

# Continuous Systems an example

Giacomo Boffi

Dipartimento di Ingegneria Civile e Ambientale, Politecnico di Milano

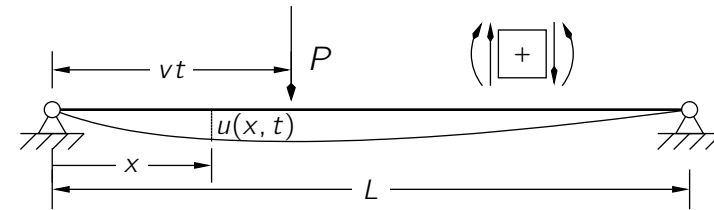
Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution

## Problem statement



A uniform beam ( $m(x) = m$ ,  $EJ(x) = EJ$ ) of length  $L$  is loaded by a moving load  $P$ , moving with constant velocity,  $v(t) = v$ , in the interval  $0 \leq t \leq t_0 = L/v = t_0$ .

Using the sign conventions indicated above, compute and plot the midspan displacement  $u(L/2, t)$  and the midspan bending moment  $M_b(L/2, t)$  as functions of time in the interval  $0 \leq t \leq t_0$  for different values of the velocity.

NB: the beam is at rest for  $t = 0$ .

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution

## Equation of motion

For an uniform beam, the equation of dynamic equilibrium is

$$m \frac{\partial^2 u(x, t)}{\partial t^2} + EJ \frac{\partial^4 u(x, t)}{\partial x^4} = p(x, t).$$

In our example, the loading function must be defined in terms of  $\delta(x)$ , the Dirac's delta distribution,

$$p(x, t) = P \delta(x - vt).$$

The Dirac's delta is a *generalized* function of one variable, defined by

$$\delta(x - x_0) \equiv 0 \quad \text{and} \quad \int f(x) \delta(x - x_0) dx = f(x_0).$$

Note that the Dirac distribution and the Kronecker's symbol  $\delta_{ij}$  are two different things.

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

## Equation of motion

The solution will be computed by separation of variables

$$u(x, t) = q(t) \phi(x)$$

and modal analysis,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\begin{aligned} \phi_n(x) &= \sin \beta_n x, & \beta_n &= \frac{n\pi}{L}, \\ m_n &= \frac{mL}{2}, & \omega_n^2 &= \beta_n^4 \frac{EJ}{m} = n^4 \pi^4 \frac{EJ}{mL^4}. \end{aligned}$$

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

## Orthogonality relationships

Continuous Systems

Giacomo Boffi

Problem statement

Solution  
Equation of motion

For an uniform beam, the orthogonality relationships are

$$m \int_0^L \phi_n(x) \phi_m(x) dx = m_n \delta_{nm},$$

$$EJ \int_0^L \phi_n(x) \phi_m^{IV}(x) dx = k_n \delta_{nm} = m_n \omega_n^2 \delta_{nm}.$$

in the equations above  $\delta$  is the Kroneker's  $\delta$  symbol, a completely different thing from Dirac's  $\delta$  distribution.

## Decoupling the EOM

Continuous Systems

Giacomo Boffi

Problem statement

Solution  
Equation of motion

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the modal series representation of  $u(x, t)$ :

$$m \sum_{m=1}^{\infty} \ddot{q}_m \phi_m + EJ \sum_{m=1}^{\infty} q_m \phi_m^{IV} = P \delta(x - vt),$$

2. every term is multiplied by  $\phi_n$  and integrated over the lenght of the beam

$$m \int_0^L \phi_n \sum_{m=1}^{\infty} \ddot{q}_m \phi_m dx + EJ \int_0^L \phi_n \sum_{m=1}^{\infty} q_m \phi_m^{IV} dx = P \int_0^L \phi_n \delta(x - vt) dx, \quad n = 1, \dots, \infty$$

3. we use the ortogonality relationships and the definition of  $\delta$ ,

$$m_n \ddot{q}(t) + k_n q(t) = P \phi_n(vt) = P \sin \frac{n\pi vt}{L}, \quad n = 1, \dots, \infty.$$

## Solutions

Continuous Systems

Giacomo Boffi

Problem statement

Solution  
Equation of motion

Considering that the initial conditions are nil for all the modal equations, with  $\bar{\omega}_n = n\pi v/L$  and  $\beta_n = \bar{\omega}_n/\omega_n$  the individual solutions are given by

$$q_n(t) = \frac{P}{k_n} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}$$

With  $k_n = m_n \omega_n^2 = \frac{mL}{2} n^4 \pi^4 \frac{EJ}{mL^4} = n^4 \pi^4 \frac{EJ}{2L^3}$ , it is

$$q_n(t) = \frac{2PL^3}{n^4 \pi^4 EJ} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}.$$

It is apparent that for  $\beta_n^2 = 1$  there is resonance.

