Continuous Systems

an example

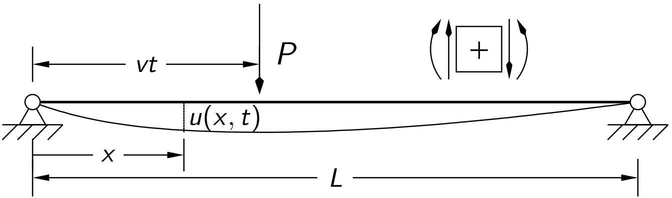
Giacomo Boffi

http://intranet.dica.polimi.it/people/boffi-giacomo

Dipartimento di Ingegneria Civile Ambientale e Territoriale
Politecnico di Milano

April 12, 2018

Problem statement



A uniform beam, (unit mass m, flexural stiffness EJ and length L) is loaded by a load P , moving with constant velocity $v(t) = v$ in the time interval $0 \le t \le t_0 = {}^L\!/_v = t_0$.

Plot the response in the interval $0 \le t \le t_0 = {}^L\!/_v$ in terms of $u({}^L\!/_2, t)$ and $M_\mathrm{b}({}^L\!/_2, t)$.

NB: the beam is at rest for $t = 0$.

Equation of motion

F or an uniform beam, the equation of dynamic equilibrium is

$$m\,\frac{\partial^2 u(x,t)}{\partial t^2} + EJ\,\frac{\partial^4 u(x,t)}{\partial x^4} = p(x,t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x,t) = P\,\delta(x - vt).$$

Equation of motion

F or an uniform beam, the equation of dynamic equilibrium is

$$m\,\frac{\partial^2 u(x,t)}{\partial t^2} + EJ\,\frac{\partial^4 u(x,t)}{\partial x^4} = p(x,t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x,t) = P\,\delta(x - vt).$$

The Dirac's delta (or distribution) is defined by

$$\delta(x - x_0) \equiv 0 \qquad \textit{and} \qquad \int f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0).$$

Equation of motion

The solution will be computed by separation of variables

$$u(x, t) = q(t)\phi(x)$$

and modal analysis,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\phi_n(x) = \sin \beta_n x, \qquad \beta_n = \frac{n\pi}{L},$$

$$m_n = \frac{mL}{2}, \qquad \omega_n^2 = \beta_n^4 \frac{EJ}{m} = n^4\pi^4 \frac{EJ}{mL^4}.$$

Orthogonality relationships

For an uniform beam, the orthogonality relationships are

$$m \int_0^L \phi_n(x)\phi_m(x)\,\mathrm{d}x = m_n\delta_{nm},$$

$$EJ \int_0^L \phi_n(x)\phi_m^{\mathrm{IV}}(x)\,\mathrm{d}x = k_n\delta_{nm} = m_n\omega_n^2\delta_{nm}.$$

Orthogonality relationships

For an uniform beam, the orthogonality relationships are

$$m \int_0^L \phi_n(x)\phi_m(x)\,\mathrm{d}x = m_n\delta_{nm},$$
$$EJ \int_0^L \phi_n(x)\phi_m^{\mathrm{IV}}(x)\,\mathrm{d}x = k_n\delta_{nm} = m_n\omega_n^2\delta_{nm}.$$

(the Kroneker's δ_{nm} is a completely different thing from Dirac's δ, OK?).

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the series representation of $u(x, t)$:

$$m \sum_{m=1}^{\infty} \ddot{q}_m \phi_m + EJ \sum_{m=1}^{\infty} q_m \phi_m^{\text{IV}} = P\, \delta(x - vt),$$

2. every term is multiplied by ϕ_n and integrated over the lenght of the beam

$$m \int_0^L \phi_n \sum_{m=1}^{\infty} \ddot{q}_m \phi_m \,\mathrm{d}x + EJ \int_0^L \phi_n \sum_{m=1}^{\infty} q_m \phi_m^{\text{IV}} \,\mathrm{d}x = \\ P \int_0^L \phi_n \delta(x - vt), \qquad n = 1, \ldots, \infty$$

3. we use the ortogonality relationships and the definition of δ,

$$m_n \ddot{q}(t) + k_n q(t) = P\, \phi_n(vt) = P \sin \frac{n\pi\, vt}{L}, \qquad n = 1, \ldots, \infty.$$

