R.C. Beams

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Axial Force

Axial Force

- Linear Model
  - tension
  - compression
- Non Linear Model
  - tension
  - compression
- Design & Requirements

Linear Model, compression

Due to bonding between concrete and steel the strains in concrete and in steel are the same, $\epsilon_c = \epsilon_s = \epsilon$ and the stresses are $\sigma_c = E_c \epsilon,\ \sigma_s = E_s \epsilon$.

From the first eq. it is $\epsilon = \sigma_c / E_c$ and substituting in the second one the steel stress is $\sigma_s = \sigma_c \cdot E_s / Ec$.

The resultant of constant stresses is $N = A_c \sigma_c + A_s \sigma_s$, with the positions $n = E_s / E_c$, $\sigma_s = n \sigma_c$ it is

$$N = A_c \sigma_c + n \cdot A_s \sigma_c = \sigma_c \cdot (A_c + n \cdot A_s) \quad \Rightarrow$$
$$\Rightarrow \quad \sigma_c = N / (A_c + n A_s),\ \sigma_s = n \sigma_c$$

The stress in concrete can be computed with reference to an homogenized area $A_\text{hom} = A_c + n \cdot A_s$.

> *The coefficient n is the* homogenization coefficient. *— For instantaneous loadings n is the ratio of the Young modules, for long duration loadings we have to take into account the long term deformation of concrete (viscosity) and, conventionally, we use $n = 15$.*

We have two possibilities,

uncracked concrete the stress in concrete is below the tension strength, the formulas seen for compression are still valid,

cracked concrete when the loading exceeds the concrete strength concrete cracks and no stresses are transmitted,

$$\sigma_c = 0, \quad N = 0 \cdot A_c + \sigma_s \cdot A_s \Rightarrow \sigma_s = N/A_s$$

It is possible that the centroids of the uncracked and the cracked section are not coincident and that a (small) bending moment is introduced when the section cracks.

In this case we are interested in the strength of the section at the ultimate limit state.

Tension — Concrete is cracked and gived no contribution, the design strength is

$$N_{Rd} = f_{yd} A_s.$$

Compression — Materials' stresses are both equal to the respective design values and the design strength is

$$N_{Rd} = f_{cd} A_c + f_{yd} A_s.$$

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No columns are exempt from bending moment, NTC (Italian Technical Norm for Constructions) prescribes that columns are verified for N and a bending moment $M_e = N \cdot e$ due to an accidental eccentricity $e = \max(20\,\mathrm{mm}, h/20)$ *where h is the column's height.*

Requirements for Compressed Members

- Strength: $N_{Ed} \leqslant N_{Rd}$.
- Minimum steel: $A_s \geqslant 0.10 N_{Ed}/f_{yd}$.
- Steel concrete ratio: $0.003 \leqslant A_s/A_c \leqslant 0.040$,
  in seismic zones: $0.010 \leqslant A_s/A_c \leqslant 0.040$.
- Minimum diameter of longitudinal rebar: $\Phi \geqslant 12\,\mathrm{mm}$.
- Maximum distance of l.r.: $d \leqslant 300\,\mathrm{mm}$.
- Min diameter of ties (stirrups): $\Phi \geqslant \max(6\,\mathrm{mm}, \Phi_{\max}/4)$.
- Max tie distance: $s \leqslant \min(250\,\mathrm{mm}, 12\Phi_{\min})$.

EC2 requires that the distance between stirrups is reduced corresponding to a beam-column node to $0.6 s_{\text{standard}}$

Ties?

Ties in columns are the same as stirrups in beams.

In beams stirrups are required to provide shear strength and in columns we have no shear... so why there is a requirement for ties/stirrups in columns?

Ties are required for

- limiting the lateral instability of compressed rebars, that are very slender beams subjected to compressive forces, when the concrete cover is damaged and
- limiting the transversal expansion of compressed concrete (Poisson effect), this leads to a marked increase in ductility when the concrete is plasticized in compression — this effect is particularly important in seismic areas.

From a preliminar structural analyisis we know only N_{Ed}, but we have to verify the column for a combination of N and M...

A convenient approach is to dimension the column so that its design strength is consistently larger than the design action.

For *normal* storey heights it's possible to see that a 15% stronger section is appropriate, so we will design our columns conservatively with a 20% larger strength, placing the most of the overstrength on the side of reinforcing steel that contributes most of the bending strength, using

$$A_c \geqslant N_{Ed}/f_{cd}, \quad A_s \geqslant 0.20 N_{Ed}/f_{yd}.$$

Tension, uncracked

- gross section $300\,\mathrm{mm} \times 500\,\mathrm{mm}$ of C25/30 concrete
- 6 Φ20 rebars, B450C steel
- $N_{Ed} = 175\,\mathrm{kN}$, tension

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$$E_c = 31\,500\,\mathrm{MPa} \Rightarrow n = 200000/31500 = 6.35,$$
$$A_{\mathrm{hom}} = 300 \cdot 500 + 6.35 \cdot 6 \cdot 10^2 \cdot \pi = 161\,970\,\mathrm{mm}^2,$$
$$\sigma_c = N/A = 175\,000\,\mathrm{N}/161\,970\,\mathrm{mm}^2 = 1.08\,\mathrm{MPa},$$
$$\sigma_s = n\sigma_c = 6.35 \cdot 1.08\,\mathrm{MPa} = 6.90\,\mathrm{MPa},$$
$$g_{ctk} = 1.8\,\mathrm{MPa}, f_{ctd} = f_{ctk}/1.5 = 1.2\,\mathrm{MPa}.$$

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Our section can resist its design action.

