THE CAUCHY PROBLEM FOR THE HYPERBOLIC-ELLIPTIC ISHIMORI SYSTEM AND SCHRÖDINGER MAPS

Carlos E. Kenig and Andrea R. Nahmod

Abstract. We show an improved local in time existence and uniqueness result for Schrödinger maps and for the hyperbolic-elliptic nonlinear system proposed by Ishimori in analogy with the 2d CCIHS chain. The proof uses fairly standard gauge geometric tools and energy estimates in combination with Kenig's version of the Koch-Tzvetkov method, to obtain a priori $L^q_t L^\infty_x$ estimates for classical solutions to certain dispersive equations.

0. Introduction

The purpose of this note is to prove a new local in time existence and uniqueness result for the $2D$ hyperbolic-elliptic Ishimori system as well as for $2D$ Schrödinger maps.

Schrödinger maps are solutions to

$$\partial_t s \,=\, J(s) \sum_k \nabla_k \partial_k s \tag{1}$$

where $s : \mathbb{R} \times \mathbb{R}^n \to \mathcal{M}$ a Kähler manifold with complex structure J, $J^2 = -I$ and metric g. By ∇_k we denote the covariant derivative on the $s^{-1}\mathcal{T}(\mathcal{M})$ bundle in the direction of x^k. These are evolution equations that arise from the same geometric considerations as wave maps. They are examples of gauge theories which are abelian in the case of Riemann surfaces- Kähler manifolds of dimension one- such as the 2-sphere $\mathbb{S}^2$ or the 2-hyperbolic space $\mathbb{H}^2$ [44]. In this note we focus on $\mathbb{S}^2$; then $J = -\mathbf{i}$ the 2×2 matrix with $1, -1$ in the off-diagonal. Physically just as wave maps are studied as examples of relativistic nonlinear field theories [22], Schrödinger maps arise naturally from the Landau–Lifshitz equations (a $U(1)$-gauge theory) governing the static as well as dynamical properties of magnetization [27][35]. To be more precise, for maps $s : \mathbb{R} \times \mathbb{R}^n \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ equation (1) takes the form

$$\partial_t s \,=\, s \times \Delta s, \qquad |s|^2 = 1 \tag{2}$$

which is the Landau-Lifshitz equation at zero dissipation when only the exchange field is retained [35][25]. In two spatial dimensions equation (2) is also known as the $2D$ classical continuous

1991 *Mathematics Subject Classification*. Primary 35J10, Secondary 45B15, 42B35.

The first author was partially supported by NSF, and at IAS by The von Neumann Fund, The Weyl Fund, The Oswald Veblen Fund and the Bell Companies Fellowship. The second author was partially supported by NSF grant DMS 0202139 and at IAS by grant DMS 0111298. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

isotropic Heisenberg spin model (2d-CCIHS); i.e. the long wave length limit of the isotropic Heisenberg ferromagnet [41] [35][25].

The hyperbolic-elliptic Ishimori system,

$$\begin{aligned} \partial_t s &= s \times \Box_{xy}\, s + \kappa(\zeta_x s_y + \zeta_y s_x) \\ \Delta\zeta &= 2s \cdot (s_x \times s_y) \end{aligned} \tag{3}$$

with $s : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, $\lim_{|x|,|y| \to \infty} s(x, y, t) = (0, 0, -1)$ and κ a real constant was proposed in 1984 by Y. Ishimori. In [12], Ishimori -seeking to show that the dynamics of topological vortices need not generally be non-integrable- introduced the system (3) in analogy with the 2d CCIHS chain, as a model having the same topological properties as the latter yet permitting topological vortices whose dynamics are integrable; the system (3) is completely integrable when $\kappa = 1$ (c.f. [12] [2] [43][26] [29]). The linearized equation of (3) in this case is the same as that of the hyperbolic-elliptic Davey-Stewartson system and has been tackled by inverse scattering methods [26] [2] [36].On the other hand, system (3) with Δ_{xy} in lieu of $\Box_{xy}$ and $\kappa = 0$ reduces to (2). The Ishimori system (3) describes the time evolution of a system of static spin vortices in the plane. The right hand side of the equation for the scalar potential function $\zeta(x, y, t)$ is the topological charge density of the system. The integer values of the topological charge Q,

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} s \cdot (s_x \times s_y)\, dxdy,$$

classify the static spin vortices [12]. Geometrically $\zeta(x, y, t)$ is a multiple of the curvature tensor and the second equation can be viewed as describing the pull back of a piece of the volume in the target $\mathbb{S}^2$.

Our main result here is that both the $2d$ Schrödinger map equation (2) into $\mathbb{S}^2$ and the Ishimori system (3) admit a local in time solution for the Cauchy initial value problem with large data in the Sobolev[1] class $H^\gamma(\mathbb{R}^2)$, $\gamma > 3/2$ suitably avoiding the north pole.[2] Uniqueness holds in $H^2(\mathbb{R}^2)$.

In the context of Schrödinger maps it was shown in [31] that one can find an appropriate frame on $s^{-1}\mathcal{T}(\mathbb{S}^2)$ so that the derivatives of the solution of (2) satisfy a certain nonlinear Schrödinger system, referred as the Modified Schrödinger Map system (MSM). Instead of repeating the argument we will describe below how the same ideas transform very similarly the Ishimori system into a nonlinear hyperbolic Schrödinger equation. In analogy with Schrödinger maps, we will refer to it as the Modified Ishimori system (MIS).

To prove our main result we rely on the latter transformation in combination with energy estimates as in [32], Strichartz estimates and a formulation devised by C. Kenig [16], [18] of the ideas in Koch-Tzvetkov's work [24], to obtain a priori estimates for classical smooth maps (see

[1] Data s_0 in the inhomogeneous Sobolev space should be thought of as $s_0 \in \dot{W}^{1,2}$ and ∂s_0, the tangent vector to $\mathbb{S}^2 \subset \mathbb{R}^3$ at s_0, being in $H^{\gamma-1}(\mathbb{R}^2)$. Note the unit normal to $\mathbb{S}^2$ at a point $u \in \mathbb{R}^3$ is u itself.

[2] More precisely we will request the range of the data to omit a small neighborhood of the north pole. This is a stronger than needed technicality that simplifies the translation back and forth between the Ishimori or Schrödinger map system and the Modified Ishimori or Schrödinger map equation. In principle, the weaker assumption of the map having degree zero should suffice; but it alone makes the translation back to the map more involved. Note that for nearly parallel spins -i.e. sufficiently small data – initially close to the south pole; none of these requirements are needed since for a short time the map will stay close to the south pole.

also Burq, Gerard and Tzvetkov [5], Bahouri and Chemin [1] and Vega [45] for earlier uses of these ideas).
Standard approximation methods and compactness then give local in time existence in $H^s(\mathbb{R}^2)$, $s > 1/2$ for the MSM and the MIS and uniqueness in $H^{s'}(\mathbb{R}^2)$, $s' > 1$. Hence as in Section 2. of [31], one obtains local existence in $H^\gamma(\mathbb{R}^2)$, $\gamma > 3/2$ and uniqueness in $H^{\gamma'}(\mathbb{R}^2)$, $\gamma' > 2$ for the Ishimori system and the Schrödinger map equation.

Independently and simultaneously to our work here, J. Kato proved for the MSM the same local in time existence result using the same method. In addition, he was able to show uniqueness in $H^1(\mathbb{R}^2)$ as opposed to $H^{s'}(\mathbb{R}^2)$, $s' > 1$ [13]. We have adapted his uniqueness argument here also for the modified Ishimori system, MSI.

Various local and global existence results for smooth Schrödinger Maps have been obtained by several authors and can be found in [41], [6], [9], [31] [32] [37] [28] and [21].

For the hyperbolic-elliptic Ishimori system there is A. Soyeur's [43] work who showed for small H^3 smooth data local and global existence. He also showed uniqueness of large data solutions in $H^4(\mathbb{R}^2)$. Global existence for small smooth data followed from Strichartz estimates (dispersive inequality) [43] plus local existence as in Shatah [38], Klainerman-Ponce [23], Sulem, Sulem and Bardos [41]. Local existence for large smooth data does not follow from the results in [21] because $\nabla\phi$ is not necessarily in L^1. Thus in this paper we also need to do a parabolic regularization in the covariant derivative equation and prove energy estimates in H^k, k large. This will be done in the Appendix and for clarity in the exposition of ideas we momentarily assume in the earlier sections existence of large data local in time smooth solutions.

Finally we note that our results in Sections 2 through 4 actually hold in any $n \geq 2$ with $H^{(n-1)/2+}$ in lieu of $H^{1/2+}$. We restrict ourselves to the more physical two dimensional case. The interested reader could easily adapt accordingly the statements to the higher dimensional set up.

Acknowledgments. The second author is grateful to Karen K. Uhlenbeck for her encouragement, generous sharing of ideas, and helpful explanations on the geometric aspects of both these systems. She also thanks Jalal Shatah for many useful conversations on Schrödinger maps.
We would like also to thank Y. Tsutsumi and J. Kato for making available to us J. Kato's unpublished manuscript.

1. The Modified Ishimori System

Consider the system (3) with $\kappa \neq 0$ any real constant. The precise value of κ will not enter but in the constants bounding the estimates. Without any loss of generality we can re-normalize $\zeta =: 1/\kappa\,\phi$ and write

$$\begin{aligned} \partial_t s &= s \times \Box_{xy}\, s + (\phi_x s_y + \phi_y s_x) \\ \Delta\phi &= 2\kappa s \cdot (s_x \times s_y) \end{aligned} \tag{4}$$

In what follows relabel for notational convenience $(t, x, y) = (x_0, x_1, x_2)$. Following [31] we start with a description in terms of the stereographic projection of $\mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ where N is the north pole. Note that for $s : \mathbb{R}^2 \to \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$, the energy of s is

$$E(s) = \frac{1}{2}\int_{\mathbb{R}^2} \sum_{j=1}^{2} \frac{|\partial s/\partial x_j|^2}{(1+|s|^2)^2} dxdy.$$

We rewrite system (4) using covariant derivatives,

$$\nabla_j := \frac{\partial}{\partial x_j} - 2\left(\frac{\partial s}{\partial x_j}\cdot \overline{s}\right)\Big/(1+|s|^2)$$

as

$$i\partial_t s = \left\{(\nabla_1 - i\frac{\partial\phi}{\partial x_2})\frac{\partial s}{\partial x_1} - (\nabla_2 + i\frac{\partial \phi}{\partial x_1})\frac{\partial s}{\partial x_2}\right\}. \tag{5}$$

Just as with Schrödinger maps we have two sets of consistency conditions, the first one follows from the Levi-Civita connection having no torsion; the second one follows from the curvature of $\{\nabla_j\}$ being the pull back of constant curvature on $\mathbb{S}^2$ by the map s [31]:

- $\nabla_j \frac{\partial s}{\partial x_k} = \nabla_k \frac{\partial s}{\partial x_j} \qquad j = 0,1,2,\ \ k = 1,2$
- $[\nabla_j, \nabla_k] = -4i\ \mathrm{Im}\ (\overline{b}_j\, b_k) \qquad j = 0,1,2,\ \ k = 1,2$

where $\quad b_j = (\frac{\partial s}{\partial x_j})/(1+|s|^2)$. Recalling now the equation for ϕ we see that

$$\Delta\phi = 2\kappa s\cdot(s_x \times s_y) = -4\kappa\ \mathrm{Im}\ (b_1\overline{b}_2)$$

Now, assuming finite energy of the map s, can can apply the 'good gauge' theorem of Uhlenbeck -existence of a global Coulomb gauge- (c.f. Theorem 1 in [31], also [44]) to ∇_j, $\frac{\partial s}{\partial x_j}$. Write

$$\begin{aligned} u_j &= (1+|s|^2)^{-1} e^{i\psi}\frac{\partial s}{\partial x_j} \\ D_j &= (1+|s|^2)^{-1} e^{i\psi}\circ\nabla_j\circ(1+|s|^2)e^{-i\psi} = \\ &= \frac{\partial}{\partial x_j} + i a_j \end{aligned} \tag{6}$$

