Project 2 - AST3310

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I. INTRODUCTION

In this scientific report we will simulate a model of a star similiar to our sun. We will find a set of differential equations that we will solve to move from the surface of the star and into its core. Our first model will be based on the sun, before we investigate how changing different initial condions affects our model so that we can complete our goals of reaching 5% of L_0, M_0 and R_0 and having a convective zone reaching 15% into our sun and a core reaching 10%

II. METHOD

With the goal of simulating the shells of our sun, we must first set our expectations and define what determines the properties of these shells. To start at we must look at when we have convection zones (CZ) and when we have radiation zones (RZ). Since we will be looking at simulations of similiar stars to the Sun, we will not that heat conduction into account.

We start by looking at what these terms mean. Radiation zones are zones where all energy is transported by photons. Photons are emitted with a given energy and shortly afterwards absorbed by particles other than the original source and then again emitted again, now with a lower energy. In this way energy is transported between particles. How far these particles travel depends on their mean free path, which can be linked to the opacity of the surrounding area. The opacity will decrease the further you get from the core and so particles get further in the direction of the surface, resulting in energy being transported to the surface. However, this mechanism is not always enough to transport all the energy that is needed. This is where convection comes in.

Convection is when energy is tranported through large scale gas motions. The hotter gas rises up while the colder gas falls down to be heated up again. The hot gas continues to rise until it reaches the surface, where other mechanisms we will not consider take over the energy transport.

We measure the energy transport as flux. And since in radiation zones no convection is needed, we can set the convective flux to zero. Since there are only two transport mechanism we consider, the sum of these two will be the total energy flux. So in order to keep track of these variables throughout the sun, we will need equations to do so.

The total flux can be written as a function of radius and luminosity at the given radius, which is defined as equation (1) with $[F_{tot}] = Wm^{-2}$ being the toal flux, $[L] = W$ the luminosity and $[r] = m$ the radius. For the radiation zone, since the convective flux $F_{con} = 0$, the radiation flux $F_{rad} = F_{tot}$.

$$F_{tot} = \frac{L(r)}{4\pi r^2} \tag{1}$$

The way we set up our equation is by using several differential equation that we solve in order to move from the surface of the star and into its core. According to the project description, these equations are more stable with respect to mass instead of radius. So in order to find F_{tot} we need to have a way to update luminosity and radius. Equation (2) shows how we find how radius changes with mass, where $[m] = kg$ is the mass of the star $[\rho] = kgm^{-3}$ is the current density and $[V] = m^3$ is the volume.

$$m = \rho V = \rho \frac{4}{3}\pi r^3$$

we take the derivative

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2 \rho \\ \frac{dr}{dm} &= \frac{1}{4\pi r^2 \rho} \end{aligned} \tag{2}$$

And equation (3) shows how we find how the luminosity changes, where $[\epsilon] = Jkg^{-1}s^{-1}$ is the total energy produced by the star per kg per second.

$$\begin{aligned} L &= \int \epsilon dm \\ \int dL &= \int \epsilon dm \\ dL &= \epsilon dm \\ \frac{dL}{dm} &= \epsilon \end{aligned} \tag{3}$$

Now we have two more parameters to consider, the density ρ and energy ϵ. The energy ϵ is dependent on the temperature $[T] = K$ and density ρ, and given these we can extract or extrapolate the corresponding value of ϵ from a table. By assuming an ideal gas, we can connect the gas pressure $[P_{gas}] = Pa$ to density ρ by the ideal gas equation (4) where $[m_u] = kg$ is the atomic mass unit and $[\mu] = *$ is the average molecular weight which is defined by equation (5) where Z_x is the fraction of total molecules of the metal x, which is completely ionized, that has i nuclenons per ion and j free particles.

$$P_{gas} = \frac{\rho k_B T}{\mu m_u} \tag{4}$$

$$\mu = \frac{1}{\sum_x \frac{2+A_x}{2A_x} Z_x} \tag{5}$$

Equation (5) comes from $\mu = \frac{\rho/n_{tot}}{m_u}$ where n_{tot} $n_{tot} = \frac{\rho}{m_u}\sum_x Z_x/A_x + \frac{1}{2}Z_x$ is the total number of particles. Using $1/A_x + 1/2 = \frac{1+A_x}{2A_x}$ we are back to equation (5).

