M623 HOMEWORK – Fall 2017

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SET 1: DUE DATE 09/14/2017

Problem 1 Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

Problem 2 Give an example of two **closed** sets $F_1, F_2 \subset \mathbb{R}^2$ such that $F_1 \cap F_2 = \emptyset$ and $\mathrm{dist}(F_1, F_2) = 0$.

Problem 3 a) Given an interval $[a, b] \subset \mathbb{R}$, construct a sequence of continuous functions $\phi_k(x)$ such that for every fixed $x \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \phi_k(x) = \begin{cases} 1 & \text{if } \quad x \in [a, b] \\ 0 & \text{if } \quad x \notin [a, b] \end{cases}$$

b) Can one construct such a sequence ϕ_k so that it also converges uniformly as $k \to \infty$? Explain and justify your answer.

Problem 4 Find $\limsup E_k$ and $\liminf E_k$ for the sequence $\{E_k\}$ defined as follows:

$$E_k := \begin{cases} [-1/k, 1] & \text{for } k \text{ odd} \\ [-1, 1/k] & \text{for } k \text{ even} \end{cases}$$

Problem 5 A continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y), \quad ;\forall\, x, y \in \mathbb{R}, \forall\, \lambda \in [0, 1]$$

Show that if ϕ is convex, then if $x_1, \ldots, x_n$ are points in $\mathbb{R}$ then

$$\phi\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \frac{\phi(x_1) + \ldots \phi(x_n)}{n}$$

More generally, show that if $\alpha_1, \ldots, \alpha_n$ is a sequence of nonnegative numbers with

$$\sum_{i=1}^{n} \alpha_i = 1$$

Then, for any n points $x_1, \ldots, x_n$ in $\mathbb{R}$ we have

$$\phi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \phi(x_i)$$

This last inequality is known as *Jensen's inequality.*

Problem 6 Let $x_1, \dots, x_n$ be all nonnegative numbers. Prove the *arithmetic-geometric mean inequality*

$$\big(x_1 x_2 \dots x_n\big)^{1/n} \le \frac{x_1 + x_2 + \dots + x_n}{n}$$

Hint Apply Jensen's inequality with a conveniently chosen convex function.

Problem 7 Compute the following Riemann integrals:

$$\int_0^1 x^k \, dx, \quad k > 0; \quad \int_0^1 x^{-k} \, dx, \ \ k \in (0,1); \quad \int_1^\infty x^{-k} \, dx, \ \ k \in (1, \infty)$$

$$\int_0^\infty e^{-ax^2} x \, dx, \quad a > 0; \quad \int_0^\infty e^{-ax^2} x^2 \, dx, \quad a > 0 \ \text{(use that} \int_{-\infty}^\infty e^{-x^2/2} \, dx = \sqrt{2\pi})$$

$$\int_a^b \cos(mx) \, dx, \quad m \in \mathbb{N}.$$

For the last one, fix a and b and investigate the limit $m \to \infty$. Does the result depend on a, b?

SET 2: Due Date 09/28/2017

From Chapter 1 (pp 37-42): 1, 2, 11, 28

Additional Problems

AI. Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has lenght $\delta 3^{-k}$, for some $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

AII. The following problem is a special case of Problem 4 in [SS, Ch1] dealing with what we call *Fat Cantor Sets.*

Construct a closed set $\mathcal{C}$ analogous to the Cantor $\frac{1}{3}$-set by removing instead at the stage k^{th} stage 2^{k-1} centrally situated open intervals each of length $\ell_k = \frac{1}{4^k}$. The set $\mathcal{C}$ is again defined as the (countably) infinity intersection of the closed sets C_k appearing at stage. k.

a) Show that $\mathcal{C}$ is compact, totally disconnected and has no isolated points (this is similar to problem 1).

b) Show that $m_*(\mathcal{C}) = \frac{1}{2}$ and conclude (with justification) that $\mathcal{C}$ is uncountable.

AIII. a) Let $A = \cup_{n=1}^\infty A_n$ with $m_*(A_n) = 0$. Use the definition of exterior measure to prove that $m_*(A) = 0$.

b) Use a) to prove that any countable set in $\mathbb{R}^d$ is measurable and has measure zero.

