Math 623
Fall 2016

Problem Set # 1

(1) Find $\limsup E_k$ and $\liminf E_k$ if the sequence $\{E_k\}$ is defined as follows,

$$E_k := \begin{cases} [-1/k, 1] & \text{for } k \text{ odd} \\ [-1, 1/k] & \text{for } k \text{ even} \end{cases}$$

(2) Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

(3) Give an example of two **closed** sets $F_1, F_2 \subset \mathbb{R}^2$ for which $\mathrm{dist}(F_1, F_2) = 0$.

(4) If $\delta = (\delta_1, \ldots, \delta_d)$ is a d-tuple of positive numbers $\delta_i > 0$, and E is a subset of $\mathbb{R}^d$, we define δE by

$$\delta E := \{(\delta_1 x_1, \ldots, \delta_d x_d) : \text{ where}(x_1, \ldots, x_d) \in E\}$$

Prove that E is measurable whenever δE is measurable, and

$$m(\delta E) = \delta_1 \ldots \delta_d m(E)$$

(5) Show that any *countable* set $E \subset \mathbb{R}$ must be a *set of measure zero.*
Hint: Think first of the case where E is actually finite. For the general case, it might be helpful to think about infinite converging series, e.g. the geometric series.

(6) Given an interval $[a, b] \subset \mathbb{R}$, construct a sequence of continuous functions $\phi_k(x)$ such that for every fixed $x \in \mathbb{R}$ we have

$$\lim_{k \to \infty} \phi_k(x) = \begin{cases} 1 \text{ if } x \in [a, b] \\ 0 \text{ if } x \notin [a, b] \end{cases}$$

Further, answer the following question: can one construct such a sequence ϕ_k so that it also converges uniformly as $k \to \infty$?.

(7) Prove that the Cantor ser constructed in Chapter 1 of SS is **totally disconnected** and **perfect**. In other words, given two distinct points $x, y \in \mathcal{C}$ there is a point $z \notin C$ that lies between x and y, and yet $\mathcal{C}$ has no isolated points.

(8) A continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda \phi(x) + (1 - \lambda)\phi(y), \ \ \forall \, x, y \in \mathbb{R}, \forall \, \lambda \in [0, 1]$$

Show that if ϕ is convex, then if $x_1, \ldots, x_n$ are points in $\mathbb{R}$ then

$$\phi\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \frac{\phi(x_1) + \ldots \phi(x_n)}{n}$$

More generally, show that if $\alpha_1, \ldots, \alpha_n$ is a sequence of nonnegative numbers with

$$\sum_{i=1}^{n} \alpha_i = 1$$

Then, for any n points $x_1, \ldots, x_n$ in $\mathbb{R}$ we have

$$\phi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \phi(x_i)$$

This last inequality is known as *Jensen's* inequality.

(9) Let $x_1, \ldots, x_n$ be all nonnegative numbers. Prove the *arithmetic-geometric* mean inequality

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Hint: Apply Jensen's inequality with a conveniently chosen convex function.

(10) Compute the following Riemann integrals:

$$\int_0^1 x^k \, dx, \quad k > 0; \quad \int_0^1 x^{-k} \, dx, \quad k \in (0,1); \int_1^\infty x^{-k} \, dx, \quad k \in (1, \infty);$$

$$\int_0^\infty e^{-ax^2} x \, dx \quad a > 0; \qquad \int_0^\infty e^{-ax^2} x^2 \, dx \quad a > 0;$$

$$\left(\text{ You may use that } \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \right)$$

$$\int_a^b \cos(mx) \, dx, \quad m \in \mathbb{N}.$$

For the last one, fix a and b and investigate the limit $m \to \infty$. Does the result depend on a, b?

(11) Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has lenght $\delta 3^{-k}$, for some $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

(12) * Assume that ϕ is twice differentiable, show that ϕ is convex if and only if $\phi''(x) \geq 0$ for all x.

(13) * Show that the union of a countable family of sets of measure zero is also a set of measure zero.

(14) * Fix numbers $a < b$ in $\mathbb{R}$ and consider $f : [a,b] \to \mathbb{R}$ a positive, **Riemann integrable** function. Then, define the set

$$E := \{(x,y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$$

Show that E is a measurable set. *Hint: Think in terms of Darboux sums.*

(15) * Theorem 1.3 in SS states that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$ is generally false. Prove the following

(a) An open disc in $\mathbb{R}^2$ is **not** the disjoint union of open rectangles.

(b) An open connected set $\Omega \subset \mathbb{R}^2$ is the disjoint union of open rectangles if and only if Ω itself is an open rectangle.