What is... it like to lower the mixed volume?

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Freie Universität Berlin

BMS Friday - What is...? Seminar
08. Feb. 2019

Outline

1 Setup

2 Talking About the Volumes

3 Mixed Volume

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2. Talking About the Volumes
3. Mixed Volume

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Ingredients

- The set $\mathcal{K}_n$ of all convex, compact sets in $\mathbb{R}^n$.
- The set $\mathcal{P}_n \subseteq \mathcal{K}_n$ of all convex polytopes in $\mathbb{R}^n$.
- The volume $V_n(K)$ of some $K \in \mathcal{K}_n$.
- The stretching λK of some $K \in \mathcal{K}_n$, by any $\lambda \geq 0$.
- The Minkowski sum $K + L$ of some $K, L \in \mathcal{K}_n$.

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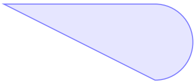
The Minkowski Sum of Two Convex Bodies

Definition

For any two $K, L \in \mathcal{K}_n$, the Minkowski sum $K + L \in \mathcal{K}_n$ is the set of every vector sum:

$$K + L := \{u + v \in \mathbb{R}^n : u \in K, v \in L\}$$



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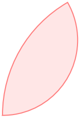

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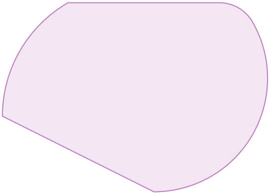

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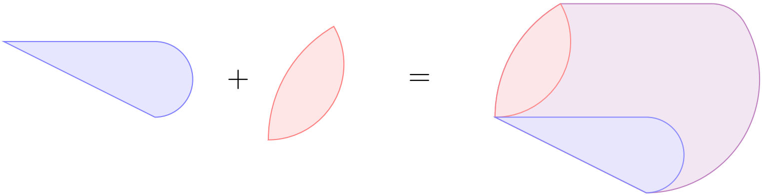

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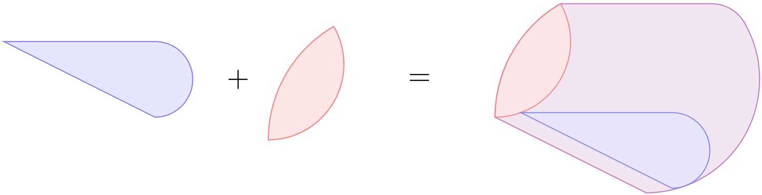

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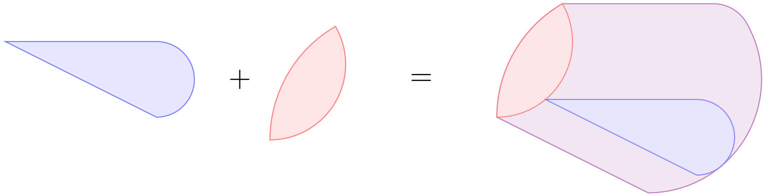

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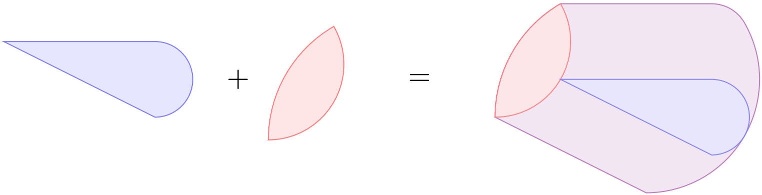

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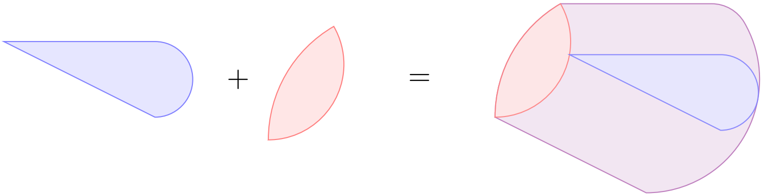

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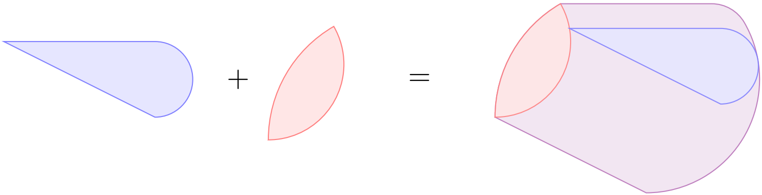

