NAME and ID#:

FINAL TAKE HOME EXAM for M425

FALL 2004

Instructions.

(1) **This exam is due on Monday December 20th <u>no later</u> than 12 noon; maximum. It has 5 questions for a total of** 100 **points and will constitute** 30% **of your grade**

(2) **You must work on this exam alone.**

(3) **You may ONLY consult the Marsden and Tromba book or the class notes. Absolutely NO other books or notes are permitted.**

(4) **You cannot discuss the problems with other people, including classmates.**

(5) **You may ask me any questions you have.**

(6) **Please write down EACH PROBLEM IN A SEPARATE PAGE AND STAPLE all relevant pages. USE THIS PAGE as COVER.**

(7) **Write down your answers <u>clearly</u>. Show all steps and work for full credit and <u>justify each and all steps</u> when using a particular theorem or result.**

1) **(a)** Suppose a particle follows the path described by

$$\mathbf{c}(t) = (2e^{t-1}, -e^{1-t}, \sin(\pi t))$$

and that at time $t_0 = 1$ it flies off on a tangent. Compute its position at time $t_1 = 2$

(b) Find the linear approximation to the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (\, x + y + e^z \,,\, y\, x^2 \,)$$

near the point (1, 1, 0)

2) **a)** Compute the directional derivative of the function $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = e^{-\,(x^2+y^2+z^2)}$$

at the point $(1, -1, 2)$ in the direction of $\mathbf{d} = (4, -4, 2)$

b) Let $\mathbf{F} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}$. Show that $\mathbf{F}$ is <u>not</u> a gradient vector field.

3) **a)** Compute and sketch the gradient vector field of the function

$$f(x, y) = \frac{x^2}{8} - \frac{y^2}{12}.$$

Make sure your sketch contains sufficent vectors in each quadrant to have a telling picture of what the flow lines would look like (do not draw or find the flow lines).

b) Let $f(x, y) = e^{xy}\sin(x + y)$ and let $\mathbf{c}(t)$ be a flow line of $\mathbf{F} = \nabla f$ with $\mathbf{c}(0) = (0, \pi/2)$. Use the chain rule plus the *definition* of a flow line to compute

$$\frac{d}{dt}\{f(\mathbf{c}(t))\}\,|_{t=0}$$

4) a) Let $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and let $\mathbf{c}(t) = (\cos^3 t, \sin^3 t, 0), \quad 0 \le t \le 2\pi$.

Prove:

(i) that $\mathbf{F} = \nabla f$ for an appropriate f.

(ii) that $\mathbf{c}$ is closed curve.

<u>Use</u> (i) and (ii) to compute

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

b) Let $u(x, y)$ be a smooth function on the plane satisfying

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

on the disk $x^2 + y^2 < 4$. Use Green's theorem to show that

$$\int_C \frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy = 0$$

where C is the unit circle $x^2 + y^2 = 1$.

5) Let

$$\mathbf{F} = 2y \sin(x^2 + y^2)\mathbf{i} - 2x \sin(x^2 + y^2)\mathbf{j}$$

and let D denote the rectangle $[0, 1] \times [0, 2]$. Use the vector form of Green's theorem known as the *Divergence theorem in the plane* to compute the normal component of $\mathbf{F}$ around the rectangle D. That is, find

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds.$$