# Dynamics of Structures 2011-2012

Second Home Assignment, due on your oral exam

The day of your oral examination you shall submit to me a printed or handwritten paper with your solutions to the following four problems.

## 1 Differential support motion

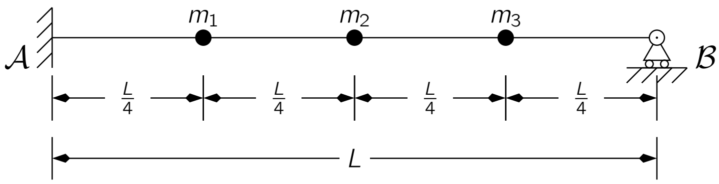



Figure 1: discrete model

The system in figure 1 is composed by a massless, uniform elastic beam with constant flexural stiffness $EJ$ and three identical supported masses $m_1 = m_2 = m_3 = m/4$ equispaced at a reciprocal distance $L/4$; the beam is clamped in $\mathcal{A}$ and simply supported in $\mathcal{B}$,

The dynamical degrees of freedom of the system are, of course, the vertical displacements of the three masses, $x_1, x_2, x_3$ and, disregarding shear and axial deformations, the stiffness matrix can be evaluated using the PVD:

$$\boldsymbol{K} = \frac{96}{17}\frac{EJ}{L^3}\begin{bmatrix} 216 & -81 & -32 \\ -81 & 58 & -5 \\ -32 & -5 & 40 \end{bmatrix}.$$

1. Compute the eigenvalues in terms of $\omega_0^2 = {}^{EJ}/_{mL^3}$.
2. Compute the normalised eigenvector matrix (i.e., $\boldsymbol{\Psi}^T \boldsymbol{M}\, \boldsymbol{\Psi} = m\, \boldsymbol{I}$).

The system is excited by an imposed vertical displacement applied in $\mathcal{B}$. To proceed, you need to use an additional degree of freedom, as in figure 2 (have you noticed $m_4$? In the optional question need arises for its value, and it is $m_4 = m/8$. Why?).

The flexibility matrix for the new 4 DOF system can be easily computed, and

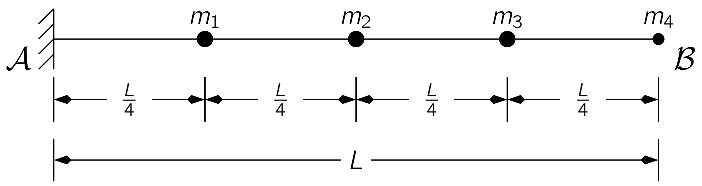



Figure 2: modified discrete model

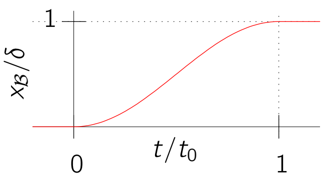
it is

$$\boldsymbol{F}_{4x4} = \frac{1}{384}\frac{L^3}{EJ}\begin{bmatrix} 2 & 5 & 8 & 11 \\ 5 & 16 & 27 & 40 \\ 8 & 27 & 54 & 81 \\ 11 & 40 & 81 & 128 \end{bmatrix}$$

3. Compute the $3 \times 1$ influence matrix $\boldsymbol{E}$.

The imposed displacement has the following analytical expression,

$$x_{\mathcal{B}}(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ \delta \times \frac{1}{2}\left(1 - \cos\frac{\pi t}{t_0}\right) & \text{for } 0 \leq t \leq t_0, \\ \delta & \text{for } t_0 \leq t, \end{cases}$$




with $t_0 = \frac{\pi}{2\omega_1}$ and $\delta = \frac{L}{500}$.

4. Solve the modal equations of motion, starting from rest conditions, in the interval $0 \leq t \leq 5t_0$ and plot your results.

5. Give the analytical representation of $x_3(t)$ for $0 \leq t \leq 5t_0$ and plot your results.

6. *Optional* Give the analytical representation of the reaction of the roller $R_{\mathcal{B}}(t)$ for $0 \leq t \leq 5t_0$ and plot your results.

## 2 Continuous system

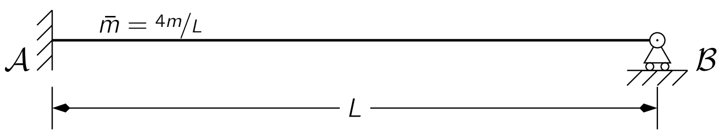



Figure 3: continuous system

The system in figure 3 is composed by a uniform elastic beam, clamped in $\mathcal{A}$ and simply supported in $\mathcal{B}$, whose unit mass is $\bar{m} = m/L$ and whose bending stiffness $EJ$ and length $L$ are the same of the beam in figure 1.

