Generalized Single Degree of Freedom Systems

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Outline

Introductory Remarks

Until now our *SDOF*'s were described as composed by a single mass connected to a fixed reference by means of a spring and a damper. While the mass-spring is a useful representation, many different, more complex systems can be studied as *SDOF* systems, either exactly or under some simplifying assumption.

1. *SDOF* rigid body assemblages, where the flexibility is concentrated in a number of springs and dampers, can be studied, e.g., using the Principle of Virtual Displacements and the D'Alembert Principle.
2. simple structural systems can be studied, in an approximate manner, assuming a fixed pattern of displacements, whose amplitude (the single degree of freedom) varies with time.

Further Remarks on Rigid Assemblages

Today we restrict our consideration to plane, 2-D systems.
In rigid body assemblages the limitation to a single shape of displacement is a consequence of the configuration of the system, i.e., the disposition of supports and internal hinges.
When the equation of motion is written in terms of a single parameter and its time derivatives, the terms that figure as coefficients in the equation of motion can be regarded as the *generalised* properties of the assemblage: generalised mass, damping and stiffness on left hand, generalised loading on right hand.

$$m^\star\ddot{x} + c^\star\dot{x} + k^\star x = p^\star(t)$$

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Further Remarks on Continuous Systems

Continuous systems have an infinite variety of deformation patterns.
By restricting the deformation to a single shape of varying amplitude, we introduce an infinity of internal contstraints that limit the infinite variety of deformation patterns, but under this assumption the system configuration is mathematically described by a single parameter, so that

- our *model* can be analysed in exactly the same way as a strict *SDOF* system,
- we can compute the *generalised* mass, damping, stiffness properties of the *SDOF model* of the continuous system.

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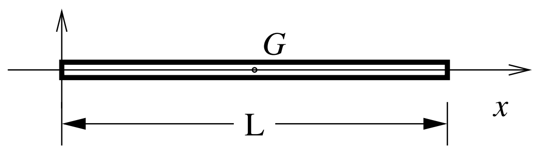
Final Remarks on Generalised *SDOF* Systems

From the previous comments, it should be apparent that everything we have seen regarding the behaviour and the integration of the equation of motion of proper *SDOF* systems applies to rigid body assemblages and to *SDOF* models of flexible systems, provided that we have the means for determining the *generalised* properties of the dynamical systems under investigation.

Assemblages of Rigid Bodies

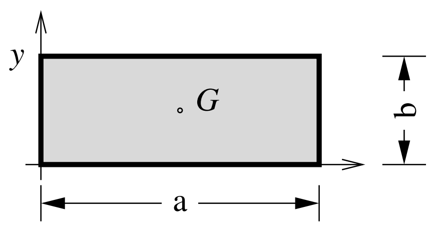
- planar, or bidimensional, rigid bodies, constrained to move in a plane,
- the flexibility is *concentrated* in discrete elements, springs and dampers,
- rigid bodies are connected to a fixed reference and to each other by means of springs, dampers and smooth, bilateral constraints (read hinges, double pendulums and rollers),
- inertial forces are distributed forces, acting on each material point of each rigid body, their resultant can be described by
    - a force applied to the centre of mass of the body, proportional to acceleration vector (of the centre of mass itself) and total mass $M = \int \mathrm{d}m$
    - a couple, proportional to angular acceleration and the moment of inertia J of the rigid body, $J = \int (x^2 + y^2)\mathrm{d}m$.

Rigid Bar



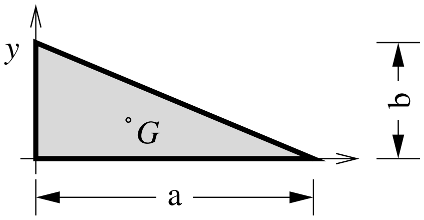
Unit mass	$\bar{m} = \text{constant},$
Length	$L,$
Centre of Mass	$x_G = L/2,$
Total Mass	$m = \bar{m}L,$
Moment of Inertia	$J = m\dfrac{L^2}{12} = \bar{m}\dfrac{L^3}{12}$

Rigid Rectangle



Unit mass	$\gamma = \text{constant},$
Sides	a, b
Centre of Mass	$x_G = a/2, \quad y_G = b/2$
Total Mass	$m = \gamma ab,$
Moment of Inertia	$J = m\dfrac{a^2 + b^2}{12} = \gamma\dfrac{a^3 b + ab^3}{12}$

Rigid Triangle



For a right triangle.

Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/3, \quad y_G = b/3$
Total Mass	$m = \gamma ab/2,$
Moment of Inertia	$J = m\dfrac{a^2+b^2}{18} = \gamma\dfrac{a^3b+ab^3}{36}$

Rigid Oval

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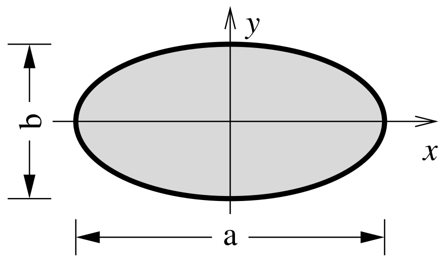
When $a = b = D = 2R$ the oval is a circle.



Unit mass	$\gamma = \text{constant},$
Axes	$a,\ b$
Centre of Mass	$x_G = y_G = 0$
Total Mass	$m = \gamma\dfrac{\pi ab}{4},$
Moment of Inertia	$J = m\dfrac{a^2+b^2}{16}$

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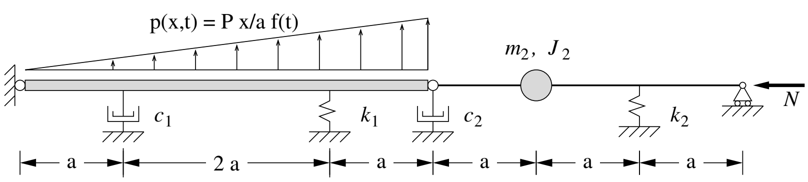
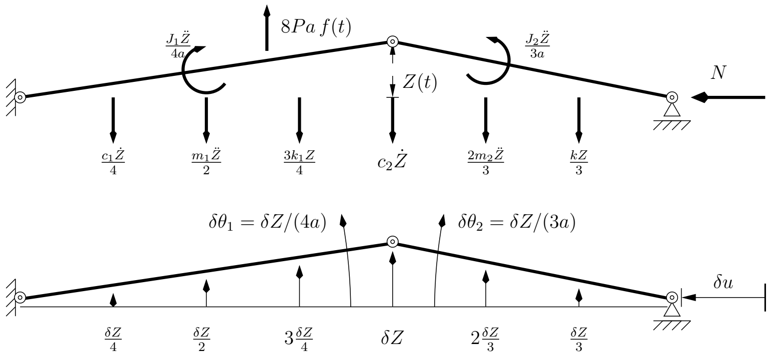


The mass of the left bar is $m_1 = \bar{m}\,4a$ and its moment of inertia is $J_1 = m_1\frac{(4a)^2}{12} = 4a^2m_1/3$.
The maximum value of the external load is $P_{\text{max}} = P\,4a/a = 4P$ and the resultant of triangular load is $R = 4P \times 4a/2 = 8Pa$

Forces and Virtual Displacements



$$u = 7a - 4a\cos\theta_1 - 3a\cos\theta_2, \quad \delta u = 4a\sin\theta_1\delta\theta_1 + 3a\sin\theta_2\delta\theta_2$$
$$\delta\theta_1 = \delta Z/(4a), \quad \delta\theta_2 = \delta Z/(3a)$$
$$\sin\theta_1 \approx Z/(4a), \quad \sin\theta_2 \approx Z/(3a)$$
$$\delta u = \left(\tfrac{1}{4a} + \tfrac{1}{3a}\right) Z\,\delta Z = \tfrac{7}{12a} Z\,\delta Z$$

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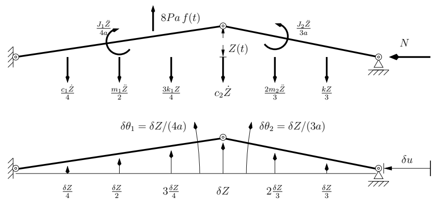
Principle of Virtual Displacements



