ETH zürich

ASL
Autonomous Systems Lab

Kinematics | **Application of Rigid Body Kinematics**

Autonomous Mobile Robots

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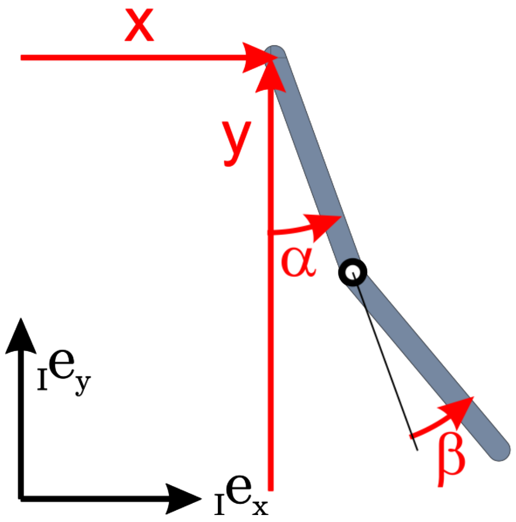
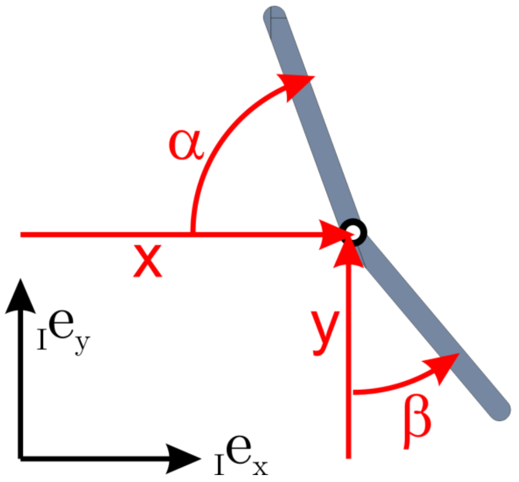
Generalized coordinates

- A set of **independent** variables that **uniquely** describe the robot's configuration



$$\mathbf{q} = \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix}$$



- Express all quantities as a function of generalized coordinates, e.g. $_I\mathbf{r}_{OP} = {_I\mathbf{r}_{OP}}(\mathbf{q})$

Jacobians

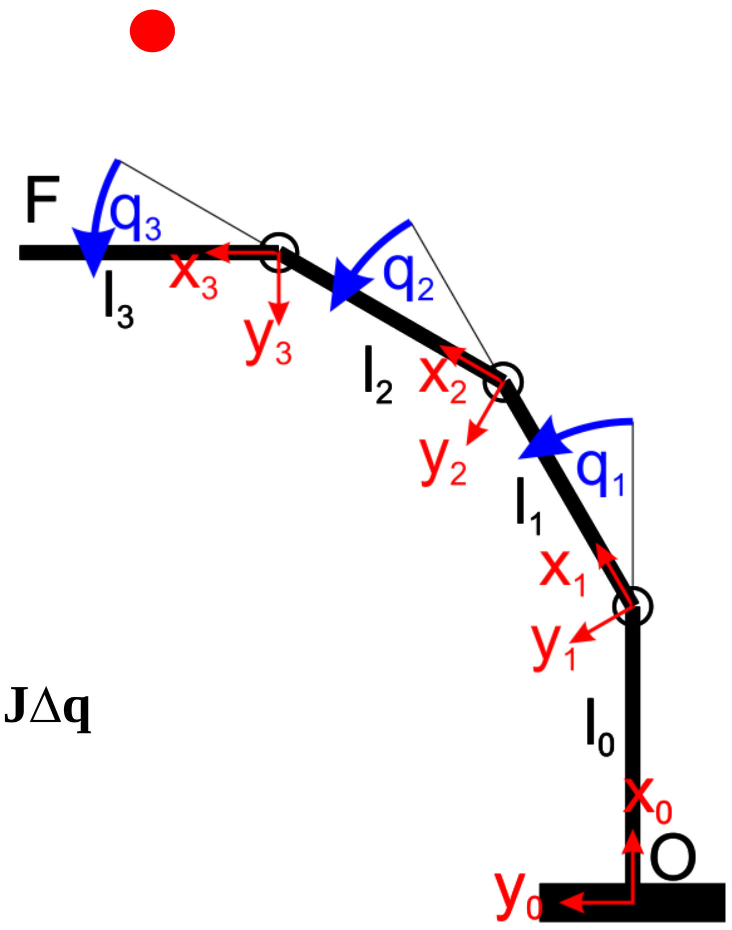
- Partial derivative of position vector $$\mathbf{J}_P = \frac{\partial \mathbf{r}_{OP}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \cdots & \frac{\partial r_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial q_1} & \cdots & \frac{\partial r_m}{\partial q_n} \end{bmatrix}$$

