CHAPTER 21

RANDOM PROCESSES

21-1 DEFINITION

A random process is a family, or ensemble, of n random variables related to a similar phenomenon which may be functions of one or more independent variables. For example, suppose n accelerometers are mounted on the frames of n automobiles for the purpose of measuring vertical accelerations as these automobiles travel over a rough country road. The recorded accelerometer signals $x_i(t)$ $(i = 1, 2, \cdots, n)$, which are functions of one independent variable, namely, time t, might look something like the waveforms shown in Fig. 21-1. Each waveform in such a process differs from all other waveforms; that is, $x_r(t) \neq x_s(t)$ for $r \neq s$. To characterize this process $x(t)$ in a probabilistic sense, it is necessary to establish the multivariate probability density function $p(x_1, x_2, \cdots, x_m)$ as defined by the relation

$$p(X_1, X_2, \cdots, X_m)dx_1\, dx_2 \cdots dx_m$$

$$\equiv \Pr(X_1 < x_1 < X_1 + dx_1, X_2 < x_2 < X_2 + dx_2, \cdots, X_m < x_m < X_m + dx_m) \tag{21-1}$$

for $m = 1, 2, \cdots$, where x_i is the random variable consisting of sample values $x_{i1}, x_{i2}, \cdots, x_{im}$ across the ensemble at time t_i. Usually in engineering fields, it is sufficient to establish only the first two of these functions, that is, $p(x_1)$ and $p(x_1, x_2)$ but with t_1 and t_2 treated as variables.

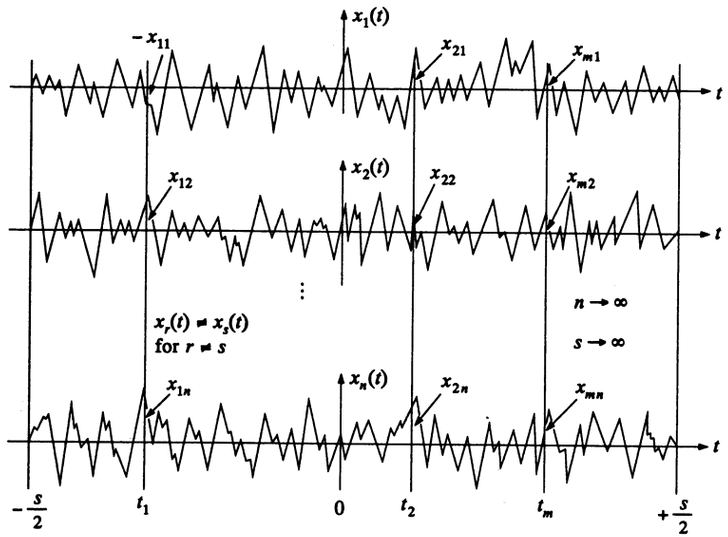


FIGURE 21-1
Random process (one independent variable).

The number of members n in the ensemble required to characterize a random process depends upon the type of process and the accuracy desired. Should it be necessary to establish the probability density functions statistically by sampling values of the random variables across the ensemble, exact results are obtained only in the limit as n approaches infinity. In practice, however, sufficient accuracy can be obtained using a finite number of members.

For some random processes, the desired probability density functions can be determined from an analysis of a single member of each process, in which case their exact characterizations are obtained only in the limit as the duration s approaches infinity. In practice these processes are always limited in duration; therefore, the characterizations obtained can only be approximate; however, engineering accuracy can usually be obtained with relatively short-duration sample waveforms.

In the above example, time t happens to be the independent variable, but it should be recognized that in general the independent variable can be any quantity.

As a second example of a random process, consider the wind drag force per unit height $p(x,t)$ acting on a tall industrial smokestack during a strong windstorm. This forcing function will contain a large steady-state or static component but will in addition contain a significant random component due to air turbulence. Clearly such turbulence produces drag forces which are not only random with respect to time t but are random with respect to the vertical space coordinate x as well. This process

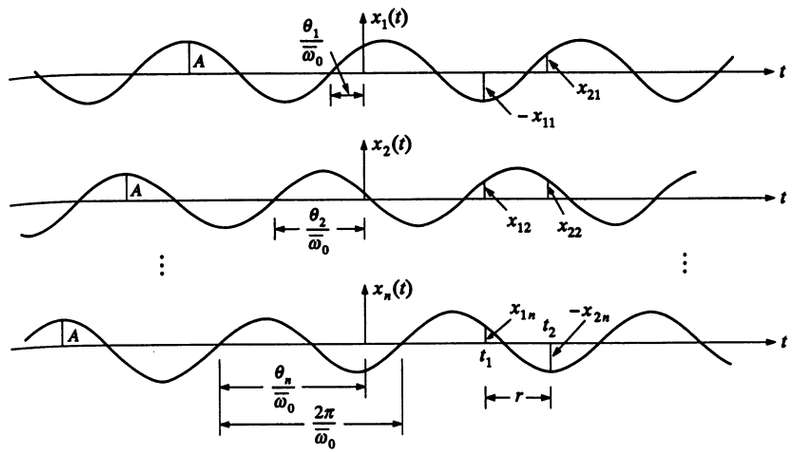


FIGURE 21-2
Random process of harmonic waveform.

therefore involves two independent variables.

The pressure fluctuations over the surface of an aircraft during flight are an example of a random process involving three independent variables, namely, time and two surface coordinates.

Obviously, the larger the number of independent variables involved in a random process, the more difficult it is to characterize the process.

21-2 STATIONARY AND ERGODIC PROCESSES

A specific random process will now be described in detail to help the reader develop a better understanding of random processes involving one independent variable. Consider the random process $x(t)$ shown in Fig. 21-2, which is defined by the relation

$$x_r(t) = A\sin(\overline{\omega}_0 t + \theta_r) \qquad r = 1, 2, \cdots, \infty \tag{21-2}$$

where $x_r(t)$ = rth member of the ensemble
A = fixed amplitude for each harmonic waveform
$\overline{\omega}_0$ = fixed circular frequency
θ_r = rth sampled value of a random phase angle θ having a uniform probability density function in the range $0 < \theta < 2\pi$ of intensity $1/2\pi$

This process shows that waveforms need not be irregular, that is, contain many frequency components, to be classified as random. Harmonic, periodic, or aperiodic

waveforms may or may not be random, depending upon whether they are fully prescribed or not. If known in a probabilistic sense only, they are defined as random. From this definition it is clear that once a random signal has been sampled, that particular waveform immediately becomes fully known and can no longer by itself be considered random; however, it still is considered part of the random process from which it was sampled. By statistically studying a sufficient number of sampled waveforms, the probability density functions for the process can be estimated, in which case any unsampled waveform becomes known in a probabilistic sense.

To establish the probability density function for random variable $x_1 \equiv x(t_1)$, a transformation relation similar to that given by Eq. (20-4) is used, namely,

$$p(x_1) = 2p(\theta)\left|\frac{d\theta}{dx_1}\right| \tag{21-3}$$

This equation differs slightly from Eq. (20-4) since the latter is valid only when $x_1 = x_1(\theta)$ and $\theta = \theta(x_1)$, its inverse relation, are single-valued functions. In this example, however, as random variable θ is allowed to change over its full range $0 < \theta < 2\pi$, random variable x_1 changes not once but twice over the range $-A < x_1 < +A$, which explains why the factor of 2 appears in Eq. (21-3). When Eq. (21-2) is substituted into Eq. (21-3) and the known information

$$p(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 < \theta < 2\pi \\ 0 & \theta < 0\,;\ 0 > 2\pi \end{cases} \tag{21-4}$$

is used, the probability density function $p(x_1)$ becomes

$$p(x_1) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - x_1^2}} & -A < x_1 < A \\ 0 & x_1 < -A\,;\ x_1 > A \end{cases} \tag{21-5}$$

Equations (21-4) and (21-5) are plotted in Fig. 21-3.

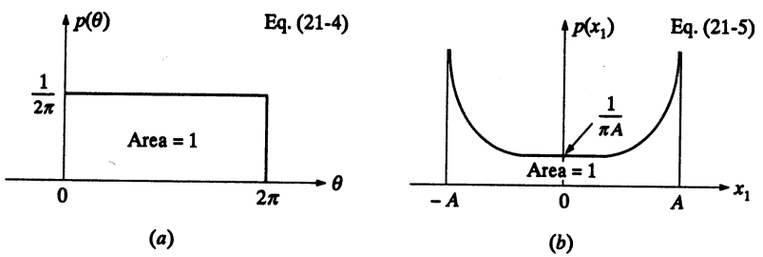


FIGURE 21-3
Probability density functions for θ and x_1, where $x_1 = A\sin(\bar{\omega}_0 t_1 + \theta)$.

The joint probability density function $p(x_1, x_2)$, where $x_1 \equiv x(t_1)$ and $x_2 \equiv x(t_2)$, can be obtained for the above process in the following manner. First, by using the appropriate trigonometric identity, x_2 can be expressed in the form

$$x_2 \equiv x(t_2) = x_1 \cos\bar{\omega}_0\tau \pm \sqrt{A^2 - x_1^2}\,\sin\bar{\omega}_0\tau \qquad -A \le x_1 \le A \tag{21-6}$$

Clearly this relation shows that for any sampled value of x_1, random variable x_2 has only two possible values with equal chances of occurring. In other words, for a given time interval $\tau = t_2 - t_1$, the conditional probability density function $p(x_2|x_1)$ consists of two Dirac delta functions, namely,

$$p(x_2|x_1) = \frac{1}{2}\left[\delta\left(x_2 - x_1\cos\bar{\omega}_0\tau + \sqrt{A^2 - x_1^2}\,\sin\bar{\omega}_0\tau\right) + \delta\left(x_2 - x_1\cos\bar{\omega}_0\tau - \sqrt{A^2 - x_1^2}\,\sin\bar{\omega}_0\tau\right)\right] \tag{21-7}$$

Substituting Eqs. (21-5) and (21-7) into the following form of Eq. (20-53)

$$p(x_1, x_2) = p(x_1)\,p(x_2|x_1) \tag{21-8}$$

leads to

$$p(x_1, x_2) = \frac{1}{2\pi\sqrt{A^2 - x_1^2}} \times \left[\delta(x_2 - x_1\cos\bar{\omega}_0\tau + \sqrt{A^2 - x_1^2}\,\sin\bar{\omega}_0\tau) + \delta(x_2 - x_1\cos\bar{\omega}_0\tau - \sqrt{A^2 - x_1^2}\,\sin\bar{\omega}_0\tau)\right] \tag{21-9}$$

which is valid in the range $-A < x_2 < +A$ and $-A < x_1 < +A$. Outside this range $p(x_1, x_2)$ equals zero.

Example E21-1. Consider the single harmonic random process defined by Eq. (21-2), namely,

$$x_r(t) = A\sin(\bar{\omega}_0 t + \theta_r) \qquad r = 1, 2, \cdots, \infty \tag{a}$$

where A is a fixed amplitude, $\bar{\omega}_0$ is a fixed circular frequency, and θ_r is the rth sampled value of a random phase angle θ having a uniform probability density function over the range $0 < \theta < 2\pi$. Defining the random variables x_1 and x_2 as

$$x_1 \equiv x(t) \qquad x_2 \equiv x(t + \tau) \tag{b}$$

characterize the form of the scatter diagram for variables x_1 and x_2 and plot the diagram for $\overline{\omega}_0\tau = 0,\ \pi/4,\ \pi/2,\ 3\pi/4$, and π.

The form of the scatter diagram can easily be obtained from Eq. (21-9) by noting that sample pairs of random variables x_1 and x_2 must satisfy the condition

$$x_2 - x_1 \cos\overline{\omega}_0\tau = \pm\sqrt{A^2 - x_1^2}\,\sin\overline{\omega}_0\tau \tag{c}$$

Squaring both sides of Eq. (c) gives

$$x_2^2 - 2\cos\overline{\omega}_0\tau\, x_1 x_2 + x_1^2 = A^2\sin^2\overline{\omega}_0\tau \tag{d}$$

This equation represents an ellipse with its major and minor axes at 45° from the x_1 and x_2 axes. To determine the dimensions of the ellipse along the major and minor axes, transform Eq. (d) to a new set of orthogonal axes u and v located on the principal axes of the ellipse; that is, use the linear transformation

$$u = \frac{1}{\sqrt{2}}(x_1 + x_2) \qquad v = \frac{1}{\sqrt{2}}(x_2 - x_1) \tag{e}$$

to obtain

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \tag{f}$$

where

$$a^2 = \frac{\sin^2\overline{\omega}_0\tau}{1 - \cos\overline{\omega}_0\tau}A^2 \qquad b^2 = \frac{\sin^2\overline{\omega}_0\tau}{1 + \cos\overline{\omega}_0\tau}A^2 \tag{g}$$

Thus it is shown that the scatter diagram is in the form of an ellipse with its principal axes at 45° from the x_1 and x_2 axes and with the ellipse dimensions along its principal axes being

$$2a = \frac{2\sin\overline{\omega}_0\tau}{\sqrt{1 - \cos\overline{\omega}_0\tau}}A \qquad 2b = \frac{2\sin\overline{\omega}_0\tau}{\sqrt{1 + \cos\overline{\omega}_0\tau}}A \tag{h}$$

as shown in Fig. E21-1. Substituting the values 0, $\pi/4$, $\pi/2$, $3\pi/4$, and π, separately, into Eqs. (h) for $\overline{\omega}_0\tau$ gives the corresponding values $\sqrt{2}\,A$, $1.31\,A$, $1.00\,A$, $0.54\,A$, and 0 for a and $0, 0.54\,A$, $1.00\,A$, $1.31\,A$, and $\sqrt{2}\,A$ for b. Plots of the scatter diagrams for each of these five cases are shown in Fig. E21-2. Note from the figure that the ellipse degenerates into a straight line for $\overline{\omega}_0\tau = 0$ and π. From the above it is clear that a straight line with positive slope of 1 will occur for $\overline{\omega}_0\tau = 0,\ 2\pi,\ 4\pi,\ 6\pi, \cdots$, a straight line with negative slope of 1 will occur for $\overline{\omega}_0\tau = \pi,\ 3\pi,\ 5\pi, \cdots$, a circle will occur for $\overline{\omega}_0\tau = \pi/2,\ 3\pi/2,\ 5\pi/2, \cdots$, and an ellipse will occur for all other values of $\overline{\omega}_0\tau$.

Usually of main interest are the mean values, mean square values, variances, the covariance, and the correlation coefficient for random variables x_1 and x_2. Using

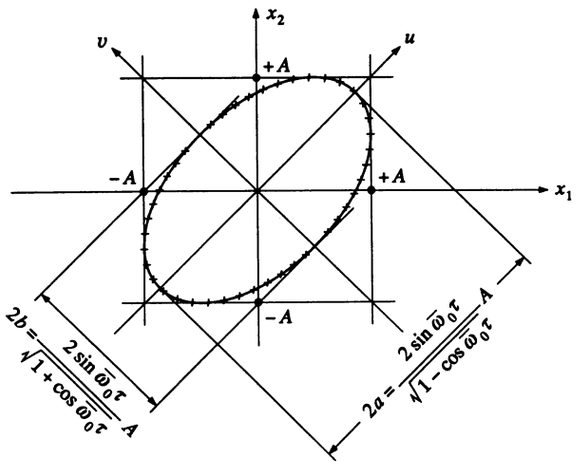


FIGURE E21-1
Scatter diagram for random variables x_1 and x_2 derived from single harmonic process, Eq. (21-2).