The virtual work of the InertialDampingElasticExternal forces:

$$\delta W_{\mathsf{I}} = -m_1\frac{\ddot{Z}}{2}\frac{\delta Z}{2} - J_1\frac{\ddot{Z}}{4a}\frac{\delta Z}{4a} - m_2\frac{2\ddot{Z}}{3}\frac{2\delta Z}{3} - J_2\frac{\ddot{Z}}{3a}\frac{\delta Z}{3a}$$
$$= -\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z}\,\delta Z$$

$$\delta W_{\mathsf{D}} = -c_1\frac{\dot{Z}}{4}\frac{\delta Z}{4} - -c_2 Z\,\delta Z = -\left(c_2 + c_1/16\right)\dot{Z}\,\delta Z$$
$$\delta W_{\mathsf{S}} = -k_1\frac{3Z}{4}\frac{3\delta Z}{4} - k_2\frac{Z}{3}\frac{\delta Z}{3} = -\left(\frac{9k_1}{16} + \frac{k_2}{9}\right) Z\,\delta Z$$
$$\delta W_{\mathsf{Ext}} = 8Pa\,f(t)\frac{2\delta Z}{3} + N\frac{7}{12a} Z\,\delta Z$$

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Principle of Virtual Displacements

For a rigid body in condition of equilibrium the total virtual work must be equal to zero

$$\delta W_{\mathsf{I}} + \delta W_{\mathsf{D}} + \delta W_{\mathsf{S}} + \delta W_{\mathsf{Ext}} = 0$$

Substituting our expressions of the virtual work contributions and simplifying δZ, the equation of equilibrium is

$$\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z} +$$
$$+ \left(c_2 + c_1/16\right)\dot{Z} + \left(\frac{9k_1}{16} + \frac{k_2}{9}\right) Z =$$
$$8Pa\,f(t)\frac{2}{3} + N\frac{7}{12a} Z$$

Principle of Virtual Displacements

Collecting Z and its time derivatives give us

$$m^\star \ddot{Z} + c^\star \dot{Z} + k^\star Z = p^\star f(t)$$

introducing the so called *generalised properties*, in our example it is

$$\begin{aligned}
m^\star &= \frac{1}{4}m_1 + \frac{4}{9}9m_2 + \frac{1}{16a^2}J_1 + \frac{1}{9a^2}J_2,\\
c^\star &= \frac{1}{16}c_1 + c_2,\\
k^\star &= \frac{9}{16}k_1 + \frac{1}{9}k_2 - \frac{7}{12a}N,\\
p^\star &= \frac{16}{3}Pa.
\end{aligned}$$

It is worth writing down the expression of $k^\star$:

$$k^\star = \frac{9k_1}{16} + \frac{k_2}{9} - \frac{7}{12a}N$$

Geometrical stiffness

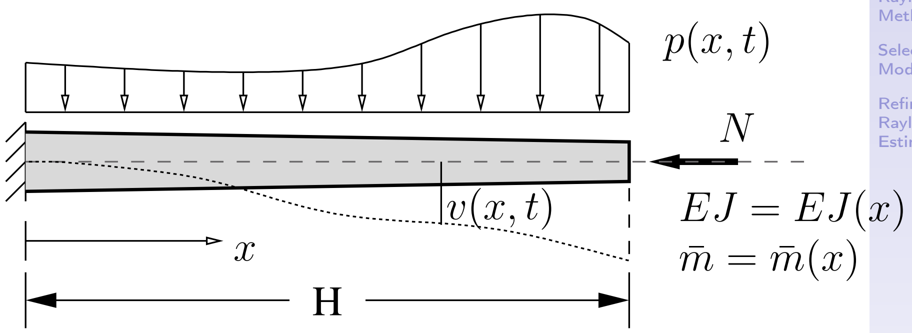
Let's start with an example...

Consider a cantilever, with varying properties $\bar{m}$ and EJ, subjected to a load that is function of both time t and position x,

$$p = p(x,t).$$

The transverse displacements v will be function of time and position,

$$v = v(x,t)$$



... and an hypothesis

To study the previous problem, we introduce an *approximate model* by the following hypothesis,

$$v(x,t) = \Psi(x)\, Z(t),$$

that is, the hypothesis of *separation of variables*
Note that $\Psi(x)$, the *shape function*, is adimensional, while $Z(t)$ is dimensionally a generalised displacement, usually chosen to characterise the structural behaviour.
In our example we can use the displacement of the tip of the chimney, thus implying that $\Psi(H) = 1$ because

$$Z(t) = v(H,t) \quad \text{and}$$
$$v(H,t) = \Psi(H)\, Z(t)$$

Principle of Virtual Displacements

For a flexible system, the PoVD states that, at equilibrium,

$$\delta W_{\mathsf{E}} = \delta W_{\mathsf{I}}.$$

The virtual work of external forces can be easily computed, the virtual work of internal forces is usually approximated by the virtual work done by bending moments, that is

$$\delta W_{\mathsf{I}} \approx \int M\, \delta\chi$$

where χ is the curvature and $\delta\chi$ the virtual increment of curvature.

