Wheeled Locomotion | Differential Kinematics

Autonomous Mobile Robots

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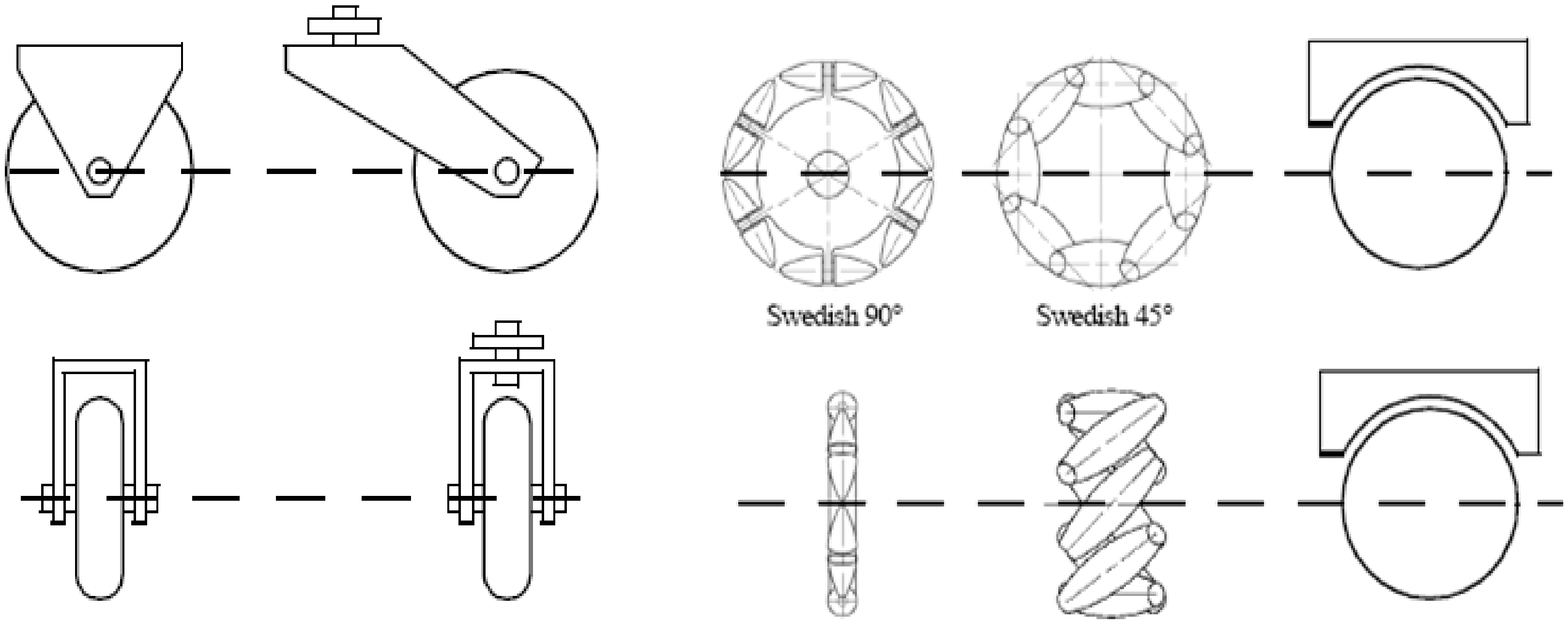
Review: Differential Kinematics

- What is the relationship between wheel speeds and platform velocity?

