WEAK SOLUTIONS & BASIC DISTRIBUTION THEORY

Consider the linear PDE of order m

$$Lu = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u \quad \text{where}$$

$$a_\alpha \in C^{|\alpha|}(\Omega) \quad \Omega \subseteq \mathbb{R}^n. \quad \alpha \in \mathbb{Z}^n \quad \alpha = (\alpha_1, \dots, \alpha_n) \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

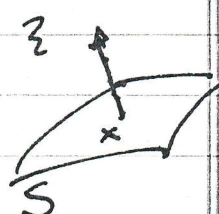
The principal part of L is the top order term

$$\sum_{|\alpha| = m} a_\alpha(x) D^\alpha \quad \text{and has}$$

principal symbol

$$\sigma_L(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \qquad \xi \in \mathbb{R}^n$$

$$\left(\xi^\alpha := \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n} \right)$$



DEFINITION: We say that $\xi \in \mathbb{R}^n$ is CHARACTERISTIC for L at $x \in \mathbb{R}^n$ if $\sigma(x, \xi) = 0$ and a hypersurface S is characteristic for L at $x \in S$ if its NORMAL VECTOR ξ at x is CHARACTERISTIC

Recall: The Cauchy-Kovalewski theorem tells you the Cauchy problem can be solved in the real analytic category for nonchar. initial surfaces S

• Adjoints and Weak Solutions:

Recall the integration by parts formula (IBP):

Let $\Omega \subseteq \mathbb{R}^n$ bounded domain with boundary $\partial\Omega$ piecewise C^1. Then (divergence theorem) for functions $u, v \in C^1(\overline{\Omega})$

$$\int_\Omega \frac{\partial u}{\partial x_k} v \, dx = -\int_\Omega u \frac{\partial v}{\partial x_k} dx + \int_{\partial\Omega} u \cdot v \, \nu_k \, d\sigma$$

where $\vec{\nu} = (\nu_1, \nu_2, \dots, \nu_n)$ is the outwards unit normal vector to $\partial\Omega$.

• When v vanishes near $\partial\Omega$ – the boundary term above vanishes too – we only need $u \in C^1(\Omega)$ and $v \in C^1_0(\Omega)$. In this case we have (more generally)

$$\int_\Omega (D^\alpha u) v \, dx = (-1)^m \int_\Omega u D^\alpha v \, dx \qquad \boxed{|\alpha| = m}$$

for all $u \in C^m(\Omega)$, $v \in C^m_0(\Omega)$.

This shows that $(u \in C^m(\Omega), v \in C^m_0(\Omega))$

$$(\dagger) \qquad \int_\Omega L u \, v \, dx = \int_\Omega u \, L' v \, dx$$

where

$$L'v = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^{\alpha}\left(a_{\alpha}(x)\, v\right)$$

$\uparrow C^{|\alpha|}(\Omega)$.

L' is called the ADJOINT of L. We can write $(\dagger)$ as

$$\langle Lu, v \rangle = \langle u, L'v \rangle \text{ for}$$

$$u \in C^m(\Omega), \quad v \in C_0^m(\Omega).$$

$\uparrow$ compact support in Ω.

Now if $Lu = f$ in Ω then we have

$$(\dagger\dagger) \quad \int_{\Omega} u L'v \, dx = \int_{\Omega} f v \, dx \quad \forall\, v \in C_0^m(\Omega).$$

Note $(\dagger\dagger)$ in itself does not require any longer any sort of regularity/smoothness on u to make sense. In fact, in general, it only requires that u and f be locally integrable in Ω; that is

u and $f \in L^1_{loc}(\Omega)$.

DEFINITION: $u \in L^1_{loc}(\Omega)$ is said to be a WEAK SOLUTION of $Lu = f$ if $(\dagger\dagger)$ holds for all $v \in C_0^m(\Omega)$.

→ In particular, if $L = \frac{\partial}{\partial x_k}$ and u is a weak solution of $\frac{\partial u}{\partial x_k} = f$ then we say that f is THE WEAK DERIVATIVE of u

NOTE: We are not saying that every locally integrable u has a weak derivative that is itself a function. In fact (as we will see below) most of the time they will be "distributions".

