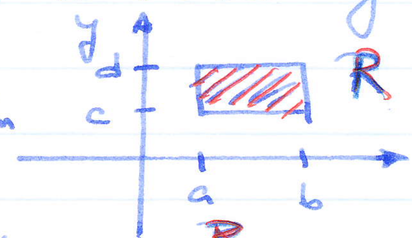
Summary of Main points in Sections 5.1, 5.2, 5.3

Note this material was covered in Multivariable Calculus so here we are just reviewing these sections quickly.

Suppose $R = [a,b] \times [c,d]$ is a rectangle in the plane $\mathbb{R}^2$ and

$f : R \to \mathbb{R}$ is a nice function

$(x,y) \to f(x,y)$

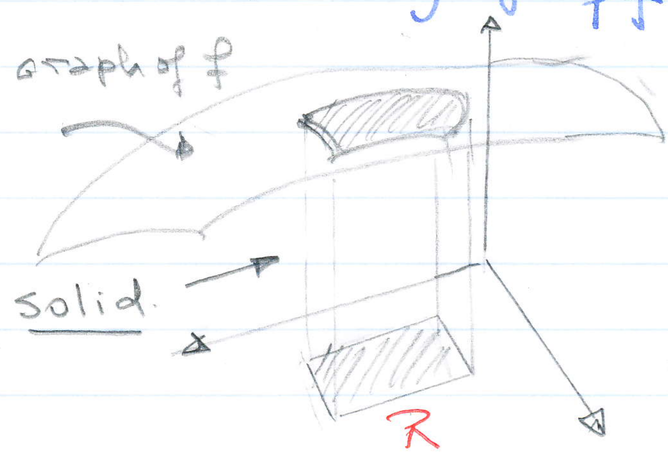


such that $f(x,y) \geq 0$ for $(x,y) \in R$.

Then $\displaystyle\iint_R f(x,y)\,dx\,dy$ = volume of solid over R and under graph of f

$$= \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$



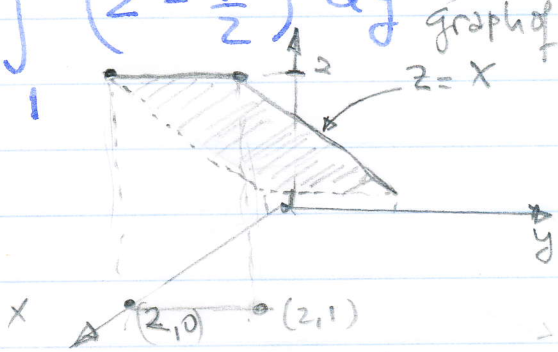
Example: 1 Let $f : [1,2] \times [1,4] \longrightarrow \mathbb{R}$

ie. $R = [1,2] \times [1,4]$ and $f(x,y) = x$

Compute $\displaystyle\iint_R f(x,y)\,dx\,dy = \int_1^4 \left[\int_1^2 x \;dx \right] dy =$

Graph of f = plane $z = x$

$$= \int_1^4 \left(\frac{x^2}{2} \Big|_{x=1}^{x=2} \right) dy = \int_1^4 \left(2 - \frac{1}{2}\right) dy$$

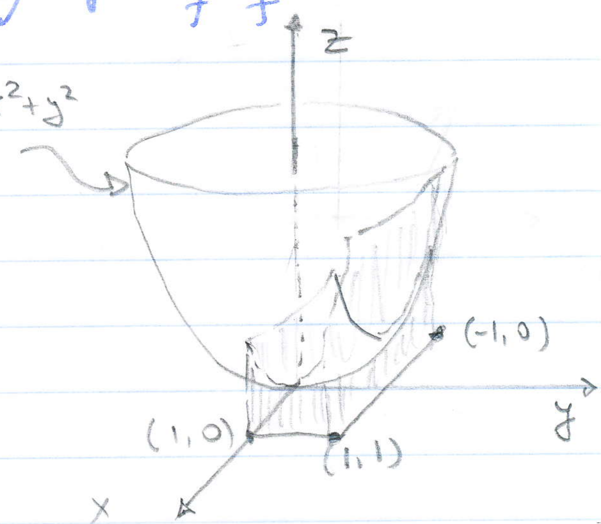


order of integration is not important could do first dx or dy.

$$= \frac{3}{2} y \Big|_1^4 = \boxed{\frac{9}{2}}.$$

② Let now $R = [-1, 1] \times [0, 1]$ and

$f(x, y) = x^2 + y^2$. Now graph of f is a paraboloid



$$\iint_R f(x,y)\, dx\, dy$$

$$= \int_{-1}^1 \int_0^1 (x^2+y^2)\, dy\, dx$$

$$= \int_{-1}^1 x^2 y + \frac{y^3}{3} \Big|_{y=0}^{y=1} dx$$

$$= \int_{-1}^1 \left(x^2 + \frac{1}{3}\right) dx = \frac{x^3}{3} + \frac{x}{3} \Big|_{-1}^1 = \boxed{\frac{4}{3}}$$

More generally in order to compute a volume of a solid we may think of it as a "union" of "slices". Then volume is like "a sum" (integral) of the area of all slices.

