8 H. Lewey's Example of a Linear Equation without Solutions

It is natural to believe that linear partial differential equations always have solutions, in fact so many that it is possible to impose additional conditions. [There are trivial *nonlinear* equations without solutions, e.g. the equation $\exp(u_x) = 0$.] Solutions may of course, cease to exist at a "singular" point, where the characteristic form vanishes, as is the case for the equation $xu_x + yu_y = 1$ which has no solution in a neighborhood of the origin, or the equation $xu_x + yu_y + u = 0$ which has only the trivial solution. (See Problem 2, p. 16). It was surprising therefore when H. Lewy (Annals of Mathematics 66 (1957), 155–158, see also [16]) constructed a linear equation without singular points that has no solution *anywhere*. His equation has the form $Lu = F$ where the linear first order differential operator L has complex-valued linear functions as coefficients, and F is a suitably chosen function of class C^∞. (For *analytic* F there would always be solutions by Cauchy–Kowalevski.) The function F in this example is not given explicitly; its existence is proved by a non-constructive argument. The single equation $Lu = F$ with complex coefficients for a complex-valued u is equivalent to a system of two equations with real coefficients for two real-valued functions.

Theorem. *Let L denote the differential operator acting on functions $u(x,y,z)$ defined by*

$$Lu = -u_x - iu_y + 2i(x + iy)u_z. \tag{1.1a}$$

There exists a function $F(x,y,z) \in C^\infty(\mathbb{R}^3)$ such that the equation

$$Lu = F(x,y,z) \tag{1.1b}$$

has no solution whose domain is an open set Ω in $\mathbb{R}^3$, with $u \in C^1(\Omega)$, and u_x, u_y, u_z Hölder continuous in Ω.

In the proof we first construct special F, for which every solution of (1.1b) must become singular at certain special points. By superposition we construct then an F such that every solution must become singular in a dense set of points. The proof is broken up into a number of lemmas.

Lemma I. *Let $\psi(z) \in C^\infty(\mathbb{R})$, where ψ is real-valued. Let for a certain $\delta > 0$ and $\zeta \in \mathbb{R}$*

$$\Omega = \{(x,y,z) | (x,y,z) \in \mathbb{R}^3;\quad x^2 + y^2 < \delta;\quad |z - \zeta| < \delta\}.$$

A solution $u \in C^1(\Omega)$ of

$$Lu = \psi'(z) \tag{1.2a}$$

can exist only if $\psi(z)$ is real analytic at ζ.

PROOF. Set

$$v(r,\theta,z) = e^{i\theta}\sqrt{r}\,u(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta, z). \tag{1.2b}$$

Then $v \in C^1$ for $0 < r < \delta$, $\theta \in \mathbb{R}$, $|z - \zeta| < \delta$. Moreover, v has period 2π in θ. One easily verifies that

$$Lu = -2v_r - \frac{i}{r}v_\theta + 2iv_z = \psi'(z).$$

The function

$$V(z,r) = \int_0^{2\pi} v(r,\theta,z)\,d\theta$$

is defined and in C^1 for $0 < r < \delta$, $|z - \zeta| < \delta$, and moreover satisfies

$$V_z + iV_r = \int_0^{2\pi}\left(v_z - \frac{1}{2r}v_\theta + iv_r\right)d\theta = -\pi i\psi'(z).$$

Now the continuity of $u(x,y,z)$ implies that $v(r,\theta,z)$ is continuous for $0 \leqslant r < \delta$, $\theta \in \mathbb{R}$, $|z - \zeta| < \delta$, and that $v(0,\theta,z) = 0$. Then $V(z,r)$ is continuous for $0 \leqslant r < \delta$, $|z - \zeta| < \delta$, and vanishes for $r = 0$. It follows that the function $W = V(z,r) + i\pi\psi(z)$ is C^1 for $0 < r < \delta$, $|z - \zeta| < \delta$, satisfies $W_z + iW_r = 0$ and is continuous for $0 \leqslant r < \delta$, $|z - \zeta| < \delta$. Thus W is an analytic function of $z + ir$ for $r > 0$, $|z - \zeta| < \delta$, which is still continuous for $r = 0$ and has vanishing real part there. Equivalently the real and imaginary parts of $W(z,r)$ are conjugate harmonics. By *reflection*, that is by $W(z,-r) = -\overline{W(z,r)}$, we can extend W as analytic function of $z + ir$ to $|r| < \delta$, $|z - \zeta| < \delta$. (See Problem 5, p. 110). It follows that $\pi\psi(z)$, the imaginary part of $W(z,0)$, is real analytic for $|z - \zeta| < \delta$. $\square$

