Kinematics | **Basics of Rigid Body Kinematics**

Autonomous Mobile Robots

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Introduction

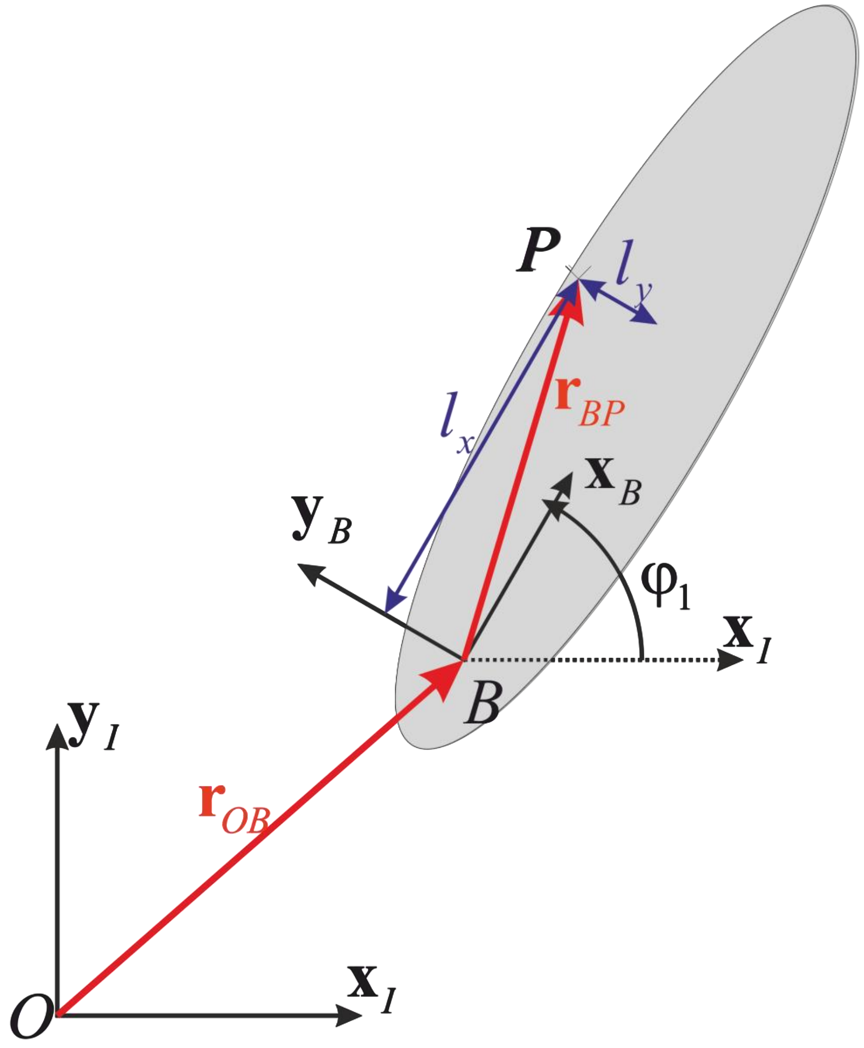
- Definition of kinematics
  - “Description of the **motion** of *points*, *bodies*, or *systems of bodies*”
  - *…* without consideration of the causes of motion (=> dynamics)
  - Required for **kinematic simulation** and **control**
- Types of motion of single bodies
  - Translation
  - Rotation
  - Combined motion

Vectors and coordinate transformation

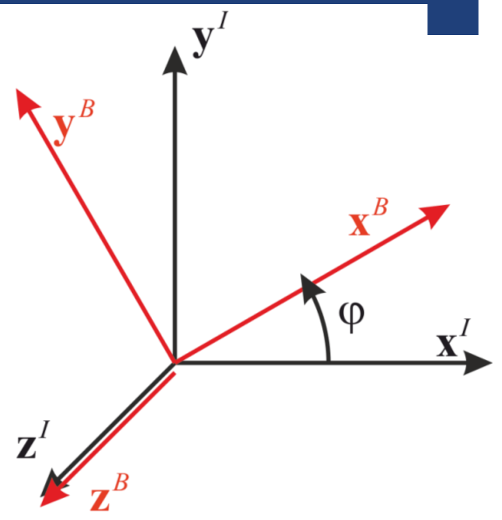
- Reference frames
  - Coordinate system **I** (inertial, not moving)
  - Coordinate system **B** (body-fixed, moving)

