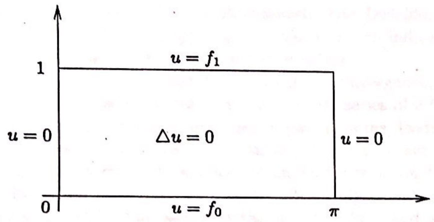


Figure 11. Dirichlet problem in a rectangle

4 Problem

1. Consider the Dirichlet problem illustrated in Figure 11.

More precisely, we look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x, y) : 0 \leq x \leq \pi, \ 0 \leq y \leq 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \quad \text{and} \quad u(x, 1) = f_1(x),$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle.

Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \quad \text{and} \quad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Compare this result with the solution of the Dirichlet problem in the strip obtained in Problem 3, Chapter 5.

2 Basic Properties of Fourier Series

> Nearly fifty years had passed without any progress on the question of analytic representation of an arbitrary function, when an assertion of Fourier threw new light on the subject. Thus a new era began for the development of this part of Mathematics and this was heralded in a stunning way by major developments in mathematical Physics.
>
> *B. Riemann*, 1854

In this chapter, we begin our rigorous study of Fourier series. We set the stage by introducing the main objects in the subject, and then formulate some basic problems which we have already touched upon earlier.

Our first result disposes of the question of uniqueness: Are two functions with the same Fourier coefficients necessarily equal? Indeed, a simple argument shows that if both functions are continuous, then in fact they must agree.

Next, we take a closer look at the partial sums of a Fourier series. Using the formula for the Fourier coefficients (which involves an integration), we make the key observation that these sums can be written conveniently as integrals:

$$\frac{1}{2\pi} \int D_N(x - y) f(y) \, dy,$$

where $\{D_N\}$ is a family of functions called the Dirichlet kernels. The above expression is the convolution of f with the function D_N. Convolutions will play a critical role in our analysis. In general, given a family of functions $\{K_n\}$, we are led to investigate the limiting properties as n tends to infinity of the convolutions

$$\frac{1}{2\pi} \int K_n(x - y) f(y) \, dy.$$

We find that if the family $\{K_n\}$ satisfies the three important properties of "good kernels," then the convolutions above tend to $f(x)$ as $n \to \infty$ (at least when f is continuous). In this sense, the family $\{K_n\}$ is an "approximation to the identity." Unfortunately, the Dirichlet kernels D_N do not belong to the category of good kernels, which indicates that the question of convergence of Fourier series is subtle.

Instead of pursuing at this stage the problem of convergence, we consider various other methods of summing the Fourier series of a function. The first method, which involves averages of partial sums, leads to convolutions with good kernels, and yields an important theorem of Fejér. From this, we deduce the fact that a continuous function on the circle can be approximated uniformly by trigonometric polynomials. Second, we may also sum the Fourier series in the sense of Abel and again encounter a family of good kernels. In this case, the results about convolutions and good kernels lead to a solution of the Dirichlet problem for the steady-state heat equation in the disc, considered at the end of the previous chapter.

1 Examples and formulation of the problem

We commence with a brief description of the types of functions with which we shall be concerned. Since the Fourier coefficients of f are defined by

$$a_n = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}\,dx, \qquad \text{for } n \in \mathbb{Z},$$

where f is complex-valued on $[0, L]$, it will be necessary to place some integrability conditions on f. We shall therefore assume for the remainder of this book that all functions are at least Riemann integrable.[1] Sometimes it will be illuminating to focus our attention on functions that are more "regular," that is, functions that possess certain continuity or differentiability properties. Below, we list several classes of functions in increasing order of generality. We emphasize that we will not generally restrict our attention to real-valued functions, contrary to what the following pictures may suggest; we will almost always allow functions that take values in the complex numbers $\mathbb{C}$. Furthermore, we sometimes think of our functions as being defined on the circle rather than an interval. We elaborate upon this below.

