5. Let $f(z)C^a(\sum)$ where $\sum$ is the unit disc $|z| < 1$ in $\mathbb{C}^1$. Let $u(x,y) = \operatorname{Re}(f(x + iy))$ for points $(x,y) \in \mathbb{R}^2$ with $x^2 + y^2 < 1$.

(a) Show that $u(x,y)$ is represented by its power series

$$\sum_{j,k=0}^{\infty} c_{jk} x^j y^k$$

for $|x| + |y| < 1$.

(b) Show that in the example $f(z) = 1/(1 - z)$, the series for $u = (1 - x)/((1 - x)^2 + y^2)$ does not converge (absolutely) for $x = y = r > \frac{1}{2}$. [Hint: Show that here

$$\sum_k c_{kk} r^{2k}$$

diverges.]

4. The Lagrange–Green Identity

We recall the Gauss divergence theorem:

$$\int_\Omega D_k u(x)\,dx = \int_{\partial\Omega} u(x)\frac{dx_k}{dn}\,dS_x = \int_{\partial\Omega} u(x)\zeta_k\,dS_x, \tag{4.1}$$

where d/dn denotes differentiation in the direction of the exterior unit normal $\zeta = (\zeta_1, \dots, \zeta_n)$ of $\partial\Omega$ and $dx = dx_1 \dots dx_n$, dS_x = surface element with integration on x. We always assume the boundary $\partial\Omega$ of our region to be sufficiently regular so that the divergence theorem applies to all $u \in C^1(\overline{\Omega})$. The theorem can be generalized to $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ by approximating Ω from the interior. More generally, we have the formula for integration by parts,

$$\int_\Omega v^{\mathrm{T}} D_k u\,dx = \int_{\partial\Omega} v^{\mathrm{T}} u \zeta_k\,dS_x - \int_\Omega (D_k v^{\mathrm{T}}) u\,dx, \tag{4.2}$$

where u, v are column vectors belonging to $C^1(\Omega)$ with T denoting transposition.

Let now L be a linear differential operator

$$Lu = \sum_{|\alpha| < m} a_\alpha(x) D^\alpha u. \tag{4.3}$$

Let u, v be column vectors and a_α be square matrices in $C^m(\overline{\Omega})$. Then by repeated application of (4.2) it follows that

$$\int_\Omega v^{\mathrm{T}} \sum_{|\alpha| < m} a_\alpha(x) D^\alpha u\,dx$$

$$= \int_\Omega \sum_{|\alpha| < m} (-1)^{|\alpha|} D^\alpha\big(v^{\mathrm{T}} a_\alpha(x)\big) u\,dx + \int_{\partial\Omega} M(v, u, \zeta)\,dS_x. \tag{4.4}$$

Here M in the surface integral is linear in the ζ_k with coefficients which are bilinear in the derivatives of v and u, the total number of differentiations in each term being at most $m-1$. The expression M is not determined uniquely but depends on the order of performing the integration by parts. This is the *Lagrange–Green identity* for L which we also write in the form

$$\int_\Omega v^{\mathrm{T}} Lu\, dx = \int_\Omega (\tilde{L}v)^{\mathrm{T}} u\, dx + \int_{\partial\Omega} M(v,u,\zeta)\, dS_x, \tag{4.5}$$

where $\tilde{L}$ is the (formally) adjoint operator to L, defined by

$$\tilde{L}v = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha\big(a_\alpha(x)^{\mathrm{T}} v\big). \tag{4.6}$$

The characteristic forms of L and $\tilde{L}$ differ at most in sign.