Solutions

Considering that

- the initial conditions are zero for all the modal equations,
- for each mode we have a *different* excitation frequency $\overline{\omega}_n = n\pi v/L$ (and also $\beta_n = \overline{\omega}_n/\omega_n$),

the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}$$

and, with $k_n = m_n\omega_n^2 = \frac{mL}{2}\, n^4\pi^4\frac{EJ}{mL^4} = n^4\pi^4\frac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2}{n^4\pi^4}\,\frac{PL^3}{EJ}\,\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}.$$

Solutions

Considering that

- the initial conditions are zero for all the modal equations,
- for each mode we have a *different* excitation frequency $\overline{\omega}_n = n\pi v/L$ (and also $\beta_n = \overline{\omega}_n/\omega_n$),

the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}$$

and, with $k_n = m_n\omega_n^2 = \dfrac{mL}{2}\, n^4\pi^4 \dfrac{EJ}{mL^4} = n^4\pi^4\dfrac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2}{n^4\pi^4}\frac{PL^3}{EJ}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}.$$

It is apparent that we have *resonance* for $\beta_n = 1$.

Critical Velocity

Let's start from $\beta_1 = {}^{\pi v/L}\!/\!_{\omega_1} = 1$ and solve for the velocity, say v_1

$$v_1 = {}^{\omega_1 L}\!/\!_{\pi}.$$

Critical Velocity

Let's start from $\beta_1 = {}^{\pi v/L}\!/\!_{\omega_1} = 1$ and solve for the velocity, say v_1

$$v_1 = {}^{\omega_1 L}\!/\!_{\pi}.$$

It is apparent that v_1 is a critical velocity $v_\text{c} = v_1 = \omega_1 L/\pi$ that gives a resonance condition for the first mode response, while for $v = 2\,v_\text{c}$ the second mode is in resonance, etc.

Critical Velocity

Let's start from $\beta_1 = {}^{\pi v/L}/_{\omega_1} = 1$ and solve for the velocity, say v_1

$$v_1 = {}^{\omega_1 L}/_{\pi}.$$

It is apparent that v_1 is a critical velocity $v_c = v_1 = \omega_1 L/\pi$ that gives a resonance condition for the first mode response, while for $v = 2\,v_c$ the second mode is in resonance, etc.
With the position $v = \kappa v_1$ it is

$$\overline{\omega}_n = \kappa n \omega_1 \quad \text{and} \quad \beta_n = n\kappa\omega_1/n^2\omega_1 = {}^{\kappa}/_{n}$$

and we can rewrite the solution as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ}\,\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(\frac{\kappa}{n}\omega_n t) - \frac{\kappa}{n}\sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}.$$

Adimensional Time Coordinate

Introducing an adimensional time coordinate ξ with $t = t_0\xi$, noting that $\omega_n = n^2\omega_1$ we can write

$$\frac{\kappa}{n}\omega_n t = \frac{\kappa}{n} n^2\omega_1\,\xi\,t_0 = \kappa n(\frac{v_c\pi}{L})\xi\frac{L}{\kappa v_c} = n\pi\xi,$$

substituting in the solution for mode n we have

$$q_n(\xi) = \frac{2}{\pi^4}\,\frac{PL^3}{EJ}\,\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right), \qquad 0 \le \xi \le 1.$$

Adimensional Time and Adimensional Position

If we denote with $\mathbb{X}(t)$ the position of the load at time t, it is $\mathbb{X}(t) = vt = \xi L$, or $\xi = {}^{\mathbb{X}}/_{L}$ and the expression $u(x, \xi) = \sum q_n(\xi)\phi_n(x)$ can be interpreted as the displacement in x when the load is positioned in ξL.

Displacement and Bending Moment

The displacement and the bending moment are given by

$$u(x,\xi) = \frac{2PL^3}{\pi^4 EJ} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right) \sin(n\pi\frac{x}{L}),$$

$$\begin{aligned} M_\mathrm{b}(x,\xi) &= -EJ\frac{\partial^2 u(x,\xi)}{\partial x^2} \\ &= \frac{2PL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2-\kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right) \sin(n\pi\frac{x}{L}). \end{aligned}$$

Normalized Midspan Deflection

If we consider the midspan deflection (bending moment) due to a static load P on the beam, the maximum deflection (bending moment) is expected when the load is placed at midspan, and it is

$$u_{\text{stat}}(L/2, 1/2) = \frac{PL^3}{48EJ} \qquad \text{and} \qquad M_{\text{b stat}}(L/2, 1/2) = \frac{PL}{4}.$$

