabla_k] &= -4i \operatorname{Im}(\bar{b}_j b_k), & j = 0, 1, 2, \quad k = 1, 2, \end{aligned}$$

where  $b_j = (\partial s / \partial x_j) / (1 + |s|^2)$ . Recalling now the equation for  $\phi$  we see that

$$\Delta \phi = 2\kappa s \cdot (s_x \times s_y) = -4\kappa \operatorname{Im}(b_1 \bar{b}_2).$$

Now, assuming finite energy of the map  $s$ , we can apply the ‘good gauge’ theorem of Uhlenbeck—existence of a global Coulomb gauge—(cf theorem 1 in [31], also [44]) to  $\nabla_j, \partial s / \partial x_j$ . Write

$$\begin{aligned} u_j &= (1 + |s|^2)^{-1} e^{i\psi} \frac{\partial s}{\partial x_j}, \\ D_j &= (1 + |s|^2)^{-1} e^{i\psi} \circ \nabla_j \circ (1 + |s|^2) e^{-i\psi} = \frac{\partial}{\partial x_j} + i a_j. \end{aligned} \quad (6)$$

Moreover, for each  $t$  there exists a unique choice of  $\psi$  such that  $a$  is a real valued 1-form

$$\operatorname{div} a = 0, \quad a \sim 0 \quad \text{at infinity}, \quad (7)$$

$$D_j u_k = D_k u_j \quad j = 0, 1, 2, \quad k = 1, 2, \quad (8)$$

$$[D_j, D_k] = [\nabla_j, \nabla_k], \quad (9)$$

$$[D_j, D_k] = i \left( \frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \right) \quad j = 0, 1, 2, \quad k = 1, 2, \quad (10)$$

$$a_k = \frac{\partial \beta_{1k}}{\partial x_1} + \frac{\partial \beta_{2k}}{\partial x_2}, \quad k = 1, 2 \quad d\beta = 0, \quad (11)$$

$$\Delta \beta_{k,j} = -4 \operatorname{Im}(u_k \bar{u}_j) \quad k, j = 1, 2, \quad (12)$$

$$\Delta a_0 = -4 \sum_{k,j=1}^2 \left[ \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \operatorname{Re}(u_k \bar{u}_j) - \frac{1}{2} \left( \frac{\partial}{\partial x_k} \right)^2 |u_j|^2 \right], \quad (13)$$

where (8)–(10) are gauge invariant equations. Note in particular that we have

$$da = -4 \operatorname{Im}(u_1 \bar{u}_2). \quad (14)$$

Now if we let

$$v = \begin{pmatrix} \frac{\partial \phi}{\partial x_2} \\ -\frac{\partial \phi}{\partial x_1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad *v = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ -\frac{\partial \phi}{\partial x_2} \end{pmatrix} = (-v_2, v_1)$$

it follows that  $v_1, v_2$  are *real valued*,  $\operatorname{div} v = 0$  and  $\operatorname{div} *v = \Delta \phi = -4\kappa \operatorname{Im}(u_1 \bar{u}_2)$ . Thus for  $k, i = 1, 2$  we have

$$\frac{\partial v_k}{\partial x_i} = -4\kappa \mathcal{R}_{ik}(\operatorname{Im}(u_1 \bar{u}_2)),$$

where  $\mathcal{R}_{ik} := (-\Delta)^{-1} \partial_{x_i} \partial_{x_k}$ , are the second order Riesz transforms. Equation (5) can now be rewritten as

$$i\partial_t s = (\nabla_1 + iv_1) \frac{\partial s}{\partial x_1} - (\nabla_2 + iv_2) \frac{\partial s}{\partial x_2}.$$

Proceeding as in the proof of theorem 2 in [31], we write

$$\begin{aligned} D_j u_j &= (1 + |s|^2)^{-1} e^{i\psi} \nabla_j \frac{\partial s}{\partial x_j}, \\ u_0 &= i(D_1 u_1 - D_2 u_2) + \frac{\partial \phi}{\partial x_1} u_2 + \frac{\partial \phi}{\partial x_2} u_1, \end{aligned}$$

from where for  $u_1$ , for example, we have  $(\partial u_1 / \partial t) + ia_0 u_1 = D_0 u_1$  and

$$D_0 u_1 = D_1 u_0 = i(D_1^2 u_1 - D_2^2 u_1) + [D_1, D_2] u_2 + D_1(v_1 u_1) - D_1(v_2 u_2).$$

Proceeding similarly with  $u_2$ , using (6), (10)–(13)—as in [31] theorem 2—we obtain a system of the form

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= i\Box u_1 + \gamma_1 \left( a_1 \frac{\partial u_1}{\partial x_2} - a_2 \frac{\partial u_1}{\partial x_1} \right) + \gamma_2 \alpha_1 u_1 + \gamma_3 \alpha_2 u_2 + \gamma_4 (a_1^2 - a_2^2) u_1 + \gamma_5 a_0 u_1, \\ \frac{\partial u_2}{\partial t} &= i\Box u_2 + \gamma_1 \left( a_1 \frac{\partial u_2}{\partial x_2} - a_2 \frac{\partial u_2}{\partial x_1} \right) + \gamma_2 \alpha_1 u_2 + \gamma_3 \alpha_2 u_1 + \gamma_4 (a_1^2 - a_2^2) u_2 + \gamma_5 a_0 u_2, \end{aligned} \quad (15)$$

with  $\Box := \Box_{x_1, x_2} = (\partial^2 / \partial x_1^2) - (\partial^2 / \partial x_2^2)$  and  $\gamma_m, m = 1, \dots, 5$  some constants that may depend on  $\kappa$  but  $\gamma_1 \in \mathbb{R}$ . We have denoted by  $\alpha_1, \alpha_2$  quadratic terms in  $u_1, u_2$  of the form

$$\alpha_1 = \mathcal{R}(\text{Im}(u_1 \overline{u_2})), \quad \alpha_2 = \text{Im}(u_1 \overline{u_2}), \quad (15b)$$

where by  $\mathcal{R}$  we generically represent an appropriate linear combination of Riesz transforms  $\mathcal{R}_{k,l}, k, l = 1, 2$ .

We refer to this system as the MIS.

For the convenience of the reader we write now the MSM system in keeping with the notation above and that in [31]. The MSM is a system of coupled nonlinear Schrödinger equations in  $R^{2+1}$  of the form:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= i\Delta u_1 - 2 \left( a_1 \frac{\partial u_1}{\partial x_1} + a_2 \frac{\partial u_1}{\partial x_2} \right) \pm \text{Im}(u_2 \overline{u_1}) u_2 - i(a_1^2 + a_2^2) u_1 - ia_0 u_1, \\ \frac{\partial u_2}{\partial t} &= i\Delta u_2 - 2 \left( a_1 \frac{\partial u_2}{\partial x_1} + a_2 \frac{\partial u_2}{\partial x_2} \right) \pm \text{Im}(u_1 \overline{u_2}) u_1 - i(a_1^2 + a_2^2) u_2 - ia_0 u_2, \end{aligned} \quad (16)$$

where just as above

$$\begin{aligned} a &= (a_1, a_2) = \left( -\frac{\partial \beta}{\partial x_2}, \frac{\partial \beta}{\partial x_1} \right), \\ \Delta \beta &= \pm 4 \text{Im}(u_1 \overline{u_2}), \\ \Delta a_0 &= \pm \sum_{k,j=1}^2 \left( 2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \text{Re}(u_k \overline{u_j}) - \partial_{x_k}^2 |u_j|^2 \right). \end{aligned} \quad (16b)$$

In what follows we will discuss the Cauchy initial value problem for the MSM and the MIS with data  $u_1(\cdot, 0) = u_0^1(\cdot)$  and  $u_2(\cdot, 0) = u_0^2(\cdot)$  in  $H^s(\mathbb{R}^2), s > 1/2$ .

From now on, we will not distinguish between  $u_1$  and  $u_2$  in our formulae, as we will primarily use their functional analytic properties. Occasionally, we will be referring to the vector  $u = (u_1, u_2)$  and the data  $u_0 = (u_0^1, u_0^2)$ . We also note that viewed as a *linear* system with *a priori* given time dependent coefficients, the coupling of the systems occurs through

one of the cubic nonlinearities; hence without any loss of generality we may schematically write the Cauchy problem for either system (15) or (16) as

$$\frac{\partial u}{\partial t} - iLu + \gamma \delta(au) = F(u, \bar{u}), \quad (17)$$

where,  $\gamma$  is a real nonzero constant and

- (i)  $u$  is either  $u_1$  or  $u_2$ ,
- (ii)  $L$  represents either  $\Delta = (\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2)$  or  $\square_{x_1 x_2} = (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2)$
- (iii)  $a = (a_1, a_2)$  is real-valued and satisfies (7), (11), (12) and (14)
- (iv)  $\delta w$  represents either  $\operatorname{div} w = (\partial w/\partial x_1) + (\partial w/\partial x_2)$  or  $\tilde{\operatorname{div}} w = (\partial w/\partial x_1) - (\partial w/\partial x_2)$
- (v)  $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$  and  $\tilde{\nabla} = (\partial/\partial x_1, -\partial/\partial x_2)$
- (vi)  $F(u, \bar{u})$  stands for the sum of the cubic-type and ‘quintic’ terms appearing in (15) and (16).

**Remark.** The derivative term in the MSM (17) has the form  $a \cdot \nabla u$  which can easily be written as  $\operatorname{div}(au)$  since  $\operatorname{div} a = 0$ . The derivative term in the MIS (16) has the form  $a \cdot \tilde{\nabla} u$ . Thus at the expense of adding an extra cubic term of the form  $((\partial a_1/\partial x_1) - (\partial a_2/\partial x_2))u$  we can view the derivative term as  $\tilde{\operatorname{div}}(au)$ .

The following additional notation will be used throughout the rest of this paper.

