Modelling energy production of stars with different temperatures

Bjørn Magnus Hoddevik
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I. INTRODUCTION

In this report we will study the energy production in a star across a wide range of temperatures as well as the corresponding gamow peaks for the different reactions that contribute to the energy production. We will study the top three PP chains as well as the top CNO chain. We will compare the energy production to the Sun and see if they line up with what we are expecting.

In order to calculate the energy production we must first find the energy produced per reation and then the reaction rate. The reaction rate we are looking for is given in terms of reactions per kg per second. Throughout we will assume the fractions of available reactants stay constant.

Also of interest is how much of the energy produced escapes through neutrinos for each of the chains.

II. METHOD

To start, we look at the different equations we need. Using Table 3.1 from the lecture notes, we have the reaction rate $\lambda_{ik} = [reactions s^{-1} (mole cm^{-3})^{-1}]$ where i and k are the two reactants.

We also need to understand why not all reactions are important. Starting with CNO, all reactions other than λ_{p14} are very fast and can be considered instanteneous in comparison. As for the two last reaction in PP3 both beta decay (B_5^8) and alpha decay (Be_4^8) are very fast reactions and can therefor also to be considered instanteneous. We still however need to remember the energy they contribute to. Therefor, we will late need the energies Q_{decay} and Q_{CNO}.

Next we start converting this over to $reactions m^3 s^{-1}$ by multiplying by Avogadro's number to get rid of the moles and divide by 10^6 to convert to m^3 instead of cm^3. Having done so, we can easily convert over to $r_{ik} = [reactions kg^{-1} s^{-1}]$ through our first important equation (1).

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik} \tag{1}$$

Where n_i and n_k are number densities of elements i and k, ρ is the density of our sun and $\delta_{ik} = 1$ if i and k are the same element and $\delta_{ik} = 0$ if not. The number densitiy n_i can be calculated from the fraction Z_i of the element i in the core of the Sun and its atomic unit mass m_i as seen in equation (2).

$$n_i = \frac{\rho Z_i}{round(m_i) m_u} \tag{2}$$

Where $round(m_i)$ returns the rounded down atomic unit mass and m_u is the weight of an atomic unit in kg. We will also need the number density of electrons which can be written as $n_{e,tot} = \frac{\rho}{2m_u}(1+X)$ as shown in the lecture notes where X is the fraction of protons in the sun.

Next we need to be able to scale the reaction rates according to supply. As evident in table I, some elements which are produced are used in several of the PP chains. This means we have to make sure that an element is not consumed faster than produced, unless that element is protons. This also means the number densities of all elements will stay the same, except for the proton. However, we still assume the proton number density stays the same, as it makes our task easier and it would take a long time for us to notice a dent in the number density due to the shear amount of protons in the sun.

In order to control how much is consumed, we sum up the reaction rates of all reactions with a common reactant. Since the second step in PP1 requires two He_2^3, we multiply this reaction rate by two. The third step in both PP2 and PP3 also have a reactant in common, meaning that when we sum up all reaction rates dependent on He_2^3, we only count the reaction rate of the second step in PP2. This explanation is easier to comprehend through equation (3) and (4).

$$r_{He_2^3,tot} = 2r_{33} + r_{34} \tag{3}$$

$$r_{Be_4^7,tot} = r_{e7} + r_{17} \tag{4}$$

Where $r_{He_2^3,tot}$ is the sum of reaction rates with He_2^3 as a reactant and $r_{Be_4^7,tot}$ is the sum of reaction rates with Be_4^7 as a reactant. And r_{33}, r_{34}, r_{e7} and r_{17} are reaction rates as denoted in the lecture notes.

Lastly, to find the scaled reaction rates, we need to understand when to scale. We scale r_{33} and r_{34} by equation (5) and r_{e7} and r_{17} by equation (6).

$$r_{ik} = \begin{cases} r_{pp} \cdot r_i / r_{He_2^3,tot} & \text{if } r_{He_2^3,tot} > r_{pp} \\ r_i & \text{else} \end{cases} \tag{5}$$

$$r_i = \begin{cases} r_{pp} \cdot r_i / r_{Be_4^7,tot} & \text{if } r_{Be_4^7,tot} > r_{34} \\ r_i & \text{else} \end{cases} \tag{6}$$

Where $ik = 33, 34$ in equation (5) and $ik = e7, 17$ in equation (6). Furthermore, we also do not want the energy contribution from PP0 to overshadow that of the later steps in each PP chain, so we also need to distribute that equal to the reaction rate of the second step in each PP chain. We will be keeping track of the reaction rate for each seperate step of every PP chain, and so we set the first step in all PP chains equal to the reaction rate of the second step in order to distribute the energy production from PP0.

