NOTES ON LINEAR 2x2 SYSTEMS OF ODE

First let's do a quick review of COMPLEX NUMBERS

The set of Complex numbers, denoted by $\mathbb{C}$ is a representation of points in the plane $\mathbb{R}^2$ which is very useful in many instances, such as when solving equations that don't have real solutions (solutions in $\mathbb{R}$).

- The imaginary unit i is defined as $\boxed{i = \sqrt{-1}} \Leftrightarrow \boxed{i^2 = -1}$
- A complex number then has the form

$\boxed{z = a + ib}$ where a, b are real numbers.

The "real part" of z - denoted by - $\boxed{\operatorname{Re} z = a}$

The "imaginary part" of z - denoted by - $\boxed{\operatorname{Im} z = b}$

NOTE : $i.b$ is multiplication of i and b

so $ib = bi$ and $z = a + ib = a + bi$

Examples : $z = 3 + 2i$, $\operatorname{Re} z = 3$ $\operatorname{Im} z = 2$

$z = -1 + i$; $\operatorname{Re} z = -1$ $\operatorname{Im} z = 1$

$z = 3 - i$, $\operatorname{Re} z = 3$ $\operatorname{Im} z = -1$

If $\operatorname{Re} z = 0$

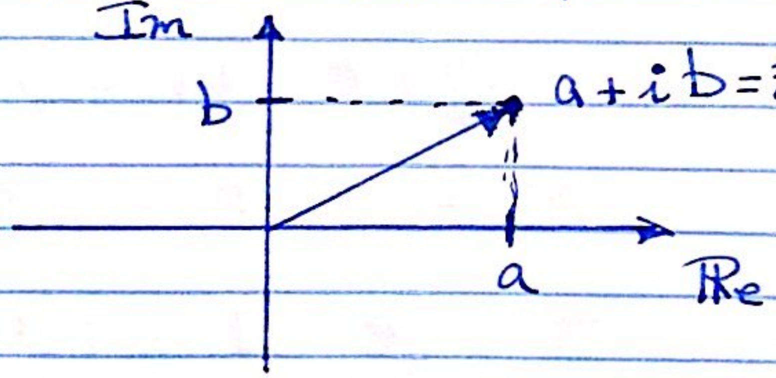
z is called PURELY IMAGINARY → $z = 7i$, $\operatorname{Re} z = 0$ $\operatorname{Im} z = 7$

Real numbers are in $\mathbb{C}$ → $z = 4$; $\operatorname{Re} z = 4$ $\operatorname{Im} z = 0$

$4 = 4 + 0i$

$z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$, $\operatorname{Re} z = \frac{1}{2}$ $\operatorname{Im} z = -\frac{\sqrt{3}}{2}$

Representation of Complex Numbers



(NOTE geometrically $a+ib$ is the point (a, b) in the plane).

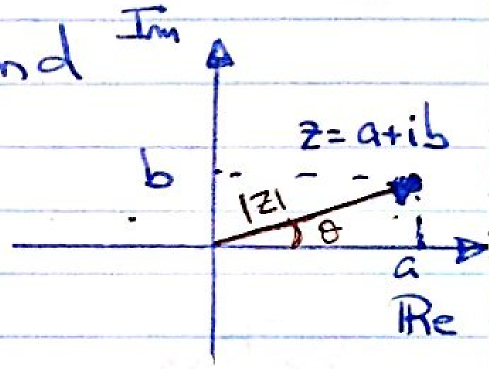
DEFINITION : (COMPLEX CONJUGATE)

Given $z = a + ib$, the complex conjugate of z is $\overline{z} = a - ib$

(*) From the representation of z we see that each z has a length and an angle associated to it. Indeed