Which is the action that leads to cracked concrete and that we use to verify the steel?

$$N_{\text{crack}} = A_{\text{hom}} f_{ctk} = 291.5\,\text{kN}$$

The steel stress is, before cracking, $n f_{ctk} = 11.43\,\text{MPa}$ and after cracking we have

$$\sigma_s = 291\,500\,\text{N}/A_s = 154.7\,\text{MPa}.$$

Same section as before,
$N_{Ed} = 1000\,\mathrm{kN}$ in compression.

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In this case (compression) we deal with a long duration loading, hence $n = 15$,

$$A_{\mathrm{hom}} = 150\,000\,\mathrm{mm}^2 + 15 \cdot 1884\,\mathrm{mm}^2 = 178\,260\,\mathrm{mm}^2,$$
$$\sigma_c = N_{Ed}/A_{\mathrm{hom}} = 1 \times 10^6\,\mathrm{N}/178\,260\,\mathrm{mm}^2 = 5.61\,\mathrm{MPa},$$
$$\sigma_s = n \cdot \sigma_c = 15 \cdot 5.61\,\mathrm{MPa} = 84.2\,\mathrm{MPa}.$$

C25/30 concrete, B450C steel, $N_{Ed} = 1750\,\mathrm{kN}$ in compression.
$f_{cd} = 14.17\,\mathrm{MPa}$, $fyd = 392.3\,\mathrm{MPa}$.

The minimum areas of concrete and steel, using our design formulas, are

$$A_c \geqslant 1750\,\mathrm{MPa}/14.17\,\mathrm{MPa} = 123\,500\,\mathrm{mm}^2$$

$$A_s \geqslant 0.2 \cdot 1750\,\mathrm{MPa}/391.3\,\mathrm{MPa} = 894\,\mathrm{mm}^2$$

Respecting *exactly* these constraints, having an additional, self-imposed constraint in the column width $b = 300\,\mathrm{mm}$ we can choose $h = 500\,\mathrm{mm}$, while for steel we can use 6 Φ14, with $A_s = 914\,\mathrm{mm}^2$

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As we are actually oversizing our column to provide bending strength, we should investigate also the possibility of using $h = 400\,\mathrm{mm}$, the concrete area is *almost* OK but the steel area is larger than the area strictly required.

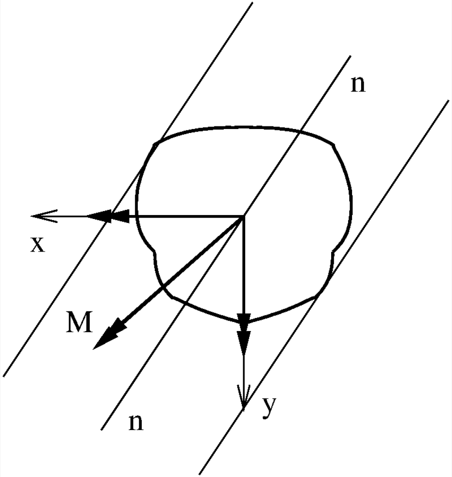
Bending Moment

Bending Moment

- Linear, uncracked.
- Linear. cracked.
- Non Linear, uncracked.
- Non Linear, cracked.

Our assumptions:

- homogenized section ($A_{hom} = n \cdot A_s$),
- tensile stress in concrete less than f_{ctd}, so that we have the contribution of all the concrete section.



C25/30 concrete, B450C steel,
$b = 300\,\mathrm{mm}$, $h = 500\,\mathrm{mm}$, $4\Phi 20 + 2\Phi 14$, $n = 6.35$, $c = 40\,\mathrm{mm}$
$M_{Ed} = 50\,\mathrm{kNm}$.

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$b = 300\,\mathrm{mm}$, $h = 500\,\mathrm{mm}$, $4\Phi 20 + 2\Phi 14$, $n = 6.35$, $c = 40\,\mathrm{mm}$
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$A_s = 1256\,\mathrm{mm}^2$, $A_s' = 308\,\mathrm{mm}^2$, $A_c = 150\,000\,\mathrm{mm}^2$,
$A_{\mathrm{hom}} = 159\,930\,\mathrm{mm}^2$
$S = 1500000 \cdot 250 + 6.35 \cdot (308 \cdot 40 + 1256 \cdot 460) = 41\,247\,000\,\mathrm{mm}^3$
$y_G = S/A = 257.9\,\mathrm{mm}$
$J = \frac{300 \cdot 500^3}{12} + 150000 \cdot (250 - 257.9)^2 + 308 \cdot (40 - 257.9)^2 + 1256 \cdot (460 - 257.9)^2$
$\Rightarrow J = 3\,552\,980\,000\,\mathrm{mm}^4$

