Introd. to PDE (Grad. Course) Math 731 (FALL).

- **Overview and Introduction. Well Posedness notion.**

What is a PDE? The key defining property is that there is more than one independent variable $x, y, \dots$ (or $x_1, x_2, \dots$).

There is also a dependent variable that is an unknown function of these variables, i.e. $u(x, y, \dots)$ (or $u(x_1, x_2, \dots)$).

Then a PDE is an identity that relates the independent variables, the dependent variable u and the partial derivatives of u;

- Notation: Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $u : \Omega \to \mathbb{R}$ or $\mathbb{C}$ $\quad x = (x_1, x_2, \dots x_n) \in \Omega$

$$u_{x_k} = \frac{\partial u}{\partial x_k} = \lim_{h \to 0} \frac{u(x + h e_k) - u(x)}{h}$$

$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \text{ etc.}$$

A multi-index $\alpha := (\alpha_1, \dots \alpha_n)$ $\quad \alpha_i \in \mathbb{Z}_{\geq 0}$

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

Then $$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$ (Note if $\alpha = 0$ $D^\alpha u = u$)

$$= \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_m}^{\alpha_m} u$$

- For $k \in \mathbb{Z}_{\geq 0}$ $D^k u := \{ D^\alpha u : |\alpha| = k \}$

Ex: $Du = \nabla u$ gradient vector
$= (\partial_{x_1} u, \ldots \partial_{x_m} u)$

$$D^2 u = \text{Hessian matrix} = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_m} \\ & & \\ \frac{\partial^2 u}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_m^2} \end{pmatrix}$$

$$|D^k u| := \Big(\sum_{|\alpha| = k} |D^\alpha u|^2 \Big)^{1/2}$$

Special Second order Operator: Laplacean of u

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = tr(D^2 u)$$

LAPLACE OPERATOR

$$= \operatorname{div}(\nabla u) = \nabla \cdot \nabla u$$

Definition 1: Let $x \in \Omega \subset \mathbb{R}^n$ $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$)

not all need to appear!!

(PDE) $F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0$

highest order that appears

is called a k^{th} order PDE where $F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given and $u : \Omega \to \mathbb{R}$ is the unknown.

Ex classical sols. satisfy it identically. Also 'weak sols' etc.

Definition 2: u is said to be a solution if it satisfies the PDE above (in an appropriate sense) at least in some region of the $(x_1 \ldots x_n)$ variables.

Remarks: (i) u may not be unique. There may be many solutions. Hence to "solve" the PDE we need to find ALL such u (within some class of functions and possibly satisfying certain auxiliary boundary/initial conditions on $\Gamma = \partial\Omega$).

(ii) By finding the solutions we could mean either obtaining simple, explicit solutions or deducing the existence and other properties of the solutions (sometimes we can find a representation of solutions).

Classification: A PDE is said to be

(a) **LINEAR** if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

(f is given; $a_\alpha(x)$ also given - could be constants) $(|\alpha| \le k)$

If $f \equiv 0$ this linear PDE is said to be **HOMOGENEOUS** (Otherwise inhomogeneous).

(b) **SEMILINEAR**: if it has the form:

in terms of b/c. f(x) may be inside a_0.

given given → f(x) could be here

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0\left(D^{k-1}u, \dots, Du, u, x\right) = 0$$

a function of possible all variables (or some of the)
~~maybe~~ (nonlinear)

(c) <u>QUASILINEAR</u> if it has the form

$$\sum_{|\alpha|=k} a_\alpha\left(D^{k-1}u, \dots, Du, u, x\right) D^\alpha u +$$

$$+ a_0\left(D^{k-1}u, \dots Du, u, x\right) = 0$$

<u>EVEN</u> if this is a linear function!

f could be here.

(d) <u>FULLY NONLINEAR</u> if it depends nonlinearly upon the <u>highest</u> order derivative.