AIV. Let $\{E_n\}_{n\geq 1}$ be a countable collection of measurable sets in $\mathbb{R}^d$. Define

$$\limsup_{n\to\infty} E_n := \{x \in \mathbb{R}^d : x \in E_n, \text{ for infinitely many } n\}$$

$$\liminf_{n\to\infty} E_n := \{x \in \mathbb{R}^d : x \in E_n, \text{ for all but finitely many } n\}$$

a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \qquad \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j$$

b) Show that

$$m(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} m(E_n)$$

$$m(\limsup_{n\to\infty} E_n) \geq \liminf_{n\to\infty} m(E_n) \quad \text{provided that } m(\bigcup_{n=1}^{\infty} E_n) < \infty$$

From Chapter 1 (pp 37-42): 5, 6, 7, 10, 16, 17, 22

<u>Hint for 17</u>: Note that

$$\{x : |f_n(x)| = \infty\} = \bigcap_{j=1}^{\infty} \{x : |f_n(x)| > \frac{j}{n}\}.$$

Hence the hypothesis implies that for each n,

$$m(\bigcap_{j=1}^{\infty} \{x : |f_n(x)| > \frac{j}{n}\}) = 0.$$

But then $\lim_{j\to\infty} m(\{x : |f_n(x)| > \frac{j}{n}\}) = 0$ (Why? Justify.). Next, follow the hint in the book from here.

<u>Bonus Problem*</u>: First do 32 (pp 44-45). (Hint for part a) consider the sets $E_k = E + r_k \subset \mathcal{N}_k$, where $\{r_k\}_{k\geq 1}$ is an enumeration of the rationals.)

Note that part b) should read "...prove that *there exists* a subset of G which is....".

Furthermore, show:

c) $\mathcal{N}^c = I \setminus \mathcal{N}$ satisfies $m_*(\mathcal{N}^c) = 1$. (Hint: argue by contradiction and use a))

d) Conclude that

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^c) \neq m_*(\mathcal{N} \cup \mathcal{N}^c)$$

SET 3: Due Date 10/19/2017

From Chapter 1 (pp 41-44): 25, 29.

Additional Problems:

A.I) Prove that a set E in $\mathbb{R}^d$ is measurable if and only if for every set A in $\mathbb{R}^d$,

$$m_*(A) = m_*(A \cap E) + m_*(A - E) \tag{1}$$

Hint: First assume E is measurable and prove (1). Then to prove the converse, to prove that (1) implies that E is measurable, assume first that $m_*(E) < \infty$. Then do the case of $m_*(E) = \infty$. For the later write $E = \bigcup_{k=1}^{\infty}[E \cap B(0,k)]$ where $B(0,k)$ is the ball centered at the origin of radius k.

A.II) Do problem 13a) in Chapter 1 page 41.

Hints for 13a) First show that for each $n \in \mathbb{N}$, the set $\mathcal{O}_n := \{x : d(x,F) < \frac{1}{n}\}$ is open.
Then show that if $x \notin F$ then since F is closed, $d(x,F) > \delta$ for some $\delta > 0$.
Finally prove that if F is closed then $F = \bigcap_{n=1}^{\infty} \mathcal{O}_n$.
Conclude.

A.III) The Baire Category Theorem states: *A complete metric space cannot be written as a countable union of nowhere dense sets.*

Recall that a set A is said to be *nowhere dense* if the interior of the closure of A is empty (that is, $(\overline{A})^o = \emptyset$).

Now, consider the rational numbers $\mathbb{Q}$ in $\mathbb{R}$. Note $\mathbb{R}$ is a complete metric space. Use the Baire Category Theorem to prove that $\mathbb{Q}$ is **not** a G_δ set. (Hint. Proceed by contradiction).

A.IV Now use **A.III**) to prove 13b) and 13c) in Chapter 1 page 41.

A.V The following relates to the proof of Theorem 4.1 page 31. Prove that the sequence of nonnegative simple functions $\{\phi_k\}_k$ that approximate pointwise f is indeed increasing, ie. $\phi_k \leq \phi_{k+1}$.

From Chapter 2 (pp 89-97): 1

SET 4: Due Date 10/26/2017

From Chapter 2 (pp 89-93): 6, 8, 9, 10, 11.

Additional Problems:

I. Fill in all details to give a full proof of Lemma 1.2 (ii) (Chapter 2 pages 53-54).

II. If a function f is integrable then we proved in Proposition 1.12 (Chapter 2) that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any set A with $m(A) \leq \delta$, we have that $\int_A |f(x)|\, dm \leq \varepsilon$ (*absolute continuity of the Lebesgue integral*).

