NOTES ON CLASSIFICATION OF CRITICAL POINTS, STABILITY AND SKETCHING THE GRAPH OF TRAJECTORIES IN PHASE PLANE.

WE consider FIRST ORDER LINEAR SYSTEMS

Y' means $\frac{dY}{dt}$

(1) $$Y' = AY \quad \text{where } Y = Y(t) = (y_1(t), y_2(t))$$

is the unknown, A is a given 2×2 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{with } \det(A) \neq 0 .$$

NOTE:

The system (1) can be also written as

$$\begin{cases} y_1'(t) = a_{11}\, y_1(t) + a_{12}\, y_2(t) \\ y_2'(t) = a_{21}\, y_1(t) + a_{22}\, y_2(t) \end{cases}$$

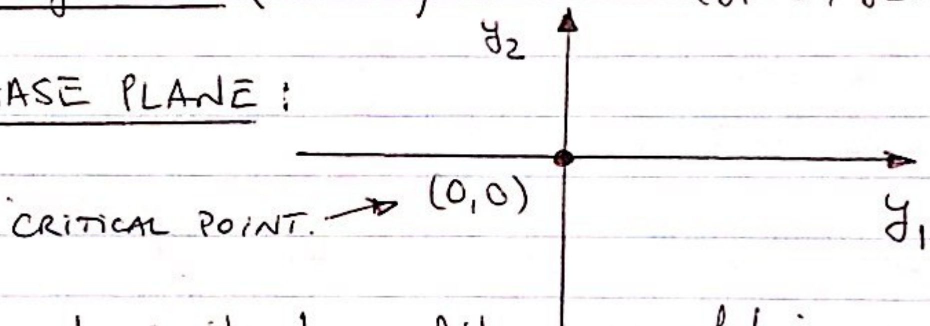
or also as

$$\underbrace{\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}}_{Y'} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}}_{Y}$$

Note that $\underline{Y = (0,0)}$ is always a solution of ① . In fact it is what we called an <u>EQUILIBRIUM SOLUTION</u> (since $Y' = 0$ when $Y = (0,0)$)

THE POINT $(0,0)$ IS ALSO CALLED <u>CRITICAL POINT</u> (in phase plane)

<u>Recall</u> <u>Solutions</u> $Y = (y_1(t), y_2(t))$ can be graphed as <u>trajectories</u> (curves) $t \longmapsto (y_1(t), y_2(t))$ in <u>PHASE PLANE</u>:



<u>FOR each</u> <u>initial condition</u> we obtain a different solution and hence <u>a different trajectory</u> in Phase plane for the system ① .
The <u>collection of all trajectories</u> for ① is called the <u>PHASE PORTRAIT associated to ①</u>.

We can <u>classify</u> the type of <u>CRITICAL POINT</u>

<u>AND</u> <u>PHASE PORTRAIT</u> we get according to

the <u>eigenvalues of A</u>

Let λ_1, λ_2 be the eigenvalues of A
(that is λ_1, λ_2 are the solutions of
$\det(A - \lambda I) = 0$).

Let $q := \lambda_1 \cdot \lambda_2$ $(= \det A)$

Let $p := \lambda_1 + \lambda_2$ $(= \operatorname{trace} A)$

Let $\Delta := p^2 - 4q$

DEFINITIONS: We say that $(0,0)$ is

- Asymptotically Stable if $q > 0$ & $p < 0$
- Stable if $q > 0$ & $p = 0$
- Unstable if $q < 0$ or $p > 0$

Additionally we say that $(0,0)$ is a

- NODE if $q > 0$ $\Delta \geq 0$
- SADDLE if $q < 0$
- CENTER if $p = 0$ $q > 0$
- SPIRAL if $p \neq 0$ $\Delta < 0$

