

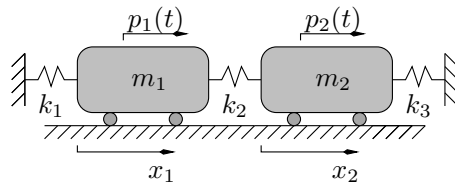
<div data-bbox="410 224 1056 313" data-label="Section-Header"> <h1>Multi Degrees of Freedom Systems</h1> <h2>MDOF</h2> </div> <div data-bbox="638 374 826 409" data-label="Text"> <p>Giacomo Boffi</p> </div> <div data-bbox="478 443 984 474" data-label="Text"> <p>http://intranet.dica.polimi.it/people/boffi-giacomo</p> </div> <div data-bbox="442 501 1021 555" data-label="Text"> <p>Dipartimento di Ingegneria Civile Ambientale e Territoriale Politecnico di Milano</p> </div> <div data-bbox="628 589 836 624" data-label="Text"> <p>March 23, 2020</p> </div>	<div data-bbox="1307 127 1406 172" data-label="Text"> <p>Multi DoF Systems</p> </div> <div data-bbox="1294 183 1422 208" data-label="Text"> <p>Giacomo Boffi</p> </div> <div data-bbox="1294 232 1406 257" data-label="Text"> <p>Introduction</p> </div> <div data-bbox="1294 268 1420 327" data-label="Text"> <p>The Homogeneous Problem</p> </div> <div data-bbox="1294 338 1372 383" data-label="Text"> <p>Modal Analysis</p> </div> <div data-bbox="1294 394 1382 416" data-label="Text"> <p>Examples</p> </div>
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Introductory Remarks

Consider an undamped system with two masses and two degrees of freedom.



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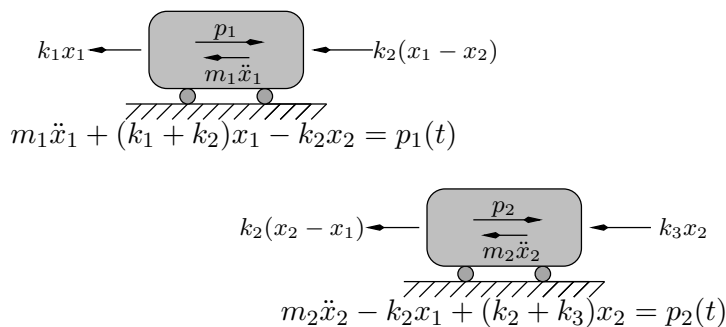
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We can separate the two masses, single out the spring forces and, using the D'Alembert Principle, the inertial forces and, finally, write an equation of dynamic equilibrium for each mass.



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The equation of motion of a 2DOF system

With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = p_1(t), \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

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The equation of motion of a 2DOF system

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Introducing the loading vector \mathbf{p} , the vector of inertial forces \mathbf{f}_I and the vector of elastic forces \mathbf{f}_S ,

$$\mathbf{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \mathbf{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \mathbf{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\mathbf{f}_I + \mathbf{f}_S = \mathbf{p}(t).$$

$\mathbf{f}_S = \mathbf{K} \mathbf{x}$

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It is possible to write the linear relationship between \mathbf{f}_S and the vector of displacements $\mathbf{x} = \{x_1 x_2\}^T$ in terms of a matrix product, introducing the so called *stiffness matrix* \mathbf{K} .

In our example it is

$$\mathbf{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{x} = \mathbf{K} \mathbf{x}$$

The stiffness matrix \mathbf{K} has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.

The stiffness matrix \mathbf{K} is hence a *square matrix* $\mathbf{K}_{\text{ndof} \times \text{ndof}}$

$\mathbf{f}_I = \mathbf{M} \ddot{\mathbf{x}}$

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Analogously, introducing the *mass matrix* \mathbf{M} that, for our example, is

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\mathbf{f}_I = \mathbf{M} \ddot{\mathbf{x}}.$$

Also the mass matrix \mathbf{M} is a square matrix, with number of rows and columns equal to the number of *DOFs*.