- Jacobian is important for kinematics
  - Linear mapping of Cartesian velocities to generalized velocities $$\dot{\mathbf{r}}_P = \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}}$$
    - Express contact constraints
    - Trajectory control
  - Linear mapping of changes in generalized coordinates to Cartesian space $$\Delta \mathbf{r}_P = \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}} \Delta \mathbf{q} = \mathbf{J}\Delta \mathbf{q}$$
    - Forward and inverse kinematics

Inverse kinematics

- Problem
  - Given a desired endeffector position $\mathbf{r}_{OF}(\mathbf{q}) = \mathbf{r}_{OF}^{goal}$
  - Determine generalized coordinates $\mathbf{q}$
  - $\mathbf{r}_{OF}(\mathbf{q})$ often not easily invertible in closed form

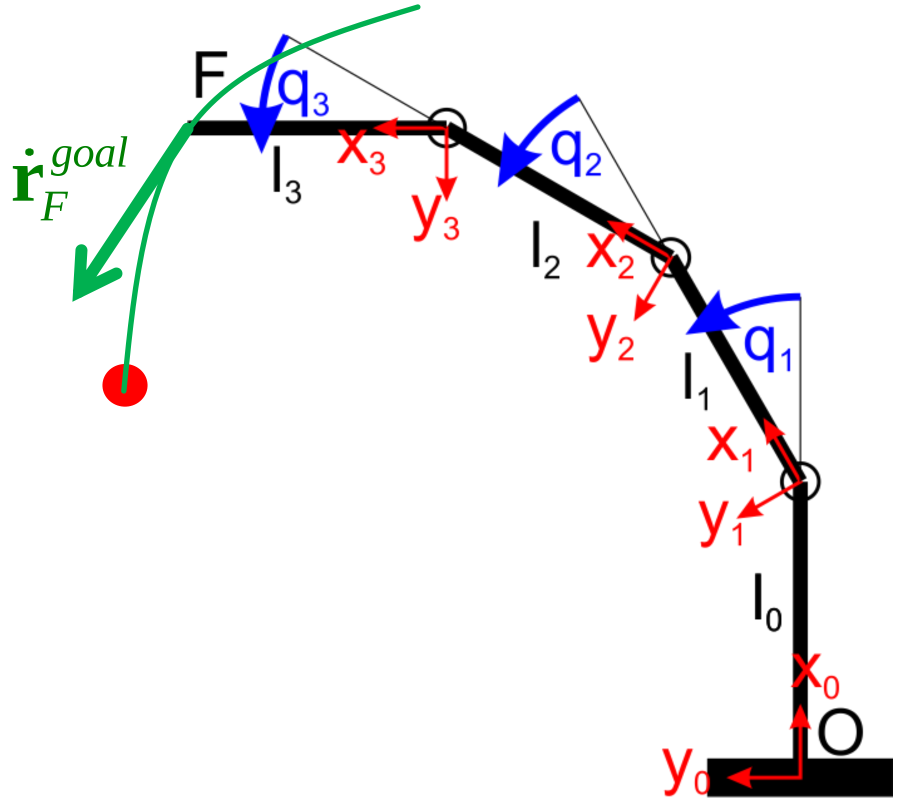
- Approach: Iteratively perform the following steps $(\mathbf{q}^i, \mathbf{r}^i)$
  1. Start from known solution (e.g. given by forward kin)
  2. Linearize $\mathbf{r}_{OF}(\mathbf{q})$ resulting in the *Jacobian* $\mathbf{J} = \left.\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}^i} \Leftrightarrow \Delta\mathbf{r} = \mathbf{J}\Delta\mathbf{q}$
  3. Invert the Jacobian to obtain $\Delta\mathbf{q} = \mathbf{J}^+\Delta\mathbf{r}$
  4. Update generalized coordinates $\mathbf{q}^{i+1} = \mathbf{q}^i + \mathbf{J}^+\left(\mathbf{r}^{goal} - \mathbf{r}^i\right)$