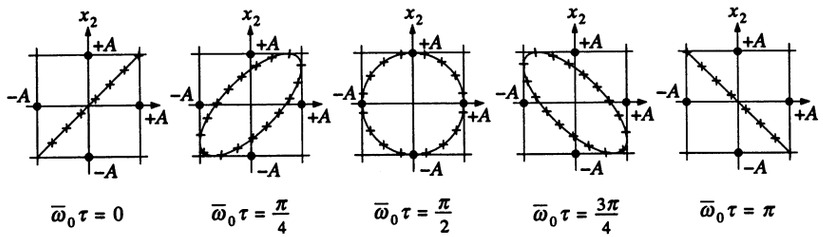


FIGURE E21-2
Scatter diagrams for five cases of the more general diagram in Fig. E21-1.

Eqs. (20-62) to (20-68) and (21-9) gives the following ensemble averages for the process:

Mean values:	$E(x_1) = E(x_2) = 0$
Mean square values:	$E(x_1^2) = E(x_2^2) = \dfrac{A^2}{2}$
Variances:	$\sigma_{x_1}^2 = \sigma_{x_2}^2 = \dfrac{A^2}{2}$
Covariance:	$\mu_{x_1x_2} = \dfrac{A^2}{2}\cos\overline{\omega}_0\tau$
Correlation coefficient:	$\rho_{x_1x_2} = \cos\overline{\omega}_0\tau$

(21-10)

The letter E has been introduced as a substitute for the bar previously placed above the random variable. It indicates that the variable has been averaged across the ensemble.

It is significant to note that all ensemble averages for this example process are independent of time t. Processes having this characteristic are defined as *stationary processes.*

It is also significant that for this process, any average obtained with respect to time t along any member r of the ensemble is exactly equal to the corresponding average across the ensemble at an arbitrary time t. Mathematically, this statement can be expressed in the form

$$\underset{r=1,2,\cdots}{\langle f(x_r)\rangle} \equiv \lim_{s\to\infty} \frac{1}{s}\int_{-s/2}^{s/2} f(x_r)\,dt = \underset{i=1,2,\cdots}{E\,[f(x_i)]} \tag{21-11}$$

where $f(x_r)$ is any function of the variable $x_r(t)$, $x_i = x(t_i)$, and where the angle brackets indicate time average. Processes having this characteristic are defined as *ergodic processes.*

It is suggested that the reader check the results given by Eq. (21-10) using Eq. (21-11) to show that the example process being considered, Eq. (21-2), is indeed ergodic; that is, show

$$\begin{aligned}
\langle x_r\rangle &= \lim_{s\to\infty}\frac{1}{s}\int_{-s/2}^{s/2} x_r(t)\,dt = 0\\
\langle x_r^2\rangle &= \lim_{s\to\infty}\frac{1}{s}\int_{-s/2}^{s/2} x_r(t)^2\,dt = \frac{A^2}{2}\\
\sigma_{x_r}^2 &= \frac{A^2}{2} \qquad r = 1,2,\cdots\\
\mu(\tau) &= \frac{A^2}{2}\cos\bar{\omega}_0\tau\\
\rho(\tau) &= \cos\bar{\omega}_0\tau
\end{aligned} \tag{21-12}$$

According to the above definitions, an ergodic process must always be stationary; however, a stationary process may or may not be ergodic.

21-3 AUTOCORRELATION FUNCTION FOR STATIONARY PROCESSES

Consider again the general random process $x(t)$ shown in Fig. 21-1, which involves one independent variable. Assume for this discussion that this process is

stationary (but not necessarily ergodic) and that it has a zero ensemble mean value, that is, $E(x) = 0$.

The covariance function $E[x(t)x(t+\tau)]$ in this case, like all ensemble averages, will be independent of time t and therefore will be a function of τ only. This function of τ will be referred to subsequently as the *autocorrelation function* and will be expressed in the form

$$R_x(\tau) = E[x(t)\,x(t+\tau)] \tag{21-13}$$

Certain important properties of the autocorrelation function should be noted, namely,

$$R_x(0) = \sigma_x^2 \qquad R_x(\tau) = R_x(-\tau) \qquad |R_x(\tau)| \le R_x(0) \tag{21-14}$$

The first of Eqs. (21-14) is obvious since $R_x(0) = E[x(t)x(t)]$ is the variance when $E[x] = 0$. The second equation is a direct result of the assumed stationarity of the process, and the third equation can readily be proved using the fact that the following mean square average must always be greater than or equal to zero:

$$E\{[x(t) \pm x(t+\tau)]^2\} = R_x(0) \pm 2R_x(\tau) + R_x(0) \ge 0 \tag{21-15}$$

or

$$|R_x(\tau)| \le R_x(0) \tag{21-16}$$

For most stationary processes, the autocorrelation function decays rapidly with increasing values of τ, thus showing a similar rapid loss of correlation of the two random variables as they are separated with respect to time. One notable exception, however, is the random process consisting of discrete harmonic waveforms, as shown in Fig. 21-2. This process has the autocorrelation function

$$R_x(\tau) = E(x_1\,x_2) = \frac{A^2}{2}\cos\bar{\omega}_0\tau \tag{21-17}$$

Clearly, regardless of the process, the two random variables $x(t)$ and $x(t+\tau)$ approach each other numerically as the time separation τ approaches zero. Therefore, these variables correlate completely in the limit as reflected by the correlation coefficient

$$\rho_x(0) = \frac{R_x(0)}{\sigma_x^2} = 1 \tag{21-18}$$

It is very significant to note that if the general process $x(t)$ being considered is stationary, has a zero mean value $E[x(t)] = 0$, and has the gaussian distribution given by Eq. (20-94), the autocorrelation function $R_x(\tau)$ completely characterizes the

process. This fact is evident since all variance and covariance functions given by Eq. (20-97) are directly related to the autocorrelation function as follows:

$$\mu_{ik} = \begin{cases} R_x(0) & i = k \\ R_x(\tau) & i \neq k \end{cases} \qquad \tau = t_k - t_i \tag{21-19}$$

For an ergodic process, the ensemble average given by Eq. (21-13) can be obtained by averaging along any single member (x_r) of the ensemble, in which case the autocorrelation function is more easily obtained using the relation

$$R_x(\tau) = \lim_{s\to\infty} \frac{1}{s} \int_{-s/2}^{s/2} x_r(t) x_r(t+\tau)\, dt \qquad r = 1, 2, \cdots \tag{21-20}$$

It should now be obvious to the reader why a gaussian ergodic process is so easily characterized in a probabilistic sense.

Example E21-2. A sample function $x_r(t)$ of random process $x(t)$ is established by assigning statistically independent sampled values of a random variable x to successive ordinates spaced at equal intervals along the time abscissa and by assuming a linear variation of the ordinates over each interval as shown in Fig. E21-3. A complete ensemble of such sample functions $(r = 1, 2, \cdots)$ can be obtained in a similar manner.

If the probability density function for x is prescribed arbitrarily, except that its mean value $\overline{x}$ is held equal to zero, and if the ordinate x_{1r} occurs at time $t = \alpha_r$, where α_r is a sampled value of a random variable α uniformly distributed over the range $0 < \alpha < \Delta\varepsilon$, determine the mean value, mean square value, and variance of $x(t)$ and the covariance of $x(t)$ and $x(t+\tau)$. What kind of random process is $x(t)$?

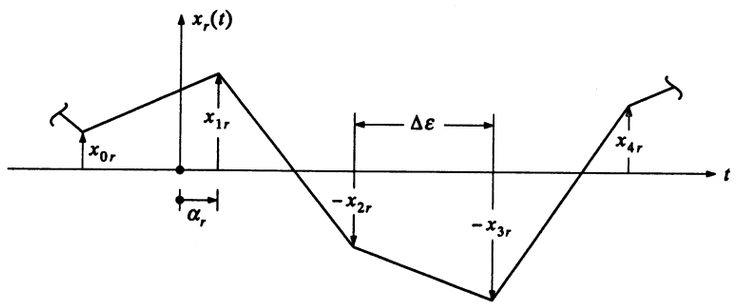


FIGURE E21-3
Sample function $x_r(t)$ from random process $x(t)$.

First, consider the above process but with all values of $\alpha_r (r = 1, 2, \cdots)$ set equal to zero, thus forcing all ordinates $x_{ir} (i, r = 1, 2, \cdots)$ to occur at time $t = (i-1)\Delta\varepsilon$. The linear variation of ordinates shown in Fig. E21-3 leads to

$$x_r(t) = \left(1 - \frac{t}{\Delta\varepsilon}\right) x_{1r} + \frac{t}{\Delta\varepsilon} x_{2r} \qquad 0 < t < \Delta\varepsilon$$

$$x_r(t+\tau) = \begin{cases} \left(1 - \dfrac{t+\tau+\Delta\varepsilon}{\Delta\varepsilon}\right) x_{0r} + \dfrac{t+\tau+\Delta\varepsilon}{\Delta\varepsilon} x_{1r} & -\Delta\varepsilon < t+\tau < 0 \\ \left(1 - \dfrac{t+\tau}{\Delta\varepsilon}\right) x_{1r} + \dfrac{t+\tau}{\Delta\varepsilon} x_{2r} & 0 < t+\tau < \Delta\varepsilon \\ \left(1 - \dfrac{t+\tau-\Delta\varepsilon}{\Delta\varepsilon}\right) x_{2r} + \dfrac{t+\tau-\Delta\varepsilon}{\Delta\varepsilon} x_{3r} & \Delta\varepsilon < t+\tau < 2\Delta\varepsilon \end{cases} \tag{a}$$

Taking the ensemble average of the first of Eqs. (a) gives

$$E[x(t)] = \left(1 - \frac{t}{\Delta\varepsilon}\right) E(x_1) + \frac{t}{\Delta\varepsilon} E(x_2)$$

However, when it is noted that

$$E(x_i) = \overline{x} = \int_{-\infty}^{\infty} x\, p(x)\, dx \qquad i = 1, 2, \cdots \tag{b}$$

the result is

$$E[x(t)] = \overline{x} = 0 \tag{c}$$

Squaring the first of Eqs. (a) and taking the ensemble average gives

$$E[x(t)^2] = \left(1 - \frac{t}{\Delta\varepsilon}\right)^2 E(x_1^2) + 2\left(1 - \frac{t}{\Delta\varepsilon}\right) \frac{t}{\Delta\varepsilon} E(x_1 x_2) + \left(\frac{t}{\Delta\varepsilon}\right)^2 E(x_2^2)$$

Making use of the relations

$$E[x_i^2] = \overline{x^2} = \int_{-\infty}^{\infty} x^2\, p(x)\, dx \qquad i, j = 1, 2, \cdots$$

$$E[x_i x_j] = 0 \qquad i \neq j \tag{d}$$

results in

$$E[x(t)^2] = \overline{x^2}\left(1 - \frac{2t}{\Delta\varepsilon} + \frac{2t^2}{\Delta\varepsilon^2}\right) \tag{e}$$

Therefore,

$$\sigma^2_{x(t)} = \overline{x^2}\left(1 - \frac{2t}{\Delta\varepsilon} + \frac{2t^2}{\Delta\varepsilon^2}\right) \tag{f}$$

From Eqs. (a) and (d)

$$E[x(t)x(t+\tau)] = \begin{cases} \left[\left(-\frac{1}{\Delta\varepsilon^2}\right)t^2 + \left(-\frac{\tau}{\Delta\varepsilon^2}\right)t + \left(\frac{\tau}{\Delta\varepsilon}+1\right)\right]\overline{x^2} \\ \qquad 0 \le t \le \Delta\varepsilon \qquad -\Delta\varepsilon \le t+\tau \le 0 \\ \left[\frac{2}{\Delta\varepsilon^2}t^2 + \left(\frac{2\tau}{\Delta\varepsilon^2} - \frac{2}{\Delta\varepsilon}\right)t + \left(1 - \frac{\tau}{\Delta\varepsilon}\right)\right]\overline{x^2} \\ \qquad 0 \le t \le \Delta\varepsilon \qquad 0 \le t+\tau \le \Delta\varepsilon \\ \left[\left(-\frac{1}{\Delta\varepsilon^2}\right)t^2 + \left(\frac{2}{\Delta\varepsilon} - \frac{\tau}{\Delta\varepsilon^2}\right)t\right]\overline{x^2} \\ \qquad 0 \le t \le \Delta\varepsilon \qquad \Delta\varepsilon \le t+\tau \le 2\Delta\varepsilon \end{cases} \tag{g}$$

Note that the covariance of $x(t)$ and $x(t+\tau)$ as given by Eq. (g) is time dependent; therefore, the random process treated above is nonstationary. Further, note that this covariance equals zero for values of τ outside the ranges indicated for Eqs. (g). The ranges indicated for the first, second, and third of Eqs. (g) are shown by the shaded regions 1, 2, and 3, respectively, in Fig. E21-4. If the origin of time $t = 0$ had been selected coincident with $x_{ir}(r = 1, 2, \cdots)$ rather than x_{1r}, as above, Eqs. (a) would obviously be of exactly the same form except that x_{0r}, x_{1r}, x_{2r}, and x_{3r} would be replaced by $x_{i-1,r}$, x_{ir}, $x_{i+1,r}$, and $x_{i+2,r}$, respectively. Thus, the covariance function $E[x(t)x(t+\tau)]$ must be periodic in time with period $\Delta\varepsilon$. This periodic behavior is also indicated in Fig. E21-4 by a repetition of the shaded regions in each interval along the time t axis.

If the probability density function $p(x)$ used in sampling values of x were gaussian in form, then the entire process $x(t)$ would be gaussian, in which case

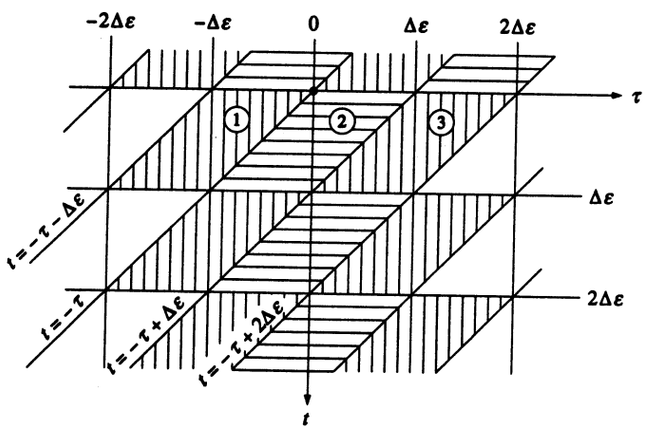


FIGURE E21-4
Regions of nonzero covariance for random variables $x(t)$ and $x(t+\tau)$.

