On a Counterexample to Fubini

Let W be a well ordered set;

$$j : [0,1] \to W$$

and let Q be defined as the set of pairs $(x, y) \in [0,1] \times [0,1]$ such that $j(x)$ preceds $j(y)$ in W (here preceds means with respect to the total order in W).

We need that

(1) Q_x should contain all of $[0,1]$ except for a countable set in $[0,1]$.
(2) Q^y should contain at most a countable set in $[0,1]$.

Hence note in particular both sections would be Borel sets.

Then the counterexample follows by considering $f(x, y) = \chi_Q(x, y)$ and observing that the integrals over $[0,1]$ w.r.t. Lebegue measure of f_x and f^y (which would be Borel functions themselves) were different (first one $= 1$; second one $= 0$)

The reason why Fubini doesn't work is because f itself is not measurable w.r.t the product Borel sigma algebra

Proving 2) above maybe tricky if $j(x)$ is "too general". One needs to actually assume a few additional things:

a) Well Ordering Principle: every nonempty set X can be "well ordered". (this relies on an equivalent form of Zorn's lemma which states that every partially ordered set has a maximally linearly ordered set i.e. given X there exists a subset E of X for which the partial order in X becomes a total or linear order in E and E is maximal in the sense that no other subset of X that is also totally ordered with respect to the same partial order strictly contains E)

b) The Continuum Hypothesis : there is no set whose cardinality is strictly bigeer than that of the natural numbers and strictly smaller than that of the reals. (In other words if a set is uncountable then its cardinality is bigger or equal than that of the reals).

As a consequence of a) we have the existence of uncountable well order sets. We need then to assume our W above is such a set.

As a consequence of b) and the fact W has cardinality at least that of $[0,1]$ we can choose $j(x)$ to be one-to-one function. Now if j(x) is chosen as above in a one-to-one fashion then this guarantees that 1) and 2) above holds as desired.