Mechanics, a Short Review

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Forces and Moments

- Forces
- Scalars and Vectors
- Sum of Forces
- Decomposition of Forces
- Statically Equivalent Systems
- Moments

We are going to explore the concepts of forces, moments and equilibrium.
A free body subjected to a single force is in motion (Newton, $\boldsymbol{F} = m\boldsymbol{a}$).
We have equilibrium (no motion) when no force is applied *or* there are different forces $\boldsymbol{F}_i$ whose sum ($\sum \boldsymbol{F}_i$) is equal to zero ($\sum \boldsymbol{F}_i = \boldsymbol{0}$).

If a body is, e.g., fixed to an axis (like a door) the axis can exert a *reactive force* that is directed in the plane of the door

- if you push the door in its plane the force is *equilibrated* by the axis' *reaction*,
- if you push perpendicular to the door's plane the door rotates around the hinges,
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We have seen in action a *moment*, $M = F \times d$, a moment induces a rotation rather than a translation but however we have no equilibrium.
To have equilibrium for a free body we must require that the sum of the moments also equals zero, $\sum M = 0$.

We will return on the formal expression of an equilibrium condition.

A *force* is a directed interaction between two bodies.

- contact forces,
- no-contact forces.

While a force is a familiar concept, its precise characterization in terms of

- magnitude
- direction
- sense

is not obvious and requres a certain level of abstraction.
Among others that contributed to the formalization of the concept of force, Leonardo da Vinci, Stevin, Galileo, Newton.

Mechanichs is interested in both scalar and vector quantities

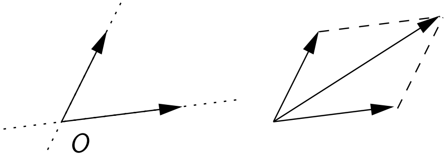
- a body has a mass (scalar)
- a body on Earth is subjected to a force, proportional to the mass and directed towards the centre of the Earth.

We represent a force with a segment on a line, the *line of action* of the force — the line of action represent the direction of the force.
The length of the segment represent, to some scale, the magnitude of the force and eventually the orientation of the segment sense of the force.
We can move the oriented segment on the line of action without changing the force it represents.

Most structural systems can be considered rigid bodies, because their deformation when subjected to ordinary forces is negligible w/r to their dimensions.

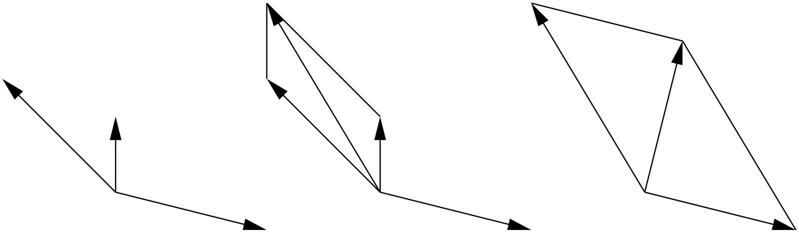
Under the assumption of rigid body we can neglect the local effects of a force and consider the force as acting anywhere along its line of action.

To study the effects of a system of forces on a rigid body we can study the effects of the sum of those forces.
As a start consider two forces, their lines of action sharing a point O.

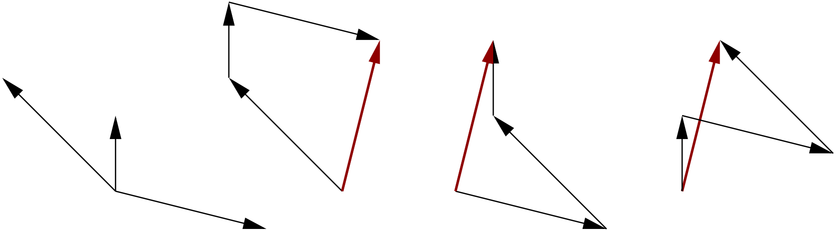


The sum of these two forces can be found *graphically* using the *parallelogram rule* — the sum is the diagonal of the parallelogram that has the two forces as two sides.

