MATH331: Solution to the Last Homework

6.2

1.
$$F(s) = \frac{3}{s^2+4} = \frac{3}{2}\frac{2}{s^2+2^2} \Longrightarrow f(t) = \frac{3}{2}\sin 2t.$$

3.
$$F(s) = \frac{2}{s^2+3s-4} = \frac{2}{(s+4)(s-1)} = \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{1}{s-(-4)} \Longrightarrow f(t) = \frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}.$$

4.
$$F(s) = \frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{6}{5}\frac{1}{s+2} + \frac{9}{5}\frac{1}{s-3} \Longrightarrow f(t) = \frac{6}{5}e^{-2t} + \frac{9}{5}e^{3t}.$$

5.
$$F(s) = \frac{2s+2}{s^2+2s+5} = \frac{2(s+1)}{(s+1)^2+2^2} \Longrightarrow f(t) = 2e^{-t}\cos 2t.$$

6.
$$F(s) = \frac{2s-3}{s^2-4} = \frac{2s-3}{(s+2)(s-2)} = \frac{7}{4}\frac{1}{s+2} + \frac{1}{4}\frac{1}{s-2} \Longrightarrow f(t) = \frac{7}{4}e^{-2t} + \frac{1}{4}e^{2t}.$$

NOTE: This can be written as $2\cosh 2t - \frac{3}{2}\sinh 2t$ since $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$.

8.
$$F(s) = \frac{8s^2-4s+12}{s(s^2+4)} = \frac{5s-4}{s^2+4} + \frac{3}{s} = 5\frac{s}{s^2+2^2} - 2\frac{2}{s^2+2^2} + 3\frac{1}{s} \Longrightarrow f(t) = 5\cos 2t - 2\sin 2t + 3.$$

13. By taking the Laplace transform of the ODE, we have
$$s^2Y(s) - s\cdot 0 - 1 - 2(sY(s)-0) + 2Y(s) = 0 \Longrightarrow (s^2-2s+2)Y(s) = 1$$
$$\Longrightarrow Y(s) = \frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1^2} \Longrightarrow y(t) = e^t\sin t.$$

14. By taking the Laplace transform of the ODE, we have
$$s^2Y(s) - s\cdot 1 - 1 - 4(sY(s)-1) + 4Y(s) = 0 \Longrightarrow (s^2-4s+4)Y(s) = s-3$$
$$\Longrightarrow Y(s) = \frac{s-2-1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2} \Longrightarrow y(t) = e^{2t} - te^{2t}.$$

20. By taking the Laplace transform of the ODE, we have
$$s^2Y(s) - s\cdot 1 - 0 + \omega^2Y(s) = \frac{s}{s^2+4} \Longrightarrow (s^2+\omega^2)Y(s) = \frac{s}{s^2+4} + s$$
$$\Longrightarrow Y(s) = \frac{s}{(s^2+4)(s^2+\omega^2)} + \frac{s}{s^2+\omega^2} = \frac{1}{\omega^2-4}\frac{s}{s^2+2^2} - \frac{1}{\omega^2-4}\frac{s}{s^2+\omega^2} + \frac{s}{s^2+\omega^2}$$
$$\Longrightarrow y(t) = \frac{1}{\omega^2-4}\cos 2t - \frac{1}{\omega^2-4}\cos\omega t + \cos\omega t = \frac{1}{\omega^2-4}\cos 2t + \frac{\omega^2-5}{\omega^2-4}\cos\omega t.$$

22. By taking the Laplace transform of the ODE, we have
$$s^2Y(s) - s\cdot 0 - 1 - 2(sY(s)-0) + 2Y(s) = \frac{1}{s+1} \Longrightarrow (s^2-2s+2)Y(s) = \frac{1}{s+1} + 1$$
$$\Longrightarrow Y(s) = \frac{1}{(s+1)\big((s-1)^2+1\big)} + \frac{1}{(s-1)^2+1} = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-3}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$
$$= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1} \Longrightarrow y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^t\cos t + \frac{7}{5}e^t\sin t.$$