Eqs. (a) would completely characterize the process in a probabilistic sense even though it is nonstationary.

The restriction placed on $\alpha_r(r = 1, 2, \cdots)$ above is now removed, and it is sampled from a uniform distribution over the range $0 < \alpha < \Delta\varepsilon$ as originally stated. Since any arbitrary time t will now occur uniformly over the intervals $\Delta\varepsilon$ looking across the ensemble, the process must be stationary and the covariance function $E[x(t)x(t+\tau)]$ is obtained by simply averaging that function as given by Eqs. (g) over time. Since the resulting function is independent of time and depends only upon the time difference τ, it becomes the autocorrelation function $R_x(\tau)$ for the process. Carrying out this averaging procedure gives

$$R_x(\tau) = \begin{cases} \frac{\overline{x^2}}{\Delta\varepsilon}\int_{-\tau-\Delta\varepsilon}^{\Delta\varepsilon}\left[\left(-\frac{1}{\Delta\varepsilon^2}\right)t^2 + \left(-\frac{\tau}{\Delta\varepsilon^2}\right)t + \left(\frac{\tau}{\Delta\varepsilon}+1\right)\right]dt \\ \qquad -2\Delta\varepsilon < \tau < -\Delta\varepsilon \\ \frac{\overline{x^2}}{\Delta\varepsilon}\left\{\int_0^{-\tau}\left[\left(-\frac{1}{\Delta\varepsilon^2}\right)t^2 + \left(-\frac{\tau}{\Delta\varepsilon^2}\right)t + \left(\frac{\tau}{\Delta\varepsilon}+1\right)\right]dt\right. \\ \qquad \left. + \int_{-\tau}^{\Delta\varepsilon}\left[\frac{2}{\Delta\varepsilon^2}t^2 + \left(\frac{2\tau}{\Delta\varepsilon^2} - \frac{2}{\Delta\varepsilon}\right)t + \left(1 - \frac{\tau}{\Delta\varepsilon}\right)\right]dt\right\} \\ \qquad -\Delta\varepsilon < \tau < 0 \\ \frac{\overline{x^2}}{\Delta\varepsilon}\left\{\int_0^{-\tau+\Delta\varepsilon}\left[\frac{2}{\Delta\varepsilon^2}t^2 + \left(\frac{2\tau}{\Delta\varepsilon^2} - \frac{2}{\Delta\varepsilon}\right)t + \left(1 - \frac{\tau}{\Delta\varepsilon}\right)\right]dt\right. \\ \qquad \left. + \int_{-\tau+\Delta\varepsilon}^{\Delta\varepsilon}\left[\left(-\frac{1}{\Delta\varepsilon^2}\right)t^2 + \left(\frac{2}{\Delta\varepsilon} - \frac{\tau}{\Delta\varepsilon^2}\right)t\right]dt\right\} \\ \qquad 0 < \tau < \Delta\varepsilon \\ \frac{\overline{x^2}}{\Delta\varepsilon}\int_0^{-\tau+2\Delta\varepsilon}\left[\left(-\frac{1}{\Delta\varepsilon^2}\right)t^2 + \left(\frac{2}{\Delta\varepsilon} - \frac{\tau}{\Delta\varepsilon^2}\right)t\right]dt \\ \qquad \Delta\varepsilon < \tau < 2\Delta\varepsilon \end{cases} \tag{h}$$

When the above integrals are completed and terms are collected, the result is

$$R_x(\tau) = \begin{cases} \left(\frac{4}{3} + \frac{2\tau}{\Delta\varepsilon} + \frac{\tau^2}{\Delta\varepsilon^2} + \frac{\tau^3}{6\Delta\varepsilon^3}\right)\overline{x^2} & -2\Delta\varepsilon \le \tau \le -\Delta\varepsilon \\ \left(\frac{2}{3} - \frac{\tau^2}{\Delta\varepsilon^2} - \frac{\tau^3}{2\Delta\varepsilon^3}\right)\overline{x^2} & -\Delta\varepsilon \le \tau \le 0 \\ \left(\frac{2}{3} - \frac{\tau^2}{\Delta\varepsilon^2} + \frac{\tau^3}{2\Delta\varepsilon^3}\right)\overline{x^2} & 0 \le \tau \le \Delta\varepsilon \\ \left(\frac{4}{3} - \frac{2\tau}{\Delta\varepsilon} + \frac{\tau^2}{\Delta\varepsilon^2} - \frac{\tau^3}{6\Delta\varepsilon^3}\right)\overline{x^2} & \Delta\varepsilon \le \tau \le 2\Delta\varepsilon \end{cases} \tag{i}$$

Because of the second of Eqs. (d), $R_x(\tau) = 0$ for $\tau \leq -2\Delta\varepsilon$ and $\tau \geq 2\Delta\varepsilon$.

If the random variable x has a normal distribution, the entire process is gaussian, in which case it is completely characterized by Eqs. (i).

21-4 POWER SPECTRAL DENSITY FUNCTION FOR STATIONARY PROCESSES

As demonstrated in Chapter 7, any sample waveform $x_r(t)$ taken from a real stationary random process having a zero mean value, that is, $E[x(t)] = 0$, can be separated into its frequency components using a standard Fourier analysis. If this waveform is represented only over the finite interval $-s/2 < t < +s/2$, the Fourier series representation can be used, namely,

$$x_r(t) = \sum_{n=-\infty}^{\infty} C_{nr} \exp(in\bar{\omega}_0 t) \tag{21-21}$$

where

$$C_{nr} = \frac{1}{s}\int_{-s/2}^{s/2} x_r(t)\exp(-in\bar{\omega}_0 t)dt$$

and where $\bar{\omega}_0 \equiv 2\pi/s$. If $x_r(t)$ is periodic, Eqs. (21-21) give an exact representation of the entire waveform provided the integration interval s is taken as one full period. Such periodic waveforms consist of discrete harmonics having circular frequencies $\bar{\omega}_0, 2\bar{\omega}_0, 3\bar{\omega}_0, \cdots$, with corresponding finite amplitudes $A_{1r} = 2|C_{1r}|$, $A_{2r} = 2|C_{2r}|$, $A_{3r} = 2|C_{3r}|, \cdots$, provided, of course, corresponding negative and positive frequency components are combined.

Usually the quantity of most interest when analyzing stationary random processes is the mean square value of $x_r(t)$ over the interval $-s/2 < t < +s/2$, which can be obtained by substituting the first of Eqs. (21-21) into the relation

$$\langle x_r(t)^2\rangle = \frac{1}{s}\int_{-s/2}^{s/2} x_r(t)^2\, dt \tag{21-22}$$

to obtain

$$\langle x_r(t)^2\rangle = \sum_{n=-\infty}^{\infty} |C_{nr}|^2 = \sum_{n=1}^{\infty} \frac{A_{nr}^2}{2} \tag{21-23}$$

When $\Delta\bar{\omega}$ represents the frequency spacing of the discrete harmonics, that is,

$$\Delta\bar{\omega} = \bar{\omega}_0 = \frac{2\pi}{s} \tag{21-24}$$

and the second of Eqs. (21-21) is used, Eq. (21-23) becomes

$$\langle x_r(t)^2\rangle = \sum_{n=-\infty}^{\infty} \frac{\left|\int_{-s/2}^{s/2} x_r(t)\exp(-in\bar{\omega}_0 t)dt\right|^2}{2\pi s}\Delta\bar{\omega} \tag{21-25}$$

If s is now allowed to approach infinity, $\Delta\bar{\omega} \to d\bar{\omega}$, $n\bar{\omega}_0 \to \bar{\omega}$, and the summation becomes an integral; thus, Eq. (21-25) is converted into the form

$$\langle x_r(t)^2\rangle = \int_{-\infty}^{\infty} S_{x_r}(\bar{\omega})\, d\bar{\omega} \tag{21-26}$$

where the function

$$S_{x_r}(\bar{\omega}) \equiv \lim_{s\to\infty} \frac{\left|\int_{-s/2}^{s/2} x_r(t)\exp(-i\bar{\omega}t)dt\right|^2}{2\pi s} \tag{21-27}$$

is defined as the *power spectral density function* for waveform $x_r(t)$ provided a limit actually exists. According to this definition, the power spectral density function is an even function when $x_r(t)$ is a real function, is positive and finite for all values of $\bar{\omega}$, and yields the mean square value of $x_r(t)$ when integrated over the entire range $-\infty < \bar{\omega} < +\infty$.

The power spectral density function for the entire stationary process $x(t)$ is obtained by simply averaging the power spectral density functions for individual members across the ensemble as follows:

$$S_x(\bar{\omega}) = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} S_{x_r}(\bar{\omega}) \tag{21-28}$$

The ensemble average of the mean square value of $x(t)$ can now be obtained by integrating $S_x(\bar{\omega})$ over the entire range $-\infty < \bar{\omega} < +\infty$.

If the random process is ergodic, each member of the ensemble will yield the same power spectral density function, in which case it is unnecessary to average across the ensemble. It is sufficient simply to generate the power spectral density function using one member. For most ergodic processes encountered in engineering, the power spectral density function given by Eq. (21-27) approaches its limit rapidly with increasing values of s, so that sufficient accuracy can usually be obtained with a relatively short sample of the waveform.

21-5 RELATIONSHIP BETWEEN POWER SPECTRAL DENSITY AND AUTOCORRELATION FUNCTIONS

Let a function $F_{x_r}(\bar{\omega})$ be defined as the Fourier transform of the time average $\langle x_r(t)x_r(t+\tau)\rangle$; that is, let

$$F_{x_r}(\bar{\omega}) \equiv \int_{-\infty}^{\infty}\left[\lim_{s\to\infty}\frac{1}{s}\int_{-s/2}^{s/2} x_r(t)x_r(t+\tau)dt\right]\exp(-i\bar{\omega}\tau)d\tau \tag{21-29}$$

Assuming that the function $F_{x_r}(\bar{\omega})$ does indeed exist, Fourier transform theory requires that the quantity in square brackets in Eq. (21-29), which is a function of τ only, decay with increasing values of $|\tau|$ so that the integral

$$I \equiv \int_{-\infty}^{\infty} \left| \lim_{s\to\infty} \frac{1}{s} \int_{-s/2}^{s/2} x_r(t)x_r(t+\tau)dt \right| d\tau \tag{21-30}$$

exists. When Eq. (21-29) is expressed in its equivalent form

$$\frac{1}{2\pi}F_{x_r}(\bar{\omega}) = \lim_{s\to\infty} \frac{1}{2\pi s} \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} x_r(t)\, x_r(t+\tau) \exp(-i\bar{\omega}\tau)\, d\tau\, dt \tag{21-31}$$

and a change of variable as defined by

$$\theta \equiv t + \tau \tag{21-32}$$

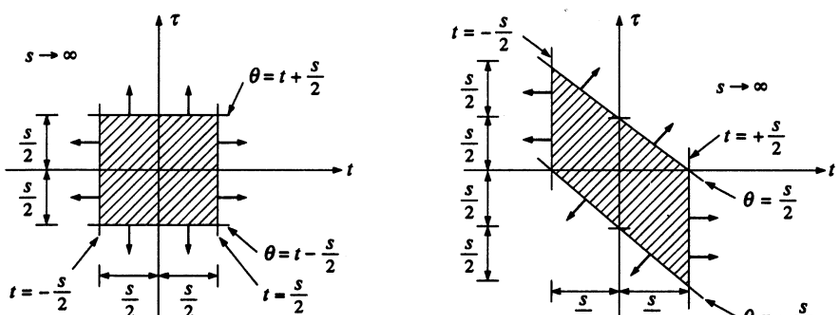
is substituted, Eq. (21-31) becomes

$$\frac{1}{2\pi}F_{x_r}(\bar{\omega}) = \lim_{s\to\infty} \frac{1}{2\pi s} \int_{-s/2}^{s/2} x_r(t) \exp(i\bar{\omega}t)dt \int_{t-s/2}^{t+s/2} x_r(\theta) \exp(-i\bar{\omega}\theta)d\theta \tag{21-33}$$

The expanding domain of integration given by Eq. (21-33) is shown in Fig. 21-4*a*. Since the function $F_{x_r}(\bar{\omega})$ can exist only when the total integrand of this equation decays rapidly with increasing values of $|\tau|$, it is valid to change the limits of the second integral as shown by the relation

$$\frac{1}{2\pi}F_{x_r}(\bar{\omega}) = \lim_{s\to\infty} \frac{1}{2\pi s} \int_{-s/2}^{s/2} x_r(t) \exp(i\bar{\omega}t)dt \int_{-s/2}^{s/2} x_r(\theta) \exp(-i\bar{\omega}\theta)d\theta \tag{21-34}$$

which simply changes the expanding domain of integration to that shown in Fig. 21-4*b*. At this point θ can be changed to t since it is serving only as a dummy time



Eq. (21-33) (*a*) Eq. (21-34) (*b*)

FIGURE 21-4
Expanding domains of integration.

variable. Equation (21-34) then can be expressed in the form

$$\frac{1}{2\pi}F_{x_r}(\bar{\omega}) = \lim_{s\to\infty} \frac{\left| \int_{-s/2}^{s/2} x_r(t) \exp(-i\bar{\omega}t)dt \right|^2}{2\pi s} \tag{21-35}$$

When Eq. (21-35) is compared with Eq. (21-27), it is clear that

$$\frac{1}{2\pi}F_{x_r}(\bar{\omega}) = S_{x_r}(\bar{\omega}) \tag{21-36}$$

If the stationary process being considered is ergodic, $F_{x_r}(\bar{\omega})$ is simply the Fourier transform of the autocorrelation function $R_x(\tau)$, and $S_{x_r}(\bar{\omega})$ equals the power spectral density for the process $S_x(\bar{\omega})$. Thus, it has been shown that for an ergodic process, the autocorrelation and power spectral density functions for the process are related through the Fourier integrals given by

$$\begin{aligned} S_x(\bar{\omega}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) \exp(-i\bar{\omega}\tau)\, d\tau \\ R_x(\tau) &= \int_{-\infty}^{\infty} S_x(\bar{\omega}) \exp(i\bar{\omega}\tau)\, d\bar{\omega} \end{aligned} \tag{21-37}$$

If the stationary process being considered is nonergodic, an additional step must be taken by averaging Eq. (21-36) across the ensemble as expressed by the relation

$$\frac{1}{2\pi} \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} F_{x_r}(\bar{\omega}) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} S_{x_r}(\bar{\omega}) \tag{21-38}$$

When Eq. (21-31) is used, it is observed that the left hand side of Eq. (21-38) is equal to $1/2\pi$ times the Fourier transform of $R_x(\tau)$. Since the right side of this same equation is $S_x(\bar{\omega})$, Eqs. (21-37) must also be valid for a nonergodic stationary process.

