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Credit risk modelling in a market model framework

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A Libor Market Model with Default Risk

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Structure of the Talk

- Libor Market Models
- Notation and Model Setup
  - the default model
  - bond prices and forward rates
- Drift restrictions in the continuous tenor case
- The forward- and survival measure
  - Girsanov's theorem
  - the forward measure
  - the survival measure
- Drift restrictions for the discrete-tenor case
- Positive recovery of par
- Default swap valuation
- Numerical implementation

Libor Market Models

- Miltersen / Sandmann / Sondermann (JoF 1997)
- Brace / Gatarek / Musiela (Math. Fin. 1997)
- Jamshidian (Finance and Stochastics 1997)

large and growing follow-up literature.

Reasons for popularity:

* positive interest rates

* automatic fitting to term structure of interest rates

* Easy calibration:
  Black formula for caplets
  excellent approximation to swaption prices

Disadvantage:
Path-dependent, relies on Monte-Carlo simulation for pricing.

Aim: Provide a natural extension of this model to incorporate default risk.

Model Setup

The Default Model

- τ : time of default
  triggered by first jump of Cox process $N(t)$
- $\lambda(t)$: default intensity
- $I(t) = \mathbf{1}_{\{\tau > t\}}$: survival indicator function

Tenor Structure

$$T_0, T_1, \ldots, T_K$$

$\kappa(t)$: The next date in the tenor structure after t.

$$T_{\kappa(t)-1} \leq t < T_{\kappa(t)}$$

Bond Prices

- Default-free zero coupon bonds
  $B(t, T_k) = B_k(t)$
- Defautable zero coupon bonds (zero recovery)
  $\overline{B}(t, T_k) = \overline{B}_k(t)$
  Defaultable bond price: $I(t)\overline{B}_k(t)$
- Default-risk factors

$$D(t, T_k) = D_k(t) = \frac{\overline{B}_k(t)}{B_k(t)}.$$

Forward Rates

The default-free forward rate over $[T_k, T_{k+1}]$:

$$F(t, T_k, T_{k+1}) = F_k(t) = \frac{1}{\delta}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right).$$

The defaultable forward rate over $[T_k, T_{k+1}]$:

$$\overline{F}(t, T_k, T_{k+1}) = \overline{F}_k(t) = \frac{1}{\delta}\left(\frac{\overline{B}_k(t)}{\overline{B}_{k+1}(t)} - 1\right).$$

The forward credit spread over $[T_k, T_{k+1}]$:

$$S(t, T_k, T_{k+1}) = S_k(t) = \overline{F}_k(t) - F_k(t).$$

The discrete forward default intensity:

$$H(t, T_k, T_{k+1}) = H_k(t) = \frac{1}{\delta}\left(\frac{D_k(t)}{D_{k+1}(t)} - 1\right).$$

Note: $S_k = H_k(1 + \delta F_k)$.

Dynamics

All dynamics are driven by d-dimensional BM W.

Lognormal diffusion for default-free forward rates ($\sigma_k^F = \text{const}$):

$$\frac{dF_k}{F_k} = \mu_k^F dt + \sigma_k^F dW$$

Lognormal diffusion for *either* credit spreads *or* intensities:

$$\frac{dH_k}{H_k} = \mu_k^H dt + \sigma_k^H dW \tag{1}$$

$$\frac{dS_k}{S_k} = \mu_k^S dt + \sigma_k^S dW, \tag{2}$$

with σ_k^H or σ_k^S constant.
(Correct choice is problem-dependent.)

Relations between volatilities:

$$\begin{aligned}
\sigma_k^H &= \sigma_k^S - \frac{\delta_k F_k}{1+\delta_k F_k}\sigma_k^F \\
\overline{F}_k \sigma_k^{\overline{F}} &= \sigma_k^F F_k + \sigma_k^S S_k \\
&= (1+\delta_k F_k) H_k \sigma_k^H + (1+\delta_k H_k) F_k \sigma_k^F .
\end{aligned}$$

Drift Restrictions Continuous Tenor

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\ln B(t,T) \\ \overline{f}(t,T) &= -\frac{\partial}{\partial T}\ln \overline{B}(t,T) \end{aligned}$$

Default-free forward rates
(Heath-Jarrow-Morton 1992):

$$df(t,T) = \sigma^f(t,T)\left(\int_t^T \sigma^f(t,s)ds\right)dt + \sigma^f(t,T)dW_Q,$$

Defaultable forward rates: (Schönbucher 1998)

$$\begin{aligned} d\overline{f}(t,T) &= \sigma^{\overline{f}}(t,T)\left(\int_t^T \sigma^{\overline{f}}(t,s)ds\right)dt + \sigma^{\overline{f}}(t,T)dW_Q, \\ \overline{f}(t,t) &= \lambda(t) + f(t,t). \end{aligned}$$

All dynamics under the spot martingale measure Q

The Girsanov Theorem

- $W_Q(t)$: a n-dimensional Q-Brownian motion
- $N(t)$: a point process
- $\lambda_Q(t)$: the Q-intensity of $N(t)$.

Measure Change processes (regularity required):

- θ: a n-dimensional predictable process
- $\phi(t)$: a nonnegative predictable process

Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \theta(t)dW_Q(t) + (\phi(t) - 1)(dN(t) - \lambda_Q(t)dt).$$

Then for the probability measure P with

$$dP(t) = L(t)dQ(t)$$

it holds that

$$dW_Q(t) - \theta(t)dt = dW_P(t)$$

defines W_P as P-Brownian motion and

$$\lambda_P(t) = \phi(t)\lambda_Q(t)$$

is the intensity of N under P.

Basic Measures

The Subjective Measure P

does not take risk premia into account:
Need to change to a pricing measure.

The Spot Martingale Measure Q

Discount factor $\beta(t) = e^{-\int_0^t r(s)ds}$.

Under the spot-martingale measure Q, the value of a random payoff X at time as seen from $t \leq T_k$ is

$$p(t) = \mathbf{E}^Q \left[\frac{\beta(T_k)}{\beta(t)} X \;\middle|\; \mathcal{F}_t \right].$$

Thus $\beta(t)p(t)$, i.e. the price $p(t)$ normalized with the Q-numeraire $1/\beta(t)$, is a Q-martingale.

The T_k Forward Measure P_k

$$p = B_k(0)\,\mathbf{E}^Q\left[\frac{\beta(T_k)B_k(T_k)}{B_k(0)}\,X\right],$$

Define the Radon-Nikodym density process

$$L_k(t) := \frac{\beta(t)B_k(t)}{B_k(0)} =: \left.\frac{dP_k}{dQ}\right|_{\mathcal{F}_t}.$$

for a change of measure from Q to a new measure P_k.

$$p = B_k(0)\,\mathbf{E}^Q\left[\,X L_k(T_k)\,\right] = B_k(0)\int_\Omega X L_k(T_k)\,dQ$$

$$= B_k(0)\int_\Omega X\frac{dP_k}{dQ}\,dQ = B_k(0)\,\mathbf{E}^{P_k}\left[\,X\,\right].$$

Default Probabilities under P_k:

$$I(0)D_k(0) == \mathbf{E}^{P_k}\left[\,I(T_k)\,\right] = P_k[\tau > T_k].$$

$I(t)D_k(t)$ is a P_k-martingale.

By Girsanov:

$$dW_k(t) := dW_Q(t) + \alpha_k(t)dt,$$

$\alpha_k(t)$ is minus the vector of the volatilities of the default-free zero-coupon bond $B_k(t)$:

$$\alpha_k(t) = \int_t^{T_k} \sigma^f(t,s)ds.$$

The default intensity is *not* affected by the change of measure, $\lambda_Q = \lambda_{P_k}$.

Recurrence Relation between the α_k:

$$\alpha_{k+1}(t) = \alpha_k(t) + \frac{\delta_k F_k(t)}{1 + \delta_k F_k(t)} \sigma_k^F(t)$$

Change from P_k to P_{k+1}:

$$\mathbf{E}^{P_k}[\, X \,] = \frac{1}{1 + \delta_k F_k(0)} \mathbf{E}^{P_{k+1}}[\, (1 + \delta_k F_k(T_k))X \,].$$

Change from $\overline{P}_k$ to P_k

$$\mathbf{E}^{\overline{P}_k}[\,X\,] = \frac{1}{D_k(0)}\mathbf{E}^{P_k}[\,I(t)D_k(t)X\,]$$
$$= \frac{B_k(0)}{\overline{B}_k(0)}\mathbf{E}^{P_k}\left[\,I(t)\frac{\overline{B}_k(t)}{B_k(t)}X\,\right].$$

The relation between the Brownian motions under the T_k forward measure and the T_k survival measure is (for $t < T_k$)

$$d\overline{W}_k(t) = dW_k(t) + \alpha_k^D(t)dt.$$

If the Cox process properties of $N(t)$ are used and X does not contain any direct reference to defaults:

$$\mathbf{E}^{\overline{P}_k}[\,X\,] = \frac{B_k(0)}{\overline{B}_k(0)}\mathbf{E}^{P_k}\left[\,e^{-\int_0^t \lambda(s)ds}\frac{\overline{B}_k(t)}{B_k(t)}X\,\right],$$
$$\mathbf{E}^{\overline{P}_k}[\,X\,] = \mathbf{E}^{P_k}\left[\,\mathcal{E}\left(\int_0^{T_k}\alpha_k^D(s)dW_k(s)\right)X\,\right]$$
$$=: \mathbf{E}^{P_k}[\,L_k^D(T_k)X\,].$$

Radon-Nikodym density:

$$\frac{d\overline{P}'_k}{dP_k}(t) = L_k^D(t) = e^{-\int_0^t \lambda(s)ds} \frac{\overline{B}_k(t)}{B_k(t)} \frac{B_k(t)}{\overline{B}_k(0)}$$

$$=: \frac{\gamma(t) D_k(t)}{D_k(0)},$$

where $\gamma(t) = e^{-\int_0^t \lambda(s)ds}$. In particular, $1/(\gamma(t)D_k(t))$ is a $\overline{P}'_k$-martingale and therefore it is also a $\overline{P}_k$-martingale.

Definition:

Independence ... between the default-free bond prices B_k and defaults means

$$\alpha_k(t)\alpha_l^D(t) = 0 \qquad \forall\, k, l$$

Drift Restrictions

Default-Free Forward Rates:

B_k/B_{k+1} is a martingale under the T_{k+1}-*forward measure.*

$$F_k = \frac{1}{\delta_k}(\frac{B_k}{B_{k+1}} - 1)$$

is a martingale under the T_{k+1}-*forward measure.*

$$dF_k(t) = F_k(t)\sigma_k^F dW_{k+1}(t).$$

Defaultable Forward Rates:

$\overline{B}_k/\overline{B}_{k+1}$ is a martingale under the T_k-*survival measure.*

$$\overline{F}_k = \frac{1}{\delta_k}(\frac{\overline{B}_k}{\overline{B}_{k+1}} - 1)$$

is a martingale under the T_{k+1}-*survival measure.*

$$d\overline{F}_k(t) = \overline{F}_k(t)\overline{\sigma}_k d\overline{W}_{k+1}(t).$$

Forward Spreads:

The dynamics of the forward spreads under the T_{k+1} survival measure are

$$dS_k = F_k \sigma_k^F \alpha_{k+1}^D dt + S_k \sigma_k^S d\overline{W}_{k+1}.$$

Forward Intensities:

Dynamics under $\overline{P}_{k+1}$:

$$\begin{aligned} dH_k =& F_k \sigma_k^F \left(\frac{1+\delta_k H_k}{1+\delta_k F_k} \alpha_{k+1}^D - \delta_k H_k \sigma_k^H \right) dt \\ &+ H_k \sigma_k^H d\overline{W}_{k+1}. \end{aligned}$$

Independence:

Under independence $\sigma^F \alpha^D = 0 = \sigma^S \sigma^F$:

$$
\begin{aligned}
dF_k &= F_k \sigma_k^F d\overline{W}_{k+1} \\
d\overline{F}_k &= \overline{F}_k \overline{\sigma}_k d\overline{W}_{k+1} \\
dS_k &= S_k \sigma_k^S d\overline{W}_{k+1} \\
dH_k &= H_k \sigma_k^H d\overline{W}_{k+1}
\end{aligned}
$$

The discrete default intensities H_k, the credit spreads S_k, the defaultable forward rates $\overline{F}_k$ and the default-free forward rates F_k, in short *all forward rates with fixing at T_k are martingales under the T_{k+1}-survival measure.*

If H_k and F_k are independent under Q, then this independence is preserved and S_k and F_k are also independent under $\overline{P}_{k+1}$.

Recovery of Par

If a defaultable coupon bond defaults in the time interval $]T_k, T_{k+1}]$ then its recovery is composed of the recovery rate π times the sum of
the notional of the bond (here normalised to 1)
and the accrued interest over $]T_k, T_{k+1}]$.

The accrued interest can be

(a) c, a constant in the case of a fixed-coupon bond with coupon c,
recovery is $\pi(1 + c)$

(b) F_k in the case of a floating rate bond
recovery is $\pi(1 + \delta_k F_k(T_k))$

The recovery payoffs occur in cash at $T_{K(\tau)}$ i.e. at the next tenor date T_{k+1} if a default was in $]T_k, T_{k+1}]$.

Further outstanding coupons have zero recovery.

Discrete Defaults Correction

Effect:
postponement of the default from somewhere in $]T_k, T_{k+1}]$ to T_{k+1}.

Approximate correction to this:

Assume that r and λ are constant over $[T_k, T_{k+1}]$.