MORE PRECISE CLASSIFICATION IN NEXT TABLE

EIGENVALUES	TYPE OF CRITICAL POINT OF (0,0)	STABILITY of (0,0)
① $\lambda_1 > \lambda_2 > 0$	NODE SOURCE	UNSTABLE
② $\lambda_1 < \lambda_2 < 0$	NODE SINK	ASYMPT. STABLE
③ $\lambda_1 < 0 < \lambda_2$	SADDLE POINT	UNSTABLE
④ $\lambda_1 = \lambda_2 > 0$	PROPER NODE (EIGENVECTORS) IMPROPER NODE (if ONLY 1 EIGENVECTOR)	UNSTABLE
⑤ $\lambda_1 = \lambda_2 < 0$	PROPER NODE (IF 2 EIGENV.) IMPROPER NODE (IF ONLY 1 EIGENVECTOR)	ASYMPT. STABLE
⑥ λ_1, λ_2 complex $\lambda_1, \lambda_2 = a \pm ib$ if $a > 0$ if $a < 0$	SPIRAL POINT	 UNSTABLE ASYMPT. STABLE
⑦ λ_1, λ_2 purely complex $(a=0)$ $\lambda_1 = ib$ $\lambda_2 = -ib$	CENTER	STABLE

Remark: Improper nodes are also called degenerate nodes

Proper nodes are also called STAR Points.

How to sketch the graph of trajectories

→ We illustrate this in some cases.

- **Cases 1, 2 and 3** (TWO real distinct eigenvalues)

Solution $Y(t) = c_1 e^{\lambda_1 t} \vec{V}_1 + c_2 e^{\lambda_2 t} \vec{V}_2$

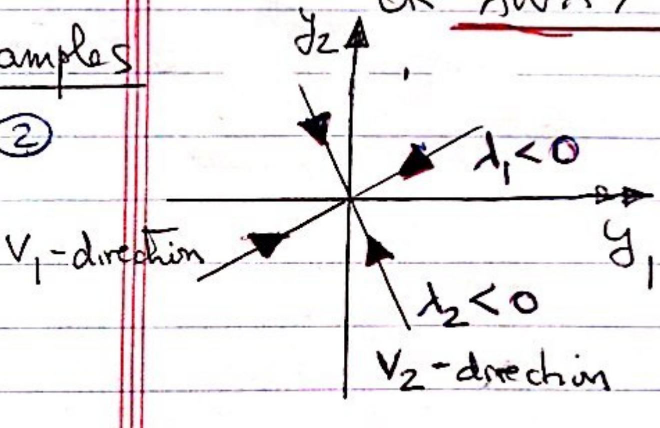
where $\vec{V}_1$ is eigenvector for λ_1

$\vec{V}_2$ is eigenvector for λ_2

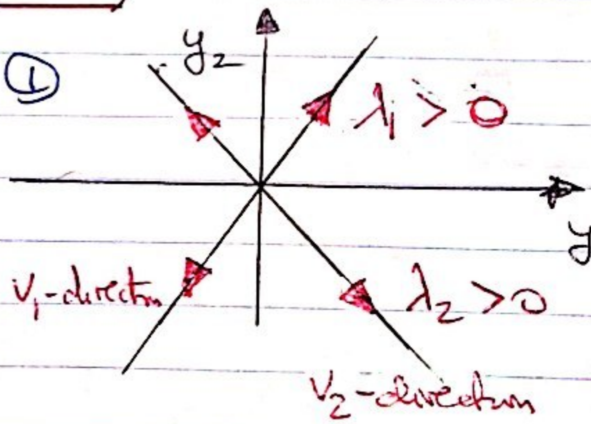
Start by drawing the eigenvector directions $\vec{V}_1$ and $\vec{V}_2$ SO THAT THEY POINT TOWARDS ($\lambda<0$) OR AWAY ($\lambda>0$) THE ORIGIN

Examples

2



1



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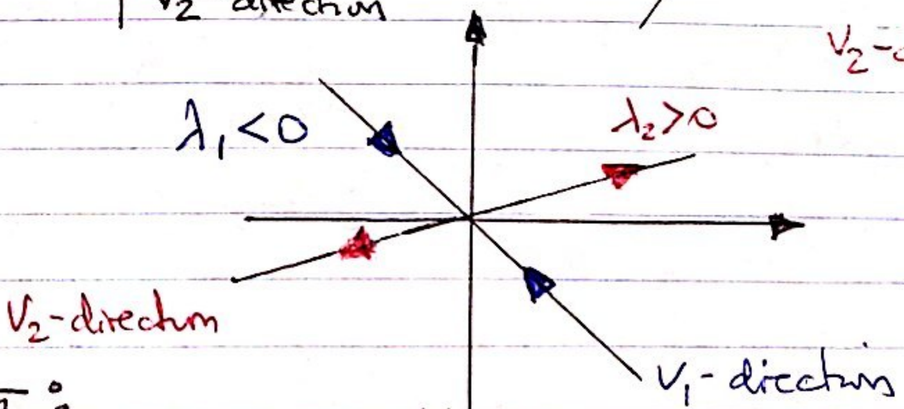