Lemma II. *Let $\psi(z) \in C^\infty(\mathbb{R})$, where ψ is real valued. Let there exist a solution $u(x,y,z)$ of class C^1 of the equation*

$$Lu = \psi'(z - 2\eta x + 2\xi y) \tag{1.3a}$$

in a neighborhood of the point (ξ,η,ζ). Then, $\psi(z)$ is real analytic at $z = \zeta$.

Proof. Set

$$U(X,Y,Z) = u(X + \xi, Y + \eta, Z + 2\eta X - 2\xi Y). \tag{1.3b}$$

Then $U(X,Y,Z)$ is of class C^1 in a neighborhood of $(0,0,\zeta)$ and satisfies

$$-U_X - iU_Y + 2i(X + iY)U_Z = \psi'(Z)$$

as is verified easily. Apply Lemma I. □

In what follows let $\psi(z)$ denote a fixed real-valued periodic function in $C^\infty(\mathbb{R})$ which is not real analytic (see problem 4, p. 69) at any real z. For a function $F(x,y,z) \in C^\infty(\mathbb{R}^3)$ and a multi-index $\alpha = (a,b,c)$ we write $D^\alpha F$ for $(\partial/\partial x)^a(\partial/\partial y)^b(\partial/\partial z)^c F$, and $|\alpha|$ for $a + b + c$. Let $Q_j = (\xi_j, \eta_j, \zeta_j)$ for $j = 1, 2, \ldots,$ denote a sequence of points which is dense in $\mathbb{R}^3$, fixed in what follows. We set

$$c_j = 2^{-j}\exp(-\rho_j) \quad \text{where } \rho_j = |\xi_j| + |\eta_j|. \tag{1.4a}$$

Finally we introduce the bounded infinite sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ of real numbers ε_j. They form a vector space, with addition and multiplication by real scalars defined in an obvious manner, which becomes a Banach space B when referred to the norm

$$\|\varepsilon\| = \sup_j |\varepsilon_j|. \tag{1.4b}$$

Lemma III. *For any $\varepsilon \in B$ the series*

$$F_\varepsilon(x,y,z) = \sum_{j=1}^{\infty} \varepsilon_j c_j \psi'(z - 2\eta_j x + 2\xi_j y) \tag{1.4c}$$

and all its formal derivatives with respect to x,y,z converge uniformly, defining a function $F_\varepsilon \in C^\infty(\mathbb{R}^3)$.

Proof. Since ψ is periodic,

$$M_k = \sup_z |\psi^{(k)}(z)| \tag{1.4d}$$

is finite for any k. Then

$$\begin{aligned}|D^\alpha \varepsilon_j c_j \psi'(z - 2\eta_j x + 2\xi_j y)| &\leqslant \|\varepsilon\| c_j M_{|\alpha|+1}\rho_j^{|\alpha|} \\ &\leqslant 2^{-j}\|\varepsilon\| M_{|\alpha|+1}\rho_j^{|\alpha|}\exp(-\rho_j) \leqslant 2^{-j}\|\varepsilon\| M_{|\alpha|+1}\left(\frac{|\alpha|}{e}\right)^{|\alpha|}.\end{aligned} \tag{1.4e}$$

This implies uniform convergence of the series for $D^\alpha F_\varepsilon$. □

Definition. For positive integers j,n let $\Omega_{j,n}$ denote the ball in $\mathbb{R}^3$ with center Q_j and radius $n^{-1/2}$, consisting of the points $P = (x,y,z)$ with

$$|P - Q_j|^2 = (x - \xi_j)^2 + (y - \eta_j)^2 + (z - \zeta_j)^2 < \frac{1}{n}. \tag{1.5a}$$