Moreover, for each t there exists a unique choice of ψ such that a is a real valued 1-form

$$\mathrm{div}\ \ a = 0, \qquad a\sim 0 \quad \text{at infinity} \tag{7}$$

$$D_j u_k = D_k u_j \qquad j = 0,1,2,\ \ k = 1,2 \tag{8}$$

$$[D_j, D_k] = [\nabla_j, \nabla_k] \tag{9}$$

$$[D_j, D_k] = i(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k}) \qquad j = 0,1,2, \quad k = 1,2, \tag{10}$$

$$a_k = \frac{\partial \beta_{1k}}{\partial x_1} + \frac{\partial\beta_{2k}}{\partial x_2},\ k = 1,2 \qquad d\beta = 0 \tag{11}$$

$$\Delta\beta_{k,j} = -4\ \mathrm{Im}\ (u_k\overline{u}_j) \quad k,j = 1,2 \tag{12}$$

$$\Delta a_0 = -4\sum_{k,j=1}^{2}\left[\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\mathrm{Re}(u_k\overline{u_j}) - \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)^2 |u_j|^2\right]. \tag{13}$$

where (8)– (10) are gauge invariant equations. Note in particular we have that

$$da = -4\ \mathrm{Im}\ (u_1\overline{u}_2) \tag{14}$$

Now if we let

$$v = \begin{pmatrix} \dfrac{\partial\phi}{\partial x_2} \\ -\dfrac{\partial\phi}{\partial x_1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad *v = (\frac{\partial\phi}{\partial x_1}, -\frac{\partial\phi}{\partial x_2}) = (-v_2, v_1),$$

it follows that v_1, v_2 are *real valued* , div $v = 0$ and div $*v = \Delta\phi = -4\kappa\ \mathrm{Im}\ (u_1\overline{u}_2)$. Thus for $k, i = 1, 2$ we have that

$$\frac{\partial v_k}{\partial x_i} = -4\kappa\mathcal{R}_{ik}(\ \mathrm{Im}\ (u_1\overline{u}_2))$$

where $\mathcal{R}_{ik} := (-\Delta)^{-1}\partial_{x_i}\partial_{x_k}$, are the second order Riesz transforms. Equation (5) can now be rewritten as

$$i\partial_t s = (\nabla_1 + iv_1)\frac{\partial s}{\partial x_1} - (\nabla_2 + iv_2)\frac{\partial s}{\partial x_2}.$$

Proceeding as in the proof of Theorem 2 in [31], we write

$$D_j u_j = (1 + |s|^2)^{-1} e^{i\psi}\nabla_j \frac{\partial s}{\partial x_j}$$
$$u_0 = i(D_1u_1 - D_2u_2) + \frac{\partial\phi}{\partial x_1}u_2 + \frac{\partial\phi}{\partial x_2}u_1$$

from where for u_1 for example, we have $\frac{\partial u_1}{\partial t} + ia_0u_1 = D_0u_1$ and

$$D_0u_1 = D_1u_0 = i(D_1^2u_1 - D_2^2u_1) + [D_1, D_2]u_2 + D_1(v_1u_1) - D_1(v_2u_2).$$

Proceeding similarly with u_2, using (6), (10)–(13) - as in [31] Theorem 2- we obtain a system of the form

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= i\Box u_1 + \gamma_1(a_1\frac{\partial u_1}{\partial x_2} - a_2\frac{\partial u_1}{\partial x_1}) + \gamma_2\alpha_1u_1 + \gamma_3\alpha_2u_2 + \gamma_4(a_1^2 - a_2^2)u_1 + \gamma_5a_0u_1 \\ \frac{\partial u_2}{\partial t} &= i\Box u_2 + \gamma_1(a_1\frac{\partial u_2}{\partial x_2} - a_2\frac{\partial u_2}{\partial x_1}) + \gamma_2\alpha_1u_2 + \gamma_3\alpha_2u_1 + \gamma_4(a_1^2 - a_2^2)u_2 + \gamma_5a_0u_2 \end{aligned} \tag{15}$$

with $\Box := \Box_{x_1,x_2} = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ and γ_m, $m = 1, \ldots 5$ some constants that may depend on κ but $\gamma_1 \in \mathbb{R}$. We have denoted by α_1, α_2 quadratic terms in u_1, u_2 of the form

$$\alpha_1 = \mathcal{R}(\ \mathrm{Im}\ (u_1\overline{u_2})), \qquad \alpha_2 = \ \mathrm{Im}\ (u_1\overline{u_2}) \tag{15b}$$

where by $\mathcal{R}$ we generically represent an appropriate linear combination of Riesz transforms $\mathcal{R}_{k,l}, k, l = 1, 2$

We refer to this system as the Modified Ishimori system.

For the convenience of the reader we write now the Modified Schrödinger Map system in keeping with the notation above and that in [31]. The MSM is a system of coupled nonlinear Schrödinger equations in R^{2+1} of the form:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= i\Delta u_1 - 2(a_1\frac{\partial u_1}{\partial x_1} + a_2\frac{\partial u_1}{\partial x_2}) \pm Im(u_2\overline{u_1})u_2 - i(a_1^2+a_2^2)u_1 - i\,a_0 u_1 \\ \frac{\partial u_2}{\partial t} &= i\Delta u_2 - 2(a_1\frac{\partial u_2}{\partial x_1} + a_2\frac{\partial u_2}{\partial x_2}) \pm Im(u_1\overline{u_2})u_1 - i(a_1^2+a_2^2)u_2 - ia_0 u_2 \end{aligned} \tag{16}$$

where just as above

$$\begin{aligned} a = (a_1,a_2) &= (-\frac{\partial\beta}{\partial x_2}, \frac{\partial\beta}{\partial x_1}) \\ \Delta\beta &= \pm 4Im(u_1\overline{u_2}), \\ \Delta a_0 &= \pm\sum_{k,j=1}^{2}(\,2\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}Re(u_k\overline{u_j}) - \partial_{x_k}^2|u_j|^2\,). \end{aligned} \tag{16b}$$

In what follows we will discuss the Cauchy initial value problem for the MSM and the MIS with data $u_1(\cdot,0)=u_0^1(\cdot)$ and $u_2(\cdot,0)=u_0^2(\cdot)$ in $H^s(\mathbb{R}^2), s>1/2$.

From now on, we will not distinguish between u_1 and u_2 in our formulas, as we will primarily use their functional analytic properties. Occasionally, we will be referring to the vector $u=(u_1,u_2)$ and the data $u_0=(u_0^1,u_0^2)$. We also note that viewed as a *linear* system with a priori given time dependent coefficients, the coupling of the systems occurs through one of the cubic nonlinearities; hence without any loss of generality we may schematically write the Cauchy problem for either system (15) or (16) as

$$\frac{\partial u}{\partial t} - iLu + \gamma\,\delta(au) \;=\; F(u,\overline{u}) \tag{17}$$

where, γ is a real nonzero constant and

(i) u is either u_1 or u_2,

(ii) L represents either $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ or $\Box_{x_1x_2} = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$

(iii) $a=(a_1,a_2)$ is real-valued and satisfies $(7),(11),(12)$ and (14)

(iv) δw represents either div $w = \frac{\partial w}{\partial x_1} + \frac{\partial w}{\partial x_2}$ or $\tilde{\text{div}}\ w = \frac{\partial w}{\partial x_1} - \frac{\partial w}{\partial x_2}$

(v) $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $\tilde{\nabla} = (\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2})$

(vi) $F(u,\overline{u})$ stands for the sum of the cubic-type and 'quintic' terms appearing in(15) and (16).

Remark. The derivative term in the MSM (17) has the form $a\cdot\nabla u$ which can easily be written as div (au) since div $a=0$. The derivative term in the MIS (16) has the form $a\cdot\tilde{\nabla}u$. Thus at the expense of adding an extra cubic term of the form $(\frac{\partial a_1}{\partial x_1} - \frac{\partial a_2}{\partial x_2})u$ we can view the derivative term as $\tilde{\text{div}}$ (au).

The following additional notation will be used throughout the rest of this article. For $1 \leq q, r \leq \infty$, we write

$$\|f\|_{L^q_t L^r_x} := \| \, \|f(\cdot,t)\|_{L^r(\mathbb{R}^2)} \, \|_{L^q(\mathbb{R})}$$

and write $\|f\|_{L^q([0,T])L^r_x}$ or simply $\|f\|_{L^q_T L^r_x}$ to denote $\| \, \|f(\cdot,t)\|_{L^r(\mathbb{R}^2)} \, \|_{L^q([0,T])}$. For $|s| \leq 1$ we denote by $J^s := (I-\Delta)^{s/2}$ the Bessel potential, and by $D^s = (-\Delta)^{s/2}$ the Riesz potential. By $|\nabla|^{-1} f(x)$ we denote the pseudo-differential operator of order -1 given by convolution with $1/|x|, x \in \mathbb{R}^2$ or on the Fourier side by point-wise multiplication by $|\xi|^{-1}$, $\xi \in \mathbb{R}^2$. Finally we write $A \lesssim B$ when there is a positive constant c such that $A \leq cB$. The meaning of this constant may change from line to line and its dependence on other parameters in the problems will be clear in the context such inequality appears.

2. Preliminaries

In this section we collect a few basic facts from Littlewood-Paley theory, recall the Leibniz rule for fractional differentiation and some linear estimates of Strichartz type associated to the free group e^{itL}.

Let $f(x)$ be a function on $\mathbb{R}^2$ and $\hat{f}(\xi)$ its Fourier transform. Consider $m(\xi)$ to be a non-negative radial bump function supported on the ball $|\xi| \leq 2$ and equal to 1 on the ball $|\xi| \leq 1$. Then for each integer k let $P_k(f)$ be the Littlewood-Paley projection operator onto frequencies $|\xi| \lesssim 2^k$. This is defined by

$$\widehat{P_k(f)}(\xi) := m(2^{-k}\xi)\hat{f}(\xi).$$

Clearly, $P_k \to I$ as $k \to \infty$ and $P_k \to 0$ as $k \to -\infty$ in the L^2 sense. The operator Q_k is the projection onto the *frequency annulus* $|\xi| \sim 2^k$ given by the formula,

$$Q_k := P_k - P_{k-1}.$$

Hence $\psi(\xi) := m(\xi) - m(2\xi)$ is supported on the annulus $1/2 \leq |\xi| \leq 2$, for all $\xi \neq 0$, $\sum_{k\in\mathbb{Z}} \psi(2^{-k}\xi) \equiv 1$, and

$$\widehat{Q_k(f)}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi).$$

The Littlewood-Paley projections are bounded operators in all the Lebesgue spaces. By telescoping the series we have

$$f \;=\; \sum_{k\in\mathbb{Z}} Q_k(f) \;=\; P_0 + \sum_{k>0} Q_k(f) \quad \text{in the sense of } L^2$$

or for any locally integrable function with decay at infinity. By construction and Plancherel,

$$||f||_2 \sim \big(\sum_k \|Q_k(f)\|_2^2\big)^{1/2} \sim \|P_0(f)\|_2 + \big(\sum_{k>0} \|Q_k(f)\|_2^2\big)^{1/2}$$

We also record the equivalence $\|D^s Q_k(f)\|_p \sim 2^{sk}\|Q_k(f)\|_p$ valid for any $1 \leq p \leq \infty$ and $s \in \mathbb{R}$.

Definition 1. *We denote by $\mathcal{B}^{\delta}_{r,p}$ be the Banach space of functions on $\mathbb{R}\times\mathbb{R}^n$ whose norm is given by*
$$\|f\|_{\mathcal{B}^{\delta}_{r,p}} := \|P_0(f)\|_{L^2_T L^r_x},+\Big(\sum_{k>0} 2^{\delta k\, p}\,\|Q_k(f)\|^p_{L^2_T L^r_x}\Big)^{1/p}$$
where $1<r\le\infty$, $1\le p<\infty$ and $\delta\ge 0$.

Remark. In what follows we will primarily use $\mathcal{B}^0_{\infty,1}$ and $\mathcal{B}^{3\varepsilon}_{\frac{1}{\varepsilon},1}$ for $\varepsilon>0$ small. Clearly, control of the $\mathcal{B}^0_{\infty,1}$ norm implies in particular, control of the $L^2_T L^\infty_x$ norm. Though not necessary, we choose at no extra cost to use these Besov-like space-time norms rather than the space-time Lebesgue norms.

