REVIEW PROBLEM SET 2

From Folland's book # 2.5 pages 68-69. Do problems 46, 47, 48, 49, 51.

Note: Problems 46 and 47 are similar to two of the counterexamples we discussed in class to the failure of Fubini-Tonelli when some of the hypothesis are not complied with.

Extra Problems . **(E.I)** Let $X = [0, 1]$ endowed with the Lebesgue measure and let $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(a) Use the trigonometric substitution $x = a \tan\theta$ to prove that

$$\int \frac{x^2 - a^2}{(x^2 + a^2)^2}\, dx = \frac{-x}{x^2 + a^2}.$$

(Hint. Recall the identities : $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$; $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$ and $\sin(2\theta) = 2\cos\theta\,\sin\theta$)

(b) Use (a) to compute the integrals

$$\int_0^1 \left[\int_0^1 f(x, y) dm(x)\right] dm(y)$$

and

$$\int_0^1 \left[\int_0^1 f(x, y) dm(y)\right] dm(x)$$

and show that they are unequal.

(c) In the example of (b) which hypotheses of Fubini-Tonelli Theorem are violated?

(E.II) Let $X \times Y = [0, 1] \times [1, \infty)$ equipped with the product Lebesgue measure. Define $f : X \times Y \to \mathbb{R}$ to be the function

$$f(x, y) = e^{-xy} - 2e^{-2xy}$$

(i) Prove that

$$\int_0^1 [\int_1^\infty f(x, y) dy] dx \neq \int_1^\infty [\int_0^1 f(x, y) dx] dy$$

(ii) Which hypothesis of Fubini's theorem f does not satisfy? Justify your answer by proving your claim rigorously.

(E.III) Let $X \times Y = [0, \infty) \times [0, \infty)$ equipped with the product Lebesgue measure. Define $f : X \times Y \to \mathbb{R}$ to be the function

$$f(x, y) = y.e^{-(1+x^2)y^2}$$

(i) Prove that

$$\int_0^\infty [\int_o^\infty f(x, y)dx]dy = \int_0^\infty [\int_0^\infty f(x, y)dy]dx$$

(ii) Use (i) to deduce that $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$