When you have many forces whose lines of action all pass in a single point, you can repeat the parallelogram construction substituting any two forces with their resultant, and repeating the procedure until only a force remains.



Alternatively you can join all the forces point to tail, the oriented segment from the first point to the last one is the resultant.



Sum of Parallel Forces

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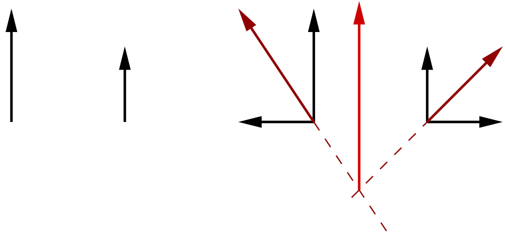

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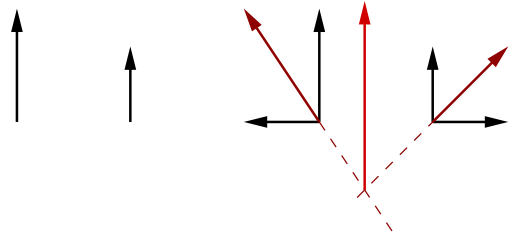
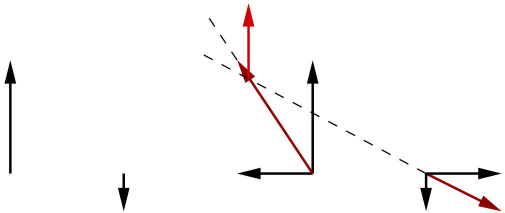
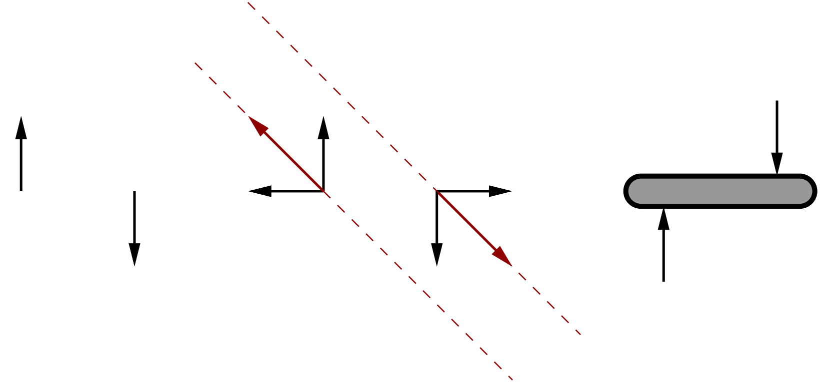


It works also when parallel forces act in opposite directions



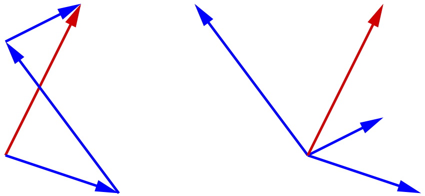
Couple of Forces



- two forces, equal and opposite, lines of action are parallel
- sum forces with zero sum: the two resultants are still parallel
- two such forces exert a *moment* on a rigid body

Decomposition of Forces

In general you can decompose a force the way you like, applying the reversal of the summation rule



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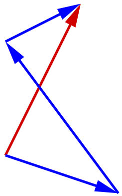

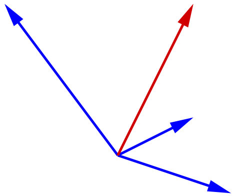

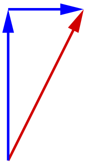

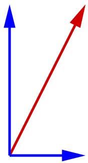

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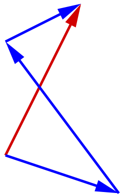
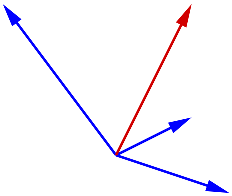
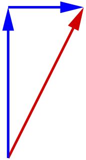
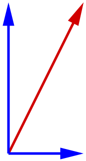