Lemma 1 (The Leibniz Rule for Fractional Derivatives).
Let $0<s<1$. Then given $1<r<\infty$ and indeces $1<p_1,q_1,p_2,q_2<\infty$,
$$\|D^s(fg)\|_r \lesssim \|D^s f\|_{p_1}\|g\|_{q_1} + \|f\|_{p_2}\|D^s g\|_{q_2}$$
$$\|J^s(fg)\|_r \lesssim \|J^s f\|_{p_1}\|g\|_{q_1} + \|f\|_{p_2}\|J^s g\|_{q_2}$$
provided $\dfrac{1}{r}=\dfrac{1}{p_1}+\dfrac{1}{q_1}=\dfrac{1}{p_2}+\dfrac{1}{q_2}$.

For a proof see for example [8], [14], [19], [3] and [7].

Remark A careful examination of the proofs shows that if in addition f,g are in L^∞, then q_1 or p_2 are allowed to be ∞ as well.

Lemma 2 (Dispersive Estimates). *Let w be a classical smooth solution to*
$$\partial_t w(t,x) - iLw(t,x) = 0, \quad t>0, \quad x\in\mathbb{R}^n$$
$$w(0,x) = f(x),$$
Then we have,

(i) The Strichartz' Estimates ([42] [15] [10]*)*
$$\|w\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}$$
provided $2\le q,\ r\le\infty$, $\dfrac{1}{q}+\dfrac{n}{2r}=\dfrac{n}{4}$; but $(q,r)\ne(2,\infty)$ when $n=2$

(ii) Let $n=2$. then for any $\varepsilon>0, T>0$, there exists $c=c(\varepsilon)>0$ such that
$$\|w\|_{L^2[0,T]L^\infty_x} \le c\,T^\varepsilon\|f\|_{H^{3\varepsilon}_x} \qquad \text{and} \qquad \|w\|_{L^2[0,T],W^{3\varepsilon,\frac{1}{\varepsilon}}} \le c\,T^\varepsilon\|f\|_{H^{3\varepsilon}_x}$$
Similar estimates hold for any $n\ge 2$.

Proof. Estimates (i) for the elliptic Schrödinger linear equation ($L=\Delta$) are well known. For the hyperbolic Schrödinger ($L=\Box_{x_1,x_2}$) they were established in this generality by Ghidaglia and Saut in their study of the Davey-Stewartson systems [10] (see also[42]). The endpoint case was settled by Keel and Tao [15].

To check (ii) we use (i) when $n=2$ for an admissible pair (q_0,r_0) such that $\frac{1}{q_0}=\frac{1}{2}-\varepsilon$ and $r_0=\frac{1}{\varepsilon}$. Then by Strichartz and Sobolev embedding
$$\|w\|_{L^{q_0}[0,T]L^\infty_x} \le \|w\|_{L^{q_0}[0,T],W^{3\varepsilon,r_0}} \le c\|f\|_{H^{3\varepsilon}_x}.$$
An application of Hölder's inequality on $[0,T]$ gives then the desired estimates since $q_0>2$.

3. A priori estimates

Proposition 1 (Energy estimates).

Let u be a classical smooth solution to (17). Then there exists an absolute constant $C > 0$, such that

$$\frac{\partial}{\partial t}\|u(t)\|_{L}^{2} \leq C_s\big(\|\nabla a(t)\|_{L^\infty}\|u(t)\|_{L^2}^{2} + \|F(t)\|_{L^2}\|u(t)\|_{L^2}\big)$$

Moreover, given $s > 0$ there exists an absolute constant $C_s > 0$, such that

$$\frac{\partial}{\partial t}\|u(t)\|_{H^s}^{2} \leq C_s\big(\|\nabla a(t)\|_{H^s}\|u(t)\|_{H^s}\|u(t)\|_{L^\infty} + \|\nabla a(t)\|_{L^\infty}\|u(t)\|_{H^s}^{2} + \|F(t)\|_{H^s}\|u(t)\|_{H^s}\big)$$

where by $\|\nabla a(t)\|_{L^\infty}$ we denote $\sup\{\,\|\dfrac{\partial a_k}{\partial x_l}\|_{L^\infty_x},\ 1\leq k,l\leq 2\,\}$.

Proof. The proof of this proposition when $L = \Delta_{x_1x_2}$ and $\delta =$ div is in [32] where in fact, L^2 conservation also holds. A careful examination of their proof when $s > 0$ shows the same argument gives the case $L = \Box_{x_1x_2}$ and $\delta = \tilde{\text{div}}$ after noting that

$$\langle \Box_{x_1x_2}u, u\rangle \;=\; -\int_{\mathbb{R}^2} |\frac{\partial u}{\partial x_1}|^2 - |\frac{\partial u}{\partial x_2}|^2\, dx_1\, dx_2;$$

whence $\text{Re}\; i\langle \Box_{x_1x_2}u, u\rangle \;=\; 0$ as well. To see the L^2 estimate above one can argue as follows. We multiply both sides of (17) by $\overline{u}$, integrate and take real part of the result. We proceed to analyze the terms that arise. Bear in mind that a is real valued; then:

$$Re\int \delta(a\,u)\overline{u}dx = Re\int \delta(a)|u|^2dx \,+\, Re\int\langle a,\tilde{\nabla}u\rangle\,\overline{u}dx$$

On the other hand, an integration by parts yields

$$\int\langle a,\tilde{\nabla}u\rangle\,\overline{u}dx \;=-\int \delta(a)|u|^2dx - \int\langle a\,u,\tilde{\nabla}\overline{u}\rangle\,dx$$

whence

$$Re\int\langle a,\tilde{\nabla}u\rangle\,\overline{u}dx = -\frac{1}{2}\int \delta(a)|u|^2\;dx.$$

We get

$$Re\int \delta(a\,u)\,\overline{u}dx = -\frac{1}{2}\int \delta(a)|u|^2dx,$$

Thus by Hölder's

$$\left|Re\int \delta(a\,u)\overline{u}dx\right| \leq C\|u\|_{L^2}^{2}\,\|\nabla a\|_{L^\infty}$$

and the desired estimate follows.

Remarks. (i) As a consequence of the energy estimates above, we have the following a priori estimate for any $0 \leq s \leq 1$:

$$\frac{\partial}{\partial t}\|u(t)\|^2_{H^s_x} \lesssim \|u(t)\|^2_{L^\infty_x}\,\|u(t)\|^2_{H^s_x} + \|\nabla a(t)\|_{L^\infty_x}\,\|u(t)\|^2_{H^s_x} + \|F(t)\|_{H^s_x}\,\|u(t)\|_{H^s_x}$$

(ii) For u a classical smooth solution (e.g. it is known that there exists $u \in H^k$ for $k > 3$ by [41] [37][9] [20] [28] [32] [40]) of (17) then the 1-form a is the solution of the elliptic problem,

$$da = \pm\ \mathrm{Im}\ (u\overline{u})$$

on each fixed time slice $\{t\} \times \mathbb{R}^2$ and it exists for each time t in the lifespan $[0, T]$ of the smooth solution in H^k. We note that da is a 2-form. The following a priori estimates then hold

$$\|da(t)\|_{L^\infty_x} \leq \|u(t)\|^2_{L^\infty_x} \qquad \|da(t)\|_{H^s_x} \leq \|u(t)\|_{H^s_x}\|u(t)\|_{L^\infty_x}, \quad 0 \leq s \leq 1$$

for every t in the lifespan of the smooth solution. The energy estimate however depends on the full ∇a and not just on da, forcing consideration of the Riesz transforms when estimating a or ∇a in terms of u.

(iii) Suppose u and v are two classical smooth solutions of (17) with the same initial data. Let a and $\mathfrak{a}$ be as above satisfying (11), (12) and (14) in terms of u and v respectively. Then if $w := u - v$ and $A = a - \mathfrak{a}$ one can see that for example, w satisfies:

$$i\partial_t w + Lw \;=\; -2i\delta(aw) \;-\; 2i\delta(Av) + F_1 - F_2$$

$$\frac{\partial}{\partial t}\|w\|^2_{L^2} \lesssim \|\nabla a\|_{L^\infty}\|w\|^2_{L^2} + \|\delta(Av)\|_{L^2}\|w\|_{L^2} + \|F_1 - F_2\|_{L^2}\|w\|_{L^2}.$$

Note that although div $a = 0$ we do not make use of it since (17) may be not just MSM but also MIS. The 'asymmetry' present in this 'difference' estimate - i.e. the inability to draw a term of the form $\|w\|^2$ in all terms on the right hand side– makes it useless to derive uniqueness for data in H^s, $s > 1/2$ solely from the 'energy method'. We will discuss this issue more in detail in Section 4.

Our next result gives a priori control on the $\mathcal{B}^{3\varepsilon}_{\frac{1}{\varepsilon},1}$ and $\mathcal{B}^0_{\infty,1}$ norms of the solution. Hence, in particular gives $L^2((0,T])L^\infty_x$ a priori control. These estimates are the key to the 1/2-derivative gain over what energy plus Sobolev would give. This result and its proof closely follows Kenig's ideas in [16].

Proposition 2. *Let u be a smooth solution of (17). Then for any $\varepsilon > 0$ and $T \in (0, 1]$ there exists a positive constant C_ε such that the following estimate holds.*

(19)

$$\|u\|_{\mathcal{B}^0_{\infty,1}} + \|u\|_{\mathcal{B}^{\tilde\varepsilon}_{r_0,1}} \;\leq C_\varepsilon\, T^\varepsilon \Big\{ \|J^{1/2+\tilde\varepsilon+\varepsilon}\, u\|_{L^\infty((0,T])L^2_x} + T^{1/2}\,(\int_0^T \|J^{1/2+\tilde\varepsilon+\varepsilon}\,(a\,u)\|^2_{L^2_x}\, dt\,)^{1/2} + \\ + \; T^{1/2}\,(\int_0^T \|J^{-1/2+\tilde\varepsilon+\varepsilon}\,F\,\|^2_{L^2_x}\, dt)^{1/2} \Big\}$$

for $\tilde{\varepsilon} = 3\varepsilon$ *and* $r_0 = \frac{1}{\varepsilon}$

Proof. We start by performing a Littlewood Paley decomposition of u. We let $\lambda := 2^k$ and write $u_\lambda := Q_k(u)$; then in -say the L^2-sense-,

$$u \,=\, P_0(u) \,+\, \sum_{k>0} u_\lambda.$$

Moreover, from equation (17),

$$\frac{\partial}{\partial t} u_\lambda - iLu_\lambda \,=\, (\delta(a\,u))_\lambda + F_\lambda = \delta(a\,u)_\lambda + F_\lambda$$

since $Q_k\delta f = \delta Q_k f$. Next we partition $[0, T]$, $T \leq 1$ into subintervals $I_j := (c_j, d_j]$, such that $|I_j| = T2^{-k} \,=\, \dfrac{T}{\lambda}$ for each fixed $k > 0$. The number of subintervals is $N := \lambda$. We write for each fixed λ:

$$\begin{aligned}
\|u_\lambda\|_{L^2((0,T])L^\infty_x} \,&=\, \left(\sum_{j=1}^{N} \|u_\lambda\|^2_{L^2(I_j)L^\infty_x}\right)^{1/2} \\
\|J^{3\varepsilon}u_\lambda\|_{L^2((0,T])L^{r_0}_x} \,&=\, \left(\sum_{j=1}^{N} \|J^{3\varepsilon}u_\lambda\|^2_{L^2(I_j)L^{r_0}_x}\right)^{1/2}
\end{aligned}$$

Now on each I_j we write

$$u_\lambda(\cdot, t) = S(t - c_j)u_\lambda(\cdot, c_j) \,+\, \int_{c_j}^{t} S(t - t')\delta(a\,u)_\lambda(t')\,dt' \,+\, \int_{c_j}^{t} S(t - t')F_\lambda(t')\,dt'$$

where we have denoted by $S(t)$ the unitary operator e^{itL}. By Lemma 2 (ii) applied to u_λ on I_j we then obtain,

$$\begin{aligned}
\|u_\lambda\|^2_{L^2(I_j)L^\infty_x} + \|J^{\tilde{\varepsilon}}u_\lambda\|^2_{L^2(I_j)L^{r_0}_x} \leq C_\varepsilon T^{2\varepsilon}\Bigg\{ &\|J^{3\varepsilon}u_\lambda(c_j)\|^2_{L^2_x} \,+\, \left(\int_{c_j}^{d_j} \|J^{1+3\varepsilon}(a\,u)_\lambda(t')\|_{L^2_x}\right)^2 + \\
&+\, \left(\int_{c_j}^{d_j} \|J^{3\varepsilon}F_\lambda(t')\|_{L^2_x}\right)^2 \Bigg\}
\end{aligned}$$

where we have used implicitly that $J^{3\varepsilon}\delta = J^{1+3\varepsilon}\mathcal{P}$ with $\mathcal{P}$ an appropriate pseudo-differential operator of order zero in x, bounded in $L^2(\mathbb{R}^2)$.

