Multi Degrees of Freedom Systems
MDOF

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Outline

Section 1

Introductory Remarks

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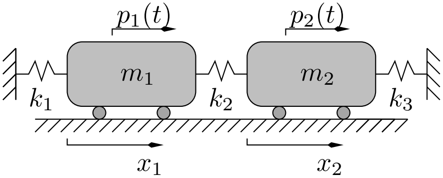
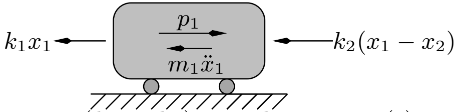
Consider an undamped system with two masses and two degrees of freedom.

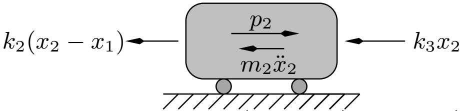


Introductory Remarks

We can separate the two masses, single out the spring forces and, using the D'Alembert Principle, the inertial forces and, finally. write an equation of dynamic equilibrium for each mass.



$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = p_1(t)$$



$$m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = p_2(t)$$

The equation of motion of a 2DOF system

With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = p_1(t), \\ m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

The equation of motion of a 2DOF system

Introducing the loading vector $\boldsymbol{p}$, the vector of inertial forces $\boldsymbol{f}_I$ and the vector of elastic forces $\boldsymbol{f}_S$,

$$\boldsymbol{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \boldsymbol{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \boldsymbol{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\boldsymbol{f}_I + \boldsymbol{f}_S = \boldsymbol{p}(t).$$

$\boldsymbol{f}_S = \boldsymbol{K}\,\boldsymbol{x}$

It is possible to write the linear relationship between $\boldsymbol{f}_S$ and the vector of displacements $\boldsymbol{x} = \left\{x_1 x_2\right\}^T$ in terms of a matrix product, introducing the so called *stiffness matrix* $\boldsymbol{K}$.

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In our example it is

$$\boldsymbol{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \boldsymbol{x} = \boldsymbol{K}\,\boldsymbol{x}$$

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The stiffness matrix $\boldsymbol{K}$ has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.
The stiffness matrix $\boldsymbol{K}$ is hence a *square matrix* $\underset{\text{ndof}\times\text{ndof}}{\boldsymbol{K}}$

$\boldsymbol{f}_I = \boldsymbol{M}\,\ddot{\boldsymbol{x}}$

Analogously, introducing the *mass matrix* $\boldsymbol{M}$ that, for our example, is

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\boldsymbol{f}_I = \boldsymbol{M}\,\ddot{\boldsymbol{x}}.$$

Also the mass matrix $\boldsymbol{M}$ is a square matrix, with number of rows and columns equal to the number of *DOF*'s.

Matrix Equation

Finally it is possible to write the equation of motion in matrix format:

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Of course it is possible to take into consideration also the damping forces, taking into account the velocity vector $\dot{\boldsymbol{x}}$ *and introducing a damping matrix* $\boldsymbol{C}$ *too, so that we can eventually write*

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$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t).$$

But today we are focused on undamped systems...

Properties of $\boldsymbol{K}$

- $\boldsymbol{K}$ is symmetrical.
  The elastic force exerted on mass i due to an unit displacement of mass j, $f_{S,i} = k_{ij}$ is equal to the force k_{ji} exerted on mass j due to an unit diplacement of mass i, in virtue of *Betti's theorem* (also known as Maxwell-Betti reciprocal work theorem).

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- $\boldsymbol{K}$ is a positive definite matrix.
  The strain energy V for a discrete system is

  $$V = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{f}_S,$$

  and expressing $\boldsymbol{f}_S$ in terms of $\boldsymbol{K}$ and $\boldsymbol{x}$ we have

  $$V = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{K}\, \boldsymbol{x},$$

  and because the strain energy is positive for $\boldsymbol{x} \neq \boldsymbol{0}$ it follows that $\boldsymbol{K}$ is definite positive.

Properties of M

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive.
Both the mass and the stiffness matrix are symmetrical and definite positive.

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Note that the kinetic energy for a discrete system can be written

$$T = \frac{1}{2} \dot{\boldsymbol{x}}^T \boldsymbol{M} \, \dot{\boldsymbol{x}}.$$

Generalisation of previous results

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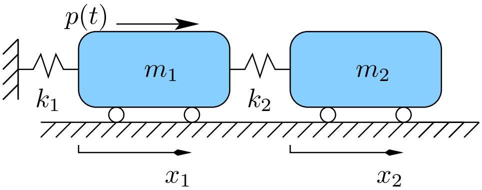
Generalisation of previous results

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with two main exceptions.