We say that a sequence of functions $\{f_n\}_{n\geq 1}$ is **equi-integrable** if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. for any set A with $m(A) \leq \delta$, we have that $\int_A |f_n(x)|\, dm \leq \varepsilon$ <u>for all</u> $n \geq 1$.

Now prove the following.

Let E be a set of finite measure, $m(E) < 1$, and let $\{f_n\} : E \to R$ be a sequence of functions which is equi-integrable. Show that if $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. x, then

$$\lim_{n\to\infty} \int_E |f_n(x) - f(x)|\, dm \;=\; 0.$$

<u>Hint</u>. Use Egorov's Theorem as in the bounded convergence theorem.

III. We say that a sequence of measurable functions $\{f_n\}_{n\geq 1}$ *converges in measure* to another measurable function f is for every $\varepsilon > 0$,

$$m(\{x \,:\, |f_n(x) \;-\; f(x)| > \varepsilon\}) \to 0, \qquad \text{as} \qquad n \to \infty$$

Prove that if a sequence of measurable function f_n converges in measure to another measurable function f then there exists a subsequence $\{f_{n_j}\}_{j\geq 1}$ which converges almost everywhere to f, that is $f_{n_j}(x) \to f(x)$ a.e. x as $j \to \infty$.

<u>Hint.</u> First show that for $\varepsilon = 2^{-j}$ one can choose n_j such that for all $n \geq n_j$

$$m(\{x \,:\, |f_n(x) \;-\; f(x)| > 2^{-j}\}) \;\geq\; 2^{-j}.$$

Next note that for each $j \geq 1$ one may choose $n_{j+1} \geq n_j$ (note this is needed to satisfy the definition of subsequence) and define $A_j := \{x \,:\, |f_{n_j}(x) \;-\; f(x)| > 2^{-j}\}$.

Use Borel-Cantelli to prove $m(\limsup_{j\to\infty} A_j) = 0$ and show this is equivalent to the desired conclusion.

IV. Suppose that $\{f_n\}_{n\geq 1}$ is a sequence of non-negative measurable function, that is $f_n \geq 0$ for all n, such that f_n converges in measure to f . Show that then

$$\int f(x)\; dm \;\leq\; \liminf_n \int f_n(x)\; dm$$

<u>Hint.</u> Let us call denote by $I = \int f dx,\ a_n = \int f_n dx,$ and $A = \liminf \int f_n dx$.

We wish to prove that $I \leq A$. Note that since A is the smallest of all limit points, there must exist a subsequence of a_{n_k} of a_n that converges to A ; i.e. $\lim_{k\to\infty} a_{n_k} = A$. In particular note that

$$A = \lim_{k\to\infty} \int f_{n_k} dx \qquad \text{and that } \ f_{n_k} \to f \text{ in measure as well}$$

Then for every subsequence $a_{n_{k_j}}$ of a_{n_k} we also have that

$$A = \lim_{j\to\infty} a_{n_{k_j}} = \lim_{j\to\infty} \int f_{n_{k_j}}\, dx$$

and $f_{n_{k_j}} \to f$ in measure as well . Next use the previous problem **III** to obtain one such (sub)subsequence for which we have a.e. convergence to f. Apply Fatou's Lemma and conclude.

V. Consider the sequence of functions $f_n(x) := \frac{n}{1+(nx)^2}$. For $a \in \mathbb{R}$ be a fixed number consider the Lebesgue integral $I_a(f_n)(x) := \int_a^\infty f_n(x)\, dm$. Compute $\lim_{n\to\infty} I_a(f_n)(x)$ in each case: i) $a = 0$ ii) $a > 0$ and iii) $a < 0$. Carefully justify your calculations (recall the transformation of integrals under dilations).

SET 5: Friday November 10th, 2017.

From Chapter 2 (pp 89-93): 2, 15, 16, 22, 23

Hint. For 2. suitably approximate f by a continuous function g with compact support.
From Chapter 2 (pp 95): 3.

Hint. Suitably use Tchebychev's inequality. For the converse consider $f_n(x) = n\chi_{[0,\frac{1}{n})}$.

I. Let f and f_n, $n \geq 1$ be measurable functions on $\mathbb{R}^d$

a) Suppose that $\mu(E) < \infty$ and that f and f_n, $n \geq 1$ are all supported on E. Prove that $f_n \to f$ a.e implies $f_n \to f$ in measure.

b) Prove that the converse of (a) is false even under the hypothesis of (a) (ie. all functions supported on E a set of finite measure)

Hint. Let $E = [0, 1]$ and consider the (double) sequence $f_{m,k}(x) = 1_{E_{m,k}}(x)$ $(m, k \in \mathbb{N})$, where $E_{m,k} := [\frac{m-1}{k}, \frac{m}{k}]$.)