We note now that by Sobolev inequality the result for $\mathcal{B}^{\tilde{\varepsilon}}_{r_0,1}$ implies the bound for $\mathcal{B}^0_{\infty,1}$. Thus we continue working with the former –though the same calculation gives it for the latter.

Cauchy-Schwarz now gives:

$$\|J^{\tilde{\varepsilon}}u_\lambda\|^2_{L^2(I_j)L^{r_0}_x} \leq C_\varepsilon T^{2\varepsilon}\left\{ \|J^{\tilde{\varepsilon}}u_\lambda(c_j)\|^2_{L^2_x} \,+\, |I_j|\|J^{1+\tilde{\varepsilon}}(a\,u)_\lambda\|^2_{L^2(I_j)L^2_x} \,+\, |I_j|\|J^{\tilde{\varepsilon}}F_\lambda\|^2_{L^2(I_j)L^2_x} \right\}$$

Next, we sum over j, $1 \le j \le N$ for λ fixed to obtain:

$$\sum_{j=1}^{N} \|J^{\tilde\varepsilon} u_\lambda\|^2_{L^2(I_j)L^{r_0}_x} \le C_\varepsilon T^{2\varepsilon}\Big\{\sum_{j=1}^{N} \|J^{\tilde\varepsilon} u_\lambda(c_j)\|^2_{L^2_x} + \sum_{j=1}^{N} |I_j|\|J^{1+\tilde\varepsilon}(a\,u)_\lambda\|^2_{L^2(I_j)L^2_x} + + \sum_{j=1}^{N} |I_j|\|J^{\tilde\varepsilon} F_\lambda\|^2_{L^2(I_j)L^2_x}\Big\}$$

$$\le C_\varepsilon T^{2\varepsilon}\Big\{N \sup_{0<t<T} \|J^{\tilde\varepsilon} u_\lambda(t)\|^2_{L^2_x} + \frac{T}{\lambda}\|J^{1+\tilde\varepsilon}(a\,u)_\lambda\|^2_{L^2((0,T])L^2_x} + \frac{T}{\lambda}\|J^{\tilde\varepsilon} F_\lambda\|^2_{L^2((0,T])L^2_x}\Big\}.$$

Hence,

$$\|J^{\tilde\varepsilon} u_\lambda\|_{L^2((0,T])L^{r_0}_x} \le C_\varepsilon T^{\varepsilon}\Big\{N^{1/2} \sup_{0<t<T} \|J^{\tilde\varepsilon} u_\lambda(t)\|_{L^2_x} + (\frac{T}{\lambda})^{1/2}\|J^{1+\tilde\varepsilon}(a\,u)_\lambda\|_{L^2((0,T])L^2_x} + + (\frac{T}{\lambda})^{1/2}\|J^{\tilde\varepsilon} F_\lambda\|_{L^2((0,T])L^2_x}\Big\}.$$

$$\le C_\varepsilon T^{\varepsilon}\Big\{\sup_{0<t<T} \|J^{1/2+\tilde\varepsilon} u_\lambda(t)\|_{L^2_x} + T^{1/2}\|J^{1/2+\tilde\varepsilon}(a\,u)_\lambda\|_{L^2((0,T])L^2_x} + T^{1/2}\|J^{-1/2+\tilde\varepsilon} F_\lambda\|_{L^2((0,T])L^2_x}\Big\}.$$

Finally, we need to sum in λ - i.e., in $k \ge 0$. We write:

$$\sum_\lambda \|J^{\tilde\varepsilon} u_\lambda\|_{L^2((0,T])L^{r_0}_x} \le C_\varepsilon T^{\varepsilon}\Big\{\sum_\lambda \sup_{0<t<T} \|J^{1/2+\tilde\varepsilon} u_\lambda(t)\|_{L^2_x} + + T^{1/2}\sum_\lambda\Big(\int_0^T \|J^{1/2+\tilde\varepsilon}(a\,u)_\lambda\|^2_{L^2_x}\Big)^{1/2} + T^{1/2}\sum_\lambda\Big(\int_0^T \|J^{-1/2+\tilde\varepsilon} F_\lambda\|^2_{L^2_x}\Big)^{1/2}\Big\}$$

$$\le C_\varepsilon T^{\varepsilon}\Big\{\sup_{0<t<T}\Big(\sum_\lambda \|J^{1/2+\tilde\varepsilon+\varepsilon} u_\lambda(t)\|^2_{L^2_x}\Big)^{1/2} + + T^{1/2}\Big(\sum_\lambda \int_0^T \|J^{1/2+\tilde\varepsilon+\varepsilon}(a\,u)_\lambda\|^2_{L^2_x}\Big)^{1/2} + T^{1/2}\Big(\sum_\lambda \int_0^T \|J^{-1/2+\tilde\varepsilon+\varepsilon} F_\lambda\|^2_{L^2_x}\Big)^{1/2}$$

$$\le C_\varepsilon T^{\varepsilon}\Big\{\sup_{0<t<T} \|J^{1/2+\tilde\varepsilon+\varepsilon} u(t)\|_{L^2_x} + + T^{1/2}\Big(\int_0^T \|J^{1/2+\tilde\varepsilon+\varepsilon}(a\,u)\|^2_{L^2_x}\Big)^{1/2} + T^{1/2}\Big(\int_0^T \|J^{-1/2+\tilde\varepsilon+\varepsilon} F\|^2_{L^2_x}\Big)^{1/2}.$$

We conclude by noting that since the larger Besov-type norm $\sum_\lambda \|J^{\tilde\varepsilon} u_\lambda\|_{L^2((0,T])L^{r_0}_x}$, controls in particular both $\|u\|_{L^2((0,T])L^\infty_x}$ and $\|u\|_{\mathcal{B}^0_{\infty,1}}$ so does the right hand side of (19) and the desired conclusions follow.

Remarks.

Given any $\varepsilon > 0$, a similar proof gives also an $L^{q_0}W^{\epsilon,r_0}$ a priori control with same right hand side and for $\epsilon > 0$ and (q_0, r_0) an admissible pair such that $4\varepsilon > \epsilon > \frac{1}{r_0} = \varepsilon$ and $\frac{1}{q_0} = \frac{1}{2} - \frac{1}{r_0}$ (c.f. the proof of Lemma 2 (ii)).

Note also that in the course of the proof one could have simply put an $L^1_{[0,T]}H_x^{1/2+\tilde{\varepsilon}+\varepsilon}$ norm on the nonlinear terms on the right hand side.

Lemma 3. *Let u be a classical smooth solution of (17) with right hand side F as in (15) or (16). Let $\varepsilon > 0$ be given and let $1 > s > 1/2 + 4\varepsilon$. There exists a positive constant $M > 0$, such that for each time t fixed in the lifespan of the smooth solution we have*

$$\begin{aligned}
&\|J^s(a\,u)\|_{L^2_x} \leq M\,\|u\|^3_{H^s_x}\\
&\|J^{s-1}F\|_{L^2_x} \leq M\left\{\|u\|^2_{H^s_x}\|u\|_{L^\infty_x} + \|u\|^4_{H^s_x}\|u\|_{L^\infty_x}\right\}
\end{aligned}$$

Proof.

$$\|J^s(a\,u)\|_{L^2_x} \lesssim \|J^s(a)\|_{L^4_x}\|u\|_{L^4_x} + \|a\|_{L^\infty_x}\|u\|_{H^s_x}$$

by Lemma 1. By the Sobolev embeddings and the boundedness of the Riesz transforms in L^p, $1 < p < \infty$ we have:

$$\|a\|_{L^\infty_x} \lesssim \|J^s(a)\|_{L^4_x} \lesssim \|J^s(u\overline{u})\|_{L^{4/3}_x} \lesssim \|J^s u\|_{L^2_x}\|u\|_{L^4_x} \lesssim \|u\|^2_{H^s_x}$$

from which the first estimate follows.

To prove the second estimate we need to take into account the special form of F as it appears in (15) and (16); in particular we pay special attention to those cubic and 'quintic' terms in which a pseudo-differential operator $\mathcal{R}$ of order zero -bounded in L^p_x, $1 < p < \infty$-. appears acting on $\mathrm{Im}\,(u\overline{u})$ such as in a_0 defined in (13) or (16b) and in α_1 defined in (15b). In fact $\mathcal{R}$ is one of the Riesz transforms or could simply be taken to be the identity operator when dealing with the simple cubic term $\mathrm{Im}\,(u\overline{u})\,u$. Recall also that $a = (a_1, a_2)$ was defined in (11)-(12) or (16b). We write generically for all of them

$$\begin{aligned}
\|J^{s-1}F\|_{L^2_x} &\lesssim \|J^{s-1}\mathcal{R}(u\overline{u})\,u\|_{L^2_x} + \|J^{s-1}(a_1^2 \pm a_2^2)\,u\|_{L^2_x}\\
&\lesssim \|\mathcal{R}(u\overline{u})\|_{L^2_x}\|u\|_{L^\infty_x} \;+\; \|(a_1^2 \pm a_2^2)\|_{L^2_x}\|u\|_{L^\infty_x}\\
&\lesssim \|u\|^2_{L^4_x}\|u\|_{L^\infty_x} + \|a\|^2_{L^4_x}\|u\|_{L^\infty_x}\\
&\lesssim \|u\|^2_{H^s_x}\|u\|_{L^\infty_x} + \|u\|^4_{H^s_x}\|u\|_{L^\infty_x}
\end{aligned}$$

since

$$\|a\|_{L^4_x} \lesssim \|\nabla a\|_{L^{4/3}_x} \lesssim \|u\overline{u}\|_{L^{4/3}_x} \lesssim \|u\|^2_{L^{8/3}_x} \lesssim \|u\|^2_{H^{1/4}_x}.$$

The constant M thus arise from all the constants absorbed in the $\lesssim$ signs. We note these depend on properties such as the L^2 boundedness of $\mathcal{R}$ and the Sobolev embedding constants. It may thus depend in particular on s, ε.

Let $\varepsilon > 0$ be given and let $s > 1/2 + 4\varepsilon$. Let $\tilde{T_1}$ be a first time in $(0, 1]$ such that

$$\sup_{0<t<\tilde{T_1}} \|u(t)\|_{H^s(\mathbb{R}^2)} \geq 2\|u_0\|_{H^s(\mathbb{R}^2)}.$$

If such $\tilde{T_1} > 0$ does not exist then we must have that $\sup_{0<t<1} \|u(t)\|_{H^s(\mathbb{R}^2)} \leq 2\|u_0\|_{H^s(\mathbb{R}^2)}$ and we will set $T_1 = 1$. Otherwise, by continuity in t of the quantity $I(t) = \sup_{0<t'<t} \|u(t')\|_{H^s(\mathbb{R}^2)}$ there exists a first $T_1 > 0$, $T_1 < \tilde{T_1}$ such that

$$\sup_{0<t<T_1} \|u(t)\|_{H^s(\mathbb{R}^2)} \; = \; 2\|u_0\|_{H^s(\mathbb{R}^2)}.$$

We then fix this T_1 for the remainder of the paper and define

$$1 > T_0 \; = \; \min\{T_1, \left(\frac{1}{9MC_\varepsilon(\|u_0\|^2_{H^s_x} + \|u_0\|^4_{H^s_x})}\right)^2\} \tag{21}$$

where $C_\varepsilon > 0$ is the constant of Proposition 2 and $M > 0$ the one of Lemma 3.