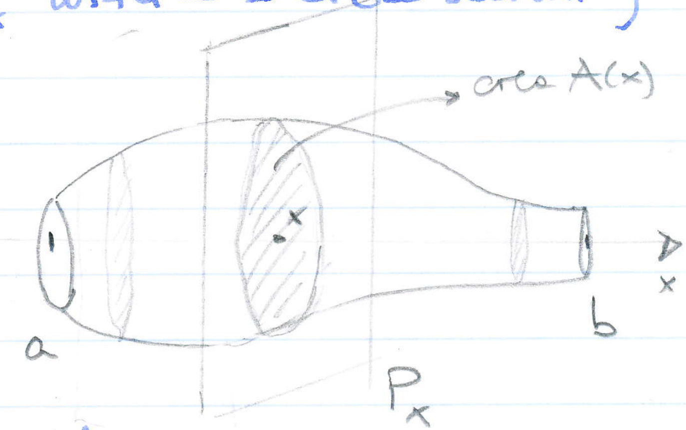
Slice Method (Cavalieri's Principle)

Let S be a solid and let P_x be a family of parallel planes for each x / $a \leq x \leq b$ such that

1. S lies in between P_a and P_b

2. The area of the slice of S cut by P_x (intersection of P_x with S = cross section) is $A(x)$.

Then $\text{vol}(S) = \int_a^b A(x)\,dx$

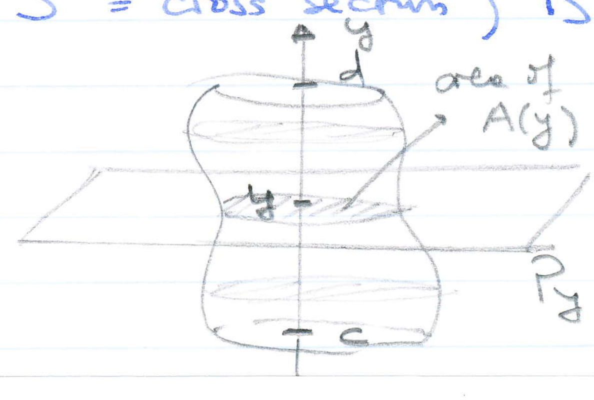


Similarly this could be done by a family P_y of parallel planes for each $c \leq y \leq d$ such that

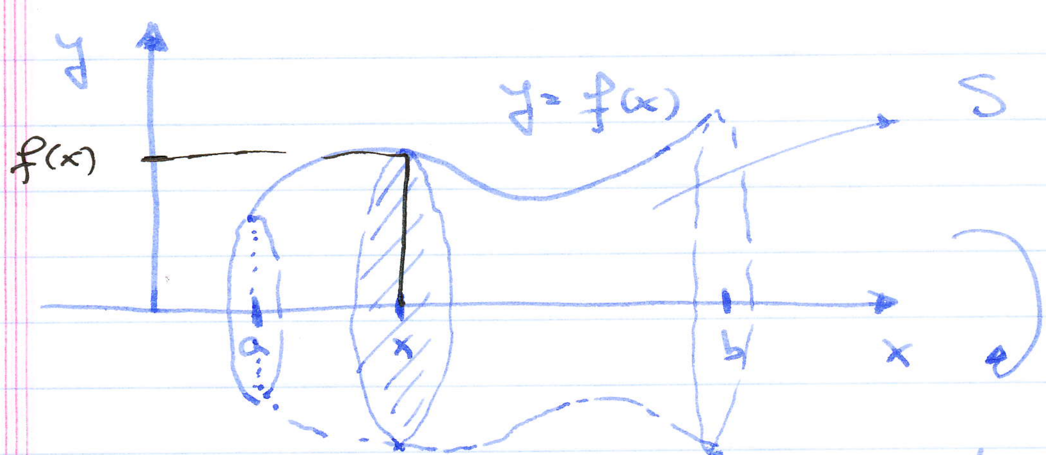
1. S lives in between P_c and P_d

2. The area of the slice of S cut by P_y (intersection of P_y with S = cross section) is $A(y)$.

Then $\text{vol}(S) = \int_c^d A(y)\,dy$



<u>Remark</u>: For general solids it might be hard to compute what $A(x)$ or $A(y)$ are. However for solids of revolution this is easier. Indeed Cavalieri principle is just the same as the "Disk method".



