

Continuous Systems

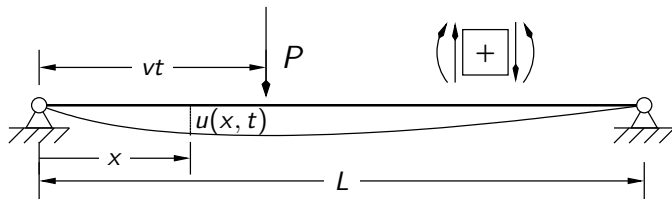
an example

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A uniform beam, (unit mass m , flexural stiffness EJ and length L) is loaded by a load P , moving with constant velocity $v(t) = v$ in the time interval $0 \leq t \leq t_0 = L/v = t_0$.

Plot the response in the interval $0 \leq t \leq t_0 = L/v$ in terms of $u(L/2, t)$ and $M_b(L/2, t)$.

NB: the beam is at rest for $t = 0$.

For a uniform beam, the equation of dynamic equilibrium is

$$m \frac{\partial^2 u(x, t)}{\partial t^2} + EJ \frac{\partial^4 u(x, t)}{\partial x^4} = p(x, t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x, t) = P \delta(x - vt).$$

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The Dirac's delta (or distribution) is defined by

$$\delta(x - x_0) \equiv 0 \quad \text{and} \quad \int f(x) \delta(x - x_0) dx = f(x_0).$$

The solution will be computed by separation of variables

$$u(x, t) = q(t)\phi(x)$$

and modal analysis,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\begin{aligned}\phi_n(x) &= \sin \beta_n x, & \beta_n &= \frac{n\pi}{L}, \\ m_n &= \frac{mL}{2}, & \omega_n^2 &= \beta_n^4 \frac{EJ}{m} = n^4 \pi^4 \frac{EJ}{mL^4}.\end{aligned}$$

For an uniform beam, the orthogonality relationships are

$$m \int_0^L \phi_n(x) \phi_m(x) dx = m_n \delta_{nm},$$

$$EJ \int_0^L \phi_n(x) \phi_m^{IV}(x) dx = k_n \delta_{nm} = m_n \omega_n^2 \delta_{nm}.$$

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(the Kroneker's δ_{nm} is a completely different thing from Dirac's δ , OK?).

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the series representation of $u(x, t)$:

$$m \sum_{m=1}^{\infty} \ddot{q}_m \phi_m + EJ \sum_{m=1}^{\infty} q_m \phi_m^{\text{IV}} = P \delta(x - vt),$$

2. every term is multiplied by ϕ_n and integrated over the lenght of the beam

$$m \int_0^L \phi_n \sum_{m=1}^{\infty} \ddot{q}_m \phi_m dx + EJ \int_0^L \phi_n \sum_{m=1}^{\infty} q_m \phi_m^{\text{IV}} dx = \\ P \int_0^L \phi_n \delta(x - vt) dx, \quad n = 1, \dots, \infty$$

3. we use the ortogonality relationships and the definition of δ ,

$$m_n \ddot{q}(t) + k_n q(t) = P \phi_n(vt) = P \sin \frac{n\pi vt}{L}, \quad n = 1, \dots, \infty.$$

Considering that

- the initial conditions are zero for all the modal equations,
- for each mode we have a *different* excitation frequency

$$\bar{\omega}_n = n\pi v/L \text{ (and also } \beta_n = \bar{\omega}_n/\omega_n),$$

the individual solutions are given by

$$q_n(t) = \frac{P}{k_n} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}$$

and, with $k_n = m_n \omega_n^2 = \frac{mL}{2} n^4 \pi^4 \frac{EJ}{mL^4} = n^4 \pi^4 \frac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2}{n^4 \pi^4} \frac{PL^3}{EJ} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}.$$

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It is apparent that we have *resonance* for $\beta_n = 1$.

Let's start from $\beta_1 = \pi v/L/\omega_1 = 1$ and solve for the velocity, say v_1

$$v_1 = \omega_1 L/\pi.$$

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Solution

**Equation of
motion**

Let's start from $\beta_1 = \pi v/L/\omega_1 = 1$ and solve for the velocity, say v_1

$$v_1 = \omega_1 L/\pi.$$

It is apparent that v_1 is a critical velocity $v_c = v_1 = \omega_1 L/\pi$ that gives a resonance condition for the first mode response, while for $v = 2 v_c$ the second mode is in resonance, etc.