Matrix Equation

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Finally it is possible to write the equation of motion in matrix format:

$$\mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{p}(t).$$

Of course it is possible to take into consideration also the damping forces, taking into account the velocity vector $\dot{\mathbf{x}}$ and introducing a damping matrix \mathbf{C} too, so that we can eventually write

$$\mathbf{M} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{p}(t).$$

But today we are focused on undamped systems...

Properties of \mathbf{K}

- \mathbf{K} is symmetrical.

The elastic force exerted on mass i due to a unit displacement of mass j , $f_{S,i} = k_{ij}$ is equal to the force k_{ji} exerted on mass j due to a unit displacement of mass i , in virtue of *Betti's theorem* (also known as Maxwell-Betti reciprocal work theorem).

- \mathbf{K} is a positive definite matrix.

The strain energy V for a discrete system is

$$V = \frac{1}{2} \mathbf{x}^T \mathbf{f}_S,$$

and expressing \mathbf{f}_S in terms of \mathbf{K} and \mathbf{x} we have

$$V = \frac{1}{2} \mathbf{x}^T \mathbf{K} \mathbf{x},$$

and because the strain energy is positive for $\mathbf{x} \neq \mathbf{0}$ it follows that \mathbf{K} is definite positive.

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Properties of \mathbf{M}

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive.

Both the mass and the stiffness matrix are symmetrical and definite positive.

Note that the kinetic energy for a discrete system can be written

$$T = \frac{1}{2} \dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}}.$$

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Generalisation of previous results

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with two main exceptions.

- 1 For a general structural system, in which not all DOFs are related to a mass, \mathbf{M} could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy is zero.
- 2 For a general structural system subjected to axial loads, due to the presence of *geometrical stiffness* it is possible that for some particular displacement vector the strain energy is zero and \mathbf{K} is *semi-definite* positive.

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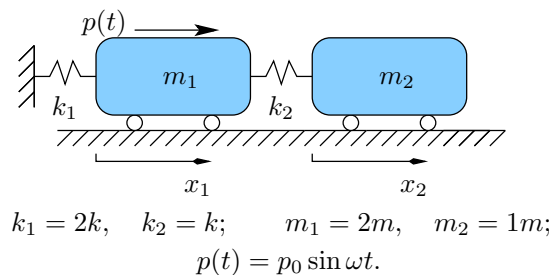
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The problem

Graphical statement of the problem



The equations of motion

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = p_0 \sin \omega t,$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0.$$

... but we prefer the matrix notation ...

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The steady state solution

We prefer the matrix notation because we can find the steady-state response of a *SDOF* system *exactly* as we found the s-s solution for a SDOF system.

Substituting $x(t) = \xi \sin \omega t$ in the equation of motion and simplifying $\sin \omega t$,

$$k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \xi - m\omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \xi = p_0 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

dividing by k , with $\omega_0^2 = k/m$, $\beta^2 = \omega^2/\omega_0^2$ and $\Delta_{st} = p_0/k$ the above equation can be written

$$\left(\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \beta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \xi = \begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \xi = \Delta_{st} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

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The steady state solution

The determinant of the matrix of coefficients is

$$\text{Det} = 2\beta^4 - 5\beta^2 + 2$$

but we want to write the polynomial in β in terms of its roots

$$\text{Det} = 2 \times (\beta^2 - 1/2) \times (\beta^2 - 2).$$

Solving for ξ/Δ_{st} in terms of the inverse of the coefficient matrix gives

$$\begin{aligned} \frac{\xi}{\Delta_{\text{st}}} &= \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{bmatrix} 1 - \beta^2 & 1 \\ 1 & 3 - 2\beta^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ &= \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix}. \end{aligned}$$