In this case $A(x)$ = area of disk whose radius is $f(x)$

$$= \pi \left[f(x) \right]^2$$

Hence if S is the solid obtained from revolving the graph of $f(x)$ about the x-axis we have

$$\text{vol}(S) = \int_a^b \underbrace{\pi \left(f(x)\right)^2}_{= A(x).} dx$$

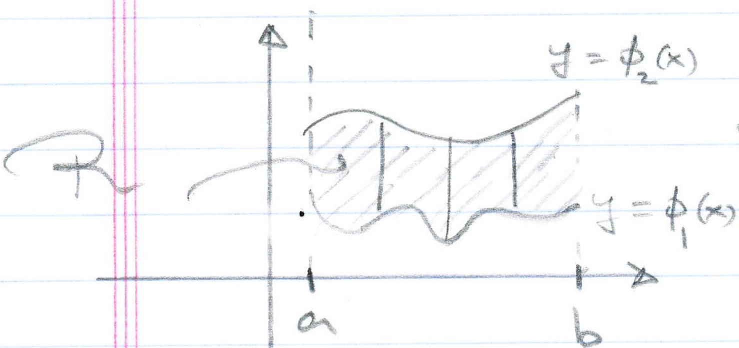
Similarly for solids of revolution about the y-axis, etc.

Double integrals over more general regions (than rectangles)

Now we must be careful with the order of integration! (unlike before)

We need the notion of "elementary regions":

- **y-simple**: these are regions R which are enclosed between 2 graphs $y = \phi_1(x)$, $y = \phi_2(x)$ and $a \le x \le b$:



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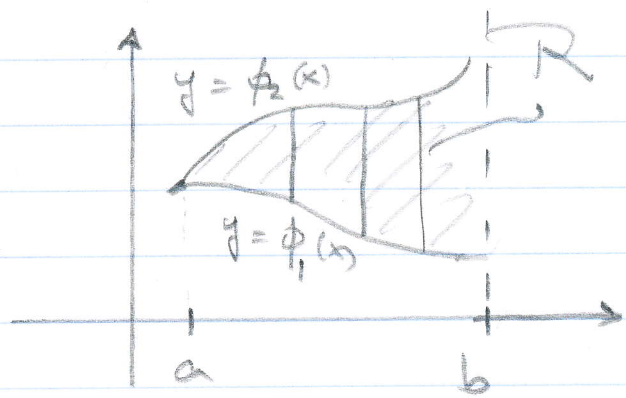


Then if we want to compute

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right] dx$$

we need to integrate y first.

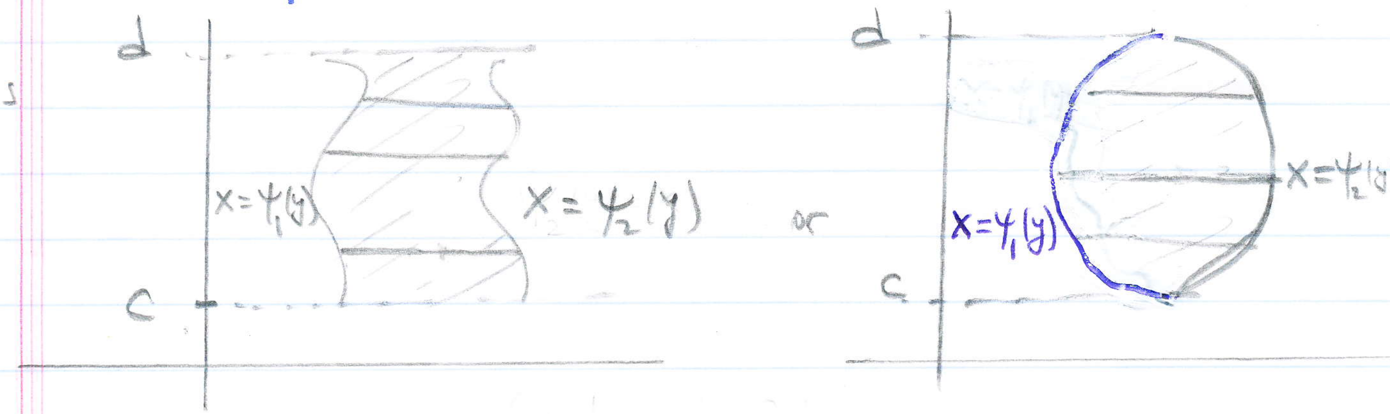
→ Remark: $\phi_1(x)$ is the fc. on bottom, $\phi_2(x)$ is the fc. on top