Lemma: If u is a weak solution to a PDE in a domain Ω where u is also C^m, then u is indeed also a CLASSICAL solution to the PDE on Ω.

Proof: Let $x \in \Omega$ and $\overset{=B}{B_\varepsilon(x)}$ a nbhd of x = open ball centered at x radius $\varepsilon > 0$) such that $\overline{B_\varepsilon(x)} \subset \Omega$. Let $v \in C_0^m(B)$. Then since $u \in C^m(\Omega)$ we can use (†) on $B \subseteq \Omega$:

$$\int_B (Lu)\, v = \int_B f v \qquad \left(u \in C^m(\Omega)\right)$$

But v is arbitrary $\Rightarrow Lu = f$ in B pointwise

Moreover, since x was also arbitrary we can repeat this for any $x \in \Omega$ to obtain that

$$Lu = f \text{ pointwise in all of } \Omega .$$ #

<u>DEFINITION</u>: The functions $v \in C_0^m(\Omega)$ are called (in this context) <u>TEST FUNCTIONS</u>

<u>Basic Distribution Theory</u>:

First note that if $v \in C_0^\infty(\Omega)$ then in particular $v \in L^\infty(\Omega)$. That is $\|v\|_{L^\infty} := \operatorname{esssup}_{x\in\Omega} |v(x)| < \infty$. Let us denote by K the support of v (compact).

Then <u>if</u> $f \in L^1_{loc}$ then f can be used to define a BOUNDED LINEAR FUNCTIONAL on $C_0^\infty(\Omega)$ as follows:

$$T_f : C_0^\infty(\Omega) \longrightarrow \mathbb{R}$$
$$v \longrightarrow \int_\Omega f(x)\, v(x)\, dx$$

Note $|T_f(v)| \leq \int_\Omega |f(x)|\,|v(x)| = \int_K |f(x)|\,|v(x)|\,dx$

$$\leq \|v\|_{L^\infty} \underbrace{\int_K |f(x)|\,dx}_{<\infty \text{ since } f\in L^1_{loc}(\Omega)} < \infty . \quad \forall v \in C_0^\infty(\Omega).$$

<u>However</u>: not every bounded linear functional over $C_0^\infty(\Omega)$ is given by a locally integrable function f! The Dirac δ_0 is one such example (more below).

<u>DEFINITION 1</u> Given $\{\varphi_n\} \subseteq C_0^\infty(\Omega)$ we say that $\varphi_n \to \varphi$ in $\underline{\mathcal{D} := C_0^\infty(\Omega)}$ if:

(a) $\exists K$ compact / $\operatorname{supp}\varphi_n \subseteq K$

(b) $\forall \alpha \in \mathbb{N}^n \cup \{0\}$,

$$D^\alpha \varphi_n \xrightarrow[n\to\infty]{} D^\alpha \varphi \text{ uniformly (in } K)$$

<u>DEFINITION 2</u> A LINEAR FUNCTIONAL $u : \mathcal{D} \to \mathbb{R}$ (or $\mathbb{C}$) is CALLED a DISTRIBUTION if it is <u>CONTINUOUS</u>.

That is if every time $\varphi_j \xrightarrow[j\to\infty]{} \varphi$ in $\mathcal{D} \Rightarrow$ $u(\varphi_j) \to u(\varphi)$ as $j \to \infty$

We denote the space of DISTRIBUTIONS (as defined above) by $\mathcal{D}'$

Remark: Note that since in the definition above we consider u LINEAR then it is enough to check ~~that~~ that if $\varphi_j \to 0$ in $\mathcal{D}$ then $u(\varphi_j) \to 0$.
(where $\varphi_j \to 0$ in $\mathcal{D}$ is as in DEF. 1).

Remark: We denote $u(\phi) = \langle u, \phi \rangle$ for any $\phi \in \mathcal{D}$. $\langle \cdot, \cdot \rangle$ = DUAL PAIRING.
Then we put the WEAK TOPOLOGY in $\mathcal{D}'$:

$$u_j \underset{j \to \infty}{\longrightarrow} u \text{ in } \mathcal{D}' \Leftrightarrow \langle u_j, \phi \rangle \underset{j \to \infty}{\longrightarrow} \langle u, \phi \rangle$$

for every $\phi \in \mathcal{D}$.