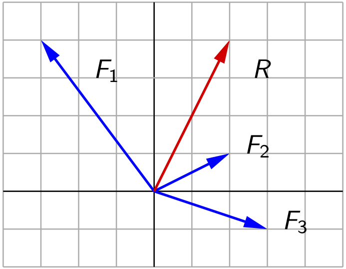






... but it is more useful to decompose a force according to its orthogonal components

because *computing* the sum of a system of forces can be done decomposing them into orthogonal components and using the *algebraic* sum of the horizontal and vertical components to find the components of the resultant.



$$F_{1X} = -3 \qquad F_{1Y} = +4$$
$$F_{2X} = +2 \qquad F_{2Y} = +1$$
$$F_{3X} = +3 \qquad F_{3Y} = -1$$

$$R_X = +2 \qquad R_Y = +4$$

$$R = \sqrt{2^2 + 4^2} = \sqrt{20} = 4.47$$

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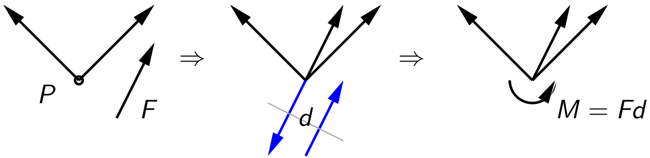
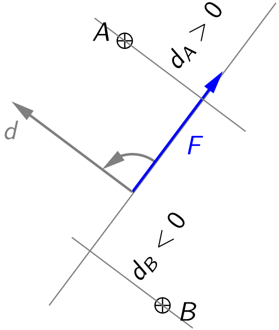
If the forces are concurrent in a point, their system is statically equivalent to the resultant force.
If the forces are non-concurrent in a point you can always find the resultant in a point *plus* a moment associated with the translation of the forces.

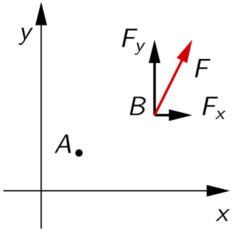


A *moment* measure the ability of a force to make a body rotate around a point — do you remember the door example?
The moment of a force around the point A is *different* from its moment with respect to a different point B.
A moment can be positive or negative, the most common convention is, a *positive moment* produces an anti-clockwise rotation.
The *value* of the moment of a force around a point is proportional to the force and to signed distance between the point and the line of action of the force, $M = F \cdot d$.



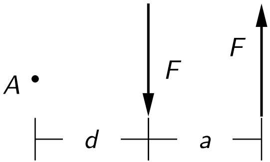
$$M_A = F \cdot d_A, \qquad M_A > 0$$
$$M_A = F \cdot d_B, \qquad M_B < 0$$

You can work out the sign convention to be used when we reason in terms of Cartesian coordinates



$$M_A = F_y \cdot (x_B - x_A) - F_x \cdot (y_B - y_A)$$

A *couple of forces* or, briefly, a *couple* has the same moment with respect to every point.



$$M_A = F \cdot (d + a) - F \cdot d = F \cdot a$$

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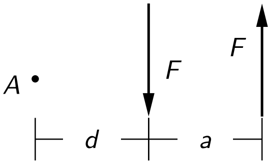

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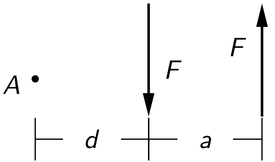

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When you write the moment with respect to a generic point, the distance from one of the forces enters both contributions with different signs and it is cancelled. a is the *arm* of the couple — by extension also the signed distance between the line of action of a force and a point is the moment arm of the force.

Equilibrium

- Particle
- Rigid Member
- Applied Forces and Reactive Forces
- Static Analysis

Equilibrium of a Particle

A *particle* is an ideal body with a mass but no dimensions, or if you want a *material point*.
Only forces that concur in the particle have an effect on it.
To have equilibrium the resultant of these vector forces must be equal 0, or in terms of Cartesian components,

$$\sum F_x = 0, \qquad \sum F_y = 0.$$

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No consideration of rotational equilibrium because all the forces are concurrent and all their moment arms are hence equal to zero...

