Beams

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Introduction

Definition:

- a prevalent dimension (the span),
- loads act perpendicular to the prevalent dimension, so no axial forces but shear and bending moment,
- the cross section dimensions are the depth (parallel to loading) and the width.

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Historical remarks:

- Galileo Galilei,
- Coulomb,
- De Saint Venant.

Beams in Buildings

Beams are almost always present in a building.
When beams form the principal structural system they are arranged in a systematic manner, usually following a hierarchical arrangement.
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The loads on the beams are progressively increasing, as well as the lengths of the beams and this leads in turn to an increase in the size of the beams in a three level system, from the secondary beams to the primary ones.

Beams in Buildings

The stresses in a beam depend on the bending moment value and on shape of the cross section.

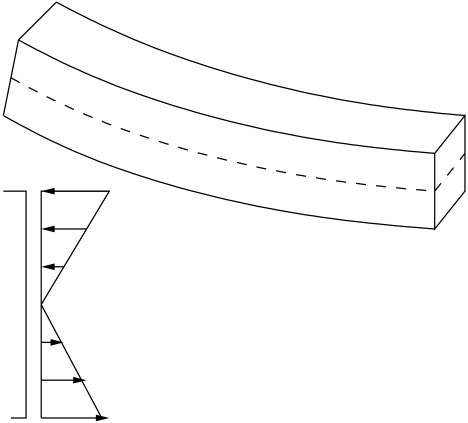
- The bending moment depends on the value of the loading, on the span of the beam and on the support conditions.
- In general, the larger the section the lower the stresses, but also the shape and its orientation w/r to loading direction is important, in particular an increase of the depth leads to an important decrease of the maximum stresses.

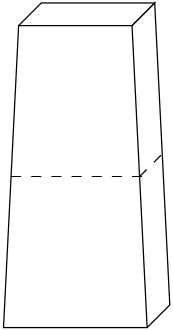
Basic Stress Distributions

At any cross section the resultant of the stresses must equal the shear force and the bending moment.

- The shear force, that acts in the direction of the depth of the section, is equilibrated by tangential stresses.
- The bending moment, that acts perpendicular to the cross section, is equilibrated by normal stresses.

Basic Stress Distributions





Basic Stress Distributions

- The moment cause the beam to bow, in our figure top fibers are shorted and bottom fibers are elongated.
- There are fibers whose length is not varied, these fibers cross the section on a line called the *neutral axis*.
- The amount of length variation depends linearly on the distance y from the neutral axis.
- The stresses, for a linear elastic material, depends on the amount of length variation(a.k.a. deformation) and are hence linearly varying, with positive and negative extremes on the borders of the cross section.

Bending Stresses

The stresses are proportional from the distance y from the neutral axis, are proportional to the bending moment and are inversely proportional to a property of the dimensions and shape of the cross section, that we call moment of inertia and denote with the symbol J. (nb dimensional analysis)

$$\sigma = My/J$$

Example

Simply supported beam, $L = 8\,\mathrm{m}$, $p = 60\,\frac{\mathrm{kN}}{\mathrm{m}}$, $M_{\max} = \frac{pL^2}{8} = 480\,\mathrm{kNm}$.
For a rectangular section it is $J = bd^3/12$ and $y_{\max} = b/2$.

The maximum stress is $\sigma_{\max} = \frac{\frac{PL^2}{8} \times \frac{d}{2}}{bd^3/12}$ and with $b = 0.3\,\mathrm{m}$, $d = 0.6\,\mathrm{m}$ it is $\sigma_{\max} = 26\,666\,667\,\frac{\mathrm{N}}{\mathrm{m}^2} = 26.67\,\mathrm{MPa}$

Bending Stresses

For a rectangular section it is

$$J = \frac{bd^3}{12}, \quad \sigma_{\max} = \frac{Md/2}{bd^3/12} = \frac{M}{bd^2/6} = M/W$$

where $W = bd^2/6$ is the *section modulus*.
It is clear that if the double the amount of material in a beam by doubling the width b the maximum stress is halved, if we double the depth d the stress is reduced to one quarter.

Example

Simply supported beam, $L = 7.2\,\mathrm{m}$, $P = 20\,\mathrm{kN}$ applied ad midspan,
$M_{\max} = PL/4 = 36\,\mathrm{kN\,m} = 36\,\mathrm{MN\,mm}$.
Rectangular section, $b = 120\,\mathrm{mm}$ and $d = 200\,\mathrm{mm}$, $J = 80 \times 10^6\,\mathrm{mm}^4$ and $W = 800 \times 10^3\,\mathrm{mm}^3$, eventually $\sigma_{\max} = 45\,\mathrm{MPa}$.