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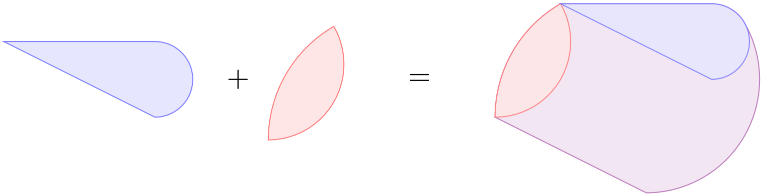

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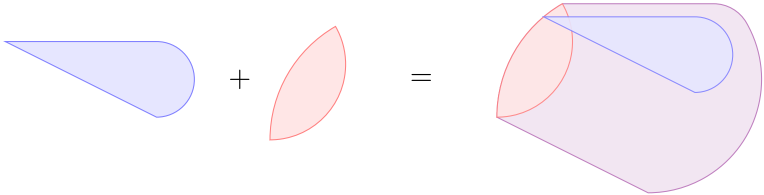

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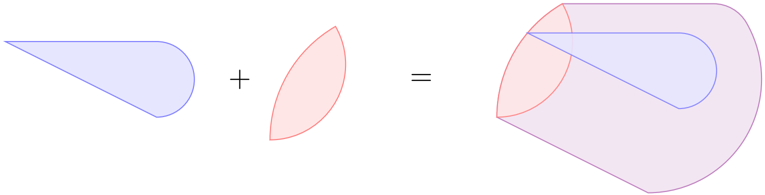

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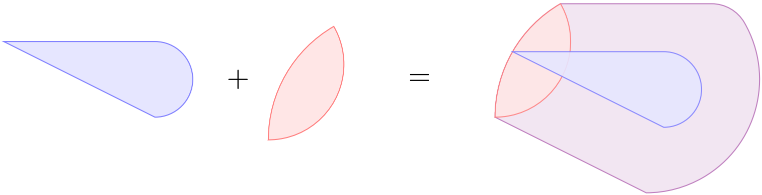

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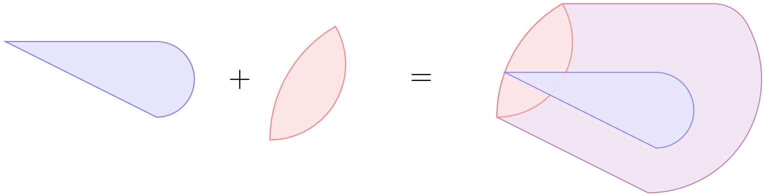

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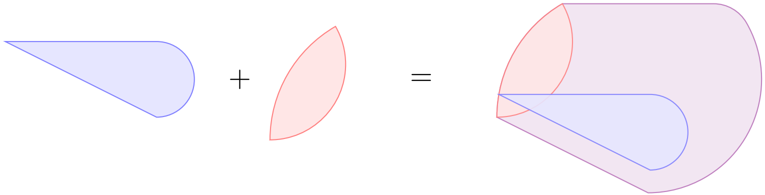

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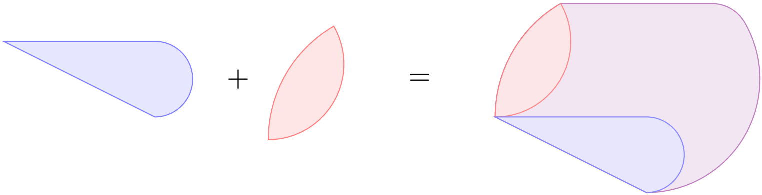

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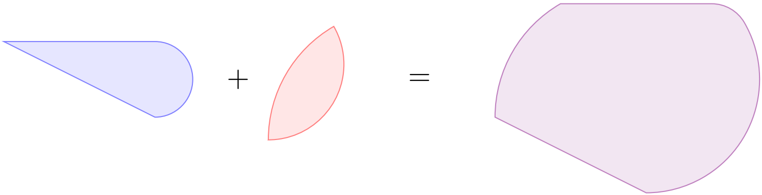

The Brunn-Minkowski Inequality

Theorem (Brunn '87, Minkowski '96)

Let $K, L \in \mathcal{K}_n$ be two compact, convex sets, then:

$$V_n(K+L)^{\frac{1}{n}} \geq V_n(K)^{\frac{1}{n}} + V_n(L)^{\frac{1}{n}}$$

Minkowski Sum Increases the Volume

Corollary

Let $K, L \in \mathcal{K}_n$ be two compact, convex sets, then:

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Proof (by picture).