Equilibrium of a Rigid Member

A planar rigid member can translate in two directions and can rotate.
For a rigid body the equilibrium conditions seen for a particle must be complemented with a rotational equilibrium

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> *The moments $M_{A,i} = F_i \cdot d_i$ are quantities with a sign, so that their sum, comprising positive and negative terms, can be equal to zero — we have already discussed the conventions of sign for the moments.*

Equilibrium of a Rigid Member

I would like to mention that the equation of equilibrium for a planar rigid body can be written also requiring that the moments of the forces acting on the member with respect to 3 different points are zero

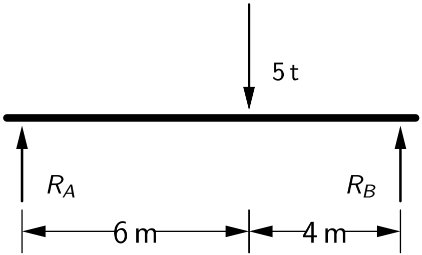
$$\sum M_A = 0, \qquad \sum M_B = 0, \qquad \sum M_C = 0$$

with the additional requirement that A, B and C are not aligned.
Translational equilibrium can be seen like writing the moment with respect to a point at an infinite distance in the direction perpendicular to the direction of translation,

$$F_{1x} - F_{2x} = F_{1x} \cdot \infty + F_{2x} \cdot (-\infty).$$

Equilibrium of a Rigid Member — Example

R_A and R_B are unknowns.



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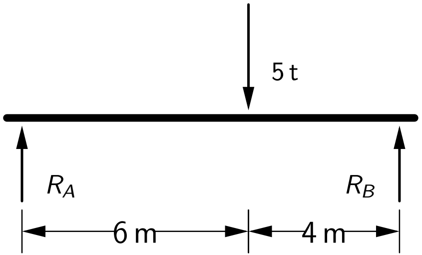

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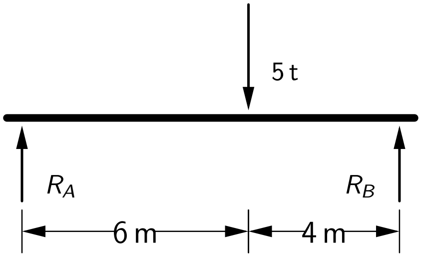

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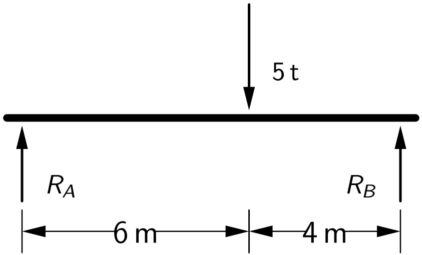

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Substituting in the vertical equilibrium, $R_A + 3\,\mathrm{t} = 5\,\mathrm{t} \Rightarrow R_A = 2\,\mathrm{t}$.

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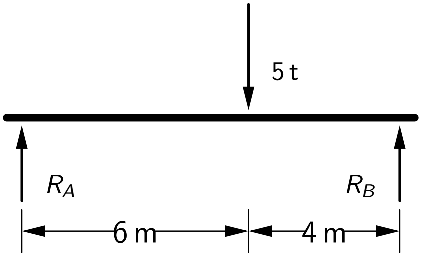

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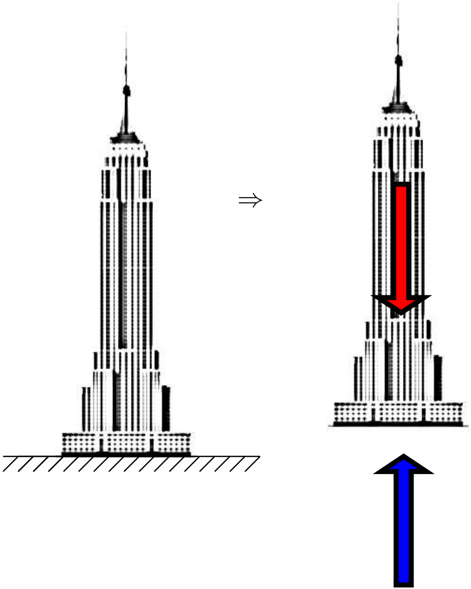
Substituting in the vertical equilibrium, $R_A + 3\,\mathrm{t} = 5\,\mathrm{t} \Rightarrow R_A = 2\,\mathrm{t}$.