A little theory

In general we have to perform our computation in two steps

1. find the position of the neutral axis,
2. find the moment of inertia with respect to the neutral axis.

A little theory

To find the position of the neutral axis

1. draw an axis parallel to the neutral one at a(n unknown) distance y_G from it,
2. the distance from the neutral axis, in terms of the distance of the temporary axis, is $y - y_G$,
3. the stress is $\sigma = (y - y_G)k$,
4. the stress resultant is $N = \int_A \sigma\, dA = k \int_A (y - y_G)\, dA = k \int_A y\, dA - k y_G A$
5. we have $N = 0$ hence
$$y_G = \frac{S_x}{A} \text{ where } S_x = \int_A y\, dA$$

Similarly $x_G = S_y / A$ with $S_y = \int_A x\, dA$.

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We have found the position of the *centroid* of the cross section w/r to two arbitrary axes.

A little theory

The moment of the stresses about the neutral axes, if now we measure the distances from it, is $M = \int_A y\sigma \, dA = k \int_A y^2 \, dA = k \, J_x$ by the definition of moment of inertia, so that we can solve for $k = M/J_x$ and eventually we can write

$$\sigma = \frac{M}{J_x} y.$$

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The expression above coincides with the expression that was shown earlier w/o an explanation.

Computing moments of inertia

The moments of inertia of many simple figures are tabulated, with respect to their own centroids. We can use these known results along with *parallel axis theorem*.

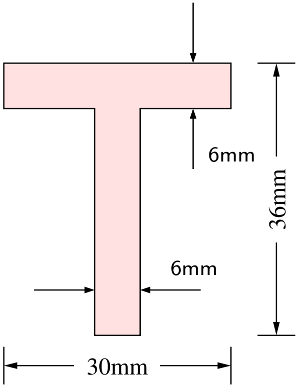
Theorem

The moment of inertia J of an area A, with respect to a axis parallel to an axis passing for its centroid is equal to the sum of the centroidal moment of inertia J' plus the moment of transport $\Delta^2 A$

$$J = J' + \Delta^2 A$$

where Δ is the distance between the two axes.

Computing moments of inertia



$$A = (6 \times 30 + 6 \times 30)\text{mm}^2 = 360\text{mm}^2$$

$$S_x = (180 \times 15 + 180 \times 33)\text{mm}^3 = (180 \times 48)\text{mm}^3$$

$$y_G = \frac{180 \times 48}{180 \times 2}\text{mm} = 24\text{mm}$$

$$\frac{J_1}{\text{mm}^4} = \frac{30^3 \times 6}{12} + 180 \times (15 - 24)^2 = (13500 + 14580) = 28080$$

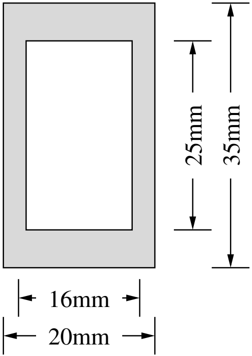
$$\frac{J_2}{\text{mm}^4} = \frac{30 \times 6^3}{12} + 180 \times (33 - 15)^2 = (540 + 14580) = 15120$$

$$J = J_1 + J_2 = 43200\text{mm}^4.$$

$$W_t = J/12\text{mm} = 3600\text{mm}^3$$

$$W_b = J/24\text{mm} = 1800\text{mm}^3$$

Computing moments of inertia



$$A = (20 \times 35 - 16 \times 25)\text{mm}^2 = 300\text{mm}^2$$

$$S_x = 10\text{mm}, \quad y_G = 17.5\text{mm}$$

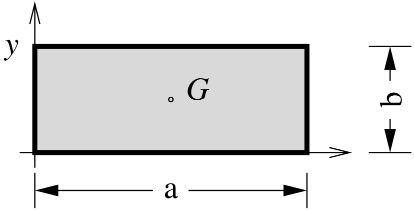
$$\frac{J_1}{\text{mm}^4} = \frac{35^3 \times 20}{12} = 71458.33$$

$$\frac{J_2}{\text{mm}^4} = \frac{25 \times 16^3}{12} = 20833.33$$

$$J = J_1 - J_2 = 50625.00\text{mm}^4.$$

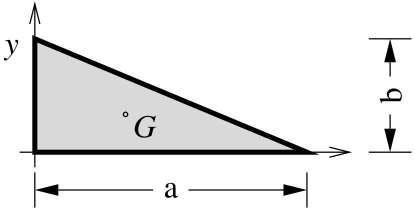
$$W = J/17.5\text{mm} = 2892.86\text{mm}^3$$

Rectangle



Sides	$a,\ b$
Centroid	$x_G = a/2, \quad y_G = b/2$
Area	$A = ab,$
Moment of Inertia	$J = \dfrac{ab^3}{12}$

Triangle

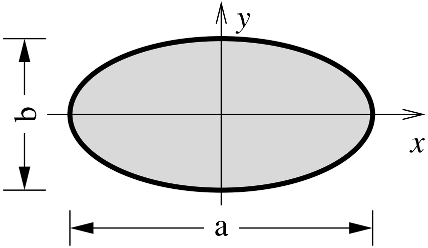


For a right triangle.