It was previously demonstrated that if a stationary process having a zero mean value is gaussian, it is completely characterized by the autocorrelation function. Now that it has been shown that the power spectral density function can be obtained by a Fourier transformation of the autocorrelation function, that function must also completely characterize such a process.

Example E21-3. Derive the power spectral density function for random process $x(t)$ as given in stationary form by Example E21-2.

Substituting Eqs. (i) of Example E21-2 into the first of Eqs. (21-37), namely,

$$S_x(\bar{\omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) \exp(-i\bar{\omega}\tau)d\tau \tag{a}$$

gives

$$S_x(\bar{\omega}) = \frac{\overline{x^2}}{2\pi}\left[\int_{-2\Delta\varepsilon}^{-\Delta\varepsilon}\left(\frac{4}{3}+\frac{2\tau}{\Delta\varepsilon}+\frac{\tau^2}{\Delta\varepsilon^2}+\frac{\tau^2}{6\Delta\varepsilon^3}\right)\exp(-i\bar{\omega}\tau)d\tau\right.$$
$$+\int_{-\Delta\varepsilon}^{0}\left(\frac{2}{3}-\frac{\tau^2}{\Delta\varepsilon^2}-\frac{\tau^3}{2\Delta\varepsilon^3}\right)\exp(-i\bar{\omega}\tau)d\tau$$
$$+\int_{0}^{\Delta\varepsilon}\left(\frac{2}{3}-\frac{\tau^2}{\Delta\varepsilon^2}+\frac{\tau^3}{2\Delta\varepsilon^3}\right)\exp(-i\bar{\omega}\tau)d\tau$$
$$\left.+\int_{\Delta\varepsilon}^{2\Delta\varepsilon}\left(\frac{4}{3}-\frac{2\tau}{\Delta\varepsilon}+\frac{\tau^2}{\Delta\varepsilon^2}-\frac{\tau^3}{6\Delta\varepsilon^3}\right)\exp(-i\bar{\omega}\tau)d\tau\right] \quad \text{(b)}$$

After integrating and collecting all terms, the result is

$$S_x(\bar{\omega}) = \frac{\overline{x^2}}{2\pi}\left\{\frac{1}{\bar{\omega}^4\,\Delta\varepsilon^3}[6-4\exp(-i\bar{\omega}\tau)-4\exp(i\bar{\omega}\tau)\right.$$
$$\left.+\exp(-2i\bar{\omega}\tau)+\exp(2i\bar{\omega}\tau)]\right\} \quad \text{(c)}$$

which can be converted to the trigonometric form

$$S_x(\bar{\omega}) = \frac{\overline{x^2}}{2\pi}\left[\frac{6-8\cos\bar{\omega}\Delta\varepsilon+2\cos 2\bar{\omega}\Delta\varepsilon}{\bar{\omega}^4\Delta\varepsilon^3}\right] \quad -\infty<\bar{\omega}<\infty \quad \text{(d)}$$

21-6 POWER SPECTRAL DENSITY AND AUTOCORRELATION FUNCTIONS FOR DERIVATIVES OF PROCESSES

When the power spectral density and autocorrelation functions for the random variable $x(t)$ are known, these same functions can easily be obtained for time derivatives of this variable such as $\dot{x}(t)$ and $\ddot{x}(t)$. To illustrate the method, consider the autocorrelation function for $x(t)$ in its most basic form, that is,

$$R_x(\tau) \equiv E[x(t)\,x(t+\tau)] \tag{21-39}$$

Differentiating with respect to τ gives

$$R_x'(\tau) = \frac{dR_x(\tau)}{d\tau} = E[x(t)\,\dot{x}(t+\tau)] \tag{21-40}$$

Since the process $x(t)$ is stationary, Eq. (21-40) can also be expressed in the form

$$R_x'(\tau) = E[x(t-\tau)\,\dot{x}(t)] \tag{21-41}$$

Differentiating once more with respect to τ gives

$$R_x''(\tau) = -E[\dot{x}(t-\tau)\,\dot{x}(t)] = -E[\dot{x}(t)\,\dot{x}(t+\tau)] \tag{21-42}$$

Since the ensemble average in Eq. (21-42) is by definition the autocorrelation function for $\dot{x}(t)$, it becomes apparent that

$$R_{\dot{x}}(\tau) = -R_x''(\tau) \tag{21-43}$$

Differentiating in the same manner two more times shows that

$$R_{\ddot{x}}(\tau) = -R_{\dot{x}}''(\tau) = R_x^{iv}(\tau) \tag{21-44}$$

The above autocorrelation functions can be expressed in the form of the second of Eqs. (21-37), namely

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\bar{\omega})\exp(i\bar{\omega}\tau)\,d\bar{\omega}$$
$$R_{\dot{x}}(\tau) = \int_{-\infty}^{\infty} S_{\dot{x}}(\bar{\omega})\exp(i\bar{\omega}\tau)\,d\bar{\omega} \tag{21-45}$$
$$R_{\ddot{x}}(\tau) = \int_{-\infty}^{\infty} S_{\ddot{x}}(\bar{\omega})\exp(i\bar{\omega}\tau)\,d\bar{\omega}$$

Substituting the first of Eq. (21-45) into Eqs. (21-43) and (21-44) gives

$$R_{\dot{x}}(\tau) = \int_{-\infty}^{\infty} \bar{\omega}^2 S_x(\bar{\omega})\exp(i\bar{\omega}\tau)\,d\bar{\omega}$$
$$R_{\ddot{x}}(\tau) = \int_{-\infty}^{\infty} \bar{\omega}^4 S_x(\bar{\omega})\exp(i\bar{\omega}\tau)\,d\bar{\omega} \tag{21-46}$$

Comparing Eqs. (21-46) with the second and third of Eqs. (21-45) shows that

$$S_{\dot{x}}(\bar{\omega}) = \bar{\omega}^2 S_x(\bar{\omega}) \qquad S_{\ddot{x}}(\bar{\omega}) = \bar{\omega}^4 S_x(\bar{\omega}) \tag{21-47}$$

Example E21-4. If random process $x(t)$ has the autocorrelation function

$$R_x(\tau) = (1-\tau^2)\,e^{-\tau^2} \tag{a}$$

find the corresponding autocorrelation functions for random processes $\dot{x}(t)$ and $\ddot{x}(t)$.

Taking derivatives of Eq. (a) gives

$$R'_x(\tau) = (2\tau^3 - 4\tau)\ e^{-\tau^2}$$

$$R''_x(\tau) = (-4\tau^4 + 14\tau^2 - 4)\ e^{-\tau^2}$$

$$R'''_x(\tau) = (8\tau^5 - 44\tau^3 + 36\tau)\ e^{-\tau^2} \qquad \text{(b)}$$

$$R^{iv}_x(\tau) = (-16\tau^6 + 128\tau^4 - 204\tau^2 + 36)\ e^{-\tau^2}$$

Thus from Eqs. (21-43) and (21-44)

$$R_{\dot{x}}(\tau) = (4\tau^4 - 14\tau^2 + 4)\ e^{-\tau^2} \qquad \text{(c)}$$

$$R_{\ddot{x}}(\tau) = (-16\tau^6 + 128\tau^4 - 204\tau^2 + 36)\ e^{-\tau^2} \qquad \text{(d)}$$

21-7 SUPERPOSITION OF STATIONARY PROCESSES

Consider a stationary process $q(t)$ which is defined as the sum of three separate stationary processes $x(t)$, $y(t)$, and $z(t)$ all of which have zero mean values. To find the autocorrelation function for this process, namely,

$$R_q(\tau) \equiv E[q(t)\, q(t+\tau)] \qquad (21\text{-}48)$$

substitute the relation

$$q(t) = x(t) + y(t) + z(t) \qquad (21\text{-}49)$$

into Eq. (21-48) to obtain

$$\begin{aligned} R_q(\tau) =& E[x(t)x(t+\tau)] + E[y(t)y(t+\tau)] + E[z(t)z(t+\tau)] \\ &+ E[x(t)y(t+\tau)] + E[y(t)z(t+\tau)] + E[x(t)z(t+\tau)] \\ &+ E[y(t)x(t+\tau)] + E[z(t)y(t+\tau)] + E[z(t)x(t+\tau)] \end{aligned} \qquad (21\text{-}50)$$

The first three ensemble averages on the right hand side of this equation are the autocorrelation functions for processes $x(t)$, $y(t)$, and $z(t)$, respectively, and the last six ensemble averages are *cross-correlation functions* (or covariance functions), which will be designated as

$$\begin{aligned} R_{xy}(\tau) &= E[x(t)y(t+\tau)] & R_{yx}(\tau) &= E[y(t)x(t+\tau)] \\ R_{yz}(\tau) &= E[y(t)z(t+\tau)] & R_{zy}(\tau) &= E[z(t)y(t+\tau)] \\ R_{xz}(\tau) &= E[x(t)z(t+\tau)] & R_{zx}(\tau) &= E[z(t)x(t+\tau)] \end{aligned} \qquad (21\text{-}51)$$

Thus, the autocorrelation function for process $q(t)$ can be expressed in terms of the autocorrelation and cross-correlation functions for $x(t)$, $y(t)$, and $z(t)$ as follows:

$$\begin{aligned} R_q(\tau) =& R_x(\tau) + R_y(\tau) + R_z(\tau) + R_{xy}(\tau) + R_{yz}(\tau) \\ &+ R_{xz}(\tau) + R_{yx}(\tau) + R_{zy}(\tau) + R_{zx}(\tau) \end{aligned} \qquad (21\text{-}52)$$

If random processes $x(t)$, $y(t)$, and $z(t)$ are uncorrelated with each other, their cross-correlation functions will equal zero, in which case

$$R_q(\tau) = R_x(\tau) + R_y(\tau) + R_z(\tau) \qquad (21\text{-}53)$$

It should be noted that for real stationary processes

$$R_{xy}(\tau) = R_{yx}(-\tau) \qquad R_{yz}(\tau) = R_{zy}(-\tau) \qquad R_{xz}(\tau) = R_{zx}(-\tau) \qquad (21\text{-}54)$$

The power spectral density function for process $q(t)$ is obtained using the first of Eqs. (21-37), that is,

$$S_q(\bar{\omega}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_q(\tau)\exp(-i\bar{\omega}\tau)d\tau \qquad (21\text{-}55)$$

Substituting Eq. (21-52) into Eq. (21-55) gives

$$\begin{aligned} S_q(\bar{\omega}) =& S_x(\bar{\omega}) + S_y(\bar{\omega}) + S_z(\bar{\omega}) + S_{xy}(\bar{\omega}) + S_{yz}(\bar{\omega}) \\ &+ S_{xz}(\bar{\omega}) + S_{yx}(\bar{\omega}) + S_{zy}(\bar{\omega}) + S_{zx}(\bar{\omega}) \end{aligned} \qquad (21\text{-}56)$$

where $S_{xy}(\bar{\omega})$, $S_{yz}(\bar{\omega})$, $\cdots$ are *cross-spectral density functions* which are related to their respective cross-correlation functions through the Fourier transform relation

$$S_{xy}(\bar{\omega}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xy}(\tau)\exp(-i\bar{\omega}\tau)\, d\tau \qquad (21\text{-}57)$$

Note that $S_{yx}(\bar{\omega})$ is the complex conjugate of $S_{xy}(\bar{\omega})$. The inverse of Eq. (21-57) is, of course,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(\bar{\omega})\exp(i\bar{\omega}\tau)\, d\bar{\omega} \qquad (21\text{-}58)$$

When the procedure of Section 21-4 is followed, the time average of the product $x_r(t)y_r(t)$ becomes

$$\langle x_r(t)y_r(t)\rangle = \int_{-\infty}^{\infty} S_{x_r y_r}(\bar{\omega})\, d\bar{\omega} \qquad (21\text{-}59)$$

where

$$S_{x_r y_r}(\bar{\omega}) \equiv \lim_{s\to\infty} \frac{\left[\int_{-s/2}^{s/2} x_r(t)\exp(-i\bar{\omega}t)dt\right]\left[\int_{-s/2}^{s/2} y_r(t)\exp(+i\bar{\omega}t)dt\right]}{2\pi s} \qquad (21\text{-}60)$$

Note that $S_{x_r y_r}(-\overline{\omega})$ is the complex conjugate of $S_{x_r y_r}(\overline{\omega})$. Therefore, only the real part of $S_{x_r y_r}(\overline{\omega})$ contributes to the integral in Eq. (21-59). If processes $x(t)$ and $y(t)$ are ergodic, $S_{x_r y_r}(\overline{\omega})$ as given by Eq. (21-60) represents the cross-spectral density for these processes. However, if processes $x(t)$ and $y(t)$ are nonergodic, the cross-spectral density function for these processes must be obtained by averaging across the ensemble, that is,

$$S_{xy}(\overline{\omega}) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} S_{x_r y_r}(\overline{\omega}) \tag{21-61}$$

21-8 STATIONARY GAUSSIAN PROCESSES: ONE INDEPENDENT VARIABLE

In engineering it is common practice to assume a gaussian, or normal, distribution for random processes. To help in establishing a rational basis for this assumption, consider a real stationary zero-mean random process $x(t)$ of the form

$$x_{jr}(t) = \sum_{n=-j}^{j} C_{nr} \exp(in\overline{\omega}_0 t) \qquad r = 1, 2, \cdots \tag{21-62}$$

where $x_{jr}(t)$ is the rth member of the ensemble which contains j discrete harmonics having frequencies $\overline{\omega}_0$, $2\overline{\omega}_0, \cdots, j\overline{\omega}_0$, and where C_{nr} represents random complex constants. For the process to have a zero-mean value, it is necessary, of course, that coefficients C_{0r} equal zero; and since it is assumed that the process contains real functions only, it is necessary that complex coefficients C_{nr} and C_{mr} be conjugate pairs when $n = -m$.