1. For a general structural system, in which not all DOFs are related to a mass, $\boldsymbol{M}$ could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy is zero.
2. For a general structural system subjected to axial loads, due to the presence of *geometrical stiffness* it is possible that for some particular displacement vector the strain energy is zero and $\boldsymbol{K}$ is *semi-definite* positive.

Graphical statement of the problem



$$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = 1m;$$
$$p(t) = p_0 \sin \omega t.$$

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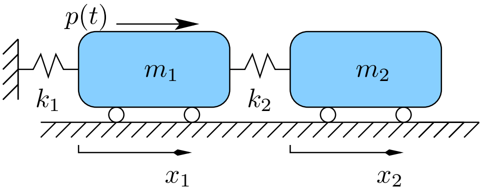

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The equations of motion

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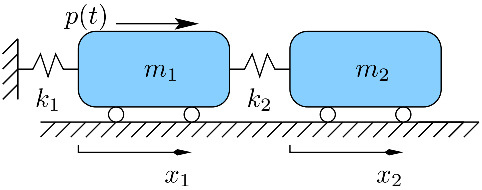

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... but we prefer the matrix notation ...

The steady state solution

We prefer the matrix notation because we can find the steady-state response of a *SDOF* system *exactly* as we found the s-s solution for a SDOF system.
Substituting $\boldsymbol{x}(t) = \boldsymbol{\xi} \sin\omega t$ in the equation of motion and simplifying $\sin\omega t$,

$$k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{\xi} - m\omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\xi} = p_0 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

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$$\left(\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \beta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \boldsymbol{\xi} = \begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \boldsymbol{\xi} = \Delta_{\mathsf{st}} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

The steady state solution

The determinant of the matrix of coefficients is

$$\text{Det} = 2\beta^4 - 5\beta^2 + 2$$

but we want to write the polynomial in β in terms of its roots

$$\text{Det} = 2 \times (\beta^2 - 1/2) \times (\beta^2 - 2).$$

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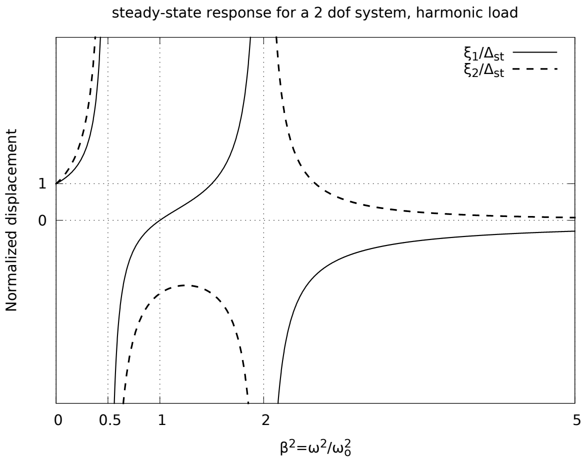
Solving for $\boldsymbol{\xi}/\Delta_{\text{st}}$ in terms of the inverse of the coefficient matrix gives

$$\begin{aligned}\frac{\boldsymbol{\xi}}{\Delta_{\text{st}}} &= \frac{1}{2(\beta^2-\frac{1}{2})(\beta^2-2)}\begin{bmatrix}1-\beta^2 & 1\\ 1 & 3-2\beta^2\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix}\\ &= \frac{1}{2(\beta^2-\frac{1}{2})(\beta^2-2)}\begin{Bmatrix}1-\beta^2\\ 1\end{Bmatrix}.\end{aligned}$$

The solution, graphically



Comment to the Steady State Solution

The steady state solution is

$$\boldsymbol{x}_{\text{s-s}} = \Delta_{\text{st}} \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix} \sin \omega t.$$

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> *We know how to compute a particular integral for a MDOF system (at least for a harmonic loading), what do we miss to be able to determine the integral of motion?*

Section 2

The Homogeneous Problem

Homogeneous equation of motion

To understand the behaviour of a *MDOF* system, we have to study the homogeneous solution.
Let's start writing the homogeneous equation of motion,

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The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* $\boldsymbol{\psi}$:

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$$\boldsymbol{x}(t) = \boldsymbol{\psi}(A\sin\omega t + B\cos\omega t).$$

Substituting in the equation of motion, we have

$$\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)\boldsymbol{\psi}(A\sin\omega t + B\cos\omega t) = \boldsymbol{0}$$

Eigenvalues

The previous equation must hold for every value of t, so it can be simplified removing the time dependency:

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This is a homogeneous linear equation, with unknowns ψ_i and the coefficients that depends on the parameter ω^2.