Another type of "elementary regions" are:

- **x-simple**: these are regions R which are enclosed between 2 graphs in y:

$x = \psi_1(y)$ and $x = \psi_2(y)$ and $c \leq y \leq d$:

Examples



Then we must integrate x first:

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right] dy$$

Remark: $\psi_1(y)$ is the fc. on the left
$\psi_2(y)$ " " " " " right.

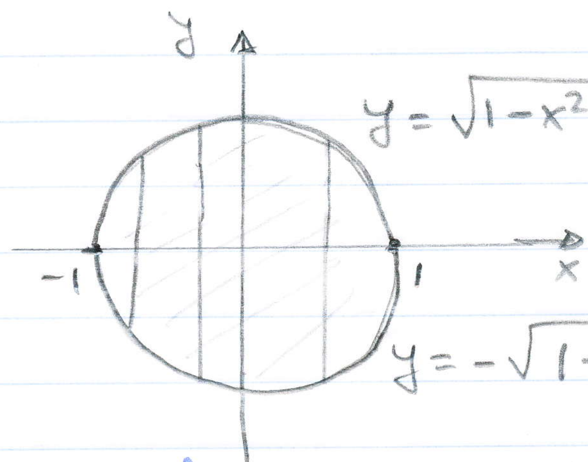
Note: Some regions can be viewed both as x-simple or y-simple. For example consider $R = \{(x,y) \,/\, x^2+y^2 = 1\}$ (circle)

Then as a y-simple: we have that

$\phi_1(x) = -\sqrt{1-x^2}$ $\quad \phi_2(x) = +\sqrt{1-x^2}$ $\quad -1 \leq x \leq 1$

y-simple region

$$\int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\right)dx$$

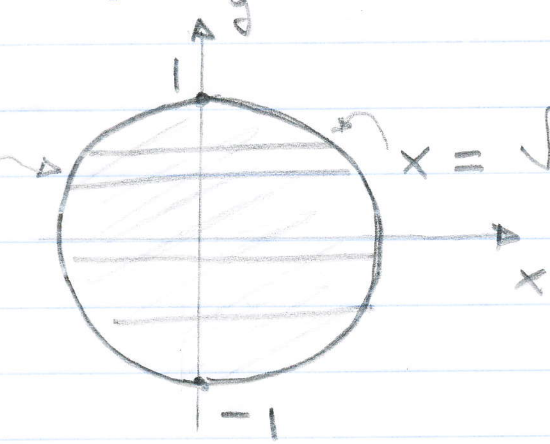


$y = \sqrt{1-x^2} = \phi_2(x)$

$y = -\sqrt{1-x^2} = \phi_1(x)$

As a x-simple : we have that

$\psi_1(y) = -\sqrt{1-y^2}$ $\qquad \psi_2(y) = \sqrt{1-y^2}$ $\quad -1 \le y \le 1$



$x = -\sqrt{1-y^2} = \psi_1(y)$ $\qquad x = \sqrt{1-y^2} = \psi_2(y)$

$$\int_{-1}^{1}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\right)dy$$

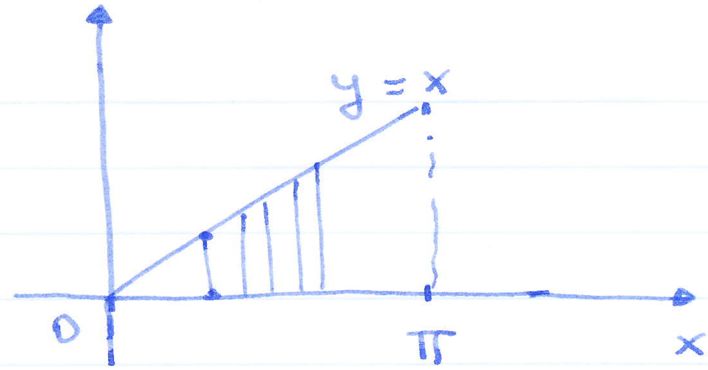
Examples:

① Integrate $f(x,y) = \sin x + 5xy^2$ over the region enclosed by $0 \le x \le \pi$ and $0 \le y \le x$.