For  $1 \leq q, r \leq \infty$ , we write

$$\|f\|_{L_t^q L_x^r} := \|\|f(\cdot, t)\|_{L^r(\mathbb{R}^2)}\|_{L^q(\mathbb{R})}$$

and write  $\|f\|_{L^q([0, T]) L_x^r}$  or simply  $\|f\|_{L_T^q L_x^r}$  to denote  $\|\|f(\cdot, t)\|_{L^r(\mathbb{R}^2)}\|_{L^q([0, T])}$ . For  $|s| \leq 1$  we denote by  $J^s := (I - \Delta)^{s/2}$  the Bessel potential, and by  $D^s = (-\Delta)^{s/2}$  the Riesz potential. By  $|\nabla|^{-1} f(x)$  we denote the pseudo-differential operator of order  $-1$  given by convolution with  $1/|x|$ ,  $x \in \mathbb{R}^2$  or on the Fourier side by point-wise multiplication by  $|\xi|^{-1}$ ,  $\xi \in \mathbb{R}^2$ . Finally we write  $A \lesssim B$  when there is a positive constant  $c$  such that  $A \leq cB$ . The meaning of this constant may change from line to line and its dependence on other parameters in the problems will be clear in the context in which such inequality appears.

### 3. Preliminaries

In this section we collect a few basic facts from Littlewood–Paley theory, recall the Leibniz rule for fractional differentiation and some linear estimates of Strichartz type associated with the free group  $e^{itL}$ .

Let  $f(x)$  be a function on  $\mathbb{R}^2$  and  $\hat{f}(\xi)$  its Fourier transform. Consider  $m(\xi)$  to be a non-negative radial bump function supported on the ball  $|\xi| \leq 2$  and equal to 1 on the ball  $|\xi| \leq 1$ . Then for each integer  $k$  let  $P_k(f)$  be the Littlewood–Paley projection operator onto frequencies  $|\xi| \lesssim 2^k$ . This is defined by

$$\widehat{P_k(f)}(\xi) := m(2^{-k}\xi) \hat{f}(\xi).$$

Clearly,  $P_k \rightarrow I$  as  $k \rightarrow \infty$  and  $P_k \rightarrow 0$  as  $k \rightarrow -\infty$  in the  $L^2$  sense. The operator  $Q_k$  is the projection onto the *frequency annulus*  $|\xi| \sim 2^k$  given by the formula,

$$Q_k := P_k - P_{k-1}.$$

Hence  $\psi(\xi) := m(\xi) - m(2\xi)$  is supported on the annulus  $1/2 \leq |\xi| \leq 2$ , for all  $\xi \neq 0$ ,  $\sum_{k \in \mathbb{Z}} \psi(2^{-k}\xi) \equiv 1$ , and

$$\widehat{Q_k(f)}(\xi) = \psi(2^{-k}\xi) \hat{f}(\xi).$$

The Littlewood–Paley projections are bounded operators in all the Lebesgue spaces. By telescoping the series we have

$$f = \sum_{k \in \mathbb{Z}} Q_k(f) = P_0 + \sum_{k > 0} Q_k(f) \quad \text{in the sense of } L^2$$

or for any locally integrable function with decay at infinity. By construction and Plancherel,

$$\|f\|_2 \sim \left( \sum_k \|Q_k(f)\|_2^2 \right)^{1/2} \sim \|P_0(f)\|_2 + \left( \sum_{k > 0} \|Q_k(f)\|_2^2 \right)^{1/2}.$$

We also record the equivalence  $\|D^s Q_k(f)\|_p \sim 2^{sk} \|Q_k(f)\|_p$  valid for any  $1 \leq p \leq \infty$  and  $s \in \mathbb{R}$ .

**Definition 1.** We denote by  $\mathcal{B}_{r,p}^\delta$  the Banach space of functions on  $\mathbb{R} \times \mathbb{R}^n$  whose norm is given by

$$\|f\|_{\mathcal{B}_{r,p}^\delta} := \|P_0(f)\|_{L_T^2 L_x^r} + \left( \sum_{k > 0} 2^{\delta k p} \|Q_k(f)\|_{L_T^2 L_x^r}^p \right)^{1/p},$$

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where  $1 < r \leq \infty$ ,  $1 \leq p < \infty$  and  $\delta \geq 0$ .

**Remark.** In what follows we will primarily use  $\mathcal{B}_{\infty,1}^0$  and  $\mathcal{B}_{1/\varepsilon,1}^{3\varepsilon}$  for  $\varepsilon > 0$  small. Clearly, control of the  $\mathcal{B}_{\infty,1}^0$  norm implies, in particular, control of the  $L_T^2 L_x^\infty$  norm. Though not necessary, we choose at no extra cost to use these Besov-like space–time norms rather than the space–time Lebesgue norms.

**Lemma 1 (the Leibniz rule for fractional derivatives).** Let  $0 < s < 1$ . Then given  $1 < r < \infty$  and indices  $1 < p_1, q_1, p_2, q_2 < \infty$ ,

$$\begin{aligned} \|D^s(fg)\|_r &\lesssim \|D^s f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|D^s g\|_{q_2}, \\ \|J^s(fg)\|_r &\lesssim \|J^s f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|J^s g\|_{q_2}, \end{aligned}$$

provided  $1/r = (1/p_1) + (1/q_1) = (1/p_2) + (1/q_2)$ .

For a proof see, for example, [3, 7, 8, 14, 19].

**Remark.** A careful examination of the proofs shows that if in addition  $f, g$  are in  $L^\infty$ , then  $q_1$  or  $p_2$  are allowed to be  $\infty$  as well.

**Lemma 2 (dispersive estimates).** Let  $w$  be a classical smooth solution to

$$\begin{aligned} \partial_t w(t, x) - iLw(t, x) &= 0, \quad t > 0, \quad x \in \mathbb{R}^n, \\ w(0, x) &= f(x). \end{aligned}$$

Then we have,

(i) The Strichartz' estimates ([10, 15, 42])

$$\|w\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2}$$

provided  $2 \leq q, r \leq \infty$ ,  $(1/q) + (n/2r) = (n/4)$ ; but  $(q, r) \neq (2, \infty)$  when  $n = 2$ .

(ii) Let  $n = 2$ . Then for any  $\varepsilon > 0$ ,  $T > 0$ , there exists  $c = c(\varepsilon) > 0$  such that

$$\|w\|_{L^2[0,T]L_x^\infty} \leq cT^\varepsilon \|f\|_{H_x^{3\varepsilon}} \quad \text{and} \quad \|w\|_{L^2[0,T], W^{3\varepsilon, 1/\varepsilon}} \leq cT^\varepsilon \|f\|_{H_x^{3\varepsilon}}.$$

Similar estimates hold for any  $n \geq 2$ .

**Proof.** Estimates (i) for the elliptic Schrödinger linear equation ( $L = \Delta$ ) are well known. For the hyperbolic Schrödinger ( $L = \square_{x_1, x_2}$ ) they were established in this generality by Ghidaglia and Saut [10] in their study of the Davey–Stewartson systems (see also [42]). The endpoint case was settled by Keel and Tao [15].

To check (ii) we use (i) when  $n = 2$  for an admissible pair  $(q_0, r_0)$  such that  $1/q_0 = (1/2) - \varepsilon$  and  $r_0 = 1/\varepsilon$ . Then by Strichartz and Sobolev embedding

$$\|w\|_{L^{q_0}[0, T]L_x^\infty} \leq \|w\|_{L^{q_0}[0, T], W^{3\varepsilon, r_0}} \leq c\|f\|_{H_x^{3\varepsilon}}.$$

An application of Hölder's inequality on  $[0, T]$  gives then the desired estimates since  $q_0 > 2$ .  $\square$

#### 4. A priori estimates

**Proposition 1 (energy estimates).** *Let  $u$  be a classical smooth solution to (17). Then there exists an absolute constant  $C > 0$ , such that*

$$\frac{\partial}{\partial t} \|u(t)\|_L^2 \leq C_s (\|\nabla a(t)\|_{L^\infty} \|u(t)\|_{L^2}^2 + \|F(t)\|_{L^2} \|u(t)\|_{L^2}).$$

Moreover, given  $s > 0$  there exists an absolute constant  $C_s > 0$ , such that

$$\frac{\partial}{\partial t} \|u(t)\|_{H^s}^2 \leq C_s (\|\nabla a(t)\|_{H^s} \|u(t)\|_{H^s} \|u(t)\|_{L^\infty} + \|\nabla a(t)\|_{L^\infty} \|u(t)\|_{H^s}^2 + \|F(t)\|_{H^s} \|u(t)\|_{H^s}),$$

where by  $\|\nabla a(t)\|_{L^\infty}$  we denote  $\sup\{\|\partial a_k / \partial x_l\|_{L_x^\infty}, 1 \leq k, l \leq 2\}$ .

**Proof.** The proof of this proposition when  $L = \Delta_{x_1, x_2}$  and  $\delta = \text{div}$  is in [32] where in fact,  $L^2$  conservation also holds. A careful examination of their proof when  $s > 0$  shows the same argument gives the case  $L = \square_{x_1, x_2}$  and  $\delta = \text{div}$  after noting that

$$\langle \square_{x_1, x_2} u, u \rangle = - \int_{\mathbb{R}^2} \left| \frac{\partial u}{\partial x_1} \right|^2 - \left| \frac{\partial u}{\partial x_2} \right|^2 dx_1 dx_2,$$

whence  $\text{Re } i \langle \square_{x_1, x_2} u, u \rangle = 0$  as well. To see the  $L^2$  estimate above one can argue as follows. We multiply both sides of (17) by  $\bar{u}$ , integrate and take real part of the result. We proceed to analyse the terms that arise. Bear in mind that  $a$  is real valued; then:

$$\text{Re} \int \delta(au) \bar{u} dx = \text{Re} \int \delta(a) |u|^2 dx + \text{Re} \int \langle a, \tilde{\nabla} u \rangle \bar{u} dx.$$

On the other hand, an integration by parts yields

$$\int \langle a, \tilde{\nabla} u \rangle \bar{u} dx = - \int \delta(a) |u|^2 dx - \int \langle au, \tilde{\nabla} \bar{u} \rangle dx,$$

whence

$$\text{Re} \int \langle a, \tilde{\nabla} u \rangle \bar{u} dx = - \frac{1}{2} \int \delta(a) |u|^2 dx.$$

We get

$$\text{Re} \int \delta(au) \bar{u} dx = - \frac{1}{2} \int \delta(a) |u|^2 dx.$$



Thus by Hölder's

$$\left| \operatorname{Re} \int \delta(au) \bar{u} \, dx \right| \leq C \|u\|_{L^2}^2 \|\nabla a\|_{L^\infty}$$

and the desired estimate follows.  $\square$

**Remarks.**

- (i) As a consequence of the energy estimates above, we have the following *a priori* estimate for any  $0 \leq s \leq 1$ :

$$\frac{\partial}{\partial t} \|u(t)\|_{H_x^s}^2 \lesssim \|u(t)\|_{L_x^\infty}^2 \|u(t)\|_{H_x^s}^2 + \|\nabla a(t)\|_{L_x^\infty} \|u(t)\|_{H_x^s}^2 + \|F(t)\|_{H_x^s} \|u(t)\|_{H_x^s}.$$

- (ii) For  $u$  a classical smooth solution (e.g. it is known that there exists  $u \in H^k$  for  $k > 3$  by [9, 20, 28, 32, 37, 40, 41] of (17) then the 1-form  $a$  is the solution of the elliptic problem,

$$da = \pm \operatorname{Im}(u \bar{u})$$

on each fixed time slice  $\{t\} \times \mathbb{R}^2$  and it exists for each time  $t$  in the lifespan  $[0, T]$  of the smooth solution in  $H^k$ . We note that  $da$  is a 2-form. The following *a priori* estimates then hold

$$\|da(t)\|_{L_x^\infty} \leq \|u(t)\|_{L_x^\infty}^2, \quad \|da(t)\|_{H_x^s} \leq \|u(t)\|_{H_x^s} \|u(t)\|_{L_x^\infty}, \quad 0 \leq s \leq 1$$

for every  $t$  in the lifespan of the smooth solution. The energy estimate however depends on the full  $\nabla a$  and not just on  $da$ , forcing consideration of the Riesz transforms when estimating  $a$  or  $\nabla a$  in terms of  $u$ .

