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CONVEX BODIES: THE BRUNN-MINKOWSKI THEORY

Second Expanded Edition

Rolf Schneider

CAMBRIDGE

CONVEX BODIES: THE BRUNN–MINKOWSKI THEORY

At the heart of this monograph is the Brunn–Minkowski theory, which can be used to great effect in studying such ideas as volume and surface area and their generalizations. In particular, the notions of mixed volume and mixed area measure arise naturally and the fundamental inequalities that are satisfied by mixed volumes are considered here in detail.

The author presents a comprehensive introduction to convex bodies, including full proofs for some deeper theorems. The book provides hints and pointers to connections with other fields and an exhaustive reference list.

This second edition has been considerably expanded to reflect the rapid developments of the past two decades. It includes new chapters on valuations on convex bodies, on extensions like the L_p Brunn–Minkowski theory, and on affine constructions and inequalities. There are also many supplements and updates to the original chapters, and a substantial expansion of chapter notes and references.

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Convex Bodies: The Brunn–Minkowski Theory

Second Expanded Edition

ROLF SCHNEIDER

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Preface to the second edition

Wie machen wirs, daß alles frisch und neu
Und mit Bedeutung auch gefällig sei?

Goethe, *Faust I*

The past 20 years have seen considerable progress and lively activity in various different areas of convex geometry. In order that this book still meet its intended purpose, it had to be updated and expanded. It remains the aim of the book to serve the newcomer to the field who wants an introduction from the very beginning, as well as the experienced reader who is either doing research in the field or is looking for some special result to be used elsewhere. In the introductory parts of the book, no greater changes have been necessary, but already here recent developments are reflected in a number of supplements. The main additions to the book are three new chapters, on valuations, on extensions and analogues of the Brunn–Minkowski theory, and on affine constructions and inequalities in the theory of convex bodies. The contents of Chapter 7 from the first edition are now found in Chapters 8 and 10 of the second edition, considerably extended. The structure of some other chapters has also been changed by, for example, dividing them into subsections, regrouping some material, or adding a new section. A few more technical proofs, which had been carried out in the first edition, have been replaced by hints to the original literature.

While the new topics added to the book all have their origins in the Brunn–Minkowski theory, their natural intrinsic developments may gradually have led them farther away. Proofs in this book are restricted to results which may have been basic for further developments, but are still close to the classical Brunn–Minkowski theory. In the remaining parts, we survey many recent results without giving proofs, but we always provide references to the sources where the proofs can be found. The section notes contain additional information.

It has become clear that this book, even with its restriction to the Brunn–Minkowski theory and its ramifications, could never be exhaustive. Therefore, I felt comfortable with being brief in the treatment of projections and sections of convex bodies and related topics, subsumed under geometric tomography, having the chance to refer instead to the book by Gardner [675], which already exists in its second edition.

The Fourier analytic approach to sections and projections is fully covered by the books of Koldobsky [1136] and of Koldobsky and Yaskin [1142]. For all questions related to projections or sections of convex bodies, these are the three books the reader should consult.

It was also comforting to learn that the important developments in another active branch of convexity, the asymptotic theory, will be the subject of forthcoming books. With an easy conscience I could, therefore, refrain from inadequate attempts to cross my borders in this direction.

This second edition would not have come into life without Erwin Lutwak's friendly persuasion and without his invaluable help. In particular, he formed a team of young collaborators who produced a LaTeX version of the first edition, thus providing a highly appreciated technical basis for my later work. Special thanks for this very useful support go to Varvara Liti and Guangxian Zhu. An advanced version of the second edition was read by Franz Schuster, and selected chapters were read by Richard Gardner, Daniel Hug, Erwin Lutwak and Gaoyong Zhang. They all helped me with many useful comments and suggestions, for which I express my sincere thanks.

Freiburg i. Br., December 2012

Preface to the first edition

The Brunn–Minkowski theory is the classical core of the geometry of convex bodies. It originated with the thesis of Hermann Brunn in 1887 and is in its essential parts the creation of Hermann Minkowski, around the turn of the century. The well-known survey of Bonnesen and Fenchel in 1934 collected what was already an impressive body of results, though important developments were still to come, through the work of A. D. Aleksandrov and others in the thirties. In recent decades, the theory of convex bodies has expanded considerably; new topics have been developed and originally neglected branches of the subject have gained in interest. For instance, the combinatorial aspects, the theory of convex polytopes and the local theory of Banach spaces attract particular attention now. Nevertheless, the Brunn–Minkowski theory has remained of constant interest owing to its various new applications, its connections with other fields, and the challenge of some resistant open problems.

Aiming at a brief characterization of Brunn–Minkowski theory, one might say that it is the result of merging two elementary notions for point sets in Euclidean space: vector addition and volume. The vector addition of convex bodies, usually called Minkowski addition, has many facets of independent geometric interest. Combined with volume, it leads to the fundamental Brunn–Minkowski inequality and the notion of mixed volumes. The latter satisfy a series of inequalities which, due to their flexibility, solve many extremal problems and yield several uniqueness results. Looking at mixed volumes from a local point of view, one is led to mixed area measures. Quermassintegrals, or Minkowski functionals, and their local versions, surface area measures and curvature measures, are a special case of mixed volumes and mixed area measures. They are related to the differential geometry of convex hypersurfaces and to integral geometry.

Chapter 1 of the present book treats the basic properties of convex bodies and thus lays the foundations for subsequent developments. This chapter does not claim much originality; in large parts, it follows the procedures in standard books such as McMullen and Shephard [1398], Roberts and Varberg [1581] and Rockafellar [1583]. Together with Sections 2.1, 2.2, 2.4 and 2.5, it serves as a general introduction to the metric geometry of convex bodies. Chapter 2 is devoted to the boundary structure of convex bodies. Most of its material is needed later, except for Section[1] 2.6, on generic boundary structure, which just rounds off the picture. Minkowski addition is the subject of Chapter 3. Several different aspects are considered here such as decomposability, approximation problems with special regard to addition, additive maps and sums of segments. Quermassintegrals, which constitute a fundamental class of functionals on convex bodies, are studied in Chapter 4, where they are viewed as specializations of curvature measures, their local versions. For these, some integral-geometric formulae are established in Section 4.5. Here I try to follow the tradition set by Blaschke and Hadwiger, of incorporating parts of integral geometry into the theory of convex bodies. Some of this, however, is also a necessary prerequisite for Section 4.6. The remaining part of the book is devoted to mixed volumes and their applications. Chapter 5 develops the basic properties of mixed volumes and mixed area measures and treats special formulae, extensions, and analogues.[2] Chapter 6, the heart of the book, is devoted to the inequalities satisfied by mixed volumes, with special emphasis on improvements, the equality cases (as far as they are known) and stability questions. Chapter 7 presents a small selection of applications. The classical theorems of Minkowski and the Aleksandrov–Fenchel–Jessen theorem are treated here, the latter in refined versions. Section 7.4 serves as an overview of affine extremal problems for convex bodies. In this promising field, Brunn–Minkowski theory is of some use, but it appears that for the solution of some long-standing open problems new methods still have to be invented.

Concerning the choice of topics treated in this book, I wish to point out that it is guided by Minkowski's original work also in the following sense. Some subjects that Minkowski touched only briefly have later expanded considerably, and I pay special attention to these. Examples are projection bodies (zonoids), tangential bodies, the use of spherical harmonics in convexity and strengthenings of Minkowskian inequalities in the form of stability estimates.