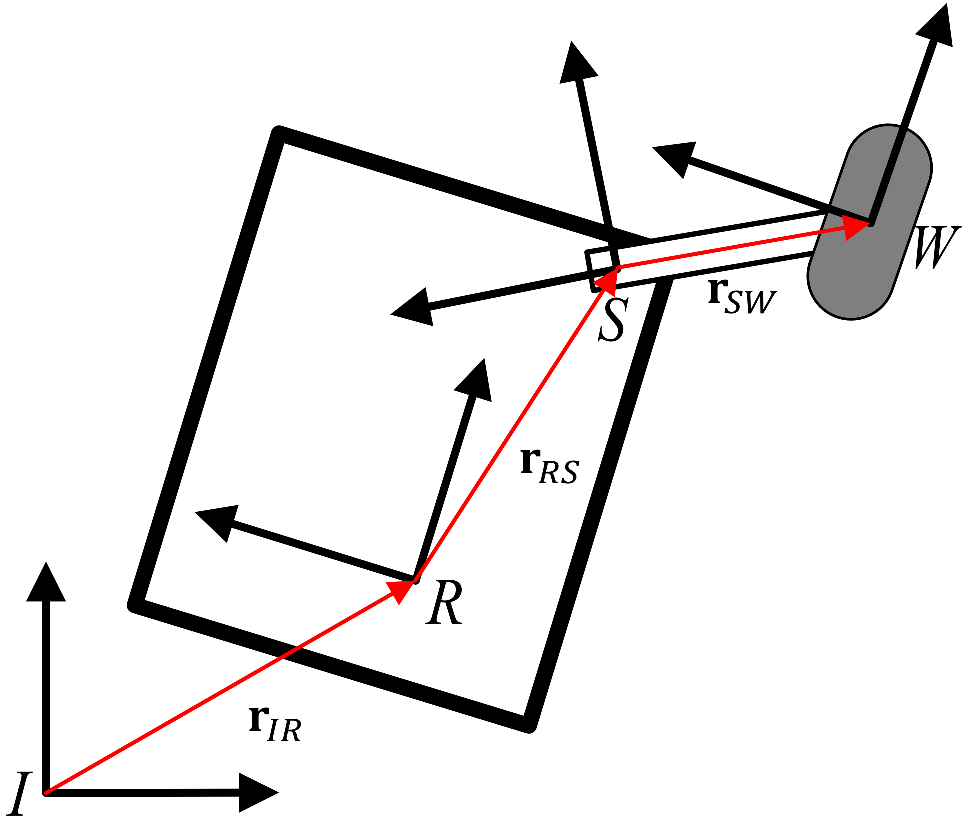


Wheeled Kinematics

- **Problem:** For a robot using several wheels, find the relationship between wheel speeds, $\dot{\varphi}_i$, and platform velocity, $\dot{\xi}_I = [\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$

- Assumptions
  - Movement on a horizontal plane
  - Point contact of the wheels
  - Wheels not deformable
  - Pure rolling, no slipping, skidding, or sliding
  - No friction for rotation around contact point
  - Wheels connected to a rigid frame (chassis)

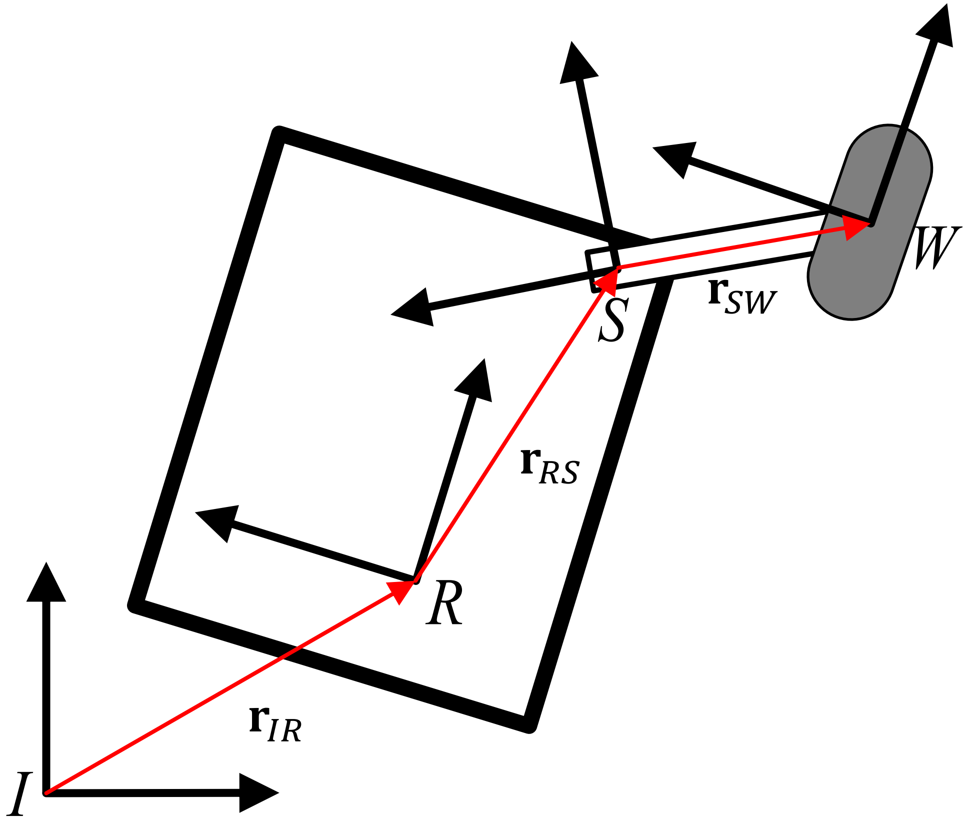
Deriving a general wheel equation



- Coordinate frames
  - I – inertial
  - R – robot
  - S – steering
  - W – wheel
- The position of the wheel in the inertial frame is