<u>Remark</u>: There is <u>no</u> general theory for solvability of <u>all</u> PDE! This is impossible as they describe a variety of diverse phenomena

<u>Remark</u>: Sometimes we distinguish one of the indep. variables as <u>time</u>. These are called EVOLUTION equations. In this case we write $u(x,t)$ instead of $u(x)$ $x \in \mathbb{R}^n$ $t \in \mathbb{R}$ or ≥ 0.

Examples:

Linear PDE : i) $\Delta u = 0$ on $\Omega \subseteq \mathbb{R}^n$ **Laplace Eq.**

(ii) Eigenvalue Eq. for Laplace : $-\Delta u = \lambda u$ $(\lambda \in \mathbb{R})$

(iii) Transport Eq : $\vec{B} \in \mathbb{R}^n$ $\vec{B} = (b_1, \dots b_n)$

NOTE : $(\nabla u, u_t) \cdot (B, 1) =$ $u_t + \underbrace{\vec{B} \cdot \nabla_x u}_{\sum_j B_j u_{x_j}} = 0$ $u = u(x,t)$

u is const on the line through (x,t) direct of $\vec{v} = (B, 1)$

$x \in \Omega \subseteq \mathbb{R}^n$

$t \geq 0$.

1-D Transport Eq. : $u_t + c\, u_x = 0$

$c =$ wave speed.

Remark: When $u = u(x,t)$ we will use ∇ for $\nabla_x = (\partial_{x_1} \dots \partial_{x_n})$. When we want to include t we will write $\nabla_{xt} = (\cancel{\partial_t}, \partial_{x_1}, \dots \partial_{x_n}, \partial_t)$

(iv) $u_x + y\, u_y = 0$ (first order PDE with nonconst coeff.) ~~axia~~ transport with $\vec{v} = (y, 1)$

(v) Liouville's Equation

Here $\sum_i \left(B_i(x)\, u(x)\right)_{x_i}$ product rule!

$u_t - \operatorname{div}(\vec{B}(x)\, u) = 0$

vector field in $\mathbb{R}^n$

$\vec{B}(x) = (B_1(x), \dots B_n(x))$

(vi) $u_t - \Delta u = 0$ Heat/Diffusion $\rightarrow$ $u_t - \operatorname{div}(A(x)\nabla u) = 0$ variable coeff. div form

matrix $n \times n$

(vii) $u_{tt} - c^2 \Delta u = 0$ Wave Eq. $c =$ wave speed constant.

p-NLW $u_{tt} - \Delta u \pm |u|^{p-1} u = 0$ (semilinear)

(viii) Klein-Gordon equation (~~non~~linear) NKG

$$u_{tt} - c^2 \Delta u + m u = 0$$

$0 = u_{tt} - c^2\Delta u + mu \pm |u|^{p-1} u$ (semilinear)

(ix) Sine-Gordon Equation (nonlinear)

$$u_{tt} - c^2 \Delta u + \sin(u) = 0 .$$ (semilinear)

(x) Airy's Eq. (Linear part of KdV eq.)

$$u_t + u_{xxx} = 0$$

$0 = u_t + u_{xxx} + \underbrace{u u_x}_{\frac{1}{2}(u^2)_x}$ (nonlinear eq.) semilinear

(xi) Schrödinger Eq.

$$i u_t + \Delta u = 0$$

p-NLS $\quad i u_t + \Delta u \pm |u|^{p-1} u = 0$ (nonlinear semilinear)

Also called "Kolmogorov forward eq." = time evol. of the probability density fc. of the vel. of a particle

(xii) Fokker-Planck Eq. (linear)

$$u_t - \sum_{i,j=1}^{n} (a_{ij}(x)\, u)_{x_i x_j} - \sum_{i=1}^{n} (b_i(x)\, u)_{x_i} = 0$$

diffusion tensor

drift vector

(xiii) Eikonal Equation: (nonlinear)

$|Du(x)| = 1$ euclidean norm

Skip

or $|Du(x)| = F(x) \quad x \in \Omega$ (w/ Dirichlet bdry cond.)

sol. $u(x)$ = shortest time needed to travel from $\partial\Omega$ to x w/ $F(x)$ = time cost at x

(xiv) Scalar Reaction/Diffusion:

$$u_t - \Delta u = f(\underline{u})$$

would have coeff in front of Δ.