The necessary prerequisites for reading this book are modest: the usual geometry of Euclidean space, elementary analysis, and basic measure and integration theory for Chapter 4. Occasionally, use is made of spherical harmonics; relevant information is collected in the Appendix. My intended attitude towards the presentation of proofs cannot be summarized better than by quoting from the preface to the book on Hausdorff measures by C. A. Rogers: 'As the book is largely based on lectures, and as I like my students to follow my lectures, proofs are given in great detail; this may bore the mature mathematician, but it will, I believe, be a great help to anyone trying to learn the subject *ab initio*.' On the other hand, some important results are stated as theorems but not proved, since this would lead us too far from the main theme, and no proofs are given in the survey sections 5.4, 6.8 and 7.4.

The notes at the end of nearly all sections are an essential part of the book. As a rule, this is where I have given references to original literature, considered questions

[1] Section numbers here refer to the first edition; they may differ in the second edition.
[2] This description of the chapters concerns the first edition. Beginning with Chapter 6, the second edition has a different structure.

of priority, made various comments and, in particular, given hints about applications, generalizations and ramifications. As an important purpose of the notes is to demonstrate the connections of convex geometry with other fields, some notes do take us further from the main theme of the book, mentioning, for example, infinite-dimensional results or non-convex sets or giving more detailed information on applications in, for instance, stochastic geometry.

The list of references does not have much overlap with the older bibliographies in the books by Bonnesen and Fenchel and by Hadwiger. Hence, a reader wishing to have a more complete picture should consult these bibliographies also, as well as those in the survey articles listed in part B of the References.

My thanks go to Sabine Linsenbold for her careful typing of the manuscript and to Daniel Hug who read the typescript and made many valuable comments and suggestions.

General hints to the literature

This book treats convex bodies with a special view to its classical part known as Brunn–Minkowski theory. To give the reader a wider picture of convex geometry, we collect here a list of textbooks and monographs, roughly ordered by category and year of publication. Each of the books listed under *'General'* has parts which can serve as an introduction to convex geometry, although at different levels and with different choices of topics in the advanced parts.

Classical

1916 Blaschke [241]: *Kreis und Kugel*
1934 Bonnesen and Fenchel [284]: *Theorie der Konvexen Körper*

General

1955 Hadwiger [908]: *Altes und Neues über Konvexe Körper*
1957 Hadwiger [911]: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*
1958 Eggleston [532]: *Convexity*
1964 Valentine [1866]: *Convex Sets*
1966 Benson [193]: *Euclidean Geometry and Convexity*
1979 Kelly and Weiss [1070]: *Geometry and Convexity*
1980 Leichtweiß [1184]: *Konvexe Mengen*
1982 Lay [1178]: *Convex Sets and Their Applications*
1993 Schneider [1717]: *Convex Bodies: The Brunn–Minkowski Theory*
1994 Webster [1928]: *Convexity*
2002 Barvinok [169]: *A Course in Convexity*
2006 Berger [200]: *Convexité dans le Plan, dans l'Espace et Au-delà*
2006 Moszyńska [1452]: *Selected Topics in Convex Geometry*
2007 Gruber [834]: *Convex and Discrete Geometry*
2010 Berger [201]: Chapter VII in *Geometry Revealed*

Convex analysis, convex functions, related topics

1951 Fenchel [570]: *Convex Cones, Sets and Functions*
1970 Rockafellar [1583]: *Convex Analysis*
1973 Roberts and Varberg [1581]: *Convex Functions*

1977 Marti [1331]: *Konvexe Analysis*

1994 Hörmander [988]: *Notions of Convexity*

2010 Borwein and Vanderwerff [305]: *Convex Functions: Constructions, Characterizations and Counterexamples*

Convex polytopes

1950 Aleksandrov [25]: *Konvexe Polyeder*

1967 Grünbaum [848]: *Convex Polytopes*

1971 McMullen and Shephard [1398]: *Convex Polytopes and the Upper Bound Conjecture*

1983 Brøndsted [338]: *An Introduction to Convex Polytopes*

1995 Ziegler [2079]: *Lectures on Polytopes*

2003 Grünbaum [849]: *Convex Polytopes,* 2nd edn

Particular aspects

1929 Bonnesen [282]: *Les Problèmes des Isopérimètres et des Isépiphanes*

1948 Aleksandrov [23]: *Die Innere Geometrie der Konvexen Flächen*

1951 Jaglom and Boltjanski [1032]: *Konvexe Figuren*

1956 Lyusternik [1310]: *Convex Figures and Polyhedra*

1958 Busemann [371]: *Convex Surfaces*

1969 Pogorelov [1538]: *Extrinsic Geometry of Convex Surfaces*

1977 Guggenheimer [867]: *Applicable Geometry: Global and Local Convexity*

1995 Gardner [672]: *Geometric Tomography*

1996 Groemer [800]: *Geometric Applications of Fourier Series and Spherical Harmonics*

1996 Thompson [1845]: *Minkowski Geometry*

1996 Zong [2081]: *Strange Phenomena in Convex and Discrete Geometry*

1998 Leichtweiß [1193]: *Affine Geometry of Convex Bodies*

2005 Koldobsky [1136]: *Fourier Analysis in Convex Geometry*

2006 Gardner [675]: *Geometric Tomography,* 2nd edn

2006 Zong [2082]: *The Cube: A Window to Convex and Discrete Geometry*

2008 Koldobsky and Yaskin [1142]: *The Interface between Convex Geometry and Harmonic Analysis*

2009 Zamfirescu [2053]: *The Majority in Convexity*

Collections

1963 Klee (ed.) [1113]: *Convexity*

1967 Fenchel (ed.) [571]: *Proceedings of the Colloquium on Convexity, Copenhagen 1965*

1979 Tölke, Wills (eds) [1848]: *Contributions to Geometry, Part 1: Geometric Convexity*

1983 Gruber, Wills (eds) [845] : *Convexity and Its Applications*

1985 Goodman, Lutwak, Malkevitch, Pollack (eds) [758]: *Discrete Geometry and Convexity*

1993 Gruber, Wills (eds) [846]: *Handbook of Convex Geometry*

1994 Bisztriczky, McMullen, Schneider, Ivić Weiss (eds) [232]: *Polytopes: Abstract, Convex and Computational*

2004 Brandolini, Colzani, Iosevich, Travaglini (eds) [328]: *Fourier Analysis and Convexity*

Conventions and notation

Here we fix our notation and collect some basic definitions. We shall work in n-dimensional real Euclidean vector space, $\mathbb{R}^n$, with origin o, standard scalar product $\langle\cdot,\cdot\rangle$, and induced norm $|\cdot|$. We do not distinguish formally between the vector space $\mathbb{R}^n$ and its corresponding affine space, although our alternating use of the words 'vector' and 'point' is deliberate and should support the reader's intuition. As long as we deal with Euclidean geometry, we also do not distinguish between $\mathbb{R}^n$ and its dual space, but use the standard scalar product to identify both spaces (being well aware of the fact that it is often considered more elegant not to make this identification).

As a rule, elements of $\mathbb{R}^n$ are denoted by lower-case letters, subsets by capitals, and real numbers by small Greek letters.

