Project 1: Quintessence field

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I. PROBLEM 1

We have:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$

and

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

The continuity equation can be written as:

$$\dot{\rho}_\phi = -H(1+\omega_\phi)\rho_\phi \tag{3}$$

where:

$$\omega_\phi(z) = \frac{p_\phi}{\rho_\phi} \tag{4}$$

From the lecture notes we have that H can be written as a function of z:

$$H = \frac{\dot{a}}{a} = \frac{1}{(1+z)} \tag{5}$$

Inserting this into equation 3 and dividing both sides by ρ_ϕ we get:

$$\frac{\dot{\rho}_\phi}{\rho_\phi} = -\frac{3[1+\omega_p hi(z)]}{(1+z)} \tag{6}$$

We integrate on both sides from 0 to z and get:

$$\ln\left(\frac{\rho_\phi(z)}{\rho_\phi(0)}\right) = \int_0^z dz' \frac{3[1+\omega_p hi(z)]}{(1+z)} \tag{7}$$

Which can be rewritten as equation (5) from project 1:

$$\rho_\phi(z) = \rho_{\phi 0} \exp\left\{\int_0^z dz' \frac{3[1+\omega_p hi(z)]}{(1+z)}\right\} \tag{8}$$

II. PROBLEM 2

We want to write equation (3) as a function of ϕ, which means we need to get an expression for $\dot{\rho}_\phi$. If we take the time derivative of equation 1 we get:

$$\begin{aligned}
\dot{\rho}_\phi &= \dot{\phi}\ddot{\phi} + \frac{dV(\phi)}{dt} \\
\dot{\rho}_\phi &= \dot{\phi}\ddot{\phi} + \frac{dV(\phi)}{d\phi}\frac{d\phi}{dt} \\
\dot{\rho}_\phi &= \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}
\end{aligned} \tag{9}$$

Inserting equation (1), (2), (4) and 9 into equation (3), we get:

$$\begin{aligned}
\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} + 3H\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right] &= 0 \\
\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} + 3H\dot{\phi}^2 &= 0 \\
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= 0
\end{aligned} \tag{10}$$

III. PROBLEM 3

We have the first Friedmann equation:

$$\dot{a}^2 + k = \frac{\kappa^2}{3}\rho a^2 \tag{11}$$

Where $\kappa^2 = 8\pi G$ and since we are in a flat universe, $k = 0$. And the second Friedmann equation:

$$\ddot{a} = -\frac{\kappa^2}{6}(\rho + 3P)a \tag{12}$$

We also have that the hubble parameter can be written as in equation (5). Which if we take the time derivative of, gives us:

$$\dot{H} = \frac{a\ddot{a} - \dot{a}^2}{a^2} \tag{13}$$

Inserting equation (11) and (12) into equation (13):

$$\begin{aligned}
\dot{H} &= \frac{a^2[-\frac{\kappa^2}{6}(\rho + 3P)] - a^2\frac{\kappa^2}{3}\rho}{a^2} \\
\dot{H} &= -\frac{\kappa^2}{6}(\rho + 3P) - \frac{\kappa^2}{3}\rho \\
\dot{H} &= -\frac{\kappa^2}{3}(\frac{3}{2}\rho + \frac{3}{2}P) \\
\dot{H} &= -\frac{\kappa^2}{2}(\rho + P)
\end{aligned} \tag{14}$$

We have that $\rho = \rho_m + \rho_r + \rho_p hi$ and $P = p_m + p_r + p_\phi$ where $p_i = \omega_i \phi_i$. Where $\omega_m = 0$ and $\omega_r = 1/3$. Giving us:

$$\dot{H} = -\frac{\kappa^2}{2}[\rho_m + (\rho_r + p_r) + (\rho_\phi + p_\phi)] \tag{15}$$

Which can be rewritten as

$$\dot{H} = -\frac{\kappa^2}{2}[\rho_m + \rho_r(1 + \omega_r) + \dot{\phi}^2] \tag{16}$$

IV. PROBLEM 4

We have that $\Omega_i = \rho_i/p_i$. Starting with Ω_ϕ, we have:

$$\Omega_\phi = \frac{\kappa^2}{3H^2}\rho_p hi \tag{17}$$

Inserting equation (1) into equation (18), we get:

$$\begin{aligned}
\Omega_\phi &= \frac{\kappa^2}{3H^2}\left(\frac{1}{2}\dot{\phi} + V(\phi)\right) \\
\Omega_\phi &= \frac{\kappa^2}{6H^2}\dot{\phi} + \frac{\kappa^2}{3H^2}V
\end{aligned} \tag{18}$$

