# On certain properties of trajectories of integer-valued random walks

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## 1. Introduction

This diploma paper is devoted to studying particular integer-valued random walks with discrete time and their trajectories. Specifically, two properties are of interest here: whether the random walk hits the lower half-plane (or, in other words, whether the ruin occurs over infinite time), and whether the random walk returns to zero infinitely often. We start studying both properties by considering the simplest symmetrical random walk, and then move on to more general cases.

Let us consider each of the properties in more detail. Studying the probability of an integer valued random walk ruining over infinite time was motivated by the article by Newman [3]. In this article the author examined an infinite series of coin tossing and the probability of the ratio of heads to tails deviating by a given amount from its asymptotical value (i.e. one). A thorough examination leads us to a very natural reformulation of this problem in terms of the ruin probabilities of an integer valued random walk. This is the form of the task that will be employed throughout this paper.

The second property considered is returning to zero infinitely often. More specifically, instead of considering all the moments of return, we count only those returns that occur at moments of time belonging to a given subset of integers (the subset is obviously infinite). This problem triggered our interest after studying the article by Chung and Erdos [6]. This article provides a generalization of the well-known Borel-Cantelli lemma that does not require the independence of random events (but obviously still requires certain additional properties). The authors illustrate possible uses of their result by applying it to two interesting examples, which leads to a simple way of examining whether the random walk returns to zero infinitely often. However, there appears to be a certain discrepancy in the way the lemma is applied, and we were not able to remove it. This motivated us to probe the problem using simpler methods.

## 2. Ruin probability

In 1960, D. Newman considered [3] the following problem.

A coin is tossed repeatedly, and the numbers of heads and tails are counted. Let $H_t$ and $T_t$ be the number of heads and tails respectively after $t$ tosses. The aim of [6] was finding the probability that

$$\max_{t=0,1,2,\ldots} \left(\frac{H_t}{T_t}\right) \leq \frac{a}{b}$$

The problem described is a special case of a more general problem concerning the ruin probability of an integer-valued random walk. Consider the following risk model:

$$U_t(u) = u + \sum_{i=1}^{t} (b - X_i), \quad t = 0, 1, 2, \ldots,$$

where $u$ is the initial capital, $b$ is the insurance premium and $X_1, X_2, \ldots$ are the sizes of the claims at the corresponding time moments $t = 1, 2, \ldots$ We restrict ourselves to the case of $X_1, X_2, \ldots$ being independent, identically distributed and integer-valued, with their values belonging to a finite set $a_1 > .. > a_m \geqslant 0$ with respective probabilities $q_1, \ldots, q_m$, $q_1 + \ldots + q_m = 1$. It is assumed below that $a_m = 0$, which can be easily achieved by choosing $b$.

Of interest is the probability that $U_t \geq 0$ holds for every $t = 0, 1, 2, \ldots.$ Fix $b, a_1, ..., a_m, q_1 .., q_m$ and denote that probability, as a function of $u$, as $p_u$.

$$p_u = \mathbf{P}\left\{\min_{t \geq 0} U_t(u) \geqslant 0\right\}$$

**Remark 2.1.** Setting $m = 2, q_1 = q_2 = \frac{1}{2}, u = 0$ and considering different $b, a_1, a_2$ we obtain the probability studied by D. Newman.

**Proposition 2.2.** *If $u < v$ then*

$$0 \leq p_u \leq p_v \leq 1.$$

**Proposition 2.3.** *For every $u < 0$ we have $p_u = 0$.*

**Theorem 2.4.** *If*

$$b - a_1 q_1 - \ldots - a_m q_m = b - EX_1 \leq 0, \tag{1}$$

*then $p_u = 0$ for every $u$.*

*Proof.* If we have a strict inequality in (1), the statement follows directly from the law of large numbers. If the left side of (1) equals zero, $U_t$ is a random walk with a zero mean and independent identically distributed steps. That means that again $\mathbf{P}\{\exists t : U_t = 0\} = 1$. $\square$

**Proposition 2.5.** *If $b \geq a_1$, then $p_u = 1$ for every $u \geq 0$.*

**Theorem 2.6.** *For every $u \geq 0$ the following recurrence holds:*

$$p_u = q_1 p_{u+b-a_1} + \ldots + q_m p_{u+b-a_m}.$$

We assume below that $a_1 > b > 0$, otherwise proposition 2.5 and theorem 2.4 give us the answer immediately.

