Recursion and Algorithm Development

- Introduction to Recursion
- Recursion Examples
- Run Time Analysis
- Search

Recursion

A recursive function is one that calls itself

What happens when we run `countdown(3)`?

```
>>> countdown(3)
3
2
1
0
-1
-2
...
-976
-977
-978
Traceback (most recent call last):
  File "<pyshell#61>", line 1, in <module>
    countdown(3)
  File "/Users/me/ch10.py", line 3, in countdown
    countdown(n-1)
...
 File "/Users/me/ch10.py", line 3, in countdown
    countdown(n-1)
  File "/Users/me/ch10.py", line 2, in countdown
    print(n)
RuntimeError: maximum recursion depth exceeded
while calling a Python object
>>>
```

```
def countdown(n):
        print(n)
        countdown(n-1)
```

ch10.py

The function calls itself repeatedly until the system resources are exhausted

- i.e., the limit on the size of the program stack is exceeded

In order for the function to terminate normally, there must be a stopping condition

Recursion

Suppose that we really want this behavior

```
>>> countdown(3)
3
2
1
Blastoff!!!
>>> countdown(1)
1
Blastoff!!!
>>> countdown(0)
Blastoff!!!
>>> countdown(-1)
Blastoff!!!
```

Recursion

```
def countdown(n):
    'counts down to 0'
    if n <= 0:
        print('Blastoff!!!')
    else:
        print(n)
        countdown(n-1)
```

ch10.py

Suppose that we really want this behavior

```
>>> countdown(3)
3
2
1
Blastoff!!!
>>> countdown(1)
1
Blastoff!!!
>>> countdown(0)
Blastoff!!!
>>> countdown(-1)
Blastoff!!!
```

If n ≤ 0
  `'Blastoff!!!'` is printed

In order for the function to terminate normally, there must be a stopping condition

Recursion

A recursive function should consist of

1. One or more base cases which provide the stopping condition of the recursion
2. One or more recursive calls on input arguments that are "closer" to the base case

Base case

Recursive step

```
def countdown(n):
    'counts down to 0'
    if n <= 0:
        print('Blastoff!!!')
    else:
        print(n)
        countdown(n-1)
```

ch10.py

This will ensure that the recursive calls eventually get to the base case that will stop the execution

Recursive thinking

```
def countdown(n):
    'counts down to 0'
    if n <= 0:
        print('Blastoff!!!')
    else:
        print(n)
        countdown(n-1)
```

ch10.py

A recursive function should consist of

1. One or more base cases which provide the stopping condition of the recursion
2. One or more recursive calls on input arguments that are “closer” to the base case

Problem with input n

To count down from n to 0 …

… we print n and then count down from n-1 to 0

Subproblem with input n-1

So, to develop a recursive solution to a problem, we need to:

1. Define one or more bases cases for which the problem is solved directly
2. Express the solution of the problem in terms of solutions to subproblems of the problem (i.e., easier instances of the problem that are closer to the bases cases)

Recursive thinking

We use recursive thinking to develop function `vertical()` that takes a non-negative integer and prints its digits vertically

First define the base case

- The case when the problem is “easy”
- When input `n` is a single-digit number

```
>>> vertical(3124)
3
1
2
4
```

To print the digits of `n` vertically …

Original problem with input `n`

… print all but the last digit of `n` and then print the last digit

Subproblem with input having one less digit than `n`

Next, we construct the recursive step

- When input `n` has two or more digits

```
def vertical(n):
    'prints digits of n vertically'
    if n < 10:
        print(n)
    else:
        vertical(n//10)
        print(n%10)
```

ch10.py

The last digit of `n`: `n%10`

So, to develop a recursive solution to a problem, we need to:

1. Define one or more bases cases for which the problem is solved directly
2. Express the solution of the problem in terms of solutions to subproblems of the problem (i.e., easier instances of the problem that are closer to the bases cases)

Exercise

Implement recursive method `reverse()` that takes a nonnegative integer as input and prints its digits vertically, starting with the low-order digit.

```
>>> vertical(3124)
4
2
1
3
```