(if $f(x)$ inhomog.)

(xv) Nonlinear Poisson eq. $\quad -\Delta u = f(\underline{u})$

(xvi) Inviscid Burgers Eq. $\quad u_t + u u_x = 0$ (quasilinear)

(xvii) <u>Minimal Surface Eq.</u> (Quasilinear)
(compute div to see)

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

second order quasil. for the graph that has the smallest surface area for a given/prescribed boundary curve (eg. soap films are minimal surfaces).

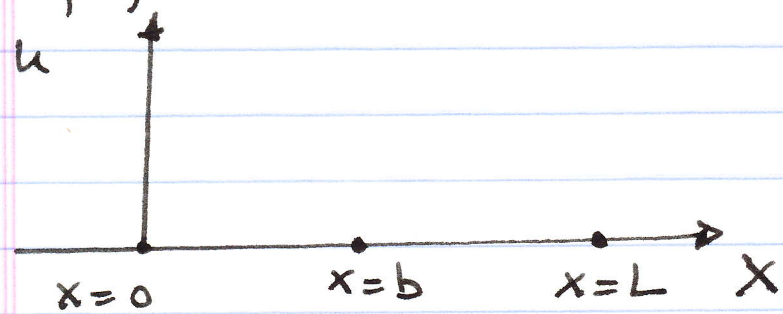
<u>Fully Nonlinear Examples</u>:

(xix) Monge-Ampère Eq. : $\det(D^2 u) = f$

(xx) p-Laplacean Eq. : $\operatorname{div}(|Du|^{p-2} Du) = 0$

Before we start studying First-Order Eqs. and the method of characteristics let's review a simple example; <u>the 1D Transport equation</u>. Let's recall first how it's derived:

Consider a fluid (water) flowing at a <u>constant rate c</u> along a horizontal pipe of fixed cross section in the positive x-axis



A substance, say a polutant is suspended in the water. <u>$u(x,t)$ = concentration in</u> grs/cm at time t position x.

Then the amount of pollutant in $[0, b]$ at time t (in grams) is

$$M = \int_0^b u(x,t)\,dx \qquad b \in [0, L)$$

At a later time, $t+h$ the same molecules of the pollutant would have move to the right by $c \cdot h$ centimeters. Hence:

$$M = \int_0^b u(x,t)\,dx = \int_{ch}^{b+ch} u(x, t+h)\,dx$$

Differentiating w.r.t. b we get

$$(*) \qquad u(b,t) = u(b+ch, t+h)$$

and differentiating $(*)$ w.r.t. h and setting $h=0$ we get

$$0 = u_t(b,t) + c\,u_x(b,t)$$

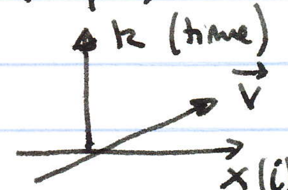
Hence the TRANSPORT EQ. $\boxed{u_t + c u_x = 0}$

Note: (1) u_t is proportional to u_x

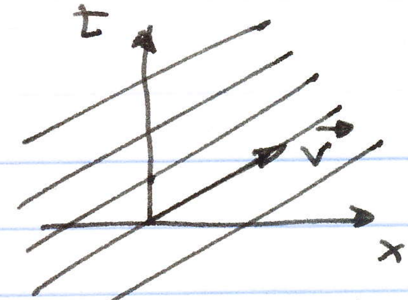
(2) $u_t + c u_x = D_{\vec{v}} u \qquad \vec{v} = c\vec{i} + \vec{k}$

$$= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \cdot (c,1) \qquad = (c,1)$$