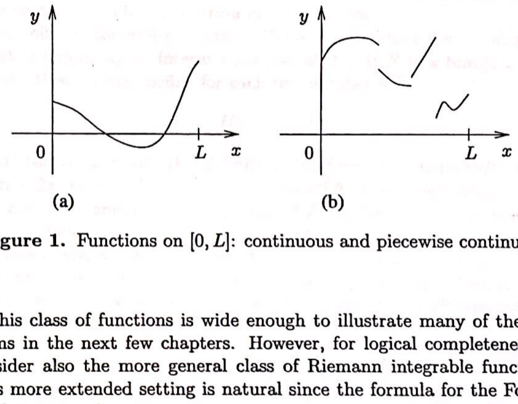
[1]Limiting ourselves to Riemann integrable functions is natural at this elementary stage of study of the subject. The more advanced notion of Lebesgue integrability will be taken up in Book III.

Everywhere continuous functions

These are the complex-valued functions f which are continuous at every point of the segment $[0, L]$. A typical continuous function is sketched in Figure 1 (a). We shall note later that continuous functions on the circle satisfy the additional condition $f(0) = f(L)$.

Piecewise continuous functions

These are bounded functions on $[0, L]$ which have only finitely many discontinuities. An example of such a function with simple discontinuities is pictured in Figure 1 (b).



(a) (b)

Figure 1. Functions on $[0, L]$: continuous and piecewise continuous

This class of functions is wide enough to illustrate many of the theorems in the next few chapters. However, for logical completeness we consider also the more general class of Riemann integrable functions. This more extended setting is natural since the formula for the Fourier coefficients involves integration.

Riemann integrable functions

This is the most general class of functions we will be concerned with. Such functions are bounded, but may have infinitely many discontinuities. We recall the definition of integrability. A real-valued function f defined on $[0, L]$ is **Riemann integrable** (which we abbreviate as **integrable**[2]) if it is *bounded*, and if for every $\epsilon > 0$, there is a subdivision $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = L$ of the interval $[0, L]$, so that if $\mathcal{U}$

[2]Starting in Book III, the term "integrable" will be used in the broader sense of Lebesgue theory.

and $\mathcal{L}$ are, respectively, the upper and lower sums of f for this subdivision, namely

$$\mathcal{U} = \sum_{j=1}^{N} [\sup_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

and

$$\mathcal{L} = \sum_{j=1}^{N} [\inf_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1}),$$

then we have $\mathcal{U} - \mathcal{L} < \epsilon$. Finally, we say that a complex-valued function is integrable if its real and imaginary parts are integrable. It is worthwhile to remember at this point that the sum and product of two integrable functions are integrable.

A simple example of an integrable function on $[0, 1]$ with infinitely many discontinuities is given by

$$f(x) = \begin{cases} 1 & \text{if } 1/(n+1) < x \leq 1/n \text{ and } n \text{ is odd,} \\ 0 & \text{if } 1/(n+1) < x \leq 1/n \text{ and } n \text{ is even,} \\ 0 & \text{if } x = 0. \end{cases}$$

This example is illustrated in Figure 2. Note that f is discontinuous when $x = 1/n$ and at $x = 0$.

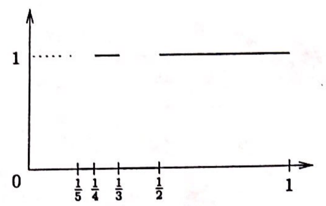


Figure 2. A Riemann integrable function

More elaborate examples of integrable functions whose discontinuities are dense in the interval $[0, 1]$ are described in Problem 1. In general, while integrable functions may have infinitely many discontinuities, these functions are actually characterized by the fact that, in a precise sense, their discontinuities are not too numerous: they are "negligible," that is, the set of points where an integrable function is discontinuous has "measure 0." The reader will find further details about Riemann integration in the appendix.

From now on, we shall always assume that our functions are integrable, even if we do not state this requirement explicitly.

Functions on the circle

There is a natural connection between 2π-periodic functions on $\mathbb{R}$ like the exponentials $e^{in\theta}$, functions on an interval of length 2π, and functions on the unit circle. This connection arises as follows.