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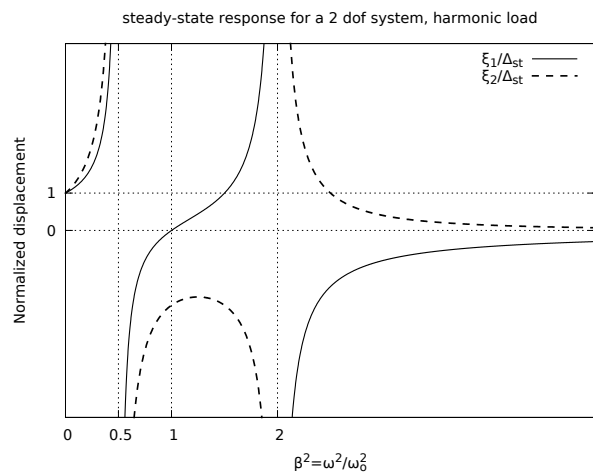
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The solution, graphically



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Comment to the Steady State Solution

The steady state solution is

$$\mathbf{x}_{s-s} = \Delta_{\text{st}} \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix} \sin \omega t.$$

As it's apparent in the previous slide, we have two different values of the excitation frequency for which the *dynamic amplification factor* goes to infinity.

For an undamped SDOF system, we had a single frequency of excitation that excites a *resonant response*, now for a *two* degrees of freedom system we have *two* different excitation frequencies that excite a resonant response.

We know how to compute a particular integral for a MDOF system (at least for a harmonic loading), what do we miss to be able to determine the integral of motion?

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Homogeneous equation of motion

To understand the behaviour of a *MDOF* system, we have to study the homogeneous solution.

Let's start writing the homogeneous equation of motion,

$$\mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{0}.$$

The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* ψ :

$$\mathbf{x}(t) = \psi(A \sin \omega t + B \cos \omega t).$$

Substituting in the equation of motion, we have

$$(\mathbf{K} - \omega^2 \mathbf{M}) \psi(A \sin \omega t + B \cos \omega t) = \mathbf{0}$$

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The previous equation must hold for every value of t , so it can be simplified removing the time dependency:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \psi = \mathbf{0}.$$

This is a homogeneous linear equation, with unknowns ψ_i and the coefficients that depends on the parameter ω^2 .

Speaking of homogeneous systems, we know that

- there is always a *trivial solution*, $\psi = \mathbf{0}$, and
- *non-trivial solutions* are possible if the determinant of the matrix of coefficients is equal to zero,

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

The *eigenvalues* of the *MDOF* system are the values of ω^2 for which the above equation (the *equation of frequencies*) is verified or, in other words, the frequencies of vibration associated with the shapes for which

$$\mathbf{K} \psi \sin \omega t = \omega^2 \mathbf{M} \psi \sin \omega t.$$

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Eigenvalues, cont.

For a system with N degrees of freedom the expansion of $\det(\mathbf{K} - \omega^2 \mathbf{M})$ is an algebraic polynomial of degree N in ω^2 .

A polynomial of degree N has exactly N roots, either real or complex conjugate.

In Dynamics of Structures those roots ω_i^2 , $i = 1, \dots, N$ are all real because the structural matrices are symmetric matrices.

Moreover, if both \mathbf{K} and \mathbf{M} are positive definite matrices (a condition that is always satisfied by stable structural systems) all the roots, all the *eigenvalues*, are strictly positive:

$$\omega_i^2 \geq 0, \quad \text{for } i = 1, \dots, N.$$

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Substituting one of the N roots ω_i^2 in the characteristic equation,

$$(\mathbf{K} - \omega_i^2 \mathbf{M}) \psi_i = \mathbf{0}$$

the resulting system of $N - 1$ linearly independent equations can be solved (except for a scale factor) for ψ_i , the eigenvector corresponding to the eigenvalue ω_i^2 .