Sides	$a,\ b$
Centroid	$x_G = a/3, \quad y_G = b/3$
Area	$A = ab/2,$
Moment of Inertia	$J = \dfrac{ab^3}{36}$

Oval

When $a = b = D = 2R$ the oval is a circle.



$$\text{Axes} \qquad a,\ b$$

$$\text{Centre of Mass} \qquad x_G = y_G = 0$$

$$\text{Area} \qquad A = \frac{\pi ab}{4} = \pi R^2,$$

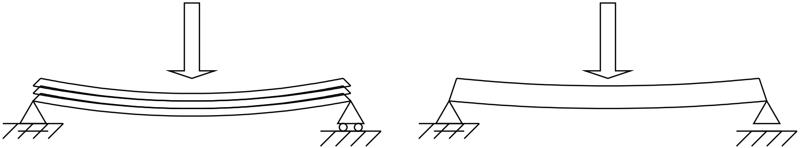
$$\text{Moment of Inertia} \qquad J = \pi \frac{ab^3}{64} = \pi \frac{R^4}{4}$$

Lateral Buckling

As was the case with trusses, one side of a beam is subjected to compression and it's subjected to the possibility of a lateral sway.
Possible remedies:

- lateral bracing
- stiffening the compressed part in the lateral direction

Shear Stresses



- On the left, a “beam” composed of three equal, separate sheets, each one behaves the same,
- on the right, the sheets have been glued and the top sheet is shortened while the bottom one is elongated.

This behavior is possible only if the sheets exchange tangential stresses at their interfaces.
These tangential stresses are the shear stresses.

Shear Stresses

E.g., for a rectangular beam the resultants of compressive and tensile stresses lie at $d/6$ from the borders, so their arm is $a = 2d/3$ and we have

$$T = C = M/a = 1.5M/d.$$

Consider now a slice of beam cut between x_1 and $x_2 = x_1 + \delta$ and separated in two halves at the neutral axis, the bottom part is subjected to two forces $T_1 = 1.5M_1/d$ and $T_2 = 1.5M_2/d$, we have an unbalanced force

$$U = 1.5\frac{M(x_1+\delta) - M(x_1)}{d}.$$

For $\delta \to dx$ it is

$$dU = 1.5dM/d = 1.5V\,dx/d,$$

the area available for equilibrating the unbalanced force using tangential stresses is $dA = b\,dx$ so eventually

$$\tau = dU/dA = 1.5V/(bd).$$

Shear Stresses

The procedure can be generalized, we know that $\sigma = My/J$ so the resultant for an area A^* is

$$F = \frac{M}{J}\int_{A^*} y\,dA = \frac{M}{J}S^*$$

where S^* is the first moment of the area A^* w/r to the neutral axis.
The variation of F over dx, $dF = S^*V\,dx/J$, is the unbalanced force to equilibrate over an area $dA = b\,dx$ so that

$$\tau = \frac{V\,S^*}{b\,J}.$$

Shear Stresses

For the T section we have previously studied, consider a maximum shear $V = 4800\,\mathrm{N}$ and consider that the flange and the web have been glued together ($b = 6\,\mathrm{mm}$).
We have to compute S^* for the flange

$$S^* = \int_{30}^{36} 30y\,dy = 5940\,\mathrm{mm}^3$$

and compute τ

$$\tau = \frac{V\,S^*}{b\,J} = \frac{4800 \times 5940}{6 \times 43200}\mathrm{MPa} = 110\mathrm{MPa}.$$

Your glue must resist to a tangential stress of 110MPa.

Deflections

Just a few formulas, E is the Young modulus of the material... for the cantilever, the concentrated load is at the free end, in the other cases the concentrated load is at midspan — the deflections are measured at the same points.

	Cantilever	Supported	Fixed Ends
P	$\frac{PL^3}{3EJ}$	$\frac{PL^3}{48EJ}$	$\frac{PL^3}{192EJ}$
w	$\frac{wL^4}{8EJ}$	$\frac{5wL^4}{384EJ}$	$\frac{wL^4}{384EJ}$

Note how much the displacements depend on the beam span, note also that different end conditions influence very much the deflections.