$$|z| = \sqrt{a^2+b^2} \quad \text{and}$$

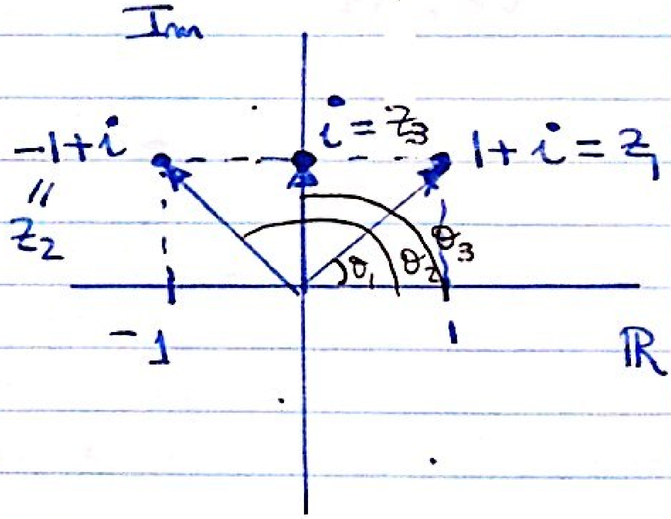
$$\theta = \arctan\left(\frac{b}{a}\right)$$



Examples: $z_1 = 1+i$

$z_2 = -1+i$

$z_3 = i$



$|z_1| = \sqrt{2} \quad \theta_1 = \frac{\pi}{4}$.

$|z_2| = \sqrt{2} \quad \theta_2 = \frac{3\pi}{4}$

$|z_3| = 1 \quad \theta_3 = \frac{\pi}{2}$

- PROPERTIES (ADDITION & MULTIPLICATION in $\mathbb{C}$)

Let $z = a+ib$ and $w = c+id$

(1) Then $z+w = (a+c)+i(b+d)$

2. $z - w = (a-c) + i(b-d)$

3. $z \cdot w = (ac - bd) + i(ad + bc)$

4. $z \cdot \bar{z} \overset{= \bar{z} \cdot z}{=} |z|^2 = a^2 + b^2$ (real).

5. $w^{-1} = \frac{1}{w} = \frac{\bar{w}}{w\bar{w}} = \frac{\bar{w}}{|w|^2}$

So $w^{-1} = \left(\frac{c}{c^2+d^2}\right) - i \frac{d}{(c^2+d^2)}$.

Note: $w \cdot w^{-1} = 1 = w^{-1} \cdot w$

Indeed: by 5 and 4 we have

$$w^{-1} w = \frac{\bar{w} \cdot w}{|w|^2} = \frac{|w|^2}{|w|^2} = 1.$$

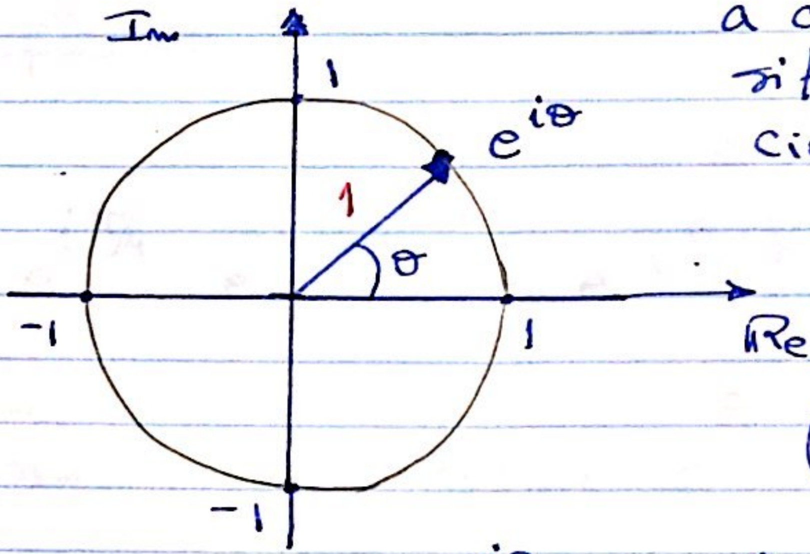
$$w w^{-1} = \frac{w \cdot \bar{w}}{|w|^2} = \frac{|w|^2}{|w|^2} = 1.$$

Returning to $*$ in page 3:

Another way to represent $z = a + ib$ is as follows:

$$z = |z| e^{i\theta}$$

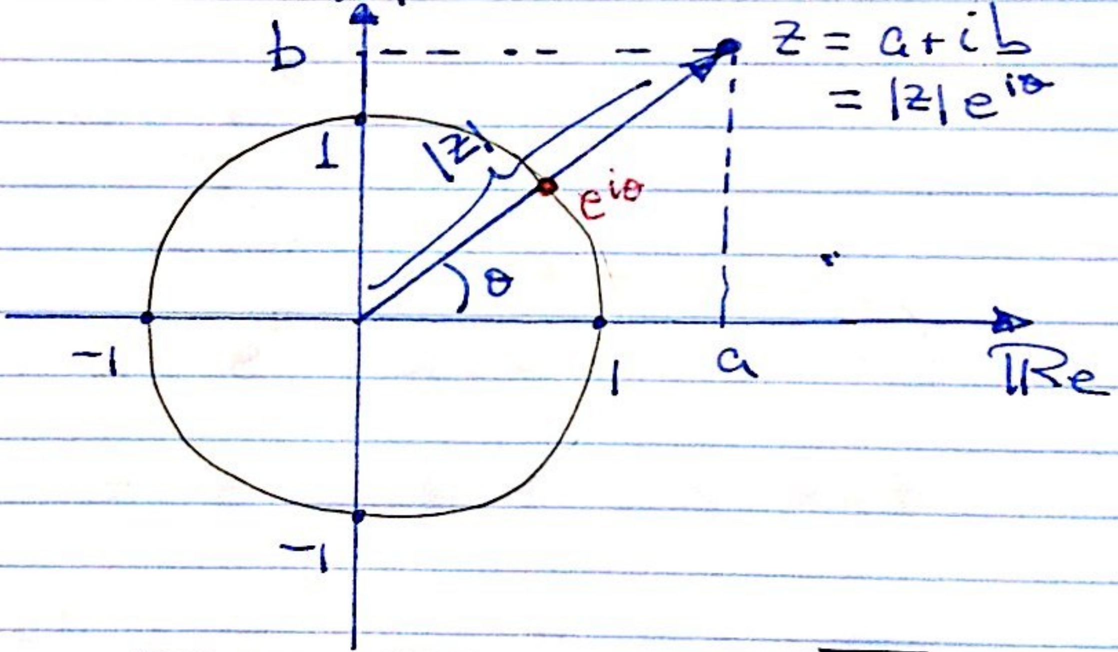
Where $e^{i\theta} := \cos\theta + i\sin\theta$ represents a complex number sitting on the circle of radius 1



$$\boxed{|e^{i\theta}| = 1}$$

(Pythagoras)
$(\cos^2\theta + \sin^2\theta = 1)$

Then: $z = |z|\, e^{i\theta} = a + ib$



Coming back to $e^{i\theta}$: ① $\overline{e^{i\theta}} = e^{-i\theta}$

② If θ_1 and θ_2 are 2 angles then

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

Examples: (i) $e^{i\pi/3}\, e^{i\pi/6} = e^{i(\pi/3 + \pi/6)}$

$= e^{i\pi/2}$

(ii) $e^{i\pi/4}\, e^{-i\pi/12} = e^{i(\pi/4 - \pi/12)} = e^{i\pi/6}$

IMPORTANT.

(I) Finally we define e^{z} where $z \in \mathbb{C}$

If $z = a + ib$

$$e^{z} = \underbrace{e^{a}}_{\text{real number}} \cdot e^{ib}$$

$$= e^{a}\left(\cos b + i \sin b\right)$$

$$= e^{a}\cos b + i\, e^{a} \sin b$$

$\operatorname{Re}(e^{z}) = e^{a}\cos b$

$\operatorname{Im}(e^{z}) = e^{a}\sin b.$

Remarks: ① If λ is a real number (scalar) and $z = a + ib$ then $\lambda z = (\lambda a) + i(\lambda b)$

② If $z = a + ib \Rightarrow iz = ia - b$ $\begin{cases} \operatorname{Re}(iz) = -b \\ \operatorname{Im}(iz) = a \end{cases}$

2x2 Linear Systems of ODE

(**) $Y' = AY$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ is a 2x2 matrix

and $Y = Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ is the unknown vector

(each entry is a function of t)

Y' means $\frac{dY}{dt} = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}$.