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We find the gas pressure P_{gas} from the total pressure $P = P_{gas} + P_{rad}$, where P_{rad} is the radiative pressure defined by equation (6).

$$P_{rad} = \frac{4\sigma T^4}{3} \tag{6}$$

We then need an equation for updating the total pressure P_{tot}, which equation (8) shows by starting with the condition for hydrostatic equilibrium.

$$\begin{aligned}\frac{dP}{dr} &= -\rho g, \quad g = G\frac{m}{r^2}\\ \frac{dP}{dm}\frac{dm}{dr} &= -\frac{\rho G m}{r^2}\\ \frac{dP}{dm} &= -\frac{\rho G m}{r^2}\frac{1}{\frac{dm}{dr}}, \quad \frac{dm}{dr} = 4\pi r^2\rho\\ \frac{dP}{dm} &= -\frac{\rho G m}{r^2}\frac{1}{4\pi r^2\rho}\\ \frac{dP}{dm} &= -\frac{Gm}{4\pi r^4}\end{aligned} \tag{7}$$

We see that both equation (4) and (6, as well as extrapolating ϵ, requires the temperature which we dont have an expression for yet. We will come back to this later, once we have touched upon convective flux.

As for the convective flux in the CZs we will need a bigger foundation first. From equation (8), or the equivalent equation 5.80 in the lecture notes, we see that we require a few more varaibles: specific heat capacity $[c_P] = Jkg^-1K^-1$, $\delta = 1$, pressure scale height $[H_P] = m$, mixing length $[l_m] = m$ and $[\xi] = *$.

$$F_{con} = \rho c_P T\sqrt{g\delta}H_P^{-3/2}\left(\frac{l_m}{2}\right)^2\xi^3 \tag{8}$$

We can estimate c_P as shown in equation (9) by assuming a monatomic and ideal gas. Pressure scale height is defined as equation (10), which can be rewriten using the condition for hydrostatic equilibrium.

$$c_P = \frac{5}{2}\frac{k_B}{\mu m_u} \tag{9}$$

$$\begin{aligned}H_P &= -P\frac{dr}{dP}\\ H_P &= P\rho g\end{aligned} \tag{10}$$

Next, mixing legth l_m can be rewritten as $l_m = \alpha_{l_m}H_P$, where we have chosen $\alpha_{l_m} = 1$.

Finally, we have ξ which requires a longer elaboration. ξ is a variable used in place of $(\nabla^* - \nabla_p)^{1/2}$, which after following Appenix **??** we can use to write equation (12). By combining this with the result from Appendix **??**, which is written in equation (13), we can solve for ξ in the resulting third degree equation. These equations contains the variable U which is defined by equation (11), where κ is the opacity at a given temperature and density. We find κ by extrapolating it from a given data table. ∇_{ad} in equation (13) and (14) for an ideal gas is given as $\nabla_{ad} = 2/5$. The ∇_{stable} can be calculated using another definition the total flux shown in equation (12).

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}} \tag{11}$$

$$\begin{aligned}F_{tot} &= \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{stable}\\ \nabla_{stable} &= \frac{3L\kappa\rho H_P}{64\sigma T^4 r^2}\end{aligned} \tag{12}$$

$$\nabla^* = \xi^2 + U\frac{4}{l_m^2}\epsilon + \nabla_{ad} \tag{13}$$

$$0 = \xi^3 + \frac{U}{l_m^2}\xi^2 + 4\left(\frac{U}{l_m^2}\right)^2\xi - \frac{U}{l_m}(\nabla_{stable} - \nabla_{ad}) \tag{14}$$

Equation (14) can be solved for ξ and gives only one real solution, meaning the rest of the solutions can be discarded. The solution gives us equation (15), where $a = 1$, $b = U/l_m^2$, $c = 4b^2$ and $d = b(\nabla_{stable} - \nabla_{ad}$.