A point on the unit circle takes the form $e^{i\theta}$, where θ is a real number that is unique up to integer multiples of 2π. If F is a function on the circle, then we may define for each real number θ

$$f(\theta) = F(e^{i\theta}),$$

and observe that with this definition, the function f is periodic on $\mathbb{R}$ of period 2π, that is, $f(\theta + 2\pi) = f(\theta)$ for all θ. The integrability, continuity and other smoothness properties of F are determined by those of f. For instance, we say that F is integrable on the circle if f is integrable on every interval of length 2π. Also, F is continuous on the circle if f is continuous on $\mathbb{R}$, which is the same as saying that f is continuous on any interval of length 2π. Moreover, F is continuously differentiable if f has a continuous derivative, and so forth.

Since f has period 2π, we may restrict it to any interval of length 2π, say $[0, 2\pi]$ or $[-\pi, \pi]$, and still capture the initial function F on the circle. We note that f must take the same value at the end-points of the interval since they correspond to the same point on the circle. Conversely, any function on $[0, 2\pi]$ for which $f(0) = f(2\pi)$ can be extended to a periodic function on $\mathbb{R}$ which can then be identified as a function on the circle. In particular, a continuous function f on the interval $[0, 2\pi]$ gives rise to a continuous function on the circle if and only if $f(0) = f(2\pi)$.

In conclusion, functions on $\mathbb{R}$ that 2π-periodic, and functions on an interval of length 2π that take on the same value at its end-points, are two equivalent descriptions of the same mathematical objects, namely, functions on the circle.

In this connection, we mention an item of notational usage. When our functions are defined on an interval on the line, we often use x as the independent variable; however, when we consider these as functions on the circle, we usually replace the variable x by θ. As the reader will note, we are not strictly bound by this rule since this practice is mostly a matter of convenience.

1.1 Main definitions and some examples

We now begin our study of Fourier analysis with the precise definition of the Fourier series of a function. Here, it is important to pin down where our function is originally defined. If f is an integrable function given on an interval $[a,b]$ of length L (that is, $b-a=L$), then the n^{th} **Fourier coefficient** of f is defined by

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x)e^{-2\pi inx/L}\,dx, \qquad n \in \mathbb{Z}.$$

The **Fourier series** of f is given formally[3] by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx/L}.$$

We shall sometimes write a_n for the Fourier coefficients of f, and use the notation

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx/L}$$

to indicate that the series on the right-hand side is the Fourier series of f.

For instance, if f is an integrable function on the interval $[-\pi,\pi]$, then the n^{th} Fourier coefficient of f is

$$\hat{f}(n) = a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta, \qquad n \in \mathbb{Z},$$

and the Fourier series of f is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

Here we use θ as a variable since we think of it as an angle ranging from $-\pi$ to π.

[3]At this point, we do not say anything about the convergence of the series.

Also, if f is defined on $[0,2\pi]$, then the formulas are the same as above, except that we integrate from 0 to 2π in the definition of the Fourier coefficients.

We may also consider the Fourier coefficients and Fourier series for a function defined on the circle. By our previous discussion, we may think of a function on the circle as a function f on $\mathbb{R}$ which is 2π-periodic. We may restrict the function f to any interval of length 2π, for instance $[0,2\pi]$ or $[-\pi,\pi]$, and compute its Fourier coefficients. Fortunately, f is *periodic* and Exercise 1 shows that the resulting integrals are independent of the chosen interval. Thus the Fourier coefficients of a function on the circle are well defined.

Finally, we shall sometimes consider a function g given on $[0,1]$. Then

$$\hat{g}(n) = a_n = \int_0^1 g(x)e^{-2\pi inx}\,dx \quad \text{and} \quad g(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}.$$

Here we use x for a variable ranging from 0 to 1.

Of course, if f is initially given on $[0,2\pi]$, then $g(x) = f(2\pi x)$ is defined on $[0,1]$ and a change of variables shows that the n^{th} Fourier coefficient of f equals the n^{th} Fourier coefficient of g.