δW_{E}

The external forces are $p(x,t)$, N and the forces of inertia f_{I}; we have, by separation of variables, that $\delta v = \Psi(x)\delta Z$ and we can write

$$\delta W_{\mathsf{p}} = \int_0^H p(x,t)\delta v\,\mathrm{d}x = \left[\int_0^H p(x,t)\Psi(x)\,\mathrm{d}x\right]\;\delta Z = p^\star(t)\,\delta Z$$

$$\begin{aligned}\delta W_{\mathsf{Inertia}} &= \int_0^H -\bar{m}(x)\ddot{v}\delta v\,\mathrm{d}x = \int_0^H -\bar{m}(x)\Psi(x)\ddot{Z}\Psi(x)\,\mathrm{d}x\;\delta Z\\ &= \left[\int_0^H -\bar{m}(x)\Psi^2(x)\,\mathrm{d}x\right]\;\ddot{Z}(t)\,\delta Z = m^\star \ddot{Z}\,\delta Z.\end{aligned}$$

The virtual work done by the axial force deserves a separate treatment...

δW_{N}

The virtual work of N is $\delta W_{\mathsf{N}} = N\delta u$ where δu is the variation of the vertical displacement of the top of the chimney.
We start computing the vertical displacement of the top of the chimney in terms of the rotation of the axis line, $\phi \approx \Psi'(x)Z(t)$,

$$u(t) = H - \int_0^H \cos\phi\,\mathrm{d}x = \int_0^H (1-\cos\phi)\,\mathrm{d}x,$$

substituting the well known approximation $cos\phi \approx 1 - \frac{\phi^2}{2}$ in the above equation we have

$$u(t) = \int_0^H \frac{\phi^2}{2}\,\mathrm{d}x = \int_0^H \frac{\Psi'^2(x)Z^2(t)}{2}\,\mathrm{d}x$$

hence

$$\delta u = \int_0^H \Psi'^2(x)Z(t)\delta Z\,\mathrm{d}x = \int_0^H \Psi'^2(x)\,\mathrm{d}x\;Z\delta Z$$

and

$$\delta W_{\mathsf{N}} = \left[\int_0^H \Psi'^2(x)\,\mathrm{d}x\;N\right]\;Z\,\delta Z = k_G^\star\,Z\,\delta Z$$

δW_{Int}

Approximating the internal work with the work done by bending moments, for an infinitesimal slice of beam we write

$$\mathrm{d}W_{\mathsf{Int}} = \frac{1}{2} M v''(x,t)\,\mathrm{d}x = \frac{1}{2} M \Psi''(x) Z(t)\,\mathrm{d}x$$

with $M = EJ(x)v''(x)$

$$\delta(\mathrm{d}W_{\mathsf{Int}}) = EJ(x)\Psi''^2(x)Z(t)\delta Z\,\mathrm{d}x$$

integrating

$$\delta W_{\mathsf{Int}} = \left[\int_0^H EJ(x)\Psi''^2(x)\,\mathrm{d}x\right] Z\delta Z = k^\star\, Z\,\delta Z$$

Remarks

- the shape function *must* respect the geometrical boundary conditions of the problem, i.e., both