Exercise

Implement recursive method `reverse()` that takes a nonnegative integer as input and prints its digits vertically, starting with the low-order digit.

```
>>> vertical(3124)
4
2
1
3
```

```
def reverse(n):
    'prints digits of n vertically starting with low-order digit'
    if n <10:               # base case: one digit number
        print(n)
    else:                   # n has at least 2 digits
       print(n%10)          # prints last digit of n
       reverse(n//10)       # recursively print in reverse all but the last digit
```

Exercise

Use recursive thinking to implement recursive function `cheers()` that, on integer input `n`, outputs `n` strings `'Hip '` followed by `'Hurray!!!'`.

```
>>> cheers(0)
Hurray!!!
>>> cheers(1)
Hip Hurray!!!
>>> cheers(4)
Hip Hip Hip Hip Hurray!!!
```

Exercise

Use recursive thinking to implement recursive function `cheers()` that, on integer input `n`, outputs `n` strings `'Hip '` followed by `'Hurray!!!'`.

```
>>> cheers(0)
Hurray!!!
>>> cheers(1)
Hip Hurray!!!
>>> cheers(4)
Hip Hip Hip Hip Hurray!!!
```

```
def cheers(n):
    if n == 0:
        print('Hurray!!!')
    else: # n > 0
        print('Hip', end=' ')
        cheers(n-1)
```

Exercise

The factorial function has a natural recursive definition:

$$n! = n \times (n-1)! \quad \text{if} \quad n > 0$$
$$0! = 1$$

Implement function `factorial()` using recursion.

Exercise

The factorial function has a natural recursive definition:

$$n! = n \times (n-1)! \quad \text{if} \quad n > 0$$
$$0! = 1$$

Implement function `factorial()` using recursion.

```
def factorial(n):
    'returns the factorial of integer n'
    if n == 0:                      # base case
        return 1
    return factorial(n-1)*n  # recursive step when n > 1
```

Program stack

The execution of recursive function calls is supported by the program stack

- just like regular function calls

line = 7
n = 31
line = 7
n = 312
line = 7
n = 3124

Program stack

```
1. def vertical(n):
2.     'prints digits of n vertically'
3.     if n < 10:
4.         print(n)
5.     else:
6.         vertical(n//10)
7.         print(n%10)
```

```
>>> vertical(3124)
```

```
n = 3124
vertical(n//10)

print(n%10)
```

vertical(3124)

```
n = 312
vertical(n//10)

print(n%10)
```

vertical(312)

```
n = 31
vertical(n//10)

print(n%10)
```

vertical(31)

```
n = 3
print(n)
```

vertical(3)

A recursive number sequence pattern

So far, the problems we have considered could have easily been solved without recursion, i.e., using iteration

We consider next several problems that are best solved recursively.

Consider function `pattern()` that takes a nonnegative integer as input and prints a corresponding number pattern

```
>>> pattern(0)
0
>>> pattern(1)
0 1 0
>>> pattern(2)
0 1 0 2 0 1 0
>>> pattern(3)
0 1 0 2 0 1 0 3 0 1 0 2 0 1 0
>>>
```

Base case: `n == 0`

Recursive step:

To implement `pattern(n)` …

… we need to execute `pattern(n-1)`,
then print `n`,
and then execute `pattern(n-1)`

```
def pattern(n):
    'prints the n-th pattern'
    if n == 0:               # base case
        print(0, end=' ')
    else:                    # recursive step: n > 0

        # to do
```

A recursive graphical pattern

```
>>> koch(0)
'F'
>>> koch(1)
'FLFRFLF'
>>> koch(2)
'FLFRFLFLFLFRFLFRFLFRFLFLFLFRFLF'
>>> koch(3)
'FLFRFLFLFLFRFLFRFLFRFLFLFLFRFLFL
FLFRFLFLFLFRFLFRFLFRFLFLFLFRFLFRF
LFRFLFLFLFRFLFRFLFRFLFLFLFRFLFLFL
FRFLFLFLFRFLFRFLFRFLFLFLFRFLF'
```

We want to develop function `koch()` that takes a nonnegative integer as input and returns a string containing pen drawing instructions

- instructions can then be used by a pen drawing app such as turtle

Base case: `n == 0`

Recursive step: `n > 0`

Run `koch(n-1)` and use obtained instructions to construct instructions for `koch(n-1)`

```
def koch(n):
    'returns directions for drawing curve Koch(n)'
    if n==0:
        return 'F'
    # recursive step: get directions for Koch(n-1)
    # use them to construct directions for Koch(n)
    return koch(n-1)+'L'+koch(n-1)+'R'+koch(n-1)+'L'+koch(n-1)
```

inefficient!