The vector $x \in \mathbb{R}^n$ is a *linear combination* of the vectors $x_1, \dots, x_k \in \mathbb{R}^n$ if $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$ with suitable $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. If such λ_i exist with $\lambda_1 + \cdots + \lambda_k = 1$, then x is an *affine combination* of $x_1, \dots, x_k$. For $A \subset \mathbb{R}^n$, $\operatorname{lin} A$ ($\operatorname{aff} A$) denotes the *linear hull* (*affine hull*) of A; this is the set of all linear (affine) combinations of elements of A and at the same time the smallest linear subspace (affine subspace) of $\mathbb{R}^n$ containing A. Points $x_1, \dots, x_k \in \mathbb{R}^n$ are *affinely independent* if none of them is an affine combination of the others, i.e., if

$$\sum_{i=1}^{k} \lambda_i x_i = o \quad \text{with } \lambda_i \in \mathbb{R} \text{ and } \quad \sum_{i=1}^{k} \lambda_i = 0$$

implies that $\lambda_1 = \cdots = \lambda_k = 0$. This is equivalent to the linear independence of the vectors $x_2 - x_1, \dots, x_k - x_1$. We may also define a map $\tau : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ by $\tau(x) := (x, 1)$; then $x_1, \dots, x_k \in \mathbb{R}^n$ are affinely independent if and only if $\tau(x_1), \dots, \tau(x_k)$ are linearly independent.

For $x, y \in \mathbb{R}^n$ we write

$$[x, y] := \{(1 - \lambda)x + \lambda y : 0 \le \lambda \le 1\}$$

for the *closed segment* and

$$[x, y) := \{(1 - \lambda)x + \lambda y : 0 \le \lambda < 1\}$$

for a *half-open segment*, both with endpoints x, y. For $A, B \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define

$$A + B := \{a + b : a \in A,\ b \in B\}, \qquad \lambda A := \{\lambda a : a \in A\},$$

and we write $-A$ for $(-1)A$, $A - B$ for $A + (-B)$ and $A + x$ for $A + \{x\}$, where $x \in \mathbb{R}^n$. The set $A + B$ is written $A \oplus B$ and called the *direct sum* of A and B if A and B are contained in complementary affine subspaces of $\mathbb{R}^n$. A set A is called *o-symmetric* (or *centred*) if $A = -A$.

By $\mathrm{cl}\, A$, $\mathrm{int}\, A$, $\mathrm{bd}\, A$ we denote, respectively, the closure, interior and boundary of a subset A of a topological space. For $A \subset \mathbb{R}^n$, the sets $\mathrm{relint}\, A$, $\mathrm{relbd}\, A$ are the relative interior and relative boundary, that is, the interior and boundary of A relative to its affine hull.

The scalar product in $\mathbb{R}^n$ will often be used to describe hyperplanes and halfspaces. A *hyperplane* of $\mathbb{R}^n$ can be written in the form

$$H_{u,\alpha} = \{x \in \mathbb{R}^n : \langle x, u \rangle = \alpha\}$$

with $u \in \mathbb{R}^n \setminus \{o\}$ and $\alpha \in \mathbb{R}$; here $H_{u,\alpha} = H_{v,\beta}$ if and only if $(v, \beta) = (\lambda u, \lambda\alpha)$ with $\lambda \neq 0$. (We warn the reader that for ‘supporting hyperplane’ we shall often say ‘support plane’, for short.) We say that u is a *normal vector* of $H_{u,\alpha}$. The hyperplane $H_{u,\alpha}$ bounds the two *closed halfspaces*

$$H^-_{u,\alpha} := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq \alpha\}, \qquad H^+_{u,\alpha} := \{x \in \mathbb{R}^n : \langle x, u \rangle \geq \alpha\}.$$

Occasionally we also use $\langle \cdot, \cdot \rangle$ to denote the scalar product on $\mathbb{R}^n \times \mathbb{R}$ which is given by

$$\langle (x, \xi), (y, \eta) \rangle = \langle x, y \rangle + \xi\eta.$$

An affine subspace of $\mathbb{R}^n$ is often called a *flat*, and the intersection of a flat with a closed halfspace meeting the flat but not entirely containing it will be called a *half-flat*. A one-dimensional flat is a *line* and a one-dimensional half-flat a *ray*.

Concerning set-theoretic notation, we point out that we use the inclusion symbol $\subset$ always in the meaning of $\subseteq$, and that for a subset $A \subset \mathbb{R}^n$ we must well distinguish between its *characteristic function*, defined by

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus A, \end{cases}$$

and its *indicator function*, given by

$$I^\infty_A(x) := \begin{cases} 0 & \text{for } x \in A, \\ \infty & \text{for } x \in \mathbb{R}^n \setminus A. \end{cases}$$

If $P(x)$ is an assertion about the elements x of a set A, we use also the notation

$$\mathbf{1}\{P(x)\} := \begin{cases} 1 & \text{if } P(x) \text{ is true}, \\ 0 & \text{if } P(x) \text{ is false}. \end{cases}$$

The following metric notions will be used. For $x, y \in \mathbb{R}^n$ and $\emptyset \neq A \subset \mathbb{R}^n$, $|x - y|$ is the *distance* between x and y and

$$d(A, x) := \inf\{|x - a| : a \in A\}$$

is the distance of x from A. For a bounded set $\emptyset \neq A \subset \mathbb{R}^n$,

$$\operatorname{diam} A := \sup\{|x - y| : x, y \in A\}$$

is the *diameter* of A. It is also denoted by $D(A)$. We write

$$B(z, \rho) := \{x \in \mathbb{R}^n : |x - z| \leq \rho\}$$

and

$$B_0(z, \rho) := \{x \in \mathbb{R}^n : |x - z| < \rho\},$$

respectively, for the closed ball and the open ball with centre $x \in \mathbb{R}^n$ and radius $\rho > 0$. The set $B^n := B(o, 1)$ is the *unit ball* and

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$$

is the *unit sphere* of $\mathbb{R}^n$.

By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff (outer) measure on $\mathbb{R}^n$, where $0 \leq k \leq n$. If A is a Borel subset of a k-dimensional flat E^k or a k-dimensional sphere $\mathbb{S}^k$ in $\mathbb{R}^n$, then $\mathcal{H}^k(A)$ coincides, respectively, with the k-dimensional Lebesgue measure of A computed in E^k, or with the k-dimensional spherical Lebesgue measure of A computed in $\mathbb{S}^k$. Hence, all integrations with respect to these Lebesgue measures can be expressed by means of the Hausdorff measure $\mathcal{H}^k$. In integrals with respect to $\mathcal{H}^n$ we often abbreviate $\mathrm{d}\mathcal{H}^n(x)$ by $\mathrm{d}x$. Similarly, in integrals over the unit sphere $\mathbb{S}^{n-1}$, instead of $\mathrm{d}\mathcal{H}^{n-1}(u)$ we write $\mathrm{d}u$. Occasionally, spherical Lebesgue measure on $\mathbb{S}^{n-1}$ is denoted by σ. The n-dimensional measure of the unit ball in $\mathbb{R}^n$ is denoted by κ_n and its surface area by ω_n, thus

$$\kappa_n = \mathcal{H}^n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)}, \quad \omega_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

(In choosing this notation, we follow Bonnesen and Fenchel [284], which has, unfortunately, not set a standard.) We use the definition $\kappa_p = \pi^{p/2}/\Gamma(1 + p/2)$ for arbitrary $p \geq 0$, not just for positive integers.