Remark: Note that every $f \in L^1_{loc}$ gives rise to a DISTRIBUTION $u_f \in \mathcal{D}'$ via the formula

$$\langle u_f, \phi \rangle = \int_\Omega f(x)\, \phi(x)\, dx \quad \phi \in \mathcal{D}$$

u_f is clearly linear. Its continuity

follows from Lebesgue Dominated Convergence (LDC) theorem:

If $\varphi_j \underset{j\to\infty}{\longrightarrow} 0$ in $\mathcal{D}$ then $\|\varphi_j\|_{L^\infty} \leq C \; \forall j$

and $u_f(\varphi_j) = \int_\Omega \varphi_j(x) f(x)\,dx = \int_K \varphi_j(x) f(x)\,dx$

$\operatorname{supp} \varphi_j \subset K$, compact for all $j \geq 1$. $\Rightarrow$

$$|\chi_K(x) f(x) \varphi_j(x)| \leq C\,|\underbrace{\chi_K(x) f(x)}_{\in L^1}|$$

($C > 0$ indep. of j)

Hence can apply LDC to pass $\lim_{j\to\infty}$ inside the integral:

$$u_f(\varphi_j) = \int_K \varphi_j(x) f(x)\,dx \to 0 \text{ as } j\to\infty.$$

#

<u>REMARK</u>: This correspondence between L^1_{loc} and distributions in $\mathcal{D}'$ is INJECTIVE if we regard as the <u>SAME</u> two functions in L^1_{loc} which are equal a.e. (this is why distributions are sometimes called "generalized functions")

Indeed: suppose $u_f = u_g$ $(f, g \in L^1_{loc})$

$$\Rightarrow \int_\Omega (f-g)(x)\,\varphi(x) = 0 \quad \forall \varphi \in \mathcal{D}$$

Hence in particular for $\varphi \in C_0^\infty \,/\, \varphi \geq 0$

$\int_\Omega \varphi\, dx = 1$ and $\operatorname{supp} \varphi \subset B(0,1)$; and

also for $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi\left(\frac{x-y}{\varepsilon}\right)$, $\varepsilon > 0$.

Let $K \subset \Omega$ be any compact. We want to show $f(x) = g(x)$ a.e. $x \in K$.:

Consider then for $\underline{\underline{x \in K}}$ the family $\varphi_\varepsilon(x)$ in $\mathcal{D}$ (as above) where we restrict ε /

$0 < \varepsilon < \operatorname{dist}(K, \Omega^c) \Rightarrow$

Note φ_ε is supported in $B(x,\varepsilon) \subset \Omega$.

<u>Now</u>: $$0 = \int_\Omega (f(y) - g(y))\,\varphi_\varepsilon(x-y)\,dy$$

$$= (f-g) * \varphi_\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} (f-g) \text{ in}$$

$L^1(K)$ (since φ_ε is an approx. to the identity)

$\Rightarrow f - g = 0$ in $L^1(K) \Rightarrow f - g = 0$ a.e x

$\Rightarrow f(x) = g(x)$ a.e. $x \in K$. #

Furthermore:

Every locally finite measure σ on $\Omega \subseteq \mathbb{R}^n$ defines a distribution by the formula:

$$\langle u_\sigma, \phi \rangle = \int_\Omega \phi(x)\, d\sigma \qquad \phi \in \mathcal{D}$$

Examples ① σ = positive Radon measure.

② Let μ_0 be the point mass at 0, that is μ_0 is the measure defined by $\mu_0(E) := \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}$

Then the "Dirac δ-function" ($\delta \in \mathcal{D}'$) is defined by

$$\langle \delta, \phi \rangle = \int \phi(x)\, d\mu_0 = \phi(0)$$

Note: similarly one can define δ_{x_0} via the point mass at $x_0 \in \mathbb{R}^n$. $\langle \delta_{x_0}, \phi \rangle = \phi(x_0)$.

PROVE IT!