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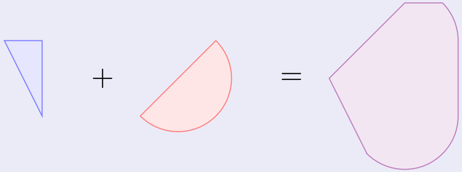

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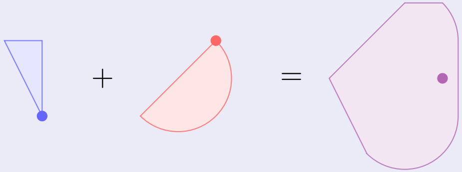

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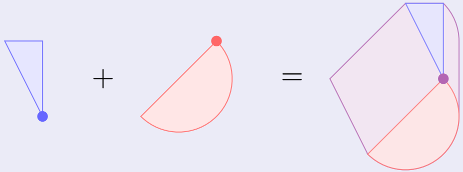


$\square$

More Handwaving

Proof of Brunn-Minkowski Inequality.

For $K = [0, x_1] \times \cdots \times [0, x_n]$ and $L = [0, y_1] \times \cdots \times [0, y_n]$:

$$\frac{V_n(K)^{\frac{1}{n}} + V_n(L)^{\frac{1}{n}}}{V_n(K+L)^{\frac{1}{n}}} = \left(\prod_{i=1}^{n} \frac{x_i}{x_i + y_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \frac{y_i}{x_i + y_i}\right)^{\frac{1}{n}}$$

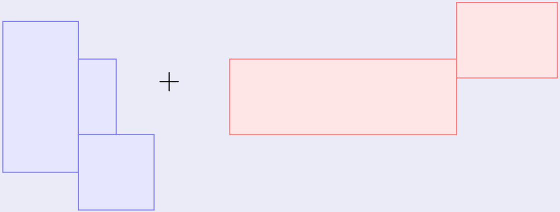
$$\leq \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_i + y_i} + \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{x_i + y_i}$$

$$= 1$$

More Handwaving

Proof (cont.)

For K, L disjoint unions of open boxes:



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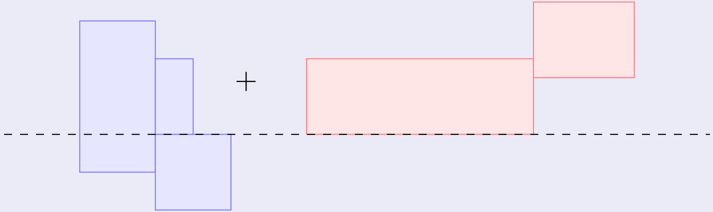

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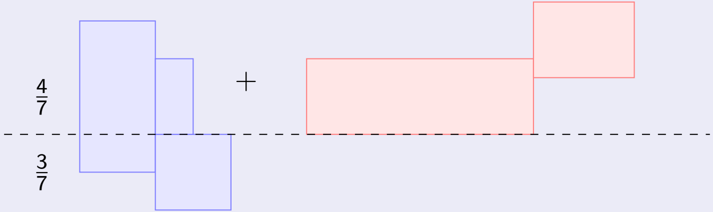

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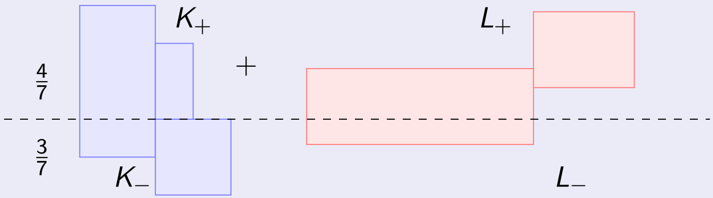

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Proof (cont.)

$$
\begin{aligned}
V_n(K+L) &\geq V_n(K_- + L_-) + V_n(K_+ + L_+) \\
&\geq \left(V_n(K_-)^{\frac{1}{n}} + V_n(L_-)^{\frac{1}{n}}\right)^n + \left(V_n(K_+)^{\frac{1}{n}} + V_n(L_+)^{\frac{1}{n}}\right)^n \\
&= V_n(K_-)\left(1 + \frac{V_n(L)^{\frac{1}{n}}}{V_n(K)^{\frac{1}{n}}}\right)^n + V_n(K_+)\left(1 + \frac{V_n(L)^{\frac{1}{n}}}{V_n(K)^{\frac{1}{n}}}\right)^n \\
&= V_n(K)\left(1 + \frac{V_n(L)^{\frac{1}{n}}}{V_n(K)^{\frac{1}{n}}}\right)^n = \left(V_n(K)^{\frac{1}{n}} + V_n(L)^{\frac{1}{n}}\right)^n
\end{aligned}
$$