$$\begin{aligned}\xi &= \frac{Z}{3\sqrt[3]{2}a} - \frac{\sqrt[3]{2}(3ac - b^2)}{Z} - \frac{b}{ac}\\ &\text{where,}\\ Z &= \sqrt[3]{\sqrt{(27a^2d + 9abc - 2b^3)^2 + 4(3ac - b^2)^3} + 27a^2d + 9abc - 2b^3}\end{aligned} \tag{15}$$

Having defined all these equations, we can proceed to talk about temperature and the requirement for convection to occur. ∇_{stable} is the required temperature gradient of the star for all energy to be transported with radiation, while ∇^* is the actual temperature gradient. Effectively, this means in convectively stable zones, $\nabla_{stable} = \nabla^*$. This is usefull for determining the stability criteria in equation (16) such that only ∇_{stable} needs to be greater than ∇_{ad}.

$$\nabla^* > \nabla_{ad} \tag{16}$$

Knowing if we are in a convectively stable region of the star lets us decide how to update our temperature parameter. In the stable regions, the temperature behaves as equation (17), which is a simplified version of equation (18) that takes into account both radiative and convective transport.

$$\frac{dT}{dm} = -\frac{3\kappa L}{256\pi^2\sigma r^4 T^3} \tag{17}$$

$$\frac{dT}{dm} = \nabla^* \frac{T}{P}\frac{dP}{dm} \tag{18}$$

We now have our set of equations for progressing through the star. Our initial conditions will first be set to measured values of the Sun in table I. These values will later be modified in order to create a model which fulfills our goals of a larger continious outer convection zone for the first 15% of R_0, a core reaching out to at least 10% of R_0 and the Luminosity L, mass m and radius r reaching 0 or at least 5% of their respective initial parameter at the end of the simulation. In order to achive this we will first study how changing the initial values of R_0, T_0, ρ_0 and P_0. In order to change P_0, we will first calculate it by summing equations (4) and (6) to find P as given by T and ρ and then setting $P_0 = aP(T,\rho)$ where a is a set value. Seeing the effects of these changing, we can tune our model more methodically until all goals are achieved.

Parameter	Value
L_0	$1 \cdot L_\odot$
R_0	$1 \cdot R_\odot$
M_0	$1 \cdot M_\odot$
ρ_0	$1.42 \cdot 10^{-7} \cdot \bar{\rho}_\odot$
T_0	5770 K
X	0.7
$Y_{He_2^3}$	10^{-10}
Y	0.29
$Z_{Li_3^7}$	10^{-7}
$Z_{Be_4^7}$	10^{-7}
$Z_{N_7^{14}}$	10^{-11}

Table I. Lists the first set of initial conditions given to our model. $\bar{\rho}_\odot$ is the average density of the sun given by $\bar{\rho}_\odot = 1.408 \cdot 10^3 kgm^-3$

Lastly, we also want to plot the relative energy production from PP1, PP2, PP3 and CNO as functions of radius. In order to do so we will extrapolate values from a table containing the relative energy production of every branch/chain at a given temperature.

III. RESULTS

Starting with a couple of checks to confirm our extrapolation correctly for both opacity κ in table II and total energy produced per kg per s ϵ in table III.

$\log_{10} T$	$\log_{10} R$ (cgs)	$\log_{10}\kappa$ expected (cgs)	$\log_{10}\kappa$ prediction (cgs)	$error_{rel}$ (%)
3.750	-6.000	-1.550	-1.546	0.259
3.755	-5.950	-1.510	-1.503	0.494
3.755	-5.800	-1.570	-1.567	0.185
3.755	-5.700	-1.610	-1.610	0.009
3.755	-5.550	-1.670	-1.673	0.199
3.770	-5.950	-1.330	-1.312	1.366
3.780	-5.950	-1.200	-1.190	0.818
3.795	-5.950	-1.020	-1.018	0.187
3.770	-5.800	-1.390	-1.374	1.155
3.775	-5.750	-1.350	-1.331	1.404
3.780	-5.700	-1.310	-1.288	1.683
3.795	-5.550	-1.160	-1.158	0.177
3.800	-5.500	-1.110	-1.114	0.359

Table II. Contains the input values of R and T, where $R = \frac{\rho}{T\cdot 10^{-6}}$, and their expected opacity κ. $\log_{10}\kappa$ (cgs) needs to later be converted to SI units.