Given a default happens in $]T_k, T_{k+1}]$, the T_{k+1}-value of π received at default and invested at r until T_{k+1} is

$$\pi' := \frac{\lambda}{\lambda + r} \frac{F(1 + \delta H)}{H(1 + \delta F)} \pi \geq \pi.$$

Use π' instead of π and work with recovery payoffs at the next tenor date T_{k+1}. Typically

$$\pi'/\pi \approx 1.005 - 1.02$$

Value of the Recovery Payoffs

- The value of 1 at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$ is under independence

$$e_k := \overline{B}_{k+1}\delta_k H_k$$

- The value of $1 + \delta_k F_k(T_k)$ at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$ is under independence

$$\overline{B}_{k+1}\delta_k S_k$$

The value of $F_k(T_k)$ at T_{k+1} if no default occurs until T_{k+1} is $\overline{B}_{k+1}F_k$.

Recovery Payoffs under Correlation

• Under correlation, the value of a payment of 1 at T_{k+1} if a default happens in $]T_k, T_{k+1}]$ is

$$
\begin{aligned}
e_k :=& \delta_k H_k \overline{B}_{k+1} \\
& + \overline{B}_k \operatorname{cov}^{P_k}\left(L_k^D(T_k)\, ,\, \frac{1}{1+\delta_k F_k(T_k)} \right) \\
\approx& \delta_k H_k \overline{B}_{k+1} \\
& + \frac{\overline{B}_k}{1+\delta_k F_k} \left(e^{(1-F_k(0)) A_{k,k}^D \sigma_k^F} - 1 \right), \\
=& \delta_k H_k \overline{B}_{k+1} \\
& + \overline{B}_{k+1}(1+\delta_k H_k) \left(e^{(1-F_k(0)) A_{k,k}^D \sigma_k^F} - 1 \right),
\end{aligned}
$$

• and the value of $1 + \delta_k F_k(T_k)$ at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$ is

$$\overline{B}_{k+1}\delta_k S_k - \delta_k \overline{B}_{k+1} \operatorname{cov}^{P_{k+1}}\Big(L^D_{k+1}(T_k)\,;\, F_k(T_k)\Big)$$
$$\approx \overline{B}_{k+1}\delta_k S_k - \overline{B}_{k+1}\delta_k F_k \left(e^{A^D_{k+1,k}\sigma^F_k} - 1\right).$$

The value of $F_k(T_k)$ at T_{k+1} if no default occurs until T_{k+1} is $\overline{B}_{k+1} F_k e^{A^D_{k+1,k}\sigma^F_k}$.

The approximated average volatility of L^D_k is

$$\int_0^{T_m} \alpha^D_{k+1}(t)dt \approx \sum_{l=0}^{k} \frac{\delta_l H_l(0)\sigma^H_l(0)}{1 + \delta_l H_l(0)} T_{l\wedge m} =: A^D_{k+1,m}.$$

With these results we can value

- defaultable fixed coupon bonds
- defaultable floating coupon bonds
- default swaps

Default Swap Valuation

Specification

- **A** pays s at T_i until T_N or default (fee stream)
- **B** pays the difference between the post-default price of the reference asset (a bond issued by **C**) and its par value at default (default payment).

The Fee

The value of the fee stream can be directly determined as

$$s \sum_{k=1}^{N} \overline{B}_k(0)$$

This valuation is valid for all fee streams of credit derivatives that pay fees until default.

The Default Payment

The value of receiving 1 at default is
$D^{\text{DDP}} = \sum_{k=0}^{N-1} e_k$.

$C(t)$: the value of the reference asset
$C_0(t)$: the value of the RA under zero recovery.

Portfolio (a):

- the reference asset C
- the default payment of the default swap $D^{\text{Def Put}}$

Payoffs: 1 at default (from the default put) and the regular payments in survival.

Porfolio (b):

- reference asset under zero recovery C_0
- payment of 1 at default D^{DDP}

$$C_0 + \sum_{k=0}^{N-1} e_k - C = D^{\text{Def Put}}.$$

For defaultable coupon bond with coupon c as reference asset:

$D^{\text{Def Put}} = (1 - \pi - c\pi) \sum_{k=0}^{N-1} e_k$.

The Default Swap Rate

$$\bar{s} = (1 - \pi(1+c)) \sum_{k=0}^{N-1} \overline{w}_k \delta_k H_k$$

(for independence), and

$$\approx (1 - \pi(1+c)) \left(\sum_{k=0}^{N-1} \overline{w}_k \delta_k H_k \right.$$

$$\left. + (1 + \delta_k H_k)(e^{(1-F_k(0))A^D_{k,k}\sigma^F_k} - 1) \right)$$

under correlation, where

$$\overline{w}_k := \overline{B}_{k+1} / \sum_{j=0}^{N-1} \overline{B}_{j+1}$$

A plain vanilla fixed-for-floating interest rate swap rate:

$$s = \sum_{k=0}^{N-1} w_k \delta_k F_k$$

with weights $w_k := B_{k+1} / \sum_{j=0}^{N-1} B_{j+1}$.

Numerical Implementation

- $C(T_k)$ be the value of a credit derivative given $\tau > T_k$
- X_k its payoff at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$

$$
\begin{aligned}
C(T_k) &= \mathbf{E}^Q \left[\frac{\beta(T_{k+1})}{\beta(T_k)} C(T_{k+1}) \right] \\
&= B_{k+1}(T_k)\, \mathbf{E}^{P_{k+1}} [\, C(T_{k+1}) I(T_{k+1}) \,] \\
&\quad + B_{k+1}(T_k)\, \mathbf{E}^{P_{k+1}} [\, C(T_{k+1})(1 - I(T_{k+1})) \,] \\
&= \dots \\
&= \overline{B}_{k+1}(T_k)\, \mathbf{E}^{\overline{P}_{k+1}} [\, C(T_{k+1}) \,] \\
&\quad + B_{k+1}(T_k)(1 - D_{k+1}(T_k)) X_{k+1}.
\end{aligned}
$$

Default Tree Simulation

Starting from T_k the simulation until T_{k+1} proceeds as follows:

- P_{k+1} survival probability until T_{k+1} is $D_{k+1}(T_k)$. This is the probability on the survival branch.
- $1 - D_{k+1}(T_k)$ probability on the default branch.
- Add the value in default weighted with default probability.
- Add any survival payoffs (with survival probability).
- Simulate F, H until T_{k+1} on the survival branch of the tree under the T_{k+1} *survival* measure $\overline{P}_{k+1}$.

The simulation takes place under the *survival* spot Libor measure.

A LIBOR MARKET MODEL WITH DEFAULT RISK

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ABSTRACT. In this paper a discrete-tenor model for default risk is developed along the lines of the Libor Market Models by Miltersen / Sandmann / Sondermann (1997) and Brace / Gatarek / Musiela (1997). The effective forward rates and effective forward credit spreads are modelled as diffusion processes with a lognormal volatility structure, recovery is modelled as a fraction of the par value of the defaulted coupon bond. No-arbitrage dynamics of the forward rates and forward spreads are derived, as well as closed-form solutions for defaultable coupon bonds, default swap rates and asset swap rates, and approximate solutions are given for options on default swaps, which can be made exact in a modified modelling framework. Furthermore, the numerical implementation of the model is discussed.

1. INTRODUCTION

In this paper a discrete-tenor model for default risk is developed along the lines of the Libor Market Models by Miltersen / Sandmann / Sondermann (1997) and Brace / Gatarek / Musiela (1997). The effective forward rates and effective forward credit spreads are modelled as diffusion processes with a lognormal volatility structure, recovery is modelled as a fraction of the par value of the defaulted coupon bond. No-arbitrage dynamics of the forward rates and forward spreads are derived, as well as closed-form solutions for defaultable coupon bonds, default swap rates and asset swap rates, and approximate solutions are given for default swaptions. Furthermore, the implementation of the model is discussed.

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This is a preliminary and incomplete version. Comments and suggestions are welcome. All errors are my own.

Practical Application of a Transition Matrix Model

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Historical Default Probabilities vs. Credit Spreads

- The defaultable bond prices of the model do not agree with observed bond prices
- Credit spreads are too low. (Particularly for higher quality debt.)
- Reasons: Risk Premia, liquidity premia
- Credit spreads within rating classes are stochastic.
- Adjustments to the historical transition matrix:
  - Risk Premia
  - Stochasticity
  - Consistency: Monotonicity
  - Eliminate NR class

Incorporating Risk Premia

Jarrow / Lando / Turnbull (1997)

The generator matrix under the martingale measure is

$$\tilde{A}(t) := U(t)A$$

where $U(t) = \text{diag}(\mu_1(t), \ldots, \mu_{K-1}(t), 1)$ strictly positive and deterministic.

- one of many possibilities
- multiplying row i with $\mu_i(t)$
- scaling all transition rates for class i with $\mu_i(t)$
- time-dependence for fitting
- need forward equations to calculate transition probabilities

Itô's Lemma for Markov Chains

$R(t)$ Markov Chain with generating matrix A.
$f(t, x, R)$ a function of time t,

$$dx = \mu_x dt + \sigma_x dW$$

and rating R. Then for $R(t) = i$:

$$\begin{aligned} df = & \frac{\partial}{\partial t} f dt + \frac{\partial}{\partial x} f dx + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} f dt \\ & + [f(t, x, R(t + dt)) - f(t, x, R(t))]. \end{aligned}$$

Itô's lemma for a scalar function.

Expectations

For pricing we need the expectation of df:

$$\mathbf{E}[\,df\,] = \frac{\partial}{\partial t} f\, dt + \frac{\partial}{\partial x} f\, \mathbf{E}[\,dx\,] + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} f\, dt + \sum_{k=1}^{K} A_{ik} f(t,x,k) dt.$$

Let $F = (F_1(t,x), \ldots, F_K(t,x))^T$ a vector valued function with:

$$F_i(t,x) = f(t,x,i)$$

$F_i(t,x)$ is the value of $f(t,x,R)$ for $R(t) = i$. Then

$$\mathbf{E}[\,dF\,] = \frac{\partial}{\partial t} F\, dt + \frac{\partial}{\partial x} F\, \mathbf{E}[\,dx\,] + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} F\, dt + AF dt.$$

Intuition for Itô

Given $R(t) = i$:

$$f(R(t+\Delta t)) = \begin{cases} f(i) & \text{with Prob} \quad 1 - \sum_{k \neq i} A_{ik} \Delta t \\ f(k) & \text{with Prob} \quad A_{ik} \Delta t \end{cases}$$

Thus

$$\begin{aligned} \mathbf{E}[\,\Delta f\,] &= \mathbf{E}[\,f(R(t+\Delta t)) - f(R(t))\,] \\ &= \sum_{k \neq i} (f(k) - f(i)) A_{ik} \Delta t \\ &= \sum_{k \neq i} f(k) A_{ik} \Delta t + f(i) \sum_{k \neq i} -A_{ik} \Delta t \\ &= \sum_{k=1}^{K} f(k) A_{ik} \Delta t. \end{aligned}$$

Bond Pricing

By construction the Markov chain is independent of the continuous processes (esp. risk-free interest rates).

Bond prices

$$
\begin{aligned}
&\overline{B}(0,T,r,R(0)=i) \\
&= \mathbf{E}\left[\, b_{0T}(1-c\mathbf{1}_{\{\tau\leq T\}})|R(0)=i \,\right] \\
&= \mathbf{E}\left[\, b_{0T}|R(0)=i \,\right]\mathbf{E}\left[\, (1-c\mathbf{1}_{\{\tau\leq T\}})|R(0)=i \,\right] \\
&= B(0,T,r)(1-c\mathbf{E}\left[\, \mathbf{1}_{\{\tau\leq T\}}|R(0)=i \,\right]) \\
&= B(0,T,r)(1-cP_{iK}(T)),
\end{aligned}
$$

where $P_{iK}(T)$ is the transition probability from i to K.

In vectors:

$$\overline{B}(0,T,r) = B(0,T,r)(1-cP_{\boldsymbol{i}K}(T)).$$

European payoff:

- $F'(t, r)$ price vector
- T maturity
- $F^*(r)$ payoff at T
- F vector of default-free securities paying off F^* for sure.

Price:

$$F'(t, r) = P(T)F(t, r)$$

General Payoffs

$$\mathbf{E}[\,dF'\,] = rF'dt$$

Yields the pricing equation: for all R:

$$0 = \frac{\partial}{\partial t}F'_R + \frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}F'_R + \mu_r\frac{\partial}{\partial r}F'_R + \sum_{k=1}^{K} A_{Rk}F'_k - rF'.$$

or in matrix notation

$$0 = \frac{\partial}{\partial t}F' + \frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}F' + \mu_r\frac{\partial}{\partial r}F' + (A - rI)F'.$$

- K functions $F'_i(t, r)$
- each satisfying coupled p.d.e.s (coupled by AF')
- not worse than $F'(t, r, \lambda)$: another continuous variable

- also holds for time- and state dependent A
- can usually separate interest-rate dependence from Markov chain via independence.

Downgrade Triggers

Hitting probability of rating k:
Delete row k from A:

$$A_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2K} \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ a_{K1} & a_{K2} & \cdots & \cdots & \cdots & a_{KK} \end{pmatrix}$$

and calculate transition probablilites $P(T)$ then

- for $j \neq i$ then $P_{ij}(T)$= probability of going from j to i without going through k.

- $P_{ik}(T)$ = probability of hitting k until T.

Stochastic Spreads

Multiply A with h where

$$dh = \mu_h dt + \sigma_h dW.$$

Then $A_h := Ah$ and

$$A_h = MD_h M^{-1} = MhDM^{-1}.$$

Transition probabilities are A_h

$$P_h(t) = \mathbf{E}\left[\exp\left\{A\int_0^t h(s)ds\right\}\right]$$
$$= M\mathbf{E}\left[\exp\left\{D\int_0^t h(s)ds\right\}\right]M^{-1}.$$

Usually h is chosen such that

$$\mathbf{E}\left[\exp\left\{\int_0^t h(s)ds\right\}\right]$$

has a closed-form representation.

