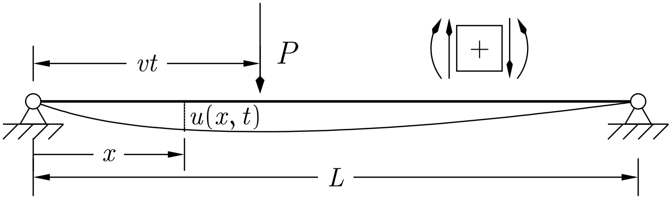
Continuous Systems

an example

Giacomo Boffi

Dipartimento di Ingegneria Strutturale, Politecnico di Milano

Problem statement



A uniform beam ($m(x) = m$, $EJ(x) = EJ$) of lenght L is loaded by a moving load P, moving with constant velocity, $v(t) = v$, in the interval $0 \le t \le t_0 = {}^{L}\!/_{v} = t_0$.

Using the sign conventions indicated above, compute and plot the midspan displacement $u({}^{L}\!/_{2}, t)$ and the midspan bending moment $M_{\mathsf{b}}({}^{L}\!/_{2}, t)$ as functions of time in the interval $0 \le t \le t_0$ for different values of the velocity.

NB: the beam is at rest for $t = 0$.

Equation of motion

For an uniform beam, the equation of dynamic equilibrium is

$$m\,\frac{\partial^2 u(x,t)}{\partial t^2} + EJ\,\frac{\partial^4 u(x,t)}{\partial x^4} = p(x,t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x,t) = P\,\delta(x - vt).$$

Equation of motion

For an uniform beam, the equation of dynamic equilibrium is

$$m\,\frac{\partial^2 u(x,t)}{\partial t^2} + EJ\,\frac{\partial^4 u(x,t)}{\partial x^4} = p(x,t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x,t) = P\,\delta(x - vt).$$

The Dirac's delta is a *generalized* function of one variable, defined by

$$\delta(x - x_0) \equiv 0 \qquad \text{and} \qquad \int f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0).$$

Note that the Dirac distribution and the Kronecker's symbol δ_{ij} are two different things.

Equation of motion

The solution will be computed by separation of variables

$$u(x,t) = q(t)\Phi(x)$$

and modal analysis,

$$u(x,t) = \sum_{n=1}^{\infty} q_n(t)\Phi_n(x)$$

Equation of motion

The solution will be computed by separation of variables

$$u(x,t) = q(t)\phi(x)$$

and modal analysis,

$$u(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\phi_n(x) = \sin\beta_n x, \qquad \beta_n = \frac{n\pi}{L},$$
$$m_n = \frac{mL}{2}, \qquad \omega_n^2 = \beta_n^4 \frac{EJ}{m} = n^4\pi^4 \frac{EJ}{mL^4}.$$

Orthogonality relationships

For an uniform beam, the orthogonality relationships are

$$m \int_0^L \phi_n(x)\phi_m(x)\,\mathrm{d}x = m_n\delta_{nm},$$
$$EJ \int_0^L \phi_n(x)\phi_m^{\mathsf{IV}}(x)\,\mathrm{d}x = k_n\delta_{nm} = m_n\omega_n^2\delta_{nm}.$$

in the equations above δ is the Kroneker's δ symbol, a completely different thing from Dirac's δ distribution.

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the modal series representation of $u(x, t)$:

$$m \sum_{m=1}^{\infty} \ddot{q}_m \phi_m + EJ \sum_{m=1}^{\infty} q_m \phi_m^{\mathsf{IV}} = P\,\delta(x - vt),$$

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the modal series representation of $u(x,t)$:

$$m \sum_{m=1}^{\infty} \ddot{q}_m \phi_m + EJ \sum_{m=1}^{\infty} q_m \phi_m^{\mathsf{IV}} = P\,\delta(x - vt),$$

2. every term is multiplied by ϕ_n and integrated over the lenght of the beam

$$m \int_0^L \phi_n \sum_{m=1}^{\infty} \ddot{q}_m \phi_m \,\mathrm{d}x + EJ \int_0^L \phi_n \sum_{m=1}^{\infty} q_m \phi_m^{\mathsf{IV}} \,\mathrm{d}x = P \int_0^L \phi_n \delta(x - vt), \qquad n = 1, \ldots, \infty$$

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the modal series representation of $u(x,t)$:

$$m\sum_{m=1}^{\infty}\ddot{q}_m\phi_m + EJ\sum_{m=1}^{\infty}q_m\phi_m^{\mathsf{IV}} = P\,\delta(x-vt),$$

2. every term is multiplied by ϕ_n and integrated over the lenght of the beam

$$m\int_0^L\phi_n\sum_{m=1}^{\infty}\ddot{q}_m\phi_m\,\mathrm{d}x + EJ\int_0^L\phi_n\sum_{m=1}^{\infty}q_m\phi_m^{\mathsf{IV}}\,\mathrm{d}x = \\ P\int_0^L\phi_n\delta(x-vt), \qquad n=1,\ldots,\infty$$