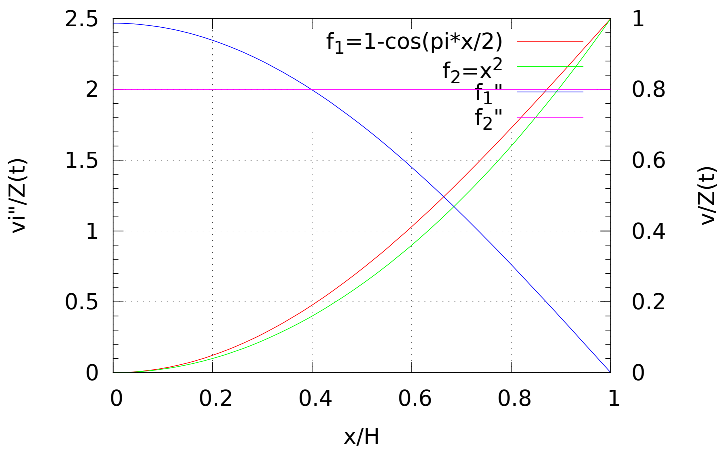
$$\Psi_1 = x^2 \qquad \text{and} \qquad \Psi_2 = 1 - \cos\frac{\pi x}{2H}$$

  are accettable shape functions for our example, as $\Psi_1(0) = \Psi_2(0) = 0$ and $\Psi_1'(0) = \Psi_2'(0) = 0$
- better results are obtained when the second derivative of the shape function at least *resembles* the typical distribution of bending moments in our problem, so that between

$$\Psi_1'' = \text{constant} \qquad \text{and} \qquad \Psi_2'' = \frac{\pi^2}{4H^2}\cos\frac{\pi x}{2H}$$

  the second choice is preferable.

Remarks



Example

Using $\Psi(x) = 1 - \cos\frac{\pi x}{2H}$, with $\bar{m} =$ constant and $EJ =$ constant, with a load characteristic of seismic excitation, $p(t) = -\bar{m}\ddot{v}_g(t)$,

$$m^\star = \bar{m}\int_0^H (1 - \cos\frac{\pi x}{2H})^2\,\mathrm{d}x = \bar{m}(\frac{3}{2} - \frac{4}{\pi})H$$

$$k^\star = EJ\frac{\pi^4}{16H^4}\int_0^H \cos^2\frac{\pi x}{2H}\,\mathrm{d}x = \frac{\pi^4}{32}\frac{EJ}{H^3}$$

$$k_G^\star = N\frac{\pi^2}{4H^2}\int_0^H \sin^2\frac{\pi x}{2H}\,\mathrm{d}x = \frac{\pi^2}{8H}N$$

$$p_g^\star = -\bar{m}\ddot{v}_g(t)\int_0^H 1 - \cos\frac{\pi x}{2H}\,\mathrm{d}x = -\left(1 - \frac{2}{\pi}\right)\bar{m}H\,\ddot{v}_g(t)$$

Vibration Analysis

- The process of estimating the vibration characteristics of a complex system is known as *vibration analysis*.
- We can use our previous results for flexible systems, based on the *SDOF* model, to give an estimate of the natural frequency $\omega^2 = k^\star/m^\star$
- A different approach, proposed by Lord Rayleigh, starts from different premises to give the same results but the *Rayleigh's Quotient* method is important because it offers a better understanding of the vibrational behaviour, eventually leading to successive refinements of the first estimate of ω^2.

Rayleigh's Quotient Method

Our focus will be on the *free vibration* of a flexible, undamped system.

- inspired by the free vibrations of a proper *SDOF* we write

$$Z(t) = Z_0 \sin\omega t \qquad \text{and}$$
$$v(x,t) = Z_0\Psi(x)\sin\omega t,$$
$$\dot{v}(x,t) = \omega\, Z_0\Psi(x)\cos\omega t.$$

- the displacement and the velocity are in quadrature: when v is at its maximum $\dot{v} = 0$, hence $V = V_{\mathsf{max}}$, $T = 0$ and when $\dot{v}$ is at its maximum it is $v = 0$, hence $V = 0$, $T = T_{\mathsf{max}}$,
- disregarding damping, the energy of the system is constant during free vibrations,

$$V_{\mathsf{max}} + 0 = 0 + T_{\mathsf{max}}$$

Rayleigh' s Quotient Method

Now we write the expressions for V_{max} and T_{max},

$$V_{\text{max}} = \frac{1}{2} Z_0^2 \int_S EJ(x)\Psi''^2(x)\,\mathrm{d}x,$$
$$T_{\text{max}} = \frac{1}{2}\omega^2 Z_0^2 \int_S \bar{m}(x)\Psi^2(x)\,\mathrm{d}x,$$

equating the two expressions and solving for ω^2 we have

$$\omega^2 = \frac{\int_S EJ(x)\Psi''^2(x)\,\mathrm{d}x}{\int_S \bar{m}(x)\Psi^2(x)\,\mathrm{d}x}.$$

Recognizing the expressions we found for $k^\star$ and $m^\star$ we could question the utility of Rayleigh's Quotient...