Remark: If $g \in L^1(\Omega)$ $\Omega \subseteq \mathbb{R}^n$ $g \geq 0$ such that ⓘ $\int_\Omega g(x)\, dx = A$; ⓘⓘ $g_\varepsilon(x) = \frac{1}{\varepsilon^n} g\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$

ⓘⓘⓘ $\int_\Omega g_\varepsilon(x)\, dx = A$ for all $\varepsilon > 0$.

Then: $g_\varepsilon \xrightarrow[\varepsilon \to 0]{} a\delta$ in $\mathcal{D}'$

Support of Distributions:

Lemma: (PROP 9.2 Folland Real & Complex p. 284)

Let $\{V_\alpha\}_\alpha$ be a collection of open sets in Ω and let $V = \bigcup_\alpha V_\alpha \quad V \subseteq \Omega$.

Then if $F, G \in \mathcal{D}'(\Omega)$ satisfy that
$F = G$ in V_α then $F = G$ on V.

(we will not prove this, it uses Uryshon's lemma)

Definition 1: If $u, v \in \mathcal{D}'(\Omega)$ we say that $u = v$ on V open if $\langle u, \phi \rangle = \langle v, \phi \rangle$ for all $\phi \in C_0^\infty(V)$

In particular, for $u \in \mathcal{D}'(\Omega)$, we say $u = 0$ in V (V open) if $\langle u, \phi \rangle = 0 \ \forall \phi \in C_0^\infty(V)$

DEFINITION 2: The SUPPORT of a distribution u is the complement of the largest open set on which $u = 0$ (in the sense above).

NOTE: Lemma above says that if $F \in \mathcal{D}'(\Omega) \Rightarrow \exists$ a maximal open subset of Ω on which $F = 0$ (namely the union of all open subsets on which $F = 0$) $\Rightarrow$ Def 2 OKAY!

Theorem: $u \in \mathcal{D}'(\Omega)$ if and only if for every compact $K \subset \Omega$, $\exists\, C_K > 0$ and $N_K \in \mathbb{N}$ such that

$$|\langle u, \phi \rangle| \leq C_K \sum_{|\alpha| \leq N_K} \| D^\alpha \phi \|_{L^\infty}$$

for all $\phi \in C_0^\infty(K)$

Proof: $\Leftarrow$) is left as homework (you do it)

$\Rightarrow$) Suppose $\exists\, K_0 \subseteq \Omega \,/\, \forall\, m, c\ \exists$ a nontrivial $\varphi_{m,c} \in C_0^\infty(K_0)$ and

$$|\langle u, \varphi_{m,c} \rangle| > c \sum_{|\alpha| \leq m} \| D^\alpha \varphi_{m,c} \|_{L^\infty} > 0$$

Then, in particular we can take $m = c$ and call $\varphi_{m,c} = \varphi_m$. Let $\psi_m = \dfrac{\varphi_m}{u(\varphi_m)}$

Then $\displaystyle\sum_{|\alpha| \leq m} \| D^\alpha \psi_m \|_{L^\infty} < \frac{1}{m}$

Letting $m \to \infty$ we then have $D^\alpha \psi_m \longrightarrow 0$ UNIFORMLY !

$\Longrightarrow$

$\psi_m \to 0$ in $\mathcal{D}$. But this is a contradiction since $u(\psi_m) \equiv 1$ $\forall m$ by linearity of u. #.

- If a distribution $u \in \mathcal{D}'$ has compact support on Ω (open and bounded) then for $\psi \in C_0^\infty(\Omega)$ / $\psi \equiv 1$ on a neighborhood of supp u and for any $\phi \in C_0^\infty(\Omega)$ we have $\langle u, \phi \rangle = \langle u, \psi\phi \rangle$

- For $K = \bar{\Omega}$ this implies that

(u is of FINITE ORDER)

$$|\langle u, \phi \rangle| \le C_K \sum_{|\alpha| \le N_K} \| D^\alpha(\psi\phi) \|_{L^\infty}$$

which by the product rule implies:

$$(*) \quad |\langle u, \phi \rangle| \le c \sum_{|\alpha| \le N} \sup_{x \in \Omega} |D^\alpha \phi(x)|$$

where $N = N_K$ and $c = c(C_K, \|D^\beta \psi\|_{L^\infty}; \beta \le \alpha)$

Remark: Not also that $\langle u, \psi\phi \rangle$ makes sense (in this case of ψ) for any $\phi \in C^\infty$ (not necessarily of compact support !!)