3. we use the ortogonality relationships and the definition of δ,

$$m_n\ddot{q}(t) + k_n q(t) \;=\; P\,\phi_n(vt) \;=\; P\,\sin\frac{n\pi\,vt}{L}, \qquad n \;=\; 1,\ldots,\infty.$$

Solutions

Considering that the initial conditions are nil for all the modal equations, with $\overline{\omega}_n = n\pi v/L$ and $\beta_n = \overline{\omega}_n/\omega_n$ the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \leq t \leq \frac{L}{v}$$

Solutions

Considering that the initial conditions are nil for all the modal equations, with $\overline{\omega}_n = n\pi v/L$ and $\beta_n = \overline{\omega}_n/\omega_n$ the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}$$

With $k_n = m_n\omega_n^2 = \frac{mL}{2}\ n^4\pi^4\frac{EJ}{mL^4} = n^4\pi^4\frac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2PL^3}{n^4\pi^4 EJ}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}.$$

Solutions

Considering that the initial conditions are nil for all the modal equations, with $\overline{\omega}_n = n\pi v/L$ and $\beta_n = \overline{\omega}_n/\omega_n$ the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}$$

With $k_n = m_n\omega_n^2 = \dfrac{mL}{2}\; n^4\pi^4\dfrac{EJ}{mL^4} = n^4\pi^4\dfrac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2PL^3}{n^4\pi^4 EJ}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n \sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}.$$

It is apparent that for $\beta_n^2 = 1$ there is resonance.

Solutions

The critical velocity $v_{\mathsf{cr},n}$ for mode n is given by $\beta_n = 1$, substituting $\omega_n = n^2\omega_1$ we have ${}^{n\pi v_{\mathsf{cr},n}/L}\!/_{n^2\omega_1} = 1$ that gives $v_{\mathsf{cr},n} = {}^{n\omega_1 L}\!/_{\pi} = n\, v_{\mathsf{cr},1} = n\, v_{\mathsf{cr}}$, where $v_{\mathsf{cr}} = {}^{\omega_1 L}\!/_{\pi}$.

With the position $v = \kappa v_{\mathsf{cr}}$ it is

$$\overline{\omega}_n = \kappa n \omega_1 \text{ and } \beta_n = n\kappa\omega_1/n^2\omega_1 = {}^{\kappa}\!/_{n}.$$

Solutions

The critical velocity $v_{\mathsf{cr},n}$ for mode n is given by $\beta_n = 1$, substituting $\omega_n = n^2\omega_1$ we have ${}^{n\pi v_{\mathsf{cr},n}/L}\!/\!_{n^2\omega_1} = 1$ that gives $v_{\mathsf{cr},n} = {}^{n\omega_1 L}\!/\!_{\pi} = n\,v_{\mathsf{cr},1} = n\,v_{\mathsf{cr}}$, where $v_{\mathsf{cr}} = {}^{\omega_1 L}\!/\!_{\pi}$.

With the position $v = \kappa v_{\mathsf{cr}}$ it is

$$\overline{\omega}_n = \kappa n \omega_1 \text{ and } \beta_n = n\kappa\omega_1/n^2\omega_1 = {}^{\kappa}\!/\!_{n}.$$

The solution can be rewritten as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ}\,\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(\frac{\kappa}{n}\omega_n t) - \frac{\kappa}{n}\sin\omega_n t\right),$$

$$\text{for } 0 \le t \le \frac{L}{v}.$$

Adimensional time

Introducing an adimensional time coordinate ξ with $t = t_0\xi$, noting that $\omega_n = n^2\omega_1$ we can write the argument of the first sine as follows:

$$\frac{\kappa}{n}\omega_n t = \kappa n \omega_1 \xi t_0 = n\xi t_0 \kappa v_{\mathsf{cr}} \pi / L = n\pi\xi \times (vt_0)/L = n\pi\xi.$$

In a similar way we have $\omega_n t = n^2\pi\xi/\kappa$.
Substituting in the equation of the modal responses the new expressions for the sine arguments, it is

$$q_n(\xi) = \frac{2PL^3}{\pi^4 EJ}\,\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)$$

for $0 \le \xi \le 1$.

Adimensional time IS adimensional position

If we denote with $\mathbb{X}(t)$ the position of the load at time t, it is $\mathbb{X}(t) = vt = \xi L$, or $\xi = {}^{\mathbb{X}}\!/_{L}$ and the expression $u(x, \xi) = \sum q_n(\xi)\phi_n(x)$ can be interpreted as the displacement in x when the load is positioned in $\mathbb{X} = \xi L$.

Analytical expressions of u and M_b

The displacement and the bending moment are given by

$$u(x,\xi) = \frac{2PL^3}{\pi^4 EJ} \sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{x}{L})}{n^2(n^2-\kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right),$$

$$\begin{aligned} M_b(x,\xi) &= -EJ\frac{\partial^2 u(x,\xi)}{\partial x^2} = \\ &= \frac{2PL}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{x}{L})}{n^2-\kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right). \end{aligned}$$

Midspan deflection and bending moment

The maximum values of the midspan deflection and bending moment are obtained when P is placed at midspan,

$$u_{\mathsf{stat}} = \frac{PL^3}{48EJ}, \qquad M_{\mathsf{b\ stat}} = \frac{PL}{4}.$$

It is convenient to normalize the responses with respect to these maxima to have an appreciation of the dynamical effects.