We denote by $E_{j,n}$ the subset of B consisting of those ε for which there exists a solution $u(P) = u(x,y,z)$ of class $C^1(\Omega_{j,n})$ of the equation

$$Lu = F_\varepsilon(x,y,z) \tag{1.5b}$$

for which

$$u(Q_j) = 0 \tag{1.5c}$$

$$|D^\alpha u(P)| \leqslant n \quad \text{for } |\alpha| \leqslant 1, \quad P \in \Omega_{j,n} \tag{1.5d}$$

$$|D^\alpha u(P) - D^\alpha u(Q)| \leqslant n|P - Q|^{1/n} \quad \text{for } |\alpha| = 1, \quad P \in \Omega_{j,n}, \quad Q \in \Omega_{j,n}. \tag{1.5e}$$

(Condition (1.5d) represents bounds for u and its first derivatives in $\Omega_{j,n}$, while (1.5e) prescribes a uniform Hölder condition on the first derivatives.)

Lemma IV. *The sets $E_{j,n}$ are closed subsets of B that are nowhere dense (i.e., have no interior points).*

PROOF. Let $\varepsilon^1, \varepsilon^2, \ldots$ be in $E_{j,n}$, let ε be in B, and

$$\lim_{k\to\infty} \|\varepsilon - \varepsilon^k\| = 0.$$

By (1.4e) with $\alpha = 0$

$$|F_\varepsilon - F_{\varepsilon^k}| \leqslant M_1 \|\varepsilon - \varepsilon^k\|.$$

Thus the F_{ε^k} converge to F_ε. Denote by u_k the solution u of $Lu = F_{\varepsilon^k}$ with the properties (1.5c,d,e). Since the u_k and their first derivatives are equi-bounded and equi-continuous in $\Omega_{j,n}$ there exists a subsequence of the u_k which converges uniformly to a function u together with its first derivatives. Then u must again satisfy (1.5c,d,e). Since also the Lu_k in the subsequence converge to Lu, the function u is a solution of (1.5b), and $u \in E_{j,n}$. This shows that $E_{j,n}$ is closed.

Let δ denote the bounded sequence all of whose elements are zero, except the j-th one which shall have the value $1/c_j$. Then

$$F_\delta = \psi'(z - 2\eta_j x + 2\xi_j y).$$

Let ε be an interior point of $E_{j,n}$. We can find a positive number θ so small that also

$$\varepsilon' = \varepsilon + \theta\delta \in E_{j,n}.$$

Let u, u' be the solutions of $Lu = F_\varepsilon$ respectively $Lu' = F_{\varepsilon'}$ with the properties guaranteed by the definition of $E_{j,n}$. Set $u'' = (u' - u)/\theta$. Then u'' is a solution of class C^1 of $Lu'' = F_\delta$ in a neighborhood of the point Q_j. This contradicts Lemma II, since ψ is not real analytic at ζ_j. □

PROOF OF THE THEOREM. Assume the theorem does not hold. There would exist for every $\varepsilon \in B$ an open set $\Omega \subset \mathbb{R}^3$ and a solution u of $Lu = F_\varepsilon$ in Ω with Hölder continuous first derivatives. Now Ω contains a point Q_j, since the sequence $Q_1, Q_2, \ldots$ is dense. Thus $\Omega_{j,n} \subset \Omega$ for all sufficiently large n. For n sufficiently large u will also satisfy (1.5d,e). It will satisfy as well (1.5c) if we replace u by $u - u(Q_j)$. But this means that $\varepsilon \in E_{j,n}$. Hence B is the union of all the $E_{j,n}$ with positive j, n. This contradicts Lemma IV, since the complete metric space B cannot be union of a countable set of closed nowhere dense subsets (Baire category argument! see [7]).

PROBLEMS

1. (a) Write equation (1.1b) as a system of two equations for the real and imaginary parts of u.
   (b) Show that the characteristic form of the system is semi-definite but not definite for all x,y,z.
   (c) Show that there are no real characteristic surfaces $\phi(x,y,z) = \text{const.}$ with grad $\phi \neq 0$.