Non-negative Super-resolution is Stable

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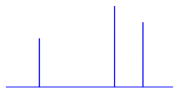
Problem

Consider the problem of localising k non-negative point sources on the interval $[0,1]$, namely finding their locations $t_1,\ldots,t_k$ and magnitudes $a_1,\ldots,a_k \geq 0$ from m noisy samples $y_1,\ldots,y_m$ which consist of the convolution of the input signal with a known kernel ϕ (e.g. Gaussian $\phi(t)=e^{-t^2/\sigma^2}$) and additive noise η bounded by δ. We use the formulation of the problem from [4].

Input signal

x is the discrete measure we want to reconstruct.

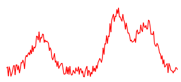


$$x(t)=\sum_{i=1}^{k} a_i \cdot \delta_{t_i},$$

t_i and a_i $(i=1,\ldots,k)$ are the locations and magnitudes of the point sources.

Measured signal

y_i are the samples we use to reconstruct x.



$$y_j=\int_{[0,1]} \phi(t-s_j)x(\mathrm{d}t)+\eta_j,$$

where s_j $(j=1,\ldots,m)$ are sampling locations and $\eta=[\eta_1,\ldots,\eta_m]^T$ with $\|\eta\|_2 \leq \delta$ is the noise.

Feasibility problem

We solve the following problem to find x from y

Find $z \geq 0$ subject to

$$\left\| y-\int_{[0,1]}\Phi(t)z(\mathrm{d}t)\right\|_2 \leq \delta \quad (1)$$

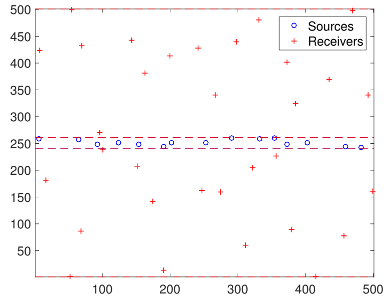
where

$y=[y_1,\ldots,y_m]^T$,

$\Phi(t)=[\phi(t-s_1),\ldots,\phi(t-s_m)]^T$.

Application example

We sample at specific points the sound emitted by ships (sources) traveling along a shipping lane in a 2D region of interest and we want to find their locations and the overall sound level, either in a static or in a dynamic setting. The problem is described in more detail in [2].



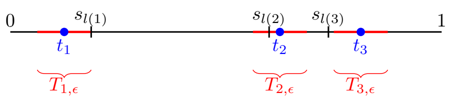
An example of a receiver and source configuration in the region of interest.

Results

Setup

Given the source locations $t_1,\ldots,t_k$, we define:

- $\Delta=\min_{i\neq j}|t_i-t_j|$ is the minimum separation of the sources
- For $\lambda\in[0,\frac{1}{2})$ and for each source t_i, there exists a sampling location $s_{l(i)}$ such that $|s_{l(i)}-t_i|<\lambda\Delta$
- $T_{i,\epsilon}=(t_i-\epsilon,t_i+\epsilon)$ and $T_\epsilon^C=[0,1]\setminus\cup_{i=1}^k T_{i,\epsilon}$ for some $0<\epsilon<\Delta/2$.



For a solution $\hat{x}$ to the Feasibility Problem (1) and the true measure x, we show that the error in $T_{i,\epsilon}$ is bounded and proportional to the noise level δ and ϵ.

Theorem 1 - General kernel

Let $\hat{x}$ be a solution of (1). If $m\geq 2k+1$, ϕ is Lipschitz continuous and, for Δ, λ and ϵ defined previously, we have

$$\phi(\lambda\Delta)\geq 2\phi(\Delta-\lambda\Delta)+\frac{2}{\Delta}\int_{\Delta-\lambda\Delta}^{1/2-\lambda\Delta}\phi(x)\,\mathrm{d}x$$

then, for all $i\in 1,\ldots,k$,

$$\left|\int_{T_{i,\epsilon}}\hat{x}(\mathrm{d}t)-a_i\right|\leq [2\left(1+\frac{\phi^\infty\|b\|_2}{\bar{f}}\right)\cdot\delta$$
$$+L\|\hat{x}\|_{TV}\cdot\epsilon]\|A^{-1}\|_\infty$$

where

- L is the Lipschitz constant of ϕ,
- $\phi^\infty=\max_{t\in[0,1]}|\phi(t)|$,
- b is the vector of coefficients of the dual certificate, and $\bar{f}$ is involved in its construction, see below,
- A is the sampling matrix defined below.