Example, continued

$$\sigma_{\text{c,top}} = 50 \times 10^6\,\mathrm{Nmm}(-257.9\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = -3.63\,\mathrm{MPa},$$
$$\sigma_{\text{c,bot}} = 50 \times 10^6\,\mathrm{Nmm}(+242.1\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = +3.41\,\mathrm{MPa},$$
$$\sigma_{\text{s,top}} = 6.3550 \times 10^6\,\mathrm{Nmm}(-217.9\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = -19.5\,\mathrm{MPa},$$
$$\sigma_{\text{s,bot}} = 6.3550 \times 10^6\,\mathrm{Nmm}(+202.1\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = +18.1\,\mathrm{MPa}.$$

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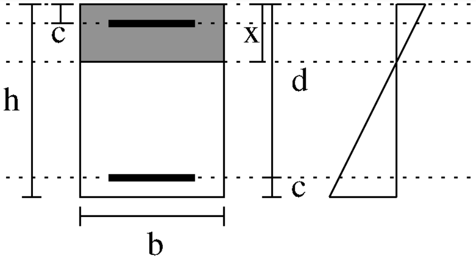
For C25/30 the characteristic tension strength in flexure is $f_{cfk} = 2.16\,\mathrm{MPa}$ hence the section is, in effect, cracked.

The moment that cause cracking is

$$\begin{aligned} M_{\text{crack}} &= J \cdot f_{cfk}/y_{\text{bot}} \\ &= \frac{3553 \times 10^6\,\mathrm{mm}^4 \cdot 2.16\,\mathrm{MPa}}{242.1\,\mathrm{mm}} \\ &= 242.1\,\mathrm{mm}. \end{aligned}$$

In this case we don’t know the position of the neutral axis, we want to determine it writing an equation of equilibrium.

- b base of rectangular section
- h height of the section
- c concrete cover (center to edge)
- d depth of tensile steel
- x (unknown) distance of the neutral axis from the compressed edge
- z arm of the internal couple
- A_s area of tensile steel
- A_s' area of compressed steel



Neutral Axis Position

In general, the neutral axis must be baricentric or, equivalently, the static moment w.r.t the neutral axis is equal to zero. For a rectangular section, using the homogenization coefficient n

$$S_x = -\frac{bx^2}{2} + (nA_s')(c-x) + (nA_s)(d-x) = 0,$$

$$\left(\frac{b}{2}\right) x^2 + n(A_s + A_s')x - n(A_s d + A_s' c) = 0,$$

$$x^2 + 2n\frac{A_s + A_s'}{b}x - 2n\frac{A_s d + A_s' c}{b} = 0.$$

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$$x^2 + 2n\frac{A_s + A_s'}{b}x - 2n\frac{A_s d + A_s' c}{b} = 0.$$

The position of the neutral axis does not depend on strength but only on geometry and n, and is determined solving a 2nd degree equation.

Considering only the positive root,

$$x = n\frac{A_s + A_s'}{b}\left(-1 + \sqrt{1 + \frac{2}{n}\,\frac{A_s bd + A_s' bc}{(A_s + A_s')^2}}\right).$$

(The result was simplified w.r.t. the standard solution of a 2nd degree equation).

Having x we have

$$J = \frac{bx^3}{3} + nA_s(d-x)^2 + nA_s'(c-x)^2,$$

$$\sigma_{c,\max} = -\frac{M}{J}x, \quad \sigma_s = n\frac{M}{J}(d-x), \quad \sigma_s' = n\frac{M}{J}(c-x),$$

$$z = \frac{M}{A_s\sigma_s} = \frac{M}{A_s\; nM(d-x)/J} = \frac{J}{nA_s(d-x)}.$$

Example

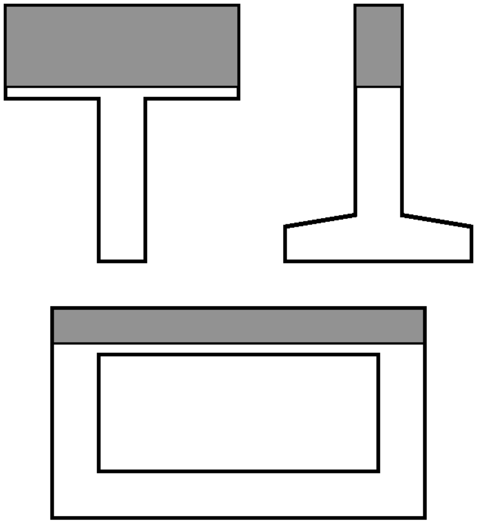
Sheet1

Data				Results	
b	300	mm	x	128.376	mm2
h	500	mm	J	1,103,956,609.838	mm4
c	40	mm	f_c	-5.814	N/mm2
n	6.35		f_s	95.376	N/mm2
As	1256	mm2	f'_s	-25.417	N/mm2
A's	308	mm2	z	417.390	mm
d	460	mm			
M	50000000	N mm			

Sections not Uniformly Rectangular

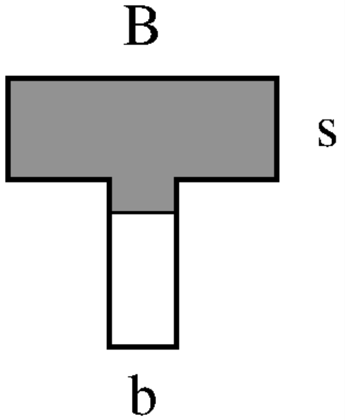
In many cases (T beams, inverted T beams, box beams) the formulas for rectangular beams are correct — as long as the neutral axis is inside the flange.