The simplest example corresponds to the Laplace operator $L=\Delta$ for scalars u and v. Then one integration by parts yields

$$\int_\Omega v\Delta u\, dx = \int_{\partial\Omega} \sum_i v u_{x_i} \zeta_i\, dS_x - \int_\Omega \sum_i v_{x_i} u_{x_i}\, dx. \tag{4.7}$$

We write this as

$$\int_\Omega v\Delta u\, dx = \int_{\partial\Omega} v \frac{du}{dn}\, dS_x - \int_\Omega \sum_i v_{x_i} u_{x_i}\, dx. \tag{4.8}$$

Integrating once more by parts we obtain

$$\int_\Omega v\Delta u\, dx = \int_\Omega u\Delta v\, dx + \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS_x. \tag{4.9}$$

5. The Uniqueness Theorem of Holmgren

It is clear from the arguments used in the proof of the Cauchy–Kowalevski theorem that an analytic Cauchy problem with data prescribed on an analytic noncharacteristic surface S has at most one *analytic* solution u, since the coefficients of the power series for u are determined uniquely. This does not exclude the possibility that other *nonanalytic* solutions of the same problem might exist. However, uniqueness can be proved for the Cauchy problem for a *linear* equation with analytic coefficients and for data (not necessarily analytic) prescribed on an analytic noncharacteristic surface S. The method of proof (due to Holmgren) makes use of the Cauchy–Kowalevski theorem and the Lagrange–Green identity. (Extension of the uniqueness theorem to *nonanalytic* equations is much more difficult).

The principle of Holmgren's uniqueness argument is simple. Let u be a solution of a first order linear system

$$Lu = \sum_{k=1}^{n} a^k(x) \frac{\partial u}{\partial x_k} + b(x)u = 0 \tag{5.1a}$$

in a "lens-shaped" region R bounded by two hypersufaces S and Z (see Fig. 3.1). Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^N$ and the a^k, b are $N \times N$ matrices. Assume that u has Cauchy data $u = 0$ on Z and that S is non-characteristic; that is, the matrix

$$A = \sum_{k=1}^{n} a^k(x)\zeta_k \tag{5.1b}$$

is non-degenerate for $x \in S$, and ζ = unit normal of S at x. Let v be a solution of the *adjoint* equation

$$\tilde{L}v = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}((a^k)^{\mathrm{T}}v) + b^{\mathrm{T}}v = 0 \quad \text{for } x \in \mathbb{R} \tag{5.1c}$$

(T for transposition) with Cauchy data

$$v = w(x) \quad \text{for } x \in S. \tag{5.1d}$$

Applying the Lagrange–Green identity (4.5) we find that

$$\int_S w^{\mathrm{T}}Au \, dS = 0. \tag{5.1e}$$

Let now Γ be the set of functions w on S for which the Cauchy problem (5.1c,d) has a solution v. If Γ is dense in $C^0(S)$ (that is if every continuous function on S can be approximated uniformly by functions in Γ) we conclude that (5.1e) hold for every $w \in C^0(S)$. But then $Au = 0$ on S, and hence also, since A is non-degenerate, $u = 0$ on S. For if $Au \neq 0$ for some $z \in S$, then also $Au \neq 0$ for all x in a neighborhood ω of z on S. We can find a continuous non-negative scalar function $\phi(x)$ on S with support in ω and with $\phi(z) > 0$. Then

$$\int_S \phi(Au)^{\mathrm{T}}(Au) \, dS > 0 \tag{5.1f}$$

for $w = \phi Au$ contrary to (5.1e). Now in the case where the matrices a^k and b are *real analytic*, and S and w are real analytic, the Cauchy–Kowalevski theorem guarantees the existence of a solution v of $\tilde{L}v = 0$ with $v = w$ on S

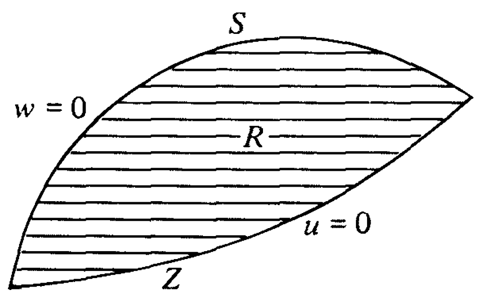

Figure 3.1

in a *sufficiently small* neighborhood of S, though we cannot be sure that that neighborhood includes all of R. To bridge the gap between S and Z and to conclude that $u = 0$ throughout R, we have to cover all of R by an analytic family of non-characteristic surfaces S_λ. Making these notions more precise we are led to the following definition of a family of hypersurfaces forming an *analytic field* of surfaces, and to the general uniqueness theorem below.

