M524 HOMEWORK –SPRING 2016

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SET 1 - DUE 02/04/16

Chapter 4 of Beals:

4A: 1, 2, 3. **4B:** 1, 2, 5, 9, 10, 12, 25. Additional ones (pp 59-60): 1, 2.

Additional Problem: Prove Theorem 4.8 (2^m test) in Chapter 4. Hint Prove first that

$$2^{m-1} a_{2^m} \leq \sum_{j=2^{m-1}+1}^{2^m} a_j \leq 2^{m-1} a_{2^{m-1}}$$

and use the first inequality when assuming that $\sum_j a_j$ converges and the second inequality when assuming that $\sum_{m=1} 2^m a_{2^m}$ converges.

Chapter 5 of Beals:

5A: 1, 2, 6, 9, 10. **5B:** 1, 2, 7.

Hints. For **5A.** 1. Consider all cases for $a \in \mathbb{R}$: positive, negative, zero.

For **5A.** 9. Consider $\lambda := \limsup_{j\to\infty} |\frac{a_{j+1}}{a_j}| < 1$ and choose a real number μ such that $\lambda < \mu < 1$. Note then that by the definition of lim sup there is an N large enough so that if $j > N$ the $|\frac{a_{j+1}}{a_j}| < \mu$. This in turn gives for any $k \geq 0$

$$|a_{N+k}| \leq \mu |a_{N+k-1}| \leq \mu a_{N+k-2} \cdots \leq \mu^k |a_N|$$

for any $k \geq 0$, which can be written (change variables $n = N + k$) as $|a_n| \leq \mu^{N-n} |a_N|$. Use the root test.

Additional problem. In light of Theorem 5.9 (p. 71) we have the following

Definition: A series $\sum_n a_n$ is said to be Abel summable (to L) if:

a) The power series $\sum_n a_n x^n$ converges for all $|x| < 1$, and

b) $f(x) := \sum_n a_n x^n \to L$ as $x \to 1^-$.
Consider

$$g(x) = \sum_{n\geq 1} (-1)^{n+1} n\, x^n.$$

i) Find the radius and interval $I \subset \mathbb{R}$ of convergence (ie. domain of g).

ii) Prove that if $h(x) = \sum_{n\geq 0}(-1)^{n+1}x^n$ then $g(x) = xh'(x)$ on $-1 < x < 1$. Justify your answer.

iii) Show that $\lim_x \to 1^- g(x) = \frac{1}{4}$

iv) Deduce that the *divergent* series $\sum_n (-1)^{n+1}n$ is Abel summable to $\frac{1}{4}$.

Note As this example shows a series that is Abel summable is not necessarily convergent. So the converse of Th. 5.9 is false. However if one add an extra condition to a_n a 'partial' result can be obtained (this is due to Tauber). Namely:

Theorem (Tauber, circa 1897) Suppose that the series $\sum_n a_n$ is Abel summable **and** that $\lim_{n\to\infty} a_n = 0$. Then $\sum_n a_n$ converges.

SET 2 - DUE 02/11/16

Chapter 6 of Beals:

6A: 1. **6B:** 1, 2, 4, 10 (see hint).

Hint. Before proving exercise 10 in 6B, show the Additional Problem 1 below.

In what follows, A is a subset of a metric space (S, d).

Additional Problem 1. Show that $\bar{A} = A^\circ \cup \partial A$. That is, show that if $p \in \bar{A}$ then $p \in A^\circ \cup \partial A$ and that if $p \in A^\circ \cup \partial A$ then $p \in \bar{A}$.

As a consequence note that one also has that $\bar{A} = A \cup \partial A$.

Additional Problem 2. Let $p \in A$. Prove that either p is a limit point of A^c or p is an interior point of A, but not both.

SET 3 - DUE 02/18/16

Chapter 6 of Beals:(cont.).

6B: 3, 13 (first part only for $[0, 1]$). **6C:** 1, 2. **6D:** 1, 2, 3, 5.

Note An alternative definition of *connected* is: A subset B of a metric space (S, d) is said to be *connected* if whenever U and V are <u>disjoint</u> <u>open</u> subsets of S

$$B \subseteq U \cup V \quad \text{implies that} \quad B \subseteq U \text{ or } B \subseteq V$$

The empty set is connected.

<u>SET 4 - Due 02/25/16</u>

Chapter 6 of Beals:(cont.).

6D: 6, 8, 9, 10,13.

<u>SET 5 - Due 03/10/16</u>

<u>Additional Problem (related to 6E*)</u>Consider the unit interval $[0,1]$ and let ξ be a fixed real number $0 < \xi < 1$. In stage 1 of the construction remove a centrally open interval of length ξ. In stage 2, remove two central open intervals each of *relative length* ξ, one in each of the remaining subintervals after stage 1. Note that each of the two subintervals has length $\frac{1-\xi}{2}$ (the total length that remains after stage 1 is $1-\xi$) so what one removes in stage 2 is two intervals **each** of length $\xi(\frac{1-\xi}{2})$. So the total removed has length $\xi(1-\xi)$ and the total length left after stage 2 is $(1-\xi)-\xi(1-\xi) = (1-\xi)^2$. Continue in this manner. Let $\mathcal{C}_\xi$ denote the set that remains after applying this procedure indefinitely and $\mathcal{C}_\xi^k$ the set that remains after completing stage k. Prove that:

a) The complement of $\mathcal{C}_\xi$ in $[0,1]$ is the union of open intervals of total length 1 (this would be the set you have removed at the end).

b) Compute the length of the set $\mathcal{C}_\xi^k$ and prove that the limit as $k \to \infty$ of the length of the set $\mathcal{C}_\xi^k$ is zero.