Rayleigh's Quotient Method

- in Rayleigh's method we know the specific time dependency of the inertial forces
  $$f_\text{I} = -\bar{m}(x)\ddot{v} = \bar{m}(x)\omega^2 Z_0 \Psi(x)\sin\omega t$$
  f_I has the same *shape* we use for displacements.
- if Ψ were the real shape assumed by the structure in free vibrations, the displacements v due to a loading $f_\text{I} = \omega^2\bar{m}(x)\Psi(x)Z_0$ should be proportional to $\Psi(x)$ through a constant factor, with equilibrium respected in every point of the structure during free vibrations.
- starting from a shape function $\Psi_0(x)$, a new shape function Ψ_1 can be determined normalizing the displacements due to the inertial forces associated with $\Psi_0(x)$, $f_\text{I} = \bar{m}(x)\Psi_0(x)$,
- we are going to demonstrate that the new shape function is a better approximation of the true mode shape

Selection of mode shapes

Given different shape functions Ψ_i and considering the true shape of free vibration Ψ, in the former cases equilibrium is not respected by the structure itself.
To keep inertia induced deformation proportional to Ψ_i we must consider the presence of additional elastic constraints. This leads to the following considerations

- the frequency of vibration of a structure with additional constraints is higher than the true natural frequency,
- the criterium to discriminate between different shape functions is: better shape functions give lower estimates of the natural frequency, the true natural frequency being a lower bound of all estimates.

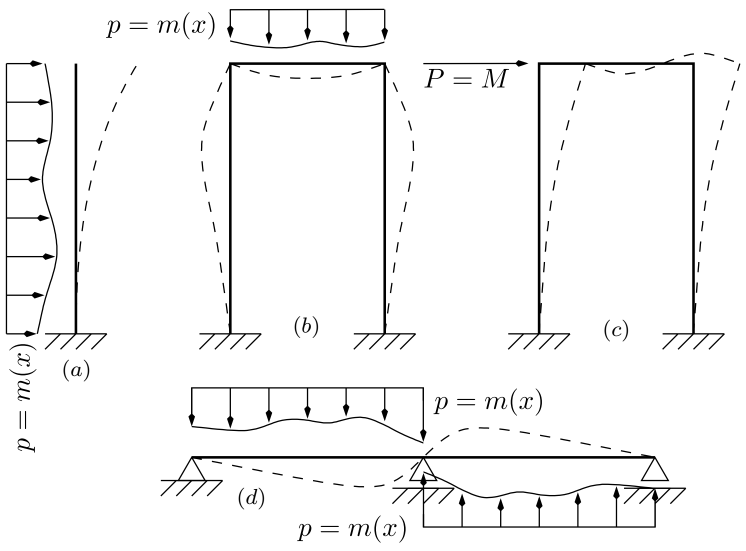
Selection of mode shapes 2

In general the selection of trial shapes goes through two steps,

1. the analyst considers the flexibilities of different parts of the structure and the presence of symmetries to devise an approximate shape,
2. the structure is loaded with constant loads directed as the assumed displacements, the displacements are computed and used as the shape function,

of course a little practice helps a lot in the the choice of a proper pattern of loading...

Selection of mode shapes 3



Refinement R_{00}

Choose a trial function $\Psi^{(0)}(x)$ and write

$$v^{(0)} = \Psi^{(0)}(x) Z^{(0)} \sin \omega t$$
$$V_{\mathsf{max}} = \frac{1}{2} Z^{(0)2} \int EJ\Psi^{(0)\prime\prime 2}\,\mathrm{d}x$$
$$T_{\mathsf{max}} = \frac{1}{2} \omega^2 Z^{(0)2} \int \bar{m}\Psi^{(0)2}\,\mathrm{d}x$$

our first estimate R_{00} of ω^2 is

$$\omega^2 = \frac{\int EJ\Psi^{(0)\prime\prime 2}\,\mathrm{d}x}{\int \bar{m}\Psi^{(0)2}\,\mathrm{d}x}.$$