Next we need to find the energy output per reaction. We do this by taking the mass difference between the reactants going in and products going out. The mass will be measured in atomic mass units m_u, allowing us to easily convert it to MeV by multiplying by $931.4941 MeV/m_u$ giving us Q_{ik}. Converting to MeV allows us to more easily validate the numbers we are getting by comparing to the lecture notes. A second advantage is that the energy of neutrinos are given in MeV also, allowing us to easily find the percentage of energy that escapes and subtracting the escaping energy to get Q'_{ik}. The escaping energy is given by equation (7). After having found the percentage, we can convert the energy produced by the step over to Joules.

$$escapingenergy_{chain} = \frac{E_{neutrinos,chain}}{E_{produced,chain}} \tag{7}$$

Where $escapingenergy_{chain}$ will be the fraction of energy that escapes.

To find the energy production of each chain we multiply the scaled reaction rate with energy produced per reaction and with the density of the sun, resulting in equation (8). As previously mentioned, we also have to remember the energies Q_{decay} and Q'_{CNO}, which will be added to last step in PP3 and used instead of Q'_{p14} for the CNO chain.

$$\epsilon_{ik}\rho = Q'_{ik} r_{ik} \rho \tag{8}$$

We then sum these up individually for each chain and divide by the total energy produced by all branches to get relative energy production by each branch.

We also want to discuss the gamow peaks of the different reactions. We are interested specifically in the gamow peaks for the temperature of the Sun $T = 1.57 \cdot 10^7$K. The gamow peaks are given by equation (**??**), which is the product of a Maxwell Boltzmann distribution and the electrostatic barrier between two charged particles with atomic numbers Z_i and Z_k.

$$gamow = exp\left(-\frac{E}{k_B T}\right) \cdot exp\left(-\sqrt{\frac{m}{2E}}\frac{Z_i Z_k e^2 \pi}{\epsilon_0 h}\right) \tag{9}$$

Where E is the energy of the particles, m is the combined mass of the two particles, e is the elementary charge, ϵ_0 is the permeability of vacuum and h is the planck constant.

Having discussed all the equations we need, we can produce our results.

III. RESULTS

Calculating the energy Q_{ik} for each PP chain reaction we get table I and table II for all steps in the CNO cycle.

Branch	Reaction	Q [MeV]
all	$3\mathrm{H}_1^1 \rightarrow He_2^3 + e^+ + \nu_e + \gamma$	6.8931
I	$2\mathrm{He}_2^3 \rightarrow He_2^4 + 2H_1^1$	12.8546
II & III	$\mathrm{He}_2^3 + He_2^4 \rightarrow Be_4^7 + \gamma$	1.5568
II	$\mathrm{Be}_4^7 + e^- \rightarrow Li_3^7 + \nu_e + \gamma$	0.8614
	$\mathrm{Li}_3^7 + H_1^1 \rightarrow 2He_2^4$	17.3295
III	$\mathrm{Be}_4^7 + H_1^1 \rightarrow B_5^8 + \gamma$	0.1131
	$\mathrm{B}_5^8 \rightarrow Be_4^8 + e + +\nu_e$	17.9846
	$\mathrm{Be}_4^8 \rightarrow 2He_2^4$	0.0931

Table I. The energy released in each step of the PP chains.

Reaction	Q [MeV]
$\mathrm{C}_6^{12} + H_1^1 \rightarrow N_7^{13} + \gamma$	12.2026
$\mathrm{N}_7^{13} \rightarrow C_6^{13} + e^+ + \nu_e$	2.1424
$\mathrm{C}_6^{13} + H_1^1 \rightarrow N_7^{14} + \gamma$	7.5451
$\mathrm{N}_7^{14} + H_1^1 \rightarrow O_8^{15} + \gamma$	7.2657
$\mathrm{O}_8^{15} \rightarrow N_7^{15} + e^+ + \nu_e$	2.7945
$\mathrm{N}_7^{15} + H_1^1 \rightarrow C_6^{12} + He_2^4$	5.3095

Table II. The energy produced by each reaction in the CNO cycle.

We also found the percentage of energy produced lost to neutrinos, as shown in table III.

From temperatures starting at $T = 10^4$K up to $T = 10^9$K we get figure 1, which shows the relative energy production for the different chains.

With a small alteration we get figure 2.

For the gamow peaks we get figure 3.

IV. DISCUSSION

We see in table II that something goes wrong in the first step in the cycle. This is most likely a result to rounding errors or incorrect masses of the elements.

In table III we see that PP3 loses the most energy out of the different chains, meaning it needs to produce more energy to componsate if it will be the primary source of energy.

Figure 1 shows how PP3 gets close to PP2 in energy production, but is not able to fully produce the same

Chain	Energy lost
PP0	3.8444%
PP1	0.9947%
PP2	4.0464%
PP3	26.1855%
CNO	4.5732%

Table III. The energy lost due to neutrinos as a percentage of the total energy produced by the chain.