Midspan deflection and bending moment

The normalized midspan displacement $\eta(\xi) = u(L/2, \xi)/u_{\mathsf{stat}}$ has the expression

$$\eta(\xi) = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right),$$

where $sin(n\pi/2) = 1, 0, -1, 0, 1, \ldots$ for $n = 1, 2, 3, 4, 5, \ldots$
Analogously, normalizing with respect to the maximum static bending moment, it is

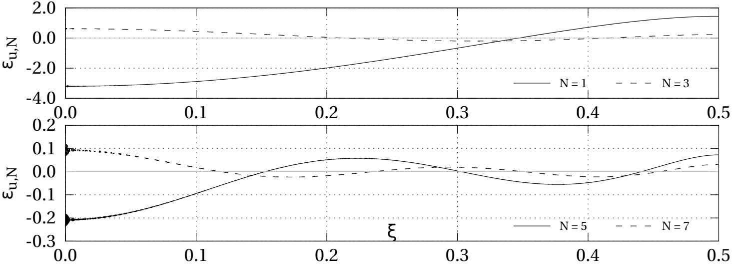
$$\mu(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin(\frac{n^2}{\kappa}\pi\xi) \right).$$

Partial sums with N terms will be denoted in the following by $\eta_N(\xi)$ and $\mu_N(\xi)$.

Error estimates

The normalized midspan statical displacement for a load P placed at $\mathbb{X} = \xi L$ is $\eta_{\mathsf{stat}}(\xi) = 3\xi - 4\xi^3$ for $0 \le \xi \le 1/2$ and we can define a percent error function (using $\kappa = 10^{-6}$ to obtain a good approximation to the static response)

$$\epsilon_{u,N}(\xi) = 100 \left(1 - \frac{\eta_N(\xi)|_{\kappa=10^{-6}}}{\eta_{\mathsf{stat}}(\xi)}\right) \qquad \text{for } 0 \le \xi \le 1/2,$$

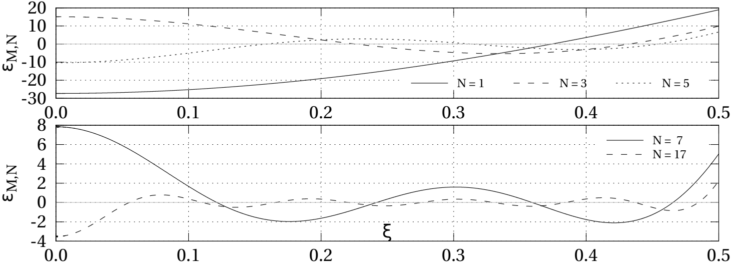


With 5 terms the approximation is in the order of $1/1000$.

Error estimates

Analogously we can use the midspan bending moment, normalized with respect to $PL/4$, $\mu_{\mathsf{stat}}(\xi) = 2\xi$ to define another percent error function

$$\epsilon_{M,N} = 100\ \left(1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\mathsf{stat}}(\xi)}\right)$$

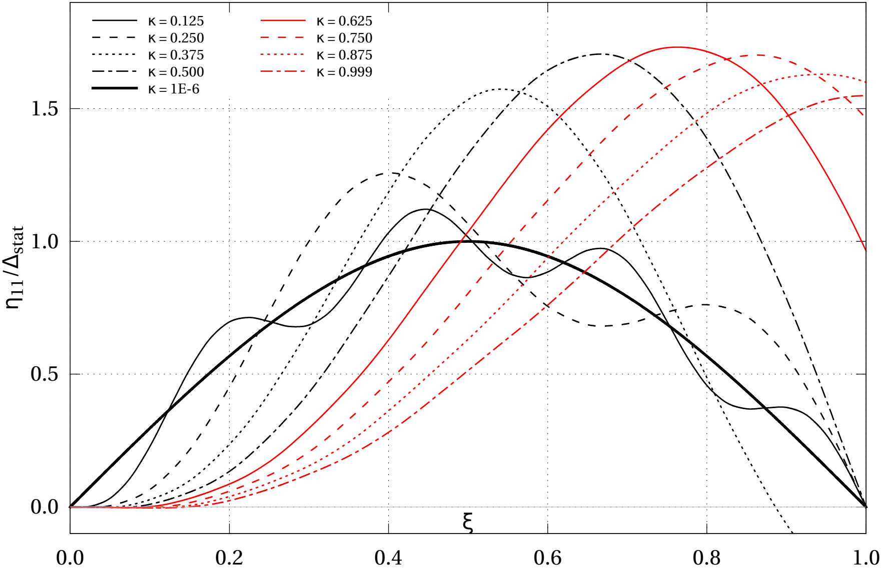


With 17 terms the approximation is in the order of 4%. As usual, worse convergence for internal forces.

The plots

Finally, we plot the normalized displacement and the normalized bending moment different values of the velocity (i.e., for different values ofκ).
Note that for the displacement I used $N = 11$ while for the bending moment I used $N = 25$.

Displacements



Bending moments