Linear maps, affine maps and isometries between Euclidean spaces are defined as usual. In particular, a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a *translation* if $\varphi(x) = x + t$ for $x \in \mathbb{R}^n$ with some fixed vector $t \in \mathbb{R}^n$, the *translation vector*. The set $A + t$ is called the *translate* of A by t. The map φ is a *dilatation* if $\varphi(x) = \lambda x$ for $x \in \mathbb{R}^n$ with some $\lambda > 0$. The set λA with $\lambda > 0$ is called a *dilatate* of A. The map φ is a *homothety* if $\varphi(x) = \lambda x + t$ for $x \in \mathbb{R}^n$ with some $\lambda > 0$ and some $t \in \mathbb{R}^n$. The set $\lambda A + t$ with $\lambda > 0$ is called a *homothet* of A. Sets A, B are called *positively homothetic* if $A = \lambda B + t$ with $t \in \mathbb{R}^n$ and $\lambda > 0$, and *homothetic* if either they are positively homothetic or one of them is a singleton (a one-pointed set). A *rigid motion* of $\mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ onto itself, and it is a *rotation* if it is an isometry fixing the origin. Every rigid motion is the composition of a rotation and a translation. A rigid motion is called *proper* if it preserves the orientation of $\mathbb{R}^n$; otherwise it is called *improper*. A rotation of $\mathbb{R}^n$ is a linear map; it preserves the scalar product and can be represented, with respect to an orthonormal basis, by an orthogonal matrix; this matrix has determinant 1 if and only if the rotation is proper. The composition of a rigid motion and a dilatation is called a *similarity*.

By $\mathrm{GL}(n)$ we denote the general linear group of $\mathbb{R}^n$ and by $\mathrm{SL}(n)$ the subgroup of volume-preserving and orientation-preserving linear mappings. The subgroup $\mathrm{SO}(n)$ is the group of proper rotations. With the topology induced by the usual matrix norm, both are topological groups, and $\mathrm{SO}(n)$ is compact. The group of proper rigid motions of $\mathbb{R}^n$ is denoted by G_n and topologized as usual. Also, the Grassmannian $G(n,k)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ and the set $A(n,k)$ of k-dimensional affine subspaces of $\mathbb{R}^n$ are endowed with their standard topologies.

The Haar measures on $\mathrm{SO}(n)$, G_n, $G(n,k)$, $A(n,k)$ are denoted, respectively, by ν, μ, ν_k, μ_k. We normalize ν by $\nu(\mathrm{SO}(n)) = 1$ and ν_k by $\nu_k(G(n,k)) = 1$. The normalizations of the measures μ and μ_k will be fixed in Section 4.4 when they are needed.

For an affine subspace E of $\mathbb{R}^n$, we denote by proj_E the orthogonal projection from $\mathbb{R}^n$ onto E. We often write $\mathrm{proj}_E A =: A \,|\, E$ for $A \subset \mathbb{R}^n$ (since A is a set, no confusion with the restriction of a function, for example $f|E$, can arise).

Some final remarks are in order. Since any k-dimensional affine subspace E of $\mathbb{R}^n$ is the image of $\mathbb{R}^k$ under some isometry, it is clear (and common practice without mention) that all notions and results that have been established for $\mathbb{R}^k$ and are invariant under isometries can be applied in E; similarly for affine-invariant notions and results.

The following notational conventions will be useful at several places. If f is a homogeneous function on $\mathbb{R}^n$, then $\bar{f}$ denotes its restriction to the unit sphere $\mathbb{S}^{n-1}$. Very often, mappings of the type $f : \mathcal{K} \times \mathbb{R}^n \to M$ will occur where $\mathcal{K}$ is some class of subsets of $\mathbb{R}^n$. In this case we usually abbreviate, for fixed $K \in \mathcal{K}$, the function $f(K,\cdot) : \mathbb{R}^n \to M$ by f_K.

We wish to point out that in definitions the word 'if' is always understood as 'if and only if'.

Finally, a remark about citations. When we list several publications consecutively, particularly in the chapter notes, we usually order them chronologically, and not in the order in which they appear in the list of references.

1

Basic convexity

1.1 Convex sets and combinations

A set $A \subset \mathbb{R}^n$ is *convex* if together with any two points x, y it contains the segment $[x, y]$, thus if

$$(1 - \lambda)x + \lambda y \in A \quad \text{for } x, y \in A,\ 0 \le \lambda \le 1.$$

Examples of convex sets are obvious; but observe also that $B_0(z, \rho) \cup A$ is convex if A is an arbitrary subset of the boundary of the open ball $B_0(z, \rho)$. As immediate consequences of the definition we note that intersections of convex sets are convex, affine images and pre-images of convex sets are convex, and if A, B are convex, then $A + B$ and λA ($\lambda \in \mathbb{R}$) are convex.

Remark 1.1.1 For $A \subset \mathbb{R}^n$ and $\lambda, \mu > 0$ one trivially has $\lambda A + \mu A \supset (\lambda + \mu)A$. Equality (for all $\lambda, \mu > 0$) holds precisely if A is convex. In fact, if A is convex and $x \in \lambda A + \mu A$, then $x = \lambda a + \mu b$ with $a, b \in A$ and hence

$$x = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu} a + \frac{\mu}{\lambda + \mu} b\right) \in (\lambda + \mu)A;$$

thus $\lambda A + \mu A = (\lambda + \mu)A$. If this equation holds, then A is clearly convex.

A set $A \subset \mathbb{R}^n$ is called a *convex cone* if A is convex and nonempty and if $x \in A$, $\lambda \ge 0$ implies $\lambda x \in A$. Thus, a nonempty set $A \subset \mathbb{R}^n$ is a convex cone if and only if A is closed under addition and under multiplication by nonnegative real numbers.

By restricting affine and linear combinations to nonnegative coefficients, one obtains the following two fundamental notions. The point $x \in \mathbb{R}^n$ is a *convex combination* of the points $x_1, \dots, x_k \in \mathbb{R}^n$ if there are numbers $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i \ge 0\ (i = 1, \dots, k),\ \sum_{i=1}^{k} \lambda_i = 1.$$

Similarly, the vector $x \in \mathbb{R}^n$ is a *positive combination* of vectors $x_1, \dots, x_k \in \mathbb{R}^n$ if

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i \geq 0 \ (i = 1, \dots, k).$$

For $A \subset \mathbb{R}^n$, the set of all convex combinations (positive combinations) of any finitely many elements of A is called the *convex hull (positive hull)* of A and is denoted by $\operatorname{conv} A$ ($\operatorname{pos} A$).

Theorem 1.1.2 *If $A \subset \mathbb{R}^n$ is* convex, *then* $\operatorname{conv} A = A$. *For an arbitrary set $A \subset \mathbb{R}^n$,* $\operatorname{conv} A$ *is the intersection of all convex subsets of $\mathbb{R}^n$ containing A. If $A, B \subset \mathbb{R}^n$, then* $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$.

Proof Let A be convex. Trivially, $A \subset \operatorname{conv} A$. By induction we show that A contains all convex combinations of any k points of A. For $k = 2$ this holds by the definition of convexity. Suppose that it holds for $k - 1$ and that $x = \lambda_1 x_1 + \dots + \lambda_k x_k$ with $x_1, \dots, x_k \in A$, $\lambda_1 + \dots + \lambda_k = 1$ and $\lambda_1, \dots, \lambda_k > 0$, without loss of generality. Then

$$x = (1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i + \lambda_k x_k \in A,$$

since

$$\frac{\lambda_i}{1 - \lambda_k} > 0, \quad \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} = 1$$

and hence

$$\sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i \in A,$$

by hypothesis. This proves that $A = \operatorname{conv} A$. For arbitrary $A \subset \mathbb{R}^n$, let $C(A)$ be the intersection of all convex sets $K \subset \mathbb{R}^n$ containing A. Since $A \subset \operatorname{conv} A$ and $\operatorname{conv} A$ is evidently convex, we have $C(A) \subset \operatorname{conv} A$. Every convex set K with $A \subset K$ satisfies $\operatorname{conv} A \subset \operatorname{conv} K = K$, hence $\operatorname{conv} A \subset C(A)$, which proves the equality.

