M624 HOMEWORK – SPRING 2015

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SET 1 - DUE 02/05/15

From Chapter 3 (pp 145-152): 11, 12, 14, 15, 16a)b), 19, 23, 32.

For 15, write each increasing fc. as a continuous one plus its associated Jump fc.

SET 2 - DUE 02/19/15

From Chapter 3 (pp 153): 4.

From Chapter 4:

Read (again!) carefully Theorem 2.2 (Riesz-Fisher) on Chaper 2 (p. 70) and compare with Theorem 1.2 Chapter 4 (p. 159). Fill in the gaps (the *why?* in class) in the proof of Th. 1.2 Ch 4) and then do:

Pb. I. For any $1 \leq p < \infty$ consider the space

$$L^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C}, \text{ measurable}, : \|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dm\right)^{\frac{1}{p}} < \infty\}.$$

Assume that $\|f\|_{L^p(\mathbb{R}^d)}$ is a norm (*challenge:* can you guess what would you need to prove the triangle inequality when $p \neq 2$?), whence $d_p(f, g) := \|f - g\|_{L^p(\mathbb{R}^d)}$ defines a metric and L^p is a metric space. **Prove** that $L^p(\mathbb{R}^d)$ is *complete*.

From Chapter 4 (pp 193-194): 1, 2, 3, 4, 5, 6, 7, 8a).

From Chapter 4 (pp 202): 2^*a)b)

SET 3 - DUE 02/26/15

From Chapter 4 (pp 195-197): 10, 11, 12, 13, 20

Pb. II. Consider the subspace $\mathcal{S}$ of $L^2([0, 1])$ spanned by the functions: 1, x, and x^3.
a) Find an orthonormal basis of $\mathcal{S}$.
b) Let $P_{\mathcal{S}}$ denote the orthogonal projection on the subspace $\mathcal{S}$, compute $P_{\mathcal{S}} x^2$.

Pb. III. Consider a function $f \in L^2([-\pi, \pi])$ whose Fourier series is $\sum_{n \in \mathbb{Z}} a_n e^{inx} = \lim_{N \to \infty} S_N(f)(x)$ - equal a.e. to $f(x)$. Show that on any subinterval $[a, b] \subset [-\pi, \pi]$ we have,

$$\int_a^b f(x) \, dx = \sum_n \int_a^b a_n \, e^{inx} \, dx.$$

In particular if $g(x) = \int_a^x f(y)dy$, the Fourier coefficients and series of $g(x)$ can be obtained from a_n, the Fourier coefficients of f.

Pb. IV. For $0 < \alpha < 1$, we say that a function f is C^α-Hölder continuous with exponent α if there exists a constant $c = c_\alpha > 0$ such that $|f(x) - f(y)| \leq c\,|x-y|^\alpha$ for all x, y. For $k \in \mathbb{N}$, we can also define the space $C^{k,\alpha}$ to be that of functions which are k-th times differentiable and whose k-th derivative is C^α-Hölder continuous (we could relabel C^α as $C^{0,\alpha}$).

Consider now f a 2π-periodic $C^{k,\alpha}$ function. If a_n are the Fourier coefficients of f, show that for some $C > 0$ independent of n,

$$|a_n| \leq \frac{C}{|n|^{k+\alpha}}$$

SET 4 - Due 03/05/15

From Chapter 4 (pp 197-202): 18, 19, 21a), 22, 24, 26.

Additional Problems (**do** but do not turn in): 29 (p 199-200) and 6* (p. 203-204). These are about Fredholm's Alternative for compact operators.

SET 5 - Due 03/12/15

From Chapter 4 (pp 196-202): 15, 23, 25, 28, 30, 32, 33.
From Chapter 6 (pp 317-322): 1, 2a), 3

SET 6 - Due 04/02/15

From Chapter 6 (pp 317-322): 5, 8, 10, 11a)b), 16a)b)

Additional Problems:

(A1) Let ν be a signed measure on $(X, \mathcal{M})$. Show that for any $E \in \mathcal{M}$

$$|\nu|(E) =$$

$$= \sup\{\sum_{k=1}^{K} |\nu(E_k)| \,:\, E_1, \dots E_K \text{ are disjoint and } E = \cup_{k=1}^{K} E_k\} \tag{1}$$

$$= \sup\{\sum_{k=1}^{\infty} |\nu(E_k)| \,:\, E_1, E_2, \dots \text{ are disjoint and } E = \cup_{k=1}^{\infty} E_k\} \tag{2}$$

$$= \sup\{|\int_E f d\nu| \,:\, |f| \leq 1\} \tag{3}$$

You may want to proceed for example by proving that (1) $\leq$ (2) $\leq$ (3) $\leq$ (1).

(A2) Let $F \in BV([a,b])$ and right continuous. Let $G(x) = |\mu_F|([a,x])$. Show that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$. To do so you may proceed by proving:

1) $T_F \leq G$ (use definition of T_F).
2) $|\mu_F(E)| \leq \mu_{T_F}(E)$ for any Borel set E (do for an interval first).
3) Show that $|\mu_F| \leq \mu_{T_F}$ and hence $G \leq T_F$ (use (A1)).

(A3) Let F and G be $BV([a,b])$ and right continuous. Let μ_F and μ_G be the corresponding signed Borel measures (recall these measures are uniquely determined by -say- $\mu_F(c,d]) = F(d) - F(c)$).

a) Show that if either F or G are continuous the following *integration by parts* formula holds:

$$\int_{(a,b]} F d\mu_G + \int_{(a,b]} G d\mu_F \;=\; F(b)G(b) - F(a)G(a)$$

b) If F and G are absolutely continuous then

$$\int_{(a,b]} FG'\,dx \;+\; \int_{(a,b]} GF'\,dx \;=\; F(b)G(b) - F(a)G(a)$$

Problems **to do** (but do not turn in): 14, 16c)d)e)f)

SET 7 - Due 04/16/15

From Chapter 1 of [SS, Vol. 4] (pp 34-43): 1, 3, 5, 6, 7, 8.

SET 8 - Due 04/30/15

From Chapter 1 of [SS, Vol. 4] (pp 36-43): 9, 12 (do this on $\mathbb{R}^n$ with Lebesgue measure), 13, 15, 16, 17, 19, 20, 34, 35.