POINT IS THAT:

- If $\lambda>0$ $e^{\lambda t}\vec{V} \to \infty$ as $t \to \infty$
- If $\lambda<0$ $e^{\lambda t}V \to 0$ as $t \to \infty$

[Note that as t moves $ce^{\lambda t}\vec{v}$ traces a line through the origin ($c>0$ and $c<0$)

② • Now if $\lambda_1 < \lambda_2 < 0$

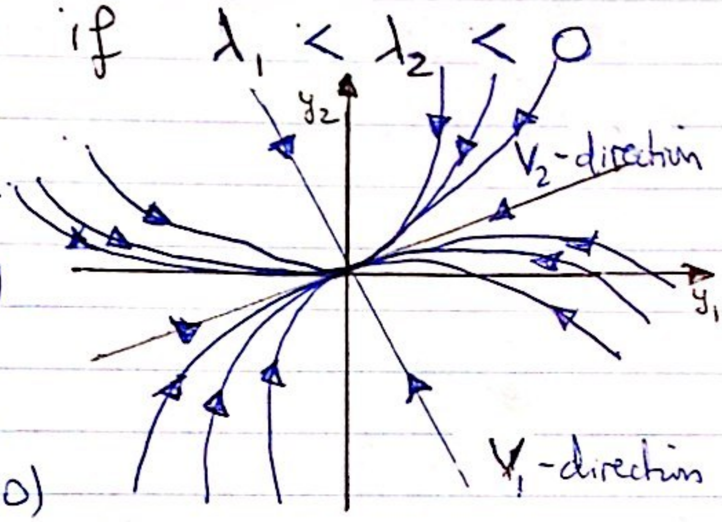
λ_2 dominates the way in which solution decays towards 0

So trajectories approach the line $e^{\lambda_2 t}\vec{V_2}$ (V_2-direction)

NODE SINK

Asymptotically stable

V_1, V_2 directions point towards (0,0)



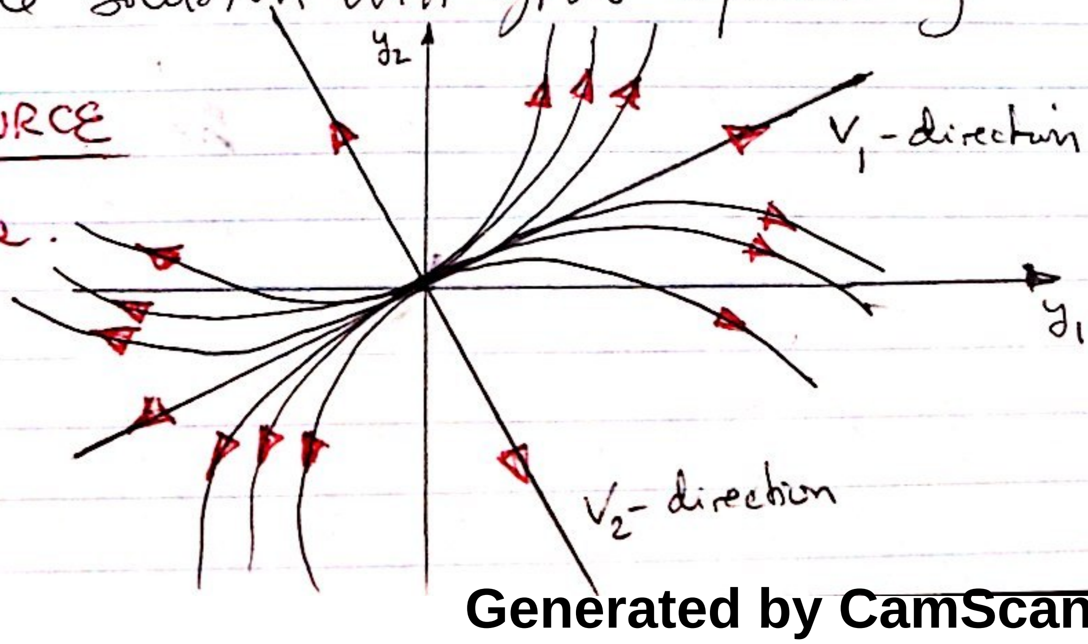
① • If $\lambda_1 > \lambda_2 > 0$. We get the same picture as above but with arrows pointing AWAY from the origin (towards ∞)

The solution will grow exponentially to ∞.