Definition. A family of hypersurfaces S_λ in $\mathbb{R}^n$ with parameter λ ranging over an open interval $\Lambda = (a,b)$ forms an *analytic field*, if the S_λ can be transformed bi-analytically into the cross sections of a cylinder whose base is the unit ball Ω in $\mathbb{R}^{n-1}$. This means that there shall exist a $1 - 1$ mapping $F: \Omega \times \Lambda \to \mathbb{R}^n$, where $x = F(y)$ is real analytic in $\Omega \times \Lambda$ and has a non-vanishing Jacobian; the S_λ for $\lambda \in \Lambda$ shall be the sets

$$S_\lambda = \{x \mid x = F(y); (y_1, \ldots, y_{n-1}) \in \Omega; y_n = \lambda\}. \tag{5.2a}$$

(Our conditions imply that the set

$$\Sigma = \bigcup_{\lambda \in \Lambda} S_\lambda, \tag{5.2b}$$

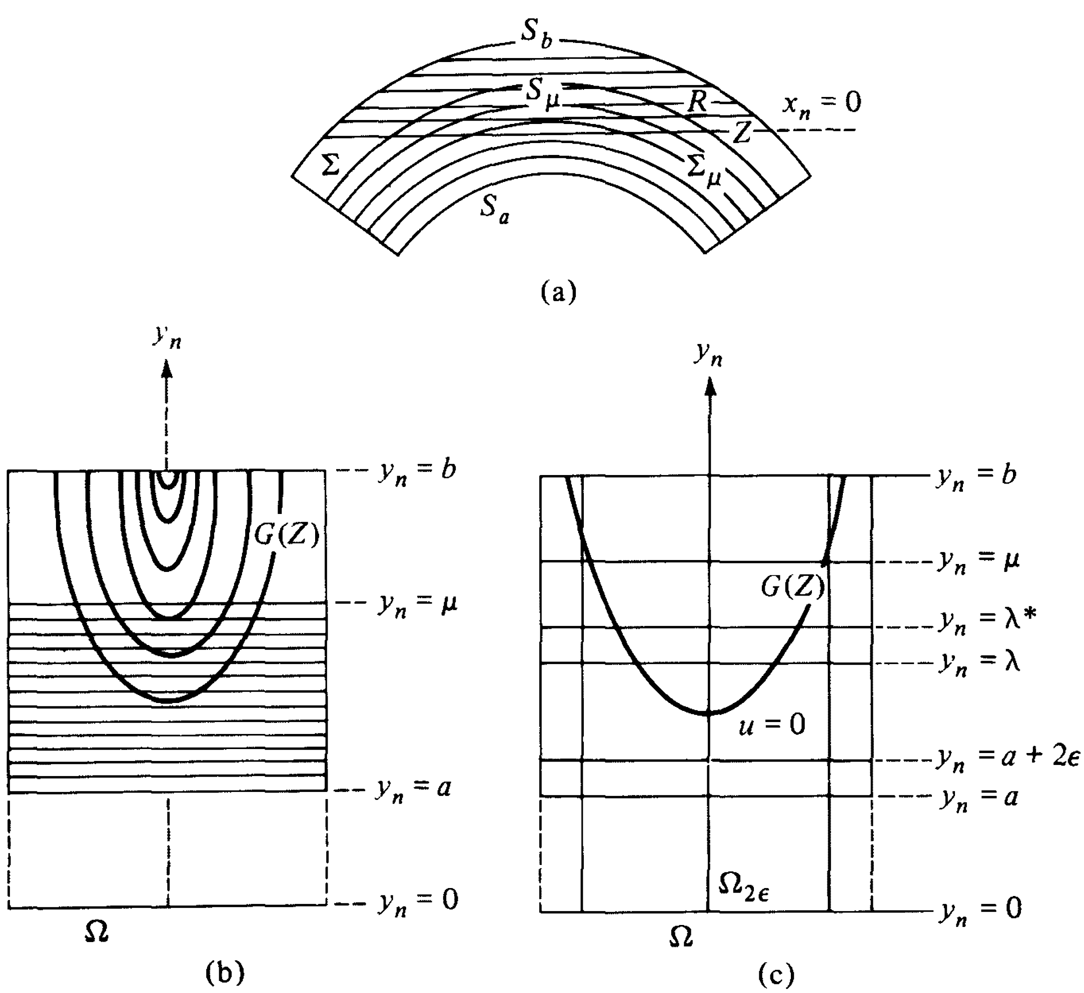

Figure 3.2

called the *support* of the field, is open, and that the transformation $x = F(y)$ has a real analytic inverse $y = G(x)$ mapping Σ onto $\Omega \times \Lambda$. In particular, $\lambda(x) = G_n(x)$ is real analytic in Σ.)

Uniqueness Theorem *(See Fig. 3.2a,b,c). Let the S_λ for $\lambda \in \Lambda = (a,b)$ form an analytic field in $\mathbb{R}^n$ with support Σ. Consider the m-th order linear system*

$$Lu = \sum_{|\alpha| \leqslant m} A_\alpha(x) D^\alpha u = 0 \tag{5.3a}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^N$, and the coefficient matrices $A_\alpha(x)$ are real analytic in Σ. Introduce the sets

$$R = \{x \mid x \in \Sigma;\, x_n \geqslant 0\} \tag{5.3b}$$

$$Z = \{x \mid x \in \Sigma;\, x_n = 0\} \tag{5.3c}$$

and for $\mu \in \Lambda$