$$\mathbf{r}_{IW} = \mathbf{r}_{IR} + \mathbf{r}_{RS} + \mathbf{r}_{SW}$$

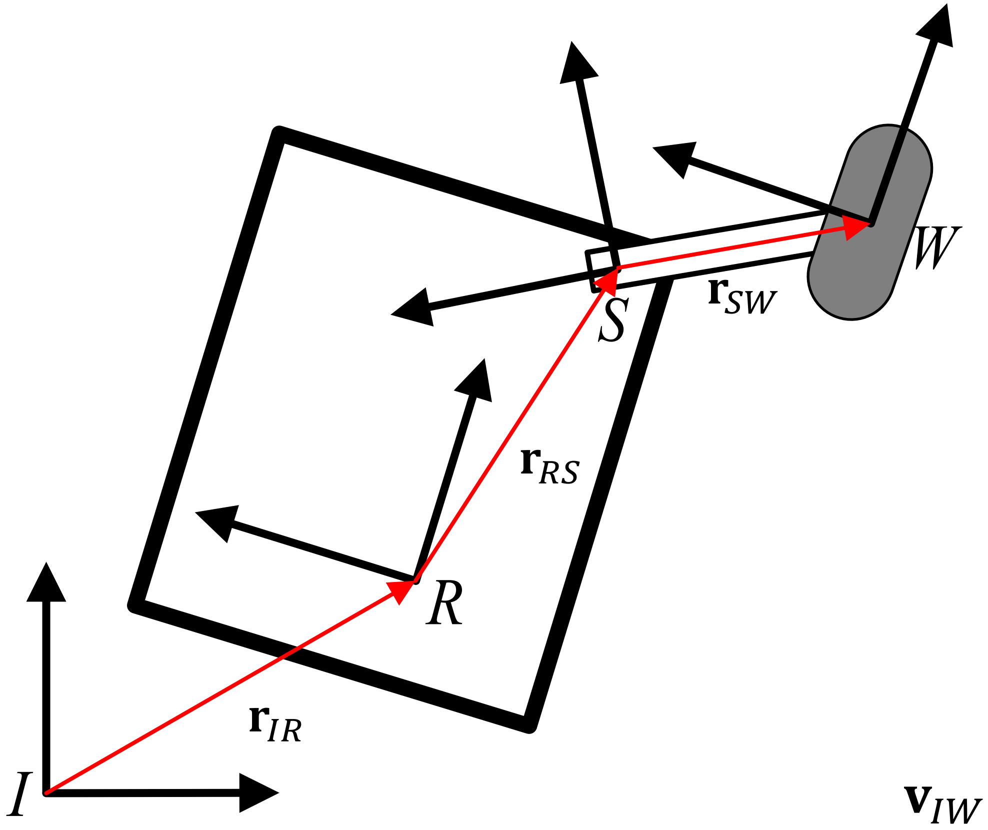
Deriving a general wheel equation



$$\mathbf{v}_{IW} = \underbrace{\mathbf{v}_{IR}}_{\text{robot velocity}} + \overbrace{\boldsymbol{\omega}_{IR} \times \underbrace{\mathbf{r}_{RS}}_{\text{steering offset}} + \boldsymbol{\omega}_{IR}}^{\text{robot angular velocity}} \times \mathbf{r}_{SW} + \overbrace{\boldsymbol{\omega}_{RS}}^{\text{steering rate}} \times \mathbf{r}_{SW}$$

wheel offset: $\mathbf{r}_{SW}$

Deriving a general wheel equation



- The position of the wheel frame is

$$\mathbf{r}_{IW} = \mathbf{r}_{IR} + \mathbf{r}_{RS} + \mathbf{r}_{SW}$$

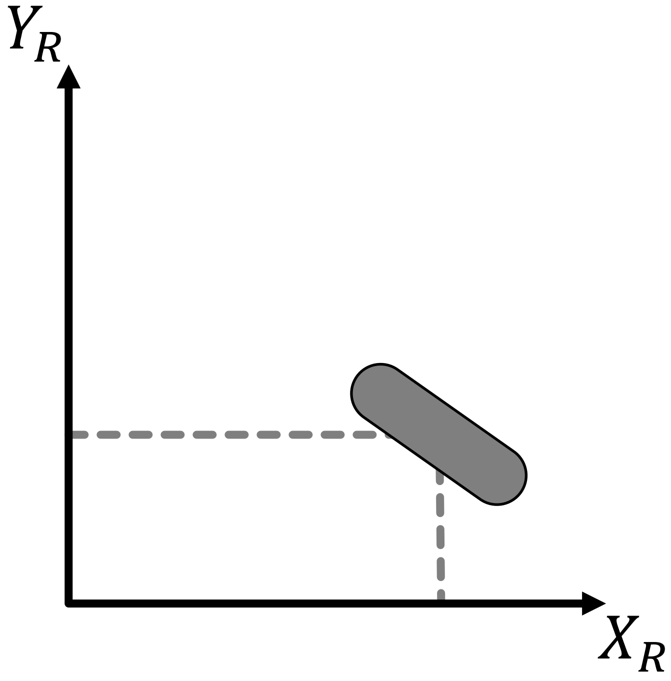
- Take the time derivative in the inertial frame and assume that

$$\mathbf{v}_{RS} = \dot{\mathbf{r}}_{RS} = \mathbf{0} \qquad \text{and} \qquad \mathbf{v}_{SW} = \dot{\mathbf{r}}_{SW} = \mathbf{0}$$

- The result is **the wheel equation**:

$$\mathbf{v}_{IW} = \mathbf{v}_{IR} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{SW} + \boldsymbol{\omega}_{RS} \times \mathbf{r}_{SW}$$