- Translation
  - Vector from *O* to *P*, expressed in **I**: ${}_I\mathbf{r}_{OP} \in \mathbb{R}^{3\times 1}$
- Rotation
  - Matrix from frame ***B*** to frame ***I***: $\mathbf{R}_{IB} \in \mathbb{R}^{3\times 3}$
- Homogeneous transformation
  - Combination of translation and rotation

$$ {}_I\mathbf{r}_{OP} = {}_I\mathbf{r}_{OB} + {}_I\mathbf{r}_{BP} = {}_I\mathbf{r}_{OB} + \mathbf{R}_{IB}\,{}_B\mathbf{r}_{BP} \qquad \begin{pmatrix} {}_I\mathbf{r}_{OP} \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{IB} & {}_I\mathbf{r}_{OB} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} {}_B\mathbf{r}_{BP} \\ 1 \end{pmatrix} $$



Rotation



- Rotation from frame **B** to **I** such that ${}_I\mathbf{r} = \mathbf{R}_{IB}\,{}_B\mathbf{r}$

$$\mathbf{R}_{IB} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inversion ${}_I\mathbf{r} = \mathbf{R}_{IB}\,{}_B\mathbf{r}, \quad {}_B\mathbf{r} = \mathbf{R}_{BI}\,{}_I\mathbf{r}, \quad \mathbf{R}_{BI} = \mathbf{R}_{IB}^T$
- Composition of rotations ${}_C\mathbf{r} = \mathbf{R}_{CI}\,{}_I\mathbf{r}, \quad \mathbf{R}_{CI} = \mathbf{R}_{CB}\mathbf{R}_{BI}$

Orientation in 3D

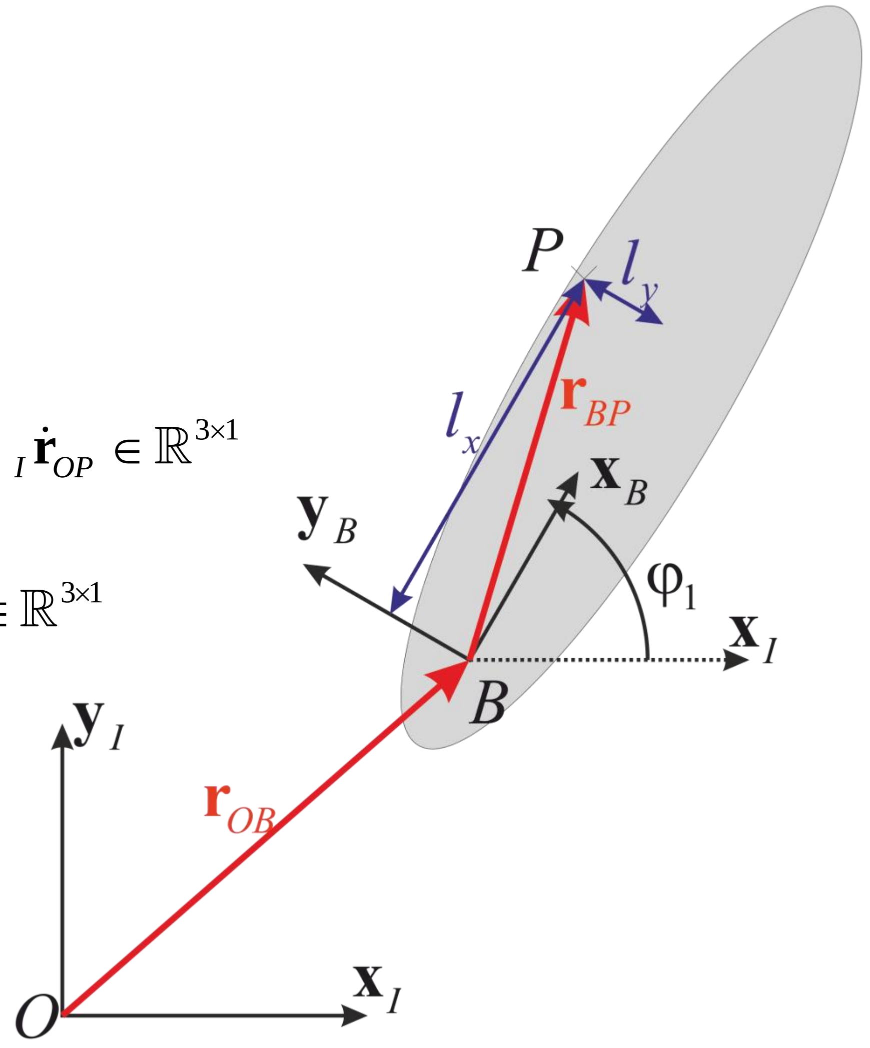
- Euler angles (z-x-z)
- Tait-Bryan/Cardan angles (z-y-x)