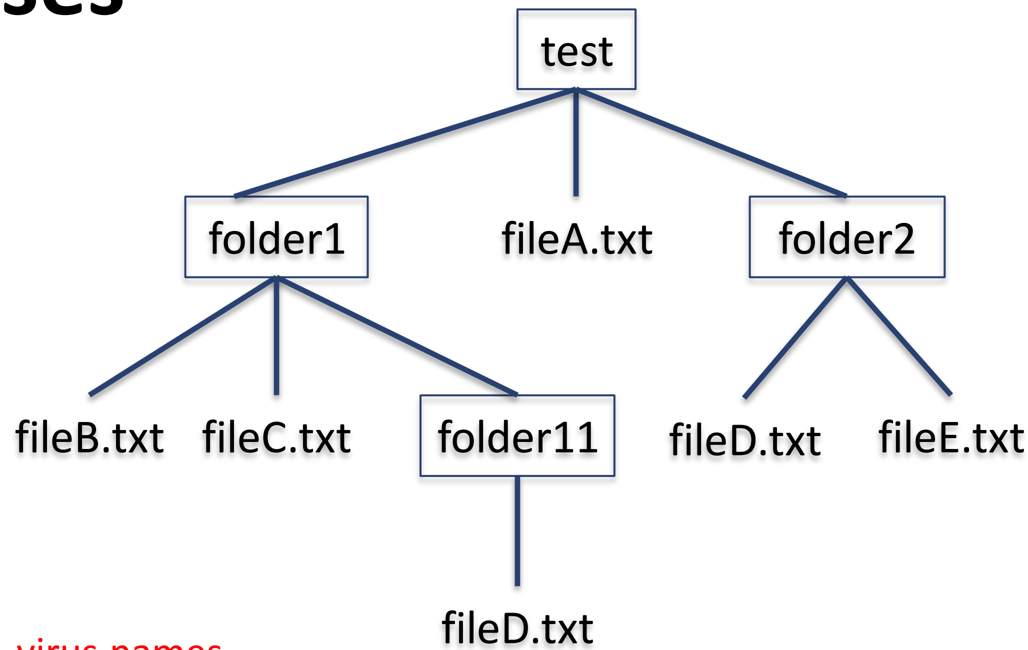
A recursive graphical pattern

```
from turtle import Screen, Turtle
def drawKoch(n):
    '''draws nth Koch curve using instructions from function koch()'''

    s = Screen()                # create screen
    t = Turtle()                # create turtle
    directions = koch(n)        # obtain directions to draw Koch(n)

    for move in directions:     # follow the specified moves
        if move == 'F':
            t.forward(300/3**n)     # forward move length, normalized
        if move == 'L':
            t.lt(60)                # rotate left 60 degrees
        if move == 'R':
            t.rt(120)               # rotate right 60 degrees
    s.bye()

def koch(n):
    'returns directions for drawing curve Koch(n)'
    if n==0:
        return 'F'
    # recursive step: get directions for Koch(n-1)
    tmp = koch(n-1)
    # use them to construct directions for Koch(n)
    return tmp+'L'+tmp+'R'+tmp+'L'+tmp
```