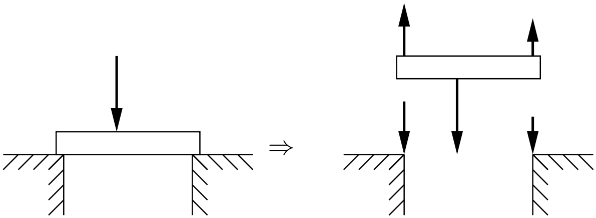
Alternatively, moment about B: $-10R_A + 20P = 0 \Rightarrow R_A = 2\,\mathrm{t}$.



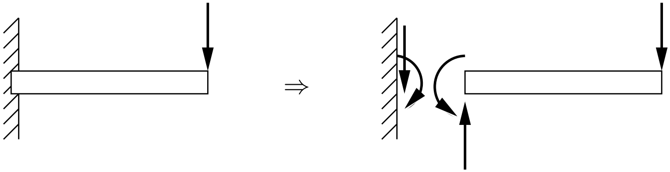
Let's start with a (large) object in equilibrium (at the left) that has a (large) weight — the red arrow — for the equilibrium of the body we must introduce another force — the blue arrow — that balances the weight of the building.

- The red arrow force is the *applied force*.
- The blue arrow force is the *reactive force*.

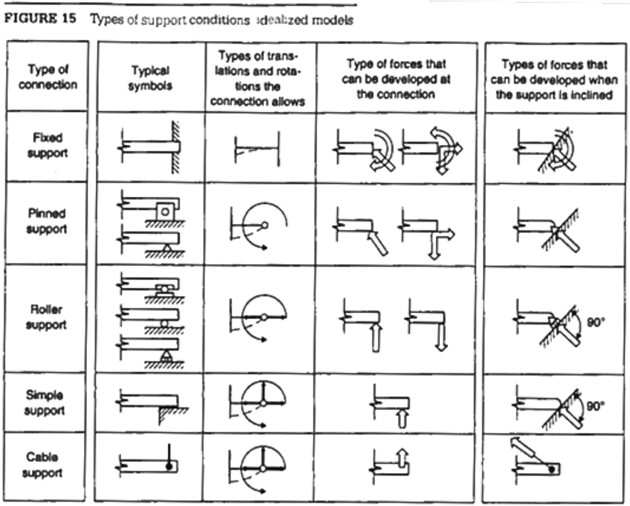
Supports:



Fixed end:



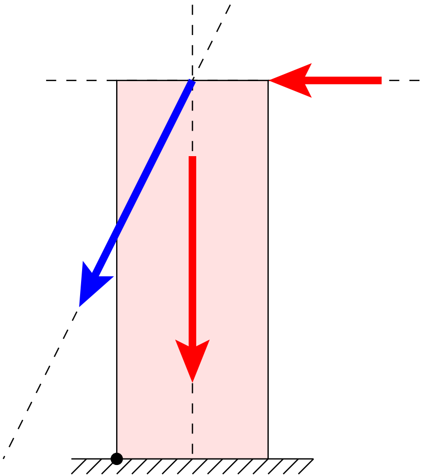
Support Classification



from *Structures*, 7th ed., Schodeck & Bechtold.

It's blackboard time...

Overturning



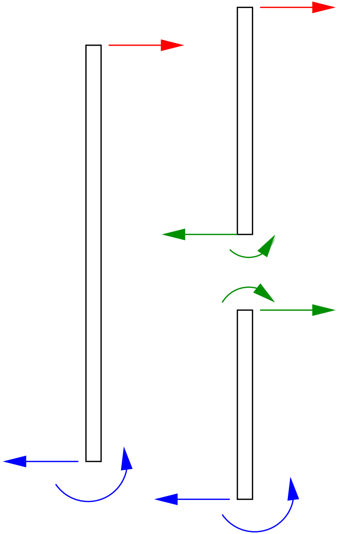
To simplify the problem and increase the margin of safety, we assume that

- the conctact between the rigid block and the supporting surface is *monolateral*, that is there is no bonding between the body and the support,
- the support can equilibrate any compressive force, applied to the body, whose line of action passes through the base of the body.