To define the randomness of coefficients C_{nr}, assume first that $|C_{nr}| = C$ (a constant) for all permissible values of n and r but that their corresponding phase angles α_{nr} are sampled values of a random variable α which has a uniform probability density function of intensity $1/2\pi$ in the range $0 < \alpha < 2\pi$. Under these conditions Eq. (21-62) can be written in the form

$$x_{jr}(t) = \sum_{n=-j}^{j} |C_{nr}| \exp\left[i(n\overline{\omega}_0 t + \alpha_{nr})\right] \tag{21-63}$$

or

$$x_{jr} = 2C \sum_{n=1}^{j} \sin(n\overline{\omega}_0 t + \theta_{nr}) \quad r = 1, 2, \cdots \tag{21-64}$$

where $\theta_{nr} = +(\pi/2) + \alpha_{nr}$. Since this process contains discrete harmonics at frequency intervals of $\overline{\omega}_0$, each ensemble member will be periodic with a period $s = 2\pi/\overline{\omega}_0$. When a new random variable $L(t)$ is defined so that

$$L_{nr}(t) = 2C \sin(n\overline{\omega}_0 t + \theta_{nr}) \tag{21-65}$$

Eq. (21-64) can be written in the form of the one-dimensional random walk:

$$x_{jr}(t) = \sum_{n=1}^{j} L_{nr}(t) \tag{21-66}$$

From Eqs. (21-5) and (21-10), it is clear that

$$p[L(t)] = \begin{cases} \dfrac{1}{\pi\sqrt{4C^2 - L^2}} & -2C < L < 2C \\ 0 & L < -2C\,;\; L > 2C \end{cases} \tag{21-67}$$

and that

$$\overline{L(t)} = 0 \qquad \sigma^2_{L(t)} = 2C^2 \tag{21-68}$$

When the one-dimensional random-walk relations given by Eq. (20-34) are used, it follows that

$$\overline{x_j(t)} = 0 \qquad \sigma^2_{x(t)} = 2jC^2 \tag{21-69}$$

At this point, apply the same limiting procedure previously used in the one-dimensional random-walk development, that is, let $\overline{\omega}_0 \to 0$, $j \to \infty$, and $C^2 \to 0$, but in such a manner that

$$n\overline{\omega}_0 \to \overline{\omega} \text{ (a variable)} \quad j\,\overline{\omega}_0 \to \overline{\omega}_1 \text{ (a constant)} \quad C^2/\overline{\omega}_0 \to S_0 \text{ (a constant)} \tag{21-70}$$

Since $\overline{\omega}_0 = 2\pi/s$, period $s \to \infty$ by this limiting procedure.

When the above relations and the second of Eqs. (21-21) are used, one finds that

$$|C_{nr}| = C = \frac{1}{s} \left| \int_{-s/2}^{s/2} x_{jr}(t) \exp(-in\overline{\omega}_0 t) dt \right| \tag{21-71}$$

It is now evident that in the limit

$$S_0 = \lim_{s\to\infty} \frac{\left| \int_{-s/2}^{s/2} x_r(t) \exp(-i\overline{\omega}t) dt \right|^2}{2\pi s} \qquad r = 1, 2, \cdots \tag{21-72}$$

where

$$x_r(t) = \lim_{\substack{j\to\infty \\ \overline{\omega}_0 \to 0}} x_{jr}(t) \tag{21-73}$$

A comparison of Eq. (21-72) with Eq. (21-27) and recognition of the limiting conditions given by the first and second of Eqs. (21-70) lead to the conclusion that ensemble member $x_r(t)$ has a uniform power spectral density function $S_{x_r}(\overline{\omega})$ of intensity S_0 over the frequency range $-\overline{\omega}_1 < \overline{\omega} < \overline{\omega}_1$ and of intensity zero outside this range and that since this power spectral density function is invariant with r, the process is ergodic; thus, the power spectral density for the entire process $x(t)$ is that function

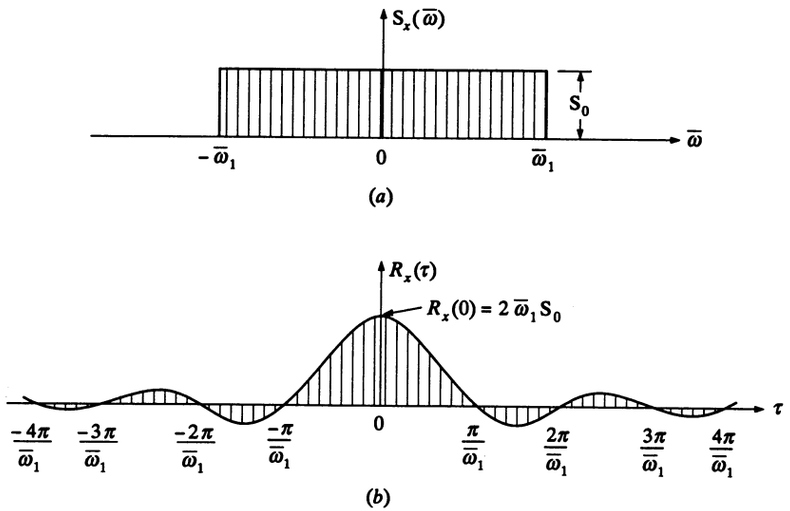


FIGURE 21-5
Power spectral density and autocorrelation functions for random process $x(t)$.

shown in Fig. 21-5*a*. Further, the earlier one-dimensional random-walk development leads to the conclusion that this random process is gaussian and that its variance [see the second of Eqs. (21-69)] is given by

$$\sigma^2_{x(t)} = 2\bar{\omega}_1 \, S_0 \tag{21-74}$$

Substituting the power spectral density function shown in Fig. 21-5*a* into the second of Eqs. (21-37), the autocorrelation function for random process $x(t)$ is found to be

$$R_x(\tau) = \frac{2S_0}{\tau} \sin \bar{\omega}_1 \tau \qquad -\infty < \tau < \infty \tag{21-75}$$

This relation is plotted in Fig. 21-5*b*.

Note that when the power spectral density function for this process becomes uniform over the entire frequency range, that is, when $\bar{\omega}_1 \to \infty$, the variance $\sigma_x^2 \to \infty$ and the autocorrelation function $R_x(\tau) \to 2\pi S_0 \, \delta(\tau)$, where $\delta(\tau)$ is a Dirac delta function located at the origin. This process, which is commonly referred to as a *white* process or simply *white noise*, can be considered as totally random since $x(t)$ is completely independent of $x(t+\tau)$ for all values of $\tau \neq 0$.

Consider again random process $x_r(t)$ but this time assume that coefficients $|C_{nr}|$ equal zero for all values of n in the range $-k < n < +k$, where $k < j$, and that they equal a constant C for all values of n in the ranges $-j \leq n \leq -k$ and

$+k \leq n \leq +j$. The same procedures as before are followed, but this time $\bar{\omega}_0 \to 0$, $k \to \infty$, $j \to \infty$, and $C^2 \to 0$ in such a manner that $n\bar{\omega}_0 \to \bar{\omega}$, $k\bar{\omega}_0 \to \bar{\omega}_1$, $j\bar{\omega}_0 \to \bar{\omega}_2$, and $C^2/\bar{\omega}_0 \to S_0$; again the process becomes gaussian in the limit and its power spectral density and autocorrelation functions are of the form

$$S_x(\bar{\omega}) = \begin{cases} S_0 & -\bar{\omega}_2 < \bar{\omega} < -\bar{\omega}_1 \; ; \; \bar{\omega}_1 < \bar{\omega} < \bar{\omega}_2 \\ 0 & \bar{\omega} < -\bar{\omega}_2 \; ; \; -\bar{\omega}_1 < \bar{\omega} < \bar{\omega}_1 \; ; \; \bar{\omega} > \bar{\omega}_2 \end{cases} \tag{21-76}$$

$$R_x(\tau) = \frac{2S_0}{\tau}(\sin \bar{\omega}_2 \tau - \sin \bar{\omega}_1 \tau) \qquad -\infty < \tau < \infty$$

To generalize one step further, consider a random process $z(t)$ defined as the sum of the statistically independent gaussian ergodic processes $x(t)$ and $y(t)$, both of which are developed separately from Eq. (21-62) using the same limiting procedure as before. From the proof given in Section 21-9, process $z(t)$ will also have a gaussian distribution.

Finally, once more use the process given by Eq. (21-63) as expressed in the equivalent form

$$x_{jr}(t) = \sum_{n=1}^{j} 2|C_{nr}| \, \sin(n\bar{\omega}_0 t + \theta_{nr}) \qquad r = 1, 2, \cdots \tag{21-77}$$

For this process assume that phase angles θ_{nr} are sampled values of random variable θ which has the uniform probability density function shown in Fig. 21-6*a* and that coefficients $|C_{nr}|$ are sampled values of a second random variable C which has an arbitrary, but prescribed, probability density as shown in Fig. 21-6*b*. When $L_{nr}(t)$ is defined by the relation

$$L_{nr}(t) \equiv 2|C_{nr}| \, \sin(n\bar{\omega}_0 t + \theta_{nr}) \tag{21-78}$$

$x_r(t)$ can again be expressed in the form

$$x_{jr}(t) \equiv \sum_{n=1}^{j} L_{nr}(t) \tag{21-79}$$

When the forms of probability density functions $p(\theta)$ and $p(C)$ are known, the probability density function for $L(t)$, as defined by Eq. (21-78), can be established if

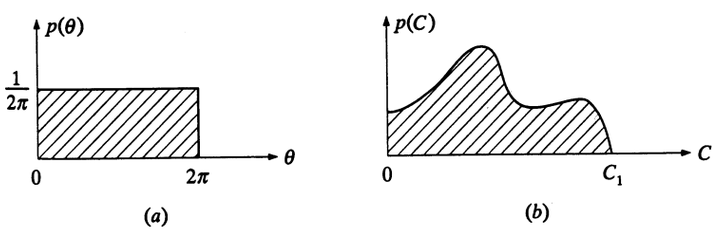


FIGURE 21-6
Probability density functions for random variables ϕ and C.

desired. For this process, however, this step is unnecessary since the mean value $\overline{L(t)}$ and the variance $\sigma^2_{L(t)}$ are the only quantities required in the random-walk development and they can be obtained without establishing the function $p[L(t)]$. From the form of Eq. (21-78), it can be reasoned that

$$\overline{L(t)} = 0 \qquad \sigma^2_{L(t)} = 2\overline{C^2} = 2\int_{-\infty}^{\infty} C^2 p(C)dC \tag{21-80}$$

This process is stationary since the variance for $L(t)$ is independent of time t.

When the one-dimensional random-walk relations given by Eq. (20-34) are used again, it follows that

$$\overline{x}_j(t) = 0 \qquad \sigma^2_{x_j(t)} = 2j\overline{C^2} \tag{21-81}$$

When Eq. (21-81) is compared with Eq. (21-69), it is clear that the same limiting procedures used previously can once again be used provided C^2 is replaced by $\overline{C^2}$. In this case

$$\mathrm{S}_0 \equiv \frac{\overline{C^2}}{\overline{\omega}_0} = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \mathrm{S}_{x_r}(\overline{\omega}) \tag{21-82}$$

where

$$\mathrm{S}_{x_r}(\overline{\omega}) = \lim_{s\to\infty} \frac{\left|\int_{-s/2}^{s/2} x_r(t)\exp(-i\overline{\omega}t)dt\right|^2}{2\pi s} \tag{21-83}$$

Although coefficients $|\mathrm{C}_{nr}|$ for this process are random in accordance with Fig. 21-b*b*, the power spetral density function as defined by Eq. (21-83) will be independent of r when the limiting procedure is applied, i.e., when $j \to \infty$ and $\overline{\omega}_0 \to 0$. If S_0 as given by Eq. (21-82) is to be finite, it is necessary that C_1 in Fig. 21-b*b* approach zero in such a way that $\overline{C^2}/\overline{\omega}_0 = \mathrm{S}_0$ is finite. If the process given by Eq. (21-77) is to have a nonuniform power spectral density function over the frequency range $j\overline{\omega}_0$, the coefficients $|\mathrm{C}_{nr}|$ would have to be dependent upon n; and, if the process is to be nonergodic, they would have to be dependent upon r.

The earlier development which restricted the nonzero-frequency components to the range $\overline{\omega}_1 < \overline{\omega} < \overline{\omega}_2$ and the development which presented the principle of superposition obviously both apply equally well to the present process involving two random variables. Therefore, it may be concluded that any stationary process $x(t)$ (whether ergodic or not) will be gaussian when its power spectral density function $\mathrm{S}_x(\overline{\omega})$ truly exists and when all the phase angles between frequency components which are randomly distributed in a uniform manner over 360° are statistically independent of each other.

When the phase angles between frequency components are not uniformly distributed over the full 360°, gaussian processes will still result in the limit; however, stationarity will no longer be maintained. For example, if the random phase angle θ for the process defined by Eq. (21-2) has a uniform probability density function

of intensity $1/\theta_1$ over the range $0 < \theta < \theta_1$, where $\theta_1 < 2\pi$, the ensemble mean square value $E[x(t)^2]$ (or variance in this case) will be time dependent. To prove this statement, substitute Eqs. (21-2) and (20-4) into Eq. (20-10) to obtain

$$E[x(t)^2] = \int_{A\sin\overline{\omega}_0 t}^{A\sin(\overline{\omega}_0 t+\theta_1)} \frac{x^2}{\theta_1\sqrt{A^2-x^2}}dx \tag{21-84}$$

After the integration is completed, this equation becomes

$$E[x(t)^2] = \frac{A^2}{2}\left\{1 - \frac{1}{2\theta_1}[\sin 2\theta_1 \cos 2\overline{\omega}_0 t - (1-\cos 2\theta_1)\sin 2\overline{\omega}_0 t]\right\} \tag{21-85}$$

which clearly shows the time dependency. Note that as $\theta_1 \to 2\pi$, the time dependency is gradually removed; that is, $E[x(t)^2] \to A^2/2$, and as $\theta_1 \to 0$, the random character of the process is gradually lost, so that $E[x(t)^2] \to A^2\sin^2\overline{\omega}_0 t$.

It is important to recognize that gaussian processes result only when the random variables involved are statistically independent.

Example E21-5. Assume random variables r_1 and r_2 as defined in Example E20-7 are used as successive discrete ordinates for all members of random process $x(t)$ given in Example E21-2. What is the joint probability density function for random variables $x(t)$ and $x(t+\tau)$?

First it should be recognized that random variables $x(t)$ and $x(t+\tau)$ are linearly related to random variables r_1 and r_2 in accordance with the first and second of Eqs. (a) in Example E21-2. Since random variables r_1 and r_2 have the normal distribution given by Eq. (f) of Example E20-7, random variables $x(t)$ and $x(t+\tau)$ must also have a normal distribution in accordance with the principle of linear transformation treated in Section 20-9. Thus the probability density functions must be of the form

$$p(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma_{x_1}}\exp\left[-\frac{(x_1-\overline{x}_1)^2}{2\sigma^2_{x_1}}\right]$$

$$p(x_1,x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2_{x_1x_2}}} \times \exp\left\{-\frac{1}{2(1-\rho^2_{x_1x_2})}\right.$$
$$\left.\times\left[\frac{(x_1-\overline{x}_1)^2}{\sigma^2_{x_1}} - \frac{2\rho_{x_1x_2}(x_1-\overline{x}_1)(x_2-\overline{x}_2)}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\overline{x}_2)^2}{\sigma^2_{x_2}}\right]\right\} \tag{a}$$

where $x_1 \equiv x(t)$ and $x_2 \equiv x(t+\tau)$. With the results in Examples E20-7 and E21-2, it is shown that

$$\overline{x_1} = \overline{x_2} = 0$$

$$\sigma_{x_1}^2 = \sigma_{x_2}^2 = R_x(0) = \frac{2}{3}\overline{x_2^2} = \frac{2}{3} \tag{b}$$

$$\rho_{x_1 x_2}(\tau) = \frac{R_x(\tau)}{R_x(0)}$$

$$= \begin{cases} 1 - \dfrac{3\tau^2}{2\Delta\varepsilon^2} + \dfrac{3|\tau|^3}{4\Delta\varepsilon^3} & -\Delta\varepsilon \le \tau \le \Delta\varepsilon \\ 2 - \dfrac{3|\tau|}{\Delta\varepsilon} + \dfrac{3\tau^2}{2\Delta\varepsilon^2} - \dfrac{|\tau|^3}{4\Delta\varepsilon^3} & -2\Delta\varepsilon \le \tau \le -\Delta\varepsilon \\ & \Delta\varepsilon \le \tau \le 2\Delta\varepsilon \\ 0 & \tau \le -2\Delta\varepsilon\ ;\ \tau \ge 2\Delta\varepsilon \end{cases} \tag{c}$$

Substituting Eqs. (b) into Eqs. (a) gives the desired probability density function.