## Solutions

Continuous Systems

Giacomo Boffi

Problem statement

Solution  
Equation of motion

The critical velocity  $v_{cr,n}$  for mode  $n$  is given by  $\beta_n = 1$ , substituting  $\omega_n = n^2 \omega_1$  we have  $n\pi v_{cr,n}/L/n^2 \omega_1 = 1$  that gives  $v_{cr,n} = n\omega_1 L/\pi = n v_{cr,1} = n v_{cr}$ , where  $v_{cr} = \omega_1 L/\pi$ .

With the position  $v = \kappa v_{cr}$  it is

$$\bar{\omega}_n = \kappa n \omega_1 \text{ and } \beta_n = n \kappa \omega_1 / n^2 \omega_1 = \kappa / n.$$

The solution can be rewritten as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left( \sin\left(\frac{\kappa}{n} \omega_n t\right) - \frac{\kappa}{n} \sin \omega_n t \right),$$

for  $0 \leq t \leq \frac{L}{v}$ .

## Adimensional time

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

Introducing an adimensional time coordinate  $\xi$  with  $t = t_0 \xi$ , noting that  $\omega_n = n^2 \omega_1$  we can write the argument of the first sine as follows:

$$\frac{\kappa}{n} \omega_n t = \kappa n \omega_1 \xi t_0 = n \xi t_0 \kappa v_{cr} \pi / L = n \pi \xi \times (v t_0) / L = n \pi \xi.$$

In a similar way we have  $\omega_n t = n^2 \pi \xi / \kappa$ .  
Substituting in the equation of the modal responses the new expressions for the sine arguments, it is

$$q_n(\xi) = \frac{2PL^3}{\pi^4 EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right)$$

for  $0 \leq \xi \leq 1$ .

## Adimensional time IS adimensional position

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

If we denote with  $\mathbb{X}(t)$  the position of the load at time  $t$ , it is  $\mathbb{X}(t) = vt = \xi L$ , or  $\xi = \mathbb{X}/L$  and the expression  $u(x, \xi) = \sum q_n(\xi) \phi_n(x)$  can be interpreted as the displacement in  $x$  when the load is positioned in  $\mathbb{X} = \xi L$ .

## Analytical expressions of $u$ and $M_b$

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

The displacement and the bending moment are given by

$$u(x, \xi) = \frac{2PL^3}{\pi^4 EJ} \sum_{n=1}^{\infty} \frac{\sin(n\pi \frac{x}{L})}{n^2(n^2 - \kappa^2)} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right),$$

$$M_b(x, \xi) = -EJ \frac{\partial^2 u(x, \xi)}{\partial x^2} =$$

$$= \frac{2PL}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi \frac{x}{L})}{n^2 - \kappa^2} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right).$$

## Midspan deflection and bending moment

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

The maximum values of the midspan deflection and bending moment are obtained when  $P$  is placed at midspan,

$$u_{\text{stat}} = \frac{PL^3}{48EJ}, \quad M_{b \text{ stat}} = \frac{PL}{4}.$$

It is convenient to normalize the responses with respect to these maxima to have an appreciation of the dynamical effects.

## Midspan deflection and bending moment

Continuous  
Systems

Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

The normalized midspan displacement  $\eta(\xi) = u(L/2, \xi)/u_{\text{stat}}$  has the expression

$$\eta(\xi) = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n^2(n^2 - \kappa^2)} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right),$$

where  $\sin(n\pi/2) = 1, 0, -1, 0, 1, \dots$  for  $n = 1, 2, 3, 4, 5, \dots$

Analogously, normalizing with respect to the maximum static bending moment, it is

$$\mu(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n^2 - \kappa^2} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right).$$

Partial sums with  $N$  terms will be denoted in the following by  $\eta_N(\xi)$  and  $\mu_N(\xi)$ .