Bond prices (similar to before):

$$\begin{aligned}\overline{B}(t,T) &= \mathbf{E}\left[\, b_{tT}(1 - c\mathbf{1}_{\{\tau \geq T\}}\,\right] \\ &= B(t,T)(1 - cP_{h\cdot K}(T-t))\end{aligned}$$

The pricing equation

$$
\begin{aligned}
0 = {} & \frac{\partial}{\partial t}F' + \frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}F' + \mu_r\frac{\partial}{\partial r}F' \\
& + \frac{1}{2}\sigma_h^2\frac{\partial^2}{\partial h^2}F' + \mu_h\frac{\partial}{\partial h}F' \\
& + (hA - rI)F'.
\end{aligned}
$$

Or (under independence):

$$
\begin{aligned}
0 = {} & \frac{\partial}{\partial t}G' + \frac{1}{2}\sigma_h^2\frac{\partial^2}{\partial h^2}G' + \mu_h\frac{\partial}{\partial h}G' \\
& + hAG'.
\end{aligned}
$$

where

$$F'(t, r, h) = B(t, T, r)G'(t, h)$$

Hedging

need K instruments to satisfy

$$V(r, R) - V(r, R_0) = \sum_{k=1}^{K} \alpha_k (F_k(r, R) - F_k(r, R_0)$$

for all R, and hedge in interest rate risk

$$\frac{\partial}{\partial r} V(r, R_0) = \sum_{k=1}^{K} \alpha_k \frac{\partial}{\partial r} F_k(r, R_0).$$

Term Structures of Credit Spreads

Necessary for fitting of the model:

- by choosing $\mu_i(t)$ appropriately (Jarrow / Lando / Turnbull)
- by choosing the recovery rate time-dependent or stochastic (Das)
- by using a full dynamic model for within-class credit spreads and rating transitions

The Full Model (Schönbucher (1999))

- K different classes, 'default' is *not* counted as a class
- $\widetilde{B}_k(t,T)$: defaultable bond prices in class k
- $\overline{f}_k(t,T)$: defaultable forward rates

$$\widetilde{B}_k(t,T) = \exp\{-\int_t^T \overline{f}_k(t,s)ds\}.$$

- $\overline{B}(t,T)$ defaultable bond (fractional recovery q)
- $R(t)$ ratings process

$$\overline{B}(t,T) = \widetilde{B}_{R(t)}(t,T).$$

- transition intensities a_{kl} can be stochastic
- Dynamics of the defaultable forward rates *within class* k:

$$d\overline{f}_k(t,T) = \overline{\alpha}_k(t,T)dt + \sum_{i=1}^{n} \overline{\sigma}_{i,k}(t,T)\, dW^i(t)$$

- Defaults and rating transitions never happen at the same time

Arbitrage-Free Dynamics

under the martingale measure

(i) The short rate spread in rating class k is

$$\lambda_k(t)q(t) = \overline{r}_k(t) - r(t).$$

thus giving the default intensity λ_k in class k.

(ii) The drift of the defaultable forward rates is restricted by

$$\overline{\alpha}_k(t,T) = \sum_{i=1}^{n} \overline{\sigma}_{i,k}(t,T)\left(\int_t^T \overline{\sigma}_{i,k}(t,s)\,ds\right) + \sum_{l=1}^{K} \frac{\widetilde{B}_l(t,T)}{\widetilde{B}_k(t,T)}(\overline{f}_k(t,T) - \overline{f}_l(t,T))\,a_{k,l}.$$

(iii) This is sufficient for $\overline{B}_{R(t)}(t,T)$ to be a martingale.

Using these dynamics with the initial term structures of the classes we have a fully fitted arbitrage-free model.

Advantages

- ✔ Appropriate model for rating transitions.
- ✔ Much data available.
- ✔ Data is from independent analysis.
- ✔ (In simplest version) relatively straightforward.
- ✔ Can be adapted to bank-internal risk-scoring methods:
  Pricing and analyzing credit risk of a loan portfolio.

Disadvantages

based in the nature of the data:

- ✗ historical data (backwards-looking)
- ✗ Real world ratings adjustments happen with delay.
- ✗ Rating agencies may define defaults differently.
- ✗ Historical default probabilities do not justify spreads observed in the market. Need risk premium adjustment to fit model (e.g. via h).
- ✗ Transition probabilities only depend on current rating (not on history). No 'ratings momentum' possible in the model.
- ✗ Data sometimes based on only very few transitions.
- ✗ Mostly US corporates.

- ✘ Spreads constant within same rating class (for h constant).

- ✘ Full model of spreads computationally very complex:
  Need at least one BM for each rating class.

- ✘ For stochastic spreads deduction of generating mx can become very complicated.

Fields of Application

- Pricing and investment tool.
- Internal risk-scoring models.
- Derivatives that condition on rating transitions.
- Not for pure credit spread derivatives.

The Forward and Backward Equations: Time Homogeneous Case

$$
\begin{aligned}
P(t+\Delta t) &= P(t)P(\Delta t) \\
&= P(t)(I+\Delta t A) \\
\frac{1}{\Delta t}(P(t+\Delta t) - P(t)) &= P(t)A \\
\frac{\partial}{\partial t}P(t) &= P(t)A
\end{aligned}
$$

The *Kolmogorov forward equation.*
Similarly

$$
\begin{aligned}
P(t+\Delta t) &= P(\Delta t)P(t) \\
&= (I+\Delta t A)P(t) \\
\frac{1}{\Delta t}(P(t+\Delta t) - P(t)) &= AP(t) \\
\frac{\partial}{\partial t}P(t) &= AP(t)
\end{aligned}
$$

The *Kolmogorov backward equation.*
Both subject to $P(0) = I$.

In Components:

Forward equation:

- $P(t+\Delta t)_{ij} =$
  the probability of going from i at time 0 to j at time $t+\Delta t$ equals

- $P(t)_{ij}(1 - \sum_{k\neq j} A_{jk}\Delta t) +$
  the probability of going to j at time t and then not moving away, plus

- $\sum_{k\neq j} P(t)_{ik}A_{kj}\Delta t$
  the probability of having gone to another state k at t and then moving to j. (summed up over all k)

$$\frac{\partial}{\partial t}P(t)_{ij} = \sum_{k=1}^{K} P(t)_{ik}A_{kj}$$

Backward equation:

- $P(t+\Delta t)_{ij} =$
  the probability of going from i at time 0 to j at time $t+\Delta t$ equals
- $(1 - \sum_{k \neq i} A_{ik} \Delta t) P(t)_{ij} +$
  the probability of staying at i until Δt and then moving to j, plus
- $\sum_{k \neq i} A_{ik} \Delta t P(t)_{kj}$
  the probability of moving to another state k in the next instant and then moving from there to j. (summed up over all k)

$$\frac{\partial}{\partial t} P(t)_{ij} = \sum_{k=1}^{K} A_{ik} P(t)_{kj}$$

The Forward and Backward Equations: General Case

Know $P(t,t) = I$.
Approximate transition matrix for the next instant Δt:

$$P(t,t+\Delta t) = I + \Delta t A(t).$$

$A(t)$ time-dependent because process time-inhomogeneous.

Then by Chapman-Kolmogorov:

$$\begin{aligned} P(t,T+\Delta t) &= P(t,T)P(T,T+\Delta t) \\ &= P(t,T)(I+\Delta t A(T)) \end{aligned}$$

$$\frac{1}{\Delta t}(P(t,T+\Delta t) - P(t,T)) = P(t,T)A(T)$$

$$\frac{\partial}{\partial T}P(t,T) = P(t,T)A(T)$$

The *Kolmogorov forward equation.*
Similarly

$$\begin{aligned} P(t,T) &= P(t,t+\Delta t)P(t+\Delta t,T) \\ &= (I+\Delta t A(t))P(t+\Delta t,T) \end{aligned}$$

$$\frac{1}{\Delta t}(P(t+\Delta t, T) - P(t, T)) = -A(t)P(t, T)$$

$$\frac{\partial}{\partial t}P(t, T) = -A(t)P(t, T)$$

The *Kolmogorov backward equation.*
Note $P(t, T) = P(T-t)$ in time-homogeneous case thus
$\frac{\partial}{\partial t}P(t, T) = -P'(T-t)$.

Solving Forward and Backward Equations

Scalar o.d.e.:

$$f'(x) = af(x), \qquad f(0) = 1$$

has solution

$$f(x) = e^{ax}.$$

Here matrix o.d.e.

$$\frac{\partial}{\partial t}P(t) = AP(t)$$

has similar solution

$$P(t) = e^{At} := \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n$$

where the matrix exponential is defined by the power series.

Solving Inhomogenous Case

Scalar o.d.e. and solution:

$$f'(t) = -a(t)f(t), \qquad f(0) = 1$$

$$f(x) = \exp\{-\int_0^t a(s)ds\}.$$

Here matrix o.d.e. (backward)

$$\frac{\partial}{\partial t}P(t,T) = -A(t)P(t,T)$$

if $A(t)$ and $\int_t^T A(s)ds$ commute

$$A(t)\left(\int_t^T A(s)ds\right) = \left(\int_t^T A(s)ds\right) A(t)$$

then the backward (and forward) equation has a similar solution

$$P(t,T) = \exp\{\int_t^T A(s)ds\}.$$

Commuting Matrices

For two matrices A and B

$$e^{A+B} = e^A e^B$$

iff A and B commute: $AB = BA$.
e.g. if A and B are similar: $A = S^{-1}BS$, or differ by a scalar.
For the Backward equation:

$$P(t+\Delta t, T) - P(t, T) \approx$$

$$\exp\{\int_{t+\Delta t}^{T} A(s)ds\} - \exp\{A(t)\Delta t + \int_{t+\Delta t}^{T} A(s)ds\}$$

$$= (I - \exp\{A(t)\Delta t\}) \exp\{\int_{t+\Delta t}^{T} A(s)ds\}$$

$$\approx (I - I - A(t)\Delta t)P(t, T)$$

and

$$\frac{1}{\Delta t}(P(t+\Delta t, T) - P(t, T)) \to -A(t)P(t, T).$$

General Solution to Matrix Decomposition

If $P(1)$ cannot be decomposed in a diagonal matrix then use its Jordan form:

$$P(1) = MJM^{-1}$$

Where J contains Jordan blocks on its diagonal:

$$B_k = \lambda_k I + N_k$$

where λ is the k-th eigenvalue of the matrix P and N_k has the following form:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Then

$$e^{B_k t} = e^{\lambda t} e^{N_k t}$$

and $e^{N_k t}$ is given by

$$\begin{pmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t^2}{2!} \\ \vdots & & \ddots & \ddots & \frac{t}{1!} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

The logarithm of J is defined by

$$\ln(B_k t) = \ln \lambda I + L$$

where L is given by

$$\begin{pmatrix} 0 & \frac{1}{\lambda} & -\frac{1}{2\lambda} & \cdots & \frac{(-1)^{k-2}}{(k-1)\lambda^{k-1}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -\frac{1}{2\lambda} \\ \vdots & & & \ddots & \frac{1}{\lambda} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

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Reduced form models

Modelling default correlation for portfolio credit risk measurement

Dr Philipp Shönbucher
UNIVERSITY OF BONN

Event researched, produced and organised by Risk Conferences

Reduced Form Models

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In this Unit You Will Learn...

model-free pricing results:

- how to extract from the credit spread curve:
  - implied default probabilities
  - implied survival probabilities
  - conditional survival probabilities
- how positive recovery is modelled: recovery in cash, in risk-free securities, in defaultable securities.
- the uses of Poisson processes, inhomogenous Poisson Processes, Cox Processes
- the models of Jarrow / Turnbull, Duffie / Singleton, Madan / Unal
- the construction of credit spread curves

Implied Probabilities

What can we derive from the term structure of credit spreads?

Assume we know:

- Default-free zero coupon bond (ZCB) curve:

  $B(t, T) =$ Price at time t of ZCB paying off 1 at T

- Defaultable zero coupon bond (ZCB) curve:

  $\overline{B}(t, T) =$ Price of defaultable ZCB

- zero recovery
- risk-neutral probabilities
- independence of [defaults and credit spreads dynamics] and [interest-rate dynamics]

Call the time of default τ

The Basic Relationship

Payoff of default-free ZCB: 1 at time T

Value of default-free ZCB:
= discounted expected payoff

$$B(t,T) = \mathbf{E}\left[\,\beta_{t,T}\,\right].$$

($\beta_{t,T}$ is the discount-factor over $[t,T]$.)

Payoff of defaultable ZCB: $\mathbf{1}_{\{\tau>T\}}$ at time T

$$\mathbf{1}_{\{\tau>T\}} = \begin{cases} 1 & \text{if default after } T\ (\tau > T), \\ 0 & \text{if default before } T\ (\tau \leq T). \end{cases}$$

Value of defaultable ZCB:
= discounted expected payoff

$$\overline{B}(t,T) = \mathbf{E}\left[\,\beta_{t,T}\mathbf{1}_{\{\tau>T\}}\,\right].$$

If there is no correlation:

$$\overline{B}(t,T) = \mathbf{E}\left[\,\beta_{t,T}\,\right]\mathbf{E}\left[\,\mathbf{1}_{\{\tau>T\}}\,\right]$$

$$= B(t,T)\mathbf{E}\left[\,\mathbf{1}_{\{\tau>T\}}\,\right]$$
$$= B(t,T)P(t,T)$$

$P(t,T)$ is the implied probability of survival in $[t,T]$.

The Implied Survival Probability

the implied survival probability is the ratio of the ZCB prices:

$$P(t,T) = \frac{\overline{B}(t,T)}{B(t,T)}$$

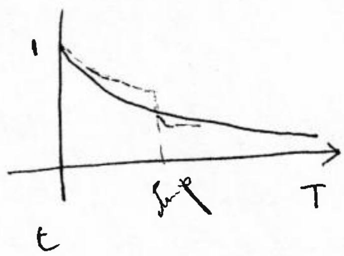
Properties:

- Default Probability = 1- Survival Probabilities

- initially at one

$$P(t,t) = 1$$

- eventually there is a default:



$$P(t,\infty) = 0$$

- $P(t,T)$ is decreasing in T.