- (iii) Suppose  $u$  and  $v$  are two classical smooth solutions of (17) with the same initial data. Let  $a$  and  $\mathfrak{a}$  be as above satisfying (11), (12) and (14) in terms of  $u$  and  $v$ , respectively. Then if  $w := u - v$  and  $A = a - \mathfrak{a}$  one can see that, for example,  $w$  satisfies:

$$\begin{aligned} i\partial_t w + Lw &= -2i\delta(aw) - 2i\delta(Av) + F_1 - F_2, \\ \frac{\partial}{\partial t} \|w\|_{L^2}^2 &\lesssim \|\nabla a\|_{L^\infty} \|w\|_{L^2}^2 + \|\delta(Av)\|_{L^2} \|w\|_{L^2} + \|F_1 - F_2\|_{L^2} \|w\|_{L^2}. \end{aligned}$$

Note that although  $\operatorname{div} a = 0$  we do not make use of it since (17) may be not just MSM but also MIS. The ‘asymmetry’ present in this ‘difference’ estimate—i.e. the inability to draw a term of the form  $\|w\|^2$  in all terms on the right-hand side—makes it useless to derive uniqueness for data in  $H^s$ ,  $s > 1/2$  solely from the ‘energy method’. We will discuss this issue in more detail in section 5.

Our next result gives *a priori* control on the  $\mathcal{B}_{1/\varepsilon, 1}^{3\varepsilon}$  and  $\mathcal{B}_{\infty, 1}^0$  norms of the solution. Hence, in particular gives  $L^2((0, T))L_x^\infty$  *a priori* control. These estimates are the key to the 1/2-derivative gain over what energy plus Sobolev would give. This result and its proof closely follows Kenig’s ideas in [16].

**Proposition 2.** *Let  $u$  be a smooth solution of (17). Then for any  $\varepsilon > 0$  and  $T \in (0, 1]$  there exists a positive constant  $C_\varepsilon$  such that the following estimate holds.*

$$\begin{aligned} \|u\|_{\mathcal{B}_{\infty, 1}^0} + \|u\|_{\mathcal{B}_{r_0, 1}^{\tilde{\varepsilon}}} &\leq C_\varepsilon T^\varepsilon \left\{ \|J^{1/2+\tilde{\varepsilon}+\varepsilon} u\|_{L^\infty((0, T))L_x^2} + T^{1/2} \left( \int_0^T \|J^{1/2+\tilde{\varepsilon}+\varepsilon}(au)\|_{L_x^2}^2 \, dt \right)^{1/2} \right. \\ &\quad \left. + T^{1/2} \left( \int_0^T \|J^{-1/2+\tilde{\varepsilon}+\varepsilon} F\|_{L_x^2}^2 \, dt \right)^{1/2} \right\} \end{aligned} \quad \text{AQ3} \quad (19)$$

for  $\tilde{\varepsilon} = 3\varepsilon$  and  $r_0 = 1/\varepsilon$ .

**Proof.** We start by performing a Littlewood–Paley decomposition of  $u$ . We let  $\lambda := 2^k$  and write  $u_\lambda := Q_k(u)$ ; then in—say the  $L^2$ -sense,

$$u = P_0(u) + \sum_{k>0} u_\lambda.$$

Moreover, from equation (17),

$$\frac{\partial}{\partial t} u_\lambda - iLu_\lambda = (\delta(au))_\lambda + F_\lambda = \delta(au)_\lambda + F_\lambda$$

since  $Q_k \delta f = \delta Q_k f$ . Next we partition  $[0, T]$ ,  $T \leq 1$  into subintervals  $I_j := (c_j, d_j]$ , such that  $|I_j| = T2^{-k} = T/\lambda$  for each fixed  $k > 0$ . The number of subintervals is  $N := \lambda$ . We write for each fixed  $\lambda$ :

$$\begin{aligned} \|u_\lambda\|_{L^2((0,T))L_x^\infty} &= \left( \sum_{j=1}^N \|u_\lambda\|_{L^2(I_j)L_x^\infty}^2 \right)^{1/2}, \\ \|J^{3\varepsilon} u_\lambda\|_{L^2((0,T))L_x^{r_0}} &= \left( \sum_{j=1}^N \|J^{3\varepsilon} u_\lambda\|_{L^2(I_j)L_x^{r_0}}^2 \right)^{1/2}. \end{aligned}$$

Now on each  $I_j$  we write

$$u_\lambda(\cdot, t) = S(t - c_j)u_\lambda(\cdot, c_j) + \int_{c_j}^t S(t - t')\delta(au)_\lambda(t') dt' + \int_{c_j}^t S(t - t')F_\lambda(t') dt',$$

where we have denoted by  $S(t)$  the unitary operator  $e^{itL}$ . By lemma 2(ii) applied to  $u_\lambda$  on  $I_j$  we then obtain

$$\begin{aligned} \|u_\lambda\|_{L^2(I_j)L_x^\infty}^2 + \|J^{\tilde{\varepsilon}} u_\lambda\|_{L^2(I_j)L_x^{r_0}}^2 &\leq C_\varepsilon T^{2\varepsilon} \left\{ \|J^{3\varepsilon} u_\lambda(c_j)\|_{L_x^2}^2 + \left( \int_{c_j}^{d_j} \|J^{1+3\varepsilon}(au)_\lambda(t')\|_{L_x^2}^2 \right)^2 \right. \\ &\quad \left. + \left( \int_{c_j}^{d_j} \|J^{3\varepsilon} F_\lambda(t')\|_{L_x^2}^2 \right)^2 \right\}, \end{aligned}$$

where we have used implicitly that  $J^{3\varepsilon} \delta = J^{1+3\varepsilon} \mathcal{P}$  with  $\mathcal{P}$  an appropriate pseudo-differential operator of order zero in  $x$ , bounded in  $L^2(\mathbb{R}^2)$ .

We note now that by Sobolev inequality the result for  $\mathcal{B}_{r_0,1}^{\tilde{\varepsilon}}$  implies the bound for  $\mathcal{B}_{\infty,1}^0$ . Thus we continue working with the former—though the same calculation gives it for the latter.

Cauchy–Schwarz now gives:

$$\|J^{\tilde{\varepsilon}} u_\lambda\|_{L^2(I_j)L_x^{r_0}}^2 \leq C_\varepsilon T^{2\varepsilon} \{ \|J^{\tilde{\varepsilon}} u_\lambda(c_j)\|_{L_x^2}^2 + |I_j| \|J^{1+\tilde{\varepsilon}}(au)_\lambda\|_{L^2(I_j)L_x^2}^2 + |I_j| \|J^{\tilde{\varepsilon}} F_\lambda\|_{L^2(I_j)L_x^2}^2 \}.$$

Next, we sum over  $j$ ,  $1 \leq j \leq N$  for  $\lambda$  fixed to obtain:

$$\begin{aligned} \sum_{j=1}^N \|J^{\tilde{\varepsilon}} u_\lambda\|_{L^2(I_j)L_x^{r_0}}^2 &\leq C_\varepsilon T^{2\varepsilon} \left\{ \sum_{j=1}^N \|J^{\tilde{\varepsilon}} u_\lambda(c_j)\|_{L_x^2}^2 + \sum_{j=1}^N |I_j| \|J^{1+\tilde{\varepsilon}}(au)_\lambda\|_{L^2(I_j)L_x^2}^2 \right. \\ &\quad \left. + \sum_{j=1}^N |I_j| \|J^{\tilde{\varepsilon}} F_\lambda\|_{L^2(I_j)L_x^2}^2 \right\} \\ &\leq C_\varepsilon T^{2\varepsilon} \left\{ N \sup_{0 < t < T} \|J^{\tilde{\varepsilon}} u_\lambda(t)\|_{L_x^2}^2 + \frac{T}{\lambda} \|J^{1+\tilde{\varepsilon}}(au)_\lambda\|_{L^2((0,T))L_x^2}^2 \right. \\ &\quad \left. + \frac{T}{\lambda} \|J^{\tilde{\varepsilon}} F_\lambda\|_{L^2((0,T))L_x^2}^2 \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \|J^{\tilde{\varepsilon}} u_\lambda\|_{L^2((0,T))L_x^{r_0}} &\leq C_\varepsilon T^\varepsilon \left\{ N^{1/2} \sup_{0 < t < T} \|J^{\tilde{\varepsilon}} u_\lambda(t)\|_{L_x^2} + \left(\frac{T}{\lambda}\right)^{1/2} \|J^{1+\tilde{\varepsilon}}(au)_\lambda\|_{L^2((0,T))L_x^2} \right. \\ &\quad \left. + \left(\frac{T}{\lambda}\right)^{1/2} \|J^{\tilde{\varepsilon}} F_\lambda\|_{L^2((0,T))L_x^2} \right\} \\ &\leq C_\varepsilon T^\varepsilon \left\{ \sup_{0 < t < T} \|J^{1/2+\tilde{\varepsilon}} u_\lambda(t)\|_{L_x^2} + T^{1/2} \|J^{1/2+\tilde{\varepsilon}}(au)_\lambda\|_{L^2((0,T))L_x^2} \right. \\ &\quad \left. + T^{1/2} \|J^{-1/2+\tilde{\varepsilon}} F_\lambda\|_{L^2((0,T))L_x^2} \right\}. \end{aligned}$$