NODE SOURCE

Unstable.

V_1 and V_2 directions point AWAY from (0, 0)



3 • If $\lambda_1 < 0 < \lambda_2$ then although

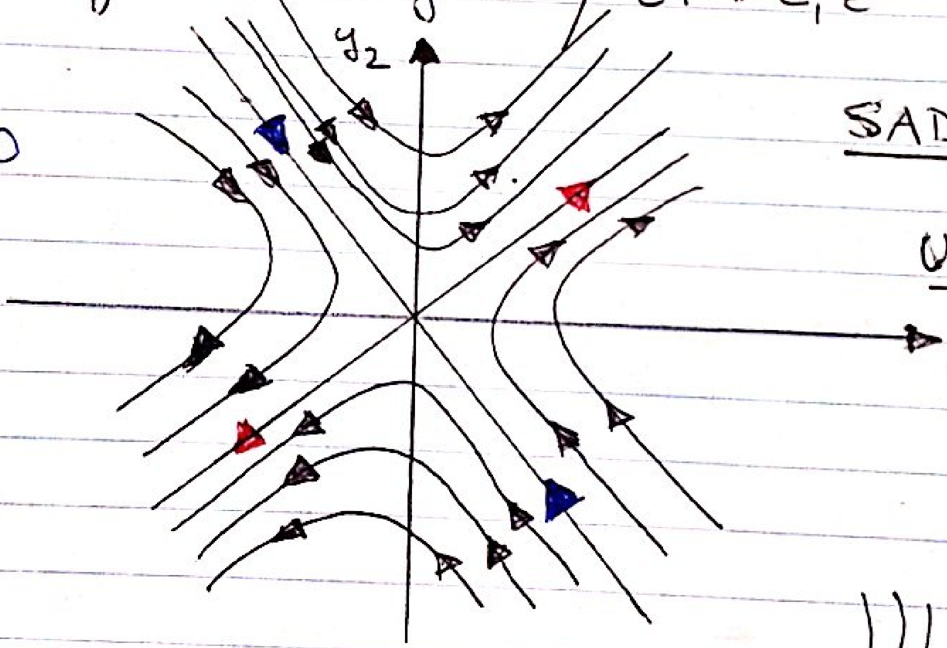
$$\left\{\begin{array}{l} c e^{\lambda_1 t} \vec{V}_1 \longrightarrow 0 \quad \text{as } t \to \infty \\ c e^{\lambda_2 t} \vec{V}_2 \longrightarrow \infty \quad \text{as } t \to \infty \end{array}\right.$$

since λ_2 dominates, $\lambda_2 > 0$ the general solutions $Y = c_1 e^{\lambda_1 t} \vec{V}_1 + c_2 e^{\lambda_2 t} \vec{V}_2 \longrightarrow \infty$

(unless of course $c_2 = 0$ in which case we get just the trajectory $t \longmapsto c_1 e^{\lambda_1 t} \vec{V}_1$ (line))

