M624 HOMEWORK – SPRING 2017

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<u>SETS 1 & 2 - DUE 02/09/2017</u>

From Chapter 3 (pp 145-152): 14, 15,19, 23, 32.

From Chapter 3 (pp 153): 4.

<u>Additional Questions (Chapter 3; left in class)</u>:

1) Explain why $J_F(y) - J_F(x) \leq \sum_{n:x<x_n\leq y} \alpha_n \leq F(y) - F(x)$ (proof of Lemma 3.13).

2) Show rigorously that $J_F(x) - F(x)$ is continuous (in proof of Lemma 3.13).

3) Rewrite explaining fully the proof of Theorem 3.14 in Chapter 3. Note you need to solve and use exercise 14 (given above in Chapter 3).

From Chapter 4:
Recall carefully the proof of both Theorem 2.2 (Riesz-Fisher) on Chaper 2 (p. 70) and the one for Theorem 1.2 Chapter 4 (p. 159) (as we did in class). Then do:

Additional Problem: For any $1 \leq p < \infty$ consider the space

$$L^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C},\ \text{measurable},: \|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dm\right)^{\frac{1}{p}} < \infty\}.$$

<u>Assume</u> that $\|f\|_{L^p(\mathbb{R}^d)}$ is a norm (*challenge:* can you guess what would you need to prove the triangle inequality when $p \neq 2$?), whence $d_p(f,g) := \|f-g\|_{L^p(\mathbb{R}^d)}$ defines a metric and L^p is a metric space. **Prove** that $L^p(\mathbb{R}^d)$ is *complete*.

From Chapter 4 (pp 193-194): 1, 2, 3, 4, 5, 6, 7, 8a).

From Chapter 4 (pp 202): 2^*a)b)

<u>SET 3 - DUE 02/23/17</u>

From Chapter 4 (pp 195-197): 10, 11, 12, 13, 20

Pb. I. Consider the subspace $\mathcal{S}$ of $L^2([0,1])$ spanned by the functions: 1, x, and x^3.

a) Find an orthonormal basis of $\mathcal{S}$.
b) Let $P_{\mathcal{S}}$ denote the orthogonal projection on the subspace $\mathcal{S}$, compute $P_{\mathcal{S}}x^2$.

Pb. II. Consider a function $f \in L^2([-\pi,\pi])$ whose Fourier series is $\sum_{n\in\mathbb{Z}} a_n e^{inx} = \lim_{N\to\infty} S_N(f)(x)$ - equal a.e. to $f(x)$. Show that on any subinterval $[a,b] \subset [-\pi,\pi]$ we have,

$$\int_a^b f(x)\,dx \;=\; \sum_n \int_a^b a_n\, e^{inx}\,dx.$$

In particular if $g(x) = \int_a^x f(y)dy$, the Fourier coefficients and series of $g(x)$ can be obtained from a_n, the Fourier coefficients of f.

Pb. III. For $0 < \alpha < 1$, we say that a function f is C^α-Hölder continuous with exponent α if there exists a constant $c = c_\alpha > 0$ such that $|f(x) - f(y)| \le c\,|x-y|^\alpha$ for all x, y. For $k \in \mathbb{N}$, we can also define the space $C^{k,\alpha}$ to be that of functions which are k-th times differentiable and whose k-th derivative is C^α-Hölder continuous (we could relabel C^α as $C^{0,\alpha}$).

Consider now f a 2π-periodic $C^{k,\alpha}$ function. If a_n are the Fourier coefficients of f, show that for some $C > 0$ independent of n,

$$|a_n| \le \frac{C}{|n|^{k+\alpha}}$$

From Chapter 4 (pp 205): 11*a)b).

SET 4 - Due 03/02/17

From Chapter 4 (pp 197-202): 18, 19, 21a), 22, 24, 26.

SET 5 - Due 03/23/17

From Chapter 4 (pp 196-202): 15, 21b), 23, 25, 28, 30, 32, 33.

Additional Problems (**Bonus**): 29 (p 199-200) and 6* (p. 203-204). These are about Fredholm's Alternative for compact operators.

SET 6 - Due 03/30/17

From Chapter 5: 1

Definition: A Fourier multiplier operator T on $\mathbb{R}^d$ is a linear operator on $L^2(\mathbb{R}^d)$ determined by a bounded function m (the multiplier) such that T is defined by the formula

$$\widehat{T(f)}(\xi) := m(\xi)\widehat{f}(\xi)$$

for all $\xi \in \mathbb{R}^d$ and any $f \in L^2(\mathbb{R}^d)$.

From Chapter 6: Read/Study the proofs in Section 1.

From Chapter 6: 1, 2a), 3.

SET 7 - Thursday April 20th

From Chapter 6 (pp 317-322): 5, 8, 10, 11a)b), 16a)b)

Additional Problems:

(A1) Let ν be a signed measure on $(X, \mathcal{M})$. Show that for any $E \in \mathcal{M}$

$$|\nu|(E) =$$

$$= \sup\{\sum_{k=1}^{K} |\nu(E_k)| \,:\, E_1, \dots E_K \text{ are disjoint and } E = \cup_{k=1}^{K} E_k\} \tag{1}$$

$$= \sup\{\sum_{k=1}^{\infty} |\nu(E_k)| \,:\, E_1, E_2, \dots \text{ are disjoint and } E = \cup_{k=1}^{\infty} E_k\} \tag{2}$$

$$= \sup\{|\int_E f d\nu| \,:\, |f| \leq 1\} \tag{3}$$

You may want to proceed for example by proving that (1) $\leq$ (2) $\leq$ (3) $\leq$ (1).

(A2) Let $F \in BV([a,b])$ and right continuous. Let $G(x) = |\mu_F|([a,x])$. Show that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$. To do so you may proceed by proving:
1) $T_F \leq G$ (use definition of T_F).
2) $|\mu_F(E)| \leq \mu_{T_F}(E)$ for any Borel set E (do for an interval first).
3) Show that $|\mu_F| \leq \mu_{T_F}$ and hence $G \leq T_F$ (use (A1)).

Problems **to do** (but do not turn in): 14, 16c)d)e)f).

From Chapter 1 of [SS, Vol. 4] (pp 34-43): 1, 3, 5, 6, 7, 8.

SET 8 - Due 04/27/15

From Chapter 1 of [SS, Vol. 4] (pp 36-43): 9, 12 (do this on $\mathbb{R}^n$ with Lebesgue measure), 13, 15, 16, 17, 19, 20, 34, 35.