MATH331: Solution to the midterm

1. (a) Multiplying both sides by $2+y^3$, we have

$$(2+y^3)\frac{dy}{dx}=2x-x^3 \Longrightarrow \int(2+y^3)dy=\int(2x-x^3)dx \Longrightarrow 2y+\frac{y^4}{4}=x^2-\frac{x^4}{4}+C.$$

Since the graph of the solution goes through $(x,y)=(0,2)$,

$$2\cdot 2+\frac{2^4}{4}=2^2-\frac{2^4}{4}+C \Longrightarrow C=8 \Longrightarrow \underline{2y+\frac{y^4}{4}=x^2-\frac{x^4}{4}+8.}$$

(b) $y'=-y^2\cos x \Longrightarrow y^{-2}y'=\cos x \Longrightarrow \int y^{-2}=-\int \cos x dx \Longrightarrow \underline{-\frac{1}{y}=-\sin x+C.}$
Note that we are assuming that $y\neq 0$ when we devide both sides by y^2. We need to consdier when $y=0$ separately. In this case, $y\equiv 0$ is also a solution.

2. (a) By diving both sides by t^3, we have $y'+\frac{4}{t}y=\frac{1}{t^3}e^{-t}$. The integrating factor is given by $\mu(t)=e^{\int \frac{4}{t}dt}=e^{4\ln t}=(e^{\ln t})^4=t^4$. Then, we have

$$\frac{d}{dt}\left(t^4y(t)\right)=te^{-t} \Longrightarrow t^4y=\int te^{-t}dt=-te^{-t}-e^{-t}+C \Longrightarrow y(t)=-\frac{1}{t^3}e^{-t}-\frac{1}{t^4}e^{-t}+\frac{C}{t^4}.$$

Using $y(1)=-2$, we have $-2=y(1)=-e^{-1}-e^{-1}+C \Longrightarrow C=-2+2e^{-1}$.

$$\Longrightarrow \underline{y(t)=-\frac{1}{t^3}e^{-t}-\frac{1}{t^4}e^{-t}+\frac{-2+2e^{-1}}{t^4}}.$$

(b) The integrating factor is given by $\mu(t)=e^{-\int \frac{2}{t}dt}=e^{-2\ln t}=(e^{\ln t})^{-2}=t^{-2}$.

$$\Longrightarrow \frac{d}{dt}(t^{-2}y)=\cos t \Longrightarrow t^{-2}y=\sin t+C \Longrightarrow y=t^2\sin t+t^2C.$$

Using $y(\pi/2)=0$. we have $0=y(\pi/2)=\frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\right)+\frac{\pi^2}{4}C \Longrightarrow C=-1. \Longrightarrow \underline{y=t^2\sin t-t^2.}$

3. (a) Let $Q(t)=$ the amount of salt (in grams) at time t. Then, we have

$$Q'(t)=2\times 0.25-2\times\frac{Q(t)}{120} \Longrightarrow Q'+\frac{1}{60}Q=0.5.$$

The integrating factor is given by $\mu(t)=e^{\int \frac{1}{60}dt}=e^{\frac{t}{60}}$.

$$\frac{d}{dt}(e^{\frac{t}{60}}Q)=0.5e^{\frac{t}{60}} \Longrightarrow e^{\frac{t}{60}}Q(t)=\int 0.5e^{\frac{t}{60}}dt=30e^{\frac{t}{60}}+C \Longrightarrow Q(t)=30+Ce^{\frac{-t}{60}}.$$

At time 0, there was no salt in the tank, i. e. $Q(0)=0$.

$$\Longrightarrow 0=Q(0)=30+C \Longrightarrow C=-30 \Longrightarrow \underline{Q(t)=30-30e^{\frac{-t}{60}}.}$$

(b) Since $\lim_{t\to\infty} e^{\frac{-t}{60}}=0$, $Q_L=30-0=\underline{30.}$
(c) $Q(200)=30-30e^{\frac{-200}{60}}\simeq \underline{28.93.}$
(d) The salt level is within 3 % of Q_L. $\Longrightarrow 0.97\times 30=29.1=30-30e^{\frac{-t}{60}}$

$$\Longrightarrow e^{\frac{-t}{60}}=0.03 \Longrightarrow -\frac{t}{60}=\ln(0.03) \Longrightarrow t=-60\ln(0.03)\simeq \underline{210.3}$$

4. (a) The characteristic equation is given by $0 = r^2 - 2r + 1 = (r-1)^2 \implies r = 1$ (multiple root). Thus, the general solution is given by $y(t) = C_1 e^t + C_2\, t\, e^t$. $\implies y'(t) = C_1 e^t + C_2 e^t + C_2\, t\, e^t$.

