M624 HOMEWORKS– Spring 2009

Prof. Andrea R. Nahmod

SET 6: Problems to do before Final Exam–Do not turn in

From Folland's book # 6.1 pages 186, 187. Do problems: 1, 3, 6, 7, 9, 10, 12..

From Folland's book # 6.3 page 196. Do problem: 31.

From Folland's book # 6.4 page 199. Do problems 38, 39.

Further Reading Suggestions: Besides your class notes on Ch. 6, you may want to read:

From Folland:

(1) Section 6.1 (all), Prop. 6.17 in Section 6.3, Propositions 6.22, 6.23 and 6.24 in Section 6.4

(2) Page 238 (Lemma 8.4 and Poposition 8.5) in Section 8.1.

From W. Rudin's book , 'Real and Complex Analysis':

(3) Theorem 3.13, 3.14 and following Remarks on pages 70 and 71.

Folland's Problem 29-Reformulated

Consider $\ell^1(\mathbb{N})$, the space of sequences of real numbers $\{a_n\}_{n\geq 1}$ such that $\|\{a_n\}_{n\geq 1}\|_{\ell^1} := \sum_{n=1}^{\infty} |a_n| < \infty$. Define $S : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ as

$$S(\{a_n\}_{n\geq 1}) = \{\frac{a_n}{n}\}_{n\geq 1}$$

(a) Prove that S is linear and continuous (bounded) from ℓ^1 into ℓ^1.

(b) Prove that S is not onto. That is show that the range of S, $\mathcal{R}(S)$ which equals the set of all sequences of real numbers $\{b_n\}_{n\geq 1}$ such that $b_n = \frac{a_n}{n}$ for some $a_n \in \ell^1(\mathbb{N})$ (or equivalently such that $\{nb_n\}_{n\geq 1} \in \ell^1(\mathbb{N})$) is a proper subset of $\ell^1(\mathbb{N})$.

(c) Prove that $\mathcal{R}(S)$ is dense in $\ell^1(\mathbb{N})$.

(d) Prove that S is not open.

Hint. To do so consider $B := \{\{a_n\}_{n\geq 1} \,/\, \|\{a_n\}_{n\geq 1}\|_{\ell^1} < 1\}$ the open ball in $\ell^1(\mathbb{N})$ and prove that $S(B)$ is not open. Note that since $0 \in S(B)$ (by linearity), it is enough to show that there exists a sequence of sequences – say $\{a^{(k)}\}_{k\geq 1}$ where for each k, $a^{(k)} := \{a_n^{(k)}\}_{n\geq 1} \in \ell^1(\mathbb{N})$ – such that for each k, $a^{(k)} \notin S(B)$ but $a^{(k)} \to 0$ as $k \to \infty$ in $\ell^1(\mathbb{N})$.

(e) Consider now the space $\mathcal{R}(S)$ endowed with the $\ell^1(\mathbb{N})$ norm. This space, namely $\mathcal{X} := (\mathcal{R}(S), \ell^1(\mathbb{N}))$ is itself a normed space which by parts (b) and (c) is not complete. If we now view $S : \ell^1(\mathbb{N}) \to \mathcal{X}$ (i.e. we restrict the codomain to its range and call this map S as well) then S is bounded, surjective and one-to-one (check).

Hence $S^{-1} : \mathcal{X} \to \ell^1(\mathbb{N})$ exists and it is defined as $S^{-1}(\{b_n\}_{n\geq 1}) := \{nb_n\}_{n\geq 1}$. Let call this $S^{-1} =: T$. Prove that T is closed but not bounded.

Q. Why doesn't this problem contradict the open mapping and closed graph theorems?

SET 5: Due April 30th, 2009

From Folland's book # 5.5 page 177. Do problems 54, 55, 57b)d) - use 22) in conjunction with 57a)- ,58, 59, 60, 61.

SET 4: Due April 23rd, 2009

From Folland's book # 5.1 page 155. Do problems 3, 6, 7, 9, 12a)-d) For this read the discussion in middle of page 153 first).

From Folland's book # 5.2 page 159. Do problems 17, 22 a).

SET 3: Due March 26th, 2009

From Folland's book # 3.5 page 107. Do problems 27, 28, 31, 32 33, 37, 42.

SET 2: Due February 26th 2009

From Folland's book # 3.3 page 94. Do problem 20.

Extra Problem 2 *: Let μ be a positive measure over $(X, \mathcal{M})$ and let $f : X \to \mathbb{C}$ be a μ-integrable function on X;– i.e. $f \in L^1(\mu)$. Define*

$$\nu(E) := \int_E f \, d\mu \qquad E \in \mathcal{M}$$

(a) *Show that ν is a complex measure on $(X, \mathcal{M})$*

(b) *In particular, show that if $f \in L^1(\mu)$ takes only real values –i.e. $f : X \to \mathbb{R}$ – then ν as defined here is a* **finite** *signed measure (here use the Extra Problem 1 above).*

Extra Problem 3: *Let μ and ν be two measures on $(X, \mathcal{M})$ defined by*

$$\mu(A) := \int_A e^{-x^2}\, dx \qquad A \in \mathcal{M}$$

$$\nu(A) = \int_A e^{-x^2+x}\, dx \qquad A \in \mathcal{M}.$$

Show that $\mu << \nu$ and compute the Radon-Nikodym derivative $\dfrac{d\mu}{d\nu}$.

From Folland's book # 3.4 page 100. Do problems 22, 23, 24, 25a).

SET 1: DUE FEBRUARY 12TH, 2009

From Folland's book # 3.1 page 88. Do problems 2, 3, 4.

From Folland's book # 3.2 page 92. Do problems 8, 9, 10, 13, 16 (correction: need both measures to be σ-finite), 17 (correction: need ν to be σ-finite also.

Extra Problem 1 : Let μ be a positive measure over $(X, \mathcal{M})$ and let f real be an *extended μ-integrable* function on X. Define

$$\nu(E) := \int_E f\, d\,\mu \qquad E \in \mathcal{M}$$

Show that ν is a *signed measure* on $(X, \mathcal{M})$