M623 HOMEWORK Part I – Fall 2014

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Some more problems (do not turn in yet)

Do problems: 8 (p. 39), 13a) (p 41), 17 (p 42), 19 (p 42), 20 (p42-43), 30 (p 44),

I. Consider the sequence of functions $f_n(x) := \frac{n}{1+(nx)^2}$. For $a \in \mathbb{R}$ be a fixed number consider the Lebesgue integral $I_a(f_n)(x) := \int_a^\infty f_n(x)\, dm$. Compute $\lim_{n\to\infty} I_a(f_n)(x)$ in each case: i) $a = 0$ ii) $a > 0$ and iii) $a < 0$. Carefully justify your calculations (recall the transformation of integrals under dilations).

II. Use the DCT to prove the following: let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions on $\mathbb{R}^d$ such that $\sum_{n=1}^\infty \int |f_n(x)|\, dm < \infty$. Show that $\sum_{n=1}^\infty f_n(x)$ *converges* a.e. $x \in \mathbb{R}^d$ to an *integrable function* and that $\sum_{n=1}^\infty \int f_n(x)\, dm = \int \sum_{n=1}^\infty f_n(x) dm$.

III. Let $0 < c < d$ be fixed real numbers and for $n \geq 1$ consider the sequence function $f_n(x) = ce^{-cnx} - de^{-dnx}$, where $x \geq 0$. Prove that:

(1) $\sum_{n=1}^\infty \int |f_n(x)|\, dm$ diverges
(2) $\sum_{n=1}^\infty \int f_n(x)\, dm = 0$
(3) $\sum_{n=1}^\infty f_n(x)$ is an integrable function on $[0,\infty)$ and that

$$\int_{[0,\infty)} \sum_{n=1}^\infty f_n(x)\, dm = \ln(\frac{d}{c}).$$

IV. Consider the function on $\mathbb{R} \times \mathbb{R}$ given by

$$f(x,y) = ye^{-(x^2+1)y^2} \quad \text{if } x \geq 0, y \geq 0 \quad \text{and} \quad 0 \quad \text{otherwise}.$$

a) Integrate $f(x,y)$ over $\mathbb{R} \times \mathbb{R}$ (justify your steps carefully)
b) Use a) to prove that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.
c) Use b) and dilation to prove that $\int_{-\infty}^\infty e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$.

V. Let $f(x)$ be a measurable function over $\mathbb{R}^{d_1}$ and $g(y)$ be a measurable function over $\mathbb{R}^{d_2}$. Prove that $F(x,y) = f(x)g(y)$ is a measurable function over $\mathbb{R}^{d_1+d_2}$.

SET 4: Due Date 10/30/14

From Chapter 2 (pp 88-97): 2, 8, 15

Additional Problems:

I. If a function f is integrable then we proved in Proposition 1.12 (Chapter 2) that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any set A with $m(A) \leq \delta$, we have that $\int_A |f(x)|\, dm \leq \varepsilon$ (*absolute continuity of the Lebesgue integral*).

We say that a sequence of functions $\{f_n\}_{n\geq 1}$ is **equi-integrable** if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. for any set A with $m(A) \leq \delta$, we have that $\int_A |f_n(x)|\, dm \leq \varepsilon$ for all $n \geq 1$.

Now prove the following.

Let E be a set of finite measure, $m(E) < 1$, and let $\{f_n\} : E \to R$ be a sequence of functions which is equi-integrable. Show that if $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. x, then

$$\lim_{n\to\infty} \int_E |f_n(x) - f(x)|\, dm \,=\, 0.$$

Hint. Use Egorov's Theorem as in the bounded convergence theorem.

II. In class we first proved the Bounded Convergence Theorem (using Egorov Theorem). Then, we proved Fatou's Lemma (using the BCT) and deduced from it the Monotone Convergence Theorem. Finally we proved the Dominated Convergence Theorem (using both BCT and MCT). Here we would like to prove these sequence of results in a different order. Namely, prove:

a) Prove Fatou's Lemma *from* the MCT by showing that for any sequence of measurable functions $\{f_n\}_{n\geq 1}$,

$$\int \liminf_{n\to\infty} f_n\, dm \,\leq\, \liminf_{n\to\infty} \int f_n\, dm.$$

Hint. Note that $\inf_{n\geq k} f_n \leq f_j$ for any $j \geq k$, whence $\int \inf_{n\geq k} f_n\, dm \,\leq\, \inf_{j\geq k} \int f_j$.

b) Now prove the DCT from Fatou's Lemma.

Hint. Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.

Problems on Fubini-Tonelli 4, 18, 19,

Some Additional Problems on Fubini coming soon.

SET 3: Due Date 10/16/14

From Chapter 2 (pp 88-97): 6, 9, 10, 11

<u>Additional Problem:</u> Fill in all details to give a full proof of Lemma 1.2 (ii) (Chapter 2 pages 53-54).

SET 2: Due Date: Thursday October 2, 2014

From Chapter 1 (pp 37-42): 18 (assume f is finite-valued), 22, 28, 32 (Part b) should say: "...prove that *there exists* a subset of G which is....").

<u>Additional Problem:</u> Suppose that A is a measurable set in $\mathbb{R}^d$ with $m(A) > 0$. Show that for any $q < m(A)$ there exist a measurable set $B \subset A$ with $m(B) = q$.

(Hint: Prove it first for the case that $m(A) = p < \infty$. Use then the intermediate value theorem for $A \cap B_R(0)$.)

SET 1: Due Date: Thursday September 18, 2014

From Chapter 1 (pp 37-42): 1, 2, 3, 4a), 5, 6, 7, 10, 11, 16 (Do **III.** below before 16)

<u>Additional Problems</u>

I. First recall that a set E in $\mathbb{R}^d$ is closed if and only if E contains all its limit points; in particular any convergent sequence in E has limit in E. Next recall that a set $K \subset \mathbb{R}^d$ is said to be compact if K is closed and bounded.

Show that the following are equivalent:

i) K is compact

ii) Any cover of K by open sets, $K \subset \cup_\alpha \mathcal{O}_\alpha$, $\mathcal{O}_\alpha$ open, contains a finite sub-cover

$$K \subset \cup_{j=1}^{M} \mathcal{O}_j, \qquad \text{for some } \; M \geq 1$$

iii) Any sequence $\{y_n\}_{n\geq 1} \subset K$, contains a convergent subsequence whose limit is in K.

II. a) Let $A = \cup_{n=1}^{\infty} A_n$ with $m_*(A_n) = 0$. <u>Use</u> the definition of exterior measure to prove that $m_*(A) = 0$.

b) Use a) to prove that any countable set in $\mathbb{R}^d$ is measurable and has measure zero.

III. Let $\{E_n\}_{n\geq 1}$ be a countable collection of measurable sets in $\mathbb{R}^d$. Define

$$\limsup_{n\to\infty} E_n := \{x \in \mathbb{R}^d \; : \; x \in E_n, \; \text{for infinitely many } \; n\}$$

$$\liminf_{n\to\infty} E_n := \{x \in \mathbb{R}^d \; : \; x \in E_n, \; \text{for all but finitely many } \; n\}$$

a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \qquad \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j$$

b) Show that

$$m(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} m(E_n)$$

$$m(\limsup_{n\to\infty} E_n) \geq \liminf_{n\to\infty} m(E_n) \quad \text{provided that } m(\bigcup_{n=1}^{\infty} E_n) < \infty$$