Hence if we define for ψ as above and u of compact support

$$\langle u, \phi \rangle = \langle u, \psi\phi \rangle \quad \forall \phi \in C^\infty$$

Then we have an extension of u to a linear functional on C^∞.

Remark: One can prove that this extension is in fact independent of the choice of ψ and is unique subject to $\langle u, \phi \rangle = 0$ whenever $\operatorname{supp}\phi \cap \operatorname{supp} u = \emptyset$.

Thus: distributions with compact support (Ω bounded) can be regarded as linear functionals on C^∞ of finite order (ie. ⊛)

Conversely, the restriction to C_0^∞ of any linear functional on C^∞ satisfying ⊛ is indeed a distribution supported Ω.

Operators on $\mathcal{D}'$:

→ Extensions of linear operators from acting on functions to acting on distributions.

Let $T : \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$ be linear and continuous. That is:
$\forall\ \varphi_j \to \varphi$ in $\mathcal{D}$ we have $T(\varphi_j) \to T(\varphi)$ in $\mathcal{D}$.

Then the DUAL OF T (denoted by T^*) is defined by $\langle Tf, g \rangle = \langle f, T^* g \rangle$

$$f, g \in \mathcal{D}.$$

Assume $T^* : \mathcal{D} \to \mathcal{D}$ is continuous.

Then we can extend $T : \mathcal{D}' \longrightarrow \mathcal{D}'$ by the formula: $\langle Tu, \phi \rangle = \langle u, T^* \phi \rangle$

$$\forall \phi \in \mathcal{D}.$$

Then: for $u \in \mathcal{D}'$ fixed Tu is a continuous linear functional acting on $\mathcal{D}$.

EXAMPLES:

① Let T be multiplication operator by a real valued function $f \in C_0^\infty$

That is $T\varphi(x) = f(x) \cdot \varphi(x)$.

Then $T = T^*$ (CHECK IT!)

and we can multiply any distribution $u \in \mathcal{D}'$ by f via the formula

$$\langle f u, \phi \rangle = \langle u, f\phi \rangle \quad \forall \phi \in \mathcal{D}$$

(note that since $f \in C_0^\infty$, $f\phi \in \mathcal{D}$ also)

2 $T = \partial^\alpha \Longrightarrow T^* = (-1)^{|\alpha|} \partial^\alpha$
(int. by parts)

Hence we can <u>differentiate distributions</u> as often as we need to obtain another distribution by the formula:

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$$

$$u \in \mathcal{D}' \quad \partial^\alpha u \in \mathcal{D}' \quad \phi \in \mathcal{D}$$

3 Combining 1 and 2 have for

$$T = \sum_{|\alpha| \le k} a_\alpha(x)\, \partial^\alpha \qquad a_\alpha \in C^\infty$$

that $$T^* \phi = \sum_{|\alpha| \le k} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \phi)$$

And so $\forall u \in \mathcal{D}'$ Tu is defined by

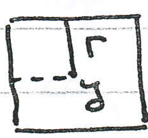
$$\langle Tu, \phi \rangle = \langle u, T^* \phi \rangle$$

Remark: If $u \in C^k$ then the distributional derivatives of u of order $\leq k$ are just the same as the regular pointwise derivatives!

Conversely: If $u \in C(\Omega)$ and the distributional derivatives $\partial^\alpha u$ are in $C(\Omega)$ for $|\alpha| \leq k$ THEN actually $u \in C^k(\Omega)$ (function!)

Sketch of the Proof: We proceed by induction on k. Suppose $k = 1$. Since being in C^k is proved in a pointwise fashion we can assume without any loss of generality that

$$\Omega = \{ x : \max |x_j - y_j| \leq r \} \quad \text{for some } y \in \mathbb{R}^n$$



For $x \in \Omega$ let

$$V(x) = \int_{y_1}^{x_1} \partial_1 u (s, x_2, \ldots, x_n)\, ds + \qquad + u(y_1, x_2, \ldots, x_n)$$

here we are using $u \in C(\Omega)$

Then V and u agree as distributions on Ω

$$\Longrightarrow u = V \text{ as functions on } \Omega$$

But $\partial_1 u$ <u>is</u> by definition of v the pointwise derivative of v with respect to the first variable.