By $k$ and $r$ we denote the numbers such that $a_1 - b \geq a_2 - b \geq ... \geq a_k - b > 0 > a_{m-r+1} - b \geq ... \geq a_m - b = -b$.

We will use the following notation for convenience:

$$c_1 = a_1 - b, ..., c_k = a_k - b; \qquad c_i > 0$$

$$b_1 = b - a_m, ..., b_r = b - a_{m-r+1}; \qquad b = b_1 \geq b_i > 0$$

From our assumptions, $k \geq 1, r \geq 1$. Now theorem 2.6 can be stated as follows:

$$p_u = q_1 p_{u-c_1} + ... + q_k p_{u-c_k} + (q_{k+1} + ... + q_{m-r}) p_u + q_{m-r+1} p_{u+b_r} + ... + q_m p_{u+b_1} \tag{2}$$

**Theorem 2.7.** *If*

$$-c_1 q_1 - ... - c_k q_k + b_r q_{m-r+1} + ... + b_1 q_m = b - EX_1 > 0, \tag{3}$$

*then* $\lim_{u\to\infty} p_u = 1$.

*Proof.* Follows from the law of large numbers. □

We will further assume that $b - EX_i = -c_1 q_1 - ... - c_k q_k + b_r q_{m-r+1} + ... + b_1 q_m > 0$.

The sequence $\{p_n\}_{n=0}^{\infty}$ is non-negative, monotone and bounded by 1. Consider the generating function

$$f(x) = \sum_{u=0}^{\infty} p_u x^u. \tag{4}$$

The generating function of the distribution of $X_i$ will be denoted by $Q_X(z) = \sum_{j=1}^{m} q_j z^{a_j}$. The characteristic polynomial of the recurrence sequence (2) is

$$P(x) = -x^b + Q_X(z) = -x^b + q_1 x^{b+c_1} + ... + q_k x^{b+c_k} + (q_{k+1} + ... + q_{m-r}) x^b + q_{m-r+1} x^{b-b_r} + ... + q_m. \tag{5}$$

**Theorem 2.8.**

$$\begin{gathered} P(x)f(x) = q_m(p_0 + p_1 x + ... + p_{b_1-1} x^{b_1-1}) + \\ q_{m-1}(p_0 x^{b_1-b_2} + p_1 x^{b_1-b_2+1} + ... + p_{b_2-1} x^{b_1-1}) + ... + \\ q_{m-r+1}(p_0 x^{b_1-b_r} + p_1 x^{b_1-b_r+1} + ... + p_{b_r-1} x^{b_1-1}) = \\ p_0(q_m + q_{m-1} x^{b_1-b_2} + ...) + p_1(q_m x + q_{m-1} x^{b_1-b_2+1} + ...) + ... + p_{b_1-1} q_m x^{b_1-1}. \end{gathered}$$

*Proof.* Consider the product of the characteristic polynomial (5) and the generating function (4). By (2), the coefficients corresponding to $x^n$ for $n \geq b$ will be reduced. The other terms have a degree less than or equal to $b - 1$ and they are listed on the right side of the theorem statement. Note that $b = b_1$, since $a_m = 0$. □

**Theorem 2.9.** *There is a polynomial $q(x)$ of degree $\leq \mu_> - c_1$ such that*

$$(x-1)f(x) = \frac{q(x)}{(x-x_1)^{\mu_1}...(x-x_s)^{\mu_s}},$$

*where $x_1, ..., x_s$ are all the roots of the characteristic polynomial* (5) *with modulus greater than one, and $\mu_1, ..., \mu_s$ are their respective multiplicities with $\mu_> = \mu_1 + ... + \mu_s$.*