Scanning for viruses

Recursion can be used to scan files for viruses

A virus scanner systematically looks at every file in the filesystem and prints the names of the files that contain a known computer virus signature



virus names

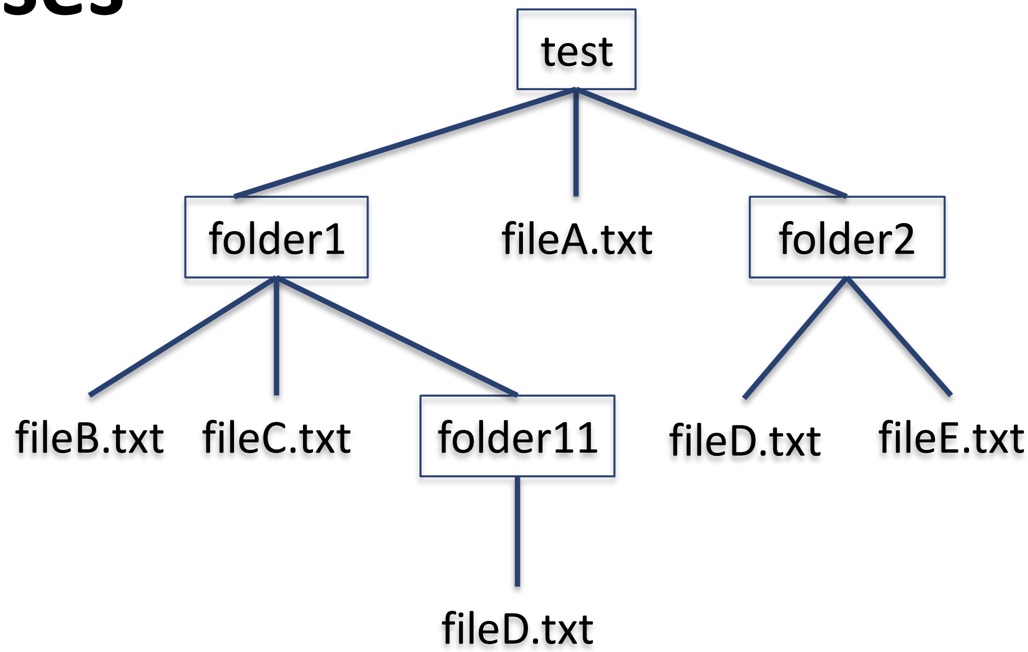
virus signatures

```
>>> signatures = {'Creeper':'ye8009g2h1azzx33',
'Code Red':'99dh1cz963bsscs3',
'Blaster':'fdp1102k1ks6hgbc'}
>>> scan('test', signatures)
test/fileA.txt, found virus Creeper
test/folder1/fileB.txt, found virus Creeper
test/folder1/fileC.txt, found virus Code Red
test/folder1/folder11/fileD.txt, found virus Code Red
test/folder2/fileD.txt, found virus Blaster
test/folder2/fileE.txt, found virus Blaster
```

pathname

dictionary mapping virus names to their signatures

Scanning for viruses



Base case: `pathname` refers to a regular file

What to do? Open the file and check whether it contains any virus signature

Recursive step: `pathname` refers to a folder

What do do? Call `scan()` recursively on every item in the folder

```
def scan(pathname, signatures):
    '''recursively scans all files
       contained, directly or indirectly,
       in the folder pathname'''

...
```