If we find otherwise, we have to take into account the variation of the section width when we determine the static moment expression.



T Beam

If, using the formula for a rectangular beam, the neutral axis is inside the flange OK, otherwise we write



$$S = Bx^2/2 - (B-b)(x-s)^2/2 + nA_s'(x-c) + nA_s(x-d).$$

For circular or trapezoidal sections it's easy to write the analytical expression of S, e.g., for an isosceles trapezoid where a is the width of the compressed edge and b the width of the tensioned edge, the contribution of concrete to S is

$$S_c = \int_0^x (a + \frac{b-a}{h} y)(x-y)\mathrm{d}y = \frac{b-a}{6h} x^3 + \frac{a}{2} x^2$$

to which we have to add the steel contributions.

Trapezoid Section

```
from scipy.optimize import newton

def S(x):
    Sc  = (b-a)*x**3/6/h + a*x**2/2
    Ssc, Sst = n * Asc * (x-c), n * Ast * (x-(h-c))
    return Sc + Ssc + Sst

a, b, h, c = 400, 200, 500, 40
Asc, Ast = 308, 1256
n = 6.35
M = 50E6

x = newton(S, h/3) ; print('x =', x)

J = (b*x**4 - a*x**4 + 4*(a*(h*x**3))) / (12*h)
J+= Asc * n * (x-c)**2 + Ast * n * (h-c-x)**2 ; print('J =', J)
z = J/(Ast*n*(h-c-x))
fc, fsc, fst = M/J*(0-x), n*M/J*(c-x), n*M/J*(h-c-x)
print('f_c = %.2f, f_sc = %.2f, f_st = %.2f'%(fc, fsc, fst))
```

Non Linear Model

Non Linear Model

- The section remains planar
- materials reach the limit state
  - the stresses in concrete are, with $\eta = \epsilon/\epsilon_{c2}$,

$$\sigma_c = -f_{cd} \begin{cases} \eta \cdot (2 - \eta) & \text{for } 0 \leqslant \eta \leqslant 1 \\ 1 & \text{for } 1 \leqslant \eta \leqslant \epsilon_{cu}/\epsilon_{c2} \end{cases}$$

  - the stresses in steel are

$$\sigma_s = E_s \cdot \epsilon, \qquad -f_{yd} \leqslant \sigma_s \leqslant +f_{yd}.$$

- The characteristics of stresses must be computed by direct integration,

$$N = \int \sigma \, dA, \; M = \int \sigma \cdot y \, dA.$$

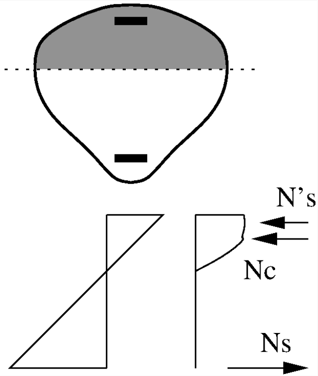
$\epsilon_{c2} = 0.0020$ *and* $\epsilon_{cu} = 0.0035$ *are defined by EC2.*

Operational assumptions

- the section is symmetrical w.r.t a vertical axis
- the bending moment acts in the same vertical plane
- neutral axis at distance x from the compressed edge
- at the limit state the deformation at the compressed edge is equal to $\epsilon_{cu} = 3.5‰$
- for a section in pure bending $N_c + N_s' = N_s$
- N_c is expressed in terms of coefficient β

$$N_c = -f_{cd} A_{c,\mathrm{comp}} \beta.$$

- The section remains planar, the diagram of deformations is linear and it is possible to compute the stresses (in concrete or in steel).
- In every strip at distance y from the neutral axis the stresses are constant and it is possible to compute the resultan force using integration.
- In the end, we must determine the position of the resultant N_c.



N_c AND COEFFICIENTS β AND κ

We start defining the area A_c, the area of concrete above the neutral axis, the area of compressed concrete.

We want to find N_c in terms of a filling coefficient β and in terms of the position of the n.a.:

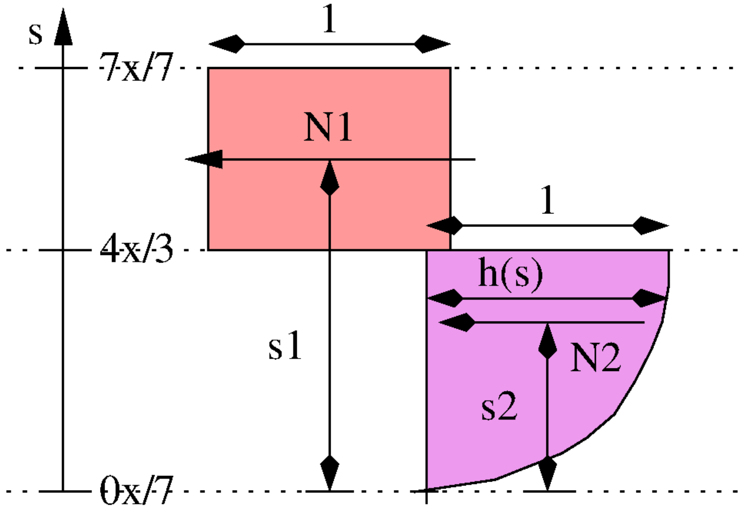
$$N_c(x) = \beta\, f_{cd}\, A_c(x)$$

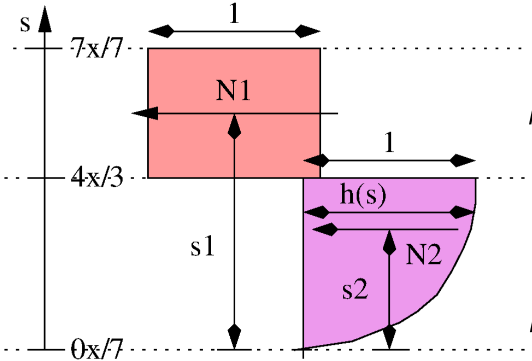
and we want to represent the moment of N_c w.r.t. the n.a. in terms of an additional coefficient κ

$$M_c(x) = N_c(x)(1-\kappa)x = \beta\, f_{cd}\, A_c(x)(1-\kappa)x.$$

The coefficient κ defines the position of the resultant measured from the compressed edge, κx.

Rectangular Section





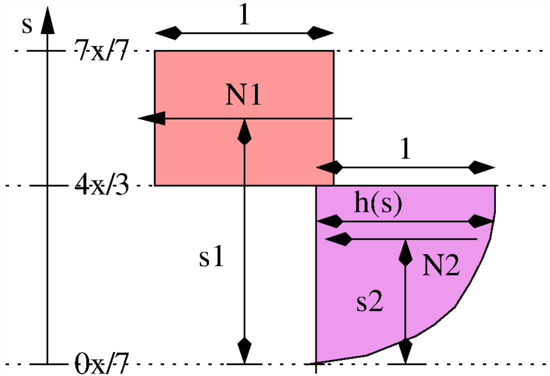
$$N_1 = \frac{3}{7}x, \quad s_1 = \frac{4}{7}x + \frac{1}{2}\frac{3}{7}x = \frac{11}{14}x$$

$$M_1 = N_1 s_1 = \frac{1}{49}\frac{33}{2}x^2$$

$$h(s) = \frac{2as - s^2}{a^2} \qquad \text{dove} \qquad a = \frac{4}{7}x$$

$$N_2 = \int_0^a h(s)\,\mathrm{d}s = \frac{2}{3}a = \frac{8}{21}x$$

$$M_2 = \int_0^a h(s)\,s\,\mathrm{d}s = \frac{5}{12}a^2 = \frac{1}{49}\frac{20}{3}x^2$$



$$N = N_1 + N_2 = \frac{17}{21}x$$

$$M = M_1 + M_2 = \frac{1}{49}\frac{139}{6}x^2$$

$$N = \beta x = \frac{17}{21}x \quad \Rightarrow \quad \beta = \frac{17}{21} = 0.809524$$

$$s_G = \frac{M}{N} = \frac{139}{238}x$$

$$s_G = (1-\kappa)x \quad \Rightarrow \quad \kappa = \frac{99}{238} = 0.415966$$

For non rectangular sections it is possible to use similar procedures, starting from the parabola-rectangle stress-strain diagram and performing the integral taking into account the variability of the width b.

Stress Block Diagram

For rectangular and non-rectangular sections it is possible to use the stress block diagram:

$$\sigma_c = \begin{cases} 0 & \epsilon \leqslant 0.2\epsilon_{cu} \\ f_{cd} & \text{otherwise} \end{cases}$$

The results obtained using the stress block are usually very good approximations to the results obtained using the parabola-rectangle diagram

$$N_s = +s\,A_s\,f_{yd}$$
$$N_s' = -s'\,A_s'\,f_{yd}$$

where s and s' are the rates of work of the reinforcement steel,

$$s = \sigma_s / f_{yd}, \quad s' = -\sigma_s' / f_{yd}.$$

Another definition is the ratio of commressed to tensile reinforcement

$$\mu = A_s' / A_s.$$

BTW the values of the stresses in steel can be deduced by the linear diagram of deformations.

For $A_s' = 0$ and assuming that $s = 1$ (this hypotesis must be verified ex-post) the equation of equilibrium, $N = 0$, is written

$$A_s f_{yd} - \beta f_{cd} b x = 0 \quad \Rightarrow \quad x = \frac{A_s f_{yd}}{\beta f_{cd} b}.$$

Having computed x the strain at the level of the reinforcement is

$$\epsilon_s = \epsilon_{cu} \frac{d - x}{x}$$

and to verify the hypotesis of yielded steel it should be

$$\epsilon_s \geqslant \epsilon_{yd}.$$

This is usually verified, except when an excessive percentage of reinforcement is present.