We can now recognize that this can be written as $\Omega_\phi = x_1^2 + x_2^2$. Next we have Ω_r:

$$\Omega_r = \frac{\kappa^2}{3H^2}\rho_r \tag{19}$$

Which we immediately recognize as $\Omega_r = x_3^2$. Last we have Ω_m, which we can find using $1 = \Omega_m + \Omega_r + \Omega_\phi$, giving us:

$$\Omega_m = 1 - x_1^2 - x_2^2 - x_3^2 \tag{20}$$

V. PROBLEM 5

Starting with equation (16), we try to write it using the dimensionless variables. We easily find that $\rho_r = \frac{3H^2}{\kappa^2}x_3^2$ and $\dot{\phi}^2 = \frac{6H^2}{\kappa^2}x_1^2$. Next we use $\omega_r = 1/3$. We can also write $\rho_m = \frac{3H^2}{\kappa^2}\Omega_m$, giving us finally:

$$\begin{aligned}
\dot{H} &= -\frac{\kappa^2}{2}\left[\frac{3H^2}{\kappa^2}\Omega_m \frac{4}{3}\frac{3H^2}{\kappa^2}x_3^2 + \frac{6H^2}{\kappa^2}x_1^2\right] \\
\dot{H} &= -\frac{1}{2}H^2\left[3(1 - x_1^2 - x_2^2 - x_3^2) + 4x_3^2 + 6x_1^2\right] \\
\frac{\dot{H}}{H^2} &= -\frac{1}{2}(3 + x_1^2 - 3x_2^2 + x_3^2)
\end{aligned} \tag{21}$$

VI. PROBLEM 6

Starting with x_1:

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{\kappa}{\sqrt{6}}\frac{H\ddot{\phi} - \dot{H}\dot{\phi}}{H^2} \\
\frac{dx_1}{dt} &= \frac{\kappa\ddot{\phi}}{\sqrt{6}H} - \frac{\dot{\phi}}{H^2}\frac{\kappa\dot{\phi}}{\sqrt{6}} \\
\frac{dx_1}{dt} &= \frac{\kappa\ddot{\phi}}{\sqrt{6}H} + H\frac{1}{2}x_1(3 + x_1^2 - 3x_2^2 + x_3^2), \quad \text{using (21)}
\end{aligned} \tag{22}$$

From the continuity equation (10), we have that $\ddot{\phi} = -V' - 3H\dot{\phi}$, using this:

$$\frac{dx_1}{dt} = H\left[\frac{\kappa(V' - 3H\dot{\phi})}{\sqrt{6}H^2} + \frac{1}{2}x_1(3 + x_1^2 - 3x_2^2 + x_3^2)\right] \tag{23}$$

We have that $\lambda = -V'/\kappa V$, $x_1 = \kappa\dot{\phi}/\sqrt{6}H$ and $x_2^2 = \kappa^2 V/3H^2$. With some jumbeling we get then:

$$\begin{aligned}
\frac{dx_1}{dt} &= H\left[\frac{\kappa^2 V V'}{\sqrt{6}\kappa V H^2} - \frac{\kappa 3\dot{\phi}}{\sqrt{6}H} + \frac{1}{2}x_1(3 + x_1^2 - 3x_2^2 + x_3^2)\right] \\
\frac{dx_1}{dt} &= H\left[\frac{\sqrt{3}\kappa^2 V}{\sqrt{2}3H^2}\frac{V'}{\kappa V} - 3x_1 + \frac{1}{2}x_1(3 + x_1^2 - 3x_2^2 + x_3^2)\right] \\
\frac{dx_1}{dt} &= H\left[\frac{\sqrt{3}}{\sqrt{2}}\lambda x_2^2 - 3x_1 + \frac{1}{2}x_1(3 + x_1^2 - 3x_2^2 + x_3^2)\right]
\end{aligned} \tag{24}$$

Lastly, we know that $\frac{df}{dN} = \frac{1}{H}\frac{df}{dt}$, giving us finally:

$$\frac{dx_1}{dN} = \frac{\sqrt{3}}{\sqrt{2}}\lambda x_2^2 - 3x_1 + \frac{1}{2}x_1(3 + x_1^2 - 3x_2^2 + x_3^2) \tag{25}$$