Proposition 3. *Let u be a classical solution of (17) with right hand side F as in (15) or (16). Let $\varepsilon > 0$ be given and let $1 > s > 1/2 + 4\varepsilon$. There exists a constant $C = C(s) > 0$, such that*

$$\|\, u \,\|_{\mathcal{B}^{3\varepsilon}_{\frac{1}{\varepsilon},1}} \; \leq \; C\, T_0^{\varepsilon}\, \|u_0\|_{H^s_x} \qquad \textit{and} \qquad \|\, u \,\|_{\mathcal{B}^{0}_{\infty,1}} \; \leq \; C\, T_0^{\varepsilon}\, \|u_0\|_{H^s_x}$$

Proof. By Proposition 2, we have that both left hand sides in the statement are less than or equal to:

$$C_\varepsilon T_0^\varepsilon \Big\{ \sup_{0<t<T_0} \|J^s u(t)\|_{L^2_x} \; + \; T_0^{1/2} \|J^s(a\, u)\|_{L^2((0,T_0])L^2_x} \; + \; T_0^{1/2} \|J^{s-1} F\|_{L^2((0,T_0])L^2_x}. \Big\}$$

Lemma 3 yield the estimates for the second and third terms. All in all we then have,

$$\begin{aligned} \|u\|_{B^{3\varepsilon}_{\frac{1}{\varepsilon},1}} \; + \; \|u\|_{B^{0}_{\infty,1}} \leq C'_\varepsilon T_0^\varepsilon \Big\{ & \|u\|_{L^\infty((0,T_0])H^s_x} \; + \; T_0 M \|u\|^3_{L^\infty((0,T_0])H^s_x} \quad + \\ & + \; T_0^{1/2} M\, \|u\|_{L^2((0,T_0])L^\infty_x} \left(\|u\|^2_{L^\infty((0,T_0])H^s_x} \; + \; \|u\|^4_{L^\infty((0,T_0])H^s_x} \right) \Big\}. \end{aligned}$$

Since $T_0 \leq T_1$, we have that $\|u\|_{L^\infty((0,T_0])H^s_x} \leq 2\|u_0\|_{H^s_x}$; and thus we can make the last term less than or equal to

$$C_\varepsilon T_0^\varepsilon \Big\{ \, (1 + 4T_0 M \|u_0\|^2_{H^s_x})\, \|u_0\|_{H^s_x} \; + \; T_0^{1/2} M \left(\|u_0\|^2_{H^s_x} \; + \; \|u_0\|^4_{H^s_x} \right) \|u\|_{L^2((0,T_0])L^\infty_x} \, \Big\}.$$

Finally by (21) we conclude from the above estimate the conclusion of Proposition 3.

4. Well posedness of the MIS and MSM

Main Theorem. *The Cauchy initial value problem associated to (17) admits a local in time solution in H^s, $s > 1/2$. More precisely, given data $u_0 \in H^s(\mathbb{R}^2)$, $s > 1/2$ there exists a time $0 < \mathbf{T} = \mathbf{T}(\|u_0\|_{H^s_x})$ and a solution to (17) such that*

$$u \in C([0, \mathbf{T}]; H^s) \quad \text{and} \quad u \in L^2_t([0, \mathbf{T}])L^\infty_x.$$

For data in H^1, the $C([0, \mathbf{T}]; H^1_x)$-solution can be shown to be unique and furthermore, the mapping $u_0 \to u \in C([0, \mathbf{T}]; H^1)$ is continuous.

Before proving the theorem we collect a few estimates for $\|F\|_{H^s}$, $s > 1/2$ that will be needed in its proof.

Lemma 4. *Let u be a classical solution of (17) with right hand side F as in (15) or (16). Let $\varepsilon > 0$ be given and let $1 > s > 1/2 + 4\varepsilon$. Let a_0 be defined by (13) and α_1 be defined by (15b). Then there exists a constant $C > 0$ such that for each fixed time t in the life span of the smooth solution,*

$$(i)\ \|F(t)\|_{H^s_x} \leq C\left\{ \left(\|u(t)\|^2_{L^\infty_x} + \|a_0(t)\|_{L^\infty_x} + \|\alpha_1(t)\|_{L^\infty_x} \right) \|u(t)\|_{H^s_x} + \|u(t)\|^5_{H^s_x} \right\}$$

$$(ii)\ \|\nabla a\|_{L^1((0,T])L^\infty_x} + \|a_0\|_{L^1((0,T])L^\infty_x} + \|\alpha_1\|_{L^1((0,T])L^\infty_x} \leq C\,\|u\|_{\mathcal{B}^0_{\infty,1}} \|u\|_{\mathcal{B}^{3\varepsilon}_{\frac{1}{\varepsilon},1}}$$

The constant C depends on properties such as the L^2 boundedness of $\mathcal{R}$ and the Sobolev embedding constants. It may thus depend in particular on s, ε.

Proof. In order to simplify the estimates we view F as essentially the sum of three terms of the form

(1) $\quad \mathrm{Im}\,(u\overline{u})\,u$
(2) $\quad a_0 u \quad$ (similarly for $\alpha_1 u$)
(3) $\quad (a_1^2 \pm a_2^2)\,u \quad$ where $a = (a_1, a_2)$ is defined by (11)-(12) or (16b)

To prove (i) we first note the H^s_x- norm of terms as in (1) are easily estimated by $\|u\|^2_{L^\infty}\|u\|_{H^s}$. Similarly, by Lemma 1 we have

$$\begin{aligned} \|a_0 u\|_{H^s_x} &\leq C\|a_0\|_{H^s}\|u\|_{L^\infty} + \|a_0\|_{L^\infty}\|u\|_{H^s} \\ &\leq C\|u\|_{H^s}\|u\|^2_{L^\infty} + \|a_0\|_{L^\infty}\|u\|_{H^s} \end{aligned}$$

since $\mathcal{R}$ commutes with J^s and it is bounded in L^2. For those terms as in (3) we proceed as follows. Again by Lemma 1 we have

$$\|J^s((a_1^2 \pm a_2^2)\,u)\|_{L^2_x} \leq C\|J^s(a_1^2 \pm a_2^2)\|_{L^4_x}\|u\|_{L^4} + \|(a_1^2 \pm a_2^2)\|_{L^\infty_x}\|u\|_{H^s}$$

As in the proof of Lemma 3,

$$\|a_1\|_{L^\infty} + \|a_2\|_{L^\infty} \lesssim \|J^s a_1\|_{L^4} + \|J^s a_2\|_{L^4} \lesssim \|u\|^2_{H^s_x}.$$

On the other hand,

$$\|J^s(a_1^2 \pm a_2^2)\|_{L^4} \leq C\Big(\|J^s a_1\|_{L^4}\|a_1\|_{L^\infty} + \|J^s a_2\|_{L^4}\|a_2\|_{L^\infty}\Big) \leq C\,\|u\|^4_{H^s}.$$

Using $H^s \hookrightarrow L^4$ and $\|(a_1^2 \pm a_2^2)\|_{L^\infty_x} \lesssim \|a_1\|^2_{L^\infty} + \|a_2\|^2_{L^\infty}$ we obtain the desired estimate for (i).

To prove (ii) we write,

$$\begin{aligned}\|\nabla a(t)\|_{L^\infty} + \|a_0(t)\|_{L^\infty} + \|\alpha_1(t)\|_{L^\infty} &\sim \|\mathcal{R}(u\,\overline{u})\|_{L^\infty}\\ &\leq C\|J^{3\varepsilon}(u\,\overline{u})\|_{L^{r_0}} \qquad \text{where } r_0 = \frac{1}{\varepsilon}\end{aligned}$$

Then,

$$\begin{aligned}&\|\nabla a\|_{L^1((0,T])L^\infty_x} + \|a_0\|_{L^1((0,T])L^\infty_x} + \|\alpha_1\|_{L^1((0,T])L^\infty_x}\\ &\sim \int_0^T \|\mathcal{R}(u\,\overline{u})(t)\|_{L^\infty_x}\,dt \leq C\int_0^T \|J^{3\varepsilon}(u)\|_{L^{r_0}}\|u\|_{L^\infty}dt\\ &\leq C\left(\int_0^T \|u\|^2_{L^\infty}\,dt\right)^{1/2}\left(\int_0^T \|J^{3\varepsilon}(u)\|^2_{L^{r_0}}\,dt\right)^{1/2}\\ &\leq C\|u\|_{L^2((0,T])L^\infty}\|J^{3\varepsilon}(u)\|_{L^2((0,T])L^{r_0}}\\ &\leq C\,\|u\|_{\mathcal{B}^0_{\infty,1}}\|u\|_{\mathcal{B}^{3\varepsilon}_{r_0,1}}\end{aligned}$$

Proposition 4. *Let u be a local smooth solution of (17). Let u_0 be in $H^s \cap H^m$, with $s > 1/2, m > 1$. Then there exists a positive constant C_s and a time $\mathbf{T} > 0$, $\mathbf{T} = \mathbf{T}(s, \|u_0\|_{H^s})$ such that*

$$\sup_{0<t<\mathbf{T}} \|u(\cdot,t)\|_{H^s} \leq C_s\|u_0\|_{H^s} \tag{22}$$

Proof. Let $\mathbf{T} = T_0$ from (21). Then since $\mathbf{T} \leq T_1 < 1$, $\sup_{0<t<\mathbf{T}} \|u(\cdot,t)\|_{H^s} \leq 2\|u_0\|_{H^s}$. We need to exhibit a positive lower bound on $\mathbf{T}$ depending solely on $\|u_0\|_{H^s}$ and absolute constants.

• Assume $\mathbf{T} = T_1$. First of all, if $T_1 = 1$ we are done. Otherwise, we proceed as follows. By the energy estimates in Proposition 1, the Remarks that follow and Lemma 4 (i) , we have

$$\frac{\partial}{\partial t}\|u(t)\|^2_{H^s} \lesssim \Big(\|u(t)\|^2_{L^\infty} + \|\nabla a(t)\|_{L^\infty} + \|a_0(t)\|_{L^\infty} + \|\alpha_1(t)\|_{L^\infty}\Big)\,\|u(t)\|^2_{H^s} + \|u(t)\|^6_{H^s}$$

Next we integrate in time from 0 to $\mathbf{T} = T_1$ and use the estimates in Lemma 4 (ii) and Proposition 3 to get

$$\begin{aligned}3\|u_0\|^2_{H^s} &= \|u(\mathbf{T})\|^2_{H^s} - \|u(0)\|^2_{H^s}\\ &\lesssim \|u\|_{\mathcal{B}^0_{\infty,1}}\,\{\|u\|_{\mathcal{B}^0_{\infty,1}} + \|u\|_{\mathcal{B}^{\tilde{\varepsilon}}_{r_0,1}}\}\,\sup_{0<t<\mathbf{T}}\|u(t)\|^2_{H^s} + \mathbf{T}\sup_{0<t<\mathbf{T}}\|u(t)\|^6_{H^s}\\ &\lesssim \{4C'_\varepsilon\mathbf{T}^{2\varepsilon}\,\|u_0\|^4_{H^s} + 32\,\mathbf{T}\,\|u_0\|^6_{H^s}\,\}\end{aligned}$$

Since $\mathbf{T} < 1$ we have that

$$1 \leq \{\frac{4}{3} C_\varepsilon^{''} \mathbf{T}^{2\varepsilon} \|u_0\|_{H^s}^2 + \frac{32}{3} C_\varepsilon^{''} \mathbf{T}^{2\varepsilon} \|u_0\|_{H^s}^4 \}.$$

Hence the lower bound,

$$\mathbf{T} \geq \frac{C_\varepsilon^{'''}}{(\|u_0\|_{H^s}^2 + \|u_0\|_{H^s}^4)^{1/2\varepsilon}} \; > \; 0.$$

• Assume $\mathbf{T} = \left(\frac{1}{9MC_\varepsilon(\|u_0\|_{H_x^s}^2 + \|u_0\|_{H_x^s}^4)} \right)^2 \; < \; T_1$ as in (21); then we automatically have the desired lower bound on $\mathbf{T}$.