Finally, we need to sum in  $\lambda$ —i.e. in  $k \geq 0$ . We write:

$$\begin{aligned} \sum_\lambda \|J^{\tilde{\varepsilon}} u_\lambda\|_{L^2((0,T))L_x^{r_0}} &\leq C_\varepsilon T^\varepsilon \left\{ \sum_\lambda \sup_{0 < t < T} \|J^{1/2+\tilde{\varepsilon}} u_\lambda(t)\|_{L_x^2} \right. \\ &\quad \left. + T^{1/2} \sum_\lambda \left( \int_0^T \|J^{1/2+\tilde{\varepsilon}}(au)_\lambda\|_{L_x^2}^2 \right)^{1/2} + T^{1/2} \sum_\lambda \left( \int_0^T \|J^{-1/2+\tilde{\varepsilon}} F_\lambda\|_{L_x^2}^2 \right)^{1/2} \right\} \\ &\leq C_\varepsilon T^\varepsilon \left\{ \sup_{0 < t < T} \left( \sum_\lambda \|J^{1/2+\tilde{\varepsilon}+\varepsilon} u_\lambda(t)\|_{L_x^2}^2 \right)^{1/2} \right. \\ &\quad \left. + T^{1/2} \left( \sum_\lambda \int_0^T \|J^{1/2+\tilde{\varepsilon}+\varepsilon}(au)_\lambda\|_{L_x^2}^2 \right)^{1/2} + T^{1/2} \left( \sum_\lambda \int_0^T \|J^{-1/2+\tilde{\varepsilon}+\varepsilon} F_\lambda\|_{L_x^2}^2 \right)^{1/2} \right\} \\ &\leq C_\varepsilon T^\varepsilon \left\{ \sup_{0 < t < T} \|J^{1/2+\tilde{\varepsilon}+\varepsilon} u(t)\|_{L_x^2} \right. \\ &\quad \left. + T^{1/2} \left( \int_0^T \|J^{1/2+\tilde{\varepsilon}+\varepsilon}(au)\|_{L_x^2}^2 \right)^{1/2} + T^{1/2} \left( \int_0^T \|J^{-1/2+\tilde{\varepsilon}+\varepsilon} F\|_{L_x^2}^2 \right)^{1/2} \right\}. \end{aligned} \tag{AQ4}$$

We conclude by noting that since the larger Besov-type norm  $\sum_\lambda \|J^{\tilde{\varepsilon}} u_\lambda\|_{L^2((0,T))L_x^{r_0}}$ , controls in particular both  $\|u\|_{L^2((0,T))L_x^\infty}$  and  $\|u\|_{B_{\infty,1}^0}$  so does the right-hand side of (19) and the desired conclusions follow.  $\square$

**Remarks.** Given any  $\varepsilon > 0$ , a similar proof gives also an  $L^{q_0} W^{\varepsilon, r_0}$  *a priori* control with same right-hand side and for  $\varepsilon > 0$  and  $(q_0, r_0)$  an admissible pair such that  $4\varepsilon > \varepsilon > 1/r_0 = \varepsilon$  and  $1/q_0 = (1/2) - (1/r_0)$  (cf the proof of lemma 2(ii)).

Note also that in the course of the proof one could have simply put an  $L_{[0,T]}^1 H_x^{1/2+\tilde{\varepsilon}+\varepsilon}$  norm on the nonlinear terms on the right-hand side.

**Lemma 3.** *Let  $u$  be a classical smooth solution of (17) with right-hand side  $F$  as in (15) or (16). Let  $\varepsilon > 0$  be given and let  $1 > s > 1/2 + 4\varepsilon$ . There exists a positive constant  $M > 0$ , such that for each time  $t$  fixed in the lifespan of the smooth solution we have*

$$\begin{aligned} \|J^s(au)\|_{L_x^2} &\leq M \|u\|_{H_x^s}^3, \\ \|J^{s-1} F\|_{L_x^2} &\leq M \{ \|u\|_{H_x^s}^2 \|u\|_{L_x^\infty} + \|u\|_{H_x^s}^4 \|u\|_{L_x^\infty} \}. \end{aligned}$$

**Proof.**

$$\|J^s(au)\|_{L_x^2} \lesssim \|J^s(a)\|_{L_x^4} \|u\|_{L_x^4} + \|a\|_{L_x^\infty} \|u\|_{H_x^s}$$

by lemma 1. By the Sobolev embeddings and the boundedness of the Riesz transforms in  $L^p$ ,  $1 < p < \infty$  we have:

$$\|a\|_{L_x^\infty} \lesssim \|J^s(a)\|_{L_x^4} \lesssim \|J^s(u\bar{u})\|_{L_x^{4/3}} \lesssim \|J^s u\|_{L_x^2} \|u\|_{L_x^4} \lesssim \|u\|_{H_x^s}^2$$

from which the first estimate follows.

To prove the second estimate we need to take into account the special form of  $F$  as it appears in (15) and (16); in particular we pay special attention to those cubic and ‘quintic’ terms in which a pseudo-differential operator  $\mathcal{R}$  of order zero—bounded in  $L_x^p$ ,  $1 < p < \infty$ —appears acting on  $\text{Im}(u\bar{u})$  such as in  $a_0$  defined in (13) or (16b) and in  $\alpha_1$  defined in (15b). In fact  $\mathcal{R}$  is one of the Riesz transforms or could simply be taken to be the identity operator when dealing with the simple cubic term  $\text{Im}(u\bar{u})u$ . Recall also that  $a = (a_1, a_2)$  was defined in (11), (12) or (16b). We write generically for all of them

$$\begin{aligned} \|J^{s-1}F\|_{L_x^2} &\lesssim \|J^{s-1}\mathcal{R}(u\bar{u})u\|_{L_x^2} + \|J^{s-1}(a_1^2 \pm a_2^2)u\|_{L_x^2} \\ &\lesssim \|\mathcal{R}(u\bar{u})\|_{L_x^2} \|u\|_{L_x^\infty} + \|(a_1^2 \pm a_2^2)\|_{L_x^2} \|u\|_{L_x^\infty} \\ &\lesssim \|u\|_{L_x^4}^2 \|u\|_{L_x^\infty} + \|a\|_{L_x^4}^2 \|u\|_{L_x^\infty} \\ &\lesssim \|u\|_{H_x^s}^2 \|u\|_{L_x^\infty} + \|u\|_{H_x^s}^4 \|u\|_{L_x^\infty} \end{aligned}$$

since

$$\|a\|_{L_x^4} \lesssim \|\nabla a\|_{L_x^{4/3}} \lesssim \|u\bar{u}\|_{L_x^{4/3}} \lesssim \|u\|_{L_x^{8/3}}^2 \lesssim \|u\|_{H_x^{1/4}}^2.$$

The constant  $M$  thus arises from all the constants absorbed in the  $\lesssim$  signs. We note these depend on properties such as the  $L^2$  boundedness of  $\mathcal{R}$  and the Sobolev embedding constants. It may thus depend in particular on  $s, \varepsilon$ .  $\square$

Let  $\varepsilon > 0$  be given and let  $s > 1/2 + 4\varepsilon$ . Let  $\tilde{T}_1$  be a first time in  $(0, 1]$  such that

$$\sup_{0 < t < \tilde{T}_1} \|u(t)\|_{H^s(\mathbb{R}^2)} \geq 2\|u_0\|_{H^s(\mathbb{R}^2)}.$$

If such  $\tilde{T}_1 > 0$  does not exist then we must have that  $\sup_{0 < t < 1} \|u(t)\|_{H^s(\mathbb{R}^2)} \leq 2\|u_0\|_{H^s(\mathbb{R}^2)}$  and we will set  $T_1 = 1$ . Otherwise, by continuity in  $t$  of the quantity  $I(t) = \sup_{0 < t' < t} \|u(t')\|_{H^s(\mathbb{R}^2)}$  there exists a first  $T_1 > 0$ ,  $T_1 < \tilde{T}_1$  such that

$$\sup_{0 < t < T_1} \|u(t)\|_{H^s(\mathbb{R}^2)} = 2\|u_0\|_{H^s(\mathbb{R}^2)}.$$

We then fix this  $T_1$  for the remainder of the paper and define

$$1 > T_0 = \min \left\{ T_1, \left( \frac{1}{9MC_\varepsilon(\|u_0\|_{H_x^s}^2 + \|u_0\|_{H_x^s}^4)} \right)^2 \right\}, \quad (21)$$

where  $C_\varepsilon > 0$  is the constant of proposition 2 and  $M > 0$  the one of lemma 3.