Deflections

A simply supported beam, $L = 6\text{m}$, has a live load $w = 3\text{kN m}^{-1}$. The section is rectangular, $b = 0.2\text{m}$ and $d = 0.4\text{m}$, the Young modulus is $E = 12\text{GPa}$. Compute the midspan displacement.

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$\Delta \approx 4\,\text{mm}$

General Principles

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- Load and Resistance Factor Design (LRFD)

Steel Beams

The design of a steel beam in most cases coincides with choosing a section in a catalogue of commonly available steel members.
The section you are going to choose has to satisfy two essential requirements in terms of strength and deformability.

- Strength: the section must be strong enough to resist the maximum value of the bending moment,
$$f_y W \geq M_{\max} \quad \rightarrow \quad W \geq M_{\max}/f_y.$$
- Deformability: the maximum deflection must be less than a fraction of the beam span, $\Delta \leq L/k$.

  The value of k depends on the load cases used to compute the deflection (live vs dead vs combination) and the function that the beam has to perform (support a roof, a floor, a wall, etc).

  E.g., for a simply supported beam subjected to a distributed load, supporting a wall
$$\Delta = \frac{5wL^4}{384EJ} \leq \frac{L}{200} \quad \rightarrow \quad J \geq \frac{1000wL^3}{384EJ}.$$

Steel beams, plastic behavior

Steel is an elastic-plastic material, if the strain exceeds a certain amount the normal stress remains constant, $\sigma = f_y$.
What happens when $M = Wf_y$ and the load is increasing?

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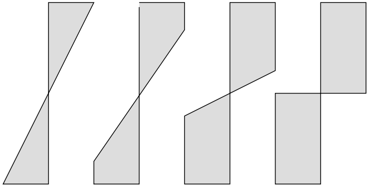

When the external bending moment increases, the outer fibres are further pulled-pushed and in a larger part of the section the stress is equal to f_y — the part of the section that is still elastic is, otoh, smaller and smaller and the elastic stiffness of the section is diminishing.
At some point, when in all the section $\sigma = \pm f_y$ the elastic stiffness is equal to 0. In terms of strength we have reached the *plastic moment*, in terms of stiffness we have formed a *plastic hinge*.

Steel beams, plastic behavior

For a rectangular beam $W = bd^2/6$ and $M_y = f_y bd^2/6$ is the maximum elastic moment. When $\sigma = \pm f_y$ we have $T = C = f_y bd/2$ and their arm is $a = d/2$, hence $M_p = f_y bd^2/4$ is the moment of plasticization, that turns the section into a plastic hinge.
For a rectangular section $M_p/M_y = 1.5$, for different shapes we have different ratios. For common I shaped sections, the ratio has an average value around 1.12, for a circular bar it's about 1.7 and it's 2.0 for a diamond shaped bar.

Shear

In most cases, when dealing with wide flange beams, T beams, channels it is possible to estimate the design value of the shear strength taking into account only the web area A_w,

$$V_{Rd} = 0.6 f_y A_w.$$

RC Principles

Concrete resists very well compression, resists very poorly tension.
A plain concrete beam when loaded will tend to crack vertically in zones of (relatively) large bending moments and diagonally in zones of high shear and eventually break down for small loadings.
To avoid the crack growt and the eventual failure we dispose some tension resistent material perpendicular to the cracks' directions, usually steel bars, longitudinal steel bars to equilibrate bending moments and vertical stirrups (U-shaped bars) or diagonally bent bars to equilibrate shear forces.

Principles of reinforcement design

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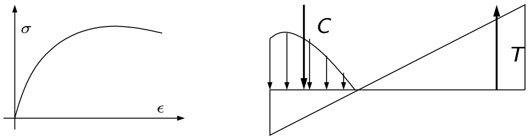
A RC beam correctly designed has the right amount of reinforcement, so that the first yielding happens in the steel and we reach a limit state in which most of the concrete is safely plasticized and the steel deformations are still under control.

In this case a possible failure of a beam is preceded by a visual increase of deflections. Further, the possibility of a plastic behavior leads to an energy dissipation that is useful to control the response to seismic excitation.

RC design at the limit state



$$T = A_s f_y, \qquad C \approx 0.8 x b f_c, \qquad x = \frac{A_s f_y}{0.8 b f_c}$$

To have a correct ductility, a good design choice is to assign appropriate values to the steel and concrete deformations.

RC Member Sizes

Min Overall Depth (U.S. practice)

Member	Simply Supported	H+F	F+F	Cantilever
Slabs	L/20	L/24	L/28	L/10
Beams	L/16	L/18	L/21	L/8

Overall depth h is the reinforcement depth d plus the bottom concrete cover.