*Proof.* Denote the polynomial on the right side in theorem 2.8 by $p_{b_1-1}(x)$. Factor the characteristic polynomial of the recurrence sequence (4) and consider the following three groups of its roots: those which have modulus greater than 1, equal to 1 or less than 1. Since $q_1 + ... + q_m = 1$, one of the roots is $x = 1$. Thus we obtain:

$$p(x) =$$

$$(x-x_1)^{\mu_1}...(x-x_s)^{\mu_s}(x-x_{s+1})^{\mu_{s+1}}...(x-x_{s+r})^{\mu_{s+r}}(x-x_{s+r+1})^{\mu_{s+r+1}}...(x-x_{s+r+l})^{\mu_{s+r+l}}(x-1)^{\mu},$$

where $|x_i| > 1$ for $i = 1, ..., s$, $|x_i| = 1$ for $i = s+1, ..., s+r$ and $|x_i| < 1$ for $i = s+r+1, ..., s+r+l$. Since $p_n \leq 1$ for every $n$, the series (3) converges at all internal points of the unit circle. This gives us:

$$p_{b_1-1}(x) = (x-x_{s+r+1})^{\mu_{s+r+1}}...(x-x_{s+r+l})^{\mu_{s+r+l}} p_{b_1-1-\mu_{s+r+1}-...-\mu_{s+r+l}}(x)$$

for some polynomial $p_{b_1-1-\mu_{s+r+1}-...-\mu_{s+r+l}}(x)$. Note that if the number $b_1 - 1 - \mu_{s+r+1} - ... - \mu_{s+r+l}$ is negative, the polynomial is identically zero.

By Chakalov's theorem [2, theorem 3.3.15], the generating function $f(x)$ has exactly one pole on the unit circle, $x = 1$ (of order 1).

This means that

$$p_{b_1-1-\mu_{s+r+1}-...-\mu_{s+r+l}}(x) =$$
$$(x-1)^{\mu-1}(x-x_{s+1})^{\mu_{s+1}}...(x-x_{s+r})^{\mu_{s+r}} p_{b_1-\mu_{s+1}-\mu_{s+2}-...-\mu_{s+r}-\mu_{s+r+1}-...-\mu_{s+r+l}}(x)$$

for some polynomial $p_{b_1-\mu-\mu_{s+1}-...-\mu_{s+r}-\mu_{s+r+1}-...-\mu_{s+r+l}}(x)$.

To sum up, we have proved that

$$(x-1)f(x) = \frac{q(x)}{(x-x_1)^{\mu_1}...(x-x_s)^{\mu_s}}$$

for some polynomial $q(x)$ of degree $\leq b_1 - \mu - \mu_{s+1} - ... - \mu_{s+r} - \mu_{s+r+1} - ... - \mu_{s+r+l} = \mu_1 + ... + \mu_s - c_1$. $\square$

**Theorem 2.10.** *Within the same assumptions $k \geq 1, r \geq 1$ u $-c_1q_1 - ... - c_kq_k + b_rq_{m-r+1} + ... + b_1q_m > 0$, we have*

$$\lim_{x\to 1-}(x-1)f(x) = -\lim_{n\to\infty} p_n \quad (= -1).$$

**Theorem 2.11.** *The equation*

$$x^{b_1} = q_1x^{b_1+c_1} + ... + q_kx^{b_1+c_k} + (q_{k+1} + ... + q_{m-r})x^{b_1} + q_{m-r+1}x^{b_1-b_r} + ... + q_m$$

*has no more than $c_1$ roots outside the unit circle (counting multiplicity).*

*Proof.* Consider the equation

$$x^{b_1} = \varepsilon(q_1 x^{b_1+c_1} + ... + q_k x^{b_1+c_k} + (q_{k+1} + ... + q_{m-r})x^{b_1} + q_{m-r+1}x^{b_1-b_r} + ... + q_m).$$

For $0 \le \varepsilon < 1$, this equation does not have any roots of modulus 1. By Rouche's theorem, it has exactly $b_1$ roots inside the unit circle for $0 \le \varepsilon < 1$. Consequently, the other $c_1$ roots are outside of the unit circle.