**Proposition 3.** *Let  $u$  be a classical solution of (17) with right-hand side  $F$  as in (15) or (16). Let  $\varepsilon > 0$  be given and let  $1 > s > 1/2 + 4\varepsilon$ . There exists a constant  $C = C(s) > 0$ , such that*

$$\|u\|_{B_{1/\varepsilon,1}^{3\varepsilon}} \leq CT_0^\varepsilon \|u_0\|_{H_x^s} \quad \text{and} \quad \|u\|_{B_{\infty,1}^0} \leq CT_0^\varepsilon \|u_0\|_{H_x^s}.$$

**Proof.** By proposition 2, we have that both left-hand sides in the statement are less than or equal to

$$C_\varepsilon T_0^\varepsilon \left\{ \sup_{0 < t < T_0} \|J^s u(t)\|_{L_x^2} + T_0^{1/2} \|J^s(au)\|_{L^2((0,T_0))L_x^2} + T_0^{1/2} \|J^{s-1}F\|_{L^2((0,T_0))L_x^2} \right\}.$$

Lemma 3 yields the estimates for the second and third terms. All in all, we then have

$$\begin{aligned} \|u\|_{B_{1/\varepsilon,1}^{3\varepsilon}} + \|u\|_{B_{\infty,1}^0} &\leq C'_\varepsilon T_0^\varepsilon \{\|u\|_{L^\infty((0,T_0)H_x^s)} + T_0 M \|u\|_{L^\infty((0,T_0)H_x^s)}^3 \\ &\quad + T_0^{1/2} M \|u\|_{L^2((0,T_0)L_x^\infty)} (\|u\|_{L^\infty((0,T_0)H_x^s)}^2 + \|u\|_{L^\infty((0,T_0)H_x^s)}^4)\}. \end{aligned}$$

Since  $T_0 \leq T_1$ , we have that  $\|u\|_{L^\infty((0,T_0)H_x^s)} \leq 2\|u_0\|_{H_x^s}$ ; and thus we can make the last term less than or equal to

$$C_\varepsilon T_0^\varepsilon \{(1 + 4T_0 M \|u_0\|_{H_x^s}^2) \|u_0\|_{H_x^s} + T_0^{1/2} M (\|u_0\|_{H_x^s}^2 + \|u_0\|_{H_x^s}^4) \|u\|_{L^2((0,T_0)L_x^\infty)}\}.$$

Finally by (21) we conclude from the above estimate the conclusion of proposition 3.  $\square$

## 5. Well posedness of the MIS and MSM

**Main theorem.** *The Cauchy initial value problem associated with (17) admits a local in time solution in  $H^s$ ,  $s > 1/2$ . More precisely, given data  $u_0 \in H^s(\mathbb{R}^2)$ ,  $s > 1/2$  there exists a time  $0 < T = T(\|u_0\|_{H_x^s})$  and a solution to (17) such that*

$$u \in C([0, T]; H^s) \quad \text{and} \quad u \in L_t^2([0, T]L_x^\infty).$$

For data in  $H^1$ , the  $C([0, T]; H_x^1)$ -solution can be shown to be unique and furthermore, the mapping  $u_0 \rightarrow u \in C([0, T]; H^1)$  is continuous.

Before proving the theorem we collect a few estimates for  $\|F\|_{H^s}$ ,  $s > 1/2$  that will be needed in its proof.

**Lemma 4.** *Let  $u$  be a classical solution of (17) with right-hand side  $F$  as in (15) or (16). Let  $\varepsilon > 0$  be given and let  $1 > s > 1/2 + 4\varepsilon$ . Let  $a_0$  be defined by (13) and  $\alpha_1$  be defined by (15b). Then there exists a constant  $C > 0$  such that for each fixed time  $t$  in the life span of the smooth solution,*

$$\begin{aligned} (i) \quad &\|F(t)\|_{H_x^s} \leq C \{(\|u(t)\|_{L_x^\infty}^2 + \|a_0(t)\|_{L_x^\infty} + \|\alpha_1(t)\|_{L_x^\infty}) \|u(t)\|_{H_x^s} + \|u(t)\|_{H_x^s}^5\}. \\ (ii) \quad &\|\nabla a\|_{L^1((0,T)L_x^\infty)} + \|a_0\|_{L^1((0,T)L_x^\infty)} + \|\alpha_1\|_{L^1((0,T)L_x^\infty)} \leq C \|u\|_{B_{\infty,1}^0} \|u\|_{B_{1/\varepsilon,1}^{3\varepsilon}}. \end{aligned}$$

The constant  $C$  depends on properties such as the  $L^2$  boundedness of  $\mathcal{R}$  and the Sobolev embedding constants. It may thus depend in particular on  $s, \varepsilon$ .

**Proof.** In order to simplify the estimates we view  $F$  as essentially the sum of three terms of the form

- (1)  $\text{Im}(u\bar{u})u$
- (2)  $a_0 u$  (similarly for  $\alpha_1 u$ )
- (3)  $(a_1^2 \pm a_2^2)u$  where  $a = (a_1, a_2)$  is defined by (11), (12) or (16b).

To prove (i) we first note the  $H_x^s$ -norm of terms as in (1) are easily estimated by  $\|u\|_{L^\infty}^2 \|u\|_{H^s}$ . Similarly, by lemma 1 we have

$$\begin{aligned} \|a_0 u\|_{H_x^s} &\leq C \|a_0\|_{H^s} \|u\|_{L^\infty} + \|a_0\|_{L^\infty} \|u\|_{H^s} \\ &\leq C \|u\|_{H^s} \|u\|_{L^\infty}^2 + \|a_0\|_{L^\infty} \|u\|_{H^s} \end{aligned}$$

since  $\mathcal{R}$  commutes with  $J^s$  and it is bounded in  $L^2$ . For those terms as in (3) we proceed as follows. Again by lemma 1 we have

$$\|J^s((a_1^2 \pm a_2^2)u)\|_{L_x^s} \leq C \|J^s(a_1^2 \pm a_2^2)\|_{L_x^4} \|u\|_{L^4} + \|(a_1^2 \pm a_2^2)\|_{L_x^\infty} \|u\|_{H^s}.$$

As in the proof of lemma 3,

$$\|a_1\|_{L^\infty} + \|a_2\|_{L^\infty} \lesssim \|J^s a_1\|_{L^4} + \|J^s a_2\|_{L^4} \lesssim \|u\|_{H_x^s}^2.$$

On the other hand,

$$\|J^s(a_1^2 \pm a_2^2)\|_{L^4} \leq C(\|J^s a_1\|_{L^4} \|a_1\|_{L^\infty} + \|J^s a_2\|_{L^4} \|a_2\|_{L^\infty}) \leq C\|u\|_{H_x^s}^4.$$

Using  $H^s \hookrightarrow L^4$  and  $\|(a_1^2 \pm a_2^2)\|_{L_x^\infty} \lesssim \|a_1\|_{L^\infty}^2 + \|a_2\|_{L^\infty}^2$  we obtain the desired estimate for (i).

To prove (ii) we write,

$$\begin{aligned} \|\nabla a(t)\|_{L^\infty} + \|a_0(t)\|_{L^\infty} + \|\alpha_1(t)\|_{L^\infty} &\sim \|\mathcal{R}(u\bar{u})\|_{L^\infty} \\ &\leq C\|J^{3\varepsilon}(u\bar{u})\|_{L^{r_0}}, \quad \text{where } r_0 = \frac{1}{\varepsilon}. \end{aligned}$$

Then,

$$\begin{aligned} \|\nabla a\|_{L^1((0,T))L_x^\infty} + \|a_0\|_{L^1((0,T))L_x^\infty} + \|\alpha_1\|_{L^1((0,T))L_x^\infty} \\ \sim \int_0^T \|\mathcal{R}(u\bar{u})(t)\|_{L_x^\infty} dt \leq C \int_0^T \|J^{3\varepsilon}(u)\|_{L^{r_0}} \|u\|_{L^\infty} dt \\ \leq C \left( \int_0^T \|u\|_{L^\infty}^2 dt \right)^{1/2} \left( \int_0^T \|J^{3\varepsilon}(u)\|_{L^{r_0}}^2 dt \right)^{1/2} \\ \leq C\|u\|_{L^2((0,T))L^\infty} \|J^{3\varepsilon}(u)\|_{L^2((0,T))L^{r_0}} \\ \leq C\|u\|_{B_{\infty,1}^0} \|u\|_{B_{r_0,1}^{3\varepsilon}}. \quad \square \end{aligned}$$

**Proposition 4.** *Let  $u$  be a local smooth solution of (17). Let  $u_0$  be in  $H^s \cap H^m$ , with  $s > 1/2, m > 1$ . Then there exists a positive constant  $C_s$  and a time  $T > 0, T = T(s, \|u_0\|_{H^s})$  such that*

$$\sup_{0 < t < T} \|u(\cdot, t)\|_{H^s} \leq C_s \|u_0\|_{H^s}. \quad (22)$$

**Proof.** Let  $T = T_0$  from (21). Then since  $T \leq T_1 < 1$ ,  $\sup_{0 < t < T} \|u(\cdot, t)\|_{H^s} \leq 2\|u_0\|_{H^s}$ . We need to exhibit a positive lower bound on  $T$  depending solely on  $\|u_0\|_{H^s}$  and absolute constants.

(i) Assume  $T = T_1$ . First of all, if  $T_1 = 1$  we are done. Otherwise, we proceed as follows. By the energy estimates in proposition 1, the remarks that follow and lemma 4(i), we have

$$\frac{\partial}{\partial t} \|u(t)\|_{H^s}^2 \lesssim (\|u(t)\|_{L^\infty}^2 + \|\nabla a(t)\|_{L^\infty} + \|a_0(t)\|_{L^\infty} + \|\alpha_1(t)\|_{L^\infty}) \|u(t)\|_{H^s}^2 + \|u(t)\|_{H^s}^6.$$

Next we integrate in time from 0 to  $T = T_1$  and use the estimates in lemma 4(ii) and proposition 3 to get

$$\begin{aligned} 3\|u_0\|_{H^s}^2 &= \|u(T)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 \\ &\lesssim \|u\|_{B_{\infty,1}^0} \{ \|u\|_{B_{\infty,1}^0} + \|u\|_{B_{r_0,1}^\varepsilon} \} \sup_{0 < t < T} \|u(t)\|_{H^s}^2 + T \sup_{0 < t < T} \|u(t)\|_{H^s}^6 \\ &\lesssim \{4C'_\varepsilon T^{2\varepsilon} \|u_0\|_{H^s}^4 + 32T\|u_0\|_{H^s}^6\}. \end{aligned}$$

Since  $T < 1$  we have

$$1 \leq \left\{ \frac{4}{3} C''_\varepsilon T^{2\varepsilon} \|u_0\|_{H^s}^2 + \frac{32}{3} C''_\varepsilon T^{2\varepsilon} \|u_0\|_{H^s}^4 \right\}.$$

Hence the lower bound,

$$T \geq \frac{C'''_\varepsilon}{(\|u_0\|_{H^s}^2 + \|u_0\|_{H^s}^4)^{1/2\varepsilon}} > 0.$$

(ii) Assume  $T = (1/(9MC_\varepsilon(\|u_0\|_{H_x^s}^2 + \|u_0\|_{H_x^s}^4)))^2 < T_1$  as in (21); then we automatically have the desired lower bound on  $T$ .  $\square$

### Proof of the main theorem.

*Existence.* The existence proof follows standard compactness and approximation arguments relying on proposition 4. Given data  $u_0$  in  $H^s$ ,  $s > 1/2$  we approximate it by a sequence  $u_0^{(j)}$  in  $H^m$ ,  $m > 1$  of smooth initial data. We can then produce for a short time  $\{u^{(j)}(t, x)\}_j$  and  $\{a^{(j)}(t, x)\}_j$  for which the *a priori* estimates in propositions 1–4 hold. Moreover, using the MIS (15) and (15b) or the MSM system (16) and (16b) we can additionally obtain appropriate estimates for  $\partial_t u^{(j)}$ , thus ensuring the compactness of the *space-time* Sobolev embeddings. We can then extract a subsequence weak-\* converging in  $L_T^\infty H_x^s \cap L_T^{q_0} W_x^{-\epsilon, r_0}$  and strongly in  $L_t^p L_{\text{loc}}^2$ , any  $p > 1$  which is enough to establish weak convergence of the nonlinear terms.