Normalizing the midspan displacement with respect to the maximum static displacement, we write

$$\Delta(\xi) = \frac{u}{u_{\text{stat}}} = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\frac{\pi}{2}).$$

Eventually we introduce a notation for the partial sum of the first N terms:

$$\Delta_N(\xi) = \frac{96}{\pi^4} \sum_{n=1}^{N} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\frac{\pi}{2}).$$

Normalized Midspan Deflection

If we consider the midspan deflection (bending moment) due to a static load P on the beam, the maximum deflection (bending moment) is expected when the load is placed at midspan, and it is

$$u_{\text{stat}}(L/2, 1/2) = \frac{PL^3}{48EJ} \qquad \text{and} \qquad M_{\text{b stat}}(L/2, 1/2) = \frac{PL}{4}.$$

Normalizing the midspan displacement with respect to the maximum static displacement, we write

$$\Delta(\xi) = \frac{u}{u_{\text{stat}}} = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right) \sin(n\frac{\pi}{2}).$$

Eventually we introduce a notation for the partial sum of the first N terms:

$$\Delta_N(\xi) = \frac{96}{\pi^4} \sum_{n=1}^{N} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right) \sin(n\frac{\pi}{2}).$$

Normalized Midspan Deflection

If we consider the midspan deflection (bending moment) due to a static load P on the beam, the maximum deflection (bending moment) is expected when the load is placed at midspan, and it is

$$u_{\text{stat}}(L/2, 1/2) = \frac{PL^3}{48EJ} \qquad \text{and} \qquad M_{\text{b stat}}(L/2, 1/2) = \frac{PL}{4}.$$

Normalizing the midspan displacement with respect to the maximum static displacement, we write

$$\Delta(\xi) = \frac{u}{u_{\text{stat}}} = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\frac{\pi}{2}).$$

Eventually we introduce a notation for the partial sum of the first N terms:

$$\Delta_N(\xi) = \frac{96}{\pi^4} \sum_{n=1}^{N} \frac{1}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\frac{\pi}{2}).$$

Normalized Midspan Bending Moment

Analogously, normalizing with respect to the maximum static bending moment, it is

$$\mu(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right) \sin(n\frac{\pi}{2}),$$

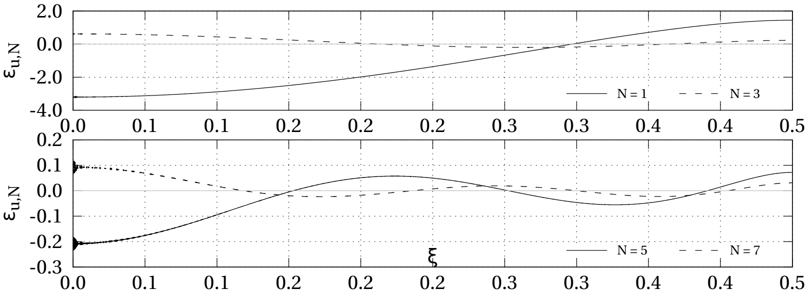
the partial sum being denoted by

$$\mu_N(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right) \sin(n\frac{\pi}{2}).$$

Error Estimates

To appreciate the approximation inherent in a truncated series, we compare the truncated series computed for $\kappa = 10^{-6}$ with the static response $\Delta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$ introducing a percent error function

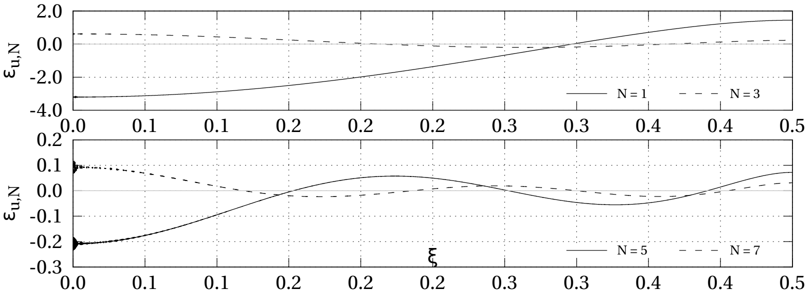
$$\epsilon_{u,N}(\xi) = 100\left(1 - \frac{\Delta_N(\xi)|_{\kappa=10^{-6}}}{\Delta_{\text{stat}}(\xi)}\right) \qquad \text{for } 0 \leq \xi \leq 1/2,$$