Example: Standard wheel



robot angular velocity

steering rate

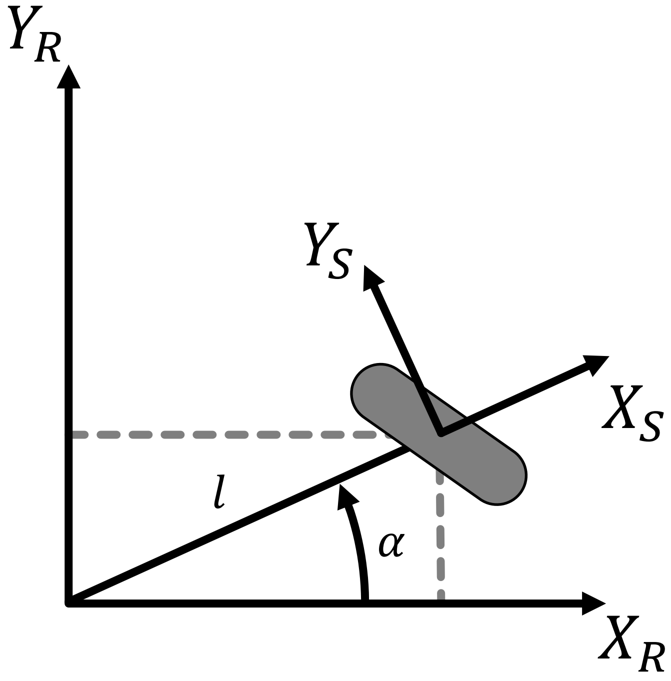
$$\mathbf{v}_{IW} = \mathbf{v}_{IR} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{SW} + \boldsymbol{\omega}_{RS} \times \mathbf{r}_{SW}$$

robot velocity

steering offset

wheel offset

Example: Standard wheel



robot angular velocity

steering rate

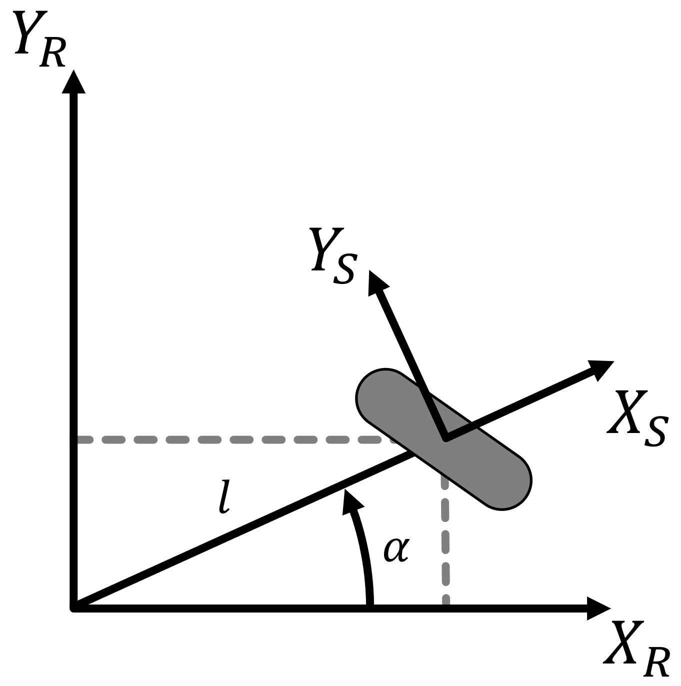
$$\mathbf{v}_{IW} = \mathbf{v}_{IR} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{SW} + \boldsymbol{\omega}_{RS} \times \mathbf{r}_{SW}$$

robot velocity

steering offset

wheel offset

Example: Standard wheel



$$\underbrace{\mathbf{v}_{IW}}_{\text{robot velocity}} = \mathbf{v}_{IR} + \overbrace{\boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS} + \boldsymbol{\omega}_{IR}}^{\text{robot angular velocity}} \times \mathbf{r}_{SW} + \boldsymbol{\omega}_{RS} \times \mathbf{r}_{SW}$$

robot angular velocity: $\boldsymbol{\omega}_{IR}$

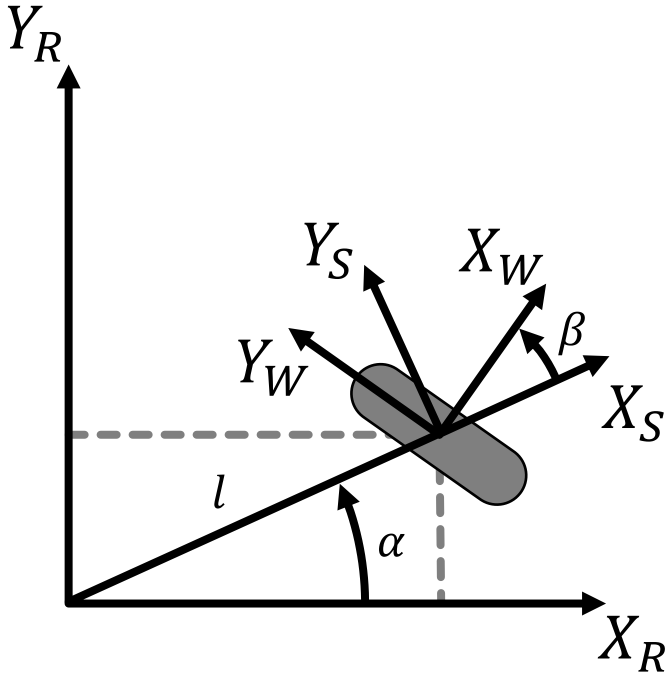
steering rate: $\boldsymbol{\omega}_{RS}$