Scanning for viruses

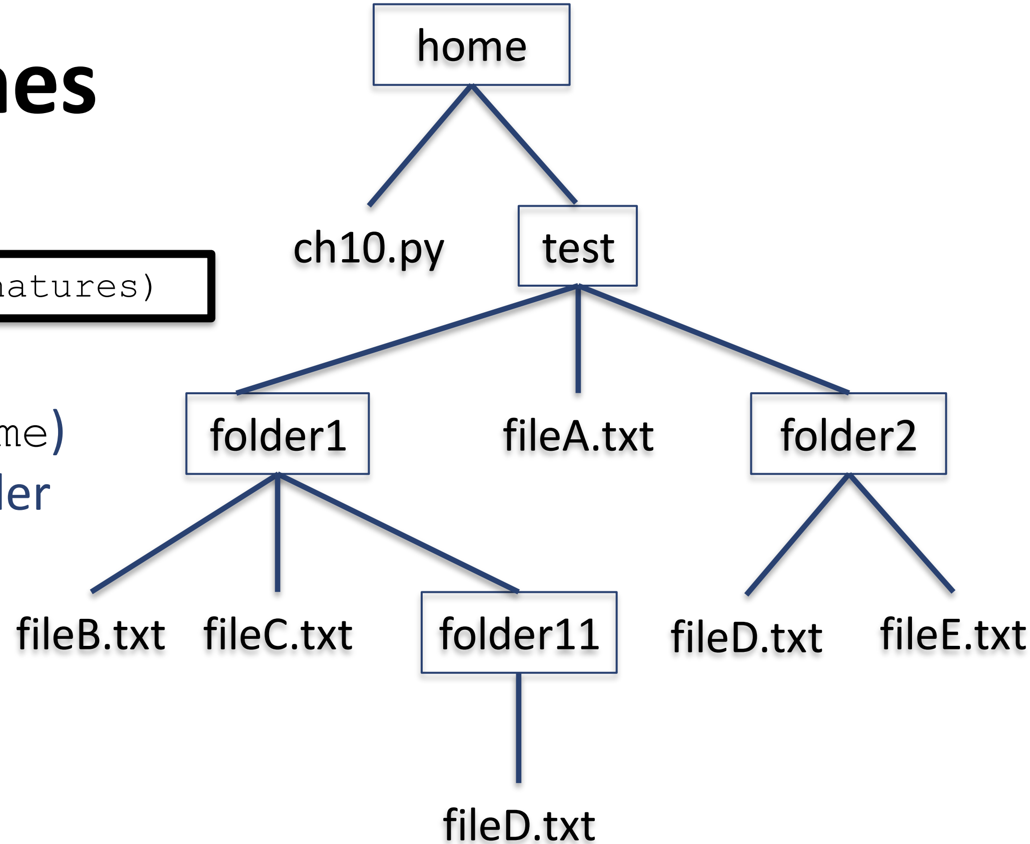
Function `isfile()` from Standard Library module `os` checks whether `pathname` is a regular file

Function `listdir()` from Standard Library module `os` returns the list of items in folder `pathname`

```
import os
def scan(pathname, signatures):

    if os.path.isfile(pathname): # base case, scan pathname
        infile = open(pathname)
        content = infile.read()
        infile.close()
        for virus in signatures:
            # check whether virus signature appears in content
            if content.find(signatures[virus]) >= 0:
                print('{}, found virus {}'.format(pathname, virus))
        return

    # pathname is a folder so recursively scan every item in it
    for item in os.listdir(pathname):
        # create pathname for item relative to current working directory
        # fullpath = pathname + '/' + item        # Mac only
        # fullpath = pathname + '\' + item        # Windows only
        fullpath = os.path.join(pathname, item) # any OS

        scan(fullpath, signatures)
```

ch10.py

Relative pathnames



When we run

```
>>> scan('test', signatures)
```

the assumption is that the current working directory is a folder (say, `home`) that contains both `ch10.py` and folder `test`

When `pathname` is `'test'` in

```
for item in os.listdir(pathname):
```

the value of `item` will (successively) be `folder1`, `fileA.txt`, and `folder2`

Why can't we make recursive call

```
scan(item, signatures)
```

?

Because `folder1`, `fifleA.txt`, and `folder2` are not in the current working directory (`home`)

The recursive calls should be made on `pathname/item` (on UNIX-like machines)

Scanning for viruses

Function `join()` from Standard Library module `os.path` joins a pathname with a relative pathname

```
import os
def scan(pathname, signatures):

    if os.path.isfile(pathname): # base case, scan pathname
        infile = open(pathname)
        content = infile.read()
        infile.close()
        for virus in signatures:
            # check whether virus signature appears in content
            if content.find(signatures[virus]) >= 0:
                print('{}, found virus {}'.format(pathname, virus))
        return

    # pathname is a folder so recursively scan every item in it
    for item in os.listdir(pathname):
        # create pathname for item relative to current working directory
        # fullpath = pathname + '/' + item      # Mac only
        # fullpath = pathname + '\' + item      # Windows only
        fullpath = os.path.join(pathname, item) # any OS

        scan(fullpath, signatures)
```

ch10.py

Fibonacci sequence

Recall the Fibonacci number sequence

1	1	2	3	5	8	13	21	34	55	...
+	+	+	+	+	+	+	+			

```
>>> rfib(0)
1
>>> rfib(1)
1
>>> rfib(2)
2
>>> rfib(5)
8
```