Vectors and coordinate transformation

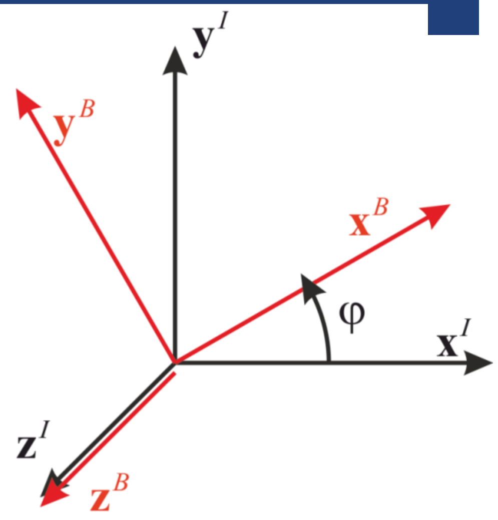
- Reference frames
  - Coordinate system **I** (inertial, not moving)
  - Coordinate system **B** (body-fixed, moving)

- Translation
  - vector from *O* to *P*, expressed in **I**: $_I\mathbf{r}_{OP} \in \mathbb{R}^{3\times1} \longrightarrow {_I}\mathbf{v}_P = {_I}\dot{\mathbf{r}}_{OP} \in \mathbb{R}^{3\times1}$
- Rotation
  - matrix from frame ***B*** to frame ***I***: $\mathbf{R}_{IB} \in \mathbb{R}^{3\times3} \longrightarrow {_I}\boldsymbol{\omega}_{IB} \in \mathbb{R}^{3\times1}$
- Homogeneous transformation
  - Combination of translation and rotation

$$_I\mathbf{v}_P = {_I}\dot{\mathbf{r}}_{OP} = {_I}\dot{\mathbf{r}}_{OB} + {_I}\boldsymbol{\omega}_{IB} \times {_I}\mathbf{r}_{BP}$$



Angular velocity



- Angular velocity of frame **B** with respect to **I** expressed in **I**

$$
{}_I\boldsymbol{\omega}_{IB} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}
$$

- Relation to rotation matrix

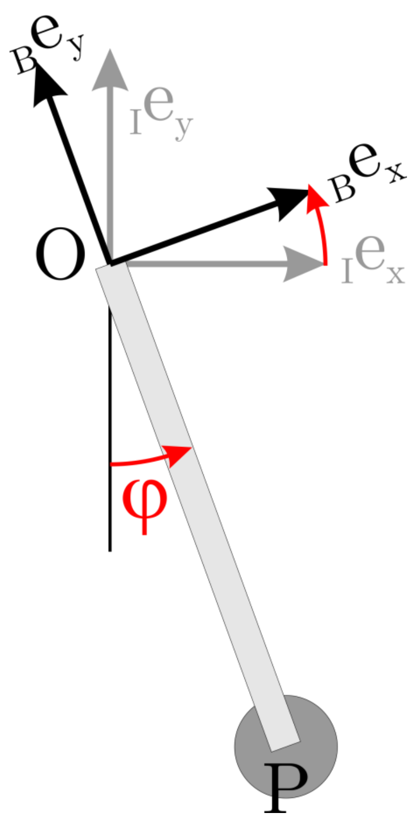
$$
{}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{IB}\mathbf{R}_{IB}^T, \qquad \hat{\boldsymbol{\omega}}_{IB} = \begin{bmatrix} 0 & -\omega_{IB}{}^z & \omega_{IB}{}^y \\ \omega_{IB}{}^z & 0 & -\omega_{IB}{}^x \\ -\omega_{IB}{}^y & \omega_{IB}{}^x & 0 \end{bmatrix}, \qquad \boldsymbol{\omega}_{IB} = \begin{pmatrix} \omega_{IB}{}^x \\ \omega_{IB}{}^y \\ \omega_{IB}{}^z \end{pmatrix}
$$