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The scale factor being arbitrary, you have to choose (arbitrarily) the value of one of the components and compute the values of all the other $N - 1$ components using the $N - 1$ linearly independent equations.

It is common to impose to each eigenvector a *normalisation with respect to the mass matrix*, so that

$$\psi_i^T \mathbf{M} \psi_i = m$$

where m represents the unit mass.

*Please understand clearly that, substituting **different eigenvalues** in the equation of free vibrations, you have **different linear systems**, leading to **different eigenvectors**.*

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The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\mathbf{x}(t) = \sum_{i=1}^N \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t).$$

In the general integral there are $2N$ unknown *constants of integration*, that must be determined in terms of the initial conditions.

Initial Conditions

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Usually the initial conditions are expressed in terms of initial displacements and initial velocities \mathbf{x}_0 and $\dot{\mathbf{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^N \boldsymbol{\psi}_i \omega_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\mathbf{x}(0) = \sum_{i=1}^N \boldsymbol{\psi}_i B_i = \mathbf{x}_0, \quad \dot{\mathbf{x}}(0) = \sum_{i=1}^N \boldsymbol{\psi}_i \omega_i A_i = \dot{\mathbf{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$\sum_{i=1}^N \boldsymbol{\psi}_{ji} B_i = x_{0,j}, \quad \sum_{i=1}^N \boldsymbol{\psi}_{ji} \omega_i A_i = \dot{x}_{0,j}, \quad j = 1, \dots, N.$$

Orthogonality - 1

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Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2 , and write the characteristic equation for each eigenvalue:

$$\mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \mathbf{M} \boldsymbol{\psi}_r$$

$$\mathbf{K} \boldsymbol{\psi}_s = \omega_s^2 \mathbf{M} \boldsymbol{\psi}_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\boldsymbol{\psi}_s^T \mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M} \boldsymbol{\psi}_r$$

$$\boldsymbol{\psi}_r^T \mathbf{K} \boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \mathbf{M} \boldsymbol{\psi}_s$$

Orthogonality - 2

The term $\psi_s^T \mathbf{K} \psi_r$ is a scalar, hence

$$\psi_s^T \mathbf{K} \psi_r = (\psi_s^T \mathbf{K} \psi_r)^T = \psi_r^T \mathbf{K}^T \psi_s$$

but \mathbf{K} is symmetrical, $\mathbf{K}^T = \mathbf{K}$ and we have

$$\psi_s^T \mathbf{K} \psi_r = \psi_r^T \mathbf{K} \psi_s.$$

By a similar derivation

$$\psi_s^T \mathbf{M} \psi_r = \psi_r^T \mathbf{M} \psi_s.$$

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Orthogonality - 3

Substituting our last identities in the previous equations, we have

$$\psi_r^T \mathbf{K} \psi_s = \omega_r^2 \psi_r^T \mathbf{M} \psi_s$$

$$\psi_r^T \mathbf{K} \psi_s = \omega_s^2 \psi_r^T \mathbf{M} \psi_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2) \psi_r^T \mathbf{M} \psi_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\psi_r^T \mathbf{M} \psi_s = 0, \quad \text{for } r \neq s.$$

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Orthogonality - 4

The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\psi_s^T \mathbf{K} \psi_r = \omega_r^2 \psi_s^T \mathbf{M} \psi_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \psi_i^T \mathbf{M} \psi_i$$

and consequently

$$\psi_i^T \mathbf{K} \psi_i = \omega_i^2 M_i.$$

M_i is the *modal mass* associated with mode no. i while $K_i \equiv \omega_i^2 M_i$ is the respective *modal stiffness*.