V_1-direction points towards 0

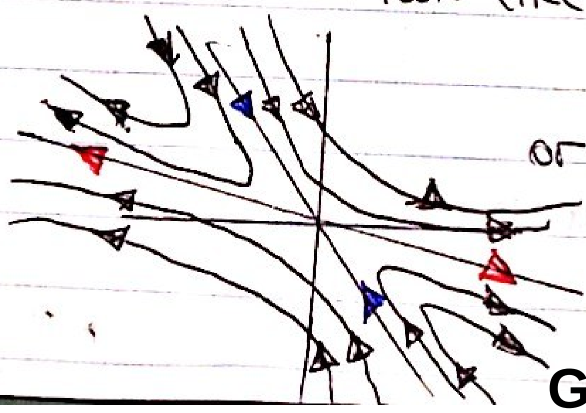
V_2-direction points AWAY from 0



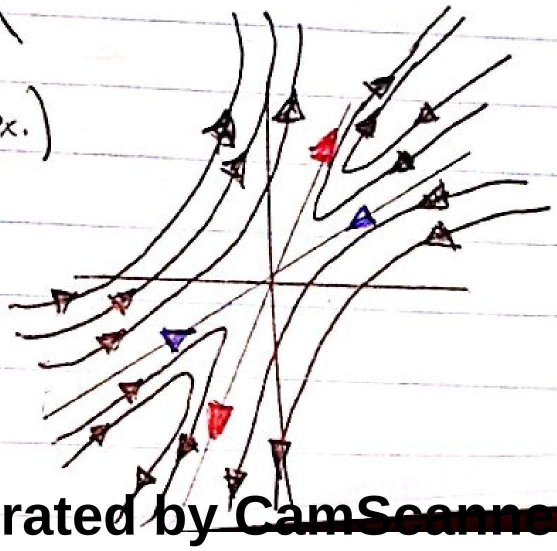
SADDLE

UNSTABLE

Note it could also look like (for ex.)



or

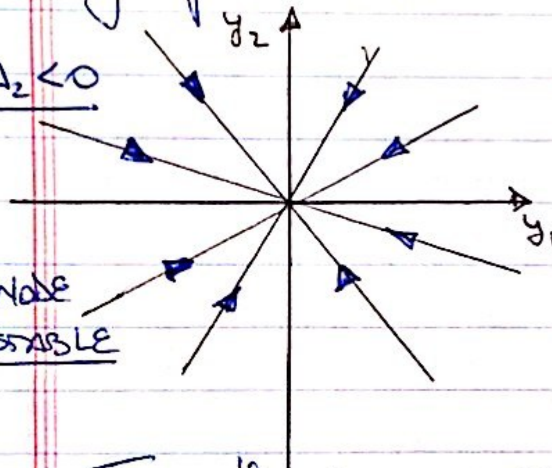


- Cases 4 and 5 in the IMPROPER (1 eigenvector) case are more complicated to graph b/c of various possibilities arising in these cases. So we don't do them (the improper or degenerate nodes) here. See book Ex. 6 pg. 145-146 Fig. 87 (Section 4.3) for an example.

4, 5 in The proper nodes or "star points" are easy to graph! Since trajectories are all lines!

if $\lambda_1 = \lambda_2 < 0$

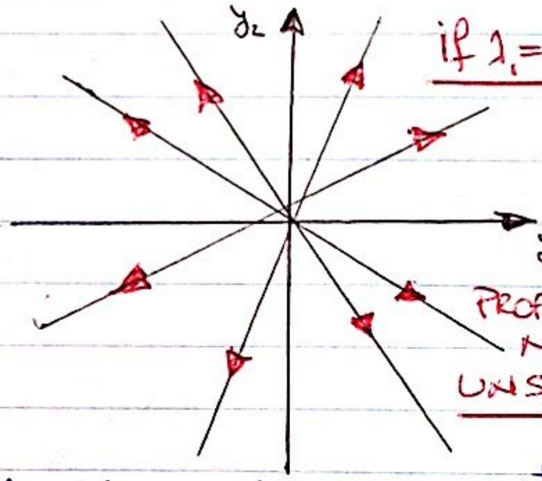
5



PROPER NODE
ASYMPT. STABLE

if $\lambda_1 = \lambda_2 > 0$

4



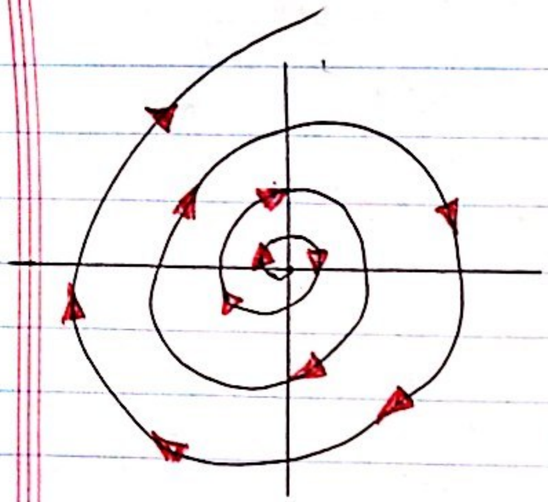
PROPER NODE
UNSTABLE

In the proper case ($\lambda_1 = \lambda_2$ real, 2 eigenvectors) the solution is $Y(t) = e^{\lambda_1 t}(c_1 \vec{V}_1 + c_2 \vec{V}_2)$ which is just another line for any c_1, c_2)

Case 6 : Now we have complex roots λ_1, λ_2 (since $\lambda_2 = \bar{\lambda}_1$ is enough to work with λ_1).