Differential kinematics

- Relation between Cartesian velocity and generalized velocities

$$\dot{\mathbf{r}}_F = \mathbf{J}_F \dot{\mathbf{q}}$$



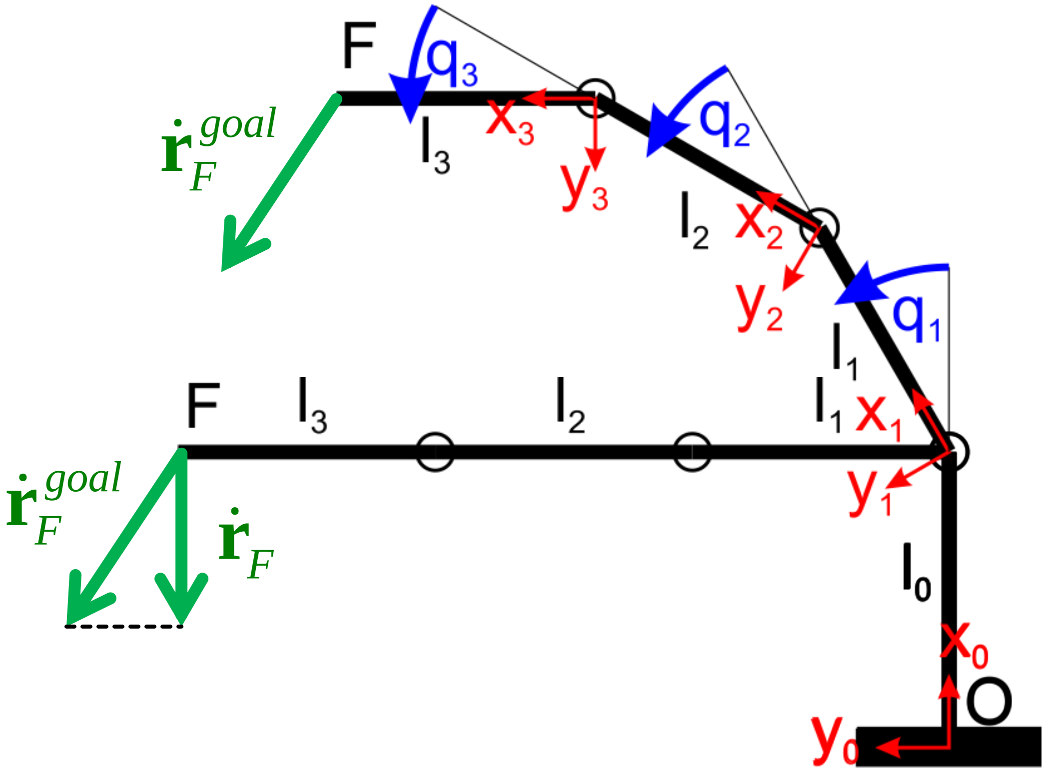
- Inverse differential kinematics
  - Given the endeffector velocity $\dot{\mathbf{r}}$
  - Determine the generalized velocities $\dot{\mathbf{q}} = \mathbf{J}^+ \dot{\mathbf{r}}_F^{goal}$

Redundancy and singularity

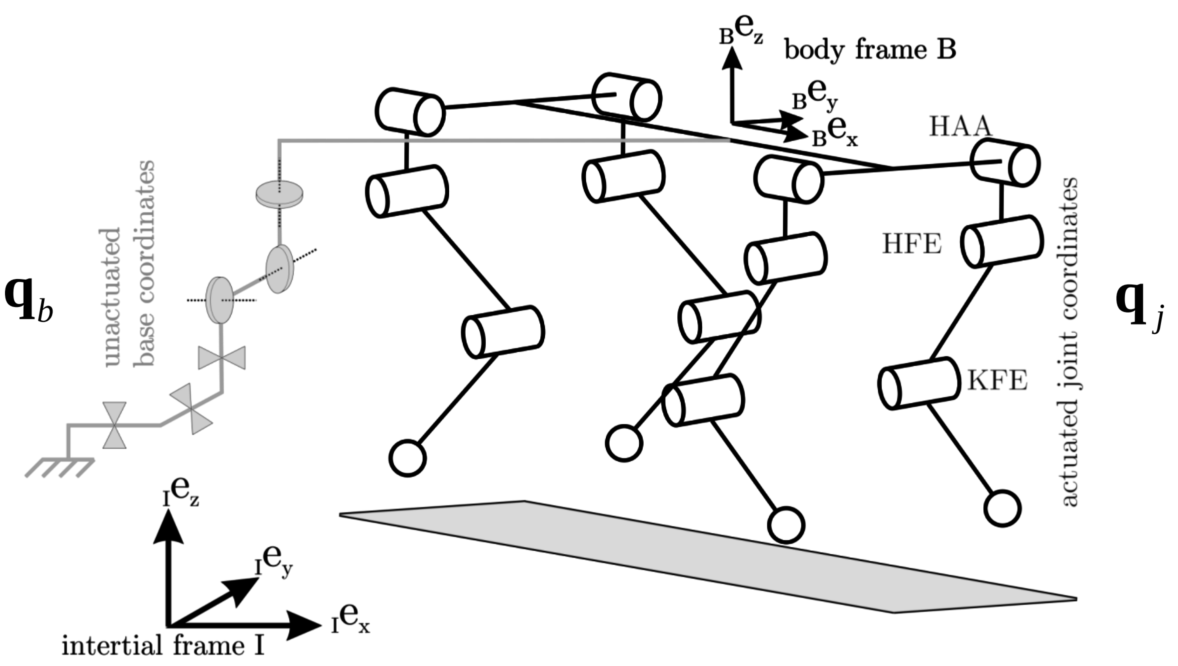
- Jacobian is often not invertible => use the Moore-Penrose pseudo inverse $\mathbf{J}^+$