To solve this equation is equivalent to solve $D_{\vec{v}} u = 0$



That is $u(x,t)$ must be constant in the direction of $\vec{v}$. The lines parallel to $\vec{v}$ have



equations $x - ct =$ constant

called <u>CHARACTERISTIC LINES</u>

and the solution $u(x,t)$ is <u>constant</u> <u>ALONG</u> each such line. hence $u(x,t)$ <u>depends</u> on $x-ct$ <u>only</u> and thus

$$u(x,t) = h(\underbrace{x-ct}_{s \in \mathbb{R} \text{ variable of } h.})$$

$h : \mathbb{R} \to \mathbb{R}$ <u>fc. of 1 variable!</u>

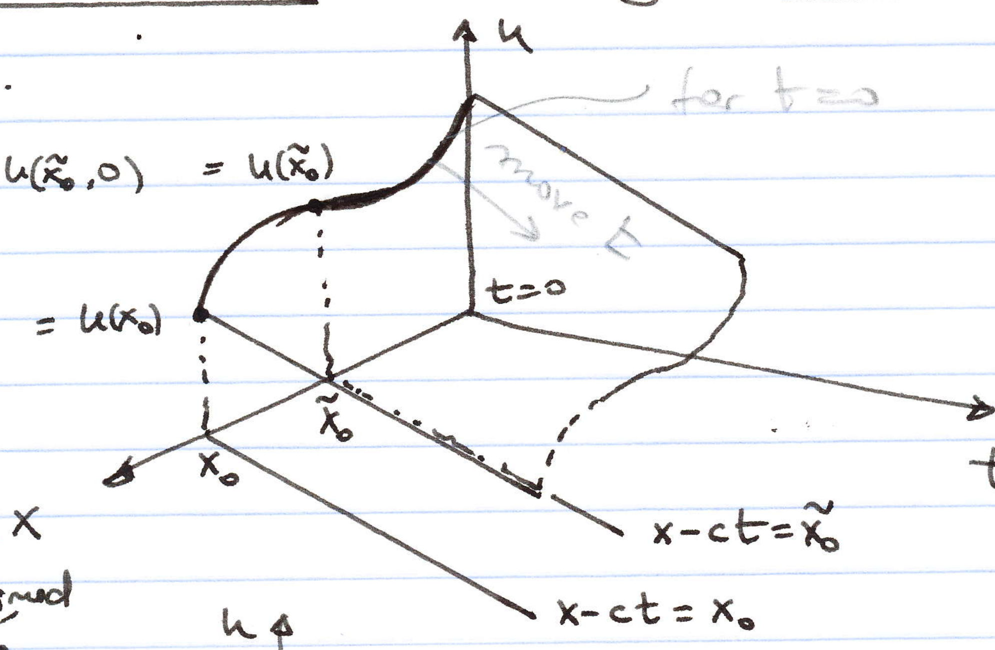
<u>Means</u>: Given (x,t) find s / $x - ct = s$ then $u(x,t) = h(s)$

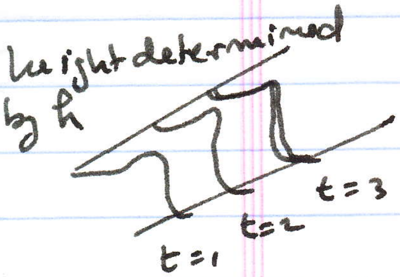
<u>Remark</u>: In the (x,t,u) space the solution defines a surface made up of parallel horizontal lines like a sheet of corrugated aluminum (roofs) This means the substance (pollutant) is <u>TRANSPORTED</u> to the right <u>at fixed speed c</u>

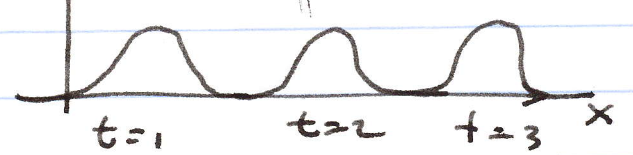
point out later; next page.

$u_0(\tilde{x}_0) = u(\tilde{x}_0, 0) = u(\tilde{x}_0)$

$u_0(x_0) = u(x_0, 0) = u(x_0)$







(movie picture)

- ## The Cauchy problem for Quasilinear Equations

~~Prob~~ (we ~~consider~~ focus on quasilinear eq. on $\mathbb{R}^2 = (x,y)$.