Double Reinforcement

The compressed steel is yielded if $\epsilon_s' \leqslant -\epsilon_{yd}$, i.e.,

$$\epsilon_{cu} \frac{x - c}{x} \leqslant -\epsilon_{yd}.$$

For B450C steel, this translates to

$$x \geqslant 2.27\, c.$$

We start assuming that both reinforcements are yielded and write

$$A_s f_{yd} - A_s' f_{yd} - \beta f_{cd}\, bx = 0 \quad \Rightarrow \quad x = \frac{(A_s - A_s') f_{yd}}{\beta f_{cd}\, b}$$

and verify that $-\epsilon_{cu} \frac{d-x}{x} \geqslant \epsilon_{yd}$ and that $x \geqslant 2.27\, c$.

When we deal with a shallow beam or a slab it is common to find that $x < 2.27\, c$ because c is large w.r.t. to the total height of the beam.

Even for large values of A_c' x is small and $x < 2.27\, c$.

If $\epsilon_s' > -\epsilon_{yd}$ the top reinforcement is still elastic and it is

$$s' = \frac{\epsilon_s'}{\epsilon_{yd}} = \frac{x-c}{x}\frac{\epsilon_{cu}}{\epsilon_{yd}}$$

and the new equation of equilibrium is

$$-f_{cd}\,\beta\,bx + \left(A_s - A_s'\frac{x-c}{x}\frac{\epsilon_{cu}}{\epsilon_{yd}}\right)f_{yd}$$

multiplying by x

$$(\beta b f_{cd})x^2 - (A_s - A_s'\frac{\epsilon_{cu}}{\epsilon_{yd}})f_{yd}\,x - A_s'\frac{\epsilon_{cu}}{\epsilon_{yd}}f_{yd}c = 0.$$

With

$$\omega = \frac{A_s}{bd}\frac{f_{yd}}{f_{cd}},\ A_s' = \mu A_s,\ \mu_1 = \mu\frac{\epsilon_{cu}}{\epsilon_{yd}}$$

the equation of equilibrium can be simplified

$$\beta x^2 - (1-\mu_1)\omega d x - \omega\mu_1 c d = 0$$

whose positive solution is

$$\frac{x}{d} = \frac{\omega}{2\beta}\left((1-\mu_1) + \sqrt{(1-\mu_1)^2 + \frac{4\beta\mu_1 c}{\omega d}}\right)$$

Finally, writing the moment of the steel forces with respect to the point of application of N_c

$$M_{Rd} = f_{yd}\left((d-\kappa x)A_s + s'(\kappa x - c)A_s'\right).$$

For the usual section,

$$x = \frac{(1256 - 308) \cdot 391.3}{0.81 \cdot 300 \cdot 14.17} = 107.7\,\mathrm{mm}.$$

Is it the top steel yielded? It's $2.27c = 91\,\mathrm{mm}$, hence we can conclude that it is yielded.

Which is the design bending strength?

$$\begin{aligned} M_{Rd} &= (1256 \cdot (460 - 0.416 \cdot 107.7) + 308 \cdot (0.414 \cdot 107.7 - 40)) \cdot 391.4 \\ &= 204.6 \times 10^6\,\mathrm{Nmm} = 204.6\,\mathrm{kNm} \end{aligned}$$

Let's change $A_s' = 782\,\mathrm{mm}^2$,

$$x = \frac{(1256 - 782) \cdot 391.3}{0.81 \cdot 300 \cdot 14.17} = 539\,\mathrm{mm} < 91\,\mathrm{mm}$$

We have to follow the other procedure...

$$\mu = \frac{7.82}{12.56} = 0.623, \quad \mu_1 = 0.623\frac{3.5}{1.96} = 1.114, \quad \omega = \frac{1256}{300 \cdot 460}\frac{391.3}{14.17} = 0.251$$

Substituting these values in the equation for the neutral axis we have $x = 72.0\,\mathrm{mm}$ and it is

$$s' = \frac{72.0 - 40}{72.0}\frac{3.5}{1.96} = 0.796$$

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It is interesting to note that in this case $M_{Rd} = 208.9\,\mathrm{kNM}$, our previous value is $204.6\,\mathrm{kNm}$. We have more than doubled A_s' to have a really small increment in the design strength.

In general, critical sections must be ductile. A good ductility ensures that plastic adaption (i.e., redistribution of bending moments) is possible and that, during sesmic excitation, large amount of energy can be dissipated.

- Deep beams,
- low reinforcements percentages,
- presence of compressed reinforcement.

It is accepted that $\epsilon_s \geqslant 1\%$ at the ultimate limit state ensures a good ductility. For our design we can start imposing $x = 0.25d$ that is equivalent to
$\epsilon_s = 3\epsilon_{cu} = 1.05\%$.

Design of a rectangular beam, single reinforcement

We want to design a ductile beam section to resist a design action M_{Ed}, if we fix in advance (see durability req.s) the concrete class we have the need to determine

- the steel area A_s and
- either the width b or the depth d of the beam.