Actually, you can think of the system of exercise 1 as a discrete model of the continuous system that you are considering now.

Using the position $\omega_0^2 = {}^{EJ}\!/_{mL^3}$,

1. find $\beta_n$, $\omega_n^2$, $\phi_n(x)$ (normalize the eigenfunctions so that $M_n = m$) for $n = 1, 2, 3$ and
2. plot $\phi_n(x)$ vs $\frac{x}{L}$ for $n = 1, 2, 3$.

Using the imposed displacement $x_{\mathcal{B}}(t)$ that was described in the previous exercise (with $\omega_1$ from exercise 1.)

3. write the modal equations of motion for for $n = 1, 2, 3$.
4. plot the three modal responses for $0 \le t \le 5t_0$ and
5. plot $v(0.75L, t)$ for $0 \le t \le 5t_0$.

For points 4 and 5 above, you can use either analytical or numerical solutions.

**Hint**

$$v_{\text{tot}} = v + v_{\text{stat}} = v + \left(\frac{3}{2}\xi^2 - \frac{1}{2}\xi^3\right) x_{\mathcal{B}}(t), \qquad \xi = {}^{x}\!/_{L},$$

and

$$p_{\text{eff}} = -m\ddot{v}_{\text{stat}}.$$

# 3 Matrix iteration

Consider a shear type building with $N$ stories, $N = 9$, numbered from the base to the top.

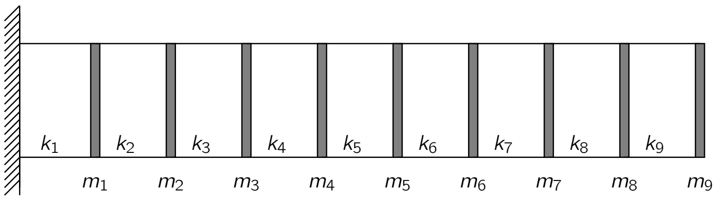



Figure 4: the shear type building of problem 3, rotated clockwise by 90°.

The storey masses $m_n$ are given by $m_n = m \times (41 - n)$ $[n = 1, \ldots, N]$ where $m$ is a unit mass.

The storey stiffnesses $k_n$ are given by $k_n = k \times (98 + 3n - n^2)$ $[n = 1, \ldots, N]$ where $k$ is a unit stiffness.

Note that the storey stiffness $k_n$ is different from the diagonal term of the stiffness matrix $k_{nn}$, $k_{nn} = k_n + k_{n+1}$.

1. Using the following initial Ritz base $\mathbf{\Phi}_0$,

$$\mathbf{\Phi}_0^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ -1 & -2 & -2 & -1 & 0 & 1 & 2 & 4 & 6 \\ 1 & 2 & 1 & 0 & -1 & -1 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 2 \end{bmatrix}$$

and the subspace iteration procedure give an estimate of $\omega_1^2$, $\omega_2^2$ and $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, the first two eigenvalues and eigenvectors of the structure, detailing the intermediate steps as follows: print the reduced matrices, the normalized structural eigenvectors and the eigenvalues for modes 1 and 2.

To check the results of the subspace iteration procedure you can compute the same eigenvalues and eigenvectors using `matlab` procedure `eig` or the analogous procedure that is offered in your programming language of choice.

2. For a *separable* load, $p = \boldsymbol{r} f(t)$, the response can be expressed in terms of the modal response and a static correction. Using the two modes that we have computed, you can write

$$\boldsymbol{x} = \boldsymbol{\psi}_1 q_1(t) + \boldsymbol{\psi}_2 q_2(t) + \left(\boldsymbol{K}^{-1} - \boldsymbol{F}_1 - \boldsymbol{F}_2\right) \boldsymbol{r} f(t).$$

Give the static correction term $\left(\boldsymbol{K}^{-1} - \boldsymbol{F}_1 - \boldsymbol{F}_2\right) \boldsymbol{r}$ for a load shape

$$\boldsymbol{r} = p_0 \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{Bmatrix}.$$

Please note that I'm not interested in the modal response.

## 4 Inelastic design

A SDOF system has a natural frequency of vibration $f_n = 3.0\,\text{Hz}$ and a yield strength $f_y = 0.50w$, where $w$ is the system's weight.

The design maximum ground acceleration is $\ddot{x}_{g0} = 0.6$ g, the amplification factor for spectral accelerations (elastic) is $\alpha_A = 2.5$ and for the spectral region of concern it is $A = \alpha_A \ddot{x}_{g0}$.

1. Find the system's required ductility $\mu$ and its peak displacement $x_m$.

2. What can be done to reduce by 10% the required ductility?