# NAME:

**MATH 624 Final Take Home Exam**

**Due**: Wednesday, May 10th 2017 no later than 5:30PM

**Instructions**

1. This exam consists of five (5) problems all counted equally for a total of $100\%$.

2. You may consult Stein-Shakarchi books III and IV, your homework or the class notes **only**. No other books or notes are permitted.

3. You should work on the problems alone; do not discuss the problems with other people or classmates. You may ask me any questions you have.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please **type** your full work and answers clearly after each problem and attached each answer to the stated problem

6. Show all your work and justify each and all steps in your proofs.

1. Let $\mu$ and $\nu$ be two positive measures on a measurable space $(X, \mathcal{M})$. Assume that $\nu$ is finite. Show that that following are equivalent:

   a) $\nu \ll \mu$ holds.

   b) For each $\{A_n\}_{n\geq 1}$ in $\mathcal{M}$ with $\lim_{n\to\infty} \mu(A_n) = 0$, we have that $\lim_{n\to\infty} \nu(A_n) = 0$.

   (c) For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $A \in \mathcal{M}$ satisfies $\mu(A) < \delta$, then $\nu(A) < \varepsilon$ holds.

2. a) Let $(X, \|\cdot\|)$ be a normed vector space. Show that the following are equivalent:

   i) $X$ is complete (hence $X$ is Banach).

   ii) If $\{x_n\}_{n\geq 1} \subseteq X$ satisfies that $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then $\sum_{n=1}^{\infty} x_n$ <u>converges in</u> $X$

   b) Let $(X, \mathcal{M})$ be a measurable space. Show that $M(X) :=$ the space of all signed **finite** measures on $(X, \mathcal{M})$ together with the norm

$$\|\mu\| = |\mu|(X)$$

   is a Banach space.

   Here, $|\mu|$ denotes the total variation of $\mu$. You may assume without proof that $M(X)$ is a vector space over $\mathbb{R}$ and that the total variation is a norm.

   <u>Hints.</u> Use part a) to prove completeness by establishing ii).

   To prove ii) suppose $\sum_n \|\mu_n\| < \infty$ and consider -for example- $\nu := \sum_{n=1}^{\infty} |\mu_n|$, a positive finite measure (why?). **Prove** that $\mu_n$ are all absolutely continuous w.r.t. $\nu$. Then use the Radon-Nikodym theorem to find $f_n \in L^1(d\nu)$ (here recall $\mu_n$ are signed **finite**). Use this to find an $f \in L^1(d\nu)$ and then a $\mu \in M(X)$ such that $\|\mu - \sum_{n=1}^{N} \mu_n\| \to 0$ as $N \to \infty$.

3. Let $X$ be a Banach space and $X^*$ its dual. Recall a sequence $\{x_n\}_{n\geq 1}$ is said to *converge weakly* to $x$ if
$$\lim_{n\to\infty} \ell(x_n) \,=\, \ell(x)$$
for any $\ell \in X^*$.

   (a) Show that convergence implies weak convergence.

   (b) Show that if $X = H$ a separable Hilbert space and $\{x_n\}_n$ is an orthonormal basis of $H$ then $x_n$ converges weakly to $0$ but it does **not** converge strongly.

   (c) Suppose $X = \ell^1(\mathbb{N})$. Show that if $x_n$ converges weakly in $\ell^1$ then it converges in $\ell^1$ (use what's the dual of $\ell^1$).

   (d) Let $f_n := n\, 1_{(0,\frac{1}{n})} \in L^p$ (for any $p \geq 1$). Show that $f_n$ converges to $0$ in measure and a.e. but it does not converge to $0$ weakly in $L^p$ for any $p$.

4. Let $f(x) = |x|$, $x \in \mathbb{R}$. Let $\mathcal{M}$ be the *smallest* $\sigma$-algebra with respect to which $f$ is measurable.

(a) Characterize $\mathcal{M}$, i.e. describe the measurable sets. Characterize also the the measurable functions with respect to $\mathcal{M}$.

Hint: For a given $J$, interval, what does the set $f^{-1}(J)$ – which must be in $\mathcal{M}$ – look like?

(b) Let $\mu$ and $\nu$ be two measures on $\mathcal{M}$ defined by

$$\mu(A) = \int_A e^{-x^2}\, dx\,, \quad A \in \mathcal{M}\,,$$

$$\nu(A) = \int_A e^{-x^2+x}\, dx\,, \quad A \in \mathcal{M}\,.$$

Show that $\mu$ is absolutely continuous with respect to $\nu$ and compute the Radon-Nikodym derivative $d\mu/d\nu$.

Hints: i) Make sure this derivative is $\mathcal{M}$-measurable.

ii) Recall $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $0 < p < 1$ and let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f$ and $g$ are positive functions then

$$\int fg d\mu \geq \left( \int f^p d\mu \right)^{1/p} \left( \int g^q d\mu \right)^{1/q} .$$

*Hint:* Use Hölder inequality for some suitable chosen functions ( call them $u$ and $v$.)