Under these hypotheses we have equilibrium, under the self weight and a lateral force, until the resultant passes into the base, if it gets outside of the base (see figure) the moment w/r to the corner will *lift* the body: **overturning**.

Internal Forces

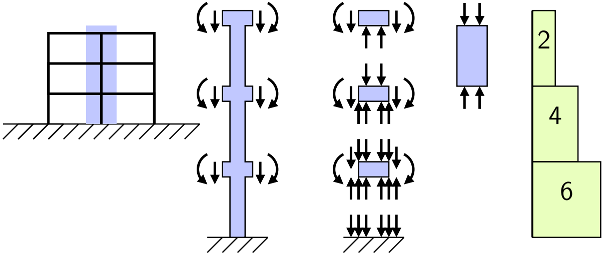
Applied forces are equilibrated by reactive forces. We extend this idea to the parts of a structure. Consider a cross section, such that a part of the structure is a free body subjected to applied and reactive loads — to have equilibrium of the free part the remaining part of the structure has to contrast with a force the sum of applied and external loads acting on the free part — and by the principle of reaction the free part transmits the reusultant to the other part of the structure.



Internal Forces

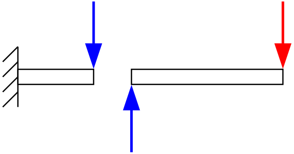
- Axial Forces
- Shear and Bending Moment
- Distribution of Shear and Moments
- Relations among Load, Shear and Moment

Consider the central column of a frame structure, you can assume that at each floor the beams transmit to the column the reactions that equilibrate their loads.
A further assumption: vertical loads are equal and the moments are equal and contrary, so that they cancel.

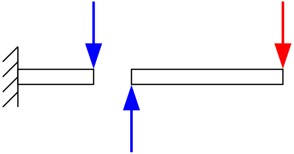


Starting from the top, our free body is the beam-column node and we need, for its equilibrium, a force directed along the column axis, hence an *axial force*, of value 2. This force, upwards into the node, is balanced by an axial force, downwards into the column segment between nodes; if we consider the segment as a free body we see that, for its equilibrium, it has to exchange an axial force of value 2 with the mid node.
So you have to take into account the axial force in the top column when you write the equilibrium for the mid node, etc etc.

We have neglected the column's self weight.



A force applied along the axis of a member is equilibrated by an axial force, a force applied transversally causes a *shear force*, that in any cross section equilibrates the transverse force.



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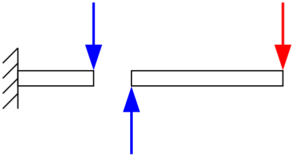
If we consider the free body loaded by the force and the shear, we recognize that is subjected to a *couple* and it is not in equilibrium, to restore equilibrium we need to apply a *moment* that acts on the cross-section.



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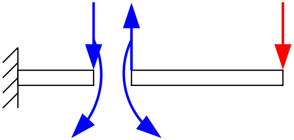

Structural Design is...

The internal forces are associated with corresponding *modes of failure*:

- an axially loaded member can be crushed or broken,
- a sheared member can be cut in two,
- a inflexed member can fail by a localized rotation,

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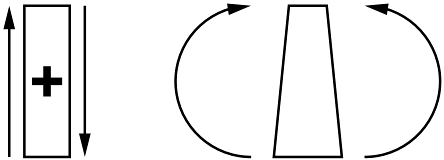
The main task of a structural designer is

- to find the structural configuration that can equilibrate *all the possible load systems* acting on a construction in an optimal way and
- to design the individual members so that their strength is always greater than the internal forces required for equilibrium.