Typically $P(t,T)$ will change over time.
(From t to $t + \Delta t$)

Two effects that change $P(t,T)$:

1. There was no default in $[t, t + \Delta t]$:
Information via the (non)-occurrence of defaults. The possibility of default in $[t, t+\Delta t]$ was previously reflected in $P(t,T)$.

2. Additional default-relevant information arriving in the meantime.

Conditional Survival Probabilities

Probability of survival in $[T_1, T_2]$
given that there was no default until T_1.:

Note: Bayes' Rule:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}.$$

Here:
$A =$ survival until T_2
$B =$ survival until T_1
$A \cap B =$ [survival until T_2] AND [survival until T_1]
$=$ [survival until T_2]

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)} = \frac{\overline{B}(t, T_2)}{B(t, T_2)} \frac{B(t, T_1)}{\overline{B}(t, T_1)}.$$

- $P(t, T, T) = 1$, unless default is 'scheduled' at T.
- Survival to T = survival to s AND survival from s to T

$$P(t, T) = P(t, s)P(t, s, T)$$

Relation to Forward Spreads

The **simply compounded forward rate** over the period $[T_1, T_2]$ as seen from t:

$$F(t, T_1, T_2) = \frac{B(t, T_1)/B(t, T_2) - 1}{T_2 - T_1}$$

Similarly for defaultable bonds:

$$\overline{F}(t, T_1, T_2) = \frac{\overline{B}(t, T_1)/\overline{B}(t, T_2) - 1}{T_2 - T_1}$$

Then the **conditional probability of default** over $[T_1, T_2]$ is given by:

$$\frac{P^{\text{def}}(t, T_1, T_2)}{T_2 - T_1} = \frac{\overline{F}(t, T_1, T_2) - F(t, T_1, T_2)}{1 + (T_2 - T_1)\overline{F}(t, T_1, T_2)}$$

[Default Probability]

=[Length of time interval] × [Spread of forward rates]

× [Discounting with defaultable forward rate]

Marginal Default Probabilities

The **continuously compounded forward rate** at T as seen from t:

$$f(t,T) = \lim_{\Delta t \searrow 0} F(t,T,T+\Delta t) = -\frac{\partial}{\partial T} \ln B(t,T)$$

$$\overline{f}(t,T) = \lim_{\Delta t \searrow 0} \overline{F}(t,T,T+\Delta t) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T)$$

The **marginal probability of default** at time T is:

$$\lim_{\Delta t \searrow 0} \frac{P^{\text{def}}(t,T,T+\Delta t)}{\Delta t} = \overline{f}(t,T) - f(t,T)$$

[Marginal default probability at time T]

=[Spread of forward rates for T]

The probability of default in $[T,T+\Delta t]$ is approximately **proportional** to the length of the interval with proportionality factor $(\overline{f}(t,T) - f(t,T))$

Influence of Recovery

Recovery c default-free bonds.

$$\overline{B}(t,T) = cB(t,T) + (1-c)B(t,T)P(t,T)$$

The probability of survival becomes:

$$P(t,T) = \frac{1}{1-c}\left(\frac{\overline{B}(t,T)}{B(t,T)} - c\right)$$

Problems if c is unknown: Cannot back out P any more from term structure alone.

Implied Probabilities: Discussion

Initial assumptions can be relaxed:

- ✔ risk-aversion:
  we are using *risk-adjusted probabilities*
  'probabilities' here are really **state prices**

- ✔ correlation of interest-rate movements and credit spread movements

- ✔ positive recovery:
  if expected recovery rate is known, can construct zero-recovery ZCB curve.

Recovery Rates

Source: Moody's

Seniority Class	Mean (%)	Standard Deviation (%)
Bank Loans	71,00	??
Senior Secured	53,80	26,86
Senior Unsecured	51,13	25,45
Senior Subordinated	38,52	23,81
Subordinated	32,74	20,18
Junior Subordinated	17,09	10,90

Recovery Models:

three different approaches for recovery of defaultable ZCB $\overline{B}$:

Payoff in default =

- 0: Zero Recovery: unrealistic
- π: a fixed cash amount π
- cB: a certain number c of default-free securities (otherwise identical)
- $(1-q)\overline{B}$: a fraction $(1-q)$ of the pre-default value of the defaultable security
- either of the above, but with the recovery rate *random*.

Note:

- no attempt is made to model the *real-world outcome* of the bankruptcy process or the **terms** of payment, all we model is the **value** of the settlement.

- models differ in the **units** in which this value is expressed: cash, default-free bonds, defaultable bonds.

- (within limits) can transform models into each other: they are mathematically equivalent

- but some are more convenient or have more realistic outcomes

- the choice of model makes no difference for securities trading at par (but we want to model defaultable ZCBs.)

Recovery as Cash

Default payoff = π: a fixed cash amount π

Problem: Payoff too high for securities below par.

Example:

- $\pi =$ 50% recovery at default.
- $r =$ 5% risk-free interest rate (constant)
- $h =$ 3% credit spread (constant)
- $T_1 =$ 10 years to maturity

Default-free bond price:

$$B(0,T) = e^{-rT} = e^{-0.5} = 0.6065$$

Defaultable bond price:

$$\overline{B}(0,T) = e^{-(r+h)T} = e^{-0.8} = 0.4493$$

but: recovery rate = 0,5: this is not compatible.

A fixed cash payoff is not compatible with a constant spread of 3%. The spread should be decreasing, and may even become **negative**.

Model suitable for *principal* of defaultable coupon bonds.

(Duffie (1998))

Decompose every defaultable security into two types of sub-claims:

- Zero-recovery claims
  that do not enter in default workout proceedings

- Par-recovery claims
  that *are* considered in default proceedings

- Defaultable securities with par recovery claims of the same amount have the same recovery in default.

- For defaultable coupon bonds:
  Par-recovery claim is the claim on the *principal* amount of the bond
  zero-recovery claims are the outstanding coupons.

Recovery in terms of risk-free bonds:

Default payoff = cB: a certain number c of default-free ZCBs

Advantage:

- Equivalent to getting only c in cash at T if there was a default before.
- Everything is expressed in $T-$forward prices.
- Can decompose into default-free ZCBs and zero-recovery ZCBs:

$$\overline{B}(t,T) = cB(t,T) + (1-c)\overline{B}_0(t,T)$$

Disadvantage: problems at constant credit spreads and long times to maturity:

In relative terms, creditors with long terms to maturity lose less than creditors with short term to maturity:

Example:

- $c =$ 50% recovery at default.
- $r =$ 5% risk-free interest rate (constant)
- $h =$ 3% credit spread (constant)
- $T_1 =$ 10 years to maturity

A creditor with $T \approx 0$ will lose 50% of the value of his portfolio.

Loss of the long-term creditor:

Default-free ZCB:	$B(0, T)$	$= 0.6065$
Defaultable ZCB:	$\overline{B}(0, T)$	$= 0.4493$
Recovery in default:	50% of 0.6065	$= 0.30325$

Loss of the long-term creditor: 33% of the value of his portfolio.

A *very* long term creditor will profit: $T = 30$

Default-free ZCB:	$B(0, T)$	$= 0.2231$
Defaultable ZCB:	$\overline{B}(0, T)$	$= 0.0907$
Recovery in default:		$= 0.1116$

A constant spread of 3% is not compatible with this recovery model. The spread should be decreasing but will not become negative.

Fractional Recovery

Fractional Recovery:

$(1-q)$ equivalent **defaultable** bonds,
defaultable bonds of the same (pre-default) rating.

Multiple Default:

- the debt is reorganized at default
- the face value of the debt is reduced by $(1-q)$
- Further defaults are possible:
  at each jump of N a default happens
- the final face value is

$$Q(T) = (1-(1-q))^{N(T)}$$

Advantage:
avoids problems of previous models.
nice representation of credit spreads.

Disadvantage:
Calculation of implied default probabilities not as easy as for recovery in terms of risk-free securities.

The usual example:

- $(1-q) = 50\%$ recovery at default.
- $r = 5\%$ risk-free interest rate (constant)
- $h = 3\%$ credit spread (constant)
- $T_1 = 10$ years to maturity

A creditor with $T \approx 0$ will lose 50% of the value of his bond.

Loss of the long-term creditor:

Default-free ZCB:	$B(0,T)$	$= 0.6065$
Defaultable ZCB:	$\overline{B}(0,T)$	$= 0.4493$
Recovery in default:	50% of 0.4493	$= 0.22465$

The long-term creditor also loses 50% of the value of his bond.

So will the very-long term creditor.

Model compatible with constant credit spreads.

Why Poisson Processes?

Ultimate goal:
A mathematical model of defaults that is realistic and tractable and useful for pricing and hedging.

Defaults are

- sudden, usually unexpected
- rare (hopefully :-)
- cause large, *discontinuous* price changes.

Require from the mathematical model the same properties.

Furthermore: Previous section

> The probability of default in a short time interval is approximately **proportional** to the length of the interval.

What is a Poisson Process?

$N(t)$ = value of the process at time t.

- Starts at zero: $N(0) = 0$
- Integer-valued: $N(t) = 0, 1, 2, \ldots$
- Increasing or constant
- Main use: marking points in time $T_1, T_2, \ldots$ the jump times of N
- Here **Default:** time of the first jump of N $\tau = T_1$
- Jump probability over small intervals proportional to that interval.
- Proportionality factor = λ

BTW: Except for the last two points, the same notation and properties apply to *Point Processes*, too.

Discrete-time approximation:

- divide $[0, T]$ in n intervals of equal length

$$\Delta t = T/n$$

- Make the jump probability in each interval $[t_i, t_i + \Delta t]$ proportional to Δt:

$$p := \mathbf{P}[\, N(t_i + \Delta t) - N(t_i) = 1 \,] = \lambda \Delta t.$$

- more exact approximation: $p = 1 - e^{-\lambda \Delta t}$

- Let $n \to \infty$ or $\Delta t \to 0$.

Important Properties

Homogeneous Poisson process with intensity λ
Jump Probabilities over interval $[t, T]$:

- No jump:

$$\mathbf{P}[\, N(T) = N(t) \,] = \exp\{-(T-t)\lambda\}$$

- n jumps:

$$\mathbf{P}[\, N(T) = N(t) + n \,] = \frac{1}{n!}(T-t)^n \lambda^n e^{-(T-t)\lambda}.$$

- Inter-arrival times

$$\mathbf{P}[\, (T_{n+1} - T_n) \in t\,dt \,] = \lambda e^{-\lambda t} dt.$$

- Expectation (locally)

$$\mathbf{E}[\, dN \,] = \lambda dt.$$

Distribution of the Time of the first Jump

T_1 time of first jump.
Distribution: $F(t) := \mathbf{P}[\, T_1 \leq t \,]$

Know probability of no jump until T:

$$\mathbf{P}[\, N(T) = 0 \,] = e^{-\lambda T}$$

= probability of $T_1 > T$. Thus

$$1 - F(t) = e^{-\lambda t}$$
$$F(t) = 1 - e^{-\lambda t}$$
$$F'(t) = f(t) = \lambda e^{-\lambda t}.$$

- T_1 is exponentially distributed with parameter λ.
- This is also the distribution of the *next* jump, given that there have been k jumps so far.
- Independently of how much time has passed so far:
  It never is 'about time a jump happened'
  or 'nothing has happened, I don't think anything will happen any more...'
- It is now $t' < t$, and no jump has happened. What is the distribution of the jump times?

$$\begin{aligned}
&\mathbf{P}[\,T_1 \leq t | T_1 \geq t'\,] \\
&= \frac{\mathbf{P}[\,T_1 \leq t\,] - \mathbf{P}[\,T_1 \leq t'\,]}{\mathbf{P}[\,T_1 \geq t'\,]} \\
&= \frac{\mathbf{P}[\,T_1 \leq t\,] - \mathbf{P}[\,T_1 \leq t'\,]}{1 - \mathbf{P}[\,T_1 \leq t'\,]} \\
&= \frac{1 - e^{-\lambda t} - 1 + e^{-\lambda t'}}{e^{-\lambda t'}} = 1 - e^{-\lambda(t-t')}
\end{aligned}$$

It's the same as before ...

Inhomogeneous Poisson Process

Imhomogeneous = with **time-dependent** intensity function $\lambda(t)$

Probability of no jumps (survival):

$$\mathbf{P}\left[\, N_T = N_t \,\right] = \exp\left\{ - \int_t^T \lambda(s) ds \right\}.$$

Probability of n jumps:

$$\mathbf{P}\left[\, N_T - N_t = n \,\right] = \frac{1}{n!} \left(\int_t^T \lambda(s) ds \right)^n e^{-\int_t^T \lambda(s) ds}.$$

Density of the time of the first jump:

$$\begin{aligned} \mathbf{P}\left[\, T_1 \in [a, b] \,\right] &= \int_a^b f(t, u) du \\ &= \int_a^b \lambda(u) e^{-\int_t^u \lambda(s) ds} du. \end{aligned}$$

Compound Poisson Process:

there is another random variable Y that is drawn at jump times.

For us:
T_1 time of default
Y recovery rate

Marker Y, distributed like $K(dy)$.
f function of $X = \sum Y_i$:

$$df = \Delta f = (f(X+Y) - f(X))dN$$

$$\mathbf{E}[\, dX \,] = \int yK(dy)\lambda dt = y^e \lambda dt$$

$$\mathbf{E}[\, df(X) \,] = \int (f(X+y) - f(X))K(dy)\lambda dt.$$

Cox Processes

default rate = Intensity of PP = credit spread.

Credit spreads are stochastic.
☞ Need stochastic intensity

- define a stochastic intensity process λ, e.g.

$$d\lambda = \mu_\lambda dt + \sigma_\lambda dW$$

- $\lambda(t)\Delta t$:
  default probability over the next time-interval $[t, t + \Delta t]$. (That's all we need to know at t.)

- at $t + \Delta t$: Intensity has changed, $\lambda(t + \Delta t) = \lambda(t) + d\lambda$ is new (local) default probability.