21-9 STATIONARY WHITE NOISE

In the previous discussion on stationary gaussian processes, white noise was defined as a process having a uniform power spectral density function of intensity S_0 over the entire frequency range $-\infty < \bar{\omega} < \infty$, which corresponds to a Dirac delta function of intensity $2\pi S_0$ at the origin for the autocorrelation function. By this definition it is clear that such processes contain frequency components of equal intensity (based on squared amplitude as a measure of intensity) over the entire frequency range thus the random variables at time t and $t+\tau$ are uncorrelated for all $\tau \neq 0$.

In subsequent developments, it will be found desirable to express white-noise processes in an equivalent but quite different manner. To develop this new type of representation, consider the random process

$$x_r(t) = \lim_{N\to\infty} \sum_{k=-N}^{N-1} a_{kr}\eta(t - k\,\Delta t - \varepsilon_r) \qquad r = 1, 2, \cdots \tag{21-86}$$

where coefficients a_{kr} are statistically independent random variables having a zero mean value and sampled in accordance with the arbitrary but prescribed probability density function $p(a)$ shown in Fig. 21-7*a*, Δt is a constant time interval, variables ε_r are statistically independent random phase parameters having the uniform probability density function shown in Fig. 21-7*b*, and $\eta(t)$ is the function defined in Fig. 21-7*c*. The rth member of this ensemble is shown in Fig. 21-7*d*. The uniformly random phase shift ε over a full interval Δt is a necessary condition for the process to be stationary.

The power spectral density function for member $x_r(t)$ can be derived by using Eq. (21-27) in its equivalent form

$$S_{x_r}(\bar{\omega}) = \lim_{N\to\infty} \frac{|Q_{x_r}(i\bar{\omega})|^2}{4\pi N\,\Delta t} \tag{21-87}$$

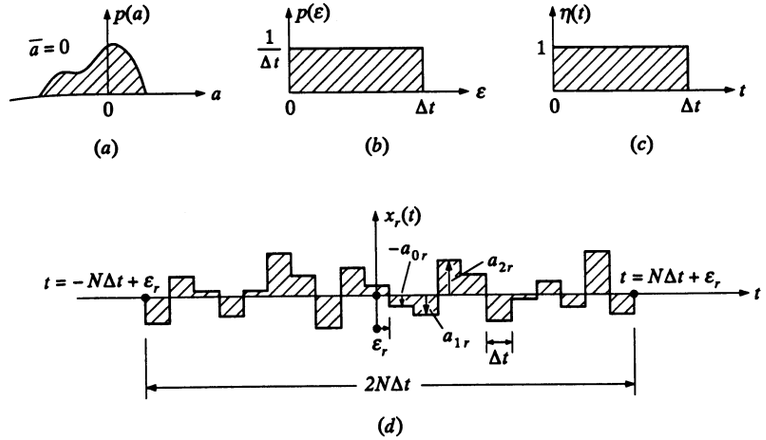


FIGURE 21-7
White-noise process, Eq. (21-86).

where

$$Q_{x_r}(i\bar{\omega}) = \int_{-N\,\Delta t+\varepsilon_r}^{N\,\Delta t+\varepsilon_r} \Big[\sum_{k=-N}^{N-1} a_{kr}\eta(t - k\,\Delta t - \varepsilon_r)\Big] \exp(-i\bar{\omega}t)dt \tag{21-88}$$

When the change of variable $\theta \equiv t - k\,\Delta t - \varepsilon_r$ is substituted into this equation and the order of summation and integration is changed, it becomes

$$Q_{x_r}(i\bar{\omega}) = \sum_{k=-N}^{N-1} a_{kr} \exp[-i\bar{\omega}(k\,\Delta t + \varepsilon_r)] \int_{-(N+k)\Delta t}^{(N-k)\Delta t} \eta(\theta) \exp(-i\bar{\omega}\theta)d\theta \tag{21-89}$$

or

$$Q_{x_r}(i\bar{\omega}) = \frac{i}{\bar{\omega}}[\exp(-i\bar{\omega}\,\Delta t) - 1] \sum_{k=-N}^{N-1} a_{kr} \exp[-i\bar{\omega}(k\,\Delta t + \varepsilon_r)] \tag{21-90}$$

Substituting this equation into Eq. (21-87) gives

$$S_{x_r}(\bar{\omega}) = \lim_{N\to\infty} \frac{1}{4\pi N\,\Delta t\,\bar{\omega}^2}[\exp(-i\bar{\omega}\,\Delta t) - 1][\exp(i\bar{\omega}\,\Delta t) - 1]$$

$$\times \sum_{k=-N}^{N-1} \sum_{j=-N}^{N-1} a_{kr}\,a_{jr} \exp[-i\bar{\omega}(k - j)\Delta t] \tag{21-91}$$

Since the process as defined is stationary but nonergodic, the power spectral density function for the process must be obtained by averaging Eq. (21-91) across the ensemble.

Since random variables a_{kr} and a_{jr} $(r = 1, 2, \cdots, \infty)$ are statistically independent, their covariances, that is, $E(a_{kr}\, a_{jr})$ for $j \neq k$, must all equal zero. Therefore, the double summation in Eq. (21-91) reduces to a single summation, which obviously equals $2N\sigma_a^2$ when averaged with respect to r across the ensemble. Thus, the power spectral density function for the process becomes

$$S_x(\overline{\omega}) = \frac{\sigma_a^2}{2\pi \Delta t\, \overline{\omega}^2}[\exp(-i\overline{\omega}\,\Delta t) - 1][\exp(i\overline{\omega}\,\Delta t) - 1] \tag{21-92}$$

or

$$S_x(\overline{\omega}) = \frac{\sigma_a^2\,\Delta t}{2\pi} \frac{\sin^2[(\overline{\omega}\,\Delta t)/2]}{[(\overline{\omega}\,\Delta t)/2]^2} \tag{21-93}$$

When $\sigma_a^2 \to \infty$ and $\Delta t \to 0$ in such a way that $\sigma_a^2\,\Delta t = C$ (a constant), this equation becomes

$$S_x(\overline{\omega}) = \frac{C}{2\pi} = S_0 \tag{21-94}$$

showing that the process becomes white noise in the limit.

As a special case of the above process, let the probability density function $p(a)$ consist of two Dirac delta functions of intensity 1/2 located at $a = \pm A$. This process becomes white noise having a uniform power spectral density function of intensity $S_0 = C/2\pi$ when $A^2 \to \infty$, $\Delta t \to 0$, and $A^2\,\Delta t \to C$.

Example E21-6. For the stationary random process defined in Example E20-5, (1) show that this process approaches white noise in the limit as $\Delta\varepsilon \to 0$ and (2) find the normalization factor C which would force this limiting process to have a constant power spectral density equal to S_0.

From the form of the autocorrelation function given in Example E21-2

$$\lim_{\Delta\varepsilon \to 0} R_x(\tau) \begin{cases} = 0 & \tau \neq 0 \\ \neq 0 & \tau = 0 \end{cases} \tag{a}$$

which suggests the form of a Dirac delta function. Integrating $R_x(\tau)$ over the infinite τ domain gives

$$\int_{-\infty}^{\infty} R_x(\tau)d\tau = 2\overline{x^2}\left\{\int_0^{\Delta\varepsilon}\left(\frac{2}{3} - \frac{\tau^2}{\Delta\varepsilon^2} + \frac{\tau^3}{2\Delta\varepsilon^3}\right)d\tau + \int_{\Delta\varepsilon}^{2\Delta\varepsilon}\left(\frac{4}{3} - \frac{2\tau}{\Delta\varepsilon} + \frac{\tau^2}{\Delta\varepsilon^2} - \frac{\tau^3}{6\Delta\varepsilon^3}\right)d\tau\right\}$$

or

$$\int_{-\infty}^{\infty} R_x(\tau)d\tau = \overline{x^2}\,\Delta\varepsilon = \Delta\varepsilon \tag{b}$$

Multiplying all discrete ordinates of process $x(t)$ by the constant $(2\pi S_0/\Delta\varepsilon)^{1/2}$ giving a new process $a(t)$, following the above procedures, would demonstrate that

$$\lim_{\Delta\varepsilon \to 0} R_a(\tau) \begin{cases} = 0 & \tau \neq 0 \\ \neq 0 & \tau = 0 \end{cases} \qquad \int_{-\infty}^{\infty} R_a(\tau)d\tau = 2\pi S_0 \tag{c}$$

thus showing that

$$R_a(\tau) \to 2\pi S_0\,\delta(\tau) \tag{d}$$

which means that process $a(t)$ approaches white noise of intensity S_0. Therefore the normalization factor C is given by

$$C = \left(\frac{2\pi S_0}{\Delta\varepsilon}\right)^{1/2} \tag{e}$$

The solution to this example can be obtained more easily by noting that the power spectral density function for the process $x(t)$, as given by Eq. (b) in Example E21-3, becomes in the limit

$$\lim_{\Delta\varepsilon \to 0} S_x(\overline{\omega}) = \frac{\overline{x^2}\,\Delta\varepsilon}{2\pi} = \frac{\Delta\varepsilon}{2\pi} \tag{f}$$

Likewise the limiting power spectral density function for process $a(t)$ would be

$$\lim_{\Delta\varepsilon \to 0} S_a(\overline{\omega}) = S_0 \tag{g}$$

again showing that the normalization factor is given by Eq. (e).

21-10 PROBABILITY DISTRIBUTION FOR MAXIMA[1]

Consider a zero-mean stationary gaussian process $x(t)$ having an arbitrary power spectral density function $S_x(p)$. A sample function taken from this process (Fig. 21-8) shows positive and negative maxima and positive and negative minima. From

[1] D. E. Cartwright and M. S. Longuet-Higgins, "The Statistical Distributions of the Maxima of a Random Function," *Proc. R. Soc.*, Ser. A, Vol. 237, pp. 212-232, 1956; A. G. Davenport, "Note on the Distribution of the Largest Value of a Random Function with Application to Gust Loading," *Proc. Inst. Civ. Eng.*, Vol. 28, pp. 187-196, 1964.

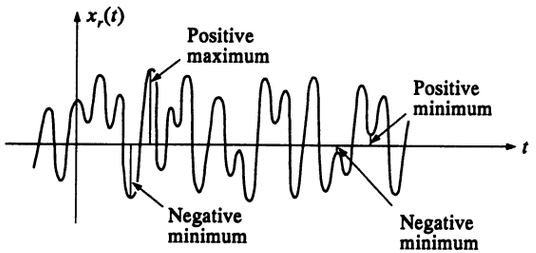


FIGURE 21-8
Sample function of process $x(t)$.

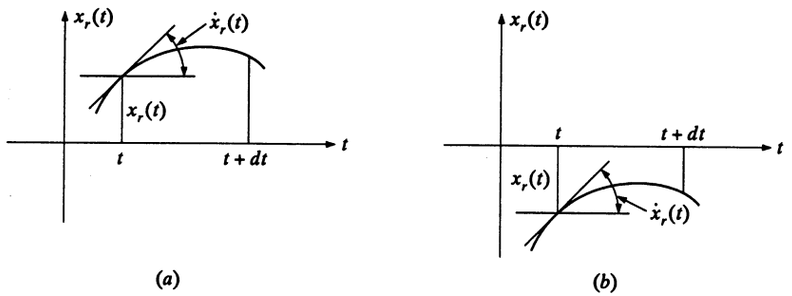


FIGURE 21-9
Maxima occurring in the time interval $(t, t+dt)$: (*a*) positive maxima; (*b*) negative maxima.

Fig. 21-9, it is clear that for a maximum (+ or −) to occur in the time interval $(t, t+dt)$, it is necessary that $\dot{x}_r(t)$ be positive and $\ddot{x}_r(t)$ be negative and that

$$0 < \dot{x}_r(t) < |\ddot{x}_r(t)|\, dt \tag{21-95}$$

With the definition of three new random variables $\zeta_1 \equiv x(t)$, $\zeta_2 \equiv \dot{x}(t)$, and $\zeta_3 \equiv \ddot{x}(t)$, the probability density function $p(\zeta_1, \zeta_2, \zeta_3)$ can be written in its normal form

$$p(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{(2\pi)^{3/2}\,|\boldsymbol{\mu}|^{1/2}} \exp\left\{ -\frac{1}{2}[\boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}}]^T \boldsymbol{\mu}^{-1} [\boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}}] \right\} \tag{21-96}$$

where $\boldsymbol{\zeta}$ is the vector $[\zeta_1, \zeta_2, \zeta_3]^T$, $\overline{\boldsymbol{\zeta}}$ is the vector $[\overline{\zeta}_1, \overline{\zeta}_2, \overline{\zeta}_3]^T = \mathbf{0}$, and $\boldsymbol{\mu}$ is the covariance matrix

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} \tag{21-97}$$

where

$$\mu_{ik} = E(\zeta_i\, \zeta_k) \tag{21-98}$$

When

$$m_n = \int_{-\infty}^{\infty} \overline{\omega}^n\, S_x(\overline{\omega})\, d\overline{\omega} \tag{21-99}$$

it is easily shown, using the techniques of derivation in Section 21-6, that

$$\boldsymbol{\mu} = \begin{bmatrix} m_0 & 0 & -m_2 \\ 0 & m_2 & 0 \\ -m_2 & 0 & m_4 \end{bmatrix} \tag{21-100}$$

Thus, Eq. (21-96) becomes

$$p(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{(2\pi)^{3/2}\,(m_2\,\Delta)^{1/2}} \times \exp\left[-1/2\left(\frac{\zeta_2^2}{m_2} + \frac{m_4\,\zeta_1^2 + 2m_2\,\zeta_1\,\zeta_3 + m_0\,\zeta_3^2}{\Delta} \right) \right] \tag{21-101}$$

where

$$\Delta \equiv m_0\, m_4 - m_2^2 \tag{21-102}$$

From Fig. 21-10 it becomes apparent that the probability of a maximum (+ or −) occurring in the range $(\zeta_1, \zeta_1 + d\zeta_1)$ during the time interval $(t, t+dt)$ is expressed by the relation

$$F(\zeta_1) d\zeta_1\, dt = \left[\int_{-\infty}^{0} p(\zeta_1, 0, \zeta_3)\, |\zeta_3|\, d\zeta_3 \right] d\zeta_1\, dt \tag{21-103}$$

Thus, it follows that the mean frequency of occurrence of maxima (+ or −) over the complete range $-\infty < \zeta_1 < \infty$ is given by

$$N_1 \equiv \int_{-\infty}^{\infty} \left[\int_{-\infty}^{0} p(\zeta_1, 0, \zeta_3)\, |\zeta_3|\, d\zeta_3 \right] d\zeta_1 \tag{21-104}$$

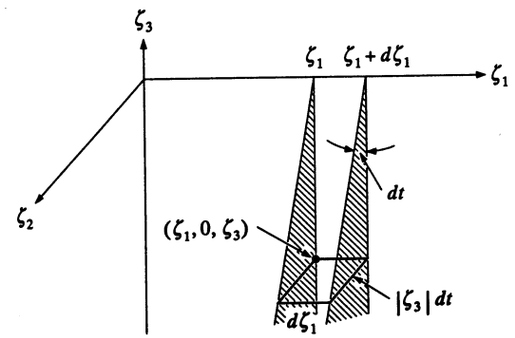


FIGURE 21-10
Shaded region satisfies the conditions for a maximum (+ or −) occurring in the range $(\zeta_1, \zeta_1 + d\zeta_1)$ and in the time interval $(t, t+dt)$.