Design Coefficient

Let's start with $N_c = \beta f_{cd} bx$, applied at a distance κx from the edge hence the lever arm, the distance from the resultants, is

$$z = d - \kappa x = (1 - \kappa\xi)d = \zeta d$$

With $\xi = 0.25$ and $\kappa = 0.416$ it is $\zeta = 0.896$ and the bending strength is

$$M = -N_c \cdot z = \beta\xi(1 - \kappa\xi)f_{cd}\, bd^2 = \frac{1}{r^2}\, bd^2$$

The dimensional design coefficient r is

$$r = \frac{1}{\sqrt{\beta\xi(1 - \kappa\xi)f_{cd}}}.$$

We must be careful to write all terms in the same units...

$$r = \frac{1}{\sqrt{\beta\xi(1-\kappa\xi)f_{cd}}}.$$

If we use, as it is customary, kNm as the unit for bending moments we must express b and d in metres and stresses in $\mathrm{kN/m^2}$ or kPa (note: $1\,\mathrm{N/mm^2}$ = $1\,\mathrm{MPa}$ = $1000\,\mathrm{kPa}$).

For $\xi = 0.25$ and, e.g., C25/30 concrete, $f_{cd} = 14.17\,\mathrm{MPa} = 14\,170\,\mathrm{kPa}$, it is

$$r = (\beta\xi(1-\kappa\xi)f_{cd})^{-0.50} = (0.81 \cdot 0.25(1-0.416 \cdot 0.25)14170)^{-0.50} = 0.0197$$

The design bending strength is

$$M_{Rd} = \frac{0.30 \cdot 0.46^2}{0.0197^2} = 163.6\,\mathrm{kNm}.$$

We can manipulate the equation of design strength

$$M_{Rd} = \frac{bd^2}{r^2}$$

using $M_{Rd} = M_{Ed}$ and solving w.r.t the other variables we have design equations for ductile sections,

$$d = \sqrt{\frac{M_{Ed}}{b}}\, r, \quad b = \frac{M_{Ed}}{d^2}\, r^2$$

and eventually

$$A_s = \frac{M_{Ed}}{z \cdot f_{yd}} \approx \frac{M_{Ed}}{0.9d \cdot f_{yd}}.$$

- $\xi = 0.25$
- $A_s' = \mu A_s$
- $c = \gamma d$
- $\epsilon_s' = -\epsilon_{cu}\frac{x-c}{c} = -\epsilon_{cu}\frac{\xi-\gamma}{\xi}$, for $\xi = 0.25 \Rightarrow \epsilon_s' = -\epsilon_{cu}(1-4\gamma)$
- top reinforcement yielded, for B450C steel, if $\gamma \leqslant 0.11$
- $s' = \mathbf{min}(1, -\epsilon_s'/\epsilon_{yd}) = \mathbf{min}(1, (1-4\gamma)\frac{\epsilon_{cu}}{\epsilon_{yd}})$
- $\sigma_s' = -s' f_{yd}$, the compression stress in top steel is a function of γ,
- $N_s' = -s'\,\mu\,N_s$
- the resultant of steel forces act at a distance $d + d_1$ from the compressed edge (below the tensile steel), from equilibrium

$$-N_s' \cdot (d + d_1 - c) = N_s \cdot d_1 \Rightarrow d_1 = d \cdot s' \cdot \mu\frac{1-\gamma}{1-s'\mu},$$

The bending moment can be written as the product of N_c by the distance between N_c and the resultant of steel forces,

$$M = \beta\, bx\, f_{cd}(d + d_1 - \kappa\xi d) = \beta\,\xi\, bd\, f_{cd}\left((1-\kappa\xi) + \frac{s'\mu(1-\gamma)}{1-s'\mu}\right) d$$

$$= bd^2\, \beta\,\xi(1-\kappa\xi) f_{cd}\left(1 + \frac{s'\mu}{1-s'\mu}\,\frac{1-\gamma}{1-\kappa\xi}\right)$$

$$= bd^2\,\frac{1}{r^2}\,\frac{1}{k^2} = bd^2\,\frac{1}{r'^2}$$

Because $1-\gamma \approx 1-\kappa\xi$ (distance of top steel to bottom steel, distange of centre of concrete comression to bottom steel) we can write, with good approximation ($< 1\%$ for reasonable values of γ),

$$\frac{1}{k^2} \approx \frac{1}{1-s'\mu}$$

For the same reasons, still $z \approx 0.9\, d$ and

$$A_s = M/(0.0\, d\, f_{yd}).$$

These parameters does not depend on the concrete class

γ	0.10	0.15	0.20
s'	1.000	0.716	0.358

γ	0.10	0.15	0.20
$\mu = 0.00$	0.896	0.896	0.896
$\mu = 0.25$	0.897	0.888	0.887
$\mu = 0.50$	0.998	0.880	0.879

Table 1: Values of ζ

	γ	0.1000	0.1500	0.2000
• C20/25	$\mu = 0.00$	0.0221	0.0221	0.0221
	$\mu = 0.25$	0.0191	0.0201	0.0212
	$\mu = 0.50$	0.0156	0.0178	0.0202
• C25/30	$\mu = 0.00$	0.0197	0.0197	0.0197
	$\mu = 0.25$	0.0171	0.0180	0.0189
	$\mu = 0.50$	0.0139	0.0160	0.0181
• C32/40	$\mu = 0.00$	0.0174	0.0174	0.0174
	$\mu = 0.25$	0.0151	0.0159	0.0167
	$\mu = 0.50$	0.0123	0.0141	0.0160

r AND r'

The reinforcement percentage is

$$\rho = \frac{A_s}{bd} = \frac{M}{0.9df_{yd}}\frac{1}{bd} = \frac{bd^2}{r^2}\frac{1}{bd^2 0.9f_{yd}}$$

solving for r

$$r = (0.9\rho f_{yd})^{-0.50}$$

or, in different words, there is a relationship between ρ and the design coefficient, $r(1.0\%) = 0.017$,

For smaller r we have larger ρ — that's ok, for better concrete (i.e., decreasing values of r we have smaller sections of concrete and consequently we need more reinforcements, in absolute terms (the section depth decreases) and as a percentage (the concrete area decreases).