Next, we do $\frac{dx_2}{dN}$:

$$\begin{aligned}
\frac{dx_2}{dt} &= \frac{\kappa}{\sqrt{3}}\frac{H\frac{1}{2}\frac{V'\dot{\phi}}{\sqrt{V}} - \sqrt{V}\dot{H}}{H^2} \\
\frac{dx_2}{dt} &= \frac{\kappa}{\sqrt{3}}\frac{H\frac{1}{2}\frac{V'\dot{\phi}}{\sqrt{V}}}{H^2} - \frac{\kappa}{\sqrt{3}}\frac{\sqrt{V}\dot{H}}{H^2} \\
\frac{dx_2}{dt} &= \frac{\kappa}{\sqrt{3}}\frac{H\frac{1}{2}\frac{V'\dot{\phi}}{\sqrt{V}}}{H^2} + H\frac{1}{2}x_2(3 + x_1^2 - 3x_2^2 + x_3^2), \quad \text{using (21)}
\end{aligned} \tag{26}$$

Next, we can rewrite the first term:

$$H\frac{1}{2}\frac{\kappa}{\sqrt{3}}\frac{\frac{V'\dot{\phi}}{\sqrt{V}}}{H^2} = -H\frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{-V'}{\kappa V}\frac{\kappa\dot{\phi}}{\sqrt{6}H}\sqrt{6} = -H\frac{\sqrt{6}}{2}x_2\lambda x_1 \tag{27}$$

Resulting in the final equation:

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + x_1^2 - 3x_2^2 + x_3^2) \tag{28}$$

From the same logic we have used so far, we can easily deduce how we arrive at the last term in $\frac{dx_3}{dN}$, and the x_3 falls out quite easily.

$$\frac{dx_3}{dN} = \frac{1}{2}x_3\frac{\dot{\rho}_r}{H\rho_r} \tag{29}$$

Next we must show that $\frac{\dot{\rho}_r}{H\rho_r} = -4$. Using the continuity equation we have that $\dot{\rho} = -3H(1+\omega_r)\rho_r$, we know that $\omega_r = 1/3$, which proves that $\frac{\dot{\rho}_r}{H\rho_r} = -4$.

VII. PROBLEM 7

For λ to be constant, it must be independent of time. A possible solution for this is $V = ce^{a\phi+b}$, which has the derivative $V' = ace^{a\phi+b} = aV$, giving:

$$\lambda = -\frac{aV}{\kappa V} = \frac{a}{\kappa} \tag{30}$$

The resulting Γ would then be, with $V'' = c^2V$

$$\Gamma = \frac{aV^2}{a^2V^2} = \frac{1}{a} \tag{31}$$

VIII. PROBLEM 8

The time derivative of λ can be written as:

$$\begin{aligned}\frac{d\lambda}{dt} &= -\frac{1}{\kappa}\left(\frac{VV'' - V'^2}{V^2}\right)V'\dot{\phi} \\ \frac{d\lambda}{dt} &= -\frac{1}{\kappa}\left(\frac{V''}{V} - \frac{V'^2}{V^2}\right)V'\dot{\phi}\end{aligned} \tag{32}$$

We can also rewrite Γ as:

$$\Gamma = \frac{V^2}{V'^2}\frac{V''}{V} = \frac{1}{\kappa^2\lambda^2}\frac{V''}{V} \tag{33}$$

Giving us:

$$\begin{aligned}\frac{d\lambda}{dt} &= -\frac{1}{\kappa}\left(\Gamma\lambda^2\kappa^2 - \lambda^2\kappa^2\right)V'\dot{\phi} \\ \frac{d\lambda}{dt} &= -\kappa\lambda^2\left(\Gamma - 1\right)V'\dot{\phi} \\ \frac{d\lambda}{dN} = \frac{1}{H}\frac{d\lambda}{dt} &= \sqrt{6}\lambda^2\left(\Gamma - 1\right)x_1\end{aligned} \tag{34}$$