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Eigenvectors are a base

The eigenvectors are linearly independent, so for every vector \mathbf{x} we can write

$$\mathbf{x} = \sum_{j=1}^N \psi_j q_j.$$

The coefficients are readily given by premultiplication of \mathbf{x} by $\psi_i^T \mathbf{M}$, because

$$\psi_i^T \mathbf{M} \mathbf{x} = \sum_{j=1}^N \psi_i^T \mathbf{M} \psi_j q_j = \psi_i^T \mathbf{M} \psi_i q_i = M_i q_i$$

in virtue of the orthogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_j = \frac{\psi_j^T \mathbf{M} \mathbf{x}}{M_j}.$$

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Eigenvectors are a base

Generalising our results for the displacement vector to the acceleration vector and expliciting the time dependency, it is

$$\begin{aligned} \mathbf{x}(t) &= \sum_{j=1}^N \psi_j q_j(t), & \ddot{\mathbf{x}}(t) &= \sum_{j=1}^N \psi_j \ddot{q}_j(t), \\ x_i(t) &= \sum_{j=1}^N \Psi_{ij} q_j(t), & \ddot{x}_i(t) &= \sum_{j=1}^N \psi_{ij} \ddot{q}_j(t). \end{aligned}$$

Introducing $\mathbf{q}(t)$, the *vector of modal coordinates* and Ψ , the *eigenvector matrix*, whose columns are the eigenvectors, we can write

$$\mathbf{x}(t) = \Psi \mathbf{q}(t), \quad \ddot{\mathbf{x}}(t) = \Psi \ddot{\mathbf{q}}(t).$$

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EoM in Modal Coordinates...

Substituting the last two equations in the equation of motion,

$$\mathbf{M} \Psi \ddot{\mathbf{q}} + \mathbf{K} \Psi \mathbf{q} = \mathbf{p}(t)$$

premultiplying by Ψ^T

$$\Psi^T \mathbf{M} \Psi \ddot{\mathbf{q}} + \Psi^T \mathbf{K} \Psi \mathbf{q} = \Psi^T \mathbf{p}(t)$$

introducing the so called *starred matrices*, with $\mathbf{p}^*(t) = \Psi^T \mathbf{p}(t)$, we can finally write

$$\mathbf{M}^* \ddot{\mathbf{q}} + \mathbf{K}^* \mathbf{q} = \mathbf{p}^*(t).$$

The vector equation above corresponds to the set of scalar equations

$$p_i^* = \sum m_{ij}^* \ddot{q}_j + \sum k_{ij}^* q_j, \quad i = 1, \dots, N.$$

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... are N independent equations!

We must examine the structure of the starred symbols.

The generic element, with indexes i and j , of the *starred* matrices can be expressed in terms of single eigenvectors,

$$\begin{aligned} m_{ij}^* &= \psi_i^T \mathbf{M} \psi_j &= \delta_{ij} M_i, \\ k_{ij}^* &= \psi_i^T \mathbf{K} \psi_j &= \omega_i^2 \delta_{ij} M_i. \end{aligned}$$

where δ_{ij} is the *Kronecker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p_i^* = \psi_i^T \mathbf{p}(t)$ we have
a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p_i^*(t), \quad i = 1, \dots, N$$

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Initial Conditions Revisited

The initial displacements can be written in modal coordinates,

$$\mathbf{x}_0 = \Psi \mathbf{q}_0$$

and premultiplying both members by $\Psi^T \mathbf{M}$ we have the following relationship:

$$\Psi^T \mathbf{M} \mathbf{x}_0 = \Psi^T \mathbf{M} \Psi \mathbf{q}_0 = \mathbf{M}^* \mathbf{q}_0.$$