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Speaking of homogeneous systems, we know that

- there is always a *trivial solution*, $\boldsymbol{\psi} = \boldsymbol{0}$, and
- *non-trivial solutions* are possible if the determinant of the matrix of coefficients is equal to zero,

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$$\det\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) = 0$$

The *eigenvalues* of the *MDOF* system are the values of ω^2 for which the above equation (the *equation of frequencies*) is verified or, in other words, the frequencies of vibration associated with the shapes for which

$$\boldsymbol{K}\boldsymbol{\psi} \sin\omega t = \omega^2 \boldsymbol{M}\boldsymbol{\psi} \sin\omega t.$$

Eigenvalues, cont.

For a system with N degrees of freedom the expansion of $\det\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)$ is an algebraic polynomial of degree N in ω^2.
A polynomial of degree N has exactly N roots, either real or complex conjugate.

In Dynamics of Structures those roots ω_i^2, $i = 1, \ldots, N$ are all real because the structural matrices are symmetric matrices.
Moreover, if both $\boldsymbol{K}$ and $\boldsymbol{M}$ are positive definite matrices (a condition that is always satisfied by stable structural systems) all the roots, all the *eigenvalues*, are strictly positive:

$$\omega_i^2 \geq 0, \qquad \text{for } i = 1, \ldots, N.$$

Eigenvectors

Substituting one of the N roots ω_i^2 in the characteristic equation,

$$\left(\boldsymbol{K} - \omega_i^2 \boldsymbol{M}\right) \boldsymbol{\psi}_i = \boldsymbol{0}$$

the resulting system of $N-1$ linearly independent equations can be solved (except for a scale factor) for $\boldsymbol{\psi}_i$, the eigenvector corresponding to the eigenvalue ω_i^2.

Eigenvectors

The scale factor being arbitrary, you have to choose (arbitrarily) the value of one of the components and compute the values of all the other $N-1$ components using the $N-1$ linearly indipendent equations.

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It is common to impose to each eigenvector a *normalisation with respect to the mass matrix*, so that

$$\psi_i^T \boldsymbol{M}\, \psi_i = m$$

where m represents the unit mass.

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where m represents the unit mass.

Please understand clearly that, substituting **different eigenvalues** *in the equation of free vibrations, you have* **different linear systems**, *leading to* **different eigenvectors**.

Initial Conditions

The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\boldsymbol{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t).$$

In the general integral there are $2N$ unknown *constants of integration*, that must be determined in terms of the initial conditions.

Initial Conditions

Usually the initial conditions are expressed in terms of initial displacements and initial velocities $\boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\boldsymbol{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \boldsymbol{x}_0, \qquad \dot{\boldsymbol{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\boldsymbol{x}}_0.$$

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The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$\sum_{i=1}^{N} \psi_{ji} B_i = x_{0,j}, \qquad \sum_{i=1}^{N} \psi_{ji} \omega_i A_i = \dot{x}_{0,j}, \qquad j = 1, \ldots, N.$$

Orthogonality - 1

Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2, and write the characteristic equation for each eigenvalue:

$$\boldsymbol{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{M}\,\boldsymbol{\psi}_r$$
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premultiply each equation member by the transpose of the *other* eigenvector

$$\begin{aligned}\boldsymbol{\psi}_s^T\boldsymbol{K}\,\boldsymbol{\psi}_r &= \omega_r^2\boldsymbol{\psi}_s^T\boldsymbol{M}\,\boldsymbol{\psi}_r\\ \boldsymbol{\psi}_r^T\boldsymbol{K}\,\boldsymbol{\psi}_s &= \omega_s^2\boldsymbol{\psi}_r^T\boldsymbol{M}\,\boldsymbol{\psi}_s\end{aligned}$$

Orthogonality - 2

The term $\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r$ is a scalar, hence

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r = \left(\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r\right)^T = \boldsymbol{\psi}_r^T \boldsymbol{K}^T\, \boldsymbol{\psi}_s$$

but $\boldsymbol{K}$ is symmetrical, $\boldsymbol{K}^T = \boldsymbol{K}$ and we have

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \boldsymbol{K}\, \boldsymbol{\psi}_s.$$