All in all, by compactness we can extract a converging subsequence whose limit will satisfy either the MIS or the MSM in the sense of distributions and the same *a priori* estimates with life span  $[0, T]$ ,  $T = T(\|u_0\|_{H_x^s}) > 0$ ,  $s > 1/2$ . For more details see, for example, [13].

*Uniqueness.* This follows essentially by the argument of Kato in [13] which is in turn based on the following sharp Trüdinger's interpolation estimate

$$\|u\|_{L^q} \leq \sqrt{\frac{q}{2}} (4\pi)^{(2-q)/2q} \|u\|_{L^2(\mathbb{R}^2)}^{2/q} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{(1-2/q)}$$

shown in [34].

Suppose  $u$  and  $v$  are two classical smooth solutions of (17) with the same initial data  $f$  in  $H^1(\mathbb{R}^2)$ . Let us denote by  $M := \|f\|_{H^1}$ . Then by our local existence result we know there exists  $0 < T = T(M)$  and  $C > 0$  such that

$$\sup_{0 < t < T} \|u\|_{H^1} \leq CM \quad \text{and} \quad \sup_{0 < t < T} \|v\|_{H^1} \leq CM.$$

Let  $a$  and  $\mathfrak{a}$  satisfy (11), (12) and (14) in terms of  $u$  and  $v$ , respectively. Then if  $w := u - v$  and  $A = a - \mathfrak{a}$ ;  $w$  is the solution to the difference equation

$$i\partial_t w + Lw = -2i\delta(aw) - 2i\delta(Av) + F_1 - F_2, \quad (23)$$

where  $F_1 = F(u)$  and  $F_2 = F(v)$  with  $F$  as in (15) and (16). More precisely, taking into account the special form of  $F$  as it appears in (15) and (16), we write generically for all of them

$$F(u) \sim \mathcal{R}(u\bar{u})u + (a_1^2 \pm a_2^2)u,$$

where  $\mathcal{R}$  of order zero—bounded in  $L_x^p$ ,  $1 < p < \infty$ —appears acting on  $\text{Im}(u\bar{u})$  such as in  $a_0$  defined in (13) or (16b) and in  $\alpha_1$  defined in (15b). In fact,  $\mathcal{R}$  is one of the Riesz transforms—or could simply be taken to be the identity operator when dealing with the simple cubic term  $\text{Im}(u\bar{u})u$ . Recall also that  $a = (a_1, a_2)$  was defined in (11), (12) or (16b). Then we essentially have

$$\begin{aligned} F(u) - F(v) &\sim \mathcal{R}(u\bar{u})u + (a_1^2 \pm a_2^2)u - \mathcal{R}(v\bar{v})v + (\mathfrak{a}_1^2 \pm \mathfrak{a}_2^2)v \\ &= \mathcal{R}(u\bar{u})w + (a_1^2 \pm a_2^2)w + (\mathcal{R}(u\bar{u}) - \mathcal{R}(v\bar{v}))v + ((a_1^2 \pm a_2^2) - (\mathfrak{a}_1^2 \pm \mathfrak{a}_2^2))v. \end{aligned}$$

Then we multiply both sides of (23) by  $\bar{w}$  integrate over  $\mathbb{R}^2$  and take the real part of the integral. We estimate the terms that arise. First the argument shown in remark (iii) in section 4 show the first term on the right-hand side of (23) is estimated by

$$\left| \text{Re} \int \delta(aw) \bar{w} \, dx \right| \lesssim \|\nabla a\|_{L_x^\infty} \|w\|_{L^2}^2.$$

For the rest of the terms we follow Kato's estimates in [13] but we treat every term; that is, do not use any cancellations coming from terms being purely imaginary as is done in [13] in the MSM context. More precisely, we need to take into account the term

$$\begin{aligned} \left| \int \delta(A)v\bar{w} \, dx \right| &\leq \|\delta(A)\|_{L^q} \|v\|_{L^p} \|w\|_{L^2} \\ &\lesssim (\|u\|_{L^p} + \|v\|_{L^p}) \|w\|_{L^2}^2 \\ &\lesssim qM^2 \|w\|_{L^2}^2 \lesssim qM^{2+4q} \|w\|_{L^2}^{2-4/q}, \end{aligned}$$

where  $(1/p) + (1/q) = 1/2$ . Similarly, we also need to consider

$$\left| \int \mathcal{R}(u\bar{u})w\bar{w} \, dx \right| \lesssim \|\mathcal{R}(u\bar{u})\|_{L^\infty} \|w\|_{L^2}^2$$

and the term

$$\left| \int (a_1^2 \pm a_2^2)w\bar{w} \, dx \right| \lesssim qM^{4+2/q} \|w\|_{L^2}^{2-4/q}$$

from Sobolev's embedding and Trüdinger's sharp interpolation lemma earlier.

All in all, we get that for any  $q > 2$

$$\left| \operatorname{Re} \int \delta(Av)\bar{w} \, dx \right| \lesssim qM^{2+4/q} (1 + M^2) \|w\|_{L^2}^{2-4/q}.$$

and

$$\left\| \int (F(u) - F(v))\bar{w} \right\|_{L^2} \lesssim qM^{2+4/q} (1 + M^2) \|w\|_{L^2}^{2-4/q} + \|\mathcal{R}(u\bar{u})\|_{L^\infty} \|w\|_{L^2}^2.$$

Relying now on lemma 4 and propositions 2–4 we obtain that

$$\partial_t \|w\|_{L^2}^2 \lesssim qM^{2+4/q} (1 + M^2) \|w\|_{L^2}^{2-4/q} + \|\nabla a\|_{L_x^\infty} \|w\|_{L^2}^2 + \|\mathcal{R}(u\bar{u})\|_{L^\infty} \|w\|_{L^2}^2.$$

Next we let  $\epsilon = 2/q$  as in [13] to get that

$$\frac{1}{\epsilon} \partial_t \|w\|_{L^2}^{2\epsilon} \lesssim \frac{1}{\epsilon} M^{2+2\epsilon} (1 + M^2) + (\|\nabla a\|_{L_x^\infty} + \|\mathcal{R}(u\bar{u})\|_{L^\infty}) \|w\|_{L^2}^{2\epsilon}.$$

Multiplying both sides by  $\epsilon$  and then integrating in time over the interval  $[0, T]$  for any  $0 < T \leq T$  we obtain the estimate

$$\|w(t)\|_{L^2}^{2\epsilon} \leq C'TM^{2+2\epsilon} (1 + M^2) + C''\epsilon M \sup_{0 < t < T} \|w(t)\|_{L^2}^{2\epsilon}$$

for some positive constants  $C', C''$ .

Next by taking first  $\sup_{0 < t < T}$  both sides and then making  $\epsilon$  sufficiently small depending on  $M$  and the underlying absolute constants, we deduce first that

$$\sup_{0 < t < T} \|w(t)\|_{L^2}^{2\epsilon} \leq c'TM^{2+2\epsilon} (1 + M^2).$$

Then by raising both sides to  $1/2\epsilon$  and choosing  $T$  so that  $(c'TM^{2+2\epsilon} (1 + M^2)) < 1$ , we finally deduce that  $\|w(t)\|_{L^2} = 0$  for all  $0 < t < T$  by letting  $\epsilon \rightarrow 0$ .

Since the choice of  $T > 0$  depends solely on fixed constants in our estimates, we can restart repeatedly the same argument to get uniqueness from  $[T, 2T]$  and so on until we cover  $[0, T]$  into a finite number of steps.

The continuity of the data to solution map follows from the Bona–Smith method [4, 17] for the time of existence  $T$  given by the energy estimate computation in proposition 4 and expressed only in terms of  $\|u_0\|_{H_x^1}$  for which we have both existence and uniqueness.  $\square$



## 6. Existence and uniqueness of Ishimori and Schrödinger maps

In general, it is not a simple issue to go from solutions of the MIS and MSM systems to the full Ishimori and Schrödinger map systems directly. The transformation formulae between a solution  $u$  and the map  $s$  are quite complex. The well-posedness results on the modified systems MIS and MSM apply to a larger class of formal solutions to the equation than those which come from the Ishimori or Schrödinger maps.

Our method of using the results on the modified map equations to show existence of the Ishimori and Schrödinger maps is the same idea used in [31], and in [33] and [39] for wave maps.

The main purpose behind the idea of fixing a particular gauge and passing to the modified systems is that of obtaining *a priori* estimates for smooth solutions and—when possible—their differences in ‘rougher’ norms and using them to pass to a limit in the full original map system in an appropriate lower regularity space associated with the ‘rougher’ norm.

The following scheme enables us then to show local in time existence for the Ishimori and Schrödinger map systems for arbitrary data in  $H^{3/2+\epsilon}$  as follows. Our uniqueness argument requires data  $H^2(\mathbb{R}^2)$ .

Assume there is local in time existence and uniqueness for data in  $H^m$ ,  $m \geq 4$  for the map directly. Such solutions exist for Schrödinger maps after the works of Sulem, Sulem and Bardos (for  $\mathbb{S}^2$ ), Ding and Wang; McGahagan; Shatah; Nahmod, Stefanov and Uhlenbeck mentioned in the introduction. For the Ishimori system it follows from the discussion in the appendix of this paper in combination with the *a priori* estimates in sections 3–5 and estimates due to Soyeur (as explained in appendix).

Such smooth solutions transform over to solutions of the complete (overdetermined) system. Our results in sections 3–5 show that the time of existence depends only on  $\|u_0\|_{H^{1/2+}}$ . So given an initial data  $s_0$  in  $H^{3/2+}$ , we approximate it by smooth  $s_0^\alpha \in H^m$ , whose solutions  $s^\alpha$  satisfy the full set of equations and consistency conditions and the *a priori* estimates satisfied by the solution to the MIS and MSM systems. These *a priori* estimates are now used to pass to a weak limit<sup>6</sup>. The solution produced by the well-posedness result in our main theorem in section 5 will be the weak limit of a subsequence of the sequence of smooth solutions in  $C([0, T]; H^{1/2+}) \cap L^2([0, T]; L_x^\infty)$  and hence also satisfy the entire set of consistency conditions as desired. A key step in the argument to produce a solution in the original map system is the following lemma.

