**Your Name :**

**Final Take-Home Math 731**

Wednesday, December 17th, 2008

You may use your class notes, the various xerox notes I distributed as well as any textbook to get hints or ideas. However and ultimately the work you hand in **should be your own** and not identical copy of someone else's work. You may also ask me any questions you need to, either in person or by email. Please **justify** all your steps properly by indicating (or stating) the result you are using. Please write each problem clearly and neatly in a separate page. The exam is due Friday December 19th no later than 2pm at my office.

(**1**) Let $D = \{x \in \mathbb{R}^n \colon |x| > 1\}$ and $u : D \to \mathbb{R}$, $u \in C^2(D) \cap C^1(\overline{D})$, $\lim_{|x|\to\infty} u(x) = 0$ such that $\Delta u = 0$.

a) Let $D_n = \{x \in \mathbb{R}^n \, : \, 1 < |x| < n\}$. Show that

$$\max_{x\in\overline{D_n}} |u(x)| = \max_{x\in\partial D_n} |u(x)| =: M_n$$

and that $M_n$ is increasing.

(b) Show that $\max_{x\in\overline{D}} |u(x)| = \max_{x\in\partial D} |u(x)|$.

(**2**) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the Laplace equation with *Neumann* boundary condition:

$$\begin{cases} \Delta u &= 0 \qquad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= g \qquad \text{on} \quad \Omega, \end{cases}$$

where $g \in C(\partial\Omega)$

(a) Prove that $\int_{\partial\Omega} g(x)\, dS = 0$ is a necessary condition for the BVP problem above to have a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$

(b) Let $\mathcal{A} = C^2(\Omega) \cap C^1(\overline{\Omega})$. Prove the Dirichlet principle for the BCP above. That is show that $u \in \mathcal{A}$ solves the BVP if and only if it is a minimum for the energy

$$E[v] = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_{\partial U} gu\, dS$$

where $v \in \mathcal{A}$ (note there are no assumptions on the boundary values of $v$).

**(3)** Let $\varphi \in C^1(\mathbb{R})$ with compact support and consider the real-valued function $u$ on the upper half-plane $\mathbb{R}^2_+ = \{x = (x_1, x_2) \, : x_2 > 0\}$ defined by

$$u(x_1, x_2) := \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{\varphi(y)}{(x_1 - y)^2 + x_2^2} \, dy$$

**(a)** What PDE and type of problem does $u$ satisfy on the upper half-plane? (be precise and explain your answer)

**(b)** Prove that for each $x = (x_1, x_2) \in \mathbb{R}^2_+$,

$$1 = \int_{\mathbb{R}} K(x, y) \, dy$$

where $K(x, y) = \frac{x_2}{\pi} \frac{1}{|x-y|^2}$, $y \in \mathbb{R} = \partial\mathbb{R}^2_+$.

**(c)** Use **(b)** to rigorously prove that for each $x^0 \in \mathbb{R}$

$$\lim_{x \to x^0, \, x \in \mathbb{R}^2_+} u(x) \; = \; \varphi(x^0).$$

(Hint. Note that by hypothesis $\varphi$ is bounded and uniformly continuous.)

**(4)**) a) Solve

$$\begin{cases} u_t - u_{xx} = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = x^2 & x \in \mathbb{R} \end{cases}$$

by first showing that $u_{xxx}$ must vanish.

b) Solve the heat equation with *convection*; i.e.

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $b \in \mathbb{R}^n$ is constant. (Hint. Consider the change of variables $y = x - tb$ or equivalently the function $v(y, t) = u(y + tb, t)$ )

**(5)** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.
<u>Formulate</u> the definition of a weak solution $u \in H^1_0(\Omega)$ of the problem

$$\begin{cases} -\Delta u & = f & \text{in } \Omega \\ u & = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$ is a given function. And <u>prove</u> the existence of a weak solution.