Let $A, B \subset \mathbb{R}^n$. Let $x \in \operatorname{conv}(A + B)$, thus

$$x = \sum_{i=1}^{k} \lambda_i (a_i + b_i) \quad \text{with } a_i \in A,\ b_i \in B,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1$$

and hence $x = \sum \lambda_i a_i + \sum \lambda_i b_i \in \operatorname{conv} A + \operatorname{conv} B$. Let $x \in \operatorname{conv} A + \operatorname{conv} B$, thus

$$x = \sum_i \lambda_i a_i + \sum_j \mu_j b_j$$

with $a_i \in A$, $b_j \in B$, $\lambda_i, \mu_j \geq 0$, $\sum \lambda_i = \sum \mu_j = 1$. We may write

$$x = \sum_{i,j} \lambda_i \mu_j (a_i + b_j)$$

and deduce that $x \in \operatorname{conv}(A + B)$. $\square$

An immediate consequence is that $\operatorname{conv}(\operatorname{conv} A) = \operatorname{conv} A$.

Theorem 1.1.3 *If $A \subset \mathbb{R}^n$ is a convex cone, then* $\operatorname{pos} A = A$. *For a nonempty set $A \subset \mathbb{R}^n$,* $\operatorname{pos} A$ *is the intersection of all convex cones in $\mathbb{R}^n$ containing A. If $A, B \subset \mathbb{R}^n$, then* $\operatorname{pos}(A + B) \subset \operatorname{pos} A + \operatorname{pos} B$.

Proof As above. □

That the last inclusion in the preceding theorem may be strict is shown by the example $A = \{a\}$, $B = \{b\}$ with linearly independent vectors $a, b \in \mathbb{R}^n$.

The following result on the generation of convex hulls is fundamental.

Theorem 1.1.4 (Carathéodory's theorem) *If $A \subset \mathbb{R}^n$ and $x \in \operatorname{conv} A$, then x is a convex combination of affinely independent points of A. In particular, x is a convex combination of $n + 1$ or fewer points of A.*

Proof The point $x \in \operatorname{conv} A$ has a representation

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } x_i \in A,\ \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1,$$

with some $k \in \mathbb{N}$, and we may assume that k is minimal. Suppose that $x_1, \dots, x_k$ are affinely dependent. Then there are numbers $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, not all zero, with

$$\sum_{i=1}^{k} \alpha_i x_i = o \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 0.$$

We can choose m such that λ_m/α_m is positive and, with this restriction, as small as possible (observe that all λ_i are positive and at least one α_i is positive). In the affine representation

$$x = \sum_{i=1}^{k} \left(\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) x_i,$$

all coefficients are nonnegative (trivially, if $\alpha_i \le 0$, otherwise by the choice of m), and at least one of them is zero. This contradicts the minimality of k. Thus, $x_1, \dots, x_k$ are affinely independent, which implies that $k \le n + 1$. □

The convex hull of finitely many points is called a *polytope*. A *k-simplex* is the convex hull of $k + 1$ affinely independent points, and these points are the *vertices* of the simplex. Thus, Carathéodory's theorem states that $\operatorname{conv} A$ is the union of all simplices with vertices in A.

Another equally simple and important result on convex hulls is the following.

Theorem 1.1.5 (Radon's theorem) *Every set of affinely dependent points (in particular, every set of at least $n + 2$ points) in $\mathbb{R}^n$ can be expressed as the union of two disjoint sets whose convex hulls have a common point.*

Proof If $x_1, \dots, x_k$ are affinely dependent, there are numbers $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, not all zero, with

$$\sum_{i=1}^{k} \alpha_i x_i = o \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 0.$$

We may assume, after renumbering, that $\alpha_i > 0$ precisely for $i = 1, \dots, j$; then $1 \le j < k$ (at least one α_i is $\neq 0$, say > 0, but not all α_i are > 0). With

$$\alpha := \alpha_1 + \cdots + \alpha_j = -(\alpha_{j+1} + \cdots + \alpha_k) > 0$$

we obtain

$$x := \sum_{i=1}^{j} \frac{\alpha_i}{\alpha} x_i = \sum_{i=j+1}^{k} \left(-\frac{\alpha_i}{\alpha}\right) x_i$$

and thus $x \in \operatorname{conv}\{x_1, \dots, x_j\} \cap \operatorname{conv}\{x_{j+1}, \dots, x_k\}$. The assertion follows. □

From Radon's theorem one easily deduces Helly's theorem, a fundamental and typical result of the combinatorial geometry of convex sets.

Theorem 1.1.6 (Helly's theorem) *Let $\mathcal{M}$ be a finite family of convex sets in $\mathbb{R}^n$. If any $n + 1$ elements of $\mathcal{M}$ have a common point, then all elements of $\mathcal{M}$ have a common point.*

Proof Let $A_1, \dots, A_k$ be the sets of $\mathcal{M}$. Suppose that $k > n + 1$ (for $k < n + 1$ there is nothing to prove, and for $k = n + 1$ the assertion is trivial) and that the assertion is proved for $k - 1$ convex sets. Then for $i \in \{1, \dots, k\}$ there exists a point

$$x_i \in A_1 \cap \cdots \cap \check{A}_i \cap \cdots \cap A_k$$

where $\check{A}_i$ indicates that A_i has been deleted. The $k \ge n+2$ points $x_1, \dots, x_k$ are affinely dependent; hence from Radon's theorem we can infer that, after renumbering, there is a point

$$x \in \operatorname{conv}\{x_1, \dots, x_j\} \cap \operatorname{conv}\{x_{j+1}, \dots, x_k\}$$

for some $j \in \{1, \dots, k - 1\}$. Because $x_1, \dots, x_j \in A_{j+1}, \dots, A_k$ we have

$$x \in \operatorname{conv}\{x_1, \dots, x_j\} \subset A_{j+1} \cap \cdots \cap A_k,$$

similarly $x \in \operatorname{conv}\{x_{j+1}, \dots, x_k\} \subset A_1 \cap \cdots \cap A_j$. □

Here is a little example (another one is Theorem 1.3.11) to demonstrate how Helly's theorem can be applied to obtain elegant results of a similar nature.

Theorem 1.1.7 *Let $\mathcal{M}$ be a finite family of convex sets in $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be convex. If any $n + 1$ elements of $\mathcal{M}$ are intersected by some translate of K, then all elements of $\mathcal{M}$ are intersected by a suitable translate of K.*

Proof Let $\mathcal{M} = \{A_1, \dots, A_k\}$. To any $n+1$ elements of $\{1, \dots, k\}$, say $1, \dots, n+1$, there are $t \in \mathbb{R}^n$ and $x_i \in A_i \cap (K+t)$, hence $-t \in K - A_i$, for $i = 1, \dots, n+1$. Thus, any $n+1$ elements of the family $\{K - A_1, \dots, K - A_k\}$ have nonempty intersection. By Helly's theorem, there is a vector $-t \in \mathbb{R}^n$ with $-t \in K - A_i$ and hence $A_i \cap (K+t) \neq \phi$ for $i \in \{1, \dots, k\}$. □

Some of these results of combinatorial convexity have 'colourful versions', of which we give a simple example.