First let's determine this region :

(7b)

This is a triangle



The x-axis is the same as the line $y=0$. So y moves between the lines $y = 0 = \phi_1(x)$ and $y = x = \phi_2(x)$ } y-simple

$$\therefore \int_0^{\pi} \left(\int_0^{x} \sin x + 5xy^2 \, dy \right) dx = \int_0^{\pi} \left(y \sin x + \frac{5}{3} x y^3 \Big|_0^x \right) dx$$

$$= \int_0^{\pi} x \sin x + \frac{5}{3} x^4 \, dx$$

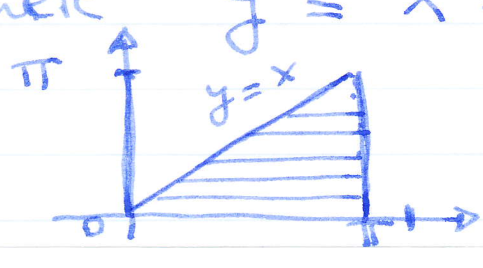
$$= \sin x - x \cos x + \frac{1}{3} x^5 \Big|_0^{\pi}$$

integ. by parts $x \sin x$

$$= \pi + \frac{\pi^5}{3}$$

<u>Note</u>: we could have also viewed the region (triangle) above as an <u>x-simple</u> <u>region</u>

where $y \le x \le \pi$ and $0 \le y \le \pi$



$\psi_1(y) = y$

$\psi_2(y) = \pi$

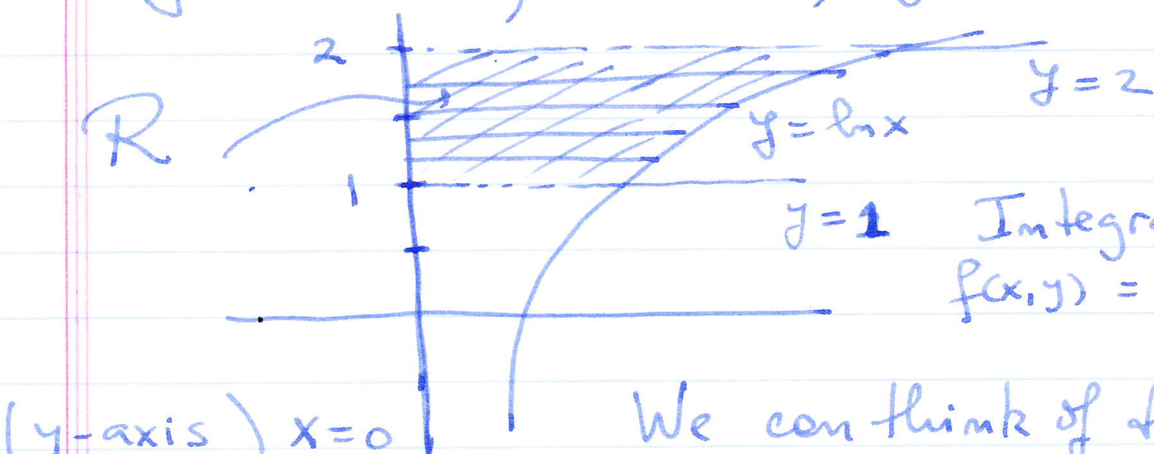
$0 \le y \le \pi$

Check that the integration as an x-simple region; namely $\displaystyle\int_0^{\pi}\left(\int_y^{\pi}(\sin x + 5xy^2)\,dx\right)dy$

gives same answer.

2. Consider the region enclosed by

$y = \ln x$, $x = 0$, $y = 1$ and $y = 2$



Integrate $f(x,y) = x+y$ over R

We can think of this as a x-simple region where

$0 \le x \le e^y$ and $1 \le y \le 2$.

(note that $y = \ln x \iff x = e^y$)

We want to integrate mony

$$\iint_R (x+y)\,dx\,dy = \int_1^2\left(\int_0^{e^y}(x+y)\,dx\right)dy$$

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$$= \int_1^2 \left(\frac{x^2}{2} + xy \Big|_{x=0}^{x=e^y} \right) dy$$

$$= \int_1^2 \frac{e^{2y}}{2} + ye^y \, dy$$

$$= \frac{e^{2y}}{4} + ye^y - e^y \Big|_1^2$$

integ. by parts ye^y

$$= \frac{e^4}{4} + 2e^2 - e^2 + \frac{e^2}{4} + e - e$$

$$= \boxed{\frac{e^4}{4} + \frac{5}{4}e^2}$$