Thus applying Rouche's theorem to the limit value of the parameter $\varepsilon = 1$ we obtain that the equation has no more than $c_1$ roots outside the unit circle. □

**Theorem 2.12.** *The equation*

$$x^{b_1} = q_1 x^{b_1+c_1} + ... + q_k x^{b_1+c_k} + (q_{k+1} + ... + q_{m-r})x^{b_1} + q_{m-r+1}x^{b_1-b_r} + ... + q_m$$

*has exactly* $c_1$ *roots outside of the unit circle, exactly* $\gcd(c_1, b_1)$ *different roots of multiplicity* 1 *and modulus* 1 *and exactly* $b_1 - \gcd(c_1, b_1)$ *roots inside the unit circle. All the roots are counted considering multiplicity.*

*Proof.* As follows from theorem 2.11, the equation in consideration has no more than $c_1$ roots outside of the unit circle. If it has less than $c_1$ roots, then the degree of the polynomial $q(x)$ in the theorem 2.9 is less than zero, meaning that the polynomial itself equals zero. We obtained a contradiction with theorem 2.10, proving the first part of the theorem.

Let $x_i$ be such a root of the polynomial (5) that $|x_i| = 1$. We have a following chain of inequalities: $1 = |x_i^{b_1}| = |q_1 x_i^{b_1+c_1} + ... + q_k x_i^{b_1+c_k} + (q_{k+1} + ... + q_{m-r})x_i^{b_1} + q_{m-r+1}x_i^{b_1-b_r} + ... + q_m| \le |q_1 x_i^{b_1+c_1}| + ... + |q_k x_i^{b_1+c_k}| + |(q_{k+1} + ... + q_{m-r})x_i^{b_1}| + |q_{m-r+1}x_i^{b_1-b_r}| + ... + |q_m| = q_1 + ... + q_k + (q_{k+1} + ... + q_{m-r}) + q_{m-r+1} + ... + q_m = 1$.

Which means that $x_i^{b_1+c_1} = 1$, ..., $x_i^{b_1+c_k} = 1$, ..., $x_i^{b_1} = 1$, $x_i^{b_1-b_r} = 1$, ... and $x_i^{b_1} = 1$. Or, equivalently, $x_i^d = 1$, where $d = \gcd(c_1, ..., c_k, b_1, ..., b_r)$. Clearly, the inverse is also valid. If $x_0^d = 1$, then $x_0$ is a root of the characteristic polynomial. Now we shall prove that $x_0$ has a multiplicity of one. Consider $x_0 p'(x_0)$. By direct computation,

$$\begin{aligned} p'(x) = &-b_1 x^{b_1-1} + (b_1 + c_1)q_1 x^{b_1+c_1-1} + ... + (b_1 + c_k)q_k x^{b_1+c_k-1} \\ &+ b_1(q_{k+1} + ... + q_{m-r})x^{b_1-1} + (b_1 - b_r)q_{m-r+1}x^{b_1-b_r-1} + ... + (b_1 - b_2)q_{m-1}x^{b_1-b_2-1}. \end{aligned}$$

Thus

$$\begin{aligned} x_0 p'(x_0) = &-b_1 + (b_1 + c_1)q_1 + ... + (b_1 + c_k)q_k + b_1(q_{k+1} + ... + q_{m-r})(b_1 - b_r)q_{m-r+1} + \\ &+ ... + (b_1 - b_2)q_{m-1} = -b_1 q_m + c_1 q_1 + ... + c_k q_k - b_r q_{m-r+1} - ... - b_2 q_{m-1} < 0. \end{aligned}$$

Meaning that the multiplicity of $x_0$ is one.