Theorem 2 - Gaussian kernel

Let $\hat{x}$ be a solution of (1) with $\phi(t)=e^{-x^2/\sigma^2}$. If $m\geq 2k+2$ and for every source we have two samples s,s' such that $|s-t_i|\leq\eta$ and $|s-s'|\leq\eta$ with $\eta\leq\sigma^2$. If $\sigma<\frac{1}{\sqrt{3}},\Delta(T)>\sigma\sqrt{\log\frac{5}{\sigma^2}}$, then:

$$\left|\int_{T_{i,\epsilon}}\hat{x}(\mathrm{d}t)-a_i\right|\leq\left[(c_1+F_2)\cdot\delta+c_2\frac{\|\hat{x}\|_{TV}}{\sigma^2}\cdot\epsilon\right]F_3,$$

where $F_2(k,\Delta(T),1/\sigma,1/\epsilon) < c_3\frac{kC_2(1/\epsilon)}{\sigma^2}\left[\frac{c_4}{\sigma^6(1-3\sigma^2)^2}\right]^k$, $F_3(\Delta(T),\sigma,\lambda) < c_5$ and the universal constants c_1,c_2,c_3,c_4,c_5 and $C_2(1/\epsilon)$ are given in [1].

Implications of Theorem 2

- Any solution consistent with measurements within δ is within $\delta+\epsilon$ of the true measure in the appropriate notion.
- As $\delta,\epsilon\to 0$, the measure x is the unique solution to (1).
- Similar results were previously known only for specific algorithms (e.g. TV-norm minimisation).

Proof ideas

Dual certificate

We use the notion of dual certificate to show that the original signal x is the unique solution of problem (1) with $\delta=0$, then extend this notion to $\delta>0$ to bound the error between the true and the estimated solution. Using ideas from [3], we show that the dual certificate exists if ϕ with different shifts forms a T-System and we construct it. The dual certificate is a function:

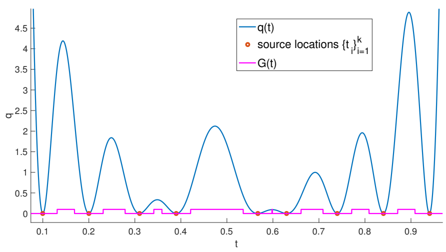
$$q(t)=\sum_{j=1}^{k} b_j\phi(t-s_j)$$

such that

- $q(t_i)=0, \forall i=1,\ldots,k$,
- $q(t)>0, \forall t\neq t_i$.
- $q(t)\geq G(t), \forall t\in[0,1]$

where, for some $\bar{f}>0$,

$$G(t)=\begin{cases}\bar{f}, & \text{if } t\in T_\epsilon^C,\\ 0, & \text{otherwise.}\end{cases}$$



The shape of the dual certificate depends on the locations of the sources and the samples.

Sampling matrix

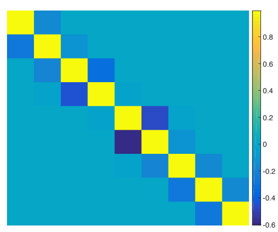
Another important idea is that of diagonally dominance of the matrices A and B defined below. The condition that A is diagonally dominant is related to how close to each source we need the samples to be for a general ϕ in Theorem 1. The determinant of the matrix B appears in the factor F_2 in Theorem 2 for the Gaussian kernel. We bound it by considering bounds on the eigenvalues of B.

$A=(a_{ij})\in\mathbb{R}^{k\times k}$

$$a_{ij}=\begin{cases}|\phi(s_{l(i)}-t_j)|, & \text{if } i=j,\\ -|\phi(s_{l(i)}-t_j)|, & \text{otherwise.}\end{cases}$$

$B=(B_{ij})\in\mathbb{R}^{2k\times 2k}$

$$B_{ij}=\begin{bmatrix}\phi(t_i-t_j) & -\phi'(t_i-t_j)\\ \phi'(t_i-t_j) & -\phi''(t_i-t_j)\end{bmatrix}$$



Example of matrix A.

Example of matrix B.

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