23. By taking the Laplace transform of the ODE, we have

$$s^2Y(s) - s\cdot 2 - (-1) + 2(sY(s) - 2) + Y(s) = \frac{4}{s+1} \Longrightarrow (s^2+2s+1)Y(s) = \frac{4}{s+1} + 2s + 3$$
$$\Longrightarrow Y(s) = \frac{4}{(s+1)^3} + \frac{2(s+1)+1}{(s+1)^2} = 2\frac{2}{(s+1)^3} + \frac{2}{s+1} + \frac{1}{(s+1)^2} \Longrightarrow y(t) = 2t^2e^{-t} + 2e^{-t} + te^{-t}.$$

6.3

7.

$$f(t) = u_2(t)(t-2)^2 \Longrightarrow F(s) = e^{-2s}\frac{2}{s^3}.$$

9.

$$f(t) = \big(u_\pi(t) - u_{2\pi}(t)\big)(t-\pi) = u_\pi(t)(t-\pi) - u_{2\pi}(t-2\pi) - \pi u_{2\pi} \Longrightarrow F(s) = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

14.

$$F(s) = e^{-2s}\frac{1}{(s+2)(s-1)} = e^{-2s}\left(\frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}\right) \Longrightarrow f(t) = \frac{1}{3}u_2(t)e^{t-2} - \frac{1}{3}u_2(t)e^{-2(t-2)}.$$

15.

$$F(s) = e^{-2s}\frac{2(s-1)}{(s-1)^2+1^2} \Longrightarrow f(t) = 2u_2(t)e^{(t-2)}\cos(t-2).$$

16.

$$F(s) = 2e^{-2s}\frac{1}{(s+2)(s-2)} = 2e^{-2s}\frac{1}{4}\left(\frac{1}{s-2} - \frac{1}{s+2}\right) \Longrightarrow f(t) = \frac{1}{2}u_2(t)\big(e^{2(t-2)} - e^{-2(t-2)}\big) = u_2(t)\sinh 2(t-2).$$

6.4

3. By taking the Laplace transform of the ODE, we have

$$s^2Y(s) - s\cdot 0 - 0 + 4Y(s) = \frac{1}{s^2+1} - e^{-2\pi t}\frac{1}{s^2+1}$$
$$\Longrightarrow Y(s) = (1-e^{-2\pi t})\frac{1}{s^2+4}\frac{1}{s^2+1} = (1-e^{-2\pi t})\frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$
$$\Longrightarrow y(t) = \frac{1}{3}(\sin t - \frac{1}{2}\sin 2t) - \frac{1}{3}u_{2\pi}(t)(\sin(t-2\pi) - \frac{1}{2}\sin 2(t-2\pi)) = \frac{1}{3}\big(1-u_{2\pi}(t)\big)(\sin t - \frac{1}{2}\sin 2t),$$

because $\sin(t-2\pi) = \sin t$ and $\sin 2(t-2\pi) = \sin 2t$.

4. By taking the Laplace transform of the ODE, we have

$$s^2Y(s) - s\cdot 0 - 0 + 4Y(s) = \frac{1}{s^2+1} + e^{-\pi t}\frac{1}{s^2+1}$$
$$\Longrightarrow Y(s) = (1+e^{-\pi t})\frac{1}{s^2+4}\frac{1}{s^2+1} = (1+e^{-\pi t})\frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$
$$\Longrightarrow y(t) = \frac{1}{3}(\sin t - \frac{1}{2}\sin 2t) + \frac{1}{3}u_\pi(t)(\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi)) = \frac{1}{3}\big(1-u_\pi(t)\big)\sin t - \frac{1}{6}\big(1+u_\pi(t)\big)\sin 2t,$$

because $\sin(t-\pi) = -\sin t$ and $\sin 2(t-\pi) = \sin 2t$.

7. By taking the Laplace transform of the ODE, we have

$$s^2Y(s) - s\cdot 1 - 0 + Y(s) = \frac{e^{-3\pi s}}{s} \Longrightarrow Y(s) = \frac{e^{-3\pi s}}{s(s^2+1)} + \frac{s}{s^2+1} = e^{-3\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{s}{s^2+1}$$
$$\Longrightarrow y(t) = u_{3\pi}(t)\big(1-\cos(t-3\pi)\big) + \cos t.$$