The other $b_1 - d$ roots obviously have to be inside the unit circle. □

**Theorem 2.13.**

$$(1-x)f(x) = \frac{(1-x_1)^{\mu_1}...(1-x_s)^{\mu_s}}{(x-x_1)^{\mu_1}...(x-x_s)^{\mu_s}},$$

*where* $x_1, ..., x_s$ *are all the roots of the characteristic polynomial* (5) *with modulus greater than one, and* $\mu_1, ..., \mu_s$ *are their respective multiplicities.*

*Proof.* By theorem 2.9, the polynomial $q(x)$ is a constant. The value of this constant is obtained from theorem 2.10. $\square$

**Example 2.14.** $p(x) = -q_1 x^{2a} + x^a - q_2$ for $q_1 + q_2 = 1$ and $0 < q_1 < q_2$

The polynomial above can be factored easily:

$$p(x) = (x^a - 1)(-q_1 x^a + q_2).$$

By theorem 2.13 we have

$$(x-1)f(x) = \frac{q_2 - q_1}{q_1 x^a - q_2}.$$

The probabilities $p_0, ..., p_n, ...$ can now be computed easily. For example, $p_0 = 1 - \frac{q_1}{q_2}$.

**Example 2.15.** $p(x) = -q_1 x^{3a} + x^{2a} - q_2$ where $q_1 + q_2 = 1$ and $0 < q_1 < 2q_2$.

The polynomial above can be factored easily:

$$p(x) = (x^a - 1)(-q_1 x^{2a} + q_2 x^a + q_2).$$

By theorem 2.13 we have

$$(x-1)f(x) = \frac{q_2 + \sqrt{q_2^2 + 4q_1 q_2} - 2q_1}{2q_1 x^a - q_2 - \sqrt{q_2^2 + 4q_1 q_2}}.$$

The probabilities $p_0, ..., p_n, ...$ can now be computed easily. For example, $p_0 = 1 - \frac{2q_1}{q_2 + \sqrt{q_2^2 + 4q_1 q_2}}$.

# 3. Ruin probability in the multidimensional case

Consider a multidimensional random walk starting at a point $(u_1, ..., u_d)$. A ruin is said to have occurred if at any moment of time for some $j$ we have $u_j(t) < 0$. To give an example, consider a company having multiple independent resources and ruining if it runs out of any single resource.

We will denote the ruin probability under consideration by $p_{u_1,...,u_d}$ The corresponding generating function is

$$f(x_1, ..., x_d) = \sum_{n_1=0}^{\infty} ... \sum_{n_d=0}^{\infty} p_{n_1,...,n_d} x_1^{n_1} ... x_d^{n_d} \tag{1}$$

Similarly to unidimensional case, the steps of the random walk $X_i$ are independent, integer-valued and finite. Let $(a_{11}, ..., a_{1d}), ..., (a_{m1}, ..., a_{md})$ be the set of possible values taken by $X_i$, with corresponding probabilities $q_1, ..., q_m$.

**Theorem 3.1.** *For $u_1 \geq 0$, ..., $u_d \geq 0$ a following recurrence holds*

$$p_{u_1,...,u_d} = q_1 p_{u_1 - a_{11},...,u_d - a_{1d}} + ... + q_m p_{u_1 - a_{m1},...,u_d - a_{md}} \tag{2}$$

The characteristic polynomial of the recurrence (2) is

$$p(x_1, ..., x_d) = (-1 + q_1 x_1^{a_{11}} ... x_r^{a_{1d}} + ... + q_m x_1^{a_{m1}} ... x_d^{a_{md}}) \cdot x_1^{c_1} ... x_d^{c_d}, \tag{3}$$

where $c_k = -\min\limits_{i \in 1..m} (a_{ik})$. The factor $x_1^{c_1} ... x_d^{c_d}$ has to be introduced to avoid terms with negative degrees.