To design a member for strength we have to know what are the internal forces it has to resist.
To know the internal forces, in every place of a member, we have to draw *diagrams of internal forces* that we can use either to find a maximum value, to design a member of constant size, or to guess a varying member shape to minimize the use of materials.

Sign Conventions

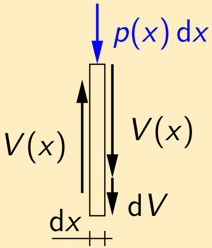
For the internal shear force in a beam, the shear is positive if, applied to a slice of beam, it rotates the slice *clockwise*.
For the internal bending moment in a horizontal beam, the moment is positive if it curves the beam elongating the bottom.
For the internal bending in a beam of generic orientation the sign is arbitrary, in Europe and other countries it is customary to represent the diagram of the moment on the elongated side of the beam.



If you consider a slice of beam in equilibrium under two equal and opposite bending moments, the elongated side is the side corresponding to the **tails** *of the moments.*

It's again blackboard time...

Load and Shear



If we consider a distributed load $p(x)$ the load on an infinitesimal slice is $p(x)\,\mathrm{d}x$ and the vertical equilibrium implies

$$V - p(x)\,\mathrm{d}x - V - \mathrm{d}V = 0 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}x} = -p(x).$$

The shear and the load distribution are related by a differential equation.

Shear and Moment

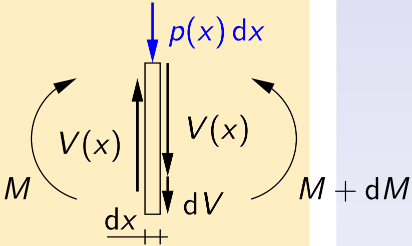
The rotation equilibrium w/r the left section is

$$-M + M + \mathrm{d}M = V\,\mathrm{d}x + \mathrm{d}^2$$

neglecting infinitesimal quantities of higher order

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V(x) \Rightarrow \frac{\mathrm{d}^2M}{\mathrm{d}x^2} = -p(x).$$

The moment and the shear are related by a differential equation.



Introduction to Stresses

Corresponding to internal forces we have internal *stresses*.
Stress, or *force intensity per unit area*, measures how a force is distributed over an area.
A force acting on a smaller area produces higher stresses, the same force acting on a larger area produces lower stresses.
The failure of structural members is due not to forces but to stresses. Knowing the stress level in a cross section you can determine if a member is safe.
Stresses are associated to *strains*, or *variation of length/deformation for unit length of material*. Strains sum up to contribute to the overall deformation of a structure.

Axial Stress

It is useful to study the stress in an axially loaded member. The axial forces that act in every cross section is not a concentrated force that acts in a point, or even in a restricted portion of the section, because this would imply that the deformations of the member are discontinuous.
The simpler distribution of stresses is a constant distribution: its resultant is $f \cdot A$ (f being the constant stress and A the area of the cross section).
For the axial equilibrium of a part, if we consider an axial load P, we eventually have

$$f \cdot A = P \Rightarrow f = \frac{P}{A}.$$

Strength of Tension Members

The mode of failure probably is pulling apart af the member at its weakest point, that in the case of a steel beam is the section interested by bolt holes.
If A is the smallest cross sectional area, or *net area* of the beam, the highest stress level is $f = P/A$.
If the material can sustain this stress level, gthe member is able to carry the load P. If we increase P also f is increasing, until the stress level exceeds the strength of the material and the memer is pulled apart.
We have reached the *failure stress level*. The failure stress level is a property of the material.
E.g., the failure stress level for certain types of steel is $235\,\mathrm{N\,mm^{-2}}$.

Design Methods

The current European approach to design is to compare two quantities, the *design strength* of a member and the *design effect* on a member, $R_d \geq E_d$

The design strength, in the simplest case, is $R_d = R_k/\gamma_M$ where R_k is a *characteristic value* of the strength, in our previous example $R_k = 235 \cdot A\,\mathrm{N}$ and γ_M ia a material safety factor (for steel, $\gamma = 1.15$).

The design effect is a combination of the effects of different actions, with different safety coefficients intervening in the definition, at the level of the amplification/reduction of actions and at the level of the combination coefficients.