robot velocity: $\mathbf{v}_{IR}$

steering offset: $\mathbf{r}_{RS}$

wheel offset: $\mathbf{r}_{SW}$

Example: Standard wheel



robot angular velocity

steering rate

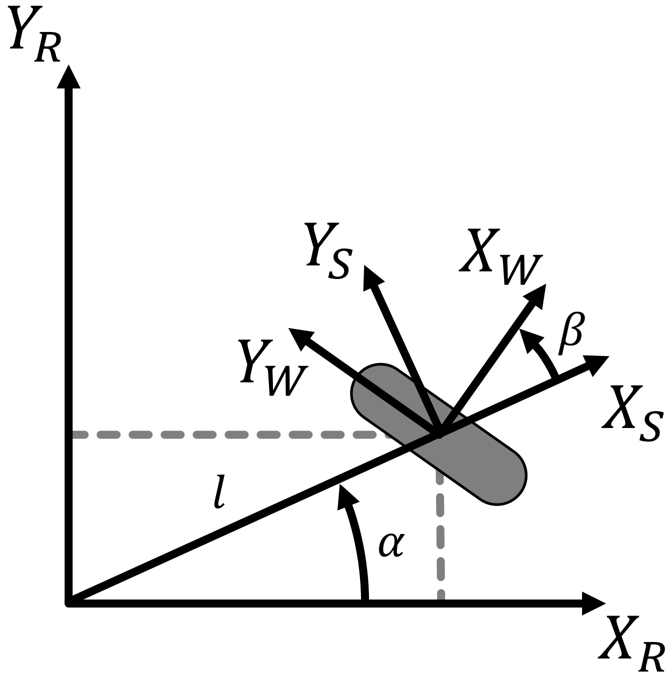
$$\mathbf{v}_{IW} = \mathbf{v}_{IR} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS} + \boxed{\boldsymbol{\omega}_{IR} \times \mathbf{r}_{SW}} + \boxed{\boldsymbol{\omega}_{RS} \times \mathbf{r}_{SW}}$$

robot velocity

steering offset

wheel offset $= 0$

Example: Standard wheel



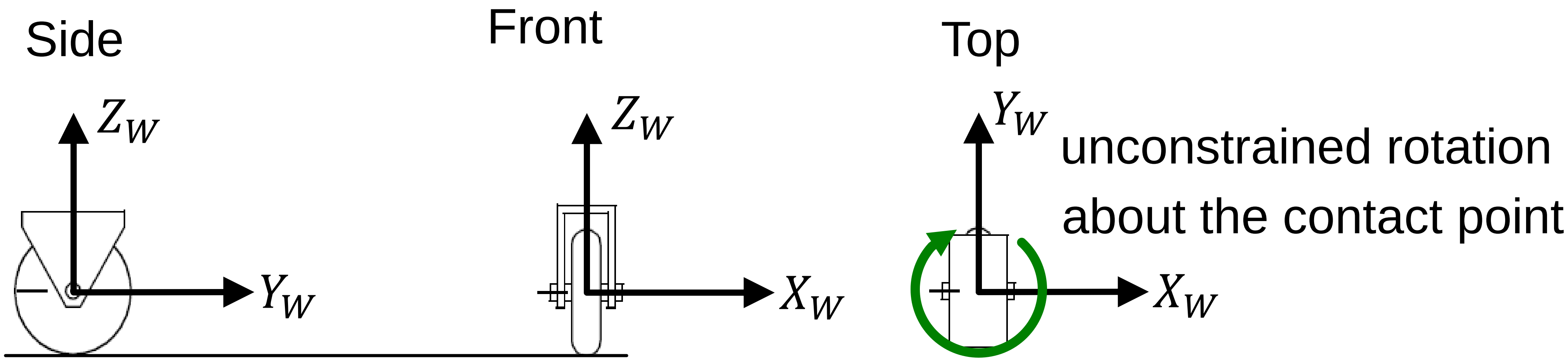
robot angular velocity

$$\mathbf{v}_{IW} = \mathbf{v}_{IR} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS}$$

robot velocity

steering offset

Constraints for a standard wheel



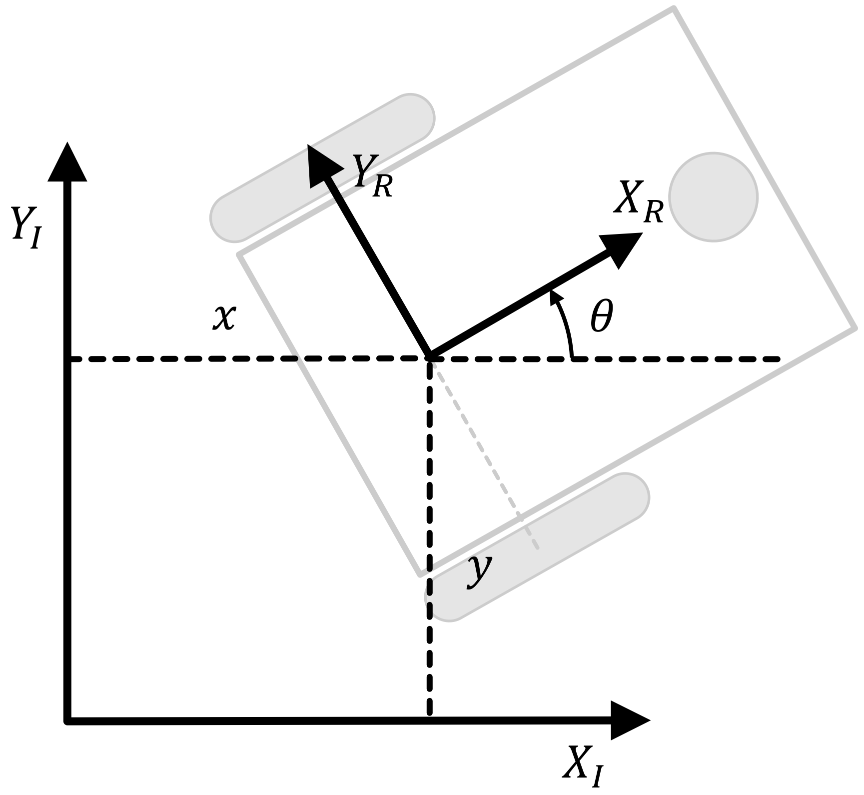
$\boxed{\dot{y}_W = r\dot{\varphi}}$ rolling constraint