Premultiplying by the inverse of \mathbf{M}^* and taking into account that \mathbf{M}^* is diagonal,

$$\mathbf{q}_0 = (\mathbf{M}^*)^{-1} \Psi^T \mathbf{M} \mathbf{x}_0 \Rightarrow q_{i0} = \frac{\psi_i^T \mathbf{M} \mathbf{x}_0}{M_i}$$

and, analogously,

$$\dot{q}_{i0} = \frac{\psi_i^T \mathbf{M} \dot{\mathbf{x}}_0}{M_i}$$

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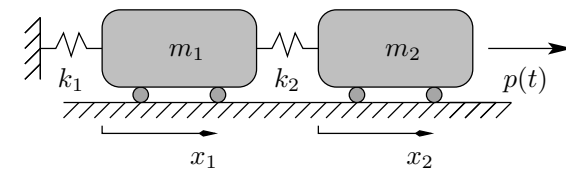
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2 DOF System



$$k_1 = 2k, \quad k_2 = 3k; \quad m_1 = 2m, \quad m_2 = 4m;$$

$$p(t) = p_0 \sin \omega t.$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin \omega t,$$

$$\mathbf{M} = m \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{K} = k \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}.$$

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Equation of frequencies

The equation of frequencies is

$$\|\mathbf{K} - \omega^2 \mathbf{M}\| = \begin{vmatrix} 5k - 2\omega^2 m & -3k \\ -3k & 3k - 4\omega^2 m \end{vmatrix} = 0.$$

Developing the determinant

$$(8m^2)\omega^4 - (26mk)\omega^2 + (6k^2)\omega^0 = 0$$

Solving the algebraic equation in ω^2

$$\omega_1^2 = \frac{k}{m} \frac{13 - \sqrt{121}}{8}, \quad \omega_2^2 = \frac{k}{m} \frac{13 + \sqrt{121}}{8};$$

$$\omega_1^2 = \frac{1}{4} \frac{k}{m}, \quad \omega_2^2 = 3 \frac{k}{m}.$$

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Eigenvectors

Substituting ω_1^2 for ω^2 in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k \left(5 - 2 \cdot \frac{1}{4} \right) \psi_{11} - 3k \psi_{21} = 0$$

while substituting ω_2^2 gives

$$k (3 - 2 \cdot 3) \psi_{12} - 3k \psi_{22} = 0.$$

Solving with the arbitrary assignment $\psi_{11} = \psi_{22} = 1$ gives the *unnormalized* eigenvectors,

$$\psi_1 = \begin{Bmatrix} +1 \\ +\frac{3}{2} \end{Bmatrix}, \quad \psi_2 = \begin{Bmatrix} -3 \\ +1 \end{Bmatrix}.$$

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Normalization

We compute first M_1 and M_2 ,

$$\begin{aligned} M_1 &= \psi_1^T \mathbf{M} \psi_1 \\ &= m \begin{Bmatrix} 1, & \frac{3}{2} \end{Bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ \frac{3}{2} \end{Bmatrix} \\ &= m \begin{Bmatrix} 2, & 6 \end{Bmatrix} \begin{Bmatrix} 1 \\ \frac{3}{2} \end{Bmatrix} = 11 m \end{aligned}$$

and, in a similar way, we have $M_2 = 22 m$; the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{11} = 3.317, \quad \alpha_2 = \sqrt{22} = 4.690.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\Psi = \begin{bmatrix} +0.30151 & -0.63960 \\ +0.45227 & +0.21320 \end{bmatrix}$$

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Modal Loadings

The modal loading is

$$\begin{aligned} p^*(t) &= \Psi^T p(t) \\ &= p_0 \begin{bmatrix} 1 & 3/2 \\ -3 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \sin \omega t \\ &= p_0 \begin{Bmatrix} 3/2 \\ 1 \end{Bmatrix} \sin \omega t \end{aligned}$$

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Modal EoM

Substituting its modal expansion for x into the equation of motion and premultiplying by Ψ^T we have the uncoupled modal equation of motion

$$\begin{cases} 11m \ddot{q}_1 + \frac{1}{4} 11m \frac{k}{m} q_1 = \frac{3}{2} p_0 \sin \omega t \\ 22m \ddot{q}_2 + 3 \frac{22m}{m} \frac{k}{m} q_2 = p_0 \sin \omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by M_i both equations, we have

$$\begin{cases} \ddot{q}_1 + \frac{1}{4} \omega_0^2 q_1 = \frac{3}{2} \frac{p_0}{11m} \sin \omega t \\ \ddot{q}_2 + 3 \omega_0^2 q_2 = \frac{p_0}{22m} \sin \omega t \end{cases}$$

