The Lebesgue-Radon-Nikodym Theorem

In what follows we assume all measures are defined on $(X, \mathcal{M})$.

Definition 1. *Let μ be a positive measure and $f : X \to [-\infty, \infty]$ a μ-measurable function. We say that f is* **extended μ-integrable** *if* <u>*at least*</u> *one of*

$$\int_X f^+ \, d\mu \qquad \text{or} \qquad \int_X f^- \, d\mu$$

is **finite**.

Remarks.

(a) Given μ a positive measure and f an extended μ-integrable function; if we define the 'set function' ν on $\mathcal{M}$ by

$$\nu(E) := \int_E f \, d\mu \qquad E \in \mathcal{M}$$

then ν *is* a signed measure (you had to prove this in the homeworks).

(b) Recall that f is **μ-integrable** when **both** are finite; which is the same as saying that $\int_X |f| \, d\mu < \infty$. In this case we write $f \in L^1(\mu)$.

(c) If ν is a signed measure and $\nu = \nu^+ - \nu^-$ is the decomposition into its positive and negative variations – both ν^+ and ν^- are positive measures– then by integration of measurable functions with respect to ν is defined by

$$\int_X f \, d\nu := \int_X f \, d\nu^+ - \int_X f \, d\nu^-.$$

Then the space of ν-integrable functions is defined by

$$L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$$

where $L^1(\nu^+)$ and $L^1(\nu^-)$ are defined as in (a).

Definition 2. *Let ν and μ be two signed measures. We say that they are* **mutually singular** *and write $\nu \perp \mu$ or $\mu \perp \nu$ if there exist $E, F \in \mathcal{M}$ such that*

(1) $E \cap F = \emptyset, \qquad E \cup F = X.$

(2) *E is null for μ and F is null for ν.*

Definition 3. *Let ν be a signed measure and μ a positive measure. We say that ν is* **absolutely continuous** *with respect to μ and write $\nu << \mu$ if*

$$\nu(E) = 0 \quad \textit{for every} \quad E \in \mathcal{M} \quad \textit{for which} \quad \mu(E) = 0$$

Notation. *Let μ be a positive measure, f be an extended μ-integrable function. In what follows we shall somehwat abuse notation and simply write $d\nu = f\, d\mu$ to refer to the signed measure ν defined by $\nu(E) = \int_X f\ d\mu$, $E \in \mathcal{M}$.*

Technical Lemma 1. *Let ν and μ be two finite positive measures. Then either*

(i) $\nu \perp \mu$ **or**

(ii) *There exists $\varepsilon > 0$ and a set $E \in \mathcal{M}$ s.t. $\mu(E) > 0$ and E is a positive set for $\nu - \varepsilon\mu$.*

Proof. Suppose ν and μ are **not** mutually singular. Then want to show there exists an $\varepsilon > 0$ and a set $E \in \mathcal{M}$ with $\mu(E) > 0$ and E a **positive set** for $\nu - \varepsilon\mu$.

Now, for each $n \in \mathbb{N}$ let $P_n \cup N_n = X$ be a Hahn decomposition for $\nu - \dfrac{1}{n}\mu$. Define

$$P = \bigcup_1^\infty P_n \qquad \text{and} \qquad N = \bigcap_1^\infty N_n = P^c.$$

Then in particular $X = P \cup N$ and N is a **negative set** for **all** $n \in \mathbb{N}$. But then the latter implies that

$$0 \le \nu(N) \le \frac{1}{n}\mu(N) \qquad \text{for all} \quad n \in \mathbb{N}.$$

By the Squeeze theorem we then have that $\nu(N) = 0$.

On the other hand, $\mu(P) \ge 0$. But since we assumed that ν and μ were not mutually singular, $\mu(P) \ne 0$ because we showed $\nu(N) = 0$ and X is the disjoint union of P and N. Hence $\mu(P) > 0$ whence there must exist an $n_0 \in \mathbb{N}$ with $\mu(P_{n_0}) > 0$ and P_{n_0} a **positive set** for $\nu - \dfrac{1}{n_0}\mu$ by deinition of P_{n_0}.

Let then $\varepsilon := \dfrac{1}{n_0}$ and $E := P_{n_0}$ to otain the desired conclusion.

Technical Lemma 2. *Let ν_1, ν_2 and η be three signed measures, such that $\nu_1 \perp \eta$ and $\nu_2 \perp \eta$. Then for any $a_1, a_2 \in \mathbb{R}$ for which $a_1\,\nu_1 + a_2\,\nu_2$ is well defined as a signed measure, we have that $(\,a_1\,\nu_1 + a_2\,\nu_2) \perp \eta$.*

Proof. First note that we only need to consider $a_j \ne 0$ and since $\nu_j \perp \eta$ then implies $a_j\nu_j \perp \eta$, for $j = 1, 2$ we can further assume WLOG that $a_j = \pm 1 for j = 1, 2$. In any of such cases we proceed as follows.