Error Estimates

To appreciate the approximation inherent in a truncated series, we compare the truncated series computed for $\kappa = 10^{-6}$ with the static response $\Delta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$ introducing a percent error function

$$\epsilon_{u,N}(\xi) = 100 \left(1 - \frac{\Delta_N(\xi)|_{\kappa=10^{-6}}}{\Delta_{\text{stat}}(\xi)} \right) \qquad \text{for } 0 \leq \xi \leq 1/2,$$

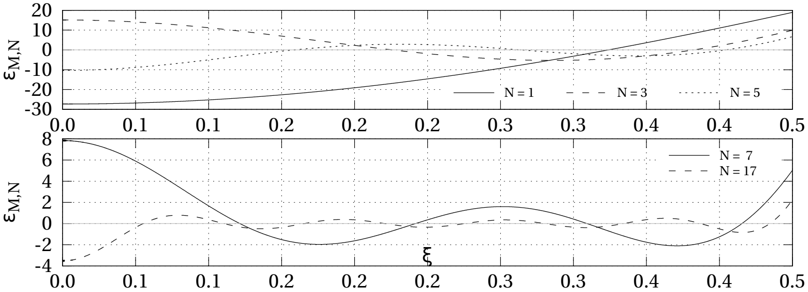


Using 4 terms ($N = 7$) the absolute error is not greater than $1/1000$.

Error Estimates

Analogously we can use the midspan bending moment, normalized with respect to $PL/4$, $\mu_{\text{stat}}(\xi) = 2\xi$ to define another percent error function

$$\epsilon_{M,N} = 100 \left(1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)}\right)$$

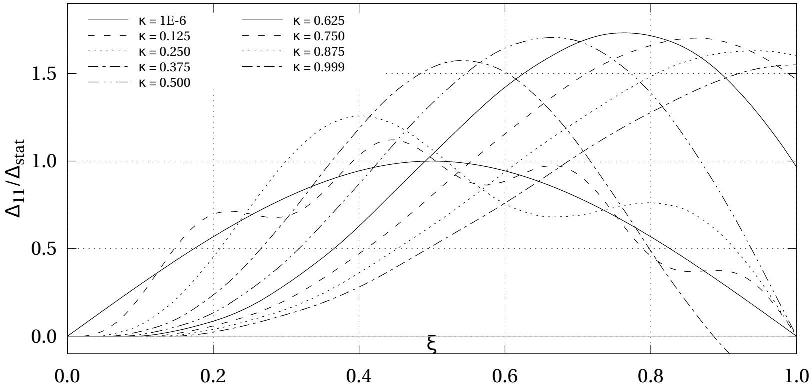


With 8 terms ($N = 17$) terms in the series, still the absolute error is greater than 3%.

The Plots

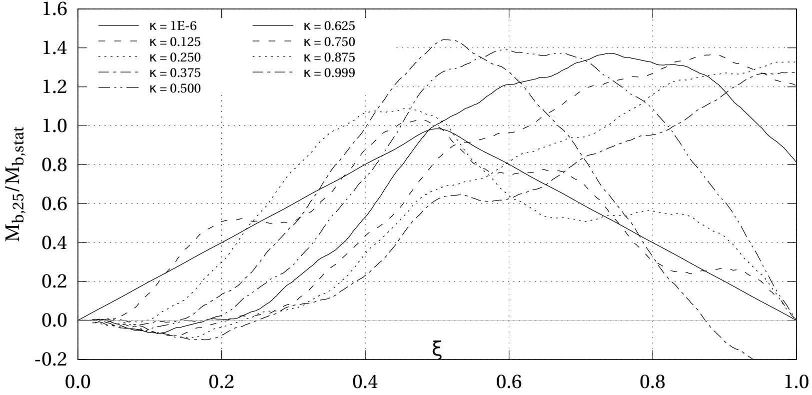
Eventually, we plot the normalized displacement and the normalized bending moment for different values of κ, i.e., for different velocities.

> *For the displacement I used* $N = 11$ *while for the bending moment I used* $N = 25$.



Normalized Midspan Displacement.

(for different velocities $v = \kappa\, v_c$)



Normalized Midspan Bending Moment.

(for different velocities $v = \kappa\, v_c$)