- Redundancy
  - Jacobian is column-rank deficient
  - Pseudo inverse minimizes $\|\dot{\mathbf{q}}\|_2$
  - Can have multiple solutions $\dot{\mathbf{q}} = \mathbf{J}^+\dot{\mathbf{r}}^{goal} + \underbrace{\left(\mathbf{I} - \mathbf{J}^+\mathbf{J}\right)}_{\mathbf{N}}\dot{\mathbf{q}}_0$

- Singularity
  - Jacobian is row-rank deficient
  - Pseudo inverse minimizes the error $\left\|\dot{\mathbf{r}}^{goal} - \dot{\mathbf{r}}\right\|_2$



Kinematic structure of mobile robots



- Generalized coordinates

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix} \begin{matrix} \text{Un-actuated base} \\ \text{Actuated joints} \end{matrix}$$

- Contact constraints
  - Footpoint is not allowed to move

$$\dot{\mathbf{r}}_F = \mathbf{J}_F \dot{\mathbf{q}} = \mathbf{0}$$

  - Choose the joint speed such that this constraint is satisfied

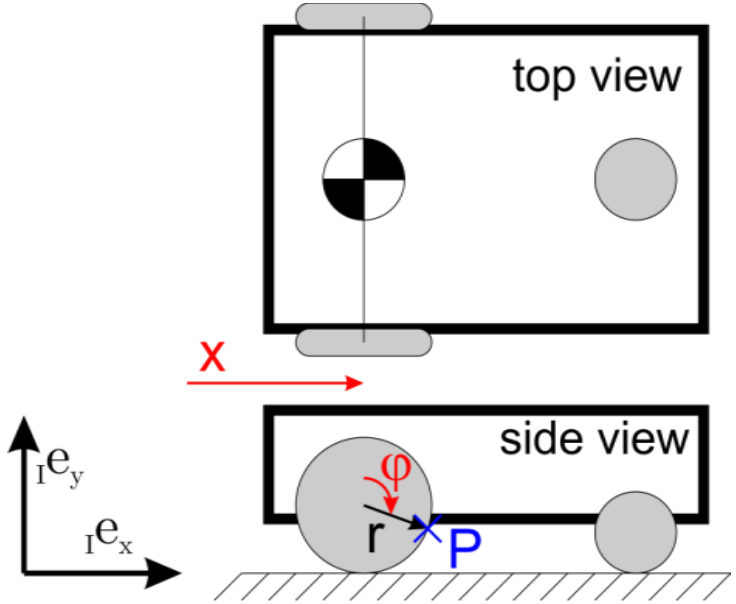
$$\dot{\mathbf{q}} = \cancel{\mathbf{J}_F^+ \dot{\mathbf{r}}_F}^{=0} + \mathbf{N}_F \dot{\mathbf{q}}_0$$

  - Move body upward

$$\dot{\mathbf{r}}_{body} = \mathbf{J}_{body} \dot{\mathbf{q}} = \mathbf{J}_{body} \mathbf{N}_F \dot{\mathbf{q}}_0$$

$$\dot{\mathbf{q}} = \cancel{\mathbf{J}_F^+ \dot{\mathbf{r}}_F}^{=0} + \mathbf{N}_F \dot{\mathbf{q}}_0 = \mathbf{N}_F \left(\mathbf{J}_{body} \mathbf{N}_F \right)^+ \dot{\mathbf{r}}_{body}$$

Kinematic structure of mobile robots



- Generalized coordinates

$$\mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

Un-actuated base

Actuated joints

- Contact constraints
  - Point on wheel $_I\mathbf{r}_{OP} = \begin{pmatrix} x + r\sin(\varphi) \\ r + r\cos(\varphi) \\ 0 \end{pmatrix}$
  - Jacobian $_I\mathbf{J}_P = \begin{bmatrix} 1 & r\cos(\varphi) \\ 0 & -r\sin(\varphi) \\ 0 & 0 \end{bmatrix}$
  - Contact constraints

$$_I\mathbf{v}_P\big|_{\varphi=\pi} = {_I\mathbf{J}_P}\big|_{\varphi=\pi}\dot{\mathbf{q}} = \begin{bmatrix} 1 & -r \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = \mathbf{0}$$

=> Rolling condition $\dot{x} - r\dot{\varphi} = 0$