By definition of mutually singular we have that $X = A_1 \cup A_1^c$ and $X = A_2 \cup A_2^c$ where $A_i \in \mathcal{M}$ be such that A_i is η-null for $i = 1, 2$ and A_i^c is ν_i-null for $i = 1, 2$.

We want to show that there exists a set $A \in \mathcal{M}$ such that $X = A \cup A^c$, a is η-null and A^c $(\nu_1 + \nu_2)$-null.

Set $A := A_1 \cup A_2$. Then clearly $A^c = A_1^c \cap A_2^c$; $A^c \subset A_i^c$, $i = 1, 2$. Hence A^c is null for *both* ν_1 and ν_2 thus it is also null for $\nu_1 + \nu_2$ so long as the latter makes sense as a signed measure (which we are assuming).

It remains to show that A is still a null set for η. For example we rewrite

$$A = (A_1 \setminus A_2) \cup (A_2 \setminus A_1) \cup (A_1 \cap A_2).$$

Then given any $F \subset A$; we use the additivity of the measure to write

$$\begin{aligned} \eta(F) &= \eta(F \cap (A_1 \setminus A_2)) + \eta(F \cap (A_2 \setminus A_1)) + \eta(F \cap (A_1 \cap A_2)) \\ &= 0 \end{aligned}$$

because each term on the right hand side vanishes since A_i are null sets for η and each set on the right hand side is a subset of A_1 or A_2. Thus concluding the proof.

Technical Lemma 3. *Let μ be a positive measure and f_1, $f_2 : X \to [-\infty, \infty]$ be extended μ-integrable functions such that*

$$\begin{aligned} &|f_1|\, d\mu \qquad \textit{and} \qquad |f_2|\, d\mu \quad \textit{are } \sigma\textit{-finite} \\ &\int_X {f_1}^-\, d\mu < \infty \quad \textit{and} \quad \int_X {f_2}^-\, d\mu < \infty \\ &\int_A f_1\, d\mu = \int_A f_2\, d\mu \quad \textit{for all } A \in \mathcal{M}. \end{aligned}$$

Then $f_1(x) = f_2(x)$ $\mu - a.e.$

Proof. By the σ-finiteness of the measures we have that

$$X = \bigcup X_i, \quad \int_{X_i} |f_1| d\mu < \infty \quad \text{and} \quad X = \bigcup {X'}_j \quad \int_{{X'}_j} |f_2| d\mu < \infty.$$

Let $Z_{i,j} := (X_i \cap {X'}_j)$ $i, j \in \mathbb{N}$. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, we can relabel $Z_{i,j} =: Z_k$; then $X = \bigcup Z_k$.

Let now $X = \bigcup_n (\bigcup_{k=1}^n Z_k) := \bigcup_n Y_n$; $Y_n \subset Y_{n+1}$. We have that

$$\int_{Y_n} |f_1|\, d\mu < \infty \quad \int_{Y_n} |f_2|\, d\mu < \infty \qquad \text{for all } n \in \mathbb{N}.$$

In other words $\chi_{Y_n} f_1$ and $\chi_{Y_n} f_2$ are in $L^1(\mu)$. On the other hand by assumption we have,

$$\int_A \chi_{Y_n}(x) f_1(x) = \int_{A \cap Y_n} f_1(x) d\mu = \int_{A \cap Y_n} f_2(x)\, d\mu = \int_A \chi_{Y_n}(x) f_2(x) \quad \text{for all } A \in \mathcal{M}$$

Then by Proposition 2.23 b. $\chi_{Y_n}(x) f_1(x) = \chi_{Y_n}(x) f_2(x)$, $\mu - a.e$ in x.

Thus $|\chi_{Y_n}(x) f_1(x) - \chi_{Y_n}(x) f_2(x),| = 0$ $\mu - a.e$ in x. By Proposition 2.16 we then have

$$\int \chi_{Y_n} |f_1(x) - f_2(x)|\, d\mu = \int_{Y_n} |f_1(x) - f_2(x)| d\mu = 0 \qquad \text{for all } n \in \mathbb{N} \tag{1}$$

Then by the MCT letting $n \to \infty$ in (1) we can conclude that $f_1(x) = f_2(x)$, $\mu - a.e.$ in $x \in X$.