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With the position $v = \kappa v_1$ it is

$$\bar{\omega}_n = \kappa n \omega_1 \quad \text{and} \quad \beta_n = n \kappa \omega_1 / n^2 \omega_1 = \kappa/n$$

and we can rewrite the solution as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin\left(\frac{\kappa}{n} \omega_n t\right) - \frac{\kappa}{n} \sin \omega_n t \right), \quad 0 \leq t \leq \frac{L}{v}.$$

Introducing an adimensional time coordinate ξ with $t = t_0 \xi$, noting that $\omega_n = n^2 \omega_1$ we can write

$$\frac{\kappa}{n} \omega_n t = \frac{\kappa}{n} n^2 \omega_1 \xi t_0 = \kappa n \left(\frac{v_c \pi}{L} \right) \xi \frac{L}{\kappa v_c} = n \pi \xi,$$

substituting in the solution for mode n we have

$$q_n(\xi) = \frac{2}{\pi^4} \frac{PL^3}{EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right), \quad 0 \leq \xi \leq 1.$$

If we denote with $\mathbb{X}(t)$ the position of the load at time t , it is $\mathbb{X}(t) = vt = \xi L$, or $\xi = \mathbb{X}/L$ and the expression $u(x, \xi) = \sum q_n(\xi) \phi_n(x)$ can be interpreted as the displacement in x when the load is positioned in ξL .

The displacement and the bending moment are given by

$$u(x, \xi) = \frac{2PL^3}{\pi^4 EJ} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\pi\frac{x}{L}\right),$$

$$\begin{aligned} M_b(x, \xi) &= -EJ \frac{\partial^2 u(x, \xi)}{\partial x^2} \\ &= \frac{2PL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\pi\frac{x}{L}\right). \end{aligned}$$

Normalized Midspan Deflection

If we consider the midspan deflection (bending moment) due to a static load P on the beam, the maximum deflection (bending moment) is expected when the load is placed at midspan, and it is

$$u_{\text{stat}}(L/2, 1/2) = \frac{PL^3}{48EJ} \quad \text{and} \quad M_{\text{b stat}}(L/2, 1/2) = \frac{PL}{4}.$$

Normalizing the midspan displacement with respect to the maximum static displacement, we write

$$\Delta(\xi) = \frac{u}{u_{\text{stat}}} = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right).$$

Eventually we introduce a notation for the partial sum of the first N terms:

$$\Delta_N(\xi) = \frac{96}{\pi^4} \sum_{n=1}^N \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right).$$

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Normalized Midspan Bending Moment

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Analogously, normalizing with respect to the maximum static bending moment, it is

$$\mu(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right),$$

the partial sum being denoted by

$$\mu_N(\xi) = \frac{8}{\pi^2} \sum_{n=1}^N \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right).$$

Error Estimates

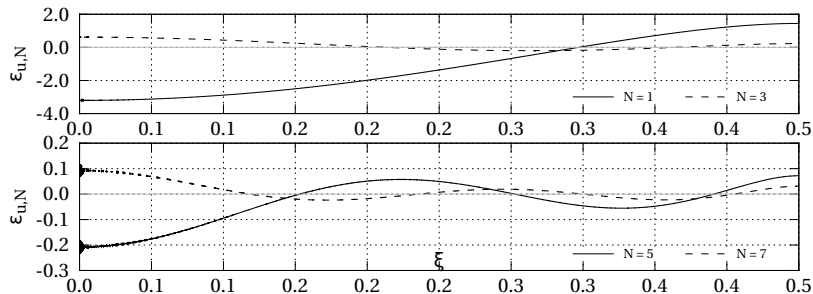
To appreciate the approximation inherent in a truncated series, we compare the truncated series computed for $\kappa = 10^{-6}$ with the static response $\Delta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$ introducing a percent error function

$$\epsilon_{u,N}(\xi) = 100 \left(1 - \frac{\Delta_N(\xi)|_{\kappa=10^{-6}}}{\Delta_{\text{stat}}(\xi)} \right) \quad \text{for } 0 \leq \xi \leq 1/2,$$

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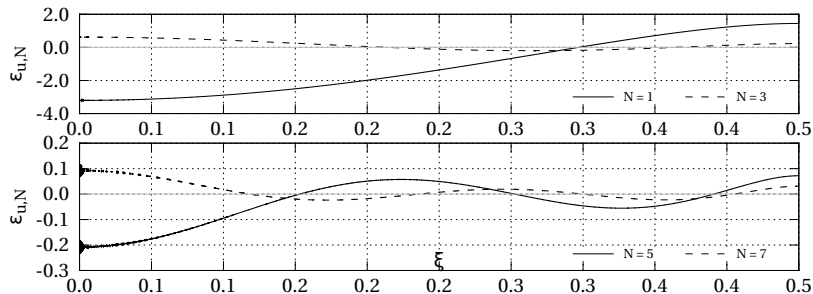
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Using 4 terms ($N = 7$) the absolute error is not greater than $1/1000$.