For the general case, approximate with boxes. $\square$

Find Highest Volume Under Constant Surface Area

Theorem (Isoperimetric Inequality)

Let $K \in \mathcal{K}_n$ be a compact, convex set, then:

$$V_n(K) \leq V_n(rB^n)$$

where B_n is the unit ball of dimension n and r is chosen so that:

$$V_{n-1}(K) = V_{n-1}(rB^n)$$

Corollary

An equivalent statement of the theorem is that for every $K \in \mathcal{K}_n$:

$$\frac{V_n(K)^{\frac{1}{n}}}{V_{n-1}(K)^{\frac{1}{n-1}}} \leq \frac{V_n(B^n)^{\frac{1}{n}}}{V_{n-1}(B^n)^{\frac{1}{n-1}}}$$

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But What is Surface Area?

Definition

Let $K \in \mathcal{K}_n$. Then define the surface area of K to be the limit:

$$V_{n-1}(K) := \lim_{\varepsilon \to 0^+} \frac{V_n(K_\varepsilon) - V_n(K)}{\varepsilon}$$

where $K_\varepsilon := K + \varepsilon B^n$.

Example

$$\begin{aligned} V_{n-1}(B^n) &= \lim_{\varepsilon \to 0^+} \frac{V_n((1+\varepsilon)B^n) - V_n(B^n)}{\varepsilon} \\ &= V_n(B^n) \lim_{\varepsilon \to 0^+} \frac{(1+\varepsilon)^n - 1}{\varepsilon} \\ &= nV_n(B^n) \end{aligned}$$

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Proof of Isoperimetric using B-M

Proof of Isoperimetric Inequality.

Let $\varepsilon > 0$. Then:

$$\frac{V_n(K+\varepsilon B^n) - V_n(K)}{\varepsilon} \geq \frac{\left(V_n(K)^{\frac{1}{n}} + \varepsilon V_n(B^n)^{\frac{1}{n}}\right)^n - V_n(K)}{\varepsilon}$$
$$= n\left(V_n(K)^{\frac{1}{n}}\right)^{n-1} V_n(B^n)^{\frac{1}{n}} + O(\varepsilon)$$

Taking the limit on both sides, for $\varepsilon \to 0^+$:

$$V_{n-1}(K) \geq nV_n(K)^{\frac{n-1}{n}} V_n(B^n)^{\frac{1}{n}}$$
$$\Rightarrow \frac{V_n(K)^{\frac{1}{n}}}{V_{n-1}(K)^{\frac{1}{n-1}}} \leq \frac{V_n(B^n)^{\frac{1}{n}}}{V_{n-1}(B^n)^{\frac{1}{n-1}}}$$

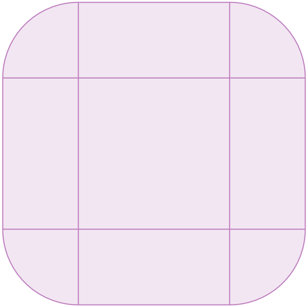
$\square$

Can We Speak In General About Linear Combinations Of Convex Bodies?

Example

Let $K = [0,1]^2$ and $L = B^2$. Then:

$$V_2(\lambda K + \mu L) = \lambda^2 + 4\lambda\mu + \pi\mu^2$$



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Theorem

Let $K_1, \ldots, K_m \in \mathcal{K}_n$. Then, the Volume

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m)$$

is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_m$

Mixed Volumes

Definition

Let $K_1, \ldots, K_n \in \mathcal{K}_n$, then their mixed Volume $V(K_1, \ldots, K_n)$ is defined to be the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ of the polynomial $V_n(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ over $n!$.