II. In Chapter 2 we first prove the Bounded Convergence Theorem (using Egorov Theorem). Then, we proved Fatou's Lemma (using the BCT) and deduced from it the Monotone Convergence Theorem. Finally we proved the Dominated Convergence Theorem (using both BCT and MCT). Here we would like to prove these sequence of results in a different order. Namely, prove:

a) Prove Fatou's Lemma *from* the MCT by showing that for any sequence of measurable functions $\{f_n\}_{n\geq 1}$,

$$\int \liminf_{n\to\infty} f_n\, dm \;\leq\; \liminf_{n\to\infty} \int f_n\, dm.$$

Hint. Note that $\inf_{n\geq k} f_n \leq f_j$ for any $j \geq k$, whence $\int \inf_{n\geq k} f_n\, dm \;\leq\; \inf_{j\geq k} \int f_j$.

b) Now prove the DCT from Fatou's Lemma.

Hint. Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.

III. Use the DCT to prove the following: let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions on $\mathbb{R}^d$ such that $\sum_{n=1}^{\infty} \int |f_n(x)|\, dm < \infty$. Show that $\sum_{n=1}^{\infty} f_n(x)$ *converges* a.e. $x \in \mathbb{R}^d$ to an *integrable function* and that $\sum_{n=1}^{\infty} \int f_n(x)\, dm = \int \sum_{n=1}^{\infty} f_n(x) dm$.

SET 6: Due Date 11/30/17

Chapter 2 (pp 89-97) (Problems on Fubini-Tonelli): 4, 18, 19.

Additional Problems:

I. Consider the function on $\mathbb{R} \times \mathbb{R}$ given by

$$f(x,y) = ye^{-(x^2+1)y^2} \quad \text{if } x \geq 0, y \geq 0 \quad \text{and} \quad 0 \quad \text{otherwise .}$$

a) Integrate $f(x,y)$ over $\mathbb{R} \times \mathbb{R}$ (justify your steps carefully)
b) Use a) to prove that $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.
c) Use b) and dilation to prove that $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$.

II. Let $f(x)$ be a measurable function over $\mathbb{R}^{d_1}$ and $g(y)$ be a measurable function over $\mathbb{R}^{d_2}$. Prove that $F(x,y) = f(x)g(y)$ is a measurable function over $\mathbb{R}^{d_1+d_2}$.

From Chapter 2 (pp 88-97): Read Cor. 3.7- 3.8, then do 7. Recall Invariance (p. 73). Read Prop 3.9 then do 21a)b)c).

Additional Problems(cont.):

III. Let s be a fixed positive number. Prove that

$$\int_0^{\infty} e^{-sx} \frac{\sin^2 x}{x}\, dx = \frac{1}{4} \log(1 + 4s^{-2})$$

by integrating $e^{-sx} \sin(2xy)$ with respect to $x \in (0,\infty)$, $y \in (0,1)$ and with respect to $y \in (0,1)$, $x \in (0,\infty)$. Justify all your steps. (**Hints.** $\cos(2\theta) = 1 - 2\sin^2\theta$. In order to do one of the integrations, either integrate by parts twice or use the definition of the appropriate trigonometric function in terms of complex exponentials.)

IV. Consider the function $f(x,y) := e^{-xy} - 2e^{-2xy}$ where $x \in [0,\infty)$ and $y \in [0,1]$.
i) Prove that for a.e. $y \in [0,1]$ f^y is integrable on $[0,\infty)$ with respect to $m_{\mathbb{R}}$.
ii) Prove that for a.e. $x \in [0,\infty)$ f^x is integrable on $[0,1]$ with respect to $m_{\mathbb{R}}$.
iii) Use Fubini to prove that $f(x,y)$ is not integrable on $[0,\infty) \times [0,1]$ with respect to $m_{\mathbb{R}^2}$.

SET 7 - DUE 12/12/17

From Chapter 3 (pp 145-146 -Section 5): 4, 5, 7, 11, 12, 15, 16a).

From Chapter 3 (pp 152- Section 6) Bonus Problem: 1 (this problem is based on a good understanding of Lemma 1.2 p.102 and of Lemma 3.9 p.128-definition of Vitali covering is just before the statement of Lemma 3.9.)