Proof of the Main Theorem.

Existence. The existence proof follows standard compactness and approximation arguments relying on Proposition 4. Given data u_0 in H^s, $s > 1/2$ we approximate it by a sequence $u_0^{(j)}$ in H^m, $m > 1$ of smooth initial data. We can then produce for a short time $\{u^{(j)}(t,x)\}_j$ and$\{a^{(j)}(t,x)\}_j$ for which the a priori estimates in Propositions 1–4 hold. Moreover, using the MIS (15)(15b) or the MSM system (16)(16 b) we can additionally obtain appropriate estimates for $\partial_t u^{(j)}$ thus ensuring the compactness of the *space-time* Sobolev embeddings. We can then extract a subsequence weak-$*$ converging in $L^\infty_{\mathbf{T}} H^s_x \cap L^{q_0}_{\mathbf{T}} W_x^{-\epsilon, r_0}$ and strongly in $L^p_t L^2_{loc}$, any $p > 1$ which is enough to establish weak convergence of the nonlinear terms.

All in all, by compactness we can extract a converging subsequence whose limit will satisfy either the MIS or the MSM in the sense of distributions and the same a priori estimates with life span $[0, \mathbf{T}]$, $\mathbf{T} = \mathbf{T}(\|u_0\|_{H^s_x}) > 0$, $s > 1/2$. For more details see for example [13].

Uniqueness. This follows essentially by the argument of J. Kato in [13] which is in turn based on the following sharp Trüdinger's interpolation estimate

$$\|u\|_{L^q} \leq \sqrt{q/2}\,(4\pi)^{(2-q)/2q}\, \|u\|_{L^2(\mathbb{R}^2)}^{2/q}\, \|\nabla u\|_{L^2(\mathbb{R}^2)}^{(1-2/q)}.$$

shown in [34].

Suppose u and v are two classical smooth solutions of (17) with the same initial data f in $H^1(\mathbb{R}^2)$. Let us denote by $M := \|f\|_{H^1}$. Then by our local existence result we know there exist $0 < \mathbf{T} = \mathbf{T}(M)$ and $C > 0$ such that

$$\sup_{0<t<\mathbf{T}} \|u\|_{H^1} \leq C\, M \quad \text{and} \quad \sup_{0<t<\mathbf{T}} \|v\|_{H^1} \leq C\, M.$$

Let a and $\mathfrak{a}$ satisfy (11), (12) and (14) in terms of u and v respectively. Then if $w := u - v$ and $A = a - \mathfrak{a}$; w is the solution to the difference equation

$$i\partial_t w + Lw \; = \; -2i\delta(aw) \; - \; 2i\delta(Av) + F_1 - F_2 \tag{23}$$

where $F_1 = F(u)$ and $F_2 = F(v)$ with F as in (15) and (16). More precisely, taking into account the special form of F as it appears in (15) and (16) we write generically for all of them

$$F(u) \sim \mathcal{R}(u\overline{u})\, u + (a_1^2 \pm a_2^2)\, u$$

where $\mathcal{R}$ of order zero -bounded in L^p_x, $1 < p < \infty$-. appears acting on $\mathrm{Im}\,(u\overline{u})$ such as in a_0 defined in (13) or (16b) and in α_1 defined in (15b). In fact $\mathcal{R}$ is one of the Riesz transforms - or could simply be taken to be the identity operator when dealing with the simple cubic term $\mathrm{Im}\,(u\overline{u})\,u$-. Recall also that $a = (a_1, a_2)$ was defined in (11)-(12) or (16b). Then we essentially have that

$$F(u) - F(v) \sim \mathcal{R}(u\overline{u})\,u + (a_1^2 \pm a_2^2)\,u - \mathcal{R}(v\overline{v})\,v + (\mathfrak{a}_1^2 \pm \mathfrak{a}_2^2)\,v$$
$$= \mathcal{R}(u\overline{u})w + (a_1^2 \pm a_2^2)\,w + \Big(\mathcal{R}(u\overline{u}) - \mathcal{R}(v\overline{v})\Big)\,v \;+\; \Big((a_1^2 \pm a_2^2) - (\mathfrak{a}_1^2 \pm \mathfrak{a}_2^2)\Big)\,v$$

Then we multiply both sides of (23) by $\overline{w}$ integrate over $\mathbb{R}^2$ and take the real part of the integral. We estimate the terms that arise. First the argument shown in Remark (iii) in Section 3 show the first term on the right hand side of (23) is estimated by

$$|Re\int \delta(a\,w)\overline{w}\,dx| \lesssim \|\nabla a\|_{L^\infty_x}\|w\|^2_{L^2}.$$

For the rest of the terms we follow J. Kato's estimates in [13] but we treat every term; that is, do not use any cancellations coming from terms being purely imaginary as is done in [13] in the MSM context. More precisely, we need to take into account the term

$$\begin{aligned}|\int \delta(A)v\overline{w}\,dx| &\le \|\delta(A)\|_{L^q}\|v\|_{L^p}\|w\|_{L^2}\\ &\lesssim (\|u\|_{L^p} + \|v\|_{L^p})\|_{L^p}\|w\|^2_{L^2}\\ &\lesssim qM^2\|w\|^2_{L^2} \lesssim qM^{2+4q}\|w\|^{2-4/q}_{L^2}\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Similarly we also need to consider

$$|\int \mathcal{R}(u\overline{u})\,w\overline{w}dx| \lesssim \|\mathcal{R}(u\overline{u})\|_{L^\infty}\|w\|^2_{L^2}$$

and the term

$$|\int (a_1^2 \pm a_2^2)\,w\overline{w}dx| \lesssim q\,M^{4+2/q}\|w\|^{2-4/q}_{L^2}$$

from Sobolev's embedding and Trüdinger's sharp interpolation lemma above.

All in all, we get that for any $q > 2$

$$|Re\int \delta(A\,v)\overline{w}\,dx| \lesssim qM^{2+4/q}(1+M^2)\|w\|^{2-4/q}_{L^2}$$

and

$$\|\int \Big(F(u) - F(v)\Big)\overline{w}\|_{L^2} \lesssim qM^{2+4/q}(1+M^2)\|w\|^{2-4/q}_{L^2} + \|\mathcal{R}(u\overline{u})\|_{L^\infty}\|w\|^2_{L^2}$$

Relying now on Lemma 4 and Propositions 2, 3 and 4 we obtain that

$$\partial_t\|w\|^2_{L^2} \lesssim qM^{2+4/q}(1+M^2)\|w\|^{2-4/q}_{L^2} \;+\; \|\nabla a\|_{L^\infty_x}\|w\|^2_{L^2} + \|\mathcal{R}(u\overline{u})\|_{L^\infty}\|w\|^2_{L^2}.$$

Next we let $\epsilon = 2/q$ as in [13] to get that

$$\frac{1}{\epsilon}\,\partial_t \|w\|_{L^2}^{2\epsilon} \lesssim \frac{1}{\epsilon} M^{2+2\epsilon}(1+M^2) \,+\, (\|\nabla a\|_{L^\infty_x} + \|\mathcal{R}(u\overline{u})\|_{L^\infty})\|w\|_{L^2}^{2\epsilon}.$$

Multiplying both sides by ϵ and then integrating in time over the interval $[0,T]$ for any $0 < T \leq \mathbf{T}$ we obtain the estimate

$$\|w(t)\|_{L^2}^{2\epsilon} \leq C'\,TM^{2+2\epsilon}(1+M^2) \,+ C''\,\epsilon\, M \sup_{0<t<T} \|w(t)\|_{L^2}^{2\epsilon},$$

for some positive constants C', C''.

Next by taking first $\sup_{0<t<T}$ both sides and then making ϵ sufficiently small depending on M and the underlying absolute constants we deduce first that

$$\sup_{0<t<T} \|w(t)\|_{L^2}^{2\epsilon} \leq c'TM^{2+2\epsilon}(1+M^2).$$

Then by raising both sides to $1/2\epsilon$ and choosing T so that $(c'TM^{2+2\epsilon}(1+M^2)) < 1$ we finally deduce that $\|w(t)\|_{L^2} = 0$ for all $0 < t < T$ by letting $\epsilon \to 0$.

Since the choice of $T > 0$ depends solely on fixed constants in our estimates, we can restart repeatedly the same argument to get uniqueness from [T, 2T] and so on until we cover $[0, \mathbf{T}]$ into a finite number of steps.

The continuity of the data to solution map follows from the Bona-Smith method [4] [17] for the time of existence $\mathbf{T}$ given by the energy estimate computation in Proposition 4 and expressed only in terms of $\|u_0\|_{H^1_x}$ for which we have both existence and uniqueness. .

5. Existence and Uniqueness of Ishimori and Schrödinger Maps

In general, it is not a simple issue to go from solutions of the MIS and MSM systems to the full Ishimori and Schrödinger map systems directly. The transformation formulas between a solution u and the map s are quite complex. The well-posedness results on the modified systems MIS and MSM apply to a larger class of formal solutions to the equation than those which come from the Ishimori or Schrödinger maps.

Our method of using the results on the modified map equations to show existence of the Ishimori and Schr̈odinger maps is the same idea used in [31], and in [33] and [39] for wave maps.

The main purpose behind the idea of fixing a particular gauge and passing to the modified systems is that of obtaining a priori estimates for smooth solutions and -when possible- their differences in 'rougher' norms and using them to pass to a limit in the full original map system in an appropriate lower regularity space associated to the 'rougher' norm.

The following scheme allow us then to show local in time existence for the Ishimori and Schrödinger map systems for arbitrary data in $H^{3/2+\epsilon}$ as follows. Our uniqueness argument requires data $H^2(\mathbb{R}^2)$.

Assume there is local in time existence and uniqueness for data in $H^m, m \geq 4$ for the map directly. Such solutions exist for Schrödinger maps after the works of Sulem, Sulem and Bardos (for $\mathbb{S}^2$), Ding and Wang; McGahahan; Shatah; Nahmod, Stefanov and Uhlenbeck mentioned in the introduction. For the Ishimori system it follows from the discussion in the appendix of this paper in combination with the a priori estimates in sections 2-4 and estimates due to A. Soyeur (as explained in our Appendix here).

Such smooth solutions transform over to solutions of the complete (overdetermined) system. Our results in sections 2-4 show that the time of existence depends only on $\|u_0\|_{H^{1/2+}}$. So given an initial data s_0 in $H^{3/2+}$, we approximate it by smooth $s_0^\alpha \in H^m$, whose solutions s^α satisfy the full set of equations and consistency conditions and the a priori estimates satisfied by the solution to the MIS and MSM systems. These a priori estimates are now used to pass to a weak limit[3]. The solution produced by the well-posedness result in our Main Theorem in section 4 will be the weak limit of a subsequence of the sequence of smooth solutions in $C([0,T);H^{1/2+}) \cap L^2([0,T);L^\infty_x)$ and hence also satisfy the entire set of consistency conditions as desired. A key step in the argument to produce a solution in the original map system is the following lemma

Lemma 5. *Estimate (22) of the solution to the modified Schrödinger or Ishimori map system implies a bound on the extrinsic $H^\alpha(\mathbb{R}^2), \alpha > 1/2$ norm of ds. In other words there exists a positive constant C such that*

$$\|ds(t)\|_{H^\alpha_x} \leq C\,\|u(t)\|_{H^\alpha_x}$$

for all t in the life span of u.