Fourier series are part of a larger family called the **trigonometric series** which, by definition, are expressions of the form $\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$ where $c_n \in \mathbb{C}$. If a trigonometric series involves only finitely many non-zero terms, that is, $c_n = 0$ for all large $|n|$, it is called a **trigonometric polynomial**; its **degree** is the largest value of $|n|$ for which $c_n \neq 0$.

The N^{th} **partial sum** of the Fourier series of f, for N a positive integer, is a particular example of a trigonometric polynomial. It is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx/L}.$$

Note that by definition, the above sum is *symmetric* since n ranges from $-N$ to N, a choice that is natural because of the resulting decomposition of the Fourier series as sine and cosine series. As a consequence, the convergence of Fourier series will be understood (in this book) as the "limit" as N tends to infinity of these symmetric sums.

In fact, using the partial sums of the Fourier series, we can reformulate the basic question raised in Chapter 1 as follows:

Problem: In what sense does $S_N(f)$ converge to f as $N \to \infty$?

Before proceeding further with this question, we turn to some simple examples of Fourier series.

EXAMPLE 1. Let $f(\theta) = \theta$ for $-\pi \leq \theta \leq \pi$. The calculation of the Fourier coefficients requires a simple integration by parts. First, if $n \neq 0$, then

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta \\ &= \frac{1}{2\pi}\left[-\frac{\theta}{in}e^{-in\theta}\right]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-in\theta}\, d\theta \\ &= \frac{(-1)^{n+1}}{in},\end{aligned}$$

and if $n = 0$ we clearly have

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\, d\theta = 0.$$

Hence, the Fourier series of f is given by

$$f(\theta) \sim \sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\theta}{n}.$$

The first sum is over all non-zero integers, and the second is obtained by an application of Euler's identities. It is possible to prove by elementary means that the above series converges for every θ, but it is not obvious that it converges to $f(\theta)$. This will be proved later (Exercises 8 and 9 deal with a similar situation).

EXAMPLE 2. Define $f(\theta) = (\pi - \theta)^2/4$ for $0 \leq \theta \leq 2\pi$. Then successive integration by parts similar to that performed in the previous example yield

$$f(\theta) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}.$$

EXAMPLE 3. The Fourier series of the function

$$f(\theta) = \frac{\pi}{\sin \pi\alpha} e^{i(\pi-\theta)\alpha}$$

on $[0, 2\pi]$ is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{n+\alpha},$$

whenever α is not an integer.

EXAMPLE 4. The trigonometric polynomial defined for $x \in [-\pi, \pi]$ by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

is called the N^{th} **Dirichlet kernel** and is of fundamental importance in the theory (as we shall see later). Notice that its Fourier coefficients a_n have the property that $a_n = 1$ if $|n| \leq N$ and $a_n = 0$ otherwise. A closed form formula for the Dirichlet kernel is

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}.$$

This can be seen by summing the geometric progressions

$$\sum_{n=0}^{N} \omega^n \quad \text{and} \quad \sum_{n=-N}^{-1} \omega^n$$

with $\omega = e^{ix}$. These sums are, respectively, equal to

$$\frac{1-\omega^{N+1}}{1-\omega} \quad \text{and} \quad \frac{\omega^{-N}-1}{1-\omega}.$$

Their sum is then

$$\frac{\omega^{-N}-\omega^{N+1}}{1-\omega} = \frac{\omega^{-N-1/2}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}} = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)},$$

giving the desired result.

EXAMPLE 5. The function $P_r(\theta)$, called the **Poisson kernel**, is defined for $\theta \in [-\pi, \pi]$ and $0 \leq r < 1$ by the absolutely and uniformly convergent series

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

This function arose implicitly in the solution of the steady-state heat equation on the unit disc discussed in Chapter 1. Note that in calculating the Fourier coefficients of $P_r(\theta)$ we can interchange the order of integration and summation since the sum converges uniformly in θ for each fixed r, and obtain that the n^{th} Fourier coefficient equals $r^{|n|}$. One can also sum the series for $P_r(\theta)$ and see that

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

In fact,

$$P_r(\theta) = \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega}^n \quad \text{with } \omega = re^{i\theta},$$

where both series converge absolutely. The first sum (an infinite geometric progression) equals $1/(1-\omega)$, and likewise, the second is $\overline{\omega}/(1-\overline{\omega})$. Together, they combine to give

$$\frac{1-\overline{\omega}+(1-\omega)\overline{\omega}}{(1-\omega)(1-\overline{\omega})} = \frac{1-|\omega|^2}{|1-\omega|^2} = \frac{1-r^2}{1-2r\cos\theta+r^2},$$

as claimed. The Poisson kernel will reappear later in the context of Abel summability of the Fourier series of a function.

