The Hahn Banach Theorem for Normed Spaces

1. ZORN'S LEMMA: Suppose that a partially ordered set S has the property that every chain (that is every totally ordered subset $\mathcal{E}$) has an upper bound in S. Then S contains at least one maximal element.

"$\leq$" is a relation. $(S, \leq)$ is partially or linearly ordered (see Folland).

Definitions ① $\mathcal{E}$ is totally ordered if $\forall\, t_1, t_2 \in \mathcal{E}$ $t_1 \leq t_2$ or $t_2 \leq t_2$

② $\mathcal{E}$ has an upper bound $u \in S$ if $t \leq u$ for all $t \in \mathcal{E}$ (u need NOT be in $\mathcal{E}$).

③ $m \in S$ is a maximal element if there is NO $x \in S$ such that $m < x$.

Theorem (HB for normed spaces)

Let X be a normed vector space and let Y be a subspace of X. Then any continuous linear functional $\ell_0 \in Y^*$ on Y can be extended to a continuous linear functional $\ell \in X^*$ on X with the same operator norm. Thus ℓ agrees with ℓ_0 on Y and $\|\ell\|_{X^*} = \|\ell_0\|_{Y^*}$

- NOTE: The extension is in general NOT unique.

The proof proceeds in two steps

Step I : The first step is to show that the conclusion of the HB theorem holds in the case when X and Y are real vector spaces and X is spanned by Y and a vector $v \notin Y$ $(v \in X)$. (i.e. Y has codimension 1).

The proof of this step proceeds exactly as the first part of the proof I gave in class (see attached notes by Folland, top of page 158) when we first extend the functional to $Y + \mathbb{R}.v$.

THIS IS ZORN'S LEMMA ARGUMENT.

Step II : Fix X, Y and l_0. Define a partial extension of l_0 to be a pair (Y', l') where Y' is an intermediate space between X and Y and l' is an extension of l_0 with the same operator norm as l_0.

The set of all partial extensions is partially ordered by declaring $(Y', l') \leq (Y'', l'')$ if $Y' \subseteq Y''$ and l'' extends l'.

Clearly every chain (totally order) of partial extensions has an upper bound. Hence, by Zorn's Lemma there must exist a MAXIMAL partial extension (Y_{max}, l_{max}).

If $Y_{max} = X$ we are done. Suppose then $Y_{max} \subsetneq X$. Then pick $v \in X \setminus Y_{max}$ and use step I to extend l_{max} further to the larger space spanned by Y_{max} and v (ie $Y_{max} + \mathbb{R} \cdot v$). But this contradicts the maximality (since this new extension will be related by "$\leq$" with (Y_{max}, l_{max}) and majorize it).

Remark: The Hahn Banach theorem also holds in the complex case. (See statement in Ex. 33 pg 43 [SS, Vol 4] For a proof see Theorem 5.7 of attached notes by Folland (pg 158).

Remark: When X is a (separable) Hilbert space one may prove the HB theorem without relying on Zorn's lemma.

Remark: Read carefully Theorem 5.8 pg 159 of Folland that collects some very important consequences and "separation properties" of the HB theorem

Finally we give a "geometric version" of the Hahn Banach Theorem:

(Usually A is convex also).

Definition: We call a set A of a real vector V space "algebraically open" if the sets $\{ \lambda \mid x + \lambda v \in A \}$ are open in $\mathbb{R}$ for all $x, v \in V$.

[Remark: every open set in a normed vector space is algebraically open].

Geometric HB Theorem: Let A, B be CONVEX subsets of a real vector space V, with A algebraically open. Then the following are equivalent:

① A and B are disjoint

② There exists a linear functional $l: V \to \mathbb{R}$ and a constant c such that $l < c$ on A and $l \geq c$ on B

(Equivalently: there is a hyperplane separating A and B with A avoiding the hyperplane entirely).