**Theorem 3.2.** *There exists a polynomial $q(x_1, ..., x_r)$ such that*

$$p(x_1, ..., x_r) f(x_1, ..., x_r) = q(x_1, ..., x_r)$$

*Proof.* Consider a product of the characteristic polynomial (3) and the generating function (1). Applying (2), we can see that the coefficients corresponding to $x_1^{n_1} ... x_r^{n_r}$ for $n_1 \geq 0, ... n_r \geq b_1$ reduce. The other terms have a negative degree for at least one of the variables, so they are equal to zero. $\square$

Thus, similarly to the unidimensional case, a generating function in the multidimensional case allows representation as a fraction of two polynomials. Unfortunately, unlike the unidimensional case, we were not able to reduce this fraction or find a simpler representation.

# 4. Returning to zero infinitely often

Let $X_i$ be independent Bernoullian random variables, $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Set $S_n = X_1 + ... + X_n$.

Consider an increasing sequence of even integers $n_k$. Our aim is to determine the conditions for $n_k$ such that $P(S_{n_k} = 0 \; infinitely \; often) = 1$

Set $\chi_n = I\{S_n = 0\}$. $S_{n_k} = 0$ infinitely often means that $\sum_{k=1}^{\infty} \chi_{n_k} = \infty$.

It is well-known that

**Proposition 4.1.** $P(\chi_n = 1) \sim \frac{1}{\sqrt{\pi n/2}}$ *for $n \to \infty$*

Together with Borel-Cantelli lemma, this proposition gives us

**Proposition 4.2.** *If $n_k \geq C \cdot k^{2+\varepsilon}$, then $P(S_{n_k} = 0 \; i.o.) = 0$*

Now we will consider the case of $n_k \sim k^2$. Let us prove the following

**Proposition 4.3.** *If $n_k = 2 \cdot C \cdot k^2$, then $P(S_{n_k} = 0 \; i.o.) = 1$*

*Proof.*

$$\sum_{k=1}^{\infty} E\chi_{n_k} \sim \sum_{k=1}^{\infty} \frac{C'}{\sqrt{n_k}} = \infty$$

Since the expectation of the series tends to infinity, what is left to prove is that $\sum_{k=1}^{N} \chi_{n_k}$ does not deviate too much from its expectation.

From the Chebyshov inequality,

$$P\left\{\left|\sum_{k=1}^{N}\chi_{n_k}-\sum_{k=1}^{N}E\chi_{n_k}\right|\geq\lambda\right\}\leq\frac{D\left(\sum_{k=1}^{N}\chi_{n_k}\right)}{\lambda^2}$$

Set $\lambda_N=\frac{1}{2}\sum_{k=1}^{N}E\chi_{n_k}$.

Thus we have to prove that

$$D\left(\sum_{k=1}^{N}\chi_{n_k}\right)=o\left(\sum_{k=1}^{N}E\chi_{n_k}\right)^2$$

Or, equivalently

$$(*)=\frac{E\left[\left(\sum_{k=1}^{N}\chi_{n_k}\right)^2\right]}{\left(\sum_{k=1}^{N}E\chi_{n_k}\right)^2}\to 1, N\to\infty$$

From the Cauchy-Schwartz inequality, $(*)\geq 1$.

Multiplying the terms within the numerator and denominator of the fraction, we obtain

$$(*)\sim\frac{\sum\limits_{j=1}^{N}E(\chi_{n_j}^2)+2\sum\limits_{j<k}^{N}E(\chi_{n_j}\chi_{n_k})}{\sum\limits_{j=1}^{N}(E\chi_{n_j})^2+2\sum\limits_{j<k}^{N}E\chi_{n_j}E\chi_{n_k}}$$

Since the series $\sum_{k=1}^{\infty}E\chi_{n_k}$ diverges, the sums on the left (with a single variable $j$) are infinitely smaller than the sums on the right (with variables $j,k$). Thus we have to consider only the pairwise products:

$$(*)\sim\frac{\sum_{j<k}^{N}E(\chi_{n_j}\chi_{n_k})}{\sum_{j<k}^{N}E\chi_{n_j}E\chi_{n_k}}$$

Now we shall prove that it is possible to replace $E(\chi_{n_j})$ with their asymptotic values within the whole sum.