Let us return to the problem formulated earlier. The definition of the Fourier series of f is purely formal, and it is not obvious whether it converges to f. In fact, the solution of this problem can be very hard, or relatively easy, depending on the sense in which we expect the series to converge, or on what additional restrictions we place on f.

Let us be more precise. Suppose, for the sake of this discussion, that the function f (which is always assumed to be Riemann integrable) is defined on $[-\pi,\pi]$. The first question one might ask is whether the partial sums of the Fourier series of f converge to f pointwise. That is, do we have

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta) \quad \text{for every } \theta? \tag{1}$$

We see quite easily that in general we cannot expect this result to be true at every θ, since we can always change an integrable function at one point without changing its Fourier coefficients. As a result, we might ask the same question assuming that f is continuous and periodic. For a long time it was believed that under these additional assumptions the answer would be "yes." It was a surprise when Du Bois-Reymond showed that there exists a continuous function whose Fourier series diverges at a point. We will give such an example in the next chapter. Despite this negative result, we might ask what happens if we add more smoothness conditions on f: for example, we might assume that f is continuously

differentiable, or twice continuously differentiable. We will see that then the Fourier series of f converges to f uniformly.

We will also interpret the limit (1) by showing that the Fourier series sums, in the sense of Cesàro or Abel, to the function f at all of its points of continuity. This approach involves appropriate averages of the partial sums of the Fourier series of f.

Finally, we can also define the limit (1) in the mean square sense. In the next chapter, we will show that if f is merely integrable, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |S_N(f)(\theta) - f(\theta)|^2\, d\theta \to 0 \quad \text{as } N\to\infty.$$

It is of interest to know that the problem of pointwise convergence of Fourier series was settled in 1966 by L. Carleson, who showed, among other things, that if f is integrable in our sense,[4] then the Fourier series of f converges to f except possibly on a set of "measure 0." The proof of this theorem is difficult and beyond the scope of this book.

2 Uniqueness of Fourier series

If we were to assume that the Fourier series of functions f converge to f in an appropriate sense, then we could infer that a function is uniquely determined by its Fourier coefficients. This would lead to the following statement: if f and g have the same Fourier coefficients, then f and g are necessarily equal. By taking the difference $f-g$, this proposition can be reformulated as: if $\hat{f}(n)=0$ for all $n\in\mathbb{Z}$, then $f=0$. As stated, this assertion cannot be correct without reservation, since calculating Fourier coefficients requires integration, and we see that, for example, any two functions which differ at finitely many points have the same Fourier series. However, we do have the following positive result.

Theorem 2.1 *Suppose that f is an integrable function on the circle with $\hat{f}(n)=0$ for all $n\in\mathbb{Z}$. Then $f(\theta_0)=0$ whenever f is continuous at the point θ_0.*

Thus, in terms of what we know about the set of discontinuities of integrable functions,[5] we can conclude that f vanishes for "most" values of θ.

Proof. We suppose first that f is real-valued, and argue by contradiction. Assume, without loss of generality, that f is defined on

Skip proof

[4] Carleson's proof actually holds for the wider class of functions which are square integrable in the Lebesgue sense.

[5] See the appendix.

$[-\pi, \pi]$, that $\theta_0 = 0$, and $f(0) > 0$. The idea now is to construct a family of trigonometric polynomials $\{p_k\}$ that "peak" at 0, and so that $\int p_k(\theta) f(\theta)\, d\theta \to \infty$ as $k \to \infty$. This will be our desired contradiction since these integrals are equal to zero by assumption.