$$\Sigma_\mu = \{x \mid x \in S_\lambda \quad \text{for some } \lambda \text{ with } a < \lambda \leqslant \mu\}. \tag{5.3d}$$

We assume that Z and all S_λ are non-characteristic, with respect to L, and that $\Sigma_\mu \cap R$ for any $\mu \in \Lambda$ is a closed *subset of the open set Σ. Let u be a solution of* (5.3a) *of class $C^m(R)$ and have vanishing Cauchy data on Z.* Then $u = 0$ in R.*

Proof. Since Z is non-characteristic it follows from the vanishing of the Cauchy data of u on Z that also $D^\alpha u = 0$ on Z for $|\alpha| = m$. Define $u(x) = 0$ at all x of Σ not belonging to R. Then the extended function u is of class C^m and a solution of (5.3a) throughout Σ. Moreover, for any $\mu \in \Lambda$ the closure of the set of points $x \in \Sigma_\mu$ where $u(x) \neq 0$ is a closed subset of Σ. We only have to prove $u \equiv 0$ in Σ, ignoring Z. We apply the mapping $x = F(y)$ associated with the analytic field. Analyticity and non-characteristic behavior are preserved. Renaming the new independent variables x instead of y, and letting (5.3a) stand for the transformed differential equations, we now have to deal with the family of surfaces

$$S_\lambda = \{x \mid (x_1, \ldots, x_{n-1}) \in \Omega;\, x_n = \lambda\} \tag{5.4a}$$

whose support is the set $\Sigma = \Omega \times \Lambda$. The S_λ are non-characteristic, that is,

$$\det A_\alpha(x) \neq 0 \quad \text{for } \alpha = (0, \ldots, 0, m). \tag{5.4b}$$

$u(x)$ is a solution of class C^m of (5.3a) for $x \in \Sigma$. Any limit x of points where $u \neq 0$ either lies in Σ or has $x_n = b$.