Substituting Eq. (21-101) for $\zeta_2 = 0$ into this relation and carrying out the double integration leads to Rice's equation

$$N_1 = \frac{1}{2\pi}\left(\frac{m_4}{m_2}\right)^{1/2} \tag{21-105}$$

Since the probability density function for maxima is by definition the ratio $F(\zeta_1)/N_1$, it can be obtained using Eqs. (21-103) and (21-105). In doing so, it is convenient to express maxima in the nondimensional form

$$\zeta \equiv \frac{\zeta_1}{m_0^{1/2}} \tag{21-106}$$

allowing its probability density function to be expressed in the form

$$p(\eta) = \frac{1}{(2\pi)^{1/2}}\left[\varepsilon e^{-\eta^2/2\varepsilon^2} + (1-\varepsilon^2)^{1/2}\,\eta e^{-\eta^2/2}\int_{-\infty}^{[\eta(1-\varepsilon^2)^{1/2}]/\varepsilon} e^{-x^2/2}dx\right] \tag{21-107}$$

where

$$\varepsilon^2 \equiv \frac{m_0 m_4 - m_2^2}{m_0 m_4} = \frac{\Delta}{m_0 m_4} \tag{21-108}$$

From Eq. (21-99) it can easily be shown that Δ is always positive; therefore, ε, as defined by Eq. (21-108), must always be in the range

$$0 < \varepsilon < 1 \tag{21-109}$$

Equation (21-107) is plotted in Fig. 21-11 for different values of ε throughout this range. Note that for a narrow-band process approaching the single harmonic process given by Eq. (21-2), $\varepsilon \to 0$, in which case Eq. (21-107) reduces to the form of a Rayleigh distribution, Eq. (20-92). When the process is white noise or band-limited white noise, as given by Eq. (21-75), $\varepsilon = 2/3$. The limiting case $\varepsilon = 1$ can be approached by superposition of a single harmonic process $y(t)$ at frequency $\bar{\omega}_2$ and a band-limited process $z(t)$ within the frequency range $-\bar{\omega}_1 < \bar{\omega} < \bar{\omega}_1$, provided that $\bar{\omega}_2/\bar{\omega}_1 \to \infty$ and $\sigma_y^2/\sigma_z^2 \to 0$. This is equivalent to placing a very high-frequency, low-amplitude "dither" signal on top of a low-frequency band-limited signal. The resulting distribution of maxima as given by Eq. (21-107) approaches the form of a gaussian distribution.

If the value of ε is to be estimated using a single sample waveform from process $x(t)$, this can easily be accomplished by first counting the total number of maxima (+ and −) N and the number of negative maxima N^- occurring in a sample function of reasonable duration. Dividing N^- by N gives the proportion r of negative maxima present in the total, which must be equal to the area under the probability density

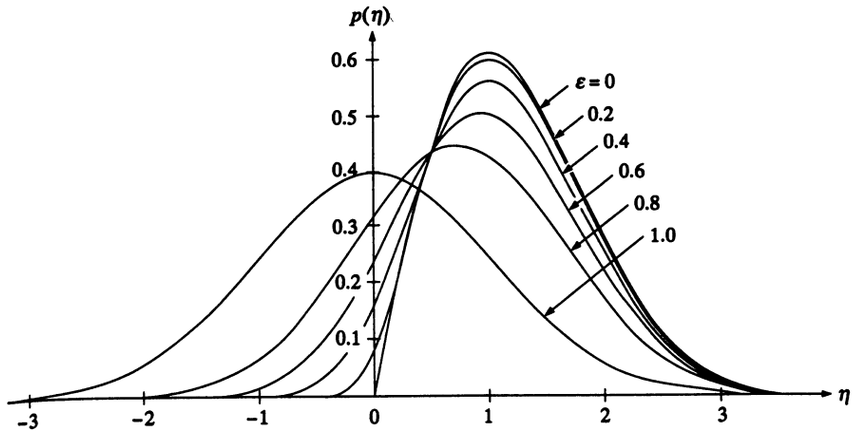


FIGURE 21-11
Probability density function for maxima for different values of ε.

function $p(\eta)$ to the left of the origin in Fig. 21-11. It can be shown that ε is approximately related to this area by

$$\varepsilon^2 = 4r(1-r) \tag{21-110}$$

Thus after $r = N^-/N$ has been determined, ε can immediately be estimated by this relation.

Example E21-7. Compute the numerical value of ε for stationary process $x(t)$ which has a uniform power spectral density function of intensity S_0 over the ranges $-\bar{\omega}_2 < \bar{\omega} < -\bar{\omega}_1$ and $\bar{\omega}_1 < \bar{\omega} < \bar{\omega}_2$ as given by Eqs. (21-76).

Substituting the first of Eqs. (21-76) into Eq. (21-99) and completing the integral for $n = 0$, 2, and 4 gives, respectively,

$$\begin{aligned} m_0 &= 2S_0(\bar{\omega}_2 - \bar{\omega}_1) \\ m_2 &= \frac{2S_0}{3}(\bar{\omega}_2^3 - \bar{\omega}_1^3) \\ m_4 &= \frac{2S_0}{5}(\bar{\omega}_2^5 - \bar{\omega}_1^5) \end{aligned} \tag{a}$$

Substituting these relations into Eq. (21-108) yields

$$\varepsilon^2 = 1 - \frac{5}{9}\frac{(1-\gamma^3)^2}{(1-\gamma)(1-\gamma^5)} \tag{b}$$

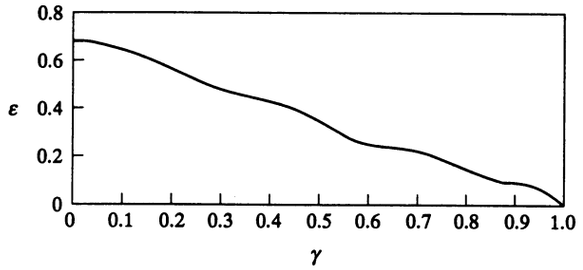


FIGURE E21-5
Parameter ε versus frequency ratio $\overline{\omega}_1/\overline{\omega}_2$.

where γ is the dimensionless frequency parameter

$$\gamma = \frac{\overline{\omega}_1}{\overline{\omega}_2} \tag{c}$$

Equation (b) is plotted in Fig. E21-5, showing that ε varies in an approximately linear fashion from a value of $2/3$ at $\gamma = 0$ to a value of zero at $\gamma = 1$, thus (from Fig. 21-11) showing how the probability density function for maxima approaches the Rayleigh distribution as the frequency bandwidth narrows.

21-11 PROBABILITY DISTRIBUTION FOR EXTREME VALUES[2]

Consider N independently observed maxima having the probability density function $p(\eta)$ given by Eq. (21-107). The probability (Pr) that all N maxima will be less than η is given by

$$\Pr\ (\text{all } N \text{ maxima} < \eta) = P(\eta)^N \tag{21-111}$$

where $P(\eta)$ is the probability distribution function for maxima as defined by

$$P(\eta) \equiv \int_{-\infty}^{\eta} p(\eta)\, d\eta \tag{21-112}$$

Obviously, the probability distribution function for the largest maximum (extreme value) must also be given by Eq. (21-111), that is,

$$P_e(\eta) = P(\eta)^N \tag{21-113}$$

[2] D. E. Cartwright and M. S. Longuet-Higgins, 'The Statistical Distribution of the Maxima of a Random Function," loc. cit.; A. G. Davenport, "Note on the Distribution of the Largest Value of a Random Function with Application to Gust Loading," loc. cit.

Taking the derivative of Eq. (21-113) gives the probability density function for the extreme value in the form

$$P_e(\eta) = N\, P(\eta)^{N-1} p(\eta) \tag{21-114}$$

For large values of N, it is quite apparent that relatively large values of η_e (extreme value) are of interest; therefore the accuracy with which the extreme-value distribution $P_e(\eta)$ can be defined depends very much on the accuracy of the function $P(\eta)$ as it approaches unity asymptotically with increasing values of η.

Using Eqs. (21-107), (21-112), and (21-113), Davenport has shown, relying in part on earlier work by Cartwright and Lonquet-Higgins, that the probability distribution function for extreme values η_e is

$$P(\eta_e) = \exp\left[-\nu\, T \exp\left(-\frac{\eta_e^2}{2}\right)\right] \tag{21-115}$$

in which ν, the mean frequency of occurrence of zero crossings with positive slope, is given by

$$\nu \equiv \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right)^{1/2} \tag{21-116}$$

The corresponding probability density function $p(\eta_e)$ can easily be obtained by differentiating Eq. (21-115) with respect to η_e.

Using the extreme-value probability distribution function given by Eq. (21-115), it has been shown by Davenport that the mean extreme value is given by the approximate relation

$$\overline{\eta}_e \doteq (2 \ln \nu\, T)^{1/2} + \frac{\gamma}{(2 \ln \nu\, T)^{1/2}} \tag{21-117}$$

in which γ is Euler's constant, equal to 0.5772, and that the standard deviation of the extreme values is given by

$$\sigma_{\eta_e} = \frac{\pi}{\sqrt{6}} \frac{1}{(2 \ln \nu\, T)^{1/2}} \tag{21-118}$$

Figure 21-12 shows a plot of the probability density function for process $x(t)$, a plot of the probability density function for maxima $\eta(\varepsilon = 2/3)$, and plots of the extreme-value probability density function for four different values of $\nu\, T(10^2, 10^3, 10^4, 10^5)$. It should be noted that the probability density functions for extreme values are sharply peaked and that the degree of peaking increases with increasing values of $\nu\, T$. Because of this characteristic, engineering designs can often be based on the mean extreme value $\overline{\eta}_e$ as expressed by Eq. (21-117), which is plotted in Fig. 21-13. It

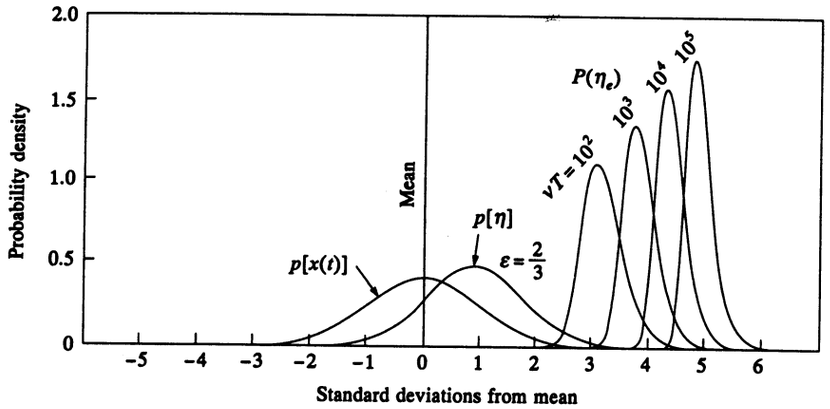


FIGURE 21-12
Probability density functions for $x(t)$, η, and η_e.

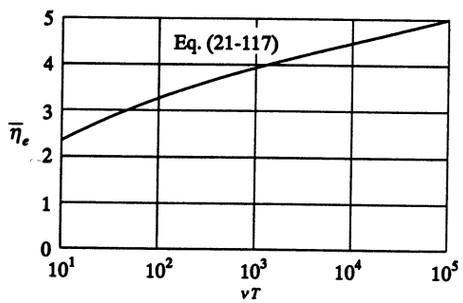


FIGURE 21-13
Normalized mean extreme-value vs. νT.

is clear from this figure that arbitrarily assuming $\overline{\eta}_e$ equal to 3, as is often done in practice, can be considerably on the unconservative side for large values of νT.

Since the general form of the probability distribution function for largest maxima $P_e(\eta)$ closely depends on the accuracy of the probability distribution function for maxima $P(\eta)$ as it nears unity with increasing values of η, other forms of $P_e(\eta)$ have been derived by making various assumptions regarding the manner in which $P(\eta)$ approaches unity. One such assumption is that $P(\eta)$ approaches unity in the manner

$$P(\eta) = 1 - e^{-\eta} \tag{21-119}$$

With this asymptotic form, the extreme-value distribution (Gumbel Type I)[3] can be

[3] E. J. Gumbel and P. G. Carlson, "Extreme Values in Aeronautics," *Jour. of Aero. Sci.*, pp. 389-398, June, 1954; E. J. Gumbel, "Probability Tables for the Analysis of Extreme-Value Data," *Natl. Bur. Stds. Appl. Math. Ser.* 22, July, 1953.

expressed as

$$P(\eta_e) = \exp\{-\exp[-\alpha(\eta_e - u)]\} \tag{21-120}$$

where α and u are constants. Since the second derivative of Eq. (21-120) vanishes for $\eta_e = u$, constant u must equal the most probable value of η_e. Equation (21-120) gives the mean and standard deviation for the extreme values in the forms

$$\overline{\eta}_e = u + \frac{\gamma}{\alpha} \tag{21-121}$$

$$\sigma_{\eta_e} = \frac{\pi}{\sqrt{6}\,\alpha} \tag{21-122}$$

where γ is Euler's constant (0.5772). From Eq. (21-122) it is clear that constant α is a measure of the dispersion of the extreme values.

If a very large number of experimental extreme values are known, the mean and standard deviation can be calculated fairly accurately, whereupon Eqs. (21-121) and (21-122) can be used to solve for α and u. However, if the number of extreme values is relatively small, a correction should be made using the procedure reported by Gumbel.

The lognormal probability density function given previously by Eq. (20-40) is often used as the probability density function for extreme values. Its use requires finding not only the mean and standard deviation of the extreme values but their median value (50 percentile) as well.