Main Theorem. *Let ν be a σ-finite signed measure and let μ be a σ-finite positive measure on $(X, \mathcal{M})$. Then there exist unique σ-finite signed measures λ and ρ on $(X, \mathcal{M})$ such that*

$$\lambda \perp \mu, \qquad \rho << \mu, \qquad \text{and} \qquad \nu = \lambda + \rho$$

Moreover, there is an extened μ-integrable function $f : X \to \mathbb{R}$ such that

$$d\rho = f\, d\mu$$

and any two such functions are equal μ-a.e.

Proof.

CASE I: Assume ν and μ are <u>both</u> <u>finite</u> and <u>positive</u> measures.

<u>*Existence*</u> Define

$$\mathcal{F} := \{ f : X \to [0, \infty] \; : \; \int_E f\, d\mu \leq \nu(E), \text{ for all } E \in \mathcal{M} \}.$$

Then

(1) $\mathcal{F} \neq \emptyset$ since $f \equiv 0 \in \mathcal{F}$.

(2) If $f, g \in \mathcal{F}$ then $h(x) := \max(f(x), g(x)) \in \mathcal{F}$. Indeed,

$$\begin{aligned} \int_E h(x) d\mu &= \int_{E \cap \{x : f(x) > g(x)\}} f(x)\, d\mu + \int_{E \setminus \{x : f(x) > g(x)\}} g(x)\, d\mu \\ &\leq \nu(\, E \cap \{x \;:\; f(x) > g(x)\}\,) + \nu(\, E \setminus \{x \;:\; f(x) > g(x)\}\,) \\ &= \nu(E) \end{aligned}$$

where the second inequality holds since $f, g \in \mathcal{F}$.

Let $\mathbf{a} := \sup_{f \in \mathcal{F}} \{ \int_X f\, d\mu \}$. Then $\mathbf{a} \leq \nu(X) < \infty$.

By definition of supremum there must exist a sequnce $\{f_n\}_{n \geq 1} \subset \mathcal{F}$ such that

$$\int_x f_n\, d\mu \to \mathbf{a} \qquad \text{as} \quad n \to \infty$$

For each $n \geq 1$ define a new sequence

$$g_n := \max\,(f_1, f_2, \ldots, f_n) \quad \text{and a function} \quad f(x) := \sup_n f_n(x).$$

Then

(1) $g_n \in \mathcal{F}$
(2) $g_n(x) \leq g_{n+1}$ and $g_n(x) \to f(x)$ as $n \to \infty$
(3) $\mathbf{a} \geq \int g_n\, d\mu \geq \int f_n\, d\mu$ by (1) and the definition of g_n.

From (3) and the Squeeze theorem (for sequences of real numbers) we then have that

$$\lim_{n\to\infty} \int g_n\, d\mu \to \mathbf{a}.$$

Moreover, by (2) and the MCT (all functions are in L^+) we can conclude that

$$f \in \mathcal{F} \qquad \text{and} \qquad \int f\, d\mu = \mathbf{a}$$

Since $\mathbf{a} < \infty$ and $f \in L^+$ the latter implies in particular that $f(x) < \infty$ $\mu - a.e.$.

Define λ so that $d\lambda := d\nu - f\, d\mu$. Then since $f \in \mathcal{F}$ we have

$$\nu(E) - \int_E f\, d\mu \geq 0 \quad \text{for all } E \in \mathcal{M}.$$

Hence λ is a positive (and finite) measure.

Next, we need to show that $\lambda \perp \mu$. We do this by contradiction. Assume λ and μ are **not** mutually singular. By the Technical Lemma 1, there exist a set $E_0 \in \mathcal{M}$ and an $\varepsilon_0 > 0$ with $\mu(E_0) > 0$ and E_0 a positive set for $\lambda - \varepsilon_0\, \mu$. But then

$$\varepsilon_0 \chi_{E_0} d\mu \leq \chi_{E_0} d\lambda \leq d\lambda = d\nu - f d\mu \longrightarrow \int_E (f + \varepsilon_0 \chi_{E_0}) d\mu \leq \int_E d\nu = \nu(E) \text{ for any } E \in \mathcal{M}$$

Thus, $(f + \varepsilon_0 \chi_{E_0}) \in \mathcal{F}$ and

$$\int_X (f + \varepsilon_0 \chi_{E_0}) d\mu = \mathbf{a} + \varepsilon_0 \mu(E_0) > \mathbf{a}$$

which contradicts the fact that $\mathbf{a}$ was the supremum of $\mathcal{F}$. We must then have that $\lambda \perp \mu$ as desired. This concludes the existence of λ, f and $d\rho := f d\mu$.