Corollary

For any $K_1, \ldots, K_m \in \mathcal{K}_n$, we can write:

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_n \in [m]} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}$$

Some Properties

Symmetric	$V(K_1, K_2, \ldots, K_n) = V(K_2, K_1, \ldots, K_n)$
Multilinear	$V(\lambda K_1 + \mu K_1', K_2, \ldots, K_n) =$ $\lambda V(K_1, K_2, \ldots, K_n) + \mu V(K_1', K_2, \ldots, K_n)$
Nomalized	$V(K, K, \ldots, K) = V_n(K)$
Monotone	$V(K_1, \ldots, K_n) \leq V(K_1', \ldots, K_n)$, if $K_1 \subseteq K_1'$

Example

In the example $K = [0,1]^2$, $L = B^2$ we had:

$$V(\lambda K + \mu L) = \lambda^2 + 4\lambda\mu + \pi\mu^2$$

And thus:

$$V(K,K) = 1 \qquad V(K,L) = V(L,K) = 2 \qquad V(L,L) = \pi$$

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Multilinear $V(\lambda K_1 + \mu K_1', K_2, \ldots, K_n) =$
$\lambda V(K_1, K_2, \ldots, K_n) + \mu V(K_1', K_2, \ldots, K_n)$

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A Special Case That Motivated Us

We started at trying to compute $V_n(K + \varepsilon B^n)$.

Definition

Let $K \in \mathcal{K}_n$ and $i \in \{0, \ldots, n\}$. The i-th Quermassintegral of K is defined to be:

$$W_i(K) = V(K, \ldots, K, B^n, \ldots, B^n)$$

where we have $n - i$ copies of K and i copies of B^n

Corollary

$$V_n(K + rB^n) = \sum_{k=0}^{n} \binom{n}{k} W_i(K) r^i$$

But How Can We Compute The Mixed Volume

Using the inclusion-exclusion formula:

Theorem

Let $K_1, \ldots, K_n \in \mathcal{K}_n$. Then:

$$V(K_1, \ldots, K_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{i_1 < \cdots < i_k} V_n(K_{i_1} + \cdots + K_{i_k})$$

Mixed Subdivision

We now make a detour through Polytopes

Definition

Let $P_1, \ldots, P_m \in \mathcal{P}_n$. A subdivision of $P_1 + P_2 + \cdots + P_m$ is a collection of cells $\mathcal{C} = \{F_1 + F_2 + \cdots + F_m : F_i \subseteq P_i\}$ partitioning the Minkowski sum $P_1 + \cdots + P_n$, where each F_i is a face of P_i.
A subdivision is called fine, if it is generic enough.

Definition

Given $P_1, \ldots, P_n$ and $\mathcal{C}$ as above, a cell $C \in \mathcal{C}$ is called mixed, if every face involved is of dimension ≤ 1.

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The Mixed Volume Equals the Sum of Volumes of the Mixed Cells

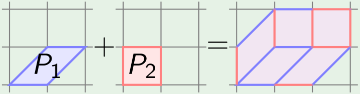
Theorem

Let $P_1, \ldots, P_n \in \mathcal{P}_n$, $\mathcal{C}$ a fine subdivision of $P_1 + \cdots + P_n$ and $\mathcal{M} \subseteq \mathcal{C}$ the set of all mixed cells of $\mathcal{C}$. Then:

$$V(P_1, \ldots, P_n) = \sum_{C \in \mathcal{M}} V_n(C)$$

Example

Let P_1, P_2 as in the picture, where the underlying lattice is $\mathbb{Z}^2$.



Then $V(P_1, P_2) = 3$, because there are 3 mixed cells in this decomposition, all having volume 1.

Finding Some Fine Subdivision

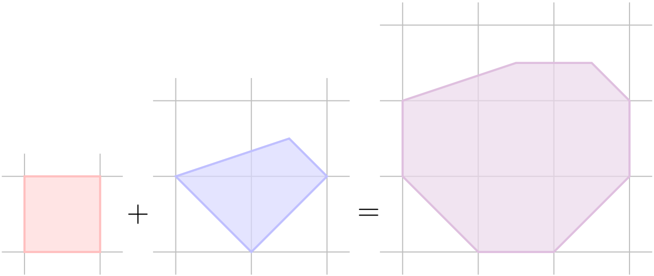
Theorem

Let $P_1, \ldots, P_n \in \mathcal{P}_n$ generic enough polytopes and $\phi_1, \ldots, \phi_n : \mathbb{R}^n \to \mathbb{R}$ generic enough affine maps. The surface of $\phi_1 P_1 + \cdots + \phi_n P_n$ is naturally partitioned in some cells $\tilde{\mathcal{C}}$. Then, the lower (resp. upper) cell subdivision