- Transformation ${}_I\boldsymbol{\omega}_{IB} = \mathbf{R}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$

- Consecutive rotations ${}_I\boldsymbol{\omega}_{IC} = {}_I\boldsymbol{\omega}_{IB} + {}_I\boldsymbol{\omega}_{BC}$

Vector differentiation

- Position and velocity in different frames
  - Inertial frame $\quad {}_I\mathbf{r} \Rightarrow {}_I(\dot{\mathbf{r}}) = \frac{d\,{}_I\mathbf{r}}{dt}$
  - Moving frame $\quad {}_B\mathbf{r} \Rightarrow {}_B(\dot{\mathbf{r}}) = \frac{d\,{}_B\mathbf{r}}{dt} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}$



- Example: single pendulum, ${}_B\mathbf{v}_P = ?$

$$ {}_I\mathbf{r}_{OP} = \begin{pmatrix} l\sin(\varphi) \\ -l\cos(\varphi) \\ 0 \end{pmatrix} \quad \Rightarrow {}_I\mathbf{v}_P = \frac{d\,{}_I\mathbf{r}_{OP}}{dt} = \begin{pmatrix} l\cos(\varphi)\dot{\varphi} \\ l\sin(\varphi)\dot{\varphi} \\ 0 \end{pmatrix} $$

$$ {}_B\mathbf{v}_P = \mathbf{R}_{BI}\,{}_I\mathbf{v}_P = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} l\cos(\varphi)\dot{\varphi} \\ l\sin(\varphi)\dot{\varphi} \\ 0 \end{pmatrix} = \begin{pmatrix} l\dot{\varphi} \\ 0 \\ 0 \end{pmatrix} $$

$$ {}_B\mathbf{r}_{OP} = \begin{pmatrix} 0 \\ -l \\ 0 \end{pmatrix} $$

$$ {}_B\mathbf{v}_P = \frac{d\,{}_B\mathbf{r}_{OP}}{dt} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{OP} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} 0 \\ -l \\ 0 \end{pmatrix} = \begin{pmatrix} l\dot{\varphi} \\ 0 \\ 0 \end{pmatrix} $$

Recapitulation

- We learned the basics for describing the motion
  - Translations $_I\mathbf{r}_{OP_1} = {_I\mathbf{r}_{OB}} + {_I\mathbf{r}_{BP_1}}$
  - Rotations $_B\mathbf{r}_{OP_1} = \mathbf{R}_{BI}\,{_I\mathbf{r}_{BP_1}}$
  - Homogeneous transformation $\begin{pmatrix} _I\mathbf{r}_{OP_1} \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{IB} & _I\mathbf{r}_{OB} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} _B\mathbf{r}_{BP_1} \\ 1 \end{pmatrix}$
  - Anglar velocities $_I\boldsymbol{\omega}_{IC} = {_I\boldsymbol{\omega}_{IB}} + {_I\boldsymbol{\omega}_{BC}}$
  - Differentiation of (position) vectors $_B\mathbf{r} \Rightarrow {_B}(\dot{\mathbf{r}}) = \frac{d\,_B\mathbf{r}}{dt} + {_B\boldsymbol{\omega}_{IB}} \times {_B\mathbf{r}}$