There is a natural recursive definition for the n-th Fibonacci number:

$$Fib(n) = \begin{cases} 1 & n = 0,1 \\ Fib(n-1) + Fib(n-2) & n > 1 \end{cases}$$

Use recursion to implement function `rfib()` that returns the n-th Fibonacci number

```
def rfib(n):
    'returns n-th Fibonacci number'
    if n < 2:                # base case
        return 1
    # recursive step
    return rfib(n-1) + rfib(n-2)
```

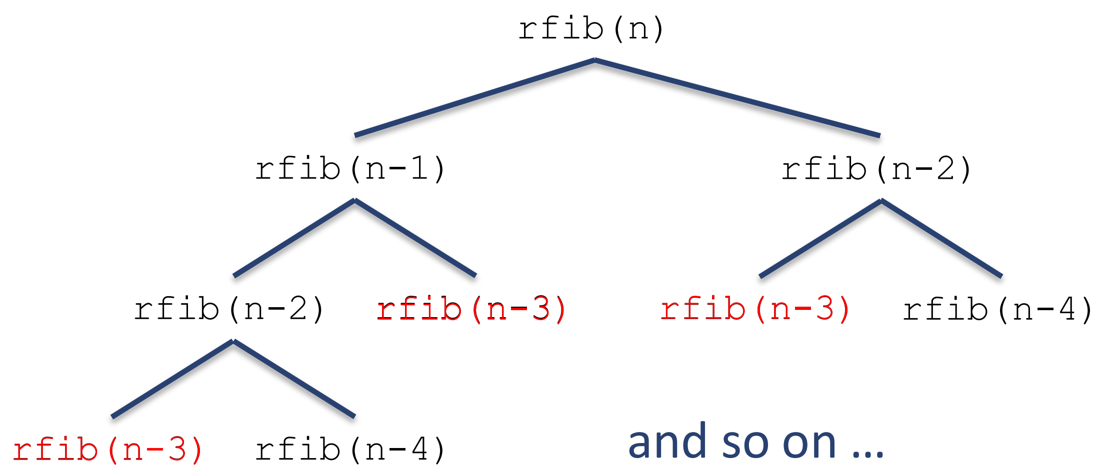
Let's test it

```
>>> rfib(20)
10946
>>> rfib(50)
(minutes elapse)
```

Why is it taking so long???

Fibonacci sequence

Let's illustrate the recursive calls made during the execution of `rfib(n)`




```
def rfib(n):
    'returns n-th Fibonacci number'
    if n < 2:            # base case
        return 1
    # recursive step
    return rfib(n-1) + rfib(n-2)
```

The same recursive calls are being made again and again

- A huge waste!

Fibonacci sequence

Compare the performance iterative `fib()` with recursive `rfib(n)`

Recursion is not always the right approach

instantaneous

```
def fib(n):
    'returns n-th Fibonacci number'
    previous = 1
    current = 1
    i = 1     # index of current Fibonacci number

    # while current is not n-th Fibonacci number
    while i < n:
        previous, current = current, previous+current
        i += 1
    return current
```

```
>>> fib(50)
20365011074
>>> fib(500)
2255915161619363308725126950360720720460113249137581905886388664184746277386868834050159870527969684986 26
>>>
```

```
def rfib(n):
    'returns n-th Fibonacci number'
    if n < 2:           # base case
        return 1
    # recursive step
    return rfib(n-1) + rfib(n-2)
```

```
>>> rfib(20)
10946
>>> rfib(50)
```

(minutes elapse)

Algorithm analysis

There are usually several approaches (i.e., algorithms) to solve a problem. Which one is the right one? the best?

Typically, the one that is fastest (for all or at least most real world inputs)

How can we tell which algorithm is the fastest?

