Lenses

Overview

A *Lens* can be defined by the sets A^+, A^-, B^+ and B^- in the relation

$$\begin{pmatrix} f^\# \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \rightleftarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$$

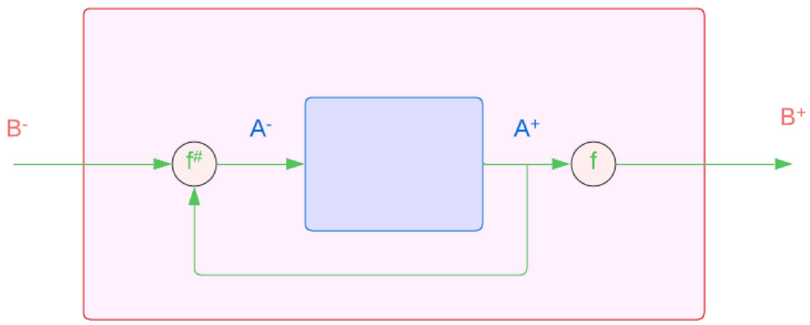
alongside the relations

$$f : A^+ \to B^+$$

$$f^\# : A^+ \times B^- \to A^-$$

We can think of f as a ***get*** operator and of $f^\#$ as a ***put*** operator. In essence, what lenses can do is allow the interaction between two different interfaces.

- f passes an object from the inner interface to the outer interface
- $f^\#$ changes the inner interface through the outer interface accounting for its current state



Lens composition

Now that we know what a lens is, let's see how might we compose them

$$\begin{pmatrix} f^\# \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \rightleftarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix} \quad \circ \quad \begin{pmatrix} B^- \\ B^+ \end{pmatrix} \rightleftarrows \begin{pmatrix} C^- \\ C^+ \end{pmatrix} : \begin{pmatrix} g^\# \\ g \end{pmatrix}$$

We want to obtain the following lens

$$\begin{pmatrix} h^\# \\ h \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \rightleftarrows \begin{pmatrix} C^- \\ C^+ \end{pmatrix}$$

In order to do this, all we need is to define what g and $g^\#$ are.

Let's start with g. We want to get from A^+ to C^+. We can get from A^+ to B^+ via f and from B^+ to C^+ using g. All we need to do here is compose them

$$h := g \circ f$$

The more difficult part is defining $g^{\#}$.

At first, we denote an object in A^+ as a^+ and A^- as a^-. This applies to B and C as well.

We see that

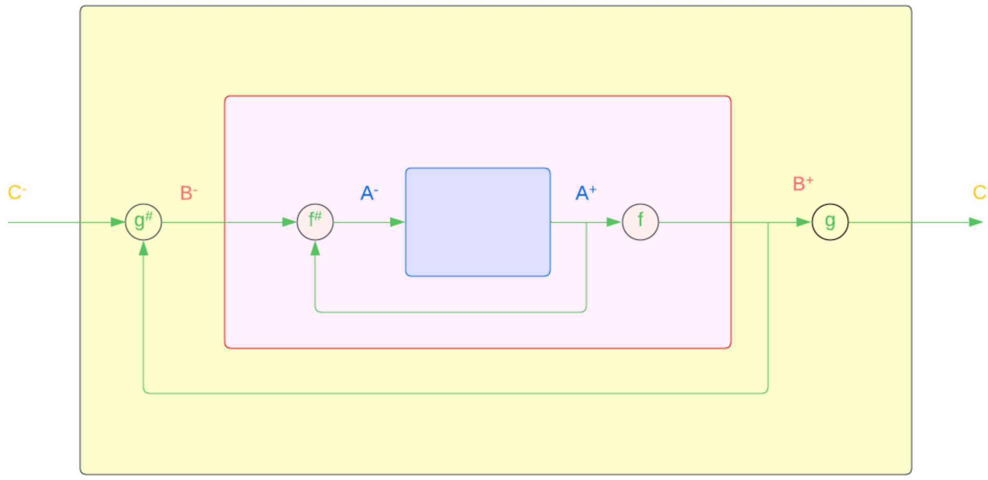
$$f(a^+) = b^+$$

$$g^{\#}(b^+, c^-) = b^-$$

Therefore, we can define $g^{\#}$ as

$$g^{\#} := (a^+, c^-) \mapsto f^{\#}(a^+, g^{\#}(f(a^+), c^-))$$

We now have a way of obtaining the composite of two lenses. Visually, this operation is equivalent to



Lens product

Another useful operation to consider for lenses is their product. Imagine an 8-bit ALU that receives 8 different inputs and can produce 8-bit results. How might we describe such a computation with a lens? More generally, assume we have

$$\mathbf{F} := \begin{pmatrix} f^{\#} \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$$

$$\mathbf{G} := \begin{pmatrix} g^{\#} \\ g \end{pmatrix} : \begin{pmatrix} C^- \\ C^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} D^- \\ D^+ \end{pmatrix}$$

How can we define the lens that takes 2 inputs and produces 2 outputs

$$\mathbf{H} = \mathbf{F} \otimes \mathbf{G}$$

where $\mathbf{H}$ can be denoted as

$$\mathbf{H} := \begin{pmatrix} h^{\#} \\ h \end{pmatrix} : \begin{pmatrix} A^- \times C^- \\ A^+ \times C^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \times D^- \\ B^+ \times D^+ \end{pmatrix}$$

We can construct h as follows

$$h := (a^+, c^+) \mapsto (f(a^+), g(c^+))$$

Creating the modify function is also straightforward

$$h^\# := ((a^+, c^+), (b^-, d^-)) \mapsto (f^\#(a^+, b^-), g^\#(c^+, d^-))$$