M524 HOMEWORK –SPRING 2015

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SET 1 - DUE 02/05/15

From Chapter 8 (Krantz's 167-168): 1, 2.

From Chapter 8 (Krantz's 171-172): 5, 6, 8, 9.

Additional Problems (assigned in class):

I. Prove all claims made in Example 8.7; that is give a full proof of Example 8.7.

II. Prove Proposition 8.11 (Theorem 2.10 in Ch 2 proves for the analogous result for sequences).

III. Find an example to prove that Theorem 8.13 fails if one only requires the $f_j' \to g$ pointwise on $I = (a, b)$ rather than uniformly as in the statement.

SET 2 - DUE 02/12/15

From Chapter 8 (Krantz's 175-176): 3, 7, 10, 11, 13

From Chapter 8 (Krantz's 180-181): 3, 4.

Read the proof of Lemma 8.22 (Krantz's p.179).

SET 3 - DUE 02/26/15

From Section 9.1 (Krantz's 188-189): 2.

From Section 9.2 (Krantz's 194): 3, 6.

From Section 9.3 (Krantz's 200): 4.

From Section 10.1 (Krantz's 210-211): 1, 9.

From Section 10.2 (Krantz's 220-221): 2, 3, 5.

SET 4 - DUE 03/12/15

From Section 11.2 (Krantz's 231-234) 3, 4a)c), 10, 13, 15, 16 (for 15 and 16, read and use 14*).

SET 4 - DUE 04/02/15

From Section 11.4 (Krantz's 251) 12a)b) (read Example 11.22 on page 247-248 first).

Additional problems (from the material we covered from *Fourier Analysis, An Introduction* Vol. I by E.M. Stein and R. Shakarchi).

(A1) Let f is a 2π-periodic integrable function on any finite interval.

(a) Prove that for any $a, b \in \mathbb{R}$

$$\int_a^b f(x)dx = \int_{a+2\pi}^{b+2\pi} f(x)dx = \int_{a-2\pi}^{b-2\pi} f(x)dx$$

(b) Prove that for any $a \in \mathbb{R}$

$$\int_{-\pi}^{\pi} f(x+a)dx = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi+a}^{\pi+a} f(x)dx$$

(A2) Suppose that $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_K = \sum_{n=1}^K b_n$ denote the partial sums of the series $\sum b_n$, and define $B_0 = 0$.

(a) Prove the *summation by parts* formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

(b) Deduce from part (a) the Dirichlet's Test for convergence of a series that states: If the partial sums $B_K = \sum_{n=1}^K b_n$ of a series $\sum_n b_n$ are bounded (that is, $|B_K| \leq C$ for some $C > 0$ independent of K) and if $\{a_n\}_n$ is a sequence of real numbers that decreases monotonically to 0 (that is $a_{n+1} \leq a_n$ and $a_n \to 0$) then the series $\sum a_n b_n$ converges.

(A3) Suppose that f is a 2π- periodic function which belongs to the class C^2 of continuous and twice differentiable functions with continuous derivatives. Show that there exists a constant $C > 0$ independent on n such that $|\widehat{f}(n)| \leq \frac{c}{|n|^2}$.

Hint Integrate twice by parts. Use periodicity.

If f is a 2π- periodic function which belongs to the class C^k of continuous and k-times differentiable functions with continuous derivatives. What can you say about $|\widehat{f}(n)|$ and how would you prove that ?

(A4) In class we discussed both the Féjer kernel associated to the Cèsearo summability of Fourier series and the Poisson kernel associated to the Abel summability of Fourier series. That the Fejer kernel is a "good kernel" follows from exercises 14, 15 and 16 in section 11.2 (pages 234-235 of Krantz). Here prove that the Poisson kernel is a good kernel. Recall that for $0 \leq r < 1$ and $-\pi \leq x < \pi$

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1 - r^2}{1 - 2r\cos x + r^2}$$

and to be a good kernel means that

1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)\,dx = 1$ for all $0 \le r < 1$

2) There exists $M > 0$ such that for all $0 \le r < 1$, $\int_{-\pi}^{\pi} |P_r(x)|\,dx \le M$

Hint: note that $P_r(x) \ge 0$ (why?) so this should be immediate from 1).

3) For every $\delta > 0$,

$$\int_{\delta \le |x| \le \pi} |P_r(x)| dx \to 0, \text{ as } r \to 1^-$$

SET 5 - DUE 04/16/15

From Section 12.1 (Krantz's 257-258) 1, 2, 3, 5, 6 and 11*

Hint for 11*: Think for example of a function defined on all of $\mathbb{R}^2$ but which is not continuous at -say- $(0,0)$ but still $\frac{\partial f}{\partial x_i}(0,0)$ exist for $i = 1, 2$. Show your work supporting each claim.

(A1). Prove by the $\varepsilon - \delta$ definition that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1 x_2$ is continuous at $(0,0)$.

From Section 12.2 (Krantz's 263) 5, 7, 8

SET 6 - DUE 04/30/15

From Section 12.3 (Krantz's 269-270) 2, 6, 9, 10

SPECIAL PROJECTS (DUE DATE: TBA.)

Work on them promptly but not turn in yet.

SP I. Prove that the *Weierstrass-type* function $F(x) : \mathbb{R} \to \mathbb{R}$ we defined in class is:

a) continuous for all $x \in \mathbb{R}$,

b) nowhere differentiable- that is differentiable for no $x \in \mathbb{R}$. To prove this, consider the sequence $z_k := x \pm \frac{4^{-k}}{2}$ where the sign $+$ or $-$ is chosen depending on x so that there is **no integer** in between $4^k z_k$ and $4^k x$ (note these two number differ at most by $\frac{1}{2}$). Then inspect and prove that the quotient

$$\left| \frac{F(z_k) - F(x)}{z_k - x} \right|$$

are bigger than 3^{k-1} for each $k \geq 1$. It might be useful to separate the difference of the series in the numerator between the sum up to k and the tail from $k+1$ to infinity.

Recall F is defined as
$$F(x) := \sum_{j=1}^{\infty} (\frac{3}{4})^j \, \psi(4^j x),$$
where
$$\psi(x) := \begin{cases} x - n & \text{if } n \leq x < n+1 \text{ when } n \text{ is even} \\ -x + (n+1) & \text{if } n \leq x < n+1 \text{ when } n \text{ is odd,} \end{cases}$$
for all $n \in \mathbb{N}$. Note that ψ is a periodic function of period 2.

SP II. Prove Exercise 2 (Section 8.3 Krantz's 175)

SP III. Prove Exercise 4* (Section 9.1 Krantz's 188).

SP IV. Prove Exercises 18* and answer 19* (with a justification) (Section 11.2, Krantz's page 235). These two problems are based on Problems 14*, 15 and 16. You may use 14* without proof. Problems 15 and 16 are part of your Homework above (Set 4).

SP V. **Verify** that $\frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function defined by $f(0) = 0$ and
$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & \text{if } \; -\pi < x < 0, \\ -\frac{pi}{2} - \frac{x}{2}, & \text{if } \; 0 < x < \pi \end{cases}$$
Note that the sawtooth function is not continuous. **Show** nonetheless that the series converges for ever x (by which we mean as usual that the partial sums $S_N(x) = \frac{1}{2i} \sum_{|n| \leq N, n \neq 0} \frac{e^{inx}}{n}$ of the series converge). In particular note that the value of the series at the origin, namely 0, is indeed the average of the values of the function $f(x)$ as x approaches the origin from the left and the right.

<u>Hint</u> Use Dirichlet's Test for convergence.