So (**) can be rewritten as

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

or as

$$\begin{cases} y_1'(t) = a_{11}\, y_1(t) + a_{12}\, y_2(t) \\ y_2'(t) = a_{21}\, y_1(t) + a_{22}\, y_2(t) \end{cases}$$

$a_{11}, a_{12}, a_{21}, a_{22}$ are real scalars (given).

<u>NOTE:</u> $Y(t) = [0, 0]$ is always a solution

- To <u>find solutions $Y(t)$</u> (<u>nonzero</u>) we proceed to find the <u>eigenvalues</u> and <u>eigenvectors</u> of A.

(I) <u>For eigenvalues</u> : solve $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} \cdot a_{22} = 0.$$

$$\lambda^2 - (a_{11} + a_{22})\lambda - a_{12} a_{22} = 0.$$

<u>Possibilities</u> : ① $\underline{\lambda_1, \lambda_2}$ <u>2 DIFFERENT REAL ROOTS</u>

② $\underline{\lambda_1 = \lambda_2}$ <u>1 double/repeated root.</u> (<u>real</u>)

③ $\underline{\lambda_1, \lambda_2}$ <u>2 complex roots</u> $(\lambda_2 = \bar{\lambda}_1)$

(λ_2 is complex conjugate of λ_1)

(II) <u>For eigenvectors</u> : Once you find the eigenvalues λ_1, λ_2 plug in $(A - \lambda I)$ and solve

plug in here

$(A - \lambda I)\vec{V} = 0$ to find eigenvector $\vec{V}$ corresponding to (each) λ_k ($k=1,2$). That is solve

$k=1,2$
$$\begin{bmatrix} a_{11} - \lambda_k & a_{12} \\ a_{21} & a_{22} - \lambda_k \end{bmatrix} \begin{bmatrix} X_1^{(k)} \\ X_2^{(k)} \end{bmatrix} = 0$$

Usually these two equations are multiples of each other.

$$\begin{cases} (a_{11} - \lambda_k) X_1^{(k)} + a_{12} X_2^{(k)} = 0 \\ a_{21} X_1^{(k)} + (a_{22} - \lambda_k) X_2^{(k)} = 0 \end{cases}$$

Pick one solution $X^{(k)} = \begin{bmatrix} X_1^{(k)} \\ X_2^{(k)} \end{bmatrix}$ for each λ_k $(k=1,2)$

In case 1 when λ_1, λ_2 are 2 different real eigenvalues, we will get 2 (real) l.i eigenvectors :

$$\lambda_1 \longrightarrow X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$$

$$\lambda_2 \longrightarrow X^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix}$$

Then the solution $Y(t)$ of $Y' = AY$ is given by

SOLUTION

$$Y(t) = c_1 e^{\lambda_1 t} X^{(1)} + c_2 e^{\lambda_2 t} X^{(2)}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix}$$

Or rewrite as

$$\begin{cases} y_1(t) = c_1 e^{\lambda_1 t} x_1^{(1)} + c_2 e^{\lambda_2 t} x_1^{(2)} \\ y_2(t) = c_1 e^{\lambda_1 t} x_2^{(1)} + c_2 e^{\lambda_2 t} x_2^{(2)} \end{cases}$$

In case (2) when $\lambda_1 = \lambda_2$ real, repeated root. (Call just $\lambda = \lambda_1 = \lambda_2$ the single eigenvalue)

Then when computing the eigenvectors could have (A) 1 eigenvalue gives 2 different l.i eigenvectors $X^{(1)}$ and $X^{(2)}$

Then solution $Y(t) = c_1 \underbrace{e^{\lambda t} X^{(1)}} + c_2 \underbrace{e^{\lambda t} X^{(2)}}$

same as in case (1) but with same $e^{\lambda t}$ in both terms.

(B) 1 eigenvalue λ gives only 1 eigenvector $\to \vec{V}$

To write the solution we need to find a new vector $\vec{w}$ that is l.i with the one we have as follows:

Recall The one we have $\vec{V}$ was found by solving
$$(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Suppose we pick as solution $\vec{V} = [v_1, v_2]$

$\to$ Then to find the other eigenvector $\vec{w}$ we solve

$$(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{solve for this}}$ $\qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{V}$

Pick one vector as solution, call it $\vec{W} = [w_1, w_2]$.

