Lemma 3.2 (Ch. 3 Section 3.1 - Stein-Shakarchi III)

Let $F:[a,b]\longrightarrow \mathbb{R}$ be a $BV[a,b]$ function.

Then for any $a\leq x\leq b$ we have:

(i) $F(x)-F(a) = P_F(a,x) - N_F(a,x)$

(Note then that $F(x) = \underbrace{\left[P_F(a,x)+F(a)\right]}_{\text{increasing}} - \underbrace{N_F(a,x)}_{\text{increasing}}$)

and

(ii) $T_F(a,x) = \underbrace{P_F(a,x)}_{\geq 0} + \underbrace{N_F(a,x)}_{\geq 0}$

Proof: (i) Given $\varepsilon>0$, $\exists$ a partition $\mathcal{P}_1$,

(by def. of supremum) $a=\tilde{t}_0<\tilde{t}_1<\ldots<\tilde{t}_{N_1}=x$ of $[a,x]$ /

$$P_F - \left(\sum_{(+)} F(\tilde{t}_j)-F(\tilde{t}_{j-1})\right) < \varepsilon \quad (\dagger)$$

(by def. of sup.) and $\exists$ (another) partition $\mathcal{P}_2$: $a=\tilde{\tilde{t}}_0<\tilde{\tilde{t}}_1<\ldots<\tilde{\tilde{t}}_{N_2}=x$

of $[a,x]$ /

$$N_F - \left(\sum_{(-)} -\left[F(\tilde{\tilde{t}}_j)-F(\tilde{\tilde{t}}_{j-1})\right]\right) < \varepsilon \quad (\dagger\dagger)$$

②

So if we let $\mathcal{P} : a = t_0 < t_1 < \cdots < t_N =$ be a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ (for ex. can take $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$) we then have that both (+) and (++) hold over $\mathcal{P}$ (check this is true)

Note also that telescoping the sums we have that

$$\sum_{(+)} \left[F(t_j) - F(t_{j-1})\right] - \left\{\sum_{(-)} -\left[F(t_j) - F(t_{j-1})\right]\right\}$$

$$= F(x) - F(a).$$

Hence, $\left| F(x) - F(a) - \left[P_F - N_F\right] \right| < 2\varepsilon$

and this establishes (i).

To prove (ii) note that for any partition $\mathcal{P} : a = t_0 < \cdots < t_N = x$ of $[a, x]$ we have:

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| =$$

$$\overset{(\dagger\dagger\dagger)}{=} \sum_{(+)} \left(F(t_j) - F(t_{j-1})\right) + \sum_{(-)} -\left[F(t_j) - F(t_{j-1})\right]$$

hence $T_F \leq P_F + N_F$. (by taking sup over all $\mathcal{P}$ of $[a,x]$) ($\sup(A+B) \leq \sup A + \sup B$.)

On the other hand,

$(\dagger\dagger\dagger)$ also implies that

$$\sum_{(+)} \left(F(t_j) - F(t_{j-1})\right) + \sum_{(-)} -\left[F(t_j) - F(t_{j-1})\right] \leq T_F \quad (*)$$

From here arguing like in the first part (using the def. of supremum for P_F and for N_F and a common refinement of the partitions) we can deduce from $(*)$ ($\forall \varepsilon > 0$ approximate) that $P_F + N_F \leq T_F$.

All in all we then have $T_F = P_F + N_F$ as desired #