(Note that when $\xi = \frac{1}{3}$ the above construction is the one that gives the Cantor 1/3 set.)

Chapter 7 of Beals:

7A: 1, 3, 4.

<u>SET 6 - Due 03/24/16</u>

7C: 1a) d)

<u>Hint</u>. Problem 1) : for the uniform part it is useful to find x_n the maximum of each f_n. For part a) a useful trick is to consider $\log[(f_n(x)]$.

7D*: Graphing Problem: Consider $I = [0, 1]$ and the Bernstein polynomials $P_n(x)$, $x \in I$, $n \in \mathbb{N}$ defined by (9) (Beals p. 95). Consider the continuous function on I defined by

$$f(x) = \frac{1}{1 + 10(x - \frac{1}{2})^2}.$$

Compute P_6, P_{10}, P_{20} and plot all three together with f on the same graph (scale suitably so the graphs are clearly visible).

Chapter 8 of Beals:

8A: 2, 3a), 5, 6 (use 5).

8B: 2 (use IVT and monotonicity).

SET 7 - DUE 04/07/16

10B: 1, 2, 3, 6.
10C: 1
11A: 1
11C: 1

SET 8 - DUE 04/14/16

12C: Additional Problem 1: Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and compactly supported (see Definition p.164) then f is (Lebesgue) integrable.

12D: 1, 3, 4, 6. 7.

Additional Problem 2.

Let f is a 2π-periodic integrable function on any finite interval.

(a) Prove that for any $a, b \in \mathbb{R}$

$$\int_a^b f(x)dx = \int_{a+2\pi}^{b+2\pi} f(x)dx = \int_{a-2\pi}^{b-2\pi} f(x)dx$$

(b) Prove that for any $a \in \mathbb{R}$

$$\int_{-\pi}^{\pi} f(x+a)dx = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi+a}^{\pi+a} f(x)dx$$

SET 9 - Due 04/26/16

13B: 1a)d), 2, 8.

13D: 2 (use Fejér's Theorem).

Additional Problem. Suppose that f is a 2π- periodic function which belongs to the class C^2 of continuous and twice differentiable functions with continuous derivatives. Show that there exists a constant $C > 0$ independent on n such that $|\widehat{f}(n)| \leq \frac{c}{|n|^2}$.

Hint Integrate twice by parts. Use periodicity.

If f is a 2π- periodic function which belongs to the class C^k of continuous and k-times differentiable functions with continuous derivatives. What can you say about $|\widehat{f}(n)|$ and how would you prove that ?

SPECIAL PROJECTS (Due date: 04/29/16)

The projects below should be typed.

Show all your work and steps clearly, justifying where appropriate what are you using.

Check your work carefully.

SP I. Do Problem 2 Section 7C (p 94).

Hint. For 2b) Prove that $d(x_{n+m}, x_n) = \sup_{x\in I} |x^{n+m} - x^n|$ does not converge to zero as $m \to \infty$ for each fixed n (so that f_n is not Cauchy). To do this consider the function $F(x) = x^{n+m} - x^n$. Note $F(x) \leq 0$ and $F(0) = 0 = F(1)$. Find the extremum $\bar{x}$ of F (which depends on n, m) and show that as $m \to \infty$ the values of $|F(\bar{x})| \to 1$.

SP II. Prove that the series

$$\sum_{j=1}^{\infty} 2^{-j/2} \sin(2^j x),$$

defines a continuous function. Prove that it is *not* differentiable at any x such that $\frac{x}{\pi} \notin \mathbb{Q}$.

Hints. First recall the M-test for series to first verify that the series is indeed absolutely convergent for each fixed x. You may then adapt a similar strategy to the one in the handwritten notes I distributed for the Weierestrass-type function to prove the continuity and the non-differentiability at each fixed x such that $\frac{x}{\pi} \notin \mathbb{Q}$.

NB: The function is in fact *nowhere differentiable* but the proof for x which are of the form $\frac{x}{\pi} = \frac{p}{2^k}$ for some $k \geq 0$ and $p \in \mathbb{Z}$ requires a more refined argument.

SP III. Verify that $\frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function defined by $f(0) = 0$ and

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & \text{if } \; -\pi < x < 0, \\ \frac{\pi}{2} - \frac{x}{2}, & \text{if } \; 0 < x < \pi \end{cases}$$

Note that the sawtooth function is not continuous. **Show** nonetheless that the series converges for ever x; by which we mean as usual that the partial sums

$$S_N(x) = \frac{1}{2i} \sum_{|n| \leq N, n \neq 0} \frac{e^{inx}}{n}$$

of the series converge. In particular note that the value of the series at the origin, namely 0, is indeed the average of the values of the function $f(x)$ as x approaches the origin from the left and the right.

Hint Use Dirichlet's Test for convergence. Check the convergence at $x = 0$ separately.

SP IV. In class we discussed the Fejér kernel associated to the (Cesàro) summability of Fourier series and proved that the Fejér kernel is a "good kernel". The project here aims at proving that the Poisson kernel defined below is also a good kernel (the Poisson kernel is associated to the 'Abel summability' of Fourier series.).

For $0 \leq r < 1$ and $-\pi \leq x < \pi$, the Poisson kernel $P_r(x)$ is defined by:

$$P_r(x) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1 - r^2}{1 - 2r\cos x + r^2}$$

and to be a good kernel means that

Prove:

1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x)\, dx = 1$ for all $0 \leq r < 1$

2) There exists $M > 0$ such that for all $0 \leq r < 1$, $\int_{-\pi}^{\pi} |P_r(x)|\, dx \leq M$

Hint: note that $P_r(x) \geq 0$ (why?) so this should be immediate from 1).

3) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |P_r(x)| dx \to 0, \text{ as } r \to 1^-$$