**Lemma 5.** *Estimate (22) of the solution to the modified Schrödinger or Ishimori map system implies a bound on the extrinsic  $H^\alpha(\mathbb{R}^2)$ ,  $\alpha > 1/2$  norm of  $ds$ . In other words, there exists a positive constant  $C$  such that*

$$\|ds(t)\|_{H_x^\alpha} \leq C \|u(t)\|_{H_x^\alpha}$$

for all  $t$  in the life span of  $u$ .

**Proof.** To begin its proof we express the modified Schrödinger or Ishimori map system and the stereographic projection in the language of moving frames. Recall we had

$$u_j = (1 + |s|^2)^{-1} e^{i\psi} \frac{\partial s}{\partial x_j}, \quad \psi = \psi(x, t)$$

and that the map  $s$  avoids the north pole. Using the conformal frame on  $\mathbb{S}^2 \setminus \{N\}$ , we write

$$ds = q^a e_a, \quad q = q_\alpha dx^\alpha,$$

<sup>6</sup> Being maps into  $\mathbb{S}^2$  we can write the second order terms in divergence form before taking weak limits.

where

$$\begin{aligned}\bar{e}_1 &:= (1 + |z|^2) \frac{\partial}{\partial x} \rightarrow \tilde{e}_1 = \bar{e}_1 \circ s, \\ \bar{e}_2 &:= (1 + |z|^2) \frac{\partial}{\partial y} \rightarrow \tilde{e}_2 = \bar{e}_2 \circ s,\end{aligned}$$

i.e.  $(\tilde{e}_1; \tilde{e}_2)$  is a smooth frame for the pull-back bundle. By a gauge transformation we can freely rotate this frame at any  $z = (x, t)$  with an  $SO(2)$  matrix  $R_a^b(z)$ . Then  $e_a = R_a^b \tilde{e}_b \circ s = R_a^b \tilde{e}_b$  is a moving frame for the pull-back bundle (cf [39]). We have

$$q_{1,j} := \left\langle \frac{\partial s}{\partial x_j}, e_1 \right\rangle, \quad q_{2,j} := \left\langle \frac{\partial s}{\partial x_j}, e_2 \right\rangle.$$

Expressed in these coordinates the modified Schrödinger or Ishimori map system becomes now a system for  $q$  in lieu of  $u$  and exactly the estimates we obtained for  $u$  hold for  $q$ .

We thus need to show that for each  $t$  fixed in the life span of  $q$ , and  $\alpha > 1/2$  as in the statement of the main theorem,

$$\|ds\|_{H_x^\alpha} \leq C \|q\|_{H_x^\alpha}.$$

We proceed as in Shatah and Struwe [39]. Consider any vector field  $W$  in  $s^*T\mathbb{S}^2$  whose coordinates in  $\{e_a\}$  are given by

$$W = Q^a e_a = Qe.$$

We know that given any  $p$ ,  $\|W\|_{L^p} \leq \|Q\|_{L^p}$  and wish to compare  $\partial_k W$  with  $\partial_k Q$ . We use the covariant derivative and second fundamental form and write

$$\begin{aligned}D_k W &= \partial_k W + B(s)(\partial_k, W), \\ (\partial_k + A)W &= (\partial_k Q_\beta^c + A_{a,k}^c Q_\beta^a) e_c,\end{aligned}$$

whence

$$\partial_k W = (\partial_k Q_\beta^c + A_{a,k}^c Q_\beta^a) e_c - B(s)(ds, Qe).$$

Thus

$$\begin{aligned}\|\partial_k W\|_{L^{8/5}} &\lesssim \|\partial_k Q\|_{L^{8/5}} + \|A\|_{L^{8/3}} \|Q\|_{L^4} + \|B\|_{L^\infty} \|ds\|_{L^2} \|Q\|_{L^8} \\ &\leq C(s) \|Q\|_{W^{1,8/5}},\end{aligned} \tag{24}$$

where  $C(s)$  is a positive constant that may be depending on the map  $s$ .

Since the map  $Q \rightarrow W$  is linear; interpolating half way between the latter estimate and  $\|W\|_{L^{8/3}} \leq \|Q\|_{L^{8/3}}$  gives the estimate

$$\|\partial^{1/2} W\|_{L^2} \lesssim \|J^{1/2} Q\|_{L^2}.$$

To obtain the desired estimate for  $\alpha > 1/2$  we choose  $p_0 = p_0(\alpha)$  such that  $1/p_0 = (3/8) - \delta$ ; with  $\delta$  depending on  $\alpha - 1/2$  and such that interpolating half way between  $\|W\|_{L^{p_0}} \leq \|Q\|_{L^{p_0}}$  and (24) yields

$$\|\partial^\alpha W\|_{L^2} \lesssim \|J^\alpha Q\|_{L^2}.$$

*Uniqueness.* Our results give a unique solution to the original map system only when the data is in the smoother  $H^2$  space. Such uniqueness should be understood as meaning that weak limits of smooth solution to the original map system are unique solutions to the same map system. The argument needed to establish the latter is somewhat more involved as it requires to show the uniqueness of the associated pull-back frames, assuming uniqueness in  $C([0, T]; H^1) \cap L^2([0, T]; L_x^\infty)$  of solutions (themselves limits of smooth solutions) to the modified map system. The frames  $\{e\}$  will have regularity comparable to that of the map  $s$ .

For Schrödinger maps such argument is done in [30]<sup>7</sup> by studying the associated frame system  $(s, u, e, a)$ . The Ishimori map system requires an adaptation of their argument. We postpone, however, its presentation until a more satisfactory uniqueness result can be achieved for the modified Ishimori map system.  $\square$

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## Appendix

We indicate in this section how to modify the arguments above in the context of Ishimori's system to obtain local in time existence of smooth—say  $H^4(\mathbb{R}^2)$ —solutions. As we mentioned earlier, uniqueness of large data solutions in  $H^4$  was showed by Soyeur.

**Lemma A (theorem 4.1 in [40]).** *Let  $u$  and  $v$  be two solutions of (A1) in  $L^\infty([0, T]; H^3(\mathbb{R}^2)) \cap L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^2))$ . If  $u(0) = v(0)$  then  $u(t) = v(t)$ , for all  $0 \leq t < T$ .*

We thus concentrate on the existence part.

Consider  $w : \mathbb{R}^2 \rightarrow \mathbb{C}$  coming from rewriting  $s : \mathbb{R}^2 \rightarrow \mathbb{S}^2$  in stereographic variables

$$w = \frac{s^1 + is^2}{1 + s^3}, \quad s = (s^1, s^2, s^3), \quad |s|^2 = 1.$$

Then the hyperbolic–elliptic Ishimori system in these coordinates looks like (cf [40])

$$\begin{aligned} iw_t + w_{xx} - w_{yy} &= \frac{2\bar{w}}{1 + |w|^2} (w_x^2 - w_y^2) + i(\phi_x w_y + \phi_y w_x), \\ \phi_{xx} + \phi_{yy} &= 4\kappa i \frac{w_x \bar{w}_y - w_y \bar{w}_x}{(1 + |w|^2)^2}, \\ w(\bar{x}, 0) &= w_0(\bar{x}), \quad \bar{x} = (x, y), \quad w(\bar{x}) \rightarrow 0, \quad \text{as } |\bar{x}| \rightarrow \infty. \end{aligned} \tag{A1}$$

*Step 1.* Recall the equation in covariant derivatives

$$i \frac{\partial s}{\partial t} = (\nabla_1 - iv_1) \frac{\partial s}{\partial x_1} - (\nabla_2 - iv_2) \frac{\partial s}{\partial x_2},$$

where  $(x_1, x_2) = (x, y)$  and  $(v_1, v_2) = (\phi_{x_2}, -\phi_{x_1})$ . And consider a ‘parabolic regularization’ of the form,

$$i \frac{\partial s}{\partial t} = (\nabla_1 - iv_1) \frac{\partial s}{\partial x_1} - (\nabla_2 - iv_2) \frac{\partial s}{\partial x_2} + i\epsilon \left( \nabla_1 \frac{\partial s}{\partial x_1} + \nabla_2 \frac{\partial s}{\partial x_2} \right). \tag{A2}$$

<sup>7</sup> They consider Schrödinger maps into a *hermitian symmetric spaces*  $\mathcal{M}^{2d}$  with first cohomology group trivial and show that a fairly rough solution to the Schrödinger map system which is the weak limit of smooth solutions solves the associated Coulomb gauged frame system in a unique fashion and vice-versa.

After a few calculations in the spirit of those by Soyeur [40] we express the latter in stereographic coordinates as

$$\begin{aligned} w_t - i(w_{xx} - w_{yy}) - \epsilon(w_{xx} + w_{yy}) &= -i \frac{2\bar{w}}{1+|w|^2} (w_x^2 - w_y^2) + \frac{2\epsilon\bar{w}}{1+|w|^2} (w_x^2 + w_y^2) \\ &\quad + (1+i\epsilon)(\phi_x w_y + \phi_y w_x), \\ \phi_{xx} + \phi_{yy} &= 4\kappa i \frac{w_x \bar{w}_y - w_y \bar{w}_x}{(1+|w|^2)^2}, \end{aligned} \quad (\text{A3})$$

$$w(\bar{x}, 0) = w_0(\bar{x}), \quad \bar{x} = (x, y), \quad w(\bar{x}) \rightarrow 0, \quad \text{as } |\bar{x}| \rightarrow \infty,$$

we write in short

$$w_t - i(w_{xx} - w_{yy}) - \epsilon(w_{xx} + w_{yy}) = F + G, \quad (\text{A4})$$

where

$$F(w) = -i \frac{2\bar{w}}{1+|w|^2} (w_x^2 - w_y^2) + \frac{2\epsilon\bar{w}}{1+|w|^2} (w_x^2 + w_y^2) \quad \text{and} \quad G = (1+i\epsilon)(\phi_x w_y + \phi_y w_x)$$

are the local and the non-local terms, respectively, on the right-hand side of (A3). Moreover for  $u \in H^2(\mathbb{R}^2)$ ,

$$\Delta\phi = 4i \left[ \left( \frac{\bar{w}w_x}{1+|w|^2} \right)_y - \left( \frac{\bar{w}w_y}{1+|w|^2} \right)_x \right].$$

We can now state two lemmas in Soyeur's [40] for which we rely on.