Uniqueness Suppose there exist another μ-integrable function f', and λ' another positive finite measure such that $d\nu = d\lambda' + f' d\mu$ as well.

Then $d\lambda - d\lambda' = (f' - f)\, d\mu$. On the other hand, by Technical Lemma 2 $\lambda - \lambda' \perp \mu$ and by definition $(f' - f)\, d\mu << d\mu$ with $(f' - f) \in L^1(\mu)$. Hence we must have that

$$d\lambda - d\lambda' = (f' - f) d\mu = 0 \longrightarrow \lambda = \lambda' \text{ and by Prop. 2.23 b. } f = f' \mu - a.e$$

CASE II: Assume ν and μ are <u>both</u> <u>σ- finite</u> <u>positive</u> measures.

<u>*Existence*</u> Let $X = \bigcup X_i \quad \mu(X_i) < \infty$ and $X = \bigcup Y_j \quad \nu(Y_j) < \infty$. For each $k \in \mathbb{N}$ define $A_{i,j} = X_i \cap Y_j$ then $\mu(A_{i,j}) = \nu(A_{i,j}) < \infty$ and since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, by relabeling we can simply write $X = \bigcup_k A_k$.

Define

$$\mu_k(E) = \mu(E \cap A_k) \qquad \nu_k(E) = \nu(E \cap A_k)\,;\ k \in \mathbb{N}.$$

By Case I, for each $k \in \mathbb{N}$ there exist unique $\lambda_k,\ f_k$ such that

$$d\nu_k \ = \ d\lambda_k \ + \ f_k \, d\mu_k \qquad \lambda_k \perp \mu_k.$$

Since $\mu_k(A_k^c) = \nu_k(A_k^c) = 0$ we have that $\lambda_k(A_k^c) = \nu_k(A_k^c) - \int_{A_k^c} f_k \, d\mu_k = 0$. Hence we may, in particular, assume that $f_k = 0$ on A_k^c.

Define

$$\lambda = \sum_k \lambda_k \qquad f = \sum_k f_k.$$

Then $d\nu = d\lambda + f d\mu,\ \lambda \perp \mu$ (you proved this in Exercise 9) and $d\lambda, d\rho := f\, d\mu$ areσ-finite as desired. Note that since $f : X \to [0,\infty]$, $\int_X f^- \, d\mu < \infty$ hence f is extended μ-integrable.

<u>*Uniqueness*</u> Follows along the same lines of Case I in conjunction with Technical Lemma 3 to conclude $f = f'$ μ-a.e in x from $d\rho = f\, d\mu$ and $d\rho' = f' \, dmu$ are σ-finite and $f, f' : X \to [0,\infty]$ extended μ-integrable.

CASE III: Assume ν is *signed* <u>σ- finite</u> and μ is <u>σ- finite</u> and <u>positive</u>

<u>*Existence*</u> Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν; where ν^+ and ν^- are positive measures. Then since ν is *signed* and σ-finite WLOG we can assume $\nu^+(X) < \infty$ and ν^- is σ-finite. Then by the previous case there exist positive functions f_+ and $f_- : X \to [0,\infty]$ and measures λ_+, λ_- such that if $d\rho_+ = f_+ d\mu$ and $d\rho_- = f_- d\mu$,

$$\nu^+ = \lambda_+ + \rho_+, \quad \nu^- = \lambda_- + \rho_-, \quad \lambda_+ \perp \mu \text{ and } \lambda_- \perp \mu$$

Since

$$\infty > \nu^+(X) = \int_X f_+ d\mu + \lambda_+(X)$$

we have that $f_+ \in L^1(\mu)$ and $\lambda_+(X) < \infty$ so that $f = f_+ - f_-$ is extended μ-integrable, $d\rho := f d\mu$ and $\lambda_+ - \lambda_-$ are signed measures, $\lambda_+ - \lambda_- \perp \mu$. This concludes the existence since,

$$\nu \ = \ \rho_+ + \lambda_+ - (\rho_- + \lambda_-)$$

<u>*Uniqueness*</u> Follows along the same lines of uniqueness in Case II.

Remark The deomposition $\nu = \lambda + \rho$ where $\lambda \perp \mu$ and $\rho << \mu$ is called the **Lebesgue decomposition** of ν w.r.t. μ.

In the case where $\nu << \mu$ the theorem says that $d\nu = f\, d\mu$ for some extended μ-integrable function f. In this case such f is called the **Radon-Nikodym derivative** of ν w.r.t. μ and it is denoted by $\frac{d\nu}{d\mu}$.