We introduce an auxiliary solution v of an adjoint Cauchy problem. Let $w(x_1, \ldots, x_{n-1})$ denote a vector (fixed for the present) whose components are polynomials. Let $v = v(x, \lambda)$ denote the solution of the adjoint equation

$$\tilde{L}v = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha (A_\alpha^T(x) v) = 0 \tag{5.4c}$$

* That means that u is of class C^m in all interior points of R, that all $D^\alpha u$ for $|\alpha| \leqslant m$ can be extended continuously to all of R, and that the extended values satisfy $D^\alpha u = 0$ on Z for $|\alpha| \leqslant m - 1$.

with Cauchy data prescribed on the plane $x_n = \lambda$:

$$D_n^k v(x_1, \ldots, x_{n-1}, \lambda, \lambda,) = 0 \quad \text{for } 0 \leqslant k < m - 1 \tag{5.4d}$$

$$D_n^{m-1} v(x_1, \ldots, x_{n-1}, \lambda, \lambda) = w(x_1, \ldots, x_{n-1}). \tag{5.4e}$$

In this problem the initial surface S_λ and hence the solution v depend on the parameter λ. We transform this Cauchy problem for v into one with data given on a fixed plane $x_n = 0$ by replacing x_n by $x_n + \lambda$ and considering λ just as an additional independent variable. Writing

$$V(x,\lambda) = v(x_1, \ldots, x_{n-1}, x_n + \lambda, \lambda) \tag{5.4f}$$

$$a_\alpha(x,\lambda) = A_\alpha(x_1, \ldots, x_{n-1}, x_n + \lambda) \tag{5.4g}$$

we have

$$\sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha (a_\alpha^T(x,\lambda) V) = 0 \tag{5.4h}$$

$$D_n^k V(x_1, \ldots, x_{n-1}, 0, \lambda) = 0 \quad \text{for } 0 \leqslant k < m - 1 \tag{5.4i}$$

$$D_n^{m-1} V(x_1, \ldots, x_{n-1}, 0, \lambda) = w(x_1, \ldots, x_{n-1}). \tag{5.4j}$$

Here the coefficient matrices $a_\alpha(x,\lambda)$ are real analytic in the set

$$(x_1, \ldots, x_{n-1}) \in \Omega; \qquad a < x_n + \lambda < b; \qquad a < \lambda < b \tag{5.4k}$$

and the initial plane $x_n = 0$ is non-characteristic with respect to (5.4h). Let Ω_ε for $0 < \varepsilon < \text{Min}(1, (b - a)/2)$ denote the closed ball of radius $1 - \varepsilon$ and center at the origin in $\mathbb{R}^{n-1}$, and let Λ_ε denote the closed interval $[a + \varepsilon, b - \varepsilon]$. The set (5.4k) has the compact subset consisting of the (x, λ) with

$$(x_1, \ldots, x_{n-1}) \in \Omega_\varepsilon; \ x_n = 0; \ \lambda \in \Lambda_\varepsilon.$$

By the general Cauchy–Kowalevski theorem on p. 77 there exists a $\delta = \delta(\varepsilon) > 0$ and a solution $V(x,\lambda)$ of (5.4h,i,j) defined for all (x,λ) with

$$(x_1, \ldots, x_{n-1}) \in \Omega_\varepsilon; \qquad |x_n| < \delta; \qquad \lambda \in \Lambda_\varepsilon.$$

It follows from (5.4f) that the Cauchy problem (5.4c,d,e) has a real analytic solution $v(x,\lambda)$ for

$$(x_1, \ldots, x_{n-1}) \in \Omega_\varepsilon; \qquad |x_n - \lambda| < \delta; \qquad \lambda \in \Lambda_\varepsilon. \tag{5.4l}$$

Take a value $\mu \in \Lambda$. Let ε be so small that

$$u(x) = 0 \quad \text{for } a < x_n < a + 2\varepsilon \tag{5.4m}$$

$$u(x) = 0 \quad \text{for } x \notin \Omega_{2\varepsilon}, \quad a < x_n < \mu. \tag{5.4n}$$