**Lemma B (lemma 1.5 in [40]).** *Let  $k \geq 2$ . Then  $F$  and  $G$  are Lipschitz maps on bounded sets of  $H^{k+2}(\mathbb{R}^2)$  with values into  $H^k(\mathbb{R}^2)$ , and moreover they satisfy the inequalities*

$$\begin{aligned} \|F(u)\|_{H^{k-1}} + \|G(u)\|_{H^{k-1}} &\leq C \|u\|_{W^{2,6}}^2 \|u\|_{H^k}, \\ \|F(u)\|_{W^{k+2,6/5}} + \|G(u)\|_{W^{k+2,6/5}} &\leq C \|u\|_{W^{k/2+2,6}}^2 \|u\|_{H^{k+3}}. \end{aligned}$$

Thus we can apply the following:

**Lemma C (lemma 2.1 in [40]; cf [11]).** *For any  $w_0 \in H^k(\mathbb{R}^2)$ ,  $k > 1$  equation (A4) possesses a unique solution  $w^\epsilon \in C([0, T_\epsilon]; H^k)$ . If the maximal time of existence  $T_\epsilon$  is finite, we have the blow up condition*

$$\limsup_{t \rightarrow T_\epsilon} \|w^\epsilon(t)\|_{H^k} = +\infty.$$

*Step 2.* Proceeding as in section 2, with the parabolic regularization to the Ishimori system (A3); we obtain an ' $\epsilon$ -modified' Ishimori equation of the form

$$w_t - i(w_{xx} - w_{yy}) - \epsilon(w_{xx} + w_{yy}) + 2\delta(aw) + 2i\epsilon\delta(aw) = H, \quad (\text{A5})$$

where  $H(w)$  contains local and nonlocal cubic and 'quintic' terms in  $u$  (all scaling like  $|u|^2 u$  in two dimensions).

*Step 3.* We next prove an energy estimate for (A5) in the spirit of the first estimate in proposition 1, section 4. We have in the notation of proposition 1.

**Lemma D.** *Let  $w^\epsilon$  be a classical smooth solution to (A5) guaranteed to exist on an interval  $[0, T^\epsilon)$  by lemma B. Then for  $s \geq 0$  there exists an absolute constant  $C_s > 0$ , such that for  $t \in [0, T^\epsilon)$*

$$\begin{aligned} \frac{\partial}{\partial t} \|w^\epsilon(t)\|_{H^s}^2 &\leq C_s (\|\nabla a^\epsilon(t)\|_{H^s} \|w^\epsilon(t)\|_{H^s} \|w^\epsilon(t)\|_{L^\infty} + \|\nabla a^\epsilon(t)\|_{L^\infty} \|w^\epsilon(t)\|_{H^s}^2 \\ &\quad + \|a^\epsilon\|_{L^\infty}^2 \|w^\epsilon(t)\|_{H^s}^2 + \|H(t)\|_{H^s} \|w^\epsilon(t)\|_{H^s}). \end{aligned}$$

**Proof.** The proof proceeds exactly as that of proposition 1 which is actually in [32]. The only exception is the estimate in the present situation of the term corresponding to the ‘low in  $a$ ’ portion in  $2i\epsilon\delta(aw)$  since now we have an ‘ $i$ ’ in front. We point out how to handle it. In what follows we will abuse notation and drop the superscript  $\epsilon$  in  $w^\epsilon$ ; that is, we will write just  $w$  for  $w^\epsilon$ . After applying the Littlewood–Paley projection operator  $Q_l$  as in [32] to both sides of the equation we obtain an equation for  $w^l := w_l^\epsilon$  of the form

$$\begin{aligned} \partial_t w_l - i\Box_{xy} w_l - \epsilon \Delta w_l &= -2\delta(a_{<l-5} w_l) - 2i\epsilon\delta(a_{<l-5} w_l) \\ &\quad - 2(1+i\epsilon)\delta([Q_l, a_{<l-5}]w_{l-2\ll l+2}) - 2(1+i\epsilon)\delta(Q_l(a_{>l-5})w) + H_l. \end{aligned}$$

Next we multiply both sides by  $2^{2sl}\bar{w}_l$  integrate and take the real part. On the left-hand side, after integration by parts we obtain

$$\partial_t \|w_l\|_{\dot{H}^s}^2 + \epsilon 2^{2sl} \int |\nabla w_l|^2 dx.$$

On the other hand of the equation we have the sum of five terms we proceed to discuss. The estimates in [32] proceed exactly in the same manner for the last 3 terms, since the presence of the constant  $2(1+i\epsilon)$  in front does not affect the arguments in [32] at all. Similarly, the first term can be estimated just as in [32]. Indeed, recall we have

$$2^{2ls} \operatorname{Re} \int \delta(a_{<l-5} w_l) \bar{w}_l dx = 2^{2ls} \operatorname{Re} \int \delta(a_{<l-5}) |w_l|^2 dx + 2^{2ls} \operatorname{Re} \int a_{<l-5} \cdot \nabla w_l \bar{w}_l dx.$$

On the other hand, an integration by parts yields

$$\int a_{<l-5} \cdot \nabla w_l \bar{w}_l dx = - \int \delta(a_{<l-5}) |w_l|^2 dx - \int a_{<l-5} w_l \cdot \nabla \bar{w}_l dx,$$

whence

$$\operatorname{Re} \int a_{<l-5} \cdot \nabla w_l \bar{w}_l dx = -\frac{1}{2} \int \delta(a_{<l-5}) |w_l|^2 dx.$$

We get

$$2^{2ls} \operatorname{Re} \int a_{<l-5} \cdot \nabla w_l \bar{w}_l dx = -\frac{2^{2ls}}{2} \int \delta(a_{<l-5}) |w_l|^2 dx.$$

Thus by Hölder’s inequality we have

$$2^{2ls} \left| \operatorname{Re} \int \delta(a_{<l-5} w_l) \bar{w}_l dx \right| \leq C \|w_l\|_{\dot{H}^s}^2 \|\nabla a\|_{L^\infty},$$

where when  $s = 0$   $\dot{H}^s$  just means  $L^2$ .

It is clear from the above that the term  $2\epsilon 2^{2ls} \operatorname{Re} i \int \delta(a_{<l-5} w_l) \bar{w}_l dx$  cannot be handled in the same manner. Instead we proceed as follows.

$$2\epsilon 2^{2ls} \operatorname{Re} i \int \delta(a_{<l-5} w_l) \bar{w}_l dx = 2\epsilon 2^{2sl} \operatorname{Re} i \int \delta(a_{<l-5}) |w_l|^2 dx + 2\epsilon 2^{2ls} \operatorname{Re} i \int a_{<l-5} \cdot \nabla w_l \bar{w}_l dx.$$

By Hölder the first term is once again bounded by  $\|w_l\|_{\dot{H}^s}^2 \|\nabla a\|_{L^\infty}$ . For the second term we note that

$$\begin{aligned} \left| 2\epsilon i 2^{2sl} \int a_{<l-5} \cdot \nabla w_l \bar{w}_l dx \right| &\leq \|a\|_{L^\infty} \int |\nabla w_l| |w_l| dx \\ &\leq 2\epsilon 2^{2sl} \|a\|_{L^\infty} \|\nabla w_l\|_{L^2} \|w_l\|_{L^2} \\ &\leq 8\epsilon \|a\|_{L^\infty}^2 \|w_l\|_{\dot{H}^s}^2 + \frac{\epsilon}{2} 2^{2sl} \|\nabla w_l\|_{L^2}^2. \end{aligned}$$

By passing the second term in the last inequality to the left-hand side we have the desired estimate.  $\square$

**Remark.** Note that similarly to the proof of lemma 3, we have that for  $s > 0$ ,

$$\|a^\epsilon(t)\|_{L^\infty}^2 \lesssim \|w^\epsilon(t)\|_{L^\infty}^2 \|w^\epsilon(t)\|_{H^s}^2.$$

*Step 4.* We note that without any loss of generality the *a priori* estimate in proposition 4 holds for the smooth local solutions  $w^\epsilon$  to (A5) and  $T := \min\{T_\epsilon, 1\}$  where  $T_\epsilon$  is the one guaranteed by lemma C. This is because the dispersive estimates in lemma 2 still hold for  $L = -i\Box_{xy} - \epsilon\Delta$ . Thus proceeding as in section 4 and 5, we have

**Theorem E.** For any  $\epsilon > 0$ ,  $w^\epsilon$ , the local solution to (A5) produced by lemma C lives on a time interval  $[0, T)$  where  $T = T(\|w_0\|_{H^{1/2+}}) > 0$  independent of  $\epsilon$ . Moreover, we have that

$$w^\epsilon \in C([0, T], H^m) \cap L_t^2([0, T])L_x^\infty.$$

Thus up to a subsequence we can pass to the limit in equation (A5) to find an  $H^m$  smooth solution  $w$  ( $m \geq 4$ ), to equation (17) or in other words to

$$w_t - i(w_{xx} - w_{yy}) - + 2\delta(aw) = H, \quad (\text{A6})$$

where  $H$  is as in (17). By the result in Soyeur [40] we know such solution is unique.

Moreover, by the energy estimates for differences in proposition 1 we have this smooth solution that  $w$  is unique in  $C([0, T], H^m)$ , ( $m \geq 4$ ).

Now the argument in section 5 applies to give the desired local in time solution to the Ishimori system in  $C([0, T]; H^{3/2+}) \cap L_{[0, T]}^2 L_x^\infty$ .

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## QUERIES

### Page 3

AQ1

Please confirm if the change in reference number [45] to [40] in the sentence 'For the hyperbolic–elliptic Ishimori system there is Soyeur's [40]...' is ok?

### Page 7

AQ2

Please check if comma is required before '+' symbol.

### Page 9

AQ3

Please provide equations (18) and (20) which is missing in the text.

### Page 11

AQ4

Please check closing curly brackets for the equation.

### Page 22

AQ5

Please provide initials for second author in Ref. [4].

### Page 22

AQ6

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### Page 22

AQ7

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### Page 23

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### Page 23

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