Theorem 1.1.8 (Coloured Radon theorem) *Let $F_1, \dots, F_{n+1}$ be two-pointed sets in $\mathbb{R}^n$. Their union has a partition into sets A, B such that each of A, B contains a point from each of the sets $F_1, \dots, F_{n+1}$ ('one of each colour') and the convex hulls of A and B have a common point.*

Proof Let $F_i = \{x_i, y_i\}$, $i = 1, \dots, n+1$. There is a non-trivial linear relation $\sum_{i=1}^{n+1} \alpha_i (x_i - y_i) = o$. Interchanging the notation for the elements of F_i where necessary, we can assume that $\alpha_i \geq 0$ for all i. After multiplication with a constant, we can also assume that $\sum_{i=1}^{n+1} \alpha_i = 1$. Then the relation

$$\sum_{i=1}^{n+1} \alpha_i x_i = \sum_{i=1}^{n+1} \alpha_i y_i$$

proves the assertion. □

Next we look at the interplay between convexity and topological properties. We start with a simple but useful observation.

Lemma 1.1.9 *Let $A \subset \mathbb{R}^n$ be convex. If $x \in \operatorname{int} A$ and $y \in \operatorname{cl} A$, then $[x, y) \subset \operatorname{int} A$.*

Proof Let $z = (1-\lambda)y + \lambda x$ with $0 < \lambda < 1$. We have $B(x, \rho) \subset A$ for some $\rho > 0$; put $B(o, \rho) =: U$. First we assume $y \in A$. Let $w \in \lambda U + z$, hence $w = \lambda u + z$ with $u \in U$. Then $x + u \in A$, hence $w = (1-\lambda)y + \lambda(x+u) \in A$. This shows that $\lambda U + z \subset A$ and thus $z \in \operatorname{int} A$.

Now we assume merely that $y \in \operatorname{cl} A$. Put $V := [\lambda/(1-\lambda)]U + y$. There is some $a \in A \cap V$. We have $a = [\lambda/(1-\lambda)]u + y$ with $u \in U$ and hence $z = (1-\lambda)a + \lambda(x-u) \in A$. This proves that $[x, y) \subset A$, which together with the first part yields $[x, y) \subset \operatorname{int} A$. □

Theorem 1.1.10 *If $A \subset \mathbb{R}^n$ is convex, then* $\operatorname{int} A$ *and* $\operatorname{cl} A$ *are convex. If $A \subset \mathbb{R}^n$ is open, then* $\operatorname{conv} A$ *is open.*

Proof The convexity of $\operatorname{int} A$ follows from Lemma 1.1.9. The convexity of $\operatorname{cl} A$ for convex A and the openness of $\operatorname{conv} A$ for open A are easy exercises. □

The union of a line and a point not on it is an example of a closed set whose convex hull is not closed. This cannot happen for compact sets, as a first application of Carathéodory's theorem shows.

Theorem 1.1.11 *If $A \subset \mathbb{R}^n$, then* $\operatorname{conv} \operatorname{cl} A \subset \operatorname{cl} \operatorname{conv} A$. *If A is bounded, then* $\operatorname{conv} \operatorname{cl} A = \operatorname{cl} \operatorname{conv} A$. *In particular, the convex hull of a compact set is compact.*

Proof The inclusion $\operatorname{conv}\operatorname{cl} A \subset \operatorname{cl}\operatorname{conv} A$ is easy to see. Let A be bounded. Then

$$\left\{(\lambda_1, \dots, \lambda_{n+1}, x_1, \dots, x_{n+1}) : \lambda_i \geq 0,\ x_i \in \operatorname{cl} A,\ \sum_{i=1}^{n+1} \lambda_i = 1\right\}$$

is a compact subset of $\mathbb{R}^{n+1} \times (\mathbb{R}^n)^{n+1}$, hence its image under the continuous map

$$(\lambda_1, \dots, \lambda_{n+1}, x_1, \dots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} \lambda_i x_i \in \mathbb{R}^n$$

is compact. By Carathéodory's theorem, this image is equal to $\operatorname{conv}\operatorname{cl} A$. Thus $\operatorname{cl}\operatorname{conv} A \subset \operatorname{cl}\operatorname{conv}\operatorname{cl} A = \operatorname{conv}\operatorname{cl} A$. □

The set $\operatorname{cl}\operatorname{conv} A$, which by Theorem 1.1.10 is convex, is called for short the *closed convex hull* of A. This is also the intersection of all closed convex subsets of $\mathbb{R}^n$ containing A.

To obtain information on the relative interiors of convex hulls, we first consider simplices.

Lemma 1.1.12 *Let $x_1, \dots, x_k \in \mathbb{R}^n$ be affinely independent; let*

$$S := \operatorname{conv}\{x_1, \dots, x_k\}$$

and $x \in \operatorname{aff} S$. Then $x \in \operatorname{relint} S$ if and only if in the unique affine representation

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \textit{with} \quad \sum_{i=1}^{k} \lambda_i = 1$$

all coefficients λ_i are positive.

Proof Clearly, we may assume that $k = n + 1$. The condition is necessary since otherwise, because the representation is unique, an arbitrary neighbourhood of x would contain points not belonging to S. To prove sufficiency, let x be represented as above with all $\lambda_i > 0$. Since $x_1, \dots, x_{n+1}$ are affinely independent, the vectors $\tau(x_1), \dots, \tau(x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ (where $\tau(x) := (x, 1)$) form a linear basis of $\mathbb{R}^n \times \mathbb{R}$, and for $y \in \mathbb{R}^n$ the coefficients $\mu_1, \dots, \mu_{n+1}$ in the affine representation

$$y = \sum_{i=1}^{n+1} \mu_i x_i \quad \text{with} \quad \sum_{i=1}^{n+1} \mu_i = 1$$

(the 'barycentric coordinates' of y) are just the coordinates of $\tau(y)$ with respect to this basis. Since coordinate functions in $\mathbb{R}^{n+1}$ are continuous, the coefficients $\mu_1, \dots, \mu_{n+1}$ depend continuously on y. Therefore, a number $\delta > 0$ can be chosen such that $\mu_i > 0$ $(i = 1, \dots, n+1)$ and thus $y \in S$ for all y with $|y - x| < \delta$. This proves that $x \in \operatorname{int} S$. □

Theorem 1.1.13 *If $A \subset \mathbb{R}^n$ is convex and nonempty, then* $\operatorname{relint} A \neq \emptyset$.

Proof Let $\dim \operatorname{aff} A = k$, then there are $k + 1$ affinely independent points in A. Their convex hull S satisfies $\operatorname{relint} S \neq \emptyset$ by Lemma 1.1.12; furthermore, $S \subset A$ and $\operatorname{aff} S = \operatorname{aff} A$. □

In view of this theorem, it makes sense to define the *dimension*, $\dim A$, of a convex set A as the dimension of its affine hull. The points of $\operatorname{relint} A$ are also called *internal* points of A.

The description of $\operatorname{relint}\operatorname{conv} A$ for an affinely independent set A, given by Lemma 1.1.12, can be extended to arbitrary finite sets.