Error Estimates

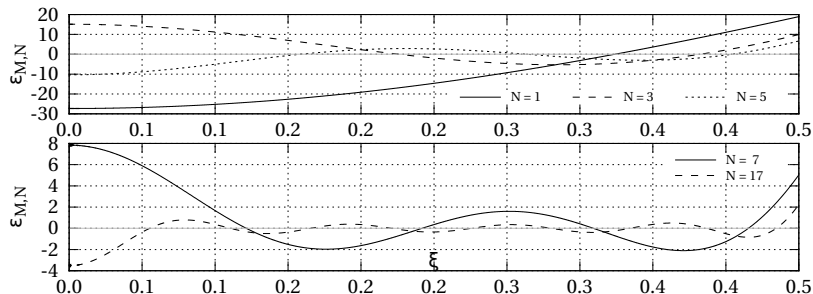
Analogously we can use the midspan bending moment, normalized with respect to $PL/4$, $\mu_{\text{stat}}(\xi) = 2\xi$ to define another percent error function

$$\epsilon_{M,N} = 100 \left(1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)} \right)$$

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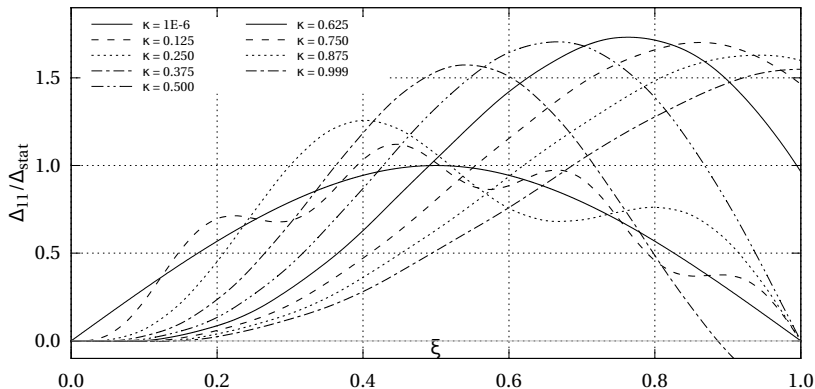
Equation of
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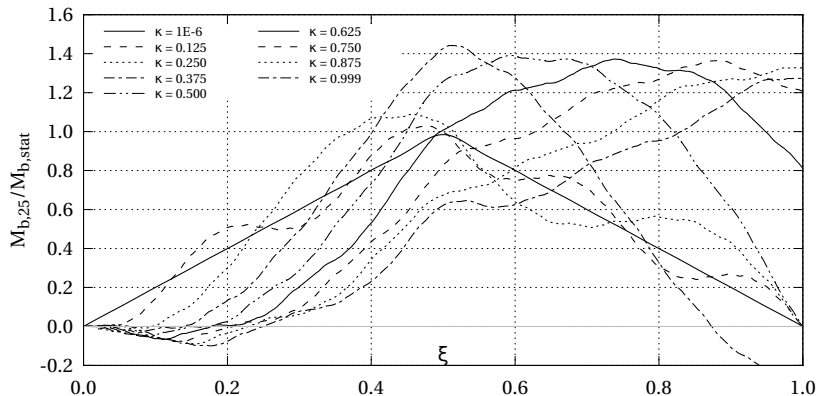
With 8 terms ($N = 17$) terms in the series, still the absolute error is greater than 3%.

Eventually, we plot the normalized displacement and the normalized bending moment for different values of κ , i.e., for different velocities.

For the displacement I used $N = 11$ while for the bending moment I used $N = 25$.



Normalized Midspan Displacement.
(for different velocities $v = \kappa v_c$)



Normalized Midspan Bending Moment.
(for different velocities $v = \kappa v_c$)