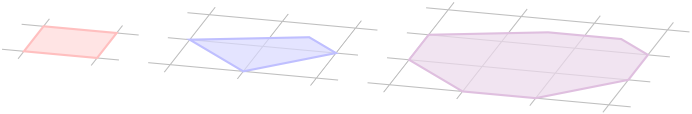
$$\mathcal{C} := \left\{ p(C) : C \in \tilde{\mathcal{C}}^- \right\}$$

is a fine subdivision of $P_1 + \cdots + P_n$, where $\tilde{\mathcal{C}}^-$ are the cells subdividing the "lower" part of the surface.

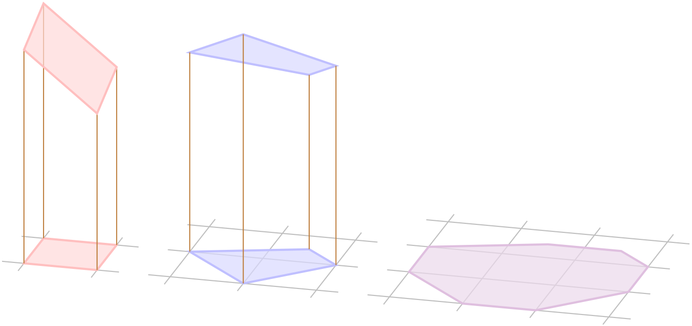
Projecting (Lowering) the Mixed Volume



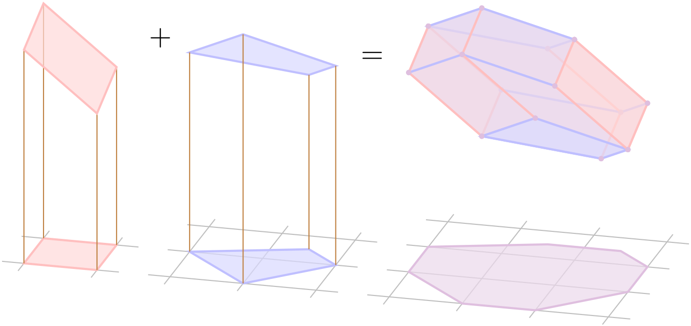
Projecting (Lowering) the Mixed Volume



Projecting (Lowering) the Mixed Volume



Projecting (Lowering) the Mixed Volume



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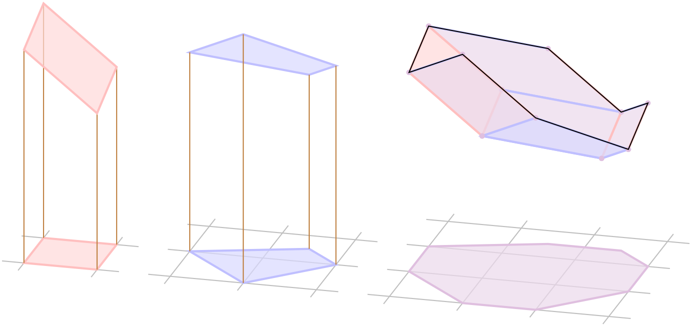
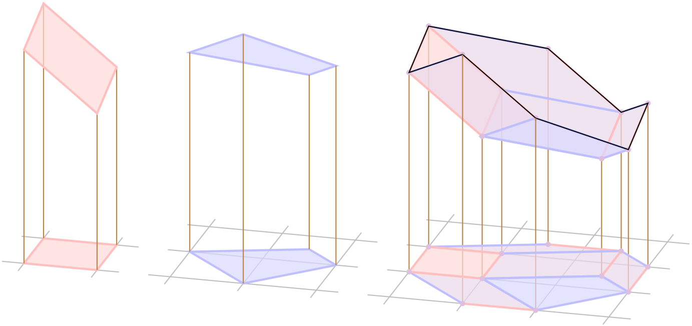
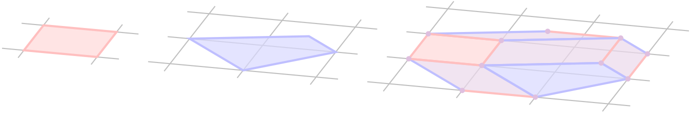


Projecting (Lowering) the Mixed Volume



Projecting (Lowering) the Mixed Volume



Projecting (Lowering) the Mixed Volume