$\log_{10} T$	$\log_{10} R$ (cgs)	$\log_{10} \epsilon$ expected (cgs)	$\log_{10} \epsilon$ prediction (cgs)	$error_{rel}$ (%)
3.750	-6.000	-87.995	-87.995	0.000
3.755	-5.950	-87.267	-87.617	0.399

Table III. Contains the input values of R and T, where $R = \frac{\rho}{T \cdot 10^{-6}}$, and their expected total energy produced by the star per unit mass per unit time ϵ. $\log_{10} \epsilon$ (cgs) needs to later be converted to SI units.

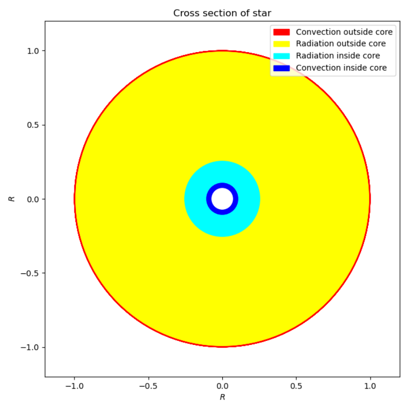

Figure 1. Shows the sun with the initial values given in table I. Both the x- and y-axis shows radius which has been normalized. The convective zones have been defined as $F_C > 0$ and the core as $L < 0.995L_\odot$

Plotting the first model, using the initial values listed in tabel I, we get figure 1. In figure 2 we see models with $R_0 = 0.8R_\odot$, $R_0 = 1.5R_\odot$, $R_0 = 2R_\odot$ and $R_0 = 5R_\odot$ respectively. In figure 3 we see models with $T_0 = 0K$, $T_0 = 10000K$, $T_0 = 25000K$ and $T_0 = 10^6 K$. In figure 4 we see models with $P_0 = 10P$ and $P_0 = 100P$. In figure 5 we see models with $\rho_0 = 1.42 \cdot 10^{-6} \cdot \bar{\rho}_\odot$ and $\rho_0 = 1.42 \cdot 10^{-5} \cdot \bar{\rho}_\odot$.

Next we have the more optimized models. The first one in figure 6 uses $L_0 = L_\odot$, $R_0 = 0.8R_\odot$, $M_0 = M_\odot$, $T_0 = 5770K$ and $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$. The second in figure 8 uses $L_0 = L_\odot$, $R_0 = 0.9R_\odot$, $M_0 = M_\odot$, $T_0 = 5770K$ and $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$.

IV. DISCUSSION

We see in both table II and III small relative errors, which means we can safely proceed with producing models.

For our first model, figure 1, using the measured parameters of the sun in table I, our simulation fails to reach the core of the sun and thus leaving a gap in the middle. The core stretches out to around 25% of the suns radius which corresponds to the suns actual core size [2] and is well within our goal of 10%. Its outer convective zone does not however approach the goal nor is it close to the 30% of our sun.

Next we look at how the different parameters affects the convective zone and core. Starting with radius, in fig 2. We see that a decrease in radius leads to a bigger core while an increase shrinks the core while extending the convective zone. Most of our governing functions are inversely proportional to radiues with some exponent, so having a larger radius effectively slows down our changes to radius, pressure and temperature, meaning it takes longer to get out of the convective zone. The shrinking of the core is not linearly proportional to the change in radius and instead it appears to shrink proportional to radius squared or cubed. An anomaly is the $R_0 = 1.5R_\odot$ model, which loses its radiative core entirely. Since no further increase in radius does this, I attribute it to my program not working entirely correctly. Since an increase in radius gives us a larger convective zone, which our initial model struggles with, this seems like a good variable to focus on. We will however need something to offset the shrinking of the core.

The second parameter that we look to change is the temperature. As we increase the surface temperature, we see that the convective zone dissapears entirely, while the core remains about the same with some small variation. The dissapearence of the convective zone can be linked to our expression of ∇_{stable} from equation 13, which says ∇_{stable} is inversely proportional to T^4. Meaning a higher temperature gives a lower ∇_{stable} so that the convection criterion $\nabla_{table} > \nabla_{ad}$ is not satisfied. A possible use for changes to T_0 would be to see if it conserves the size of the core when increasing R_0, this however seems improbable.