By a similar derivation

$$\boldsymbol{\psi}_s^T \boldsymbol{M}\, \boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s.$$

Orthogonality - 3

Substituting our last identities in the previous equations, we have

$$\boldsymbol{\psi}_r^T \boldsymbol{K}\, \boldsymbol{\psi}_s = \omega_r^2 \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s$$
$$\boldsymbol{\psi}_r^T \boldsymbol{K}\, \boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s$$

subtracting member by member we find that

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We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s = 0, \qquad \text{for } r \neq s.$$

Orthogonality - 4

The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \boldsymbol{M}\, \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

Orthogonality - 4

The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \boldsymbol{M}\, \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_i$$

and consequently

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M_i is the *modal mass* associated with mode no. i while $K_i \equiv \omega_i^2 M_i$ is the respective *modal stiffness*.

Section 3

Modal Analysis

Eigenvectors are a base

The eigenvectors are linearly independent, so for every vector $\boldsymbol{x}$ we can write

$$\boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j .$$

The coefficients are readily given by premultiplication of $\boldsymbol{x}$ by $\boldsymbol{\psi}_i^T \boldsymbol{M}$, because

$$\boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_j q_j = \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_i q_i = M_i q_i$$

in virtue of the ortogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_j = \frac{\boldsymbol{\psi}_j^T \boldsymbol{M}\, \boldsymbol{x}}{M_j}.$$

Eigenvectors are a base

Generalising our results for the displacement vector to the acceleration vector and expliciting the time dependency, it is

$$\boldsymbol{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\boldsymbol{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t),$$

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t), \qquad \ddot{x}_i(t) = \sum_{j=1}^{N} \psi_{ij} \ddot{q}_j(t).$$

Introducing $\boldsymbol{q}(t)$, the *vector of modal coordinates* and $\boldsymbol{\Psi}$, the *eigenvector matrix*, whose columns are the eigenvectors, we can write

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}\, \boldsymbol{q}(t), \qquad \ddot{\boldsymbol{x}}(t) = \boldsymbol{\Psi}\, \ddot{\boldsymbol{q}}(t).$$

EoM in Modal Coordinates...

Substituting the last two equations in the equation of motion,

$$\boldsymbol{M}\,\boldsymbol{\Psi}\,\ddot{\boldsymbol{q}} + \boldsymbol{K}\,\boldsymbol{\Psi}\,\boldsymbol{q} = \boldsymbol{p}(t)$$

premultiplying by $\boldsymbol{\Psi}^T$

$$\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{\Psi}\,\ddot{\boldsymbol{q}} + \boldsymbol{\Psi}^T\boldsymbol{K}\,\boldsymbol{\Psi}\,\boldsymbol{q} = \boldsymbol{\Psi}^T\boldsymbol{p}(t)$$

introducing the so called *starred matrices*, with $\boldsymbol{p}^\star(t) = \boldsymbol{\Psi}^T\boldsymbol{p}(t)$, we can finally write

$$\boldsymbol{M}^\star\,\ddot{\boldsymbol{q}} + \boldsymbol{K}^\star\,\boldsymbol{q} = \boldsymbol{p}^\star(t).$$

The vector equation above corresponds to the set of scalar equations

$$p_i^\star = \sum m_{ij}^\star \ddot{q}_j + \sum k_{ij}^\star q_j, \qquad i = 1, \ldots, N.$$

... are N independent equations!

We must examine the structure of the starred symbols.
The generic element, with indexes i and j, of the *starred* matrices can be expressed in terms of single eigenvectors,

$$\begin{aligned} m^\star_{ij} &= \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_j &&= \quad \delta_{ij} M_i, \\ k^\star_{ij} &= \boldsymbol{\psi}_i^T \boldsymbol{K}\, \boldsymbol{\psi}_j &&= \omega_i^2 \delta_{ij} M_i. \end{aligned}$$

where δ_{ij} is the *Kroneker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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Substituting in the equation of motion, with $p^\star_i = \boldsymbol{\psi}_i^T \boldsymbol{p}(t)$ we have
a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p^\star_i(t), \qquad i = 1, \ldots, N$$

Initial Conditions Revisited

The initial displacements can be written in modal coordinates,

$$\boldsymbol{x}_0 = \boldsymbol{\Psi}\,\boldsymbol{q}_0$$

and premultiplying both members by $\boldsymbol{\Psi}^T\boldsymbol{M}$ we have the following relationship:

$$\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{x}_0 = \boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{\Psi}\,\boldsymbol{q}_0 = \boldsymbol{M}^\star\boldsymbol{q}_0.$$