- Conditional on the realisation of the intensity process, the Cox process is an inhomogeneous Poisson process.

Locally:

$$P[N_{t+\Delta t} - N_t = 1] = \lambda(t)\Delta t.$$

Cox Processes: The Conditioning-Trick

The Gods are gambling in a certain sequence:

- First, the full path of the intensity $\lambda(t)$ is drawn from all possible paths for $\lambda(t)$.
- Then they take this $\lambda(t)$ and *use it as intensity for an inhomogeneous Poisson process* N. They draw the jumps of $N(t)$ according to this distribution.
- Then the information is revealed to the mortals: At time t they may only know $\lambda(s)$ and $N(s)$ for s up to t.

The Conditioning Trick:

First, pretend you knew the path of λ, what would the price be? (Depending on λ, of course.)

Then average over the possible paths of λ.

$$\mathbf{E}[\,X(N)\,] = \mathbf{E}[\,\mathbf{E}[\,X(N) \mid \lambda(t) \;\; \forall t\,]\,]$$

Properties of Cox Processes

Probability of no jumps (survival):

$$\mathbf{P}\left[\, N_T = N_t \,\right] = \mathbf{E}\left[\, \exp\left\{ - \int_t^T \lambda(s) ds \right\} \right].$$

Probability of n jumps:

$$\mathbf{P}\left[\, N_T - N_t = n \,\right] = \mathbf{E}\left[\, \frac{1}{n!} \left(\int_t^T \lambda(s) ds \right)^n e^{-\int_t^T \lambda(s) ds} \right].$$

Density of the time of the first jump:

$$\begin{aligned} \mathbf{P}\left[\, T_1 \in [a,b] \,\right] &= \int_a^b f(t,u) du \\ &= \int_a^b \mathbf{E}\left[\, \lambda(u) e^{-\int_t^u \lambda(s) ds} \,\right] du. \end{aligned}$$

The expectations always only refer to the realisation of λ.

The Jarrow / Turnbull Model

(JoF 50(1), p. 53ff.)

The discrete-time model:

- equivalent recovery model
- binomial tree for the spot interest rate
- constant intensity of default

The continuous-time model:

- one-factor HJM for risk-free term structure of interest rates
- defaultable forward rates: same volatility as default-free
- ... but at a spread over risk-free
- jump to default-free term structure at default (equivalent recovery)
- default triggered by constant intensity Poisson Process

Defaultable Zero Coupon Bond:

$$\begin{aligned}\overline{B}(t,T) &= \mathbf{E}\left[\, (1-(1-c)\mathbf{1}_{\{\tau<T\}})\beta_{t,T} \;\middle|\; \mathcal{F}_t \,\right] \\ &= (1-(1-c)e^{-\lambda(T-t)})B(t,T)\end{aligned}$$

The Duffie / Singleton Model

(JoF 52, pp. 1287 ff., 1997)

- Fractional recovery model: $1-q$ of pre-default value

- default-free short term interest rate and default intensity are stochastic:
  $dr = \ldots, \qquad d\lambda = \ldots$
  Dynamics can even be more general than Cox process dynamics.

- Defaultable zero coupon bond

$$\overline{B}(t,T) = \mathbf{E}\left[\exp\{-\int_t^T \underbrace{r(s)+q\lambda(s)}_{\tilde{r}(s)}ds\}\right]$$

short term defaultable interest rate

- This model works with all common specifications for risk-free interest rates and default intensity, e.g. square-root models like

$$dr = a(k-r)dt + \sigma\sqrt{r}dW_1$$

$$d\lambda = a'(k'-\lambda)dt + \sigma'\sqrt{\lambda}dW_2$$

The Madan / Unal Model

(Review of Derivatives Research 2, pp. 121 ff., 1998)

- Equivalent recovery model
- Default intensity depends on firm's value
- dependence on cumulated equity returns via nonlinear function
- stochastic recovery in default
- estimation using certificates of deposits

Defaultable Zero Coupon Bond:

$$\overline{B}(t,T) = \mathbf{E}\left[\,(1-(1-c)\mathbf{1}_{\{\tau<T\}})\beta_{t,T} \mid \mathcal{F}_t\,\right]$$
$$= B(t,T) - (1-c^e)\mathbf{E}\left[\,e^{-\int_t^T r(s)+\lambda(s,V(s))ds}\,\right]$$

Term Structure Modelling of Defaultable Bonds

(Schönbucher, Review of Derivatives Research 2, pp. 161 ff., 1998)

$$f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T)$$
$$\overline{f}(t,T) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T)$$

Default-free forward rates (Heath-Jarrow-Morton 1992):

$$df(t,T) = \sigma(t,T) \left(\int_t^T \sigma(t,s)ds \right) dt + \sigma(t,T)dW_Q,$$

Defaultable forward rates: (Schönbucher 1998)

$$d\overline{f}(t,T) = \overline{\sigma}(t,T) \left(\int_t^T \overline{\sigma}(t,s)ds \right) dt + \overline{\sigma}(t,T)dW_Q,$$

$$\overline{f}(t,t) = q\lambda(t) + f(t,t).$$

sufficient for absence of arbitrage.

The Parameters influencing Default Intensities

Duffee (1999), Zhang (2000)
find as influence factors for the default intensities in a Duffie / Singleton model:

- default-free interest rates:
  ambiguous results, no clear sign of correlation
- share price of firm:
  significant negative connection
- Equity / debt ratio
  even more significant than share prices

The Steps to a Credit Spread Curve

① construct the default-free ZCB curve

② adjust defaultable bond prices for recovery

③ adjust defaultable bond prices for default-free term structure

④ find a best-fitting credit spread curve

The Starting Point

Assume we are given:

- the default-free term structure of interest rates ZCB for all maturities T_i

$$B(t, T_i)$$

- a set $(\widehat{B}_1, \widehat{B}_2, \dots \widehat{B}_K)$ of defaultable, (fixed) coupon bonds of different maturity, but same issuer.

- The k-th defaultable coupon bond:
  - $\widehat{B}_k$ price at time t
  - $\widehat{C}_j^k$: the j-th coupon
  - T_j^k: the payment date of $\widehat{C}_j^k$
  - T_J^k: maturity of the defaultable coupon bond.

Adjusting for Recovery

Depending on recovery model:

- fractional recovery: no adjustment necessary
- risk-free bonds recovery: adjust:

Default payoff:
c equivalent default-free bonds.

$$\widehat{\overline{B}} = c\widehat{B} + (1-c)\widehat{\overline{B}}_0$$

$$\widehat{\overline{B}}_0 = \frac{1}{1-c}(\widehat{B} - c\widehat{\overline{B}})$$

where $\widehat{\overline{B}}_0$ is the zero-recovery defaultable bond. This is the bond we are interested in. We will consider this bond from now on.

Random recovery:
use the expectation of c instead of c.

The price of a defaultable coupon bond

The payments are discounted with the (yet unknown) defaultable ZCBs $\overline{B}(T)$

$$\widehat{\overline{B}} = \sum_{j=1}^{J} \widehat{\overline{C}}_j \overline{B}(T_j) + \overline{B}(T_J).$$

The defaultable ZCB:

$$\overline{B}(T) = P(T)B(T)$$

Knowing the term structure $P(T)$ of the spread-factors will enable us to recover the spread structure.

Equivalent Bonds

The equivalent bond:

- $\widetilde{\overline{C}}_j := \widehat{\overline{C}}(T_j)B(T_j)$: coupon at T_j.
- $\widetilde{\overline{M}} := \overline{B}(T_j)$: principal repayment at T_J
- $P(T_J)$: discount factors defining the new 'term structure of interest rates' (in fact this is really the spread structure).

The prices of the defaultable bond and the equivalent bond coincide:

$$\widehat{\overline{B}} = \sum_{j=1}^{J} \widehat{\overline{C}}_j B(T_j) P(T_j) + B(T_J) P(T_J)$$

$$= \sum_{j=1}^{J} \widetilde{\overline{C}}_j P(T_j) + P(T_J) \widetilde{\overline{M}}$$

Approximation of the Spread Curve

The spread factor for maturity T is given by

$$P(t,T) = \exp\{-\int_t^T h(t,s)\,ds\}$$

where $h(t,s)$ is the forward spread. Approximate the spread curve using basis functions $g_n(T-t)$:

$$h(t,T) = \alpha_1 g_1(T-t) + \alpha_2 g_2(T_t) + \ldots + \alpha_N g_N(T-t)$$

Spread Curve Construction:
find the weights α_n that fit best to the bond prices.

Useful notation:

$$G_n(T-t) := \int_t^T g(s-t)ds$$

$$h(t,T) = \sum_n \alpha_n G_n(T-t)$$

$$P(t,T;\alpha) = \exp\{-\sum_n \alpha_n G_n(T-t)\}$$

Good Choices of Basis Functions:

- Close to Principal Components
  g_1 'level'; g_2 'slope'; g_3 'curvature'
  should be orthogonal functions

- Localised Basis Functions:
  $g_n > 0$ only around T_n,
  'hat' or cubic (or 4th order) spline functions
  improved speed for optimisation, no structure is pre-defined, need many bonds

The Optimisation Problem

Price error of bond k
assuming weights $\alpha = (\alpha_1, \ldots, \alpha_N)$:
Difference of actual price and price resulting from α

$$\epsilon_k(\alpha) = \widehat{\overline{B}}_k - \sum_{j=1}^{J} \overline{\tilde{C}}_j^k P(T_j; \alpha) + P(T_J; \alpha)\overline{\tilde{M}}^k$$

Find the weights α that minimize the squared sum of the price errors:

$$\min_{\alpha} \sum_{k=1}^{K} \epsilon_k^2$$

- Can achieve perfect pricing when N (number of weights) = K (number of bonds).
- May be sensible to have $N \leq 4$. Need at least as many bonds as factors $N \leq K$.
- Can give weights to the individual bonds (accounting for liquidity etc.)

Filling Gaps in the Curve

Steps:

① collect similar bonds

- issuers of similar credit rating, industry, region
- use weights for degree of similarity
- adjust for risk-free term structure
- adjust for recovery

② pre-process the (few) bonds by the issuer in question ('real' bonds vs. similar bonds)

③ modify the optimisation problem:

- minimize the weighted sum of the pricing errors of the similar bonds
- weights = weights of similarity
- **constraint:** such that the 'real' bonds are priced exactly.

④ may adjust weights by sign of the pricing error: if a similar bond has a lower rating, only give weight to errors that would underprice this bond.

Incorporation of Exisiting Interest-Rate Models

Modelling Alternatives:

- Independence of defaults and interest rates:
  - Build a model for default intensities alone: Tree / Simulation.
    Same structure as for an interest-rate model.
  - Use the default-free interest rates to additionally discount the payoffs in the default intensity model.

- Correlation between defaults and interest-rates:
  - Need combined model for both: interest-rates and defaults
  - Defaults / default likelihoods add another dimension / factor to the model
  - Tree Models: Two-dimensional tree with branches to default
  - Simulation Models: Need to simulate interest-rates and defaults *at the same time*. Can increase convergence using branches to default.

Implied Default Probabilities with Correlation

Correlation between level of default-free interest rates and defaults / default probabilities.

- Without default model, it is impossible to exactly back out the implied default probabilities.
- (a): The effect of correlation is of a smaller order of magnitude. For small correlation no problem.
- (b): Need for implied default *probability* is diminished: We want the *values of payoffs* in default or survival. This can still be given.
- Viewed correctly, the implied survival probabilities can still be calculated under the appropriate *forward probability measure*.
- *Direction* of the effect of correlation can be derived intuitively.
- If correlation is important, a full model must be constructed.

Modelling Default Correlation for Portfolio Credit Risk Management

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In this Unit You Will Learn...

- ...where and when default correlation is relevant
- ...about the problems when modelling joint defaults
- ...how different models perform with respect to default correlation
- ...the intuition and methods behind JP Morgan's Credit Metrics
- ...the intuition and methods behind CSFP's Credit Risk+
- ...strengths and weaknesses of these two models

Where is Default Correlation Relevant?

- Letter of Credit backed debt,
  Credit Guarantees
- Counterparty risk
  especially in credit derivatives
  or with several counterparties
- Portfolios of defaultable bonds
- basket structures (especially beyond first-to-default)
- When trying to diversify credit risk away.

Causes of Default Correlation

- direct relationship between both parties
  e.g. one firm creditor of another
- using same inputs
- general state of a certain industry / region
- general state of the economy

Historical examples USA:

- Oil Industry:
  22 companies defaulted 1982-1986
- Railroad Conglomerates:
  One default each year 1970-1977
- Airlines:
  3 defaults 1970-1971
  5 defaults 1989-1990

- Thrifts: (Savings and Loan Crisis)
  19 defaults 1989-1990

- Casinos / Hotel Chains:
  10 defaults 1990

- Retailers:
  >20 defaults 1990-1992

- Construction / Real Estate:
  4 defaults 1992

- . . .

Default Correlations: General

Default probability of A: p_A
Default probability of B: p_B

(can be implied or estimated)

Not yet sufficient to determine

- joint default probability p_{AB}
- conditional default probabilities $p_{A|B}$ and $p_{B|A}$
- correlation ϱ_{AB} between default events $\mathbf{1}_{\{A\}}$ and $\mathbf{1}_{\{B\}}$.

Need at least one of the above to calculate the others.

Connection via Bayes' rule:

$$p_{A|B} = \frac{p_{AB}}{p_B}, \qquad p_{B|A} = \frac{p_{AB}}{p_A}$$

and definition of correlation

$$\varrho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1 - p_A)p_B(1 - p_B)}}.$$

Why Correlations?

Default correlations are very important because default probabilities are very small. ϱ_{AB} can have a much larger effect than usual (e.g. for equities etc).

Orders of magnitude:

$\varrho_{AB} = \varrho = O(1)$ is not very small,
$p_A = p_B = p \ll 1$ small.

