Math 623. End of Section 1. Chapter 3

Last class we finished the proof of Theorem 1.3 thus establishing that if $f \in L^1(\mathbb{R}^d)$ then

$$(\dagger) \quad \lim_{\substack{m(B)\to 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{a.e } x$$

collection of all open balls B containing x

- In particular we proved that the limit on the L.H.S. of ($\dagger$) exists a.e x.

- Note that applying the Theorem to $|f|$ we get (as we discussed before) that $f^*(x) \geq |f(x)|$ a.e x.

Definition: We define the space of "locally integrable functions" as

$$L^1_{loc}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ or } \mathbb{C} \text{ msrble.} \,/\, \int_B |f(x)|\,dx < \infty \text{ for all balls } B \right\}$$

- Note one can define the same space replacing all balls B by any compact set K.

Remark: $L^1_{loc}(\mathbb{R}^d) \supsetneq L^1(\mathbb{R}^d)$

strictly larger space! Example $f(x) = e^{|x|} \in L^1_{loc}$, $\notin L^1$

Next note that the same proof we did for Theorem 1.3 holds if instead of requiring $f \in L^1(\mathbb{R}^d)$ we only require $f \in L^1_{loc}(\mathbb{R}^d)$. Indeed. we have

Theorem 1.4: Let $f \in L^1_{loc}(\mathbb{R}^d)$ then

$$\lim_{\substack{m(B)\to 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{a.e. } x$$

(as before)

Some important consequences

Definition (Lebesgue density): If E is msrble. and $x \in \mathbb{R}^d$; we say that <u>x is a point of Lebesgue density of E</u> if

$$\lim_{\substack{m(B)\to 0 \\ B \ni x}} \frac{m(B \cap E)}{m(B)} = 1$$

What this says is that small balls around x are almost entirely covered by E. More precisely $\forall\, \alpha < 1$ close to 1 and every ball of sufficiently small radius $\ni x$ we have

$$m(B \cap E) \geq \alpha\, m(B)$$

Thus E covers at least a proportion α of B.

Applying Theorem 1.4 to χ_E we get

Corollary 1.5 : Suppose E is a msrbl. subset of $\mathbb{R}^d$. Then:

i) Almost every $x \in E$ is a point of density of E

ii) Almost every $x \notin E$ is <u>NOT</u> a point of density of E.

<u>Definition (Lebesgue set)</u> If $f \in L^1_{loc}(\mathbb{R}^d)$

$\mathcal{L}_f$ (Lebesgue set of f) consists of all $\bar{x} \in \mathbb{R}^d$ for which $f(\bar{x}) < \infty$ and $\displaystyle\lim_{\substack{m(B)\to 0\\ B \ni \bar{x}}} \frac{1}{m(B)} \int |f(y) - f(\bar{x})|\, dy = 0$

<u>This is always assumed!</u>

<u>Remarks</u> : ① If f is continuous at $\bar{x}$ then $\bar{x} \in \mathcal{L}_f$.

② If $\bar{x} \in \mathcal{L}_f \Rightarrow \displaystyle\lim_{\substack{m(B)\to 0\\ B \ni x}} \frac{1}{m(B)} \int_B f(y)\, dy = f(\bar{x})$

Important Insight

<u>Corollary 1.6</u> : If $f \in L^1_{loc}(\mathbb{R}^d)$ then a.e $x \in \mathcal{L}_f$.

$$\overbrace{}^{\in L^1_{loc}}$$

<u>Proof</u>: Apply Theorem 1.4 to $|f(y)-r| \ \forall r \in \mathbb{Q}$

Then $\exists E_r$, $m(E_r)=0$ and for $x \notin E_r$

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)} \int_B |f(y)-r|\, dy = |f(x)-r|$$

If $E := \bigcup_{r\in\mathbb{Q}} E_r \Rightarrow m(E)=0$ and for

$\bar{x} \notin E$ $(f(\bar{x})<\infty)$, given $\varepsilon>0$ $\exists r \in \mathbb{Q}$ s.t

$|f(\bar{x})-r| < \varepsilon$. Since:

$$\frac{1}{m(B)} \int_B |f(y)-f(\bar{x})|\, dy \leq$$

$$\frac{1}{m(B)} \int_B |f(y)-r|\, dy + |f(\bar{x})-r| \quad \Rightarrow$$

$$\limsup_{\substack{m(B)\to 0\\ B\ni \bar{x}}} \frac{1}{m(B)} \int_B |f(y)-f(\bar{x})|\, dy \leq 2\varepsilon$$

$$\therefore \bar{x} \in \mathcal{L}_f \qquad \#$$

<u>Remark</u>: $\mathcal{L}_f$ can be generalized to averages over more general families that "shrink regularity" to x.

(by this we mean that contain x and have bounded eccentricity), and even more general notions of averages given by convolutions with approximations to the identity (as in Section 2 Chapter 3).

- More precisely: a collection $\{\mathcal{U}_\alpha\}_\alpha$ is said to shrink regularly ("nicely") to $\bar{x}$ (or has a bounded eccentricity at $\bar{x}$) if $\exists\, c>0$ such that for each $\mathcal{U}_\alpha$ $\exists$ a ball B with $\bar{x} \in B$, $\mathcal{U}_\alpha \subset B$ and $m(\mathcal{U}_\alpha) \geq c\, m(B)$.

Thus, $\mathcal{U}_\alpha$ is contained in a ball B but its measure is comparable to the measure of B.

Examples: ① Set of all open cubes containing $\bar{x}$ shrinks regularly to $\bar{x}$.

② HOWEVER (CAVEAT!) in $\mathbb{R}^d$, $d \geq 2$ the collection of all open rectangles that contain $\bar{x}$ does NOT shrink regularly to $\bar{x}$: consider very thin rectangles. For such families the existence of limit a.e. and weak-type (1,1) inequality as in Theorem 1.1 (differentiation theorem) doesn't hold; in fact it fails.

Corollary 1.7: Suppose $f \in L^1_{loc}(\mathbb{R}^d)$. If $\{U_\alpha\}_\alpha$ shrinks regularly to $\bar{x}$ then

$$\lim_{\substack{m(U_\alpha) \to 0 \\ U_\alpha \ni x}} \frac{1}{m(U_\alpha)} \int_{U_\alpha} f(y)\, dy = f(\bar{x}) \text{ for}$$

all $\bar{x} \in \mathcal{L}_f$.

Proof: if $\bar{x} \in B$ with $U_\alpha \subset B$ and $m(U_\alpha) \geq c\, m(B) \Rightarrow$

$$0 \leq \frac{1}{m(U_\alpha)} \int_{U_\alpha} |f(y) - f(\bar{x})|\, dy \leq \frac{1}{c\, m(B)} \underbrace{\int_B |f(y) - f(\bar{x})|\, dy}_{\downarrow \; 0}$$

(Use Squeeze Theorem). #

We end by recalling the definition of the CONVOLUTION of two msrb. functions f and g:

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

$$\underset{\text{(Change variables)}}{=} \int_{\mathbb{R}} f(y)\, g(x-y)\, dy = g * f(x).$$