$\boxed{\dot{x}_W = 0}$ no-sliding constraint

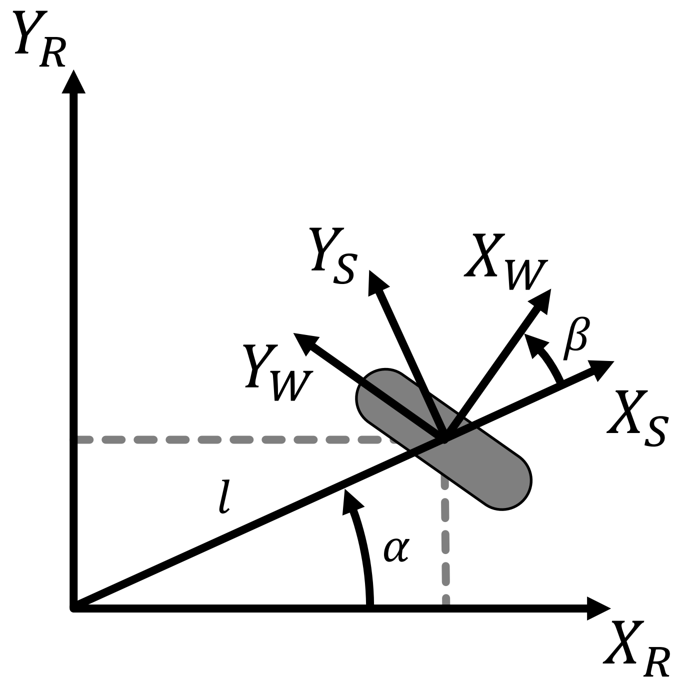
$\boxed{\dot{z}_W = 0}$ planar motion assumption

Notation

- State: $\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$
- State velocity: $\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$
- $R(a) = \begin{bmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $R(a)R(b) = R(a+b)$
- $\dot{\xi}_R = R(\theta)\dot{\xi}_I$