Then the solution $Y(t)$ of $Y' = AY$ is

$$\boxed{Y(t) = c_1 e^{\lambda t}\vec{V} + c_2\left[te^{\lambda t}\vec{V} + e^{\lambda t}\vec{W}\right]}$$

eigenvector for λ

New vector we found.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2\left(te^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + e^{\lambda t}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)$$

In case 3 we we have 2 complex (conjugate) roots λ_1, λ_2 $(\lambda_2 = \bar{\lambda}_1)$

We proceed as follows:

Suppose $\lambda_1 = \alpha + i\beta$

(then $\lambda_2 = \alpha - i\beta$) $\quad$ (α, β real)

As before find the eigenvector for λ_1 by solving

$$(A - \lambda_1 I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

solve for $[x_1, x_2]$

x_1, x_2 complex numbers.

Now since λ_1 is complex a solution will give us an eigenvector for λ_1 $\vec{V} = [V_1, V_2]$ where now V_1, V_2 are complex numbers!

(NOTE: the eigenvector for λ_2 will just be $\overline{\vec{V}} = [\overline{V_1}, \overline{V_2}]$. But we don't need it)

To write the solution $Y(t)$ with real components $[y_1(t), y_2(t)]$ we consider $e^{\lambda_1 t}\vec{V}$ and take its Re and Im parts:

Suppose $v_1 = a_1 + i b_1 \quad v_2 = a_2 + i b_2$

and $\lambda_1 = \alpha + i\beta$

Then $e^{\lambda_1 t} = e^{\alpha t} e^{i\beta t}$

$$= e^{\alpha t}\left(\cos(\beta t) + i \sin(\beta t)\right)$$

and $e^{\lambda_1 t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = e^{\lambda_1 t}\begin{bmatrix} a_1 + i b_1 \\ a_2 + i b_2 \end{bmatrix}$

$$= e^{\alpha t}\left(\cos(\beta t) + i \sin(\beta t)\right)\begin{bmatrix} a_1 + i b_1 \\ a_2 + i b_2 \end{bmatrix}$$

$$\underbrace{\mathrm{Re}\left(e^{\lambda_1 t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)}_{\text{call } X^{(1)}(t)\text{''}} = \begin{bmatrix} e^{\alpha t}\left(a_1 \cos(\beta t) - b_1 \sin(\beta t)\right) \\ e^{\alpha t}\left(a_2 \cos(\beta t) - b_2 \sin(\beta t)\right) \end{bmatrix}$$

2 l.i vectors

call $X^{(2)}(t)$ "

$$\mathrm{Im}\left(e^{\lambda_1 t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} e^{\alpha t}\left(a_1 \sin(\beta t) + b_1 \cos(\beta t)\right) \\ e^{\alpha t}\left(a_2 \sin(\beta t) + b_2 \cos(\beta t)\right) \end{bmatrix}$$

Then the solution (expressed now in terms of real numbers/functions) is given by

$$Y(t) = C_1 X^{(1)}(t) + C_2 X^{(2)}(t)$$

$$X^{(1)}(t) = \mathrm{Re}\left(e^{\lambda_1 t}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right)$$

$\alpha = \mathrm{Re}\,\lambda_1$

$$= e^{\alpha t}\begin{bmatrix} a_1 \cos(\beta t) - b_1 \sin(\beta t) \\ a_2 \cos(\beta t) - b_2 \sin(\beta t) \end{bmatrix}$$

$$X^{(2)}(t) = \mathrm{Im}\left(e^{\lambda_1 t}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right)$$

$\alpha = \mathrm{Re}\,\lambda_1$

$$= e^{\alpha t}\begin{bmatrix} a_1 \sin(\beta t) + b_1 \cos(\beta t) \\ a_2 \sin(\beta t) + b_2 \cos(\beta t) \end{bmatrix}$$

where $\begin{cases} V_1 = a_1 + i b_1 \\ V_2 = a_2 + i b_2 \end{cases}$ $[V_1, V_2]$ eigenvector for λ_1

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