Joint default probability:

$$p_{AB} = p_A p_B + \varrho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)}$$
$$p_{AB} \approx p^2 + \varrho p \quad \approx \varrho p$$

Conditional default probability:

$$p_{A|B} = p_A + \varrho_{AB}\sqrt{\frac{p_A}{p_B}(1-p_A)(1-p_B)}$$
$$p_{A|B} \approx \varrho.$$

Correlation **dominates** joint default probabilities and conditional default probabilities.

Market Variables and Data Sources

- Actual rating and default correlations
  Advantages: Objective, direct
  Disadvantages: Sparse data sets, long time ranges, need to aggregate

- Spread correlations
  Advantages: Reflect information in markets
  Disadvantages: Data quality problems, liquidity, availability

- Equity correlations:
  Advantages: liquid, easily available, good quality data
  Disadvantages: link to credit quality less obvious, needs a lot of pre-processing.

Missing Data

need to find structural models:

- to reduce dimensionality
  (dimension easily >1000)
- to compensate missing data
- to reduce the degrees of freedoms in the default correlation specification.

and most importantly:

The specification of *full* joint default probabilities is too complex:

For N names have

- 2^N joint default events
- N marginal distributions
  (individual default probabilities)

Note:

- This is different from normally distributed random variables (there the $N(N-1)/2$ elements of the correlation matrix are sufficient).
- Individual default modelling and correlation modelling can be separated.

Firm's Value Models

Simplest example:

- Firm A: Value of assets V_A
- Firm B: Value of assets V_B
- defaults only at T if $V_A < K$ or $V_B < K$
- V_A and V_B follow Brownian motions

$$dV_A = \sigma_A dW_A, \qquad dV_B = \sigma_B dW_B.$$

- Correlation directly: $dW_A \, dW_B = \varrho dt$
- Rewrite: $dV_A = \sigma_A dW_1$ and

$$dV_B = \sigma_B(\varrho dW_1 + \sqrt{1 - \varrho^2}\, dW_2)$$

Default Correlation Firm's Value Models

Let $K < 0$ and $T = 1$
and $V_A(0) = V_B(0) = 0$
and $\sigma_A = \sigma_B = 1$.
The final values are distributed as:

$V_A(T) \sim N(0, 1)$
$V_B(T) \sim N(0, 1)$

The individual default probabilities are

$p_A = N(K)$ and
$p_B = N(K)$

Joint probabilities: note

$$dV_B = \varrho dV_A + \sqrt{1-\varrho^2}\, dW_2$$

$$V_B(T) = \varrho V_A(T) + \sqrt{1-\varrho^2}\, \epsilon.$$

where $\epsilon \sim N(0,1)$.

Now *all* default correlations are possible:

- $\varrho = 1$ $\rightarrow$ defaults perfectly correlated
- $\varrho = -1$ $\rightarrow$ defaults mutually exclusive
- $\varrho = 0$ $\rightarrow$ defaults independent
- $\varrho > 0$ $\rightarrow$ defaults positively correlated: To a significant extent.

Example Firm's Value Model

Japanese Banks, $p_1 = 1,33\%$

Source: Dt. Bank, Units: %

p_2	FV Corr.	$p_{1\wedge 2}$	$p_{2\mid 1}$	$p_{2\mid 1}/p_2$	Default Corr.
0,02	48	0,006	0,45	23	3,54
0,02	65	0,012	0,90	45	7,24
0,26	48	0,052	3,91	15	8,32
0,26	65	0,087	6,54	25	14,32

Firm's Value Model: Uniform Portfolio

simplest Case: same default probability $p_i = p$, exposure $L_i = 1$, correlations

$\varrho_{ij} = \varrho$
$C := N^{-1}(p)$

Decompose firm's values $V_i = \sqrt{\varrho}F_0 + \sqrt{1-\varrho}F_i$
systematic factor F_0, individual factor: F_i

Loss Probability $P[L \le k] =$

$$\sum_{m=0}^{k} \binom{M}{m} \int_{-\infty}^{\infty} P[F_0 \in dy]\, P\left[F_i \le \frac{C - \sqrt{\varrho y}}{\sqrt{1-\varrho}}\right]^m \left(1 - P\left[F_i \le \frac{C - \sqrt{\varrho y}}{\sqrt{1-\varrho}}\right]\right)^{M-m} dy$$

Lucas

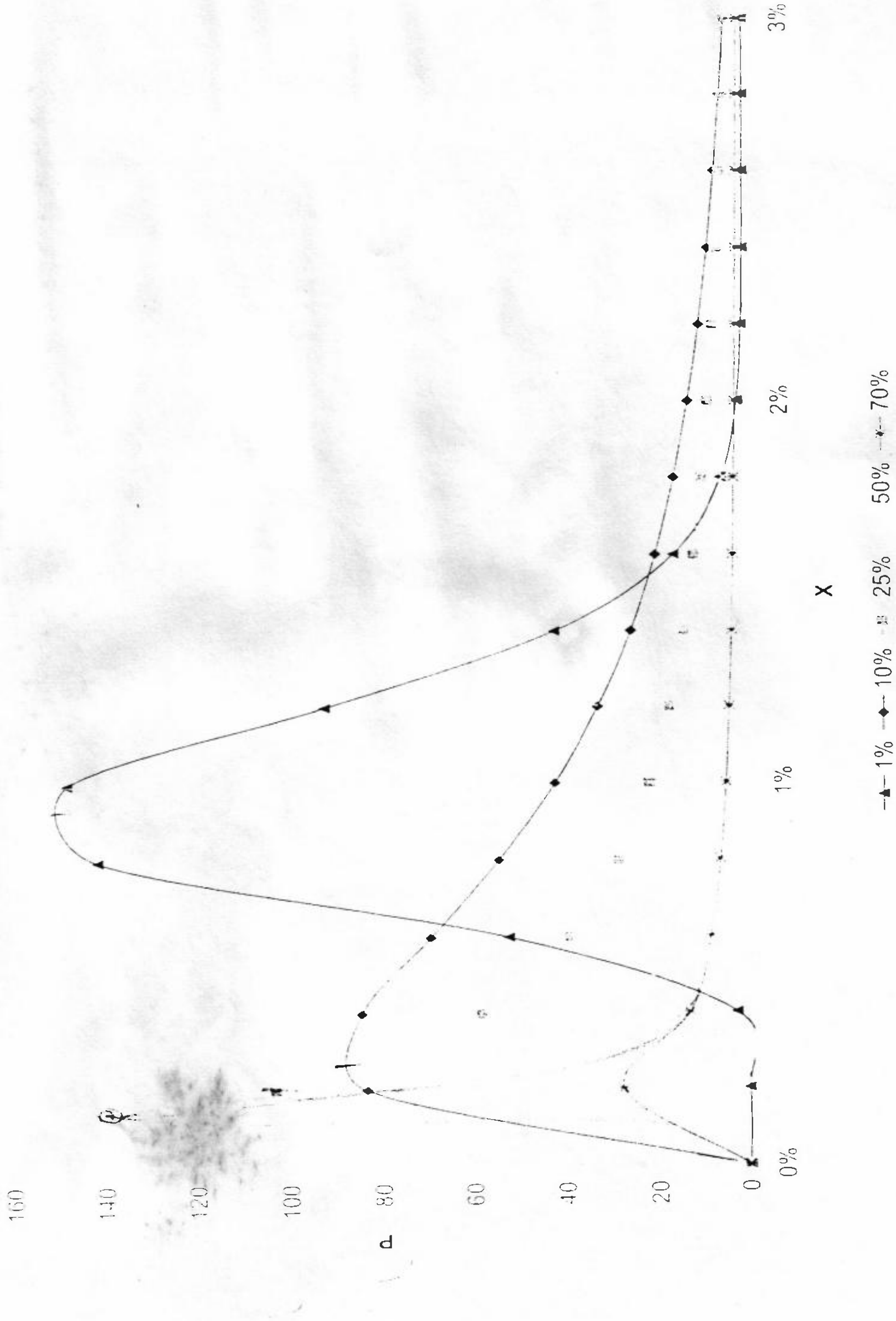
$$= \sum_{m=0}^{k} \binom{M}{m} \int_{-\infty}^{\infty} n(y)(N(\frac{C - \sqrt{\varrho}y}{\sqrt{1-\varrho}}))^m (1 - N(\frac{C - \sqrt{\varrho}y}{\sqrt{1-\varrho}}))^{M-m} dy$$

In the limit:

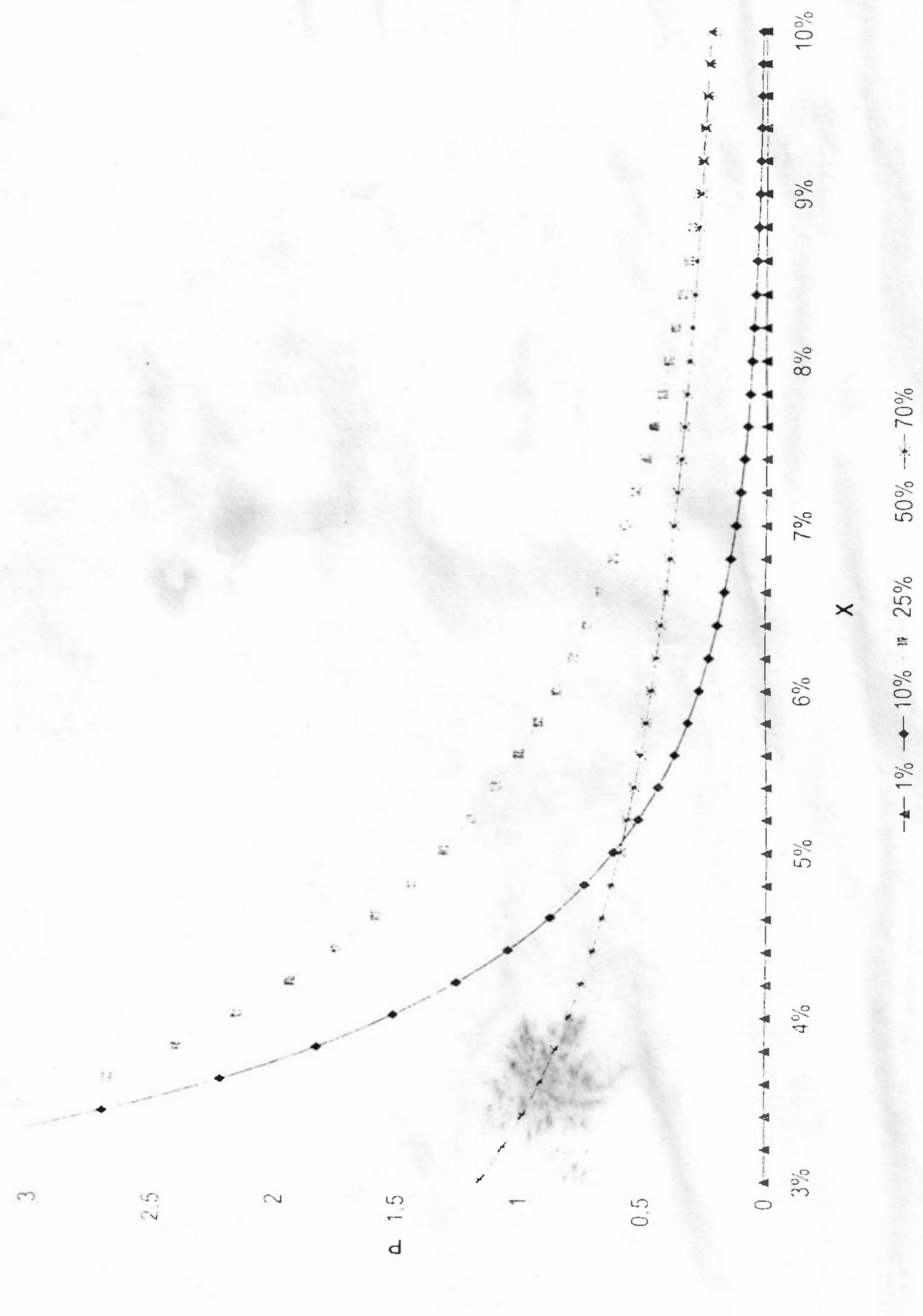
In a large portfolio of loans of the same size (=1) with the same risk (p) to firms, whose values are pairwise correlated with ϱ: The probability of losing more than a proportion x of the loans is given by:

$$P = N\left(\frac{1}{\sqrt{\varrho}}\left(N^{-1}(p) - \sqrt{1-\varrho}\, N^{-1}(x)\right)\right)$$

$$p = \sqrt{\frac{1-\varrho}{\varrho}} \exp\left\{\frac{1}{2}(N^{-1}(x)) - \frac{1}{2\varrho}\left(N^{-1}(p) - \sqrt{1-\varrho}\, N^{-1}(x)\right)^2\right\}$$



P
0
20
40
60
80
100
120
140
160
X
0%
1%
2%
3%
1%
10%
25%
50%
70%



p
3
2.5
2
1.5
1
0.5
0
3%
4%
5%
6%
7%
8%
9%
10%
X
1%
10%
25%
50%
70%

Credit Metrics

Credit risk measurement and management methodology by JP Morgan.

Features:

- Modelling framework specifically for large portfolios of defaultable bonds
- based on credit ratings and transition probabilities
- generation of loss distribution
- calculation of distribution parameters, VAR
- based on simulations, no analytical tractability
- Aim: Risk assessment, not pricing.

Methodology

Events := Changes in rating (including default)

- Probability of event: from transition matrix
- Payoff: (a) Default:
  historical recoveries for this seniority and rating
- Payoff: (b) Up/Downgrade:
  using prevailing credit spread curves for rating class.
- No attempt to link rating change probabilities to credit spreads.

This generates the loss distribution for a single bond.
Uncertainty in all inputs can be taken into account.

Default Correlations

In a portfolio need *joint* rating transition probabilities.
Rest of valuation as before.