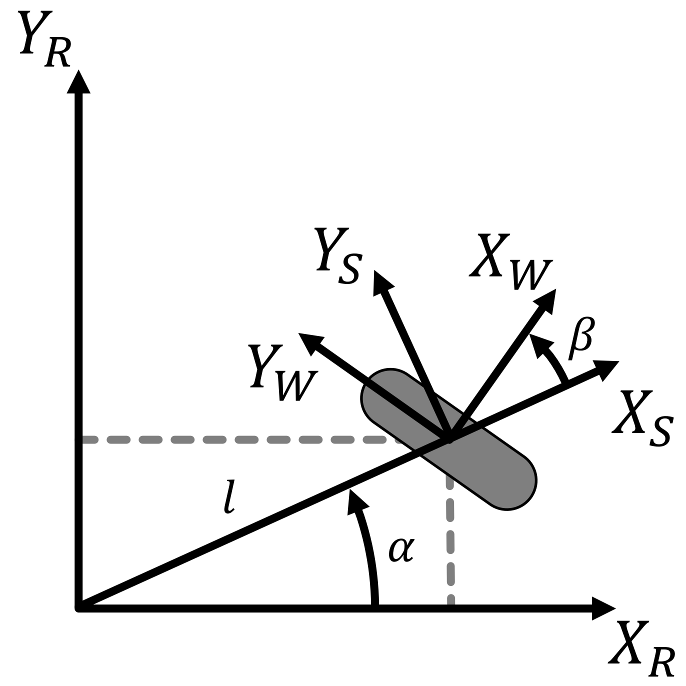


Example: Standard wheel



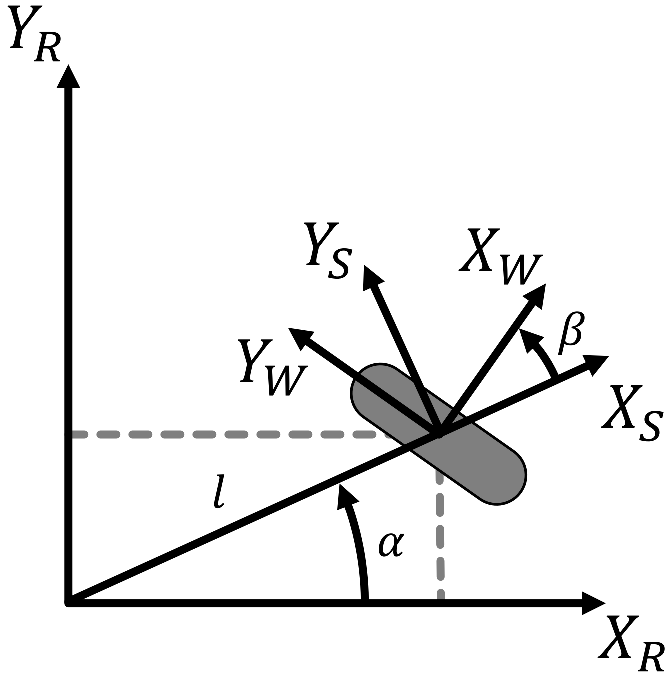
- Start with the general equation for a standard wheel
  - $\mathbf{v}_{IW} = \mathbf{v}_{IR} + \boldsymbol{\omega}_{IR} \times \mathbf{r}_{RS}$
- Express this equation in the wheel frame
  - ${}_W\mathbf{v}_{IW} = {}_W\mathbf{v}_{IR} + {}_W\boldsymbol{\omega}_{IR} \times {}_W\mathbf{r}_{RS}$
- The left hand side is known
  - ${}_W\mathbf{v}_{IW} = \begin{bmatrix} 0 \\ \dot{\varphi} r \\ 0 \end{bmatrix}$ - no-sliding constraint
    - rolling constraint
    - planar assumption

Example: Standard wheel



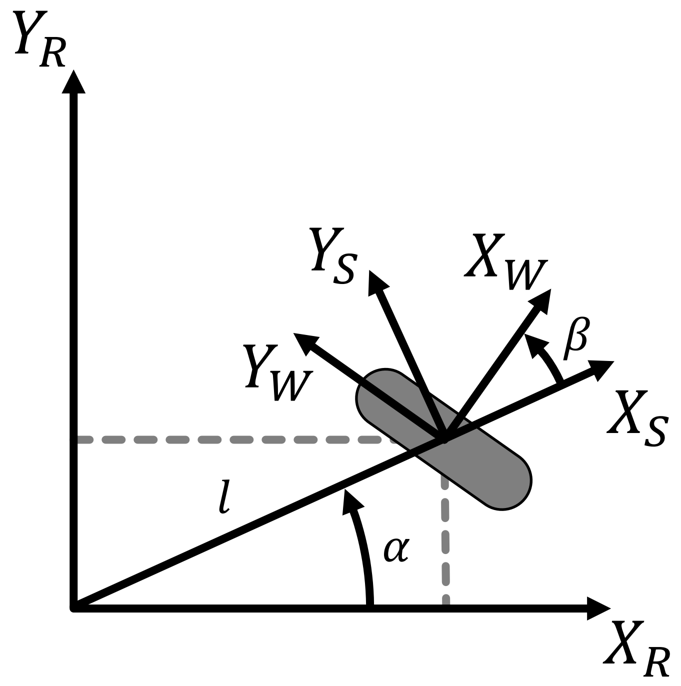
- Expand terms
  - $_W\mathbf{v}_{IW} = {}_W\mathbf{v}_{IR} + {}_W\boldsymbol{\omega}_{IR} \times {}_W\mathbf{r}_{RS}$
  - $_W\mathbf{v}_{IR} = R_{WS}R_{SR}R_{RI}\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix}$

Example: Standard wheel



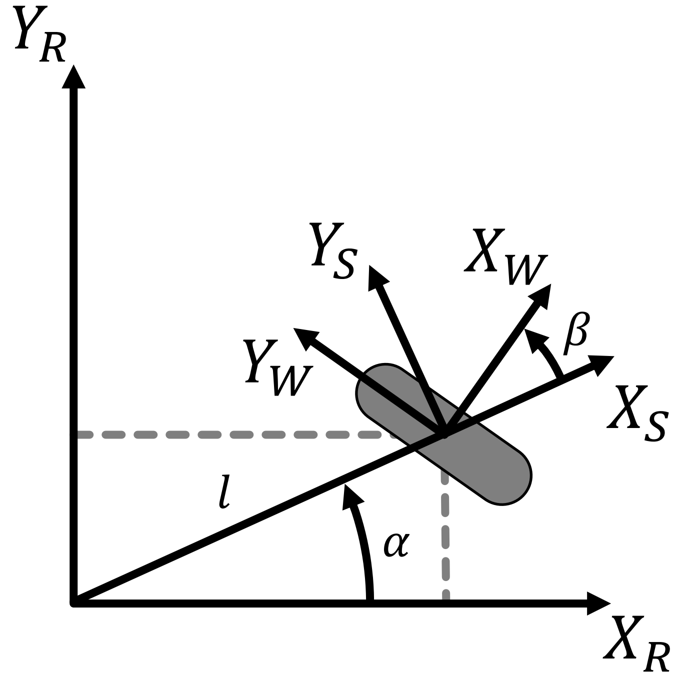
- Expand terms
  - ${}_W\mathbf{v}_{IW} = {}_W\mathbf{v}_{IR} + {}_W\boldsymbol{\omega}_{IR} \times {}_W\mathbf{r}_{RS}$
  - ${}_W\mathbf{v}_{IR} = R(\alpha + \beta)R(\theta)\begin{bmatrix}\dot{x} \\ \dot{y} \\ 0\end{bmatrix}$