**Lemma 4.4.** *Consider two sequences, $a_n,b_n>0,\frac{a_n}{b_n}\to 1$. If the series $\sum a_n$ diverges, then*

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}a_n}{\sum_{k=1}^{n}b_n}=1$$

*Proof.* Set $\varepsilon>0,N_0:\forall n>N_0$ we have $1-\varepsilon<\frac{a_n}{b_n}<1+\varepsilon$. Consider $N_1>N_0:$

$$\sum_{k=N_0+1}^{N_1}a_n>\frac{1}{\varepsilon}\sum_{k=1}^{N_0}a_n,\ \sum_{k=N_0+1}^{N_1}b_n>\frac{1}{\varepsilon}\sum_{k=1}^{N_0}b_n$$

Then for all $n > N_1$ we have

$$\frac{\sum_{k=1}^{n} a_n}{\sum_{k=1}^{n} b_n} \leq \frac{(1+\varepsilon)\sum_{k=N_0+1}^{n} a_n}{(1-\varepsilon)\sum_{k=N_0+1}^{n} b_n} \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2$$

By setting $\varepsilon \to 0$ we obtain

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_n}{\sum_{k=1}^{n} b_n} \leq 1$$

The inverse is proved similarly. □

Divide the sum into two parts:

$$\sum_{j<k}^{N} E\chi_{n_j} E\chi_{n_k} = \sum_{j=1}^{N} A_j$$

$$A_j := \sum_{k=j+1}^{N} E\chi_{n_j} E\chi_{n_k}$$

Applying lemma 4.4 and proposition 4.1 to every $A_j$ as a sum itself, we obtain

$$A_j \sim E\chi_{n_j} \cdot \sum_{k=j+1}^{N} \frac{1}{\sqrt{\pi n_k/2}}, N \to \infty$$

Now apply the same lemma to the whole sum, with $A_j$ being its members:

$$\sum_{j<k}^{N} E\chi_{n_j} E\chi_{n_k} \sim \sum_{j<k}^{N} E\chi_{n_j} \cdot \frac{1}{\sqrt{\pi n_k/2}}, N \to \infty$$

Executing the same procedure, but using $k$ as a grouping variable, we obtain:

$$\sum_{j<k}^{N} E\chi_{n_j} E\chi_{n_k} \sim \sum_{j<k}^{N} \frac{1}{\sqrt{\pi n_j/2}\sqrt{\pi n_k/2}}, N \to \infty$$

The asymptotic value for the numerator is calculated by the same method.
We have obtained the asymptotic form for the sum:

$$(*) \sim \frac{\sum_{j<k}^{N} \frac{1}{\sqrt{n_j(n_k-n_j)}}}{\sum_{j<k}^{N} \frac{1}{\sqrt{n_j \cdot n_k}}}$$

**Remark 4.5.** The argument above is correct for all the cases that suffice the requirements that the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n_k}}$ diverges and $n_{k+1} - n_k \to \infty$ when $k \to \infty$

Let us return to examining the case of $n_k \sim k^2$.

**Lemma 4.6.** *For every positive integer $m$ there exists a constant $C = C(m)$ such that*

$$\sum_{k=j+1}^{mj} \frac{1}{\sqrt{k^2 - j^2}} \leq C \cdot \sqrt{\ln j}$$

*Proof.* By the Cauchy-Schwartz inequality and the integral test,

$$\sum_{k=j+1}^{mj} \frac{1}{\sqrt{k^2 - j^2}} \leq \sqrt{mj \cdot \sum_{k=j+1}^{mj} \frac{1}{k^2 - j^2}} = \sqrt{mj \cdot \sum_{k=j+1}^{mj} \left( \frac{1}{k-j} - \frac{1}{k+j} \right) \cdot \frac{1}{2j}}$$
$$\sim \sqrt{\frac{m}{2} \left( \ln (mj - j) + \ln (mj + j) - ln2j \right)} = C(m) \cdot \sqrt{\ln j}$$