Theorem 1.1.14 *Let $x_1, \dots, x_k \in \mathbb{R}^n$, let $P := \operatorname{conv}\{x_1, \dots, x_k\}$ and $x \in \mathbb{R}^n$. Then $x \in \operatorname{relint} P$ if and only if x can be represented in the form*

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \textit{with } \lambda_i > 0 \ (i = 1, \dots, k), \ \sum_{i=1}^{k} \lambda_i = 1.$$

Proof We may clearly assume that $\dim P = n$. Suppose that $x \in \operatorname{int} P$. Put

$$y := \sum_{i=1}^{k} \frac{1}{k} x_i,$$

then $y \in P$. Since $x \in \operatorname{int} P$, we can choose $z \in P$ for which $x \in [y, z)$. There are representations

$$z = \sum_{i=1}^{k} \mu_i x_i \quad \text{with } \mu_i \geq 0, \ \sum_{i=1}^{k} \mu_i = 1,$$

$$x = (1 - \lambda) y + \lambda z \quad \text{with } 0 \leq \lambda < 1,$$

which gives

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i = (1 - \lambda)\frac{1}{k} + \lambda \mu_i > 0, \ \sum_{i=1}^{k} \lambda_i = 1.$$

Conversely, suppose that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i > 0, \ \sum_{i=1}^{k} \lambda_i = 1.$$

We may assume that $x_1, \dots, x_{n+1}$ are affinely independent. Put $\lambda_1 + \cdots + \lambda_{n+1} =: \lambda$ and

$$y := \sum_{i=1}^{n+1} \frac{\lambda_i}{\lambda} x_i.$$

Lemma 1.1.12 gives $y \in \operatorname{int}\operatorname{conv}\{x_1, \dots, x_{n+1}\} \subset \operatorname{int} P$. If $k = n + 1$, then $x = y \in \operatorname{int} P$. Otherwise, put

$$z := \sum_{i=n+1}^{k} \frac{\lambda_i}{1 - \lambda} x_i.$$

Then $z \in P$ and $x \in [y, z) \subset \operatorname{int} P$, by Lemma 1.1.9. $\square$

Theorem 1.1.15 *Let $A \subset \mathbb{R}^n$ be convex. Then*

(a) $\operatorname{relint} A = \operatorname{relint} \operatorname{cl} A$,

(b) $\operatorname{cl} A = \operatorname{cl} \operatorname{relint} A$,

(c) $\operatorname{relbd} A = \operatorname{relbd} \operatorname{cl} A = \operatorname{relbd} \operatorname{relint} A$.

Proof We may assume that $\dim A = n$. Part (a): trivially, $\operatorname{int} A \subset \operatorname{int} \operatorname{cl} A$. Let $x \in \operatorname{int} \operatorname{cl} A$. Choose $y \in \operatorname{int} A$. There is $z \in \operatorname{cl} A$ with $x \in [y, z)$ and Lemma 1.1.9 shows that $x \in \operatorname{int} A$. Part (b): trivially, $\operatorname{cl} A \supset \operatorname{cl} \operatorname{int} A$. Let $x \in \operatorname{cl} A$. Choose $y \in \operatorname{int} A$. By Lemma 1.1.9 we have $[y, x) \subset \operatorname{int} A$, hence $x \in \operatorname{cl} \operatorname{int} A$. Part (c): $\operatorname{bd} \operatorname{cl} A = \operatorname{cl} (\operatorname{cl} A) \setminus \operatorname{int} (\operatorname{cl} A) = \operatorname{cl} A \setminus \operatorname{int} A = \operatorname{bd} A$, using (a). Then $\operatorname{bd} \operatorname{int} A = \operatorname{cl} (\operatorname{int} A) \setminus \operatorname{int} (\operatorname{int} A) = \operatorname{cl} A \setminus \operatorname{int} A = \operatorname{bd} A$, using (b). □

We end this section with a definition of the central notion of this book. A nonempty, compact, convex subset of $\mathbb{R}^n$ is called a *convex body*. (Thus, in our terminology, a convex body need not have interior points. We warn the reader that many authors reserve the term ‘body’ for sets with interior points. However, we prefer to avoid endless repetitions, in this book, of the expression ‘nonempty, compact, convex subset’.) By $\mathcal{K}^n$ we denote the set of all convex bodies in $\mathbb{R}^n$ and by $\mathcal{K}^n_n$ the subset of convex bodies with interior points (thus, the lower index n stands for the dimension of the bodies). For $\emptyset \neq A \subset \mathbb{R}^n$ we write $\mathcal{K}(A)$ for the set of convex bodies contained in A and $\mathcal{K}_n(A) = \mathcal{K}(A) \cap \mathcal{K}^n_n$. Further, $\mathcal{P}^n$ denotes the set of nonempty polytopes in $\mathbb{R}^n$ and $\mathcal{P}^n_n = \mathcal{P}^n \cap \mathcal{K}^n_n$ is the subset of n-dimensional polytopes.

Notes for Section 1.1

1. The early history of the theorems of Carathéodory, Radon and Helly, and many generalizations, ramifications and analogues of these theorems, forming an essential part of combinatorial convexity, can be studied in the survey article of Danzer, Grünbaum and Klee [464], which is still strongly recommended. Various results related to Carathéodory’s theorem can be found in Reay [1561]. Sufficient conditions on a compact set in $\mathbb{R}^n$ to have Carathéodory number less than $n + 1$, and related results, were given by Bárány and Karasëv [146].

   The proof of the coloured Radon theorem 1.1.8 given here is due to Soberón [1796] (see Bárány and Larman [149] for more history).

   An important extension of Radon’s theorem was proved by Tverberg [1858, 1859]:

   Theorem (Tverberg) Every set of at least $(k - 1)(n + 1) + 1$ points in $\mathbb{R}^n$ (where $k \geq 2$) can be partitioned into k subsets whose convex hulls have a common point.

   A survey is given by Eckhoff [526]. There one also finds hints about versions of the theorems of Carathéodory, Radon and Helly in the abstract setting of so-called convexity spaces. Later surveys on Helly’s and related theorems are due to Eckhoff [528] and Wenger [1958]. For a proof of Tverberg’s theorem and information about later developments, such as the Coloured Tverberg theorem, see Matoušek [1362], §8.3.

   For more recent ‘colourful versions’ of theorems in combinatorial convexity, we refer to Arocha, Bárány, Bracho, Fabila and Montejano [77] and to Blagojević, Matschke and Ziegler [235, 236].

   Another variant of the classical theorems of combinatorial convexity are such ‘with tolerance’, first introduced by Montejano and Oliveros [1447]. The following example is due to Soberón and Strausz [1797]:

Theorem Every set S of $(r+1)(k-1)(n+1)+1$ points in $\mathbb{R}^n$ (where $k \geq 2$) has a partition in k sets $A_1, \dots, A_k$ such that, for any $C \subset S$ of at most r points, the convex hulls of $A_1 \setminus C, \dots, A_k \setminus C$ have a common point.

2. It is clear how a version of Carathéodory's theorem for convex cones is to be formulated and how it can be proved. A common generalization, a version of Carathéodory's theorem for 'convex hulls of points and directions', is given by Rockafellar [1583], Theorem 17.1.

1.2 The metric projection

In this section, $A \subset \mathbb{R}^n$ is a fixed nonempty closed convex set. To each $x \in \mathbb{R}^n$ there exists a unique point $p(A, x) \in A$ satisfying

$$|x - p(A, x)| \leq |x - y| \quad \text{for all } y \in A.$$

In fact, for suitable $\rho > 0$ the set $B(x, \rho) \cap A$ is compact and nonempty, hence the continuous function $y \mapsto |x-y|$ attains a minimum on this set, say at y_0; then $|x-y_0| \leq |x-y|$ for all $y \in A$. If also $y_1 \in A$ satisfies $|x-y_1| \leq |x-y|$ for all $y \in A$, then $z := (y_0+y_1)/2 \in A$ and $|x-z| < |x-y_0|$, except if $y_0 = y_1$. Thus, $y_0 =: p(A, x)$ is unique.