We always have a little top steel, at least because it's needed to support the stirrups, so we design our beam considering at least $\mu = 0.25$.

W.r.t. s', for $d/c \geqslant 9$ it is $s' = 1.0$, for $d/c = 7$ we have still a good contribution from compressed steel as $s' = 0.72$ but the work rate is rapidly decreasing, $s'(5) = 0.36$ or, in other words, deep beams are better from this point of view.

We can start our design choosing a reasonable value of ρ (1%, 1.5% maybe) and the corresponding value of r.

We have fixed, say, the concrete class, $\rho = 0.015$ and $b = 300\,\mathrm{mm}$, known M_{Ed} we can easily determine the depth d and the steel area A_s (using $z = 0.9d$), we must still determine the needed amount of the compressed steel.

As a first step, one must compute the design strength using r (or r' for the μ value associated to the compressed steel already present). If $bd^2 > M_{Ed}r^2$ (or $bd^2 > M_{Ed}r'^2$) the compressed steel present in the beam is enough, otherwise we must determine the needed additional steel.

First we determine the *unbalanced moment*

$$\Delta M = M_{Ed} - bd^2/r^2,$$

next the rate of work $s' = \mathbf{min}(1, (25 - 100\gamma)/14)$ and finally

$$\Delta A_s' = \frac{\Delta M}{(d - c)s' f_{yd}}.$$

Example 1

$M_{Ed} = 220\,\mathrm{kNm}$, C25/30, $b = 0.3\,\mathrm{m}$, determine d, A_s and possibly A_s'.

With $r' = 0.018$ it is $d = r'\sqrt{M/b} = 0.018\sqrt{220/0.3} = 0.49\,\mathrm{m}$

Considering a concrete cover $c = 40\,\mathrm{mm}$ and sticking with multiples of $10\,\mathrm{cm}$ we choose $h = 600\,\mathrm{mm}$ and $d = 560\,\mathrm{mm}$

The steel area is $A_s = 220 \times 10^6\,\mathrm{Nmm}/(0.9 \cdot 560\,\mathrm{mm} \cdot 391\,\mathrm{N/mm^2} = 1115.5\,\mathrm{mm^2}$ corresponding to $\rho = 0.62\%$ and the design strength (neglecting the possible presence of top steel) is

$$M_{Rd} = bd^2/r^2 = 0.30 \cdot 0.56^2/0.0197^2 = 242\,\mathrm{kNm} > M_{Ed}.$$

No additional top steel is needed.

Everything is the same but we accept the idea of more reinforcement, $\rho \approx 1\%$ and consequently $r' = 0.016$.

The depth should be $d = 0.47\,\mathrm{m}$ but we choose $h = 500\,\mathrm{mm}$ and $d = 460\,\mathrm{mm}$.

The steel area is $A_s = 1360\,\mathrm{mm}^2$ and the bending strength is $M_{Rd} = 163.6\,\mathrm{kNm} < M_{Ed}$, hence we need to dimension the compressed steel,

$$\Delta M = 220\,\mathrm{kNm} - 163.6\,\mathrm{kNm} = 56.4\,\mathrm{kNm}$$

and, because $s' = 1$ $(\gamma \approx 1/12 < 1/9)$ we have

$$A_s' = \frac{56.4 \times 10^6\,\mathrm{Nmm}}{(460\,\mathrm{mm} - 40\,\mathrm{mm})391.3\,\mathrm{N/mm}^2} = 340\,\mathrm{mm}^2$$

Example 3

A shallow beam, $h = 240\,\mathrm{mm}$, $c = 40\,\mathrm{mm}$ and $M_{Ed} = 120\,\mathrm{kNm}$.

Using $r' = 0/019$, with $d = 200\,\mathrm{mm}$, $\gamma = 0.20$ and $s' = 0.358$ we have

$$b = M \cdot r'^2/d^2 = 120 \cdot 0.019^2/0.2^2 = 1100\,\mathrm{mm}$$

. The tensile steel area is

$$A_s = 120 \times 10^6\,\mathrm{Nmm}/(0.9\,200\,\mathrm{mm}\,391.3\,\mathrm{N/mm^2}) = 1700\,\mathrm{mm^2}$$

and the no compression reinforcement design strength is

$$M_{Rd} = 113.4\,\mathrm{kNm} < M_{Ed}$$

, the unbalanced moment is $\Delta M = 6.6\,\mathrm{kNm}$ and eventually

$$A_s' = \frac{6.6 \times 10^6\,\mathrm{Nmm}}{160\,\mathrm{mm}0.358\,391.3\,\mathrm{N/mm^2}} = 300\,\mathrm{mm^2}.$$