Since f is continuous at 0, we can choose $0 < \delta \leq \pi/2$, so that $f(\theta) > f(0)/2$ whenever $|\theta| < \delta$. Let

$$p(\theta) = \epsilon + \cos\theta,$$

where $\epsilon > 0$ is chosen so small that $|p(\theta)| < 1 - \epsilon/2$, whenever $\delta \leq |\theta| \leq \pi$. Then, choose a positive η with $\eta < \delta$, so that $p(\theta) \geq 1 + \epsilon/2$, for $|\theta| < \eta$. Finally, let

$$p_k(\theta) = [p(\theta)]^k,$$

and select B so that $|f(\theta)| \leq B$ for all θ. This is possible since f is integrable, hence bounded. Figure 3 illustrates the family $\{p_k\}$. By

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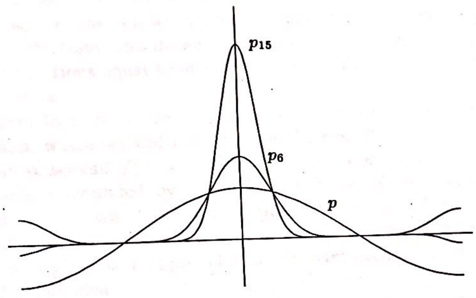

Figure 3. The functions p, p_6, and p_{15} when $\epsilon = 0.1$

construction, each p_k is a trigonometric polynomial, and since $\hat{f}(n) = 0$ for all n, we must have

$$\int_{-\pi}^{\pi} f(\theta) p_k(\theta)\, d\theta = 0 \quad \text{for all } k.$$

However, we have the estimate

$$\left| \int_{\delta \leq |\theta|} f(\theta) p_k(\theta)\, d\theta \right| \leq 2\pi B (1 - \epsilon/2)^k.$$

Also, our choice of δ guarantees that $p(\theta)$ and $f(\theta)$ are non-negative whenever $|\theta| < \delta$, thus

$$\int_{\eta \leq |\theta| < \delta} f(\theta) p_k(\theta)\, d\theta \geq 0.$$

Finally,

$$\int_{|\theta| < \eta} f(\theta) p_k(\theta)\, d\theta \geq 2\eta \frac{f(0)}{2} (1 + \epsilon/2)^k.$$

Therefore, $\int p_k(\theta) f(\theta)\, d\theta \to \infty$ as $k \to \infty$, and this concludes the proof when f is real-valued. In general, write $f(\theta) = u(\theta) + iv(\theta)$, where u and v are real-valued. If we define $\overline{f}(\theta) = \overline{f(\theta)}$, then

$$u(\theta) = \frac{f(\theta) + \overline{f}(\theta)}{2} \quad \text{and} \quad v(\theta) = \frac{f(\theta) - \overline{f}(\theta)}{2i},$$

and since $\hat{\overline{f}}(n) = \overline{\hat{f}(-n)}$, we conclude that the Fourier coefficients of u and v all vanish, hence $f = 0$ at its points of continuity. The idea of constructing a family of functions (trigonometric polynomials in this case) which peak at the origin, together with other nice properties, will play an important role in this book. Such families of functions will be taken up later in Section 4 in connection with the notion of convolution. For now, note that the above theorem implies the following.

Corollary 2.2 *If f is continuous on the circle and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.*

The next corollary shows that the problem (1) formulated earlier has a simple positive answer under the assumption that the series of Fourier coefficients converges absolutely.

Yes

Corollary 2.3 *Suppose that f is a continuous function on the circle and that the Fourier series of f is absolutely convergent, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then, the Fourier series converges uniformly to f, that is,*

$$\lim_{N \to \infty} S_N(f)(\theta) = f(\theta) \quad \textit{uniformly in } \theta.$$

Proof. Recall that if a sequence of continuous functions converges uniformly, then the limit is also continuous. Now observe that the assumption $\sum |\hat{f}(n)| < \infty$ implies that the partial sums of the Fourier

skip proof

series of f converge absolutely and uniformly, and therefore the function g defined by

$$g(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}$$

SKIP

is continuous on the circle. Moreover, the Fourier coefficients of g are precisely $\hat{f}(n)$ since we can interchange the infinite sum with the integral (a consequence of the uniform convergence of the series). Therefore, the previous corollary applied to the function $f - g$ yields $f = g$, as desired.