Lastly, we look at changes to density ρ_0 and pressure P_0. Not surprising, they seem to have the exact same effect. This is due to how our simulation calculates P_0. By summing equation (4) and equation (6) we get a pressure P which we use to define $P_0 = aP$. For the initial ρ_0 defined in table I, P_{gas} becomes the dominant of the two, meaning increasing ρ_0 is effectively the same as increasing the pressure. The opposite is also true, as ρ is updated using the pressure. The resulting change to our model is interesting also, unlike a change to radius, the core does not shrink massively compared to the increase to the convective zone.

Having looked at how all the parameters affect our model, a balance between a lower radius and higher density appears to be a possible solution to our goals. We therefor start by leaving temperature unchanged and de-

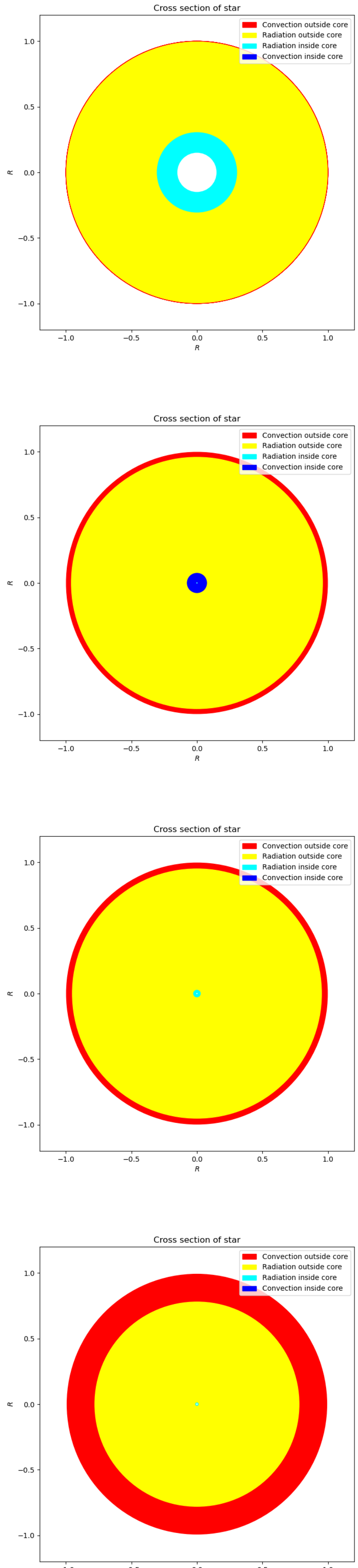

Figure 2. Shows the sun with the same initial values except for radius $R_0 = 0.8R_\odot$, $R_0 = 1.5R_\odot$, $R_0 = 2R_\odot$ and $R_0 = 5R_\odot$