Proof. To begin its proof we express the modified Schrödinger or Ishimori map system and the stereographic projection in the language of moving frames. Recall we had

$$u_j = (1+|s|^2)^{-1}\, e^{i\psi}\frac{\partial s}{\partial x_j} \qquad \psi = \psi(x,t)$$

and that the map s avoids the north pole. Using the conformal frame on $\mathbb{S}^2 \setminus \{N\}$, we write

$$ds = q^a e_a; \qquad q = q_\alpha dx^\alpha$$

where

$$\begin{aligned}\bar{e}_1 &:= (1+|z|^2)\frac{\partial}{\partial x} \to \tilde{e}_1 = \bar{e}_1 \circ s\\ \bar{e}_2 &:= (1+|z|^2)\frac{\partial}{\partial x} \to \tilde{e}_2 = \bar{e}_2 \circ s\end{aligned}$$

i.e. $(\tilde{e}_1;\tilde{e}_2)$ is a smooth frame for the pull-back bundle. By a gauge transformation we can freely rotate this frame at any $z=(x,t)$ with an $SO(2)$ matrix $R_a^b(z)$. Then $e_a = R_a^b\bar{e}_b \circ s = R_a^b\tilde{e}_b$ is a moving frame for the pull-back bundle (c.f. [39]). We have

$$q_{1,j} := \langle \frac{\partial s}{\partial x_j}, e_1\rangle \qquad\qquad q_{2,j} := \langle \frac{\partial s}{\partial x_j}, e_2\rangle.$$

Expressed in these coordinates the modified Schrödinger or Ishimori map system becomes now a system for q in lieu of u and exactly the estimates we obtained for u hold for q.

[3]Being maps into $\mathbb{S}^2$ we can write the second order terms in divergence form before taking weak limits.

We thus need to show that for each t fixed in the life span of q, and $\alpha > 1/2$ as in the statement of the Main Theorem,

$$\|ds\|_{H^\alpha_x} \le C\|q\|_{H^\alpha_x}$$

We proceed as in Shatah and Struwe [39]. Consider any vector field W in $s^*\mathcal{T}\mathbb{S}^2$ whose coordinates in $\{e_a\}$ are given by

$$W = Q^a e_a = Qe.$$

We know that given any p, $\|W\|_{L^p} \le \|Q\|_{L^p}$ and wish to compare $\partial_k W$ with $\partial_k Q$. We use the covariant derivative and second fundamental form and write

$$\begin{aligned} D_k W &= \partial_k W + B(s)(\partial_k, W) \\ (\partial_k + A)W &= (\partial_k Q^c_\beta + A^c_{a,k} Q^a_\beta)\, e_c \quad \text{whence} \\ \partial_k W &= (\partial_k Q^c_\beta + A^c_{a,k} Q^a_\beta) e_c - B(s)(d\,s, Q\,e) \end{aligned}$$

Thus

$$\begin{aligned} \|\partial_k W\|_{L^{8/5}} &\lesssim \|\partial_k Q\|_{L^{8/5}} + \|A\|_{L^{8/3}}\|Q\|_{L^4} + \|B\|_{L^\infty}\|ds\|_{L^2}\|Q\|_{L^8} \\ &\le C(s)\,\|Q\|_{W^{1,8/5}} \end{aligned} \tag{24}$$

where $C(s)$ is a positive constant that maybe depending on the map s.

Since the map $Q \to W$ is linear; interpolating half way between the latter estimate and $\|W\|_{L^{8/3}} \le \|Q\|_{L^{8/3}}$ gives the estimate

$$\|\partial^{1/2} W\|_{L^2} \lesssim \|J^{1/2} Q\|_{L^2}$$

To obtain the desired estimate for $\alpha > 1/2$ we choose $p_0 = p_0(\alpha)$ such that $\frac{1}{p_0} = \frac{3}{8} - \delta$; with δ depending on $\alpha - 1/2$ and such that interpolating half way between $\|W\|_{L^{p_0}} \le \|Q\|_{L^{p_0}}$ and (24) yields

$$\|\partial^\alpha W\|_{L^2} \lesssim \|J^\alpha Q\|_{L^2}$$

Uniqueness. Our results give a unique solution to the original map system only when the data is in the smoother H^2 space. Such uniqueness should be understood as meaning that weak limits of smooth solution to the original map system are unique solutions to the same map system. The argument needed to establish the latter is somewhat more involved as it requires to show the uniqueness of the associated pull-back frames, assuming uniqueness in $C([0,T); H^1) \cap L^2([0,T); L^\infty_x)$ of solutions (themselves limits of smooth solutions) to the modified map system. The frames $\{e\}$ will have regularity comparable to that of the map s.

For Schrödinger maps such argument is done in [30][4] by studying the associated frame system (s, u, e, a). The Ishimori map system requires an adaptation of their argument. We postpone however its presentation until a more satisfactory uniqueness result can be achieved for the modified Ishimori map system.

[4] They consider Schrödinger maps into a *hermitian symmetric spaces* $\mathcal{M}^{2d}$ with first cohomology group trivial and show that a fairly rough solution to the Schrödinger map system which is the weak limit of smooth solutions solves the associated Coulomb gauged frame system in a unique fashion and vice-versa.

Appendix

We indicate in this section how to modify the arguments above in the context of Ishimori's system to obtain local in time existence of smooth -say $H^4(\mathbb{R}^2)$- solutions. As we mentioned above, uniqueness of large data solutions in H^4 was showed by A. Soyeur.

Lemma A (Theorem 4.1 in [40]**).** *Let u and v be two solutions of (A.1) in $L^\infty([0,T);H^3(\mathbb{R}^2)\cap L^\infty([0,T);W^{2,\infty}(\mathbb{R}^2))$. If $u(0)=v(0)$ then $u(t)=v(t)$, for all $0\le t<T$.*

We thus concentrate in the existence part.

Consider $w:\mathbb{R}^2\to\mathbb{C}$ coming from rewriting $s:\mathbb{R}^2\to\mathbb{S}^2$ in stereographic variables

$$w=\frac{s^1+is^2}{1+s^3}\qquad s=(s^1,s^2,s^3)\quad |s|^2=1.$$

Then the hyperbolic-elliptic Ishimori system in these coordinates looks like (c.f.[40])

$$\begin{aligned} iw_t+w_{xx}-w_{yy}&=\frac{2\bar w}{1+|w|^2}(w_x^2-w_y^2)+i(\phi_x w_y+\phi_y w_x)\\ \phi_{xx}+\phi_{yy}&=4\kappa i\frac{w_x\bar w_y-w_y\bar w_x}{(1+|w|^2)^2}\\ w(\bar x,0)&=w_0(\bar x)\qquad \bar x=(x,y)\qquad w(\bar x)\to 0 \text{ as } |\bar x|\to\infty \end{aligned}\tag{A.1}$$

Step 1. Recall the equation in covariant derivatives

$$i\frac{\partial s}{\partial t}=(\nabla_1-iv_1)\frac{\partial s}{\partial x_1}-(\nabla_2-iv_2)\frac{\partial s}{\partial x_2}$$

where $(x_1,x_2)=(x,y)$, and $(v_1,v_2)=(\phi_{x_2},-\phi_{x_1})$. And consider a 'parabolic regularization' of the form,

$$i\frac{\partial s}{\partial t}=(\nabla_1-iv_1)\frac{\partial s}{\partial x_1}-(\nabla_2-iv_2)\frac{\partial s}{\partial x_2}+i\epsilon(\nabla_1\frac{\partial s}{\partial x_1}+\nabla_2\frac{\partial s}{\partial x_2})\tag{A.2}$$

After a few calculations in the spirit of those by Soyeur [40] we express the latter in stereographic coordinates as

$$\begin{aligned} w_t-i(w_{xx}-w_{yy})-\epsilon(w_{xx}+w_{yy})&=-i\frac{2\bar w}{1+|w|^2}(w_x^2-w_y^2)+\frac{2\epsilon\bar w}{1+|w|^2}(w_x^2+w_y^2)+\\ &\quad+(1+i\epsilon)(\phi_x w_y+\phi_y w_x)\\ \phi_{xx}+\phi_{yy}&=4\kappa i\frac{w_x\bar w_y-w_y\bar w_x}{(1+|w|^2)^2}\\ w(\bar x,0)&=w_0(\bar x)\quad \bar x=(x,y),\ w(\bar x)\to 0 \text{ as } |\bar x|\to\infty \end{aligned}\tag{A.3}$$

we write in short

$$w_t-i(w_{xx}-w_{yy})-\epsilon(w_{xx}+w_{yy})=F+G\tag{A.4}$$

where

$$F(w) = -i\frac{2\bar{w}}{1+|w|^2}(w_x^2 - w_y^2) + \frac{2\epsilon\bar{w}}{1+|w|^2}(w_x^2 + w_y^2 \quad \text{and} \quad G = (1+i\epsilon)(\phi_x w_y + \phi_y w_x)$$

are the local and the nonlocal terms respectively on the right hand side of (A.3). Moreover for $u \in H^2(\mathbb{R}^2)$,

$$\Delta\phi = 4i\left[\left(\frac{\bar{w}w_x}{1+|w|^2}\right)_y - \left(\frac{\bar{w}w_y}{1+|w|^2}\right)_x\right].$$

We can now state two Lemmas in Soyeur's [40] we rely on:

Lemma B (Lemma 1.5 in [40] **).** *Let $k \geq 2$. Then F and G are Lipschitz maps on bounded sets of $H^{k+2}(\mathbb{R}^2)$ with values into $H^k(\mathbb{R}^2)$, and moreover they satisfy the inequalities*

$$\|F(u)\|_{H^{k-1}} + \|G(u)\|_{H^{k-1}} \leq C\|u\|^2_{W^{2,6}}\|u\|_{H^k}$$
$$\|F(u)\|_{W^{k+2,6/5}} + \|G(u)\|_{W^{k+2,6/5}} \leq C\|u\|^2_{W^{k/2+2,6}}\|u\|_{H^k+3}$$

Thus we can apply:

Lemma C (Lemma 2.1 in [40]**; c.f.** [11]**).** *For any $w_0 \in H^k(\mathbb{R}^2), k > 1$ equation (A.4) possesses a unique solution $w^\epsilon \in C([0, T_\epsilon[; H^k)$. If the maximal time of existence T_ϵ is finite. we have the blow up condition*

$$\limsup_{t \to T_\epsilon} \|w^\epsilon(t)\|_{H^k} = +\infty.$$

Step 2. Proceeding as in section 1, with the parabolic regularization to the Ishimori system (A.3); we obtain an 'ϵ-modified' Ishimori equation of the form

$$w_t - i(w_{xx} - w_{yy}) - \epsilon(w_{xx} + w_{yy}) + 2\delta(aw) + 2i\epsilon\delta(aw) = H \tag{A.5}$$

where $H(w)$ contains local and nonlocal cubic and 'quintic' terms in u (all scaling like $|u|^2u$ in 2 dimensions).

Step 3. We next prove an energy estimate for (A.5) in the spirit of the first estimate in Proposition 1, section 3 above. We have in the notation of Proposition 1.