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Particular Integral

We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 (\omega_1^2 - \omega^2) \sin \omega t = \frac{3}{22} \frac{p_0}{k} \frac{k}{m} \sin \omega t$$

solving for C_1

$$C_1 = \frac{3}{22} \Delta \frac{\omega_0^2}{\omega_1^2 - \omega^2}$$

and, analogously,

$$C_2 = \frac{1}{22} \Delta \frac{\omega_0^2}{\omega_2^2 - \omega^2}$$

with $\Delta = p_0/k$.

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Integrals

The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + C_1 \sin \omega t, \\ q_2(t) = A_2 \sin \omega_2 t + B_2 \cos \omega_2 t + C_2 \sin \omega t, \end{cases}$$

and, for a system initially at rest, it is

$$\begin{cases} q_1(t) = C_1 (\sin \omega t - \beta_1 \sin \omega_1 t), \\ q_2(t) = C_2 (\sin \omega t - \beta_2 \sin \omega_2 t), \end{cases}$$

where $\beta_i = \omega/\omega_i$

We are interested in structural degrees of freedom, too...

$$\begin{cases} x_1(t) = (\psi_{11} q_1(t) + \psi_{12} q_2(t)) \\ x_2(t) = (\psi_{21} q_1(t) + \psi_{22} q_2(t)) \end{cases}$$

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The response in modal coordinates

To have a feeling of the response in modal coordinates, let's say that the frequency of the load is $\omega = 2\omega_0$, hence $\beta_1 = \frac{2.0}{\sqrt{1/4}} = 4$ and $\beta_2 = \frac{2.0}{\sqrt{3}} = 1.15470$.



In the graph above, the responses are plotted against an adimensional time coordinate α with $\alpha = \omega_0 t$, while the ordinates are adimensionalised with respect to $\Delta_{st} = \frac{p_0}{k}$

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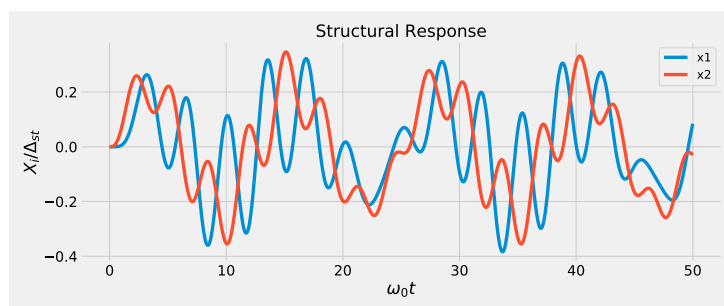
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The response in structural coordinates

Using the same normalisation factors, here are the response functions in terms of $x_1 = \psi_{11}q_1 + \psi_{12}q_2$ and $x_2 = \psi_{21}q_1 + \psi_{22}q_2$:



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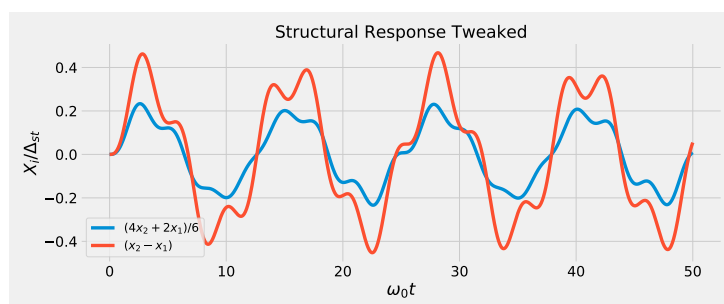
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The response in structural coordinates

And the displacement of the centre of mass plotted along with the difference in displacements.



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