Those coefficients are different for the different risks that we can deal with when we consider different types of structural failures.

> *If a particular failure is associate with large damage (high cost, loss of lives) we can accept only a low probability that the failure occurs during the life of the construction — on the other hand, if a different type of failure can be remedied with a relatively low cost, we can accept a higher probability for said minor failure.*

Combination Factors, an example

Different actions, e.g., *dead loads* and *live loads*, have different levels of uncertainity.
Dead loads are known with a greater level of confidence and enter the combination for effects leading to collapse with a smaller coefficient, typically 1.3.
Live loads are known with a smaller level of confidence and enter the combination with a larger coefficient, typically 1.5.

A steel rod is tensioned by a dead load of 100 kN ($\approx$ 5 Mercedes E Class). Design against collapse.

1. The design effect is an axial, tension force in the rod, $E_d = 1.3 \cdot 100\,\mathrm{kN} = 130\,\mathrm{kN}$.
2. The design strength is $R_d = A \cdot 235\,\mathrm{N\,mm^{-2}}/1.15 = A \cdot 204\,\mathrm{N\,mm^{-2}}$.
3. The condition $R_d = A \cdot 204\,\mathrm{N\,mm^{-2}} \geq 130\,\mathrm{kN}$ implies $A \geq 130\,000/204\,\mathrm{mm^2} = 638\,\mathrm{mm^2}$. A circular rod with $\phi = 30$ has an area $A = 706\,\mathrm{mm^2}$.

Mechanical Properties of Materials

A simple introduction to some of the properties of materials that are relevant for structural design.

Mechanical Properties of Materials

A simple introduction to some of the properties of materials that are relevant for structural design.

- Load Deformation Properties of Materials
- Elastic Behavior
- Strength
- Other Properties
- Deformations in Axially Loaded Members

When we apply a load to a member it changes its shape and dimensions. It is deformed. For many materials (e.g., steel) deformations can be categorized as *elastic deformations* and *plastic deformations*.

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When a member is increasingly loaded, initially the deformations are small, are proportional to the amount of loading and *they are reversible*. If you remove the cause of loading, the member returns at its original shape and dimensions.

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Plastic Behaviour

As the load increase the member material enters the *plastic range*: the stresses are so high to produce a permanent change of the internal structure of the material. When these changes had happened the original state cannot be restored exactly and the unloaded member has *permanent deformations*.

Plastic deformations are usually large w/r to elastic ones and are no more proportional to the loading values.

Large plastic deformation eventually lead the member to be pulled apart.

The deformation of a body is described in terms of *strain* (symbol ϵ), $\epsilon = \Delta S/S$ the ratio between the change in size w/r to the original size S.

Hooke's Law

The relationship between strain an stress was first postulated by Robert Hooke (XVII century), Hooke's law states that the ratio of the stress to the strain is constant.

$$f/\epsilon = \text{const} = E.$$

The constant E depends on the characteristics of the material and it's called the *modulus of elasticity* (or Young's Modulus).

Axial Strain

A member subjected to a tensile force elongates a certain amount. If L is the member length and ΔL is the change of length, the strain is $\epsilon = \Delta L/L$.

Typical E Values

For steel, $E = 206\,000\,\mathrm{N\,mm^{-2}}$, for aluminium $E = 78\,000\,\mathrm{N\,mm^{-2}}$, a typical value for concrete is $E = 30\,000\,\mathrm{N\,mm^{-2}}$ and a typical value for wood is $E = 11\,000\,\mathrm{N\,mm^{-2}}$. Beware that for concrete and woof there is a large variability of E,

Strength is used to refer to the load-bearing capacity of a material.
Different materials behave differently when they reach their strength limit.
Steel, when experimentally studied in a load-testing machine exhibits a *yelding stress*, that is at a certain stress level the strain increase without a substantial increase of the load.

When the strain exceeds a certain level, the load level increases again until it reaches a maximum value, the so called *ultimate strength*.
After the ultimate strength the steel deforms extremely fast, its section diminishes (necking) and finally pulls apart (ruptures).