## Error estimates

Continuous  
Systems

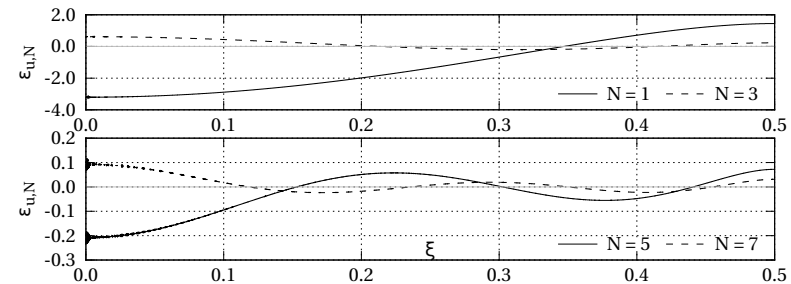
Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

The normalized midspan statical displacement for a load  $P$  placed at  $\mathbb{X} = \xi L$  is  $\eta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$  for  $0 \leq \xi \leq 1/2$  and we can define a percent error function (using  $\kappa = 10^{-6}$  to obtain a good approximation to the static response)

$$\epsilon_{u,N}(\xi) = 100 \left( 1 - \frac{\eta_N(\xi)|_{\kappa=10^{-6}}}{\eta_{\text{stat}}(\xi)} \right) \quad \text{for } 0 \leq \xi \leq 1/2,$$



With 5 terms the approximation is in the order of  $1/1000$ .

## Error estimates

Continuous  
Systems

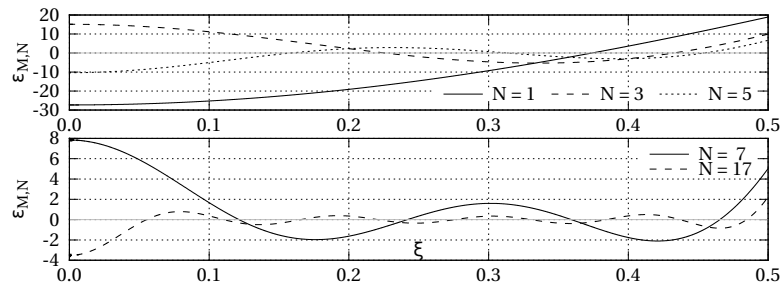
Giacomo Boffi

Problem  
statement

Solution  
Equation of motion

Analogously we can use the midspan bending moment, normalized with respect to  $PL/4$ ,  $\mu_{\text{stat}}(\xi) = 2\xi$  to define another percent error function

$$\epsilon_{M,N} = 100 \left( 1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)} \right)$$



With 17 terms the approximation is in the order of 4%. As usual, worse convergence for internal forces.

## The plots

Continuous  
Systems

Giacomo Boffi

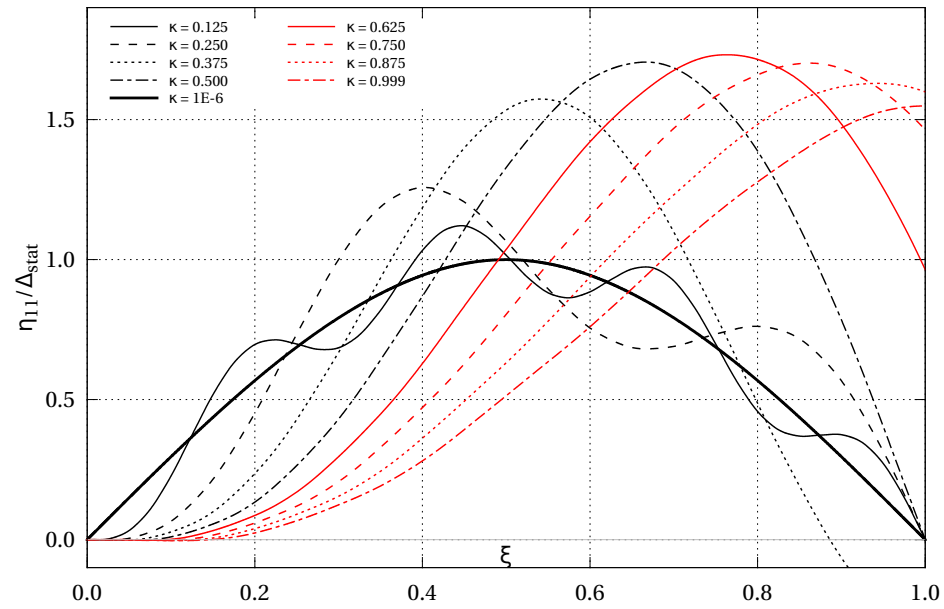
Problem  
statement

Solution  
Equation of motion

Finally, we plot the normalized displacement and the normalized bending moment different values of the velocity (i.e., for different values of  $\kappa$ ).

Note that for the displacement I used  $N = 11$  while for the bending moment I used  $N = 25$ .

## Displacements



## Bending moments