□

Replace $n_k$ with their asymptotic values, fix an arbitrary number $m$ and divide the sum into two parts.

$$(*) \sim \frac{\sum_{j<k}^{N} \frac{1}{j\sqrt{k^2-j^2}}}{\sum_{j<k}^{N} \frac{1}{jk}} = \frac{\sum_{j=1}^{N} \left( \sum_{j<k<mj} + \sum_{k \geq mj}^{N} \right)}{\dots} = (1) + (2)$$

Apply the lemma 4.6 and the integral test to the first part

$$(1) \leq C \cdot \frac{\ln N \sqrt{\ln N}}{\ln^2 N} = \overline{o}(1), N \to \infty$$

Consider (2). Within this sum, we have $k \geq mj$, so $\frac{1}{\sqrt{k^2-j^2}} \leq \frac{1}{\sqrt{k^2 - \frac{k^2}{m^2}}} = \frac{m}{\sqrt{m^2-1}} \cdot \frac{1}{k}$ We obtained an upper bound: $(2) \leq \frac{m}{\sqrt{m^2-1}}$.

Combining our results for both parts, we obtain that $(*) \leq \frac{m}{\sqrt{m^2-1}} + \overline{o}(1)$ for every $m$ and $N \to \infty$.

Setting $m \to \infty$, we obtain: $(*) = 1 + \overline{o}(1)$, which completes the proof.

□

**Proposition 4.7.** *If a sequence of even integers satisfies $n_k = C \cdot k^2 + \overline{o}(k^2)$, then $P(S_{n_k} = 0 \text{ } i.o.) = 1$*

*Proof.* The proof of the proposition 4.3 utilized the fact that $P(\chi_n = 1) = \frac{1}{\sqrt{\pi n/2}}(1+\overline{o}(1))$ to calculate asymptotic values. But since

$$P(\chi_{n+\overline{o}(n)} = 1) = \frac{1}{\sqrt{\pi(n + \overline{o}(n))/2}}(1 + \overline{o}(1)) = \frac{1}{\sqrt{\pi n/2}}(1 + \overline{o}(1))$$

the same expressions for the asymptotic values are applicable. □

**Remark 4.8.** The example from the article [6] states that it is sufficient for $P(S_{n_k} = 0 \; i.o.) = 1$ that the series $\sum\limits_k \frac{1}{\sqrt{n_k}}$ diverges and there exists a universal constant $A$ such that $n_{k+1} - n_k > A\sqrt{n_k}$

It can be easily seen that our proposition 4.7 is a special case of the statement above. However, the proof given in [6] seems to have a certain discrepancy. More precisely, the proof begins with dividing the sum into two parts (similarly to the way described in this paper), a smaller set of close indicators and a bigger set of indicators that do not have strong correlations. The discrepancy arises when the upper bound for the smaller set is introduced. The inequality [6], (21) seems to miss the factor corresponding to the number of elements in the smaller set. When this factor is added to the inequality, one of the conditions of lemma's applicability is no longer satisfied.

## 5. Summary

As far as ruin probability is concerned, 2.13 gives a complete algorithm for calculating the ruin probability for all possible finite integer-valued distributions of a single step. Unfortunately, it was only after we obtained this result that we were able to consider it from a different perspective and realize that it was actually a ruin probability problem (whereas the initial form of the problem considered binomial random variables and an inequality on them). Examining existing papers on the topic of ruin probability led us to two articles, [4] and [5], that already contained the same result.

To the best of our knowledge, calculating multi-dimensional ruin probability is a task yet to be solved and is thus of great appeal. However, calculating this probability requires examining certain multivariate complex functions, an obstacle we were not able to overcome.

While considering the problem of a random walk returning to zero at given moments of time, we were not able to find any articles on the subject except the one already mentioned, [6]. Although the result of that article looks very plausible, we were neither able to remove an existing discrepancy, nor to obtain a new proof. The result that we did obtain (statement 4.7) is of rather special form, leaving room for further study.

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