Premultiplying by the inverse of $\boldsymbol{M}^\star$ and taking into account that $\boldsymbol{M}^\star$ is diagonal,

$$\boldsymbol{q}_0 = (\boldsymbol{M}^\star)^{-1}\,\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{x}_0 \quad \Rightarrow \quad q_{i0} = \frac{\boldsymbol{\psi}_i^T\boldsymbol{M}\,\boldsymbol{x}_0}{M_i}$$

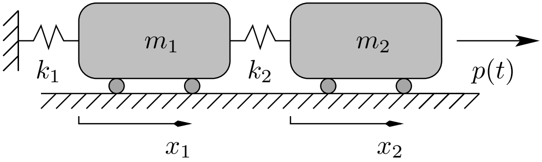
and, analogously,

$$\dot{q}_{i0} = \frac{\boldsymbol{\psi}_i{}^T\boldsymbol{M}\,\dot{\boldsymbol{x}}_0}{M_i}$$

Section 4

Examples

2 DOF System



$$k_1 = 2k, \quad k_2 = 3k; \qquad m_1 = 2m, \quad m_2 = 4m;$$
$$p(t) = p_0 \sin \omega t.$$

$$\boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \; \boldsymbol{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin \omega t,$$

$$\boldsymbol{M} = m \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \; \boldsymbol{K} = k \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}.$$

Equation of frequencies

The equation of frequencies is

$$\left\|\boldsymbol{K} - \omega^2\boldsymbol{M}\right\| = \begin{Vmatrix} 5k - 2\omega^2 m & -3k \\ -3k & 3k - 4\omega^2 m \end{Vmatrix} = 0.$$

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Developing the determinant

$$(8m^2)\,\omega^4 - (26mk)\,\omega^2 + (6k^2)\,\omega^0 = 0$$

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Developing the determinant

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Solving the algebraic equation in ω^2

$$\omega_1^2=\frac{k}{m}\frac{13-\sqrt{121}}{8}, \qquad \omega_2^2=\frac{k}{m}\frac{13+\sqrt{121}}{8};$$

$$\omega_1^2=\frac{1}{4}\frac{k}{m}, \qquad \omega_2^2=3\,\frac{k}{m}.$$

Eigenvectors

Substituting ω_1^2 for ω^2 in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k\,(5 - 2 \cdot \frac{1}{4})\psi_{11} - 3k\psi_{21} = 0$$

while substituting ω_2^2 gives

$$k\,(3 - 2 \cdot 3)\psi_{12} - 3k\psi_{22} = 0.$$

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Solving with the arbitrary assignment $\psi_{11}=\psi_{22}=1$ gives the *unnormalized* eigenvectors,

$$\boldsymbol{\psi}_1=\begin{Bmatrix}+1\\+\frac{3}{2}\end{Bmatrix},\quad \boldsymbol{\psi}_2=\begin{Bmatrix}-3\\+1\end{Bmatrix}.$$

Normalization

We compute first M_1 and M_2,

$$\begin{aligned} M_1 &= \boldsymbol{\psi}_1^T \boldsymbol{M}\, \boldsymbol{\psi}_1 \\ &= m \left\{1, \quad \tfrac{3}{2}\right\} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ \frac{3}{2} \end{Bmatrix} \\ &= m \left\{2, \quad 6\right\} \begin{Bmatrix} 1 \\ \frac{3}{2} \end{Bmatrix} = 11\, m \end{aligned}$$

and, in a similar way, we have $M_2 = 22\, m$; the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{11} = 3.317, \qquad \alpha_2 = \sqrt{22} = 4.690.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.30151 & -0.63960 \\ +0.45227 & +0.21320 \end{bmatrix}$$

Modal Loadings

The modal loading is

$$
\begin{aligned}
\boldsymbol{p}^\star(t) &= \boldsymbol{\Psi}^T\,\boldsymbol{p}(t)\\
&= p_0 \begin{bmatrix} 1 & 3/2 \\ -3 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \sin\omega t\\
&= p_0 \begin{Bmatrix} 3/2 \\ 1 \end{Bmatrix} \sin\omega t
\end{aligned}
$$