Similarly one can argue for the other $\partial_j u$ $j \geq 2$ to conclude that $u \in C^1(\mathbb{R})$

Inductive step is done similarly.

<u>EXAMPLE.</u> Find $H'(x)$ where $H(x)$ is the HEAVISIDE function:

$$H(x) := \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \qquad H : \mathbb{R} \to \mathbb{R}$$

$$\langle H', \varphi \rangle \underset{\text{by DEF}}{=} -\langle H, \varphi' \rangle$$

<u>$\varphi \in C_0^\infty$</u>

$$= -\int_0^\infty \varphi'(x)\, dx$$

$$= \varphi(0) - \lim_{x \to \infty} \varphi(x)$$

$$= \varphi(0) \quad \left(\begin{array}{c}\text{using } \varphi \text{ has} \\ \text{compact support}\end{array}\right)$$

But $\varphi(0) = \langle \delta, \varphi \rangle$

$$\therefore \; H' = \delta \quad \left(\begin{array}{c}\text{distributional derivative} \\ \text{of } H \text{ is Dirac delta} \\ \text{at } 0\end{array}\right)$$

Homeworks: 1 Find $(\operatorname{sgn} x)'$

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

2 Find $\delta^{(k)}$ for $k \geq 1$

3 Find $(\ln |x|)'$ $(x \in \mathbb{R})$

- Translations and Reflections of distributions

(Transl.) Let $\tau_x \phi(y) := \phi(y-x)$. Then

$\tau^* \phi = \tau_{-x} \phi \implies$ for $u \in \mathcal{D}'$

$$\langle \tau_x u, \phi \rangle = \langle u, \tau_{-x} \phi \rangle \quad \forall \phi \in \mathcal{D}$$

(Refl.) Let $\tilde{\phi}(x) := \phi(-x)$ then if

$T\phi = \tilde{\phi}$ we have that $T^* = T$ and so

for $u \in \mathcal{D}'$: $\langle \tilde{u}, \phi \rangle = \langle u, \tilde{\phi} \rangle$

$\forall \phi \in \mathcal{D}$.

CONVOLUTIONS :

DEF: Let $\psi \in C_0^\infty$, $\phi \in C_0^\infty$ and let T be the operator defined by $T\phi = \phi * \psi$

Then $T^*\phi = \phi * \tilde{\psi}$. So for $u \in \mathcal{D}'$ and $\psi \in C_0^\infty$ we can define $u * \psi$ as

$$(1) \quad \langle u * \psi, \phi \rangle = \langle u, \phi * \tilde{\psi} \rangle$$

On the other hand note that for $g \in L^1_{loc}$

$$g * \psi = \int g(y)\, \psi(x-y)\, dy$$

$$= \int g(y)\, \tilde{\psi}(y-x)\, dy$$

$$= \int g(y)\, \tau_x \tilde{\psi}(y)\, dy$$

DEF: And so we could also define $u * \psi$ pointwise as a continuous function by

$$(2) \quad u * \psi(x) = \langle u, \tau_x \tilde{\psi} \rangle$$

As such $u * \psi \in C^\infty$.

One can prove that definitions (1) and (2) of $u * \psi$ agree!

Example: $\delta \in \mathcal{D}'$, $\psi \in C_0^\infty$ then

$$\delta * \psi(x) = \psi(x)$$

Remark: The usefulness of distributional derivatives is that it allows us for example to define FUNCTION SPACES such as

$$H^1(\Omega) := \{ f : \Omega \to \mathbb{R} \ / \ f \in L^2(\Omega) \text{ and } \underbrace{\nabla f}_{\text{distributional derivative of } f \in L^2} \in L^2(\Omega) \}$$

NORMED SPACE

$$\|f\|_{H^1} = \left(\int_\Omega |f(x)|^2 + |\nabla f(x)|^2 \, dx \right)^{1/2}$$

(SOBOLEV SPACE)