Refinement R_{01}

We try to give a better estimate of V_{max} computing the external work done by the inertial forces,

$$p^{(0)} = \omega^2 \bar{m}(x) v^{(0)} = Z^{(0)} \omega^2 \Psi^{(0)}(x)$$

the deflections due to $p^{(0)}$ are

$$v^{(1)} = \omega^2 \frac{v^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \frac{Z^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \bar{Z}^{(1)},$$

where we write $\bar{Z}^{(1)}$ because we need to keep the unknown ω^2 in evidence. The maximum strain energy is

$$V_{\mathsf{max}} = \frac{1}{2} \int p^{(0)} v^{(1)}\, \mathrm{d}x = \frac{1}{2} \omega^4 Z^{(0)} \bar{Z}^{(1)} \int \bar{m}(x) \Psi^{(0)} \Psi^{(1)}\, \mathrm{d}x$$

Equating to our previus estimate of T_{max} we find the R_{01} estimate

$$\omega^2 = \frac{Z^{(0)}}{\bar{Z}^{(1)}} \frac{\int \bar{m}(x) \Psi^{(0)} \Psi^{(0)}\, \mathrm{d}x}{\int \bar{m}(x) \Psi^{(0)} \Psi^{(1)}\, \mathrm{d}x}$$

Refinement R_{11}

With little additional effort it is possible to compute T_{max} from $v^{(1)}$:

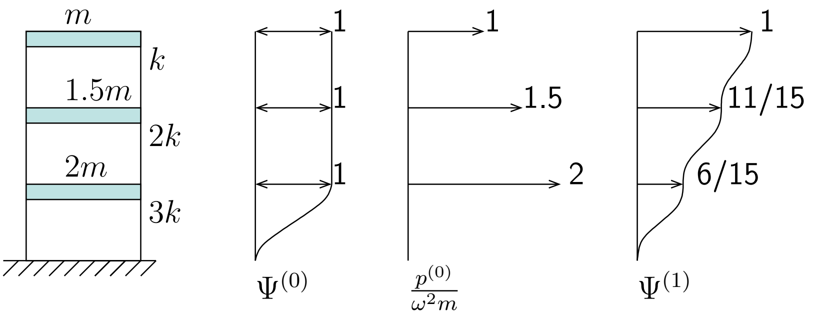
$$T_{\mathsf{max}} = \frac{1}{2} \omega^2 \int \bar{m}(x) v^{(1)2}\, \mathrm{d}x = \frac{1}{2} \omega^6 \bar{Z}^{(1)2} \int \bar{m}(x) \Psi^{(1)2}\, \mathrm{d}x$$

equating to our last approximation for V_{max} we have the R_{11} approximation to the frequency of vibration,

$$\omega^2 = \frac{Z^{(0)}}{\bar{Z}^{(1)}} \frac{\int \bar{m}(x) \Psi^{(0)} \Psi^{(1)}\, \mathrm{d}x}{\int \bar{m}(x) \Psi^{(1)} \Psi^{(1)}\, \mathrm{d}x}.$$

Of course the procedure can be extended to compute better and better estimates of ω^2 but usually the refinements are not extended beyond R_{11}, because it would be contradictory with the quick estimate nature of the Rayleigh's Quotient method and also because R_{11} estimates are usually very good ones.

Refinement Example



$$T = \frac{1}{2}\omega^2 \times 4.5 \times m\, Z_0^2$$

$$V = \frac{1}{2} \times 1 \times 3k\, Z_0^2$$

$$\omega^2 = \frac{3}{9/2}\frac{k}{m} = \frac{2}{3}\frac{k}{m}$$

$$v^{(1)} = \frac{15}{4}\frac{m}{k}\omega^2 \Psi^{(1)}$$

$$\bar{Z}^{(1)} = \frac{15}{4}\frac{m}{k}$$

$$V^{(1)} = \frac{1}{2} m \frac{15}{4}\frac{m}{k}\omega^4 \left(1 + 33/30 + 4/5\right)$$

$$= \frac{1}{2} m \frac{15}{4}\frac{m}{k}\omega^4 \frac{87}{30}$$

$$\omega^2 = \frac{\frac{9}{2} m}{m \frac{87}{8}\frac{m}{k}} = \frac{12}{29}\frac{k}{m} = 0.4138 \frac{k}{m}$$