Lemma D. *Let w^ϵ be a classical smooth solution to (A.5) guaranteed to exist on an interval $[0, T^\epsilon)$ by Lemma B above. Then for $s \geq 0$ there exists an absolute constant $C_s > 0$, such that for $t \in [0, T^\epsilon)$*

$$\frac{\partial}{\partial t}\|w^\epsilon(t)\|^2_{H^s} \leq C_s\Big(\|\nabla a^\epsilon(t)\|_{H^s}\|w^\epsilon(t)\|_{H^s}\|w^\epsilon(t)\|_{L^\infty} + \|\nabla a^\epsilon(t)\|_{L^\infty}\|w^\epsilon(t)\|^2_{H^s} + $$
$$+ \|a^\epsilon\|^2_{L^\infty}\|w^\epsilon(t)\|^2_{H^s} + \|H(t)\|_{H^s}\|w^\epsilon(t)\|_{H^s}\Big)$$

Proof. The proof proceeds exactly as that of Proposition 1 which is actually in [32]. The only exception is the estimate in the present situation of the term corresponding to the 'low in a' portion in $2i\epsilon\delta(aw)$ since now we have an 'i' in front. We point out how to handle it. In what follows we will abuse notation and drop the superscript ϵ in w^ϵ; that is we will write just w for w^ϵ. After applying the Littlewood-Paley projection operator Q_l as in [32] to both sides of the equation we obtain an equation for $w^l := w^\epsilon_l$ of the form

$$\begin{aligned}\partial_t w_l - i\Box_{xy} w_l - \epsilon\Delta w_l = -2\delta(a_{<l-5}w_l) - 2\epsilon i\delta(a_{<l-5}w_l) - \\ - 2(1+i\epsilon)\delta([Q_l,\, a_{<l-5}]w_{l-2<.<l+2}) - 2(1+i\epsilon)\delta(Q_l(a_{>l-5})w) + H_l\end{aligned}$$

Next we multiply both sides by $2^{2sl}\bar{w}_l$ integrate and take the real part. On the left hand side, after integration by parts we obtain

$$\partial_t \|w_l\|^2_{\dot{H}^s} + \epsilon 2^{2sl}\int |\nabla w_l|^2 dx$$

On the other hand of the equation we have the sum of five terms we proceed to discuss. The estimates in [32] proceed exactly in the same manner for the last 3 terms, since the presence of the constant $2(1+i\epsilon)$ in front does not affect the arguments in [32] at all. Similarly, the first term can be estimated just as in [32]. Indeed, recall we have

$$2^{2ls} Re\int \delta(a_{<l-5}w_l)\overline{w_l}dx = 2^{2ls} Re\int \delta(a_{<l-5})|w_l|^2 dx + 2^{2ls} Re\int a_{<l-5}\cdot\nabla w_l\,\overline{w_l}dx$$

On the other hand, an integration by parts yields

$$\int a_{<l-5}\cdot\nabla w_l\,\overline{w_l}dx \;= -\int \delta(a_{<l-5})|w_l|^2 dx - \int a_{<l-5}\,w_l\cdot\nabla\overline{w_l}\,dx$$

whence

$$Re\int a_{<l-5}\cdot\nabla w_l\,\overline{w_l}dx = -\frac{1}{2}\int \delta(a_{<l-5})|w_l|^2 dx.$$

We get

$$2^{2ls} Re\int a_{<l-5}\cdot\nabla w_l\,\overline{w_l}dx = -\frac{2^{2ls}}{2}\int \delta(a_{<l-5})|w_l|^2 dx,$$

Thus by Hölder's inequality we have

$$2^{2ls}\left| Re\int \delta(a_{<l-5}w_l)\overline{w_l}dx\right| \le C\|w_l\|^2_{\dot{H}^s}\|\nabla a\|_{L^\infty}.$$

where when $s=0$ $\dot{H}^s$ just means L^2.

It is clear from the above that the term $2\epsilon 2^{2ls} Re\, i\int \delta(a_{<l-5}w_l)\overline{w_l}dx$ cannot be handled in the same manner. Instead we proceed as follows.

$$2\epsilon 2^{2ls} Re\, i\int \delta(a_{<l-5}w_l)\overline{w_l}dx = 2\epsilon 2^{2sl} Re\, i\int \delta(a_{<l-5})|w_l|^2 dx + 2\epsilon 2^{2ls} Re\, i\int a_{<l-5}\cdot\nabla w_l\,\overline{w_l}dx$$

By Hölder the first term is once again bounded by $\|w_l\|^2_{\dot{H}^s}\|\nabla a\|_{L^\infty}$. For the second term we note that

$$\begin{aligned}|2\epsilon i 2^{2sl}\int a_{l-5}\cdot\nabla w_l\overline{w_l}dx| &\le \|a\|_{L^\infty}\int |\nabla w_l||w_l|dx \\ &\le 2\epsilon 2^{2sl}\|a\|_{L^\infty}\|\nabla w_l\|_{L^2}\|w_l\|_{L^2} \\ &\le 8\epsilon\|a\|^2_{L^\infty}\|w_l\|^2_{\dot{H}^s} + \frac{\epsilon}{2}2^{2sl}\|\nabla w_l\|^2_{L^2}\end{aligned}$$

By passing the second term in the last inequality to the left hand side we have the desired estimate.

Remark. Note that similarly to the proof of Lemma 3, we have that for $s > 0$,

$$\|a^\epsilon(t)\|_{L^\infty}^2 \lesssim \|w^\epsilon(t)\|_{L^\infty}^2 \|w^\epsilon(t)\|_{H^s}^2$$

Step 4. We note that without any loss of generality the a priori estimate in Proposition 4 holds for the smooth local solutions w^ϵ to (A.5) and $T := \min\{T_\epsilon, 1\}$ where T_ϵ is the one guaranteed by Lemma C. This is because the dispersive estimates in Lemma 2 still hold for $L = -i\Box_{xy} - \epsilon\Delta$. Thus proceeding as in Section 3 and 4 we have

Theorem E. *For any $\epsilon > 0$, w^ϵ, the local solution to (A.5) produced by Lemma C lives on a time interval $[0, \mathbf{T})$ where $\mathbf{T} = \mathbf{T}(\|w_0\|_{H^{1/2+}}) > 0$ independent of ϵ. Moreover, we have that*

$$w^\epsilon \in C([0, \mathbf{T}), H^m) \cap L_t^2([0, \mathbf{T}))L_x^\infty.$$

Thus up to a subsequence we can pass to the limit in equation (A.5) to find an H^m smooth solution w $(m \geq 4)$, to equation (17) or in other words to

$$w_t - i(w_{xx} - w_{yy}) - +2\delta(aw) = H \tag{A.6}$$

where H is as in (17). By the result in Soyeur [40] we know such solution is unique.

Moreover, by the energy estimates for differences in Proposition 1 we have that this smooth solution w is unique in $C([0, \mathbf{T}), H^m)$, $(m \geq 4)$.

Now the argument in Section 5 applies to give the desired local in time solution to the Ishimori system in $C(([0, \mathbf{T}); H^{3/2+}) \cap L^2_{[0,\mathbf{T})} L_x^\infty$.

References

1. H. Bahouri and J-Y Chemin, *Equations d'ondes quasilinéaires et inegalités de Strichartz*, Amer. J. of Math. **121** (1999), 1337–1377.
2. R. Beals and R. Coifman, *Proc. Conference on Nonlinear Evolution Equations: Integrability and Spectral Methods*, University of Manchester, U.K., 1990, pp. 15–23.
3. A. Bényi and R. Torres, *Symbolic calculus and the transposes of bilinear pseudo-differential operators*, Comm.Partial Diff. Eqs. **28** (2003), 1161–1181.
4. J. Bona and Smith, *The initial value problem for the Korteweg-de Vries equation*, Philos.Trans.Royal Soc. London, Ser. A **278** (1975), 555-601.
5. N. Burq, P. Gerard and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. of Math. **126** (2004), 569–605..
6. N. Chang, J. Shatah and K. Uhlenbeck, *Schrödinger maps*, Comm. Pure Appl. Math. **53** (2000), 590–602.
7. R. Coifman, *Personal Notes on Course 'Nonlinear Fourier Analysis'*, at Yale Universiy, 1988.
8. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-differentiels.*, Astérisque, 57 Société Math. de France, 1978.
9. W. Ding and Y. Wang, *Local Schrödinger flow into Kähler manifolds*, Sci. China Ser. A **44** (2001), 1446–1464..
10. J-M. Ghidaglia and J-C. Saut, *On the initial value problem for the Davey-Stewartson systems*, Nonlinearity **3** (1990), 475–506.
11. D. Henry, *Geometric theory of semilinear parabolic equations*, vol. 840, Lecture Notes in Mathematics, Springer Verlag, 1981.
12. Y. Ishimori, *Multi-vortex solutions of a two dimensional nonlinear wave equation*, Progr.Theor.Phys. **72** (1984), 33–37.
13. J. Kato, *Existence and uniqueness of the solutions to the modified Schrödinger map*, Preprint (2004).
14. T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations.*, Comm. Pure Appl. Math. **41, No. 7** (1988), 891–907.

15. M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. Math. J. **120** (1998), 955–980.
16. C. Kenig, *On the local and global well posedness theory for the KP-I equation*, Ann. Inst. H. Poincar? Anal. Non Linéaire **21** (2004), 827–838.
17. C. Kenig, *The Cauchy problem for quasilinear Schrödinger equations.*, IAS/Park City Lecture Notes, 2003.
18. C. Kenig and K. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math.Res.Letters **10** (2003), 879–895..
19. C. Kenig, G. Ponce and L. Vega, *Well posedness and scattering results for the generalized Korteweg-de Vries equation via de contraction principle*, Comm. Pure Appl. Math. **46, no. 4** (1993), 527–620..
20. C. Kenig, G. Ponce and L. Vega, *The Cauchy problem for the quasilinear Schrödinger equation*, Invent. Math. **158** (2004), 343–388.
21. C. Kenig, G. Ponce and L. Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations,*, Invent. Math. **134** (1998), 489–545.
22. S. Klainerman, *Progr. Nonlinear Differential Equations Appl. 29*, Birkhaüser, Basel, 1997, pp. 29–69..
23. S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations.*, Comm. Pure Appl. Math. **36** (1983), 133–141..
24. H. Koch and N. Tzvetkov, *Local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$*, IMNR **26** (2003), 1449–1464.
25. S. Komineas and N. Papanicolaou, *Topology and dynamics of ferromagnetics media*, Phys. D **99, no. 1** (1996), 81–107..
26. B.G. Konopelchenko and B.T. Matkarimov, *On the inverse scattering transform for the Ishimori equation*, Physics Letters A **135, no. 3** (1989), 183–189..
27. L.D. Landau and E.M. Lifshitz, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies,*, Reproduced in Collected papers of L.D. Landau, Gordon and Breach, New York (1965, 1967)., Phys. A (Soviet Union) **153, no. 8** (1935).
29. L. Martina, O.K. Pashaev, and G. Soliani, *Static vortex solutions and singular auxiliary field in the Ishimori model*, Inverse Problems **10** (1994), 7–10.
28. H. McGahagan, *Some existence and Uniqueness results for Schrd̈inger Maps and Landau-Lifshitz-Maxwell Equations*, Ph.D. Thesis, Courant Institute for Mathematical Sciences, 2004.
30. A. Nahmod, J. Shatah, L. Vega and C. Zeng, *Schrödinger maps into Hermitian Symmetric spaces and their associated Frame Systems.*, In preparation (2004).
31. A. Nahmod, A. Stefanov and K. Uhlenbeck, *On Schrödinger Maps*, Comm. Pure and Applied Math. **56, no. 1** (2003), 114–151..
32. A. Nahmod, A. Stefanov and K. Uhlenbeck, *Erratum, 'On Schrödinger Maps'*, Comm. Pure and Applied Math. **57** (2004), 833–839..
33. A. Nahmod, A. Stefanov and K. Uhlenbeck, *On the well-posedness of the wave map problem in high dimensions*, Comm.Anal.Geom. **11** (2003), 49–83.
34. T. Ogawa, *A proof of Trüdinger's inequality and its applicatio to nonlinear Schrödinger equation*, Nonlinear Anal. **14** (1990), 765–769..
35. N. Papanicolaou and T. N. Tomaras, *Dynamics of magnetic vortices*, Nuclear Phys. B **360, no. 2-3** (1991), 425–462.
36. D. Pelinovsky and D. Wu, *Gauge transformation and spectral decomposition for the Ishimori-II equations*, J. Phys. A **36** (2003), 5557–5574.
37. J. Shatah, *Personal Communication.*
38. J. Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Differential Equations **46** (1982), 409–425..
39. J. Shatah and M. Struwe, *The Cauchy problem for wave maps*, Int. Math. Res. Not. **11** (2002), 555–571..
40. A. Soyeur, *The Cauchy problem for the Ishimori equations*, J. Functional Analysis **105** (1992), 233–255..
41. C. Sulem, P.J. Sulem and Bardos, *On the continuous limit for a system of classical spins*, Comm. Math. Phys. **107** (1986), 431–454..
42. R. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations.*, Duke Math. J. **44** (1977), 705–714..
43. L.Y. Sung, *The Cauchy problem for the Ishimori equations*, J. Funct. Anal. **139** (1996), 29–67.
44. K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982), 31–42.
45. L. Vega, *El multiplicador de Schrödinger y los operadores de restricción*, Tesis Doctoral, Universidad Autónoma de Madrid, 1988.

C.E. Kenig, School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540 and

Department of Mathematics, University of Chicago, Chicago, IL, 60637
E-mail address: cek@math.ias.edu and cek@math.uchicago.edu

A. R. Nahmod, School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540 and Department of Mathematics and Statistics, Lederle GRT, University of Massachusetts, Amherst, MA 01003-4515
E-mail address: nahmod@math.ias.edu and nahmod@math.umass.edu