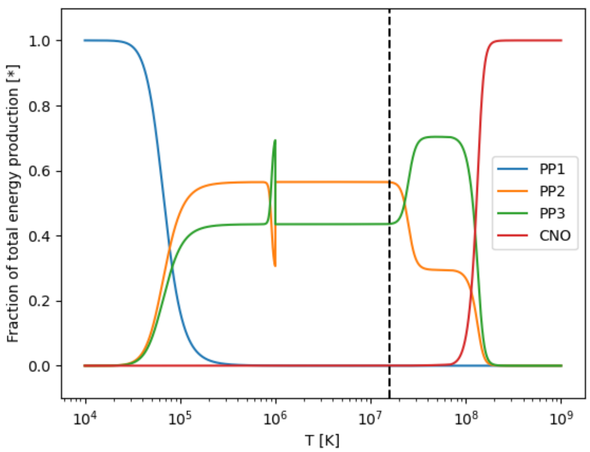

Figure 1. Shows the relative energy production along the y-axis for temperatures ranging from $T = 10^4$K up to $T = 10^9$K along the logarithmic x-axis. The black striped line indicates the temperature of the Sun.

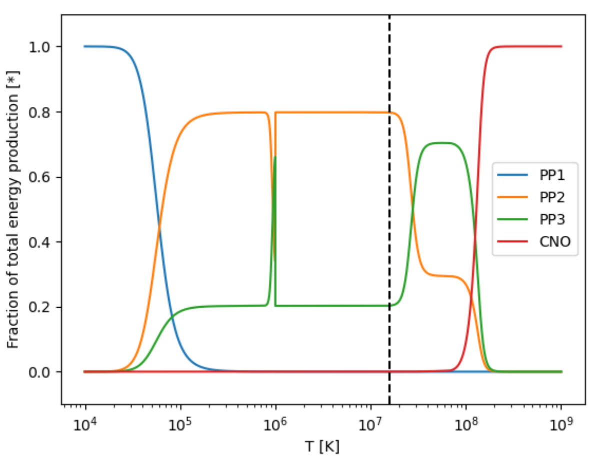

Figure 2. Shows the relative energy production along the y-axis for temperatures ranging from $T = 10^4$K up to $T = 10^9$K along the logarithmic x-axis. The black striped line indicates the temperature of the Sun. There has been made a small change to the algorithm, which makes it differ from figure 1.

amount for most temperatures. A point of further investigation could be looking at the relative energy production with the inclusion of neutrinos. More over, we see a sharp spike around $T = 10^6$K, which allignes with when there is no longer an upper limit of electron capture. In a more accurate model, perhaps this upper limit should be lowered slightly in order to ensure a smoother transition without a massive spike. The most interesting part of the graph perhaps is its innaccuracies. In the sun, PP1 is responsible for, according to the lecture notes, 83.30% of the energy production, while it is here responsible for 0%. This model is therefore not good at simulating the sun. A point for further investigation then is how it compares to stars with other core temperatures.

Figure 2 shows what I expected to find, after confering with other students. This graph was replicated by simply multiplying by 4 at a step in the calculation of reaction rates for PP2 and PP3. This indicates that there may be some oversight in how I calculated these.

Figure 3 shows the gamow peaks of the different reactions of interest. We see that PP0 is wide, but short and is located much further left than the others. In the core of the sun, PP0 reaction rates are higher for particles with much lower energies than for other reactions. Meaning, since there is little overlap between PP0 and other PP branches, they are not in resonance.

V. CONCLUSION

In this report we have learnt that our model is missing some more complex interaction as it fails to predict the relative energy production of temperatures around the core of the Sun. Making sure reactions do not consume more than produced also proved difficult, as I did not get the right combination of conditions correct for over a week.

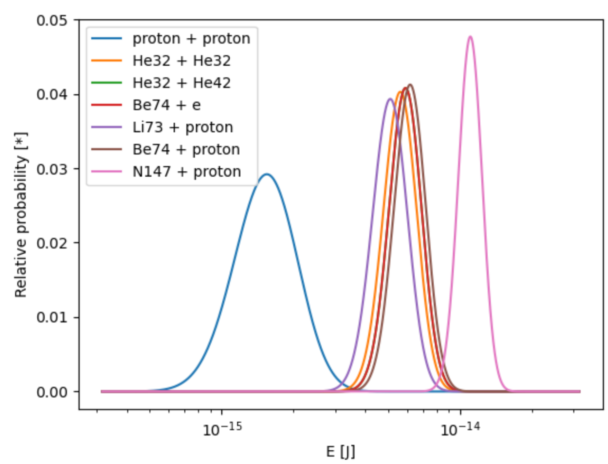

Figure 3. Shows the gamow peaks in the range $E \in [10^{-15.5}, 10^{-13.5}]$

. Since nothing interesting happens outside this range, it has been cropped.