Using a Firm's Value Model to reach probabilities of joint events

- the value of firm A's assets at T is $V_A(T)$
  the value of firm B's assets at T is $V_B(T)$
- both are normally distributed (e.g. generated by BMs)
- can infer (individually) 'rating change thresholds' from ratings transition probabilities
- now specify correlation between V_A and V_B
- calculate joint transition probabilities

Industries

To get correlations:

- first calculate correlations between different industries and countries
  from equity data

- divide firms up over different industries

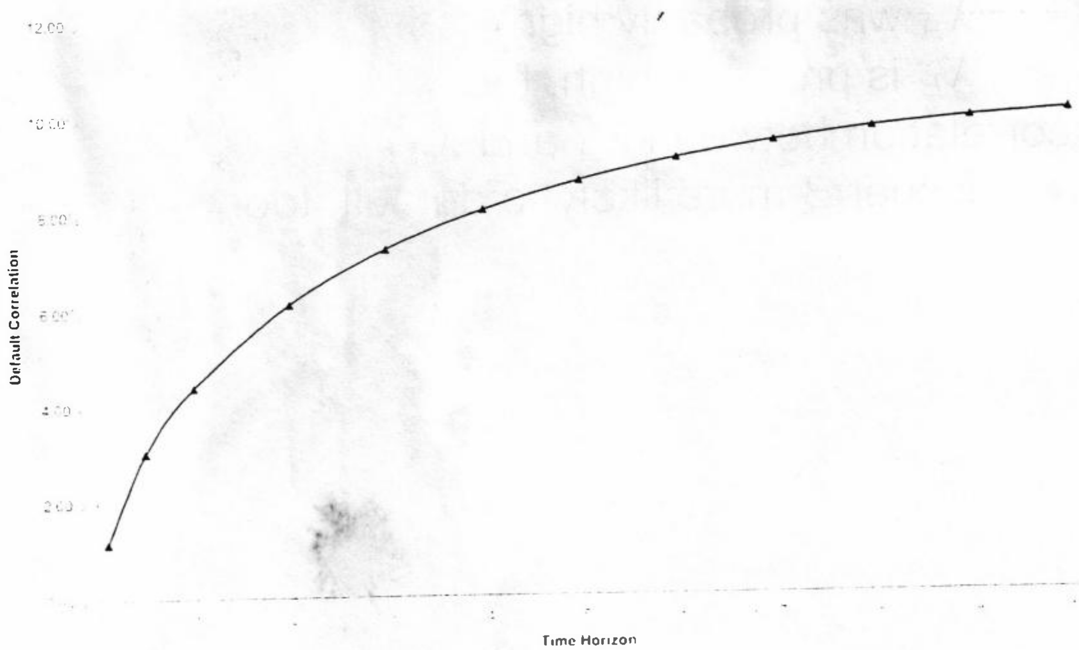
- firm's value := weighted sum of 'industry values' weighted with firm's industry participation

- get correlations between obligors.

The Influence of Maturity

The default correlation in a firm's value model depends on the time horizon.

- 2 firms
- individual 1-year def prob: 5%
- firm's value correlation: 40%



Default Correlations: Intensity Models

Issuer A: Intensity λ_A

Issuer B: Intensity λ_B

λ_A and λ_B are stochastic correlated
but given the intensities the defaults are still independent.

Correlation introduced indirectly via:

Given issuer A defaults
$\to$ λ_A was probably high
$\to$ λ_B is probably high, too
(correlation between λ_A and λ_B)
$\to$ issuer B more likely to default, too.

Example: Intensity Models

Then

$$
\begin{aligned}
p_{AB} &= \mathbf{E}\left[\, \mathbf{1}_{\{A\}}\mathbf{1}_{\{B\}} \,\right] \\
&= \mathbf{E}\left[\, \mathbf{E}\left[\, \mathbf{1}_{\{A\}}\mathbf{1}_{\{B\}} | \lambda_A \,\right] \,\right] \\
&= \mathbf{E}\left[\, (1 - e^{-\int_0^T \lambda_A ds})\mathbf{1}_{\{B\}} \,\right] \\
&= \mathbf{E}\left[\, (1 - e^{-\int_0^T \lambda_A ds})(1 - e^{-\int_0^T \lambda_B ds}) \,\right] \\
&= 1 - (1 - p_A) - (1 - p_B) + \mathbf{E}\left[\, e^{-\int_0^T (\lambda_A + \lambda_B) ds} \,\right] \\
&= p_A + p_B + \mathbf{E}\left[\, e^{-\int_0^T (\lambda_A + \lambda_B) ds} \,\right] - 1.
\end{aligned}
$$

The joint *survival* probability is

$$
\mathbf{E}\left[\, 1 - e^{-\int_0^T \lambda_A + \lambda_B ds} \,\right].
$$

Intensities add up for joint survival:
$\lambda = \lambda_A + \lambda_B$
can be viewed as the intensity of the process triggering a default time.

Most extreme correlation if both intensities are perfectly correlated: $\lambda_A = \lambda_B = \lambda$. Then:

$$p_{AB} = 2p + \mathbf{E}\left[e^{-2\int_0^T \lambda ds} \right] - 1.$$

Then the correlation is

$$\begin{aligned}
\varrho &= \frac{2p + \mathbf{E}\left[e^{-2\int_0^T \lambda ds} \right] - 1 - p^2}{p(1-p)} \\
&= \frac{\mathbf{E}\left[e^{-2\int_0^T \lambda ds} \right] - (1-p)^2}{p(1-p)} \\
&= \frac{\text{Var}\left(e^{-\int_0^T \lambda ds} \right)}{p(1-p)} \\
&\approx O(p)
\end{aligned}$$

because Var $\left(e^{-\int_0^T \lambda ds} \right)$ is of order p^2.

The default correlation that can be reached with correlated credit spreads is of the same order of magnitude as the default probabilities.
This may be too low. OK in very large portfolios.

Credit Risk+

Modelling framework developed by Credit Suisse Financial Products.

Features:

- Modelling framework specifically for large portfolios of defaultable bonds
- spreadsheet implementation
- calculation of required capital: VAR at certain levels
- generation of loss distribution
- Aim: Risk assessment, not pricing.

Credit Risk+: Modelling Approach

Defaults are triggered by a Poisson-type process. The jumps of the Poisson processes are *independent*, the only source of correlation is via the intensity.

Inputs: For each debtor need:

- Intensity = 'Default Rate'.
  either observed credit spread
  or credit rating $\rightarrow$ default rate
- default rates are random variables with
  default rate means and
  default rate volatilities
- recovery rates
- 'sector' to which the debtor belongs
  $\rightarrow$ correlation

Credit Risk+: Sectors

- each debtor is assigned a sector
- default rates are identical over sectors
- defaults are independent across sectors
- defaults within a sector are only connected via the common default rate variability.

Thus default correlation will typically be very low.

This is due to the fact that the Poisson jumps are independent. The indirect common intensity can only induce very weak correlation:

'*. . . one would expect default correlations to typically be of the same order of magnitude as default probabilities themselves.*'
(CSFP: Credit Risk+ Technical Document p. 57)
i.e. in the order of 0.5 % – 3%.

Credit Risk+ is likely to **underestimate** the default correlation and thus the worst-case losses.

Stress Events in Intensity Models

Duffie / Singleton *Simulating Correlated Defaults* (1998)

Instead of correlating default intensities λ_A, λ_B:

- joint default events
  - N'_A with λ'_A: firm A defaults *alone*
  - N'_B with λ'_B: firm B defaults *alone*
  - N'_{AB} with λ'_{AB}: firms A *and* B default *together*
- default of each subportfolio is directly triggered by a jump process of its own
- need consistent specification of joint and individual default trigger processes

Consistency:

$$\lambda_A = \lambda'_A + \lambda'_{AB}$$

$$\lambda_B = \lambda'_B + \lambda'_{AB}$$

Question:
How to distribute the intensity weight over the subsets?!

Answer:
Need structural default model to fit.
E.g. joint probabilities from a firm's value model.

Conclusion

- The effect of default correlation can be very large
(sometimes as large as the default probabilities themselves)
- Default correlation especially important for worst cases and VAR.
- Take care when applying intensity-based models: Spread correlation does not lead to similar default correlation.
- stress-testing for default correlation is a good idea.
- Structural models of default correlation are needed to fill gaps in available data.

Risk
CONFERENCES
A DIVISION OF RISK PUBLICATIONS

Practical application/bank perspective of credit theory to model credit risk

Joseph Pimbley
SUMITOMO BANK CAPITAL MARKETS

Practical Application of Credit Theory for Bank Loan Portfolios

Risk Training Course

Boston

12 June 2000

OUTLINE

- Risk-Adjusted Return on Capital
- BIS Regulatory Rules
- Adding Shareholder Value
- Next Generation Bank Business Model

RAROC and SVA *

Risk-adjusted return on capital:

$$RAROC = \frac{\text{Revenue} - \text{Expenses} - \text{Expected Loss}}{\text{Capital}}$$

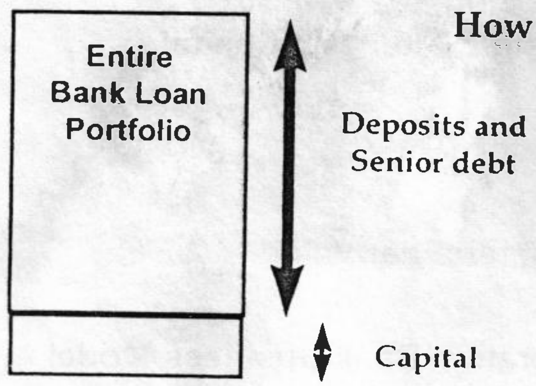
for entire portfolio OR individual obligor

Shareholder value:

$$SVA = \text{Capital} * (\text{RAROC} - \text{Cost of Capital})$$

* **(without taxes !)**

How much capital ?



One answer: Deposit "tranche" should be Aa3 credit quality as measured by (*Moody's*) expected loss

How much capital ?

- Aa3 expected loss is 0.18 bps pa

- Other view: specify Aa3 <u>default probability</u> of 0.30 bps pa

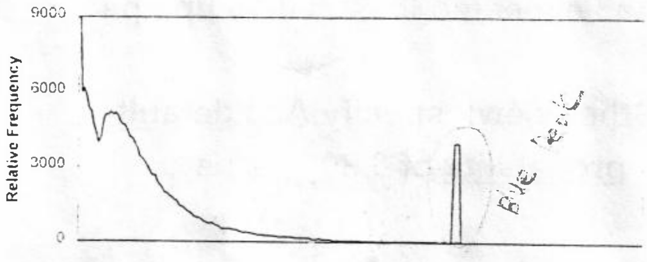
Very simple issue .... but quite important !

Capital Calculation

- Simulate correlated defaults for all obligors to get the "loss density function"

- Determine the capital level to give the target expected loss

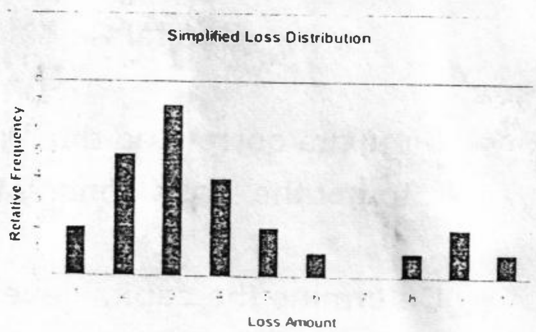
Loss Density Function



Blue Peak is Capital + Expected Loss

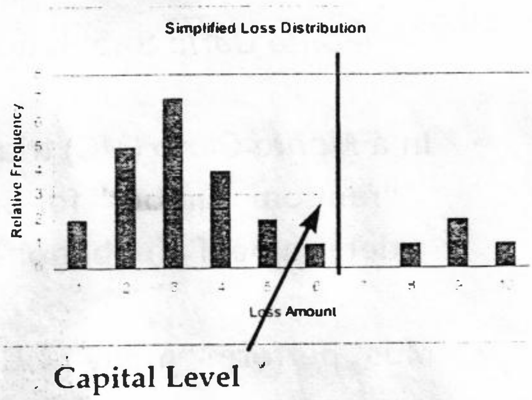
Probability of loss is 0.33 % per annum

Loss Amount	Number of Trials
1	2
2	5
3	[illegible]
4	4
5	[illegible]
6	[illegible]
7	0
8	[illegible]
9	[illegible]
10	[illegible]
Total trials	25



Example: Imagine a simplified loss distribution. There are 25 Monte Carlo trials that produce loss values from 1 to 10.

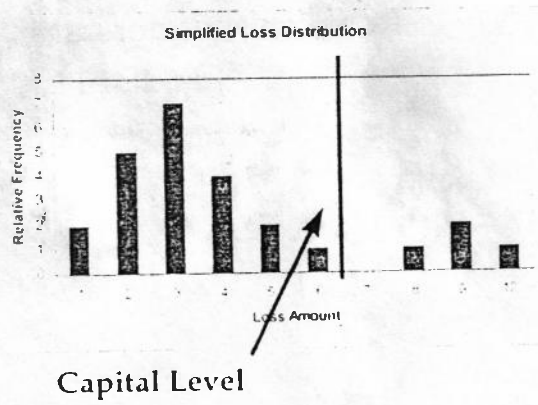
Loss Amount	Number of Trials
1	2
2	5
3	7
4	4
5	2
6	1
7	0
8	1
9	2
10	1
Total trials	25



If the capital level is 6.5, then the probability of default is the number of trials that exceed 6.5 divided by the total number of trials:
4/25 = 16%

SBCM *Capital Calculation*

Loss Amount	Number of Trials
1	2
2	5
3	7
4	4
5	2
6	1
7	0
8	1
9	2
10	1
Total trials	25



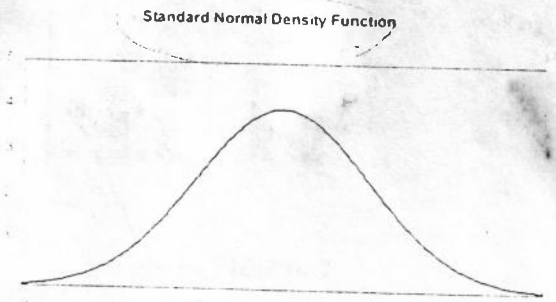
If the capital level is 6.5, the expected loss is

$$\frac{1}{25}\sum_{i=1}^{25}\max(0, L_i - 6.5) = \frac{10}{25} = 0.40$$

Monte Carlo Simulation for Capital

- In a *Monte Carlo* (MC) trial, generate a "random number" for each obligor to determine if an obligor defaults
- Must perform "many" MC trials to get an accurate result

Random Numbers are "chosen" from this distribution

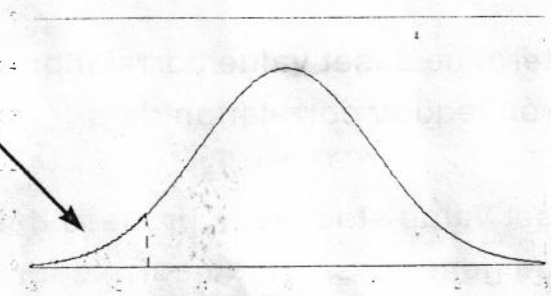


So .... the random numbers are both positive and negative and less and less likely to occur at values away from zero

If an obligor has a default probability of 5% ....