Note: $\nabla f \in L^2$ means that $\exists \ v \in L^2$ (function) such that

$$\langle \nabla f, \phi \rangle = \int v(x) \phi(x) \, dx \qquad \forall \phi \in \mathcal{D}$$

by def.

$$\| \quad (-1) \int_\Omega f \, \nabla \phi \, dx$$

PROPOSITION:

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$. Then:

1. $u * \varphi \in C^\infty(\mathbb{R}^n)$ and

$$\operatorname{supp} u * \varphi \subset \overline{\operatorname{supp} u + \operatorname{supp} \varphi}$$

2. $\partial^\alpha (u * \varphi) = \partial^\alpha u * \varphi = u * \partial^\alpha \varphi$

3. (Homework) $u * \varphi^\varepsilon \xrightarrow[\text{as } \varepsilon \to 0]{} u$ in $\mathcal{D}'$

$\varphi^\varepsilon(x) := \varepsilon^{-n} \varphi(x/\varepsilon)$ is a mollifier/approx. to identity.

4. $(u * \varphi) * \psi = u * (\varphi * \psi)$.

Remark: The convolution of 2 distributions does NOT make sense in general. However, if one of them has COMPACT SUPPORT then, YES, it makes sense and is a distribution:

$u \in \mathcal{D}'(\mathbb{R}^n)$ $\operatorname{supp} u$ COMPACT ; and

$v \in \mathcal{D}'(\mathbb{R}^n)$ $\Longrightarrow$

$$\boxed{\langle u * v, \varphi \rangle := \langle v, \tilde{u} * \varphi \rangle}$$

Moreover: $\partial^\alpha (u * v) = \partial^\alpha u * v = u * \partial^\alpha v$.

To show the REMARK above, we need to show that if $u \in \mathcal{D}'$ has compact support and $v \in \mathcal{D}'$ $\Rightarrow$ $u * v \in \mathcal{D}'$:

First note that if $u \in \mathcal{D}'$ has compact support $\Rightarrow$ $\tilde{u} \in \mathcal{D}'$ and also has compact support $\Rightarrow$

$$\exists\, c, m \;/\; |\tilde{u}(\varphi)| \leq c \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Suppose $\varphi_n \longrightarrow 0$ in $\mathcal{D}(\mathbb{R}^n)$ $\operatorname{supp}\varphi_n \subset K$ (K compact) $\forall n \geq 1$. We WTS. that

$$\langle u * v, \varphi_n \rangle \longrightarrow 0$$

$$\| $$

$$\langle v, \tilde{u} * \varphi_n \rangle; \left(\tilde{u} * \varphi_n \text{ has compact support}\right)$$

So fix α $\rightarrow$

$$|D^\alpha(\tilde{u} * \varphi_n)(x)| = |\tilde{u} * D^\alpha \varphi_n(x)| = |\langle \tilde{u}, \tau_x(\widetilde{D^\alpha \varphi_n})\rangle|$$

$$\leq c \sum_{|\beta| \leq m} \|D^\beta(\tau_x(\widetilde{D^\alpha \varphi_n}))\|_\infty$$

$\downarrow$ 0 unif. in x b/c $\varphi_n \rightarrow 0$ in $\mathcal{D}$.

$\Rightarrow$ 0 unif. in x.

#

Proof of Proposition :

① WTS. $u * \phi$ is C^∞. first show is continuous at $x \in \mathbb{R}^n$

Recall $u * \phi(x) = \langle u, \tau_x \tilde{\phi} \rangle$

So $u * \phi(x+h) = \langle u, \tau_{x+h} \tilde{\phi} \rangle$. Then

$u * \phi$ is cont. at x if we can show $\tau_{x+h} \tilde{\phi}$ converge to $\tau_x \tilde{\phi}$ in $\mathcal{D}(\mathbb{R}^n)$ as $h \to 0$ $\Big\}$ (†)

- $\tau_{x+h} \tilde{\phi}(y) = \tilde{\phi}(y - (x+h)) = \phi(x+h-y)$
- $\operatorname{supp} \tau_{x+h} \tilde{\phi} \subseteq K \subseteq \mathbb{R}^n$ Kompact $\forall |h| < 1$
- $|\nabla \phi|$ is bounded since $\phi \in C_0^\infty(\mathbb{R}^n)$.

