Math 623
Fall 2016

Problem Set # 2

(1) Suppose that $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

(2) An alternative definition of measurability for a set E is: "E is measurable if for any $\varepsilon > 0$ there is a **closed** set $F \subset E$ with $m_*(E - F) < \varepsilon$". Prove that this definition of measurability is equivalent to the one in the text.

(3) Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n := \{x : d(x, E) < 1/n\}$$

Provide a proof for the following two assertions:
(a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

(b) The conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

(4) Given a collection of sets $F_1, F_2, \ldots, F_n$, construct another collection of sets $F_1^*, F_2^*, \ldots, F_N^*$ wit $N = 2^n - 1$, so that

$$\bigcup_{k=1}^{n} F_k = \bigcup_{j=1}^{N} F_j^*$$

so that the collection $\{F_j^*\}_j$ is made out of pairwise disjoint sets and such that for any k we have $F_k = \bigcup_{F_j^* \subset F_k} F_j^*$ for every k.

(5) **(The Borel-Cantelli Lemma).** Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Show that $E := \limsup E_k$ is a measurable set and that $m(E) = 0$.

Hint: Note that $\limsup E_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

(6) Let $E \subset \mathbb{R}^2$ be a measurable set with $m(E) < \infty$. For every $t \in \mathbb{R}$, define

$$E_t := E \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le t\}$$

Then
1) Show that the function $f(t) := |E_t|$ is a continuous for every $t \in \mathbb{R}$
2) Justify the limits $\lim_{t\to-\infty} f(t) = 0$, $\lim_{t\to\infty} f(t) = m(E)$
3) Show that for every number $c \in (0, m(E))$ there exists t_c such that $m(E_{t_c}) = c$.

(7) Suppose $E_i \subset \mathbb{R}^2$ $(i = 1, 2)$ are a pair of nonempty compact sets with $E_1 \subset E_2$ and $0 < m(E_1) < m(E_2)$. Prove: $\forall c$ such that $m(E_1) < c < m(E_2)$ $\exists$ E such that $E_1 \subset E \subset E_2$ and $m(E) = c$.

(8) Show any open set $E \subset \mathbb{R}^d$ can be written as the union of closed cubes, so that $E = \bigcup Q_i$ with the following properties

  (a) The Q_i are non-overlaping, i.e. their interiors are disjoint.

  (b) There are positive constants $0 < c < C$ so that

$$cm(Q_i)^{1/d} \leq d(Q_i, E^c) \leq Cm(Q_i)^{1/d}$$

(Note that for a cube Q, $m(Q)^{1/d}$ is the same as lenght of any of its edges).

Hint: Review the proof in Stein Shakarchi for the fact that any open set is a union of almost disjoint closed cubes.

(9) Let $E \subset \mathbb{R}$ be a measurable set with $0 < m(E) < \infty$. Show that for every $\alpha \in (0, 1)$ there is an open interval I such that

$$m(E \bigcap I) \geq \alpha m(I).$$

(10) * Let $f : \mathbb{R} \to \mathbb{R}$ be a function having a continuous derivative, show that its graph, that is the set $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$ is a subset of the plane with measure zero.

(11) * Show that a σ-algebra with infinitely many sets cannot be countable. *Hint: Show first that if the σ-algebra is infinite then it contains a countable sequence of pairwise disjoint sets. Then recall how one can show* $[0, 1]$ *is uncountable by using a binary represention.*

(12) * Suppose that E is measurable with $m(E) < \infty$ and

$$E = E_1 \bigcup E_2, \quad E_1 \bigcap E_2 = \emptyset$$

Suppose that $m(E) = m_*(E_1) + m_*(E_2)$, then show E_1, E_2 are both measurable.

Note: In particular, this would show that if $E \subset Q$, where Q is a finite cube, then E is measurable if and only if $m(Q) = m_*(E) + m_*(Q \setminus E)$.

(13) * Construct a measurable subset $E \subset [0, 1]$ such that for every subinterval I, both $E \cap I$ and $I \setminus E$ have positive measure. *Hint: Take a Cantor-type subset of* $[0, 1]$ *with positive measure (see previous problem), and on each subinterval of the complement of this set, construct another such set, and so on.*