Using $y_1(0) = 1$ and $y_1'(0) = 0$, we have $\begin{cases} 1 = y_1(0) = C_1 \\ 0 = y_1'(0) = C_1 + C_2 \end{cases} \implies \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases} \implies \underline{y_1(t) = e^t - t\, e^t}$

Using $y_2(0) = 0$ and $y_2'(0) = 1$, we have $\begin{cases} 0 = y_2(0) = C_1 \\ 1 = y_2'(0) = C_1 + C_2 \end{cases} \implies \begin{cases} C_1 = 0 \\ C_2 = 1 \end{cases} \implies \underline{y_2(t) = t\, e^t}$

Also, we have $W\big(y_1, y_2\big)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = \underline{1.}$

(b) By dividing each term by $t(t-6)$ we have $y'' + \frac{5}{t-6} y' + \frac{6t}{t(t-6)} y = \frac{3t}{t(t-6)}$. Thus, $p(t)$ is continuous *except* at $t = 6$ and $q(t)$ and $g(t)$ are continuous *except* at $t = 0$ and $t = 6$. Since the initial conditions are given at $t = 4$, the longest time interval of existence is $\underline{0 < t < 6.}$

5. (a) The characteristic equation is given by $0 = 2r^2 - 3r + 1 = (2r-1)(r-1) \implies r = 1, \frac{1}{2}$. Thus, the general solution is given by $y(t) = C_1 e^t + C_2 e^{\frac{1}{2}t}$. $\implies y'(t) = C_1 e^t + \frac{1}{2} C_2 e^{\frac{1}{2}t}$.

Using $y(0) = 2$ and $y'(0) = 1/2$, we have $\begin{cases} 2 = y(0) = C_1 + C_2 \\ 1/2 = y'(0) = C_1 + \frac{1}{2} C_2 \end{cases} \implies \begin{cases} C_1 = -1 \\ C_2 = 3 \end{cases}$

$$\implies \underline{y(t) = -e^t + 3e^{\frac{1}{2}t}.}$$

(b) The characteristic equation is given by $0 = r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2 \implies r = \frac{1}{2}$ (multiple root). Thus, the general solution is given by $y(t) = C_1 e^{\frac{1}{2}t} + C_2\, t\, e^{\frac{1}{2}t}$. $\implies y'(t) = \frac{1}{2} C_1 e^{\frac{1}{2}t} + C_2 e^{\frac{1}{2}t} + \frac{1}{2} C_2\, t\, e^{\frac{1}{2}t}$.

Using $y(2) = 0$ and $y'(2) = 1$, we have $\begin{cases} 0 = y(2) = C_1 e + 2C_2 e \\ 1 = y'(2) = \frac{1}{2} C_1 e + 2C_2 e \end{cases} \implies \begin{cases} C_1 e = -2 \\ C_2 = 1 \end{cases} \implies \begin{cases} C_1 = -2e^{-1} \\ C_2 = e^{-1} \end{cases}$

$$\implies y(t) = -2e^{-1} e^{\frac{1}{2}t} + e^{-1}\, t\, e^{\frac{1}{2}t} = \underline{-2e^{\frac{1}{2}(t-2)} + t\, e^{\frac{1}{2}(t-2)}.}$$

(c) The characteristic equation is given by $r^2 + 4r + 5 = 0 \implies r = -2 \pm i$. Thus, the general solution is given by $y(t) = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$.

$$\implies y'(t) = -2C_1 e^{-2t} \cos t + C_1 e^{-2t} \sin t - 2C_2 e^{-2t} \sin t + C_2 e^{-2t} \cos t.$$

Using $y(0) = 1$ and $y'(0) = 0$, we have $\begin{cases} 1 = y(0) = C_1 \\ 0 = y'(0) = -2C_1 + C_2 \end{cases} \implies \begin{cases} C_1 = 1 \\ C_2 = 2 \end{cases}$

$$\implies \underline{y(t) = y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t.}$$

The graph of part (c) should show some oscillations due to $\cos t$ and $\sin t$.