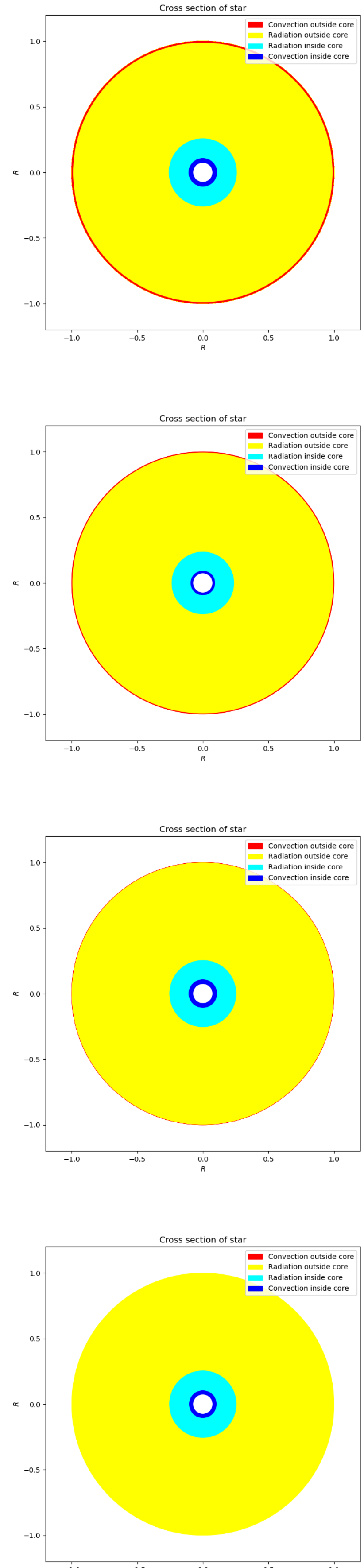

Figure 3. Shows the sun with the same initial values except for temperature $T_0 = 0K$, $T_0 = 10^5K$, $T_0 = 2.5 \cdot 10^5K$ and $T_0 = 10^6K$

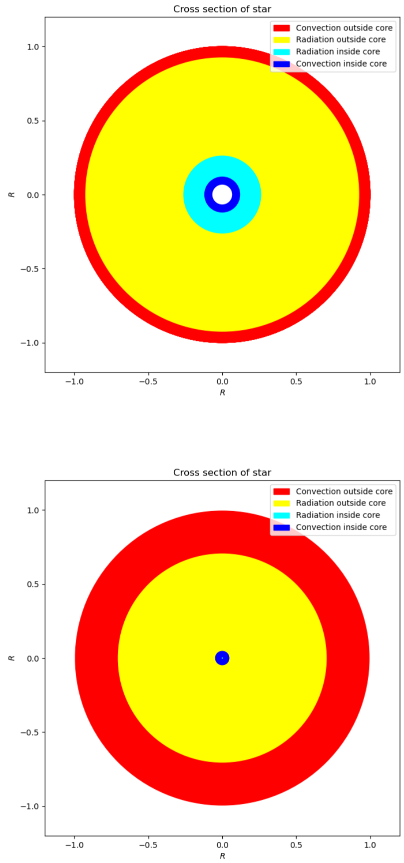

Figure 4. Uses the same initial values except for $P_0 = 10P$ in the first plot and $P_0 = 100P$ in the second.

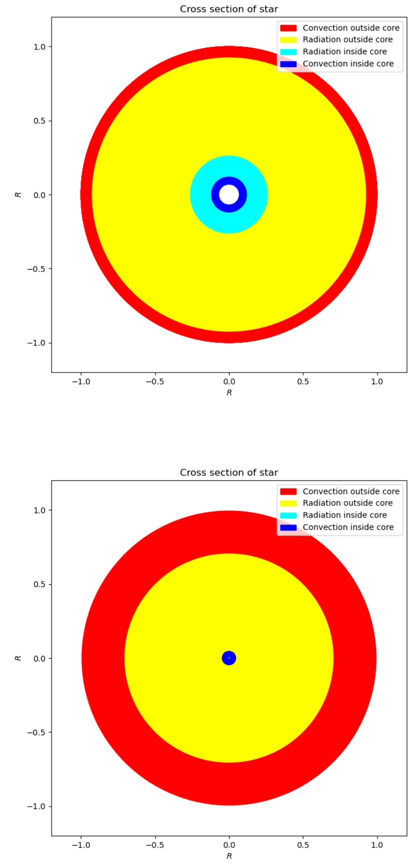

Figure 5. Uses the same initial values except for $\rho_0 = 10(1.42 \cdot 10^{-7})\bar{\rho}_\odot$ in the first plot and $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$ in the second.

creasing radius to $R_0 = 0.8R_\odot$ and $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$. We then get figure 6. Which fulfills our second and third criteria, but fails the first. The radius only closes in on $0.8\%R_0$ when we needed it to approach at least 5%. Since the radius of the core reaches well beyond our required 25%, a slightly larger radius will be the next step. With $R_0 = 0.9R_\odot$ we just touch the 5% goal and the convection zone remains large enough. However the luminosity goal is not reached. This gave us figure 8. Next we make a small change to the mass of the sun, such that $M_0 = 0.9M_\odot$ and keep the other parameters the same. This completes all goals we set and give us figure **??**, where we can se we lose the inner convection zone.