Example: Standard wheel



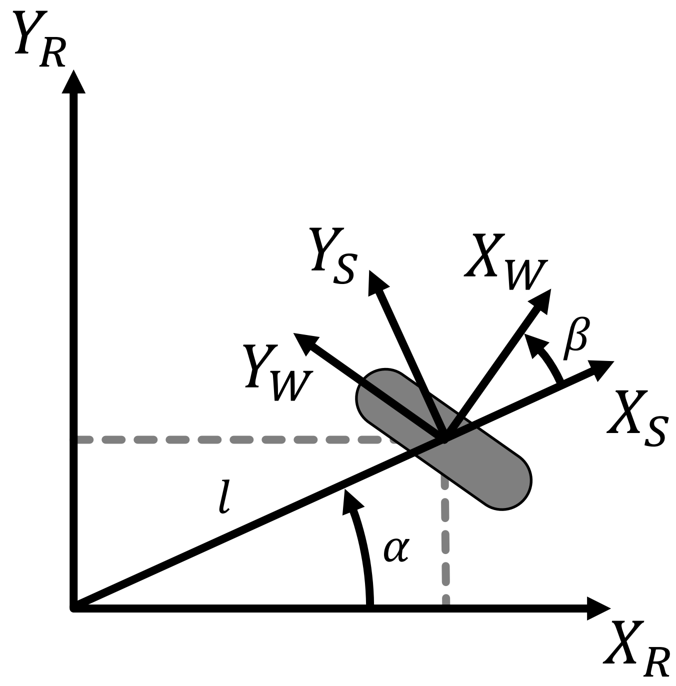
- Expand terms
  - $_W\mathbf{v}_{IW} = {_W\mathbf{v}_{IR}} + {_W\boldsymbol{\omega}_{IR}} \times {_W\mathbf{r}_{RS}}$
  - $_W\mathbf{v}_{IR} = R(\alpha+\beta)R(\theta)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix}$
  - $_W\boldsymbol{\omega}_{IR} \times {_W\mathbf{r}_{RS}} = \begin{bmatrix}0 & -\dot{\theta} & 0\\ \dot{\theta} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}l\cos\beta\\ -l\sin\beta\\ 0\end{bmatrix}$

Example: Standard wheel



- Expand terms
  - $_W\mathbf{v}_{IW} = {}_W\mathbf{v}_{IR} + {}_W\boldsymbol{\omega}_{IR} \times {}_W\mathbf{r}_{RS}$
  - $_W\mathbf{v}_{IR} = R(\alpha + \beta)R(\theta)\begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix}$
  - $_W\boldsymbol{\omega}_{IR} \times {}_W\mathbf{r}_{RS} = \begin{bmatrix} l\sin\beta \\ l\cos\beta \\ 0\end{bmatrix}\dot{\theta}$

Example: Standard wheel



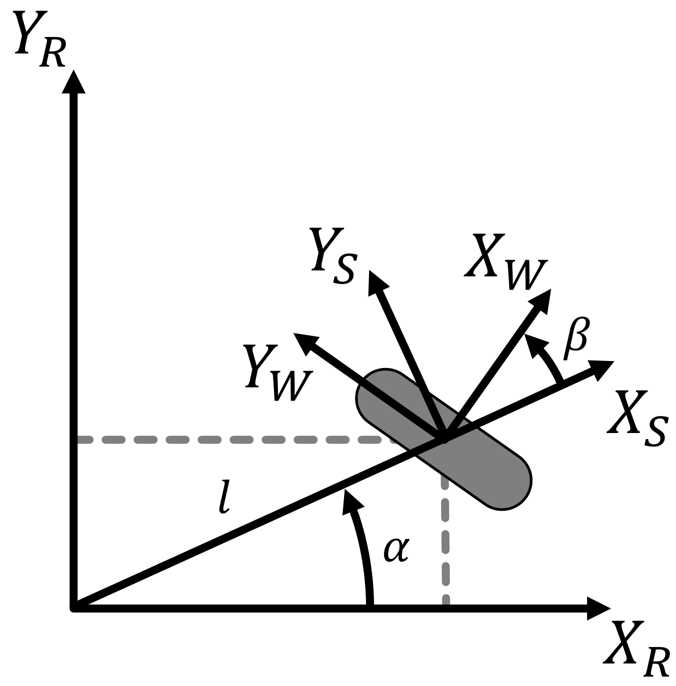
- Rolling constraint

$$[-\sin\alpha+\beta \quad \cos\alpha+\beta \quad l\cos\beta]R(\theta)\dot{\xi}_I-\dot{\varphi}r=0$$

- No-sliding constraint

$$[\cos\alpha+\beta \quad \sin\alpha+\beta \quad l\sin\beta]R(\theta)\dot{\xi}_I=0$$

Example: Standard wheel



- Rolling constraint

$$J_1(\beta_s)R(\theta)\dot{\xi}_I - \dot{\varphi}r = 0$$

- No-sliding constraint

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0$$

Preview: Deriving forward and inverse differential kinematics

- Next topic: Given a **specific robot configuration**, how can we stack the equations of each wheel to derive the **forward** and **inverse differential kinematics** for the robot chassis.