Sometimes $(x,y) \to (x,t)$

Recall from ODE theory: Fix $x \in \mathbb{R}$ and consider

$$(1) \begin{cases} \dfrac{du}{dt} = f(\cancel{x}, t, u) \\ u(x,0) = u_0(x) \quad \text{given initial data} \end{cases}$$

Def Lip: $\exists M > 0$ $|F(x) - F(x')| \leq M|x - x'|$

F Lip $\Rightarrow$ $F' \exists$ a.e. $\|F'\|_\infty \leq M$.

Then provided f is continuous in t and Lip. in u (1) can be uniquely solved, at least for $|t| < \delta$ (small t). The sol. may also exist globally or may blow up in finite time.

Now if f and u_0 (given) are continuous in x also then (x not fixed any more)

Cauchy Initial Value Problem
$$\begin{cases} \dfrac{du}{dt} = f(x,t,u) \\ u(x,0) = u_0(x) \end{cases}$$
can be thought of as a PDE w/o u_x appearing.

and the solution $u : \mathbb{R}^2 \to \mathbb{R}$ $u = u(x,t)$ will be continuous in x and t.

Geometrically the graph $z = u(x,t)$ is a surface in $\mathbb{R}^3$ that contains the curve $(x, 0, u_0(x))$. This surface may exist for all $t > 0$ or blow up at some finite time t_0 (~~with~~ which may depend on x).

Transport Equation Revisited (to motivate method of characteristics)

Let $u_0 : \mathbb{R} \to \mathbb{R}$ be C^1 and consider the following initial value problem. Let $a \in \mathbb{R}$ fixed.

$$(\text{PDE}) \quad \begin{cases} u_t + a u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

To find a solution we may try to reduce the PDE to an ODE by looking for a sol. along some curve $x(t)$

That is we want u /

$$\frac{d}{dt} u(x(t), t) = a u_x + u_t$$

$$= \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dt} \frac{dt}{dt}$$

By chain rule this $\Rightarrow$ $\frac{dx}{dt} = a$ $\Longrightarrow$

$x := x(t) = at + x_0$ where x_0 is the x-intercept

ie. $\boxed{x - at = x_0}$

! Along this curve we have $u_t = 0$ (ie u is constant along $x(t)$)

and since u has x-intercept $u_0(x_0)$ (ie. at $t=0$) we must have

$$u(x,t) = u_0(x_0)$$
$$= u_0(x - at)$$

go back to pic. 2 previous pages

This reduction of PDE to an ODE along curves is called method of characteristics

$$\begin{cases} x = at + x_0 & \text{characteristic curves for} \\ u_t + a u_x = 0. \end{cases}$$

It applies more generally to semilinear transport eqs $u_t + a u_x = f(u) \longrightarrow$ ODE $u_t = f(u)$ along $x(t) = at + x_0$ and can therefore be integrated easily provided f is "nice".

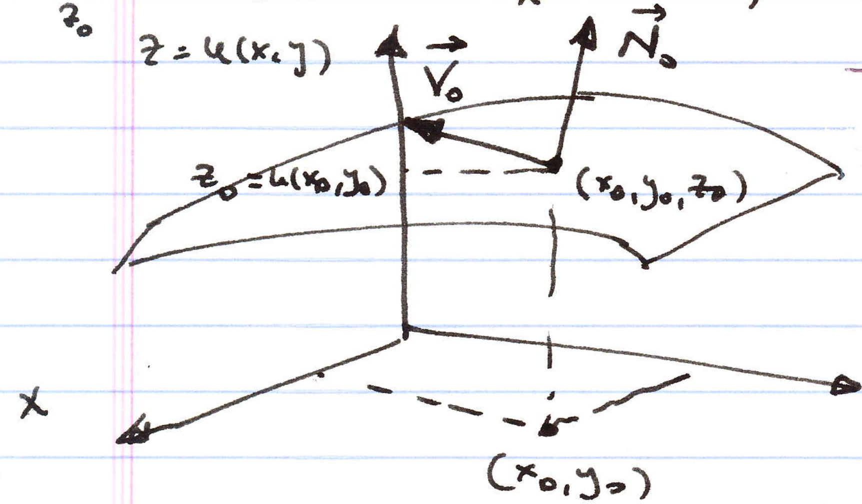
The method also applies to quasilinear PDE's:

We focus ~~first~~ on $\mathbb{R}^2$: Method of Characteristics.