Since we have reached our goals, we can move on to looking at how different parameters change as a function of radius, which is shown in figure 9. For the main parameters we see that ρ and P behave similarily as expected. Their values increase sharply as it moves through the upper convection zone. This means that even though we changed ρ_0 by two magnitudes of 10, the change quickly grows larger. A point of further optimization is tying the total mass of the sun and its density diretly, meaning

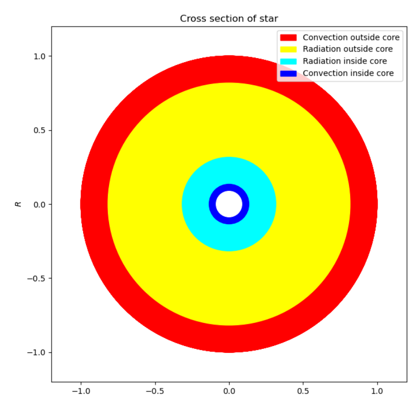

Figure 6. Shows the sun with parameters $L_0 = L_\odot$, $R_0 = 0.8R_\odot$, $M_0 = M_\odot$, $T_0 = 5770K$ and $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$.

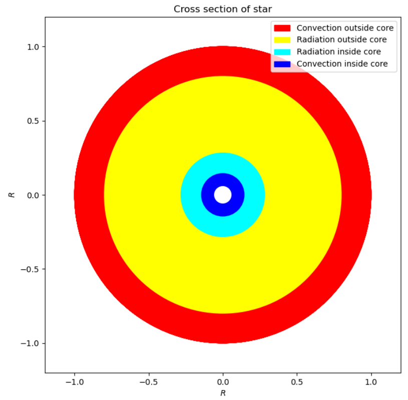

Figure 7. Shows the sun with parameters $L_0 = L_\odot$, $R_0 = 0.9R_\odot$, $M_0 = M_\odot$, $T_0 = 5770K$ and $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$.

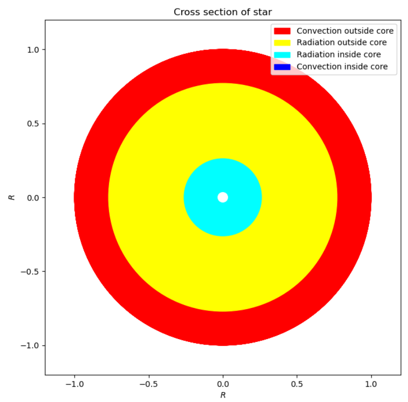

Figure 8. Shows the sun with parameters $L_0 = L_\odot$, $R_0 = 0.9R_\odot$, $M_0 = 0.9M_\odot$, $T_0 = 5770K$, $\rho_0 = 100(1.42 \cdot 10^{-7})\bar{\rho}_\odot$.

we dont end up with a more massive sun than what our initial parameters are. Further we see that luminosity has a sharp drop off towards the center. Unfortunately it seems the luminosity does not approach 0 nor 5%. Investigation further is needed to determine if this is realistic. The temperature rises quickly from the surface, with a small spike in temperature when entering the core. As for the mass we see that the majority of the mass is located within the core as expected.

Moving on to the flux, we see that F_con is 0 everywhere except for in the convective zone. Next the relative energy production by each of the branches. We see that PP1 is only dominant on the surface and otherwise PP2 is the dominant. This is not accurate, but is an expected result after project 2. Looking at the nablas, we see that ∇_{stable} and ∇^* start out with a spike and then drop below ∇_{ad}. It barely goes into the convective regime again before entering the core, but barely stays out, before dropping again.