In this way, a map $p(A, \cdot) : \mathbb{R}^n \to A$ is defined; it is called the *metric projection* or *nearest-point map* of A. It will play an essential role in Chapter 4, when the volume of local parallel sets is investigated. It also provides a simple approach to the basic support and separation properties of convex sets (see the next section), as used by Botts [309] and by McMullen and Shephard [1398].

We have $|x - p(A, x)| = d(A, x)$. For $x \in \mathbb{R}^n \setminus A$ we denote by

$$u(A, x) := \frac{x - p(A, x)}{d(A, x)}$$

the unit vector pointing from the nearest point $p(A, x)$ to x and by

$$R(A, x) := \{p(A, x) + \lambda u(A, x) : \lambda \geq 0\}$$

the ray through x with endpoint $p(A, x)$.

Theorem 1.2.1 *The metric projection is contracting, that is,*

$$|p(A, x) - p(A, y)| \leq |x - y| \quad \textit{for } x, y \in \mathbb{R}^n.$$

Proof We may assume that $v := p(A, y) - p(A, x) \neq o$. The function f defined by $f(t) := |x - (p(A, x) + tv)|^2$ for $t \in [0, 1]$ has a minimum at $t = 0$, hence $f'(0) \geq 0$. This gives $\langle x - p(A, x), v \rangle \leq 0$. Similarly we obtain $\langle y - p(A, y), v \rangle \geq 0$. Thus, the segment $[x, y]$ meets the two hyperplanes that are orthogonal to v and that go through $p(A, x)$ and $p(A, y)$, respectively. Now the assertion is obvious. □

Lemma 1.2.2 *Let $x \in \mathbb{R}^n \setminus A$ and $y \in R(A, x)$; then $p(A, x) = p(A, y)$.*

Proof With the notation and auxiliary results of the previous proof, we have $\langle x - p(A, x), v\rangle \le 0$ and $\langle y - p(A, y), v\rangle \ge 0$. Since $y \in R(A, x)$, the first inequality yields $\langle y - p(A, x), v\rangle \le 0$ and together with the second this gives $v = o$. □

Lemma 1.2.3 *Let S be the boundary of a ball containing A in its interior. Then* $p(A, S) = \mathrm{bd}\, A$.

Proof The inclusion $p(A, S) \subset \mathrm{bd}\, A$ is clear. Let $x \in \mathrm{bd}\, A$. For $i \in \mathbb{N}$ choose x_i in the ball bounded by S such that $x_i \notin A$ and $|x_i - x| < 1/i$. From Theorem 1.2.1 we have

$$|x - p(A, x_i)| = |p(A, x) - p(A, x_i)| \le |x - x_i| < \frac{1}{i}.$$

The ray $R(A, x_i)$ meets S in a point y_i and we have $p(A, y_i) = p(A, x_i)$, hence $|x - p(A, y_i)| < 1/i$. A subsequence $(y_{i_j})_{j\in\mathbb{N}}$ converges to a point $y \in S$. From $\lim p(A, y_i) = x$ and the continuity of the metric projection we see that $x = p(A, y)$. Thus $\mathrm{bd}\, A \subset p(A, S)$. □

The existence of a unique nearest-point map is characteristic of convex sets. We prove this result here to complete the picture, although no use will be made of it.

Theorem 1.2.4 *Let $A \subset \mathbb{R}^n$ be a closed set with the property that to each point of $\mathbb{R}^n$ there is a unique nearest point in A. Then A is convex.*

Proof Suppose A satisfies the assumption but is not convex. Then there are points x, y with $[x, y] \cap A = \{x, y\}$, and one can choose $\rho > 0$ such that the ball $B = B((x + y)/2, \rho)$ satisfies $B \cap A = \emptyset$. By an elementary compactness argument, the family $\mathcal{B}$ of all closed balls B' containing B and satisfying $(\mathrm{int}\, B') \cap A = \emptyset$ contains a ball C with maximal radius. By this maximality, there is a point $p \in C \cap A$, and, by the assumed uniqueness of nearest points in A, it is unique. If $\mathrm{bd}\, B$ and $\mathrm{bd}\, C$ have a common point, let this (unique) point be q; otherwise let q be the centre of B. For sufficiently small $\varepsilon > 0$, the ball $C + \varepsilon(q - p)$ includes B and does not meet A. Hence, the family $\mathcal{B}$ contains an element with greater radius than that of C, a contradiction. □

Note for Section 1.2

1. Theorem 1.2.4 was found independently (in a more general form) by Bunt [354] and Motzkin [1453]; it is usually associated with the name of Motzkin. In general, a subset A of a metric space is called a Chebyshev set if for each point of the space there is a unique nearest point in A. There are several results and interesting open problems concerning the convexity of Chebyshev sets in normed linear spaces. For more information, see Valentine [1866], Chapter VII, Marti [1331], Chapter IX, Vlasov [1894] and §6 of the survey article by Burago and Zalgaller [356].

1.3 Support and separation

The simplest support and separation properties of convex sets seem intuitively obvious, and they are easy to prove. Nevertheless, their numerous applications make them a basic tool in convexity.

Let $A \subset \mathbb{R}^n$ be a subset and $H \subset \mathbb{R}^n$ a hyperplane and let H^+, H^- denote the two closed halfspaces bounded by H. We say that H *supports* A at x if $x \in A \cap H$ and either $A \subset H^+$ or $A \subset H^-$. Further, H is a *support plane* of A or *supports* A if H supports A at some point x, which is necessarily a boundary point of A. If $H = H_{u,\alpha}$ supports A and $A \subset H^-_{u,\alpha} = \{y \in \mathbb{R}^n : \langle y, u\rangle \leq \alpha\}$, then $H^-_{u,\alpha}$ is called a *supporting halfspace* of A and u is called an *outer* (or *exterior*) *normal vector* of both $H_{u,\alpha}$ and $H^-_{u,\alpha}$. If, moreover, $H_{u,\alpha}$ supports A at x, then u is an *outer normal vector* of A at x. A flat E *supports* A at x if $x \in A \cap E$ and E lies in some support plane of A.

Lemma 1.3.1 *Let $A \subset \mathbb{R}^n$ be nonempty, convex and closed and let $x \in \mathbb{R}^n \setminus A$. The hyperplane H through $p(A, x)$ orthogonal to $u(A, x)$ supports A.*

Proof If $z \in K$ and $v := z - p(A, x)$, the argument in the proof of Theorem 1.2.1 (with $p(A, y)$ replaced by z) gives $\langle x - p(A, x), v\rangle \leq 0$. This yields the assertion. □

Theorem 1.3.2 *Let $A \subset \mathbb{R}^n$ be convex and closed. Then through each boundary point of A there is a support plane of A. If $A \neq \emptyset$ is bounded, then to each vector $u \in \mathbb{R}^n \setminus \{o\}$ there is a support plane to A with outer normal vector u.*

Proof Let $x \in \operatorname{bd} A$. First let A be bounded. By Lemma 1.2.3 there is a point $y \in \mathbb{R}^n \setminus A$ such that $x = p(A, y)$. By Lemma 1.3.1 the hyperplane through $p(A, y) = x$ orthogonal to $y - x$ supports A at x.

If A is unbounded, there exists a support plane H of $A \cap B(x, 1)$ through x; let H^- be the corresponding supporting halfspace of $A \cap B(x, 1)$. If there is a point $z \in A \setminus H^-$, then $[z, x] \subset A$, but $[z, x) \cap B(x, 1) \not\subset H^-$, a contradiction. Hence