Then by the MVT, $\exists\, \xi_0$ in between $x-y$ and $x+h-y$:

$$|\phi(x+h-y) - \phi(x-y)| \leq |\nabla \phi(\xi_0)\, h|$$
$$\leq \|\nabla \phi\|_\infty |h|$$

$$\Rightarrow \phi(x+h-y) \xrightarrow[h \to 0]{} \phi(x-y) \text{ uniformly in } y \quad (\text{for } x \in \mathbb{R}^n \text{ fixed})$$

Similarly we can show

$$\partial^\alpha \tau_{x+h} \tilde{\phi}(y) \longrightarrow \partial^\alpha \tau_x \tilde{\phi}(y) \text{ uniformly in } y \ (x \text{ fixed}) \Rightarrow$$

we have convergence in $\mathcal{D}(\mathbb{R}^n) \Rightarrow (\dagger)$.

Next we WTS $u * \phi$ is differentiable. We do this for $|\alpha| = 1$ and $\partial^\alpha = \frac{\partial}{\partial x_j}$. Higher order ones are done by induction.

So, WLOG assume $j=1$ ie. $\frac{\partial}{\partial x_1}$; and let $e_1 = (1, 0, \dots 0)$.

Once again note $\frac{\partial}{\partial x_1}(u * \phi)(x) = \lim_{h \to 0}$

$$\frac{u * \phi(x + he_1) - u * \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\langle u, \tau_{x+he_1}\tilde{\phi}\rangle - \langle u, \tau_x \tilde{\phi}\rangle \right)$$

$$= \lim_{h \to 0} \left\langle u, \frac{\tau_{x+he_1}\tilde{\phi} - \tau_x \tilde{\phi}}{h} \right\rangle$$

$\therefore$ need to show $(*)$ $\quad \frac{\tau_{x+he_1}\tilde{\phi} - \tau_x\tilde{\phi}}{h} \xrightarrow[h \to 0]{} \frac{\partial}{\partial x_1} \tau_x \tilde{\phi}$ $\underline{\underline{\text{in}}}$ $\underline{\underline{\mathcal{D}(\mathbb{R}^n)}}$

- Again note $\operatorname{supp} \tau_{x+he_1}\tilde{\phi} \subseteq K$ $\underline{\forall |h| < 1}$

$K \subseteq \mathbb{R}^n$ compact;

And by the MVT ; for ξ_0 a value in between $x-y$ and $x-y+he_1$ $\exists\ \xi'$ in between ξ_0 and $x-y$ such that

$$\left| \frac{\partial \phi}{\partial x_1}(\xi_0) - \frac{\partial \phi}{\partial x_1}(x-y) \right| \leq \left| \nabla \frac{\partial \phi}{\partial x_1}(\xi') \cdot (\xi_0 - (x-y)) \right|$$

$$\leq \| \partial^2 \phi \|_{L^\infty} |h| \longrightarrow 0 \text{ as } h \to 0$$

uniformly in y
(for x fixed)

Similarly one can prove that higher order deriv. of $(*)$ also converge uniformly.

- This also proves that $\frac{\partial}{\partial x_1}(u * \phi) = \frac{\partial u}{\partial x_1} * \phi$
- Do the other one : $\frac{\partial}{\partial x_1}(u * \phi) = u * \frac{\partial \phi}{\partial x_1}$ (check)

Finally to check supports :

If $\operatorname{supp} u \cap \operatorname{supp} \tau_x \tilde{\phi} = \emptyset \Rightarrow$

$$u * \phi(x) = \langle u, \tau_x \tilde{\phi} \rangle = 0$$

$\therefore$ if $u * \phi(x) \neq 0 \Rightarrow \exists\ z \in \operatorname{supp} u \cap \operatorname{supp} \tau_x \tilde{\phi}$

$\Rightarrow z \in \operatorname{supp} u$ & $\phi(x-z) \neq 0 \Rightarrow x = z + (x-z)$
$\in \operatorname{supp} u + \operatorname{supp} \phi$ $\#$