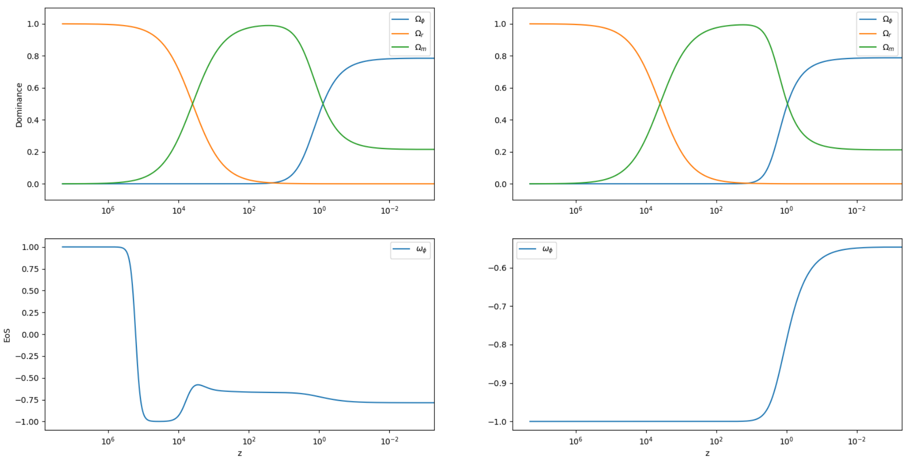

Figure 1. Shows the inverse power-law on the right and exponential potential on the right. On the top is the evolution of densities throughout the history of the universe as a function of z on a logarithmic time scale with $z \in [10^7, 0]$. On the bottom is the EoS parameter ω_ϕ

IX. PROBLEM 9

Code is delivered seperately.

We use a ODE solver to solve the differential equations given in equation (19-22) in the project. The densities can then easily be calculated. In order to find the EoS parameter ω_ϕ we find an analytical expression for equation (4), using $p_\phi = p_\rho - 2V$:

$$\begin{aligned} \omega_\phi &= \frac{p_\phi}{\rho_\phi} \\ \omega_\phi &= \frac{\rho_\phi - 2V}{\rho_\phi} \end{aligned} \tag{35}$$

Where $V = 3H^2 x_2^2/\kappa^2$. For the exponential potential we have a constant λ, as discussed previously, where $a = \kappa 3/2$ which gives $\lambda = 3/2$. This gives us figure 1

X. PROBLEM 10

To find the hubble parameter we use equation (7) in the project and divide each side by H_0^2, to get a dimensionless estimate. We have all we need to solve this analytically. We also need to find the integral in the equation, which we do by solving it with respect to N instead of z. We do this by substitution, with $z = e^{-N} - 1$ giving us $dz = -e^{-N} dN$, resulting in the intergal:

$$\int_0^z dz' \frac{3(1+\omega_\phi)}{(1+z')} = -\int_0^N 3(1+\omega_\phi) dN' \tag{36}$$

Solving this analytically we get figure 2, which also contains an estimate for the ΛCDM model. We solved this one analytically using an equation from the lecture notes:

$$\frac{H^2}{H_0^2} = \Omega_{m0} * (1+z) * *3 + (1 - \Omega_{m0}) \tag{37}$$

Where $\Omega_{m0} = 0.3$

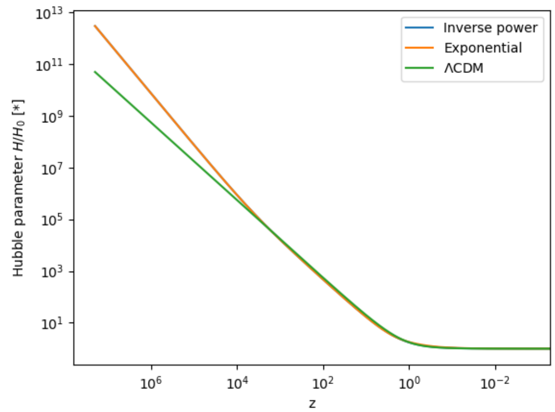

Figure 2. Shows the evolution of the Hubble parameter over time as a function of z. Both the x- and y-axis are logarithmic, with $z \in [10^7, 0]$

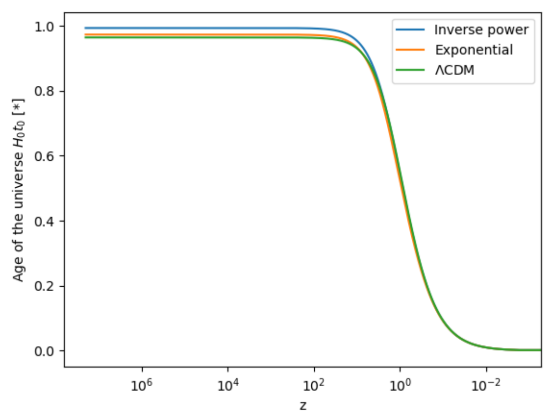

Figure 3. Shows the dimensionless age of the Universe $H_0 t_0$ as a function of z. The x-axis is logarithmic with $z \in [10^7, 0]$
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XI. PROBLEM 11