Consider (1) $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$
where a, b, c are continuous in x, y, u.

If $u(x,y)$ is a solution of (1) then $z = u(x,y)$ form a graph describing a surface with normal vector

at $(x_0, y_0, \underbrace{u(x_0,y_0)}_{z_0})$ $\vec{N}_0 = (-u_x(x_0,y_0), -u_y(x_0,y_0), 1) \rightarrow$ from (1)



The vector $\vec{V}_0$ is tangent to the surface (lies in tg. plane) and has coordinates
$\vec{V}_0 = (a, b, c)$ at (x_0, y_0, z_0)
$\vec{V}_0 \perp \vec{N}_0$

i.e. $\vec{V}_{(x,y,z)} = (a(x,y,z), b(x,y,z), c(x,y,z))$ $\quad \vec{V} : \mathbb{R}^3 \to \mathbb{R}^3$
defines a vector field in $\mathbb{R}^3$ to which graphs of solutions must be tangent ~~to~~ at each point.

- Surfaces that are tangent at each point to a given vector field in $\mathbb{R}^3$ = integral surfaces of the vector field. (recall from ODE's curves that are tangent to v.f. are called integral curves.) .

- Thus to find a solution of (1) we should try to find the integral surfaces associated to the coeff. (a, b, c) But need to be more specific because there may be many integral surfaces :

CAUCHY PROBLEM : Given a curve Γ in $\mathbb{R}^3$, can we find a solution u of (1) whose graph contains $\Gamma \subseteq \mathbb{R}^3$?

Special Case : if $\Gamma = (x, 0, h(x))$ in the xz plane
graph of h.
then the Cauchy problem is called the IVP with ID $h(x)$ and the y-variable is "time" (evol. eq.)

Idea: To construct an integral surface we think of it as the (smooth) union of characteristic curves that are the integral curves associated to the v.f. $\vec{V}$.

DEFINITION (Charact. Curve) $\chi(x(t), y(t), z(t))$ is a characteristic curve if it satisfies the following ODE system:

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = a(x,y,z) \\[2mm] \dfrac{dy}{dt} = b(x,y,z) \\[2mm] \dfrac{dz}{dt} = c(x,y,z) \end{cases}$$

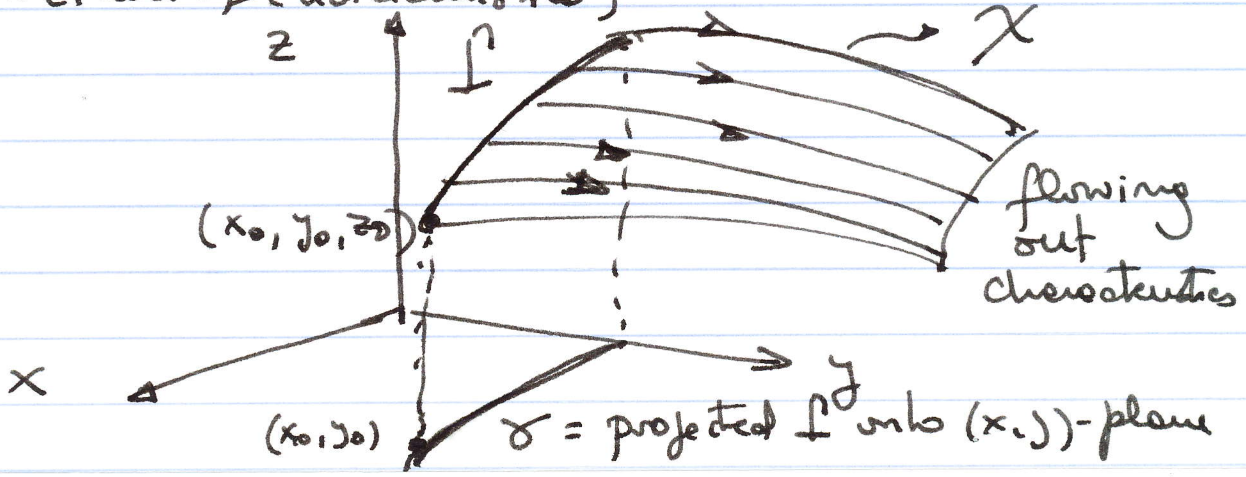
for $|t-t_0|$ small and ID: $\begin{cases} x(t_0) = x_0 = f(s) \\ y(t_0) = y_0 = g(s) \\ z(t_0) = z_0 = h(s) \end{cases}$ later