In case 6 (SPIRAL) $\operatorname{Re} \lambda_1 = a \neq 0$. $\lambda_1 = a + ib$

We have then 2 possibilities depending on the sign of $\operatorname{Re} \lambda_1$:

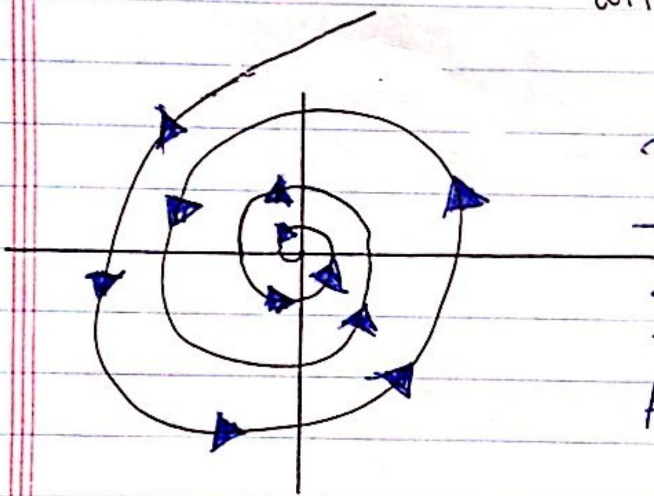


$\mathrm{Re}\,\lambda_1 = a > 0$

SPIRAL POINT / SPIRAL SOURCE

UNSTABLE

(solution $Y(t) \to \infty$ oscillating with increasing amplitud).



$\mathrm{Re}\,\lambda_1 = a < 0$

SPIRAL POINT / SPIRAL SINK

Asympt. STABLE

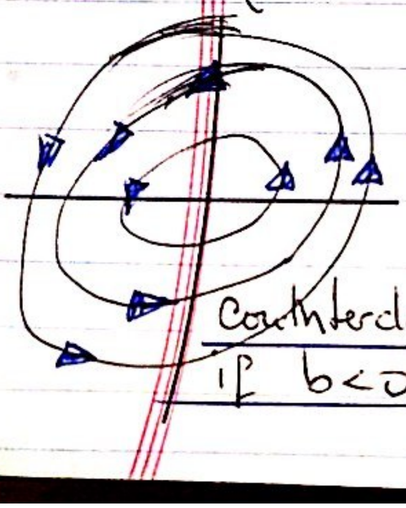
Finally Case 7 Now λ_1, λ_2 are purely imaginary

That is $\lambda_1 = ib \quad \lambda_2 = \bar{\lambda}_1 = -ib$

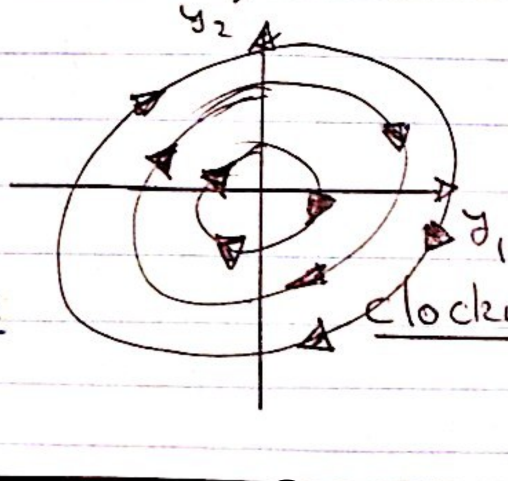
$(a = 0 = \mathrm{Re}\,\lambda_1)$.

CENTER

STABLE (Both cases)



Counterclockwise if $b<0$



Clockwise if $b>0$