Let λ, λ^* denote values in the interval $(a + \varepsilon, \mu)$ with $|\lambda - \lambda^*| < \delta$. Apply the Lagrange–Green identity (4.5) to the slice of $\Omega \times \Lambda$ bounded by the planes

$x_n = \lambda$ and $x_n = \lambda^*$. We find for $v = v(x, \lambda)$ that

$$\begin{aligned} I(\lambda) &= \int_{x_n=\lambda} w^T A_\alpha(x) u(x)\, dx_1 \cdots dx_{n-1} \\ &= \int_{x_n=\lambda^*} M(v, u, \zeta)\, dx_1 \cdots dx_{n-1} \end{aligned} \tag{5.4o}$$

where $\alpha = (0, \ldots, 0, m)$ and the integrations are extended over Ω_ε. Now the last integral in (5.4o) depends on λ only through v, and hence* is a real analytic function of λ for $\lambda \in (a + \varepsilon, \mu)$, $|\lambda - \lambda^*| < \delta$. Hence $I(\lambda)$ is real analytic in λ for $\lambda \in (a + \varepsilon, \mu)$. By (5.4m) $I(\lambda) = 0$ for $\lambda \in (a + \varepsilon, a + 2\varepsilon)$. Hence $I(\lambda) = 0$ for $\lambda \in (a + \varepsilon, \mu)$. Because of the arbitrariness of μ and ε it follows that $I(\lambda) = 0$ for $\lambda \in (a, b)$. Since w is an arbitrary polynomial vector it follows that $A_\alpha(x)u(x) = 0$ for $x \in \Sigma$. But then also $u(x) = 0$ in Σ by (5.4b). □

From the general theorem just proved one can derive uniqueness theorems for "curved" initial surfaces Z by applying analytic deformations.†

Let L be a linear m-th order differential operator acting on functions $u(x_1, \ldots, x_n)$, and let Z be an $(n - 1)$-dimensional manifold in $\mathbb{R}^n$. A closed set $R \subset \mathbb{R}^n$ is called a *domain of determinacy* for Z (with respect to L) if every solution u of class C^m of $Lu = 0$ in R vanishes if its Cauchy data on Z vanish.‡ The uniqueness theorem just proved permits to construct domains of determinacy with the help of suitable non-characteristic analytic fields, as will be shown by examples.

Let Z be a ball in the plane $x_n = 0$:

$$Z = \left\{ x \,\middle|\, \sum_{k=1}^{n-1} x_k^2 < r^2;\ x_n = 0 \right\}. \tag{5.5}$$

Let L have real analytic coefficients in a neighborhood of Z in $\mathbb{R}^n$, and let Z be noncharacteristic with respect to L. Then the lens-shaped set

$$R = \left\{ x \,\middle|\, 0 \leqslant x_n \leqslant \varepsilon \left(r^2 - \sum_{k=1}^{n-1} x_k^2 \right) \right\}$$

is a domain of determinacy for Z for all sufficiently small positive ε. To see this one only has to consider the analytic field formed by the portions of paraboloids

$$S_\lambda = \left\{ x \,\middle|\, x_n = \lambda + \varepsilon \left(r^2 - \sum_{k=1}^{n-1} x_k^2 \right);\ \sum_{k=1}^{n-1} x_k^2 \leqslant r^2 \right\}$$

* We can differentiate the last integral arbitrarily often with respect to λ and see that it belongs to some class $C_{M,r}$ uniformly in λ.

† Such theorems can also be derived directly for curved Z by the arguments applied above, imbedding Z into an analytic non-characteristic field. Here Z itself need not be analytic. Also the assumption that Z is non-characteristic can be replaced by the weaker requirement that the set of non-characteristic points is dense on Z.

‡ Z may be said to contain the *domain of dependence* of any point of R in the sense of pp. 5, 41, though a precise domain of dependence for the Cauchy problem cannot be defined in many instances, for example for elliptic equations.