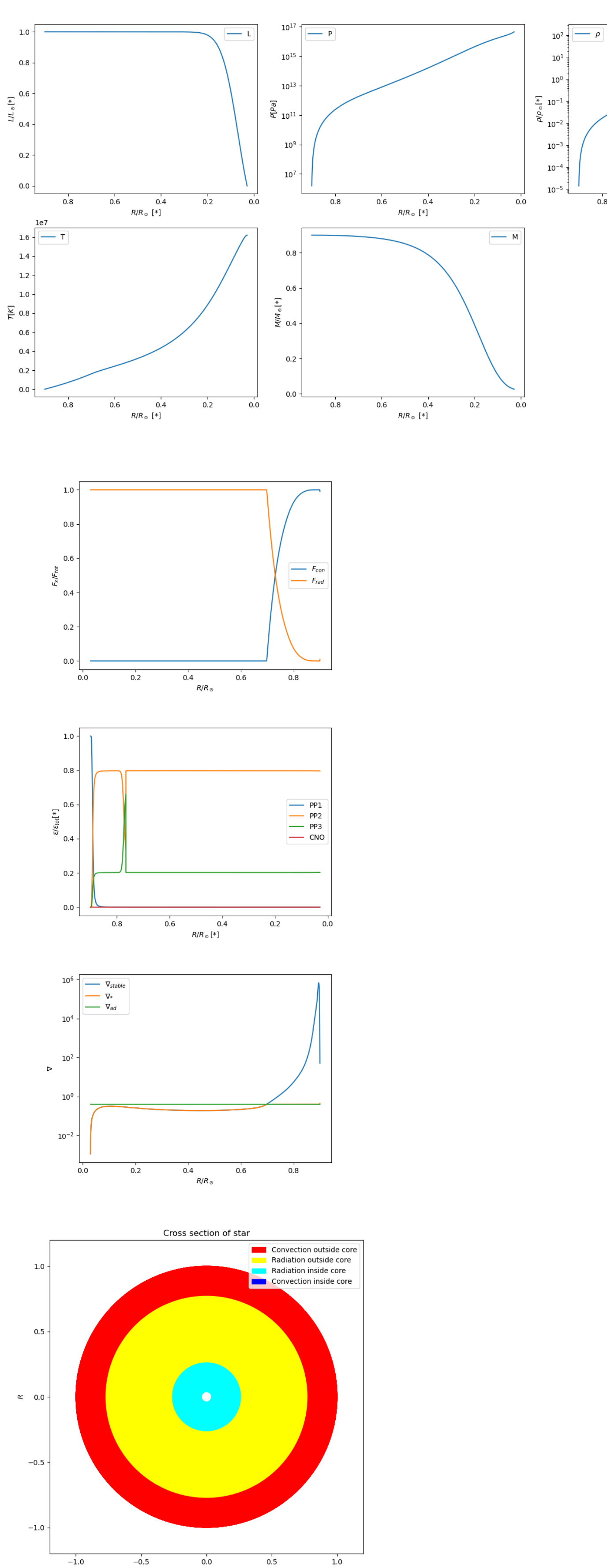

Figure 9. In order from top to bottom, it shows the main parameters ($m/M_\odot$, T, $L/L_\odot$, $\rho/\rho_\odot$ and P) as functions of radius $r/R_\odot$, F_{con} and F_{rad} as functions of radius, relative

V. CONCLUSION

I have learnt a great deal from this project, mostly just how quickly a simulation of the sun becomes. The layers of equations that go into describing five relatively easy equations is astounding. It also means there are a lot of opportunities for mistakes. Most of my trouble came from getting the equations exactly correct. The fact that $\nabla_* = \nabla_{stable}$ outside of the convective zone was obvious once I was made aware, but was not intuitive.

[1] [García *et al.*(2007)]2007Sci...316.1591GGarcía, R.A., Turck-Chièze, S., Jiménez-Reyes, S.J., Ballot, J., Pallé, P.L., Eff-Darwich, A., and, ...: 2007, *Science* **316**, 1591. doi:10.1126/science.1140598.

[2] Christensen-Dalsgaard, J., Däppen, W., Ajukov, S. V., Anderson, E. R., Antia, H. M., Basu, S., Baturin, V. A., Berthomieu, G., Chaboyer, B., Chitre, S. M., Cox, A. N., Demarque, P., Donatowicz, J., Dziembowski, W. A., Gabriel, M., Gough, D. O., Guenther, D. B., Guzik, J. A., Harvey, J. W., Hill, F., Houdek, G., Iglesias, C. A., Kosovichev, A. G., Leibacher, J. W., Morel, P., Proffitt, C. R., Provost, J., Reiter, J., Rhodes Jr., E. J., Rogers, F. J., Roxburgh, I. W., Thompson, M. J., Ulrich, R. K., 1996. The current state of solar modeling. Science, 272, 1286 - 1292.