(leave on board)

Remark 1: If the graph $z = u(x,y)$ is a smooth surface S that is the union of such characteristic curves, then at each (x_0, y_0, z_0) the tangent plane contains the vector $\vec{V}(x_0, y_0, z_0)$. Hence S must be an integral surface.

Q: How to obtain a smooth union of characteristic curves?

Remark 2: If the given curve is noncharacteristic (ie. if Γ is nowhere tangent to the v.f. $\vec{V}$) then a simple procedure to solve the Cauchy problem is to flow out each point of Γ along the characteristic curve through that point, thereby sweeping out an integral surface (method of characteristics)



Analytically this construction of an integral surface containing Γ can be achieved by first writing Γ as the graph of a curve $(f(s), g(s), h(s))$ parametrized by s and then solving eqs. (*) for each s fixed using

$$\left\{\begin{array}{l} x_0 = f(s) \\ y_0 = g(s) \\ z_0 = h(s) \end{array}\right. \quad \left.\begin{array}{l} x_0 = f(s) \\ y_0 = g(s) \end{array}\right\} := \gamma(s) \quad \text{as initial conditions}$$

This gives an integral surface S parametrized by s and t.

Remark 3 : It is also true that the integral surface S of the v.f. $\vec{V} = (a, b, c)$ is ALWAYS a union of characteristic curves (by uniqueness of ODE for $|t - t_0|$ small). Since the solutions of the characteristic eqs. are unique we find the integral surface is unique. Hence we have:

Related to HMWK 1

Theorem: If Γ is noncharacteristic then the v.f. $\vec{V} = (a, b, c)$ admits a unique integral surface S containing Γ.

Remark 4: To find the solution u as a function of (x, y) we need to replace s, t by expressions of x, y (go back). We'll see more on this later but when the integral surface S is the graph of a fc. ~~this can be achieved~~ this can be achieved exactly.

need to solve for s, t in terms of x, y (not always easy)

All in all then the method of characteristic produces a formula for the solution of (1) provided ~~[illegible]~~ we can

(1) ~~[illegible]~~ Solve (*)

(2) Solve for s, t in terms of x, y

We'll discuss this further in the semilinear case (in 2-variables still) See book[2] pg. 15 for how to generalize to n-dim + [1] 1.1.d) for quasilinear case

- Semilinear case in 2D:

can dep on u!

$$(S) \qquad a(x,y)\, u_x + b(x,y)\, u_y = c(x,y,u)$$

Γ parametrized by $(f(s)\ g(s)\ h(s))$. The CE (*) become then; Fix s:

$$(**)\quad \begin{cases} \dfrac{dx}{dt} = a(x,y) & x(s,0) = f(s) \\ \dfrac{dy}{dt} = b(x,y) & y(s,0) = g(s) \\ \dfrac{dz}{dt} = c(x,y,z) & z(s,0) = h(s) \end{cases}$$

for $|t|$ small $(t_0 = 0)$.

Note (**) forms a system in itself decoupled from z. So it can be solved independently to obtain a curve $(x(t), y(t))$ in the (xy) plane.