Standard Normal Density Function with "5% Tail"

5% probability in this region



.... we say the obligor has defaulted if the random number is in the tail region shown (less than -1.64)

(i) asset value (ii) ~~random numbers~~ = borrower numbers.

Same details

→ Default probability P_D.

(*)

SBCM *CapitalCalculation*

Default Probability Assignment

Rating	Default Probability
Aaa	0.00005[illegible]
Aa1	0.00057[illegible]
Aa2	0.00130[illegible]
Aa3	0.00302[illegible]
A1	0.00591[illegible]
A2	0.01087[illegible]
A3	[illegible]
Baa1	[illegible]
Baa2	[illegible]
Baa3	0.42000[illegible]
Ba1	[illegible]
Ba2	[illegible]
Ba3	[illegible]
B1	[illegible]
B2	[illegible]
B3	[illegible]
Caa	[illegible]

Default Correlation Assignment: General Industry Method

- Determine asset value correlations from equity correlation data
- Asset value stochastic process then the generates default correlations
- Some assumptions are weak and nobody has quantified the accuracy of this view

Default Correlation Assignment: "Modified Moody's Method"

- There are 32 industry groups by sector
- Assume a fixed <u>inter</u> - industry correlation

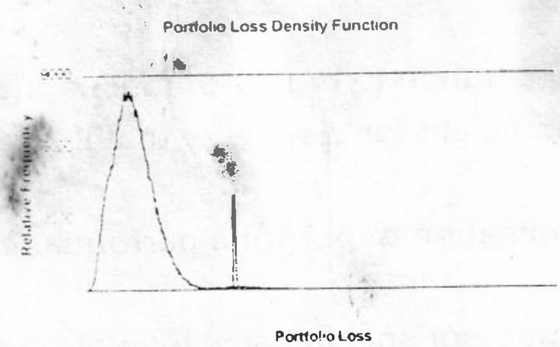
  and a fixed <u>intra</u> - industry correlation

Default Correlation Assignment: "Moody's Method"

- There are 32 industry groups by sector

- For obligors in the same industry, reduce the true number of obligors to a smaller value in an *ad hoc* manner and treat this reduced number as zero correlation

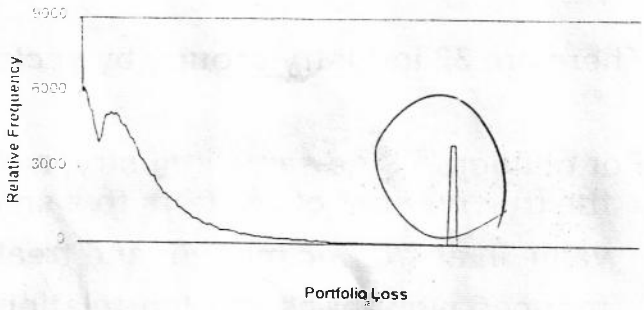
How important is correlation ?



Blue Peak is Capital + Expected Loss
Very important !
Probability of loss is 1.00 % per annum

Loss Density Function

Portfolio Loss Density Function



Blue Peak is Capital + Expected Loss

Probability of loss is 0.33 % per annum

Value of Capital Modeling

- Demonstration to regulator that capital is sufficient (or discovery that it is not !)
- Determination of portfolio performance
- Evaluation of specific new loan or hedge
- Evaluation of new strategies

RAROC and SVA *

Risk-adjusted return on capital:

$$RAROC = \frac{\text{Revenue} - \text{Expenses} - \text{Expected Loss}}{\text{Capital}}$$

for entire portfolio OR individual obligor

Shareholder value added:

$$SVA = \text{Capital} * (\text{RAROC} - \text{Cost of Capital})$$

*** (without taxes !)**

Risk Contribution

- Given the portfolio RAROC, what are the individual RAROCs of all obligors ?
- Given the total capital for the portfolio, how much is due to each of the portfolio obligors ?

Risk Contribution: Industry Solution (?)

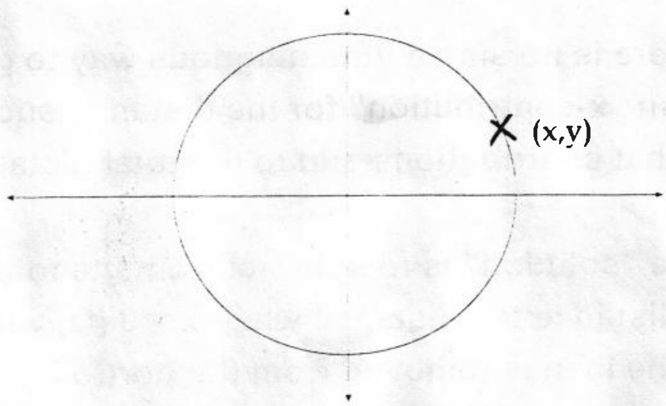
- Apportion capital to each loan such that the sum of all “contributions” is the total capital ... a “quadratic form”

$$\frac{\sum_{k} \rho_{ki} E_i E_k \sigma_i \sigma_k}{\sum_{k,i} \rho_{ki} E_i E_k \sigma_i \sigma_k} C$$

Risk Contribution: Industry Solution (?)

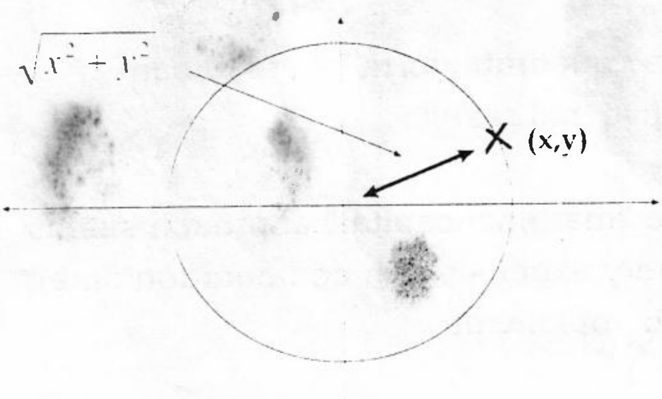
- Problem: this quadratic form (KMV) seems to be just a guess ... there’s no reason it “should” be right other than a “differential optimization” argument of KMV
- The more obvious solution is to run the capital calculation, take out one particular obligor, and run the calculation again. The difference is the “contribution” of the obligor.

Risk Contribution: Industry Solution (?)



What's the distance from the origin to the point (x,y) ?

Risk Contribution



What are the "contributions" of x and y ?

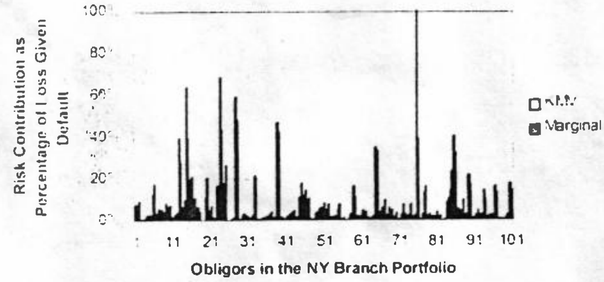
Risk Contribution: Solution (?)

- There is no single, unambiguous way to get an "x-contribution" for the distance such that contributions add to the total distance
- The "solution" is to subtract from the total distance the distance when x = 0 (ie, when the loan is removed from the portfolio)

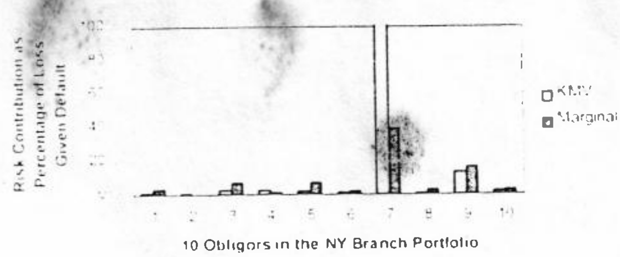
Risk Contribution: Why such a big deal ?

- The "quadratic form" method can give bad results
- The "marginal capital" approach seems very expensive (in computation time) to implement
- Portfolio decisions depend on the outcome !

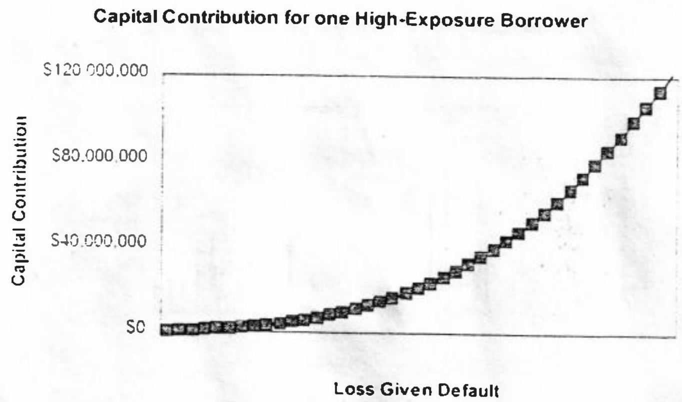
Risk Contribution (100 loans)



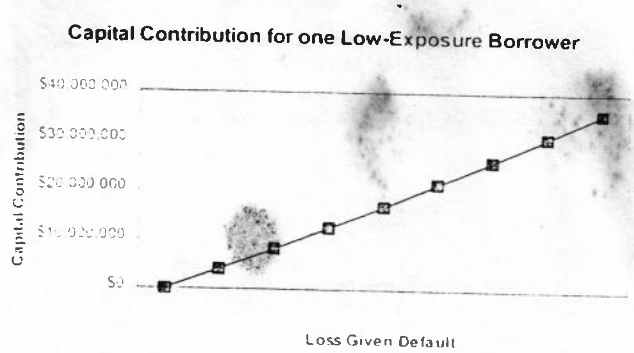
Risk Contribution (10 loans)



Large Exposure (Low credit risk)



Small Exposure (High credit risk)

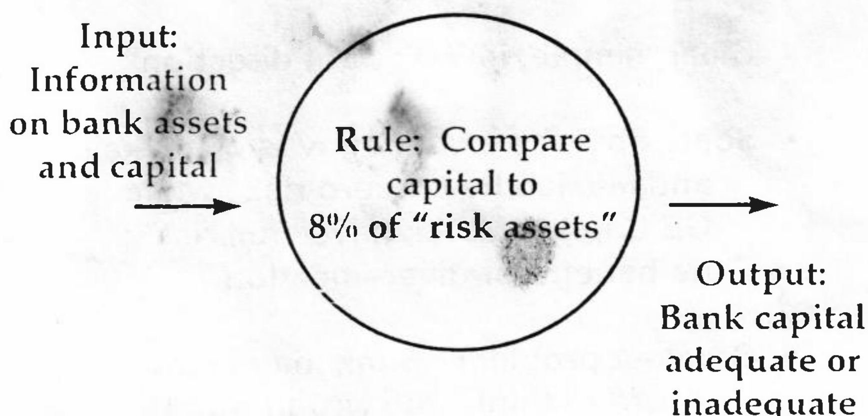


BIS Capital Adequacy

* 1988 Basle Accord defines the amount and type of capital a bank must have for international operations
* Goal is to protect depositors (?)
* Is it a model ?

SBCM *BIS Regulatory Rules*

Example: BIS capital adequacy



(Some extra complication regarding "type" of capital)

Some BIS "risk weightings"

0%	20%	100%
OECD gov't obligations	OECD bank obligations	Everything else .... eg, corporate obligations
Unfunded commitments < 1 year		
June 1999 proposal to revise		

50%
Unfunded commitments > 1 year

BIS Capital Adequacy: a good model ?

- Clear, simple, in the "right direction"
- Some obvious errors: gov'ts of Turkey and Mexico have "zero risk" while GE Capital, Merck have "full risk"; no benefit for diversification
- Greatest problem: Bank managers "learn to think" the wrong way !

BIS Capital Adequacy Problem

The model is so powerful - due to its simplicity and the importance of regulatory approval - that it is the primary (only?!) risk/return measurement tool at second-tier banks .... it creates a culture of poor decision-making

One Lesson: models can measure business performance and therefore "drive" the business (for better or worse)

BIS Capital Adequacy: Potential Improvements

* More and better risk designations
* External or internal ratings instead of OECD and bank classifications
* Short, unfunded commitments will have 20% risk weighting (why?)
* Securitizations treated differently
* No recognition of diversification

Return on regulatory capital should not play a significant role in the guidance of a major bank

Example: Bank profitability

Hypothetical situation: You're president of a large bank. You need to specify a goal, a strategy to achieve the goal, and a measurement to determine success or failure. Here are some possibilities:

Goal/Measurement #1

Goal:	"Have a good year"
Measurement:	< Unspecified >
Problem:	Too vague, no measurement