Example E21-8. The extreme values of 50 sample members of random process $x(t)$ have been measured giving the following numerical values, the absolute values of which have been arranged in order of rank:

0.82	1.14	-1.54	1.97	2.67
-0.90	1.16	1.60	1.99	-2.74
0.98	-1.20	-1.64	-2.02	-2.98
-1.03	1.29	-1.67	-2.09	3.33
-1.06	-1.39	1.70	2.11	3.50
1.08	-1.44	1.75	2.13	-3.63
1.10	1.46	-1.77	-2.23	3.85
-1.11	1.48	1.84	2.37	-4.07
-1.12	-1.50	-1.90	-2.51	-4.18
-1.13	1.51	-1.93	2.60	4.33

Assuming the positive and negative extreme values have the same Gumbel Type I probability distribution, generate its proper relation in the form of Eq. (21-120).

An approximation of the distribution function can be obtained using Eqs. (21-121) and (21-122). When the signs of the measured extreme values are ignored, the result is

$$\bar{x}_e = \frac{1}{50}\sum_{i=1}^{50} x_{e_i} = 1.97 \qquad \sigma_{x_e}^2 = \frac{1}{50}\sum_{i=1}^{50}(x_{e_i} - \bar{x}_e)^2 = 0.839 \tag{a}$$

Using Eqs. (21-121) and (21-122) gives

$$\alpha = \frac{\pi}{\sqrt{6}\,\sigma_{x_e}} = 1.40 \qquad u = \bar{x}_e - \frac{0.577}{\alpha} = 1.56 \tag{b}$$

Substituting Eqs. (b) into Eq. (21-120) results in

$$P(x_e) \doteq \exp\left\{-\exp\left[-1.40(x_e - 1.56)\right]\right\} \tag{c}$$

Using the correction as given by Gumbel for the case of 50 sample values gives the more accurate expression

$$P(x_e) = \exp\left\{-\exp\left[-1.27(x_e - 1.54)\right]\right\} \tag{d}$$

21-12 NONSTATIONARY GAUSSIAN PROCESSES

A stationary process has previously been defined as one for which all ensemble averages are independent of time; therefore, a nonstationary process is one for which these same ensemble averages are time dependent. Thus the ensemble average $E[x(t)x(t+\tau)]$, which completely characterizes a nonstationary gaussian process $x(t)$, will be dependent upon time t as well as the time interval τ.

In engineering, a nonstationary process $x(t)$ can often be represented fairly well using the quasi-stationary form

$$x(t) = \zeta(t)\, z(t) \tag{21-123}$$

where $\zeta(t)$ is a fully prescribed function of time and $z(t)$ is a stationary process. If $z(t)$ is a gaussian process, $x(t)$ will also be gaussian, in which case the covariance function

$$E[x(t)x(t+\tau)] = \zeta(t)\,\zeta(t+\tau)\,R_z(\tau) \tag{21-124}$$

completely characterizes the process.

The above characterization of nonstationary gaussian processes involving one independent variable can be extended directly to processes involving more than one independent variable.

21-13 STATIONARY GAUSSIAN PROCESS: TWO OR MORE INDEPENDENT VARIABLES

All the stationary gaussian processes characterized previously involved one independent variable which was considered to be time t. The basic concepts developed for these processes will now be extended to stationary gaussian processes involving two or more independent variables. To illustrate this extension, suppose the variable of interest is random not only with respect to time but with respect to certain space coordinates as well. For example, consider the wind drag force per unit height acting on a tall industrial smokestack during a strong windstorm, as described in Section 21-1. This loading involves two independent variables, x and t.

To characterize the random component of drag $p(x,t)$ in a probabilistic sense, it is necessary to establish probability density functions involving random variables $p(x,t)$ and $p(\alpha, t+\tau)$, where α and τ are dummy space and time variables, respectively. If the process is gaussian, these probability density functions will be completely known provided the covariance function as given by the ensemble average $E[p(x,t)p(\alpha,t+\tau)]$ can be defined. If the process is stationary, this ensemble average will be independent of time but will depend upon the time difference τ, in which case the covariance function defined by the relation

$$R_p(x,\alpha,\tau) \equiv E[p(x,t)p(\alpha,t+\tau)] \tag{21-125}$$

completely characterizes the process.

Assuming the above process is ergodic, that is, the mean wind velocity remains constant for all members of the ensemble, the cross-spectral density function for the rth member, that is,

$$\mathrm{S}_{p_r}(x,\alpha,\bar{\omega}) \equiv \lim_{s\to\infty} \frac{\left[\int_{-s/2}^{s/2} p_r(x,t)\exp(-i\bar{\omega}t)dt\right]\left[\int_{-s/2}^{s/2} p_r(\alpha,t)\exp(+i\bar{\omega}t)dt\right]}{2\pi s} \tag{21-126}$$

will also characterize the process. This cross-spectral density function is related to the covariance function through the Fourier transform relations

$$\begin{aligned} \mathrm{S}_p(x,\alpha,\bar{\omega}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_p(x,\alpha,\tau)\exp(-i\bar{\omega}\tau)\,d\tau \\ R_p(x,\alpha,\tau) &= \int_{-\infty}^{\infty} \mathrm{S}_p(x,\alpha,\bar{\omega})\exp(i\bar{\omega}\tau)\,d\bar{\omega} \end{aligned} \tag{21-127}$$

Extending the above characterizations to stationary gaussian processes involving more than two independent variables is straightforward. For example, to characterize a field potential $\Phi(x,y,z,t)$ which is random with respect to time and each space coordinate, one must establish either the covariance function

$$R_\Phi(x,y,z,\alpha,\beta,\gamma,\tau) \equiv E[\Phi(x,y,z,t)\Phi(\alpha,\beta,\gamma,t+\tau)] \tag{21-128}$$

or the corresponding cross-spectral density function $S_\Phi(x, y, z, \alpha, \beta, \gamma, \overline{\omega})$. Terms α, β, and γ are dummy variables for x, y, and z, respectively.

If the field potential $\Phi(x, y, z, t)$ is homogeneous, the covariance and cross-spectral density functions depend only on the differences in coordinates, that is, on

$$X \equiv x - \alpha \qquad Y \equiv y - \beta \qquad Z \equiv z - \gamma \tag{21-129}$$

The process is then characterized either by the function $R_\Phi(X, Y, Z, \tau)$ or by the function $S_\Phi(X, Y, Z, \overline{\omega})$.

If the potential function $\Phi(x, y, z, t)$ happens to be isotropic as well as homogeneous, the covariance and cross-spectral density functions will depend only upon the distance between points, that is, the distance

$$\rho \equiv [(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2]^{1/2} \tag{21-130}$$

in which case the process will be characterized by either $R_\Phi(\rho, \tau)$ or $S_\Phi(\rho, \overline{\omega})$.

PROBLEMS

21-1. Show that the Fourier transform of an even function and of an odd function are real and imaginary, respectively.

21-2. Find the Fourier transform of each function $x(t)$ shown in Fig. P21-1.

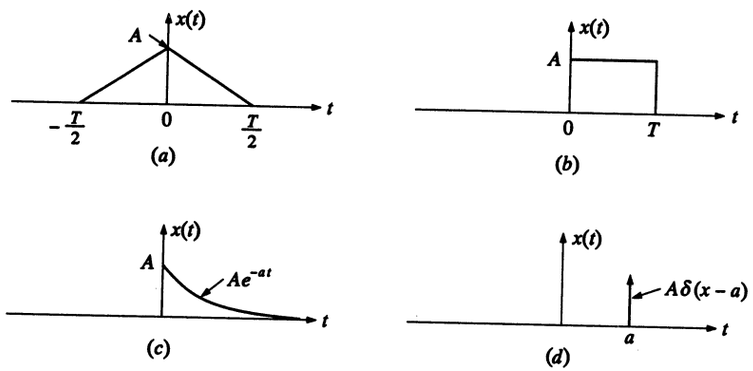


FIGURE P21-1
Functions $x(t)$ referred to in Prob. 21-2.

21-3. Consider the function $x(t) = A \cos at$ in the range $-T/2 < t < T/2$ and $x(t) = 0$ outside this range. Find and sketch the Fourier transform $X(\overline{\omega})$ when (*a*) $T = \pi/a$, (*b*) $T = 3\pi/a$, (*c*) $T = 5\pi/a$, and (*d*) $T \to \infty$.

21-4. Evaluate the integral

$$I = \int_1^\infty \left[\int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right] dx$$

by integrating first with respect to y and then with respect to x. Then reverse the order of integration and reevaluate the integral. Finally evaluate the limit L of integral I by integrating over the finite domain and then taking the limit as follows:

$$L = \lim_{T \to \infty} \left\{ \int_1^T \left[\int_1^T \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right] dx \right\}$$

Noting that the integrand in integral I is antisymmetric about the line $x = y$, which form of integration would you recommend for engineering applications?

21-5. Evaluate the integral

$$I = \int_{-\infty}^\infty \frac{\sin^2 x}{x^2} dx$$

21-6. Consider the stationary random process $x(t)$ defined by

$$x_r(t) = \sum_{n=1}^{10} A_{nr} \cos(n\overline{\omega}_0 t + \theta_{nr}) \qquad r = 1, 2, \cdots$$

where $x_r(t)$ = rth member of ensemble
A_{nr} = sample values of random variable A
$\overline{\omega}_0$ = fixed circular frequency
θ_{nr} = sample values of random phase angle θ having a uniform probability density function in range $0 < \theta < 2\pi$ of intensity $1/2\pi$

If random variable A is gaussian having a known mean value $\overline{A}$ and a known variance σ_A^2, find the ensemble mean value of $x(t)$ and the ensemble variance of $x(t)$. Is process $x(t)$ a gaussian process?

21-7. Derive the autocorrelation function for the stationary random process $x(t)$ defined in Prob. 21-6.

21-8. Derive the power spectral density function for the stationary random process $x(t)$ defined in Prob. 21-6, assuming that Dirac delta functions are permitted in the answer.

21-9. A stationary random process $x(t)$ has the autocorrelation function

$$R_x(\tau) = A \exp(-a|\tau|)$$

where A and a are real constants. Find the power spectral density function for this process.

21-10. Consider a random process $x(t)$ which takes the value $+A$ or $-A$, with equal probability, throughout each interval $n\Delta\varepsilon < t < (n+1)\Delta\varepsilon$ of each member of the process, where n is an integer running frm $-\infty$ to $+\infty$. Find and plot the ensemble covariance function $E[x(t)x(t+\tau)]$. Is this process stationary or nonstationary?

21-11. If the origin of time, that is, $t = 0$, for each member of process $x(t)$ defined in Prob. 21-10 is selected randomly over an interval $\Delta\varepsilon$ with uniform probability of occurrence, what is the covariance function $E[x(t)x(t+\tau)]$? Is this process stationary or nonstationary?

21-12. Assuming that you find the process in Prob. 21-11 stationary, what are the autocorrelation and power spectral density functions for this process? Use Eqs. (21-35) and (21-38) in finding the power spectral density function.

21-13. Show that the autocorrelation and power spectral density functions obtained in Prob. 21-12 are Fourier transform pairs in accordance with Eqs. (21-37).

21-14. Each member of a stationary random process $x(t)$ consists of a periodic infinite train of triangular pulses, as shown in Fig. P21-2. All members of the process are identical except for phase, which is a random variable uniformly distributed over the interval $(0, T)$. Assuming that the period T is not less than $2a$, where a is the duration of a single pulse, find the autocorrelation function for this process.

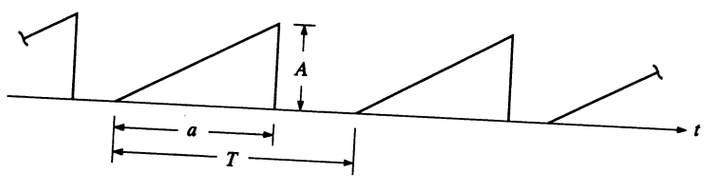


FIGURE P21-2
One sample member of process $x(t)$ referred to in Prob. 21-14.

21-15. Each member of a random process $x(t)$ consists of the superposition of rectangular pulses of duration $\Delta\varepsilon$ and of constant intensity A which are located in a random fashion with respect to time as shown in Fig. P21-3*a*. Each value of ε_n is sampled in accordance with the uniform probability density function $p(\varepsilon)$ given in Fig. P21-3*b*. What is the ensemble value $\overline{x(t)}$ for this process? Is this process stationary or nonstationary?

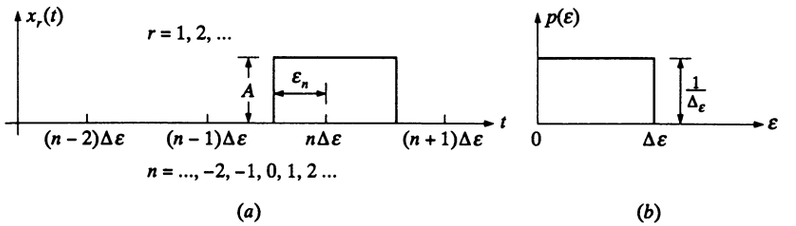


FIGURE P21-3
One sample pulse in member $x_r(t)$ of process $x(t)$ and probability density function for random variable ε referred to in Prob. 21-15.

21-16. Assume the autocorrelation and power spectral density functions $R_{xx}(\tau)$ and $S_{xx}(\overline{\omega})$ for a stationary random process $x(t)$ are known. Derive the expressions for $S_{x\dot{x}}(\overline{\omega})$, $S_{\dot{x}x}(\overline{\omega})$, $S_{x\ddot{x}}(\overline{\omega})$, $S_{\ddot{x}x}(\overline{\omega})$, $S_{\dot{x}\ddot{x}}(\overline{\omega})$, $S_{\ddot{x}\dot{x}}(\overline{\omega})$ and $R_{x\dot{x}}(\tau)$, $R_{\dot{x}x}(\tau)$, $R_{x\ddot{x}}(\tau)$, $R_{\ddot{x}x}(\tau)$, $R_{\dot{x}\ddot{x}}(\tau)$, $R_{\ddot{x}\dot{x}}(\tau)$ in terms of $S_{xx}(\overline{\omega})$ and $R_{xx}(\tau)$, respectively.

21-17. Considering two stationary random processes $x(t)$ and $y(t)$, show that $S_{yx}(\overline{\omega})$ is the complex conjugate of $S_{xy}(\overline{\omega})$.

21-18. Two stationary random processes $x(t)$ and $y(t)$ have the joint probability density function

$$p[x(t)y(t+\tau)] = \frac{1}{2ab\tau\sqrt{1-c^2}} \exp\left[-\frac{1}{2(1-c^2)}\left(\frac{x^2}{a^2} - \frac{2cxy}{ab} + \frac{y^2}{b^2}\right)\right]$$

Define a, b and c in terms of the appropriate autocorrelation and/or cross-correlation functions for processes $x(t)$ and $y(t)$. What is the corresponding joint probability density function $p[\dot{x}(t)\dot{y}(t+\tau)]$? Define the coefficients in this function in terms of the appropriate autocorrelation and/or cross-correlation functions for processes $x(t)$ and $y(t)$.