What conditions on f would guarantee the absolute convergence of its Fourier series? As it turns out, the smoothness of f is directly related to the decay of the Fourier coefficients, and in general, the smoother the function, the faster this decay. As a result, we can expect that relatively smooth functions equal their Fourier series. This is in fact the case, as we now show.

In order to state the result concisely we introduce the standard **"O" notation**, which we will use freely in the rest of this book. For example, the statement $\hat{f}(n) = O(1/|n|^2)$ as $|n| \to \infty$, means that the left-hand side is bounded by a constant multiple of the right-hand side; that is, there exists $C > 0$ with $|\hat{f}(n)| \le C/|n|^2$ for all large $|n|$. More generally, $f(x) = O(g(x))$ as $x \to a$ means that for some constant C, $|f(x)| \le C|g(x)|$ as x approaches a. In particular, $f(x) = O(1)$ means that f is bounded.

Corollary 2.4 *Suppose that f is a twice continuously differentiable function on the circle. Then*

$$\hat{f}(n) = O(1/|n|^2) \quad \text{as } |n| \to \infty,$$

so that the Fourier series of f converges absolutely and uniformly to f.

Proof. The estimate on the Fourier coefficients is proved by integrating by parts twice for $n \neq 0$. We obtain

$$\begin{aligned}
2\pi\hat{f}(n) &= \int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta \\
&= \left[f(\theta)\cdot\frac{-e^{-in\theta}}{in}\right]_0^{2\pi} + \frac{1}{in}\int_0^{2\pi} f'(\theta)e^{-in\theta}\,d\theta \\
&= \frac{1}{in}\int_0^{2\pi} f'(\theta)e^{-in\theta}\,d\theta \\
&= \frac{1}{in}\left[f'(\theta)\cdot\frac{-e^{-in\theta}}{in}\right]_0^{2\pi} + \frac{1}{(in)^2}\int_0^{2\pi} f''(\theta)e^{-in\theta}\,d\theta \\
&= \frac{-1}{n^2}\int_0^{2\pi} f''(\theta)e^{-in\theta}\,d\theta.
\end{aligned}$$

The quantities in brackets vanish since f and f' are periodic. Therefore

$$2\pi|n|^2|\hat{f}(n)| \le \left|\int_0^{2\pi} f''(\theta)e^{-in\theta}\,d\theta\right| \le \int_0^{2\pi} |f''(\theta)|\,d\theta \le C,$$

where the constant C is independent of n. (We can take $C = 2\pi B$ where B is a bound for f''.) Since $\sum 1/n^2$ converges, the proof of the corollary is complete.

Incidentally, we have also established the following important identity:

$$\widehat{f'}(n) = in\hat{f}(n), \qquad \text{for all } n \in \mathbb{Z}.$$

If $n \neq 0$ the proof is given above, and if $n = 0$ it is left as an exercise to the reader. So if f is differentiable and $f \sim \sum a_n e^{in\theta}$, then $f' \sim \sum a_n in e^{in\theta}$. Also, if f is twice continuously differentiable, then $f'' \sim \sum a_n (in)^2 e^{in\theta}$, and so on. Further smoothness conditions on f imply even better decay of the Fourier coefficients (Exercise 10).

There are also stronger versions of Corollary 2.4. It can be shown, for example, that the Fourier series of f converges absolutely, assuming only that f has one continuous derivative. Even more generally, the Fourier series of f converges absolutely (and hence uniformly to f) if f satisfies a **Hölder condition** of order α, with $\alpha > 1/2$, that is,

$$\sup_\theta |f(\theta + t) - f(\theta)| \le A|t|^\alpha \quad \text{for all } t.$$

For more on these matters, see the exercises at the end of Chapter 3.