- theoretical analysis
- experimental analysis

```
import time
def timing(func, n):
    'runs func on input n'

    start = time.time()          # take start time
    func(n)                      # run func on n
    end = time.time()            # take end time

    return end - start           # return execution time
```

```
>>> timing(fib, 30)
1.1920928955078125e-05
>>> timing(rfib, 30)
0.7442440986633301
```

0.00000119… seconds

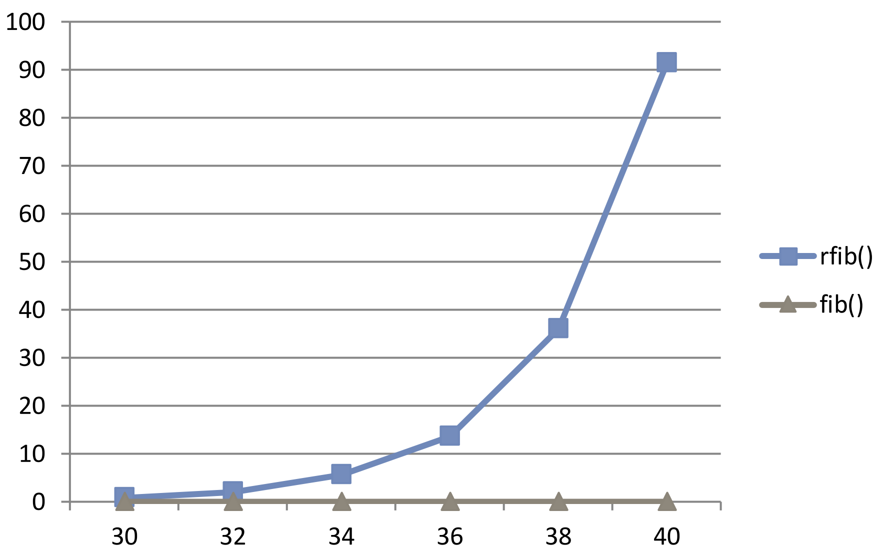
0.744… seconds

using a range of input values

- e.g., 30, 32, 34, …, 40

Algorithm analysis

```
def timingAnalysis(func, start, stop, inc, runs):
    '''prints average run-times of function func on inputs of
       size start, start+inc, start+2*inc, ..., up to stop'''
    for n in range(start, stop, inc):  # for every input size n
        acc=0.0                        # initialize accumulator
        for i in range(runs):       # repeat runs times:
            acc += timing(func, n)     # run func on input size n
                                       # and accumulate run-times
        # print average run times for input size n
        formatStr = 'Run-time of {}({}) is {:.7f} seconds.'
        print(formatStr.format(func.__name__,n,acc/runs))
```




```
>>> timingAnalysis(rfib, 30, 41, 2, 5)
Run-time of rfib(30) is 0.7410099 seconds.
Run-time of rfib(32) is 1.9761698 seconds.
Run-time of rfib(34) is 5.6219893 seconds.
Run-time of rfib(36) is 13.5359141 seconds.
Run-time of rfib(38) is 35.9763714 seconds.
Run-time of rfib(40) is 91.5498876 seconds.
>>> timingAnalysis(fib, 30, 41, 2, 5)
Run-time of fib(30) is 0.0000062 seconds.
Run-time of fib(32) is 0.0000072 seconds.
Run-time of fib(34) is 0.0000074 seconds.
Run-time of fib(36) is 0.0000074 seconds.
Run-time of fib(38) is 0.0000082 seconds.
Run-time of fib(40) is 0.0000084 seconds.
```

Searching a list

Consider list method `index()` and list operator `in`

```
>>> lst = random.sample(range(1,100), 17)
>>> lst
[9, 55, 96, 90, 3, 85, 97, 4, 69, 95, 39, 75, 18, 2, 40, 71, 77]
>>> 45 in lst
False
>>> 75 in lst
True
>>> lst.index(75)
11
>>>
```

How do they work? How fast are they? Why should we care?

- the list elements are visited from left to right and compared to the target; this search algorithm is called sequential search
- In general, the running time of sequential search is a linear function of the list size
- If the list is huge, the running time of sequential search may take time; there are faster algorithms if the list is sorted

Searching a sorted list

How can search be done faster if the list is sorted?

