Solution to 1(c)

Problem c. A set E in $\mathbb{R}^d$ is measurable if and only if for every set A in $\mathbb{R}^d$,

$$m_*(A) = m_*(A \cap E) + m_*(A - E) \tag{1}$$

Proof. 1. ($\Rightarrow$)

We know that $m_*(A) \leq m_*(A \cap E) + m_*(A - E)$ is always true by subadditivity.

So we only need to prove that $m_*(A) \geq m_*(A \cap E) + m_*(A - E)$.

Since E is measurable, then for any ϵ, we can find an open set O and a closed set F, such that $F \subset E \subset O$, and $m(O - F) \leq \epsilon$.

Let Q be an arbitrary open set containing A. Then we have $A \cap E \subset Q \cap O$, $A - E \subset Q - F$

Therefore by monotonicity, definition of measure, additivity, we have

$$\begin{align}
& m_*(A \cap E) + m_*(A - E) \tag{2}\\
\leq\ & m_*(Q \cap O) + m_*(Q - F) \tag{3}\\
=\ & m(Q \cap O) + m(Q - F) \tag{4}\\
=\ & m(Q \cap O) + m(Q \cap F^c) \tag{5}\\
=\ & m(Q \cap O) + m(Q \cap F^c \cap O^c) + m(Q \cap F^c \cap O) \tag{6}\\
\leq\ & m(Q \cap O) + m(Q \cap O^c) + m(F^c \cap O) \tag{7}\\
=\ & m(Q \cap O) + m(Q \cap O^c) + m(O - F) \tag{8}\\
\leq\ & m(Q) + \epsilon \tag{9}
\end{align}$$

Taking the infimum over all open Q containing A, then

$$m_*(A \cap E) + m_*(A - E) \leq \inf_{Q \supset A} m(Q) + \epsilon = m_*(A) + \epsilon \tag{10}$$

Here ϵ is arbitrary, hence we have $m_*(A \cap E) + m_*(A - E) \leq m_*(A)$.

2. ($\Leftarrow$)

   (a) Assume that $m_*(E) < \infty$. By observation 3 in SS, $m_*(E) = \inf m_*(O)$, where the infimum is taken over all open sets O containing E. Then for any ϵ, we can find an open set O, such that $E \subset O$ and $m_*(E) \leq m_*(O) \leq m_*(E) + \epsilon$. By (1), we know that $m_*(O - E) = m_*(O) - m_*(O \cap E) = m_*(O) - m_*(E) \leq \epsilon$. (Here the subtraction makes sense because of the assumption $m_*(E) < \infty$.) Therefore, E is measurable.

(b) If $m_*(E) = \infty$, we write $E = \cup_{k=1}^{\infty}[E \cap B(0,k)]$, where $B(0,k)$ is the ball centered at the origin of radius k. It suffices to prove that $E \cap B(0,k)$ is measurable, since a countable union of measurable sets is measurable. By part (a), it suffices to show that $E \cap B(0,k)$ satisfies (1). We know that E satisfies (1) and $B(0,k)$ is measurable, so it satisfies (1) as well.

More generally, we show that if two sets E_1, E_2 satisfy (1), then the intersection $E_1 \cap E_2$ satisfies (1).

Then by subadditivity, (1), we have

$$
\begin{aligned}
&m_*(A \cap (E_1 \cap E_2)) + m_*(A - (E_1 \cap E_2)) && (11)\\
&= m_*(A \cap (E_1 \cap E_2)) + m_*(A \cap (E_1 \cap E_2)^c) && (12)\\
&= m_*(A \cap (E_1 \cap E_2)) + m_*(A \cap (E_1^c \cup E_2^c)) && (13)\\
&= m_*(A \cap (E_1 \cap E_2)) + m_*((A \cap E_1^c) \cup (A \cup E_2^c)) && (14)\\
&= m_*(A \cap (E_1 \cap E_2)) + m_*((A \cap E_1^c) \cup (A \cap E_2^c - A \cap E_1^c)) && (15)\\
&\leq m_*(A \cap (E_1 \cap E_2)) + m_*(A \cap E_1^c) + m_*(A \cap E_2^c - A \cap E_1^c) && (16)\\
&= m_*(A \cap (E_1 \cap E_2)) + m_*(A \cap E_1^c) + m_*(A \cap E_2^c \cap E_1) && (17)\\
&= m_*(A \cap E_1 \cap E_2) + m_*(A \cap E_1 \cap E_2^c) + m_*(A \cap E_1^c) && (18)\\
&= m_*(A \cap E_1) + m_*(A \cap E_1^c) && (19)\\
&= m_*(A) && (20)
\end{aligned}
$$

Hence, $E_1 \cap E_2$ satisfies (1), so $E \cap B(0,k)$ satisfies (1) as desired.

$\square$