Modal EoM

Substituting its modal expansion for $\boldsymbol{x}$ into the equation of motion and premultiplying by $\boldsymbol{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} 11\,m\,\ddot{q}_1 \ + \dfrac{1}{4}\,11\,m\,\dfrac{k}{m}\,q_1 = \dfrac{3}{2}p_0\sin\omega t \\ 22\,m\,\ddot{q}_2 \ + 3\,22\,m\,\dfrac{k}{m}\,q_2 = p_0\sin\omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by M_i both equations, we have

$$\begin{cases} \ddot{q}_1 + \dfrac{1}{4}\,\omega_0^2 q_1 = {}^3\!/_2\,\dfrac{p_0}{11m}\sin\omega t \\ \ddot{q}_2 + 3\,\omega_0^2 q_2 = \dfrac{p_0}{22m}\sin\omega t \end{cases}$$

Particular Integral

We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 \left(\omega_1^2 - \omega^2\right) \sin \omega t = \frac{3}{22} \frac{p_0}{k} \frac{k}{m} \sin \omega t$$

solving for C_1

$$C_1 = \frac{3}{22} \Delta \frac{\omega_0^2}{\omega_1^2 - \omega^2}$$

and, analogously,

$$C_2 = \frac{1}{22} \Delta \frac{\omega_0^2}{\omega_2^2 - \omega^2}$$

with $\Delta = {}^{p_0}/_{k}$.

Integrals

The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin\omega_1 t + B_1 \cos\omega_1 t + C_1 \sin\omega t,\\ q_2(t) = A_2 \sin\omega_2 t + B_2 \cos\omega_2 t + C_2 \sin\omega t,\end{cases}$$

and, for a system initially at rest, it is

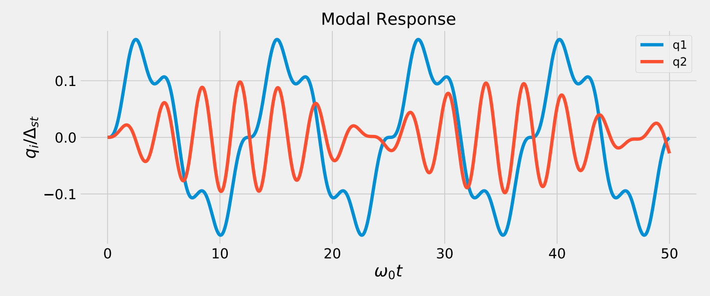
$$\begin{cases} q_1(t) = C_1 \ (\sin\omega t - \beta_1 \sin\omega_1 t)\,,\\ q_2(t) = C_2 \ (\sin\omega t - \beta_2 \sin\omega_2 t)\,,\end{cases}$$

where $\beta_i = {}^{\omega}/_{\omega_i}$
We are interested in structural degrees of freedom, too...

$$\begin{cases} x_1(t) = (\psi_{11}\, q_1(t) + \psi_{12}\, q_2(t))\\ x_2(t) = (\psi_{21}\, q_1(t) + \psi_{22}\, q_2(t))\end{cases}$$

The response in modal coordinates

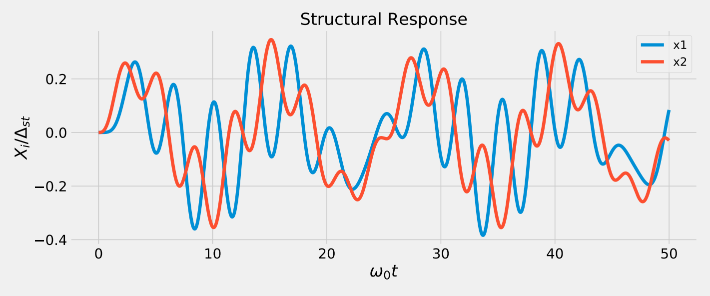
To have a feeling of the response in modal coordinates, let's say that the frequency of the load is $\omega = 2\omega_0$, hence $\beta_1 = \frac{2.0}{\sqrt{1/4}} = 4$ and $\beta_2 = \frac{2.0}{\sqrt{3}} = 1.15470$.



In the graph above, the responses are plotted against an adimensional time coordinate α with $\alpha = \omega_0 t$, while the ordinates are adimensionalised with respect to $\Delta_{\mathrm{st}} = \frac{p_0}{k}$

The response in structural coordinates

Using the same normalisation factors, here are the response functions in terms of $x_1 = \psi_{11}q_1 + \psi_{12}q_2$ and $x_2 = \psi_{21}q_1 + \psi_{22}q_2$:



The response in structural coordinates

And the displacement of the centre of mass plotted along with the difference in displacements.