To find the dimensionless time we use this equation from the lecture notes and multiply each side with H_0:

$$\begin{aligned} t_0 &= \int_0^{a_0} \frac{da}{aH} \\ H_0 t_0 &= \int_0^{a_0} \frac{H_0}{H} \frac{da}{a} \end{aligned} \tag{38}$$

We have $N = \ln a/a_0$, giving us $dN = da/a$, meaning we can find the age of the universe by using the integral:

$$H_0 t_0 = -\int_0^N \frac{H_0}{H} dN \tag{39}$$

Where H_0/H is the inverse of the dimensionless Hubble parameter we found in the previous problem. Using this we get figure 3 We see that for a given H_0, the inverse power model gives the oldest universe.

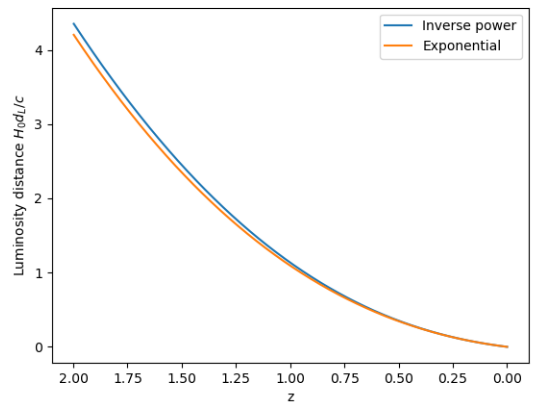

Figure 4. Shows the two models for the given interval.

XII. PROBLEM 12

Searching through the lecture notes, we find the expression:

$$d_L = \frac{c(1+z)}{H_0} \int_0^z \frac{H_0}{H} dz \tag{40}$$

Since we are now dealing with only small values of z, we do not have to convert over to N. We solve this and get figure 4.

XIII. PROBLEM 13

Using equation (41) I find that the inverse power-law model has a $\chi^2 = 452.73$ and the exponential potential model has a $\chi^2 = 483.51$. Meaning that the inverse power-law model fits the data better of the two. This is after extrapolating data to test our moddel also for $z \in [1.3, 2]$.

$$\chi^2 = \sum_i^N \frac{(O_i - E_i)^2}{E_i} \tag{41}$$

Where O_i is the observed value at the same redshift z as what we expected E_i. I was unable to use the way of finding χ^2 as provided in the project. This may be due to how I extrapolate my data. From figure 5 we see that the models do perform somewhat well, at least until the extrapolated data does not grow quickly.

XIV. PROBLEM 14

I produced some code, but it seems not to work as inteded. I was unable to find why, but in general, my code is not able to produce an accurate ΛCDM model for luminosity distance. From what I have heard from others though, is that they get something very close to $\Omega_{m0} = 0.3$. Which would probably make it a better fit than either of the quintessence field models. I think it is a fair comparisson as all models are supposed to do good for the later universe, which is what we are looking at now. Of course we can argue that this introduces some overfitting into our data and it wont do as well as the other models in earlier stages of the universe because of this overfitting. However, since the value ends up so close to what we already have looked at, this seems implausible.

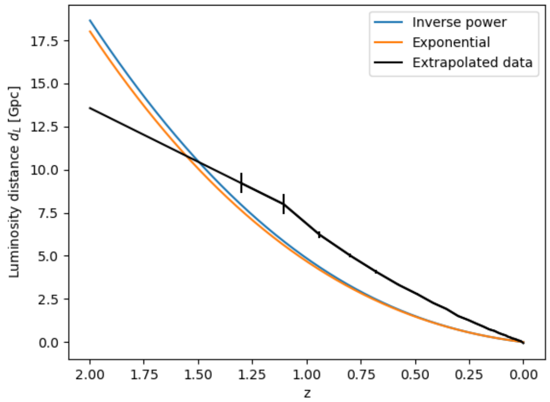

Figure 5. Shows the models compared to the actual data, including its error bars.