Suppose we search for 75

3 comparisons instead of 12

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	9	18	39	40	55	69	71	75	77	85	90	95	96	97

Suppose we search for 45

5 comparisons instead of 17

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	9	18	39	40	55	69	71	75	77	85	90	95	96	97

Algorithm idea: Compare the target with the middle element of a list

- Either we get a hit
- Or the search is reduced to a sublist that is less than half the size of the list

Let's use recursion to describe this algorithm

Searching a list

Suppose we search for target 75

								mid		mid		mid				
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	9	18	39	40	55	69	71	75	77	85	90	95	96	97

The recursive calls will be on sublists of the original list

```
def search(lst, target, i, j):
    '''attempts to find target in sorted sublist lst[i:j];
       index of target is returned if found, -1 otherwise'''
```

Algorithm:

1. Let `mid` be the middle index of list `lst`
2. Compare `target` with `lst[mid]`
   - If `target > lst[mid]` continue search of `target` in sublist `lst[mid+1:]`
   - If `target < lst[mid]` continue search of `target` in sublist `lst[?:?]`
   - If `target == lst[mid]` return `mid`

Binary search

```
def search(lst, target, i, j):
    '''attempts to find target in sorted sublist lst[i:j];
       index of target is returned if found, -1 otherwise'''
    if i == j:                           # base case: empty list
        return -1                        # target cannot be in list

    mid = (i+j)//2                       # index of median of l[i:j]

    if lst[mid] == target:               # target is the median
        return mid
    if target < lst[mid]:                # search left of median
        return search(lst, target, i, mid)
    else:                                # search right of median
        return search(lst, target, mid+1, j)
```

Algorithm:

1. Let `mid` be the middle index of list `lst`
2. Compare `target` with `lst[mid]`
   - If `target > lst[mid]` continue search of `target` in sublist `lst[mid+1:]`
   - If `target < lst[mid]` continue search of `target` in sublist `lst[?:?]`
   - If `target == lst[mid]` return `mid`

Comparing sequential and binary search

Let’s compare the running times of both algorithms on a random array

```
def binary(lst):
    'chooses item in list lst at random and runs search() on it'
    target = random.choice(lst)
    return search(lst, target, 0, len(lst))



def linear(lst):
    'choose item in list lst at random and runs index() on it'
    target = random.choice(lst)
    return lst.index(target)
```

But we need to abstract our experiment framework first

Comparing sequential and binary search

```
import time
def timing(func, n):
    'runs func on input returned by buildInput'
    funcInput = buildInput(n)  # obtain input for func

    start = time.time()        # take start time
    func(funcInput)            # run func on funcInput
    end = time.time()          # take end time

    return end - start         # return execution time


# buildInput for comparing Linear and Binary search
def buildInput(n):
    'returns a random sample of n numbers in range [0, 2n)'
    lst = random.sample(range(2*n), n)
    lst.sort()
    return lst
```

function `timing()` used for the factorial problem

generalized function `timing()` for arbitrary problems

But we need to abstract our experiment framework first

Comparing sequential and binary search

```
>>> timingAnalysis(linear, 200000, 1000000, 200000, 20)
Run time of linear(200000) is 0.0046095
Run time of linear(400000) is 0.0091411
Run time of linear(600000) is 0.0145864
Run time of linear(800000) is 0.0184283
>>> timingAnalysis(binary, 200000, 1000000, 200000, 20)
Run time of binary(200000) is 0.0000681
Run time of binary(400000) is 0.0000762
Run time of binary(600000) is 0.0000943
Run time of binary(800000) is 0.0000933
```

Exercise

Consider 3 functions that return `True` if every item in the input list is unique and `False` otherwise

Compare the running times of the 3 functions on 10 lists of size 2000, 4000, 6000, and 8000 obtained from the below function `buildInput()`

Exercise

Consider 3 functions that return `True` if every item in the input list is unique and `False` otherwise

Compare the running times of the 3 functions on 10 lists of size 2000, 4000, 6000, and 8000 obtained from the below function `buildInput()`

```
def dup1(lst):
    for item in lst:
        if lst.count(item) > 1:
            return True
    return False

def dup2(lst):
    lst.sort()
    for index in range(1, len(lst)):
        if lst[index] == lst[index-1]:
            return True
    return False

def dup3(lst):
    s = set()
    for item in lst:
        if item in s:
            return False
        else:
            s.add(item)
    return True
```

```
import random
def buildInput(n):
    'returns a list of n random integers in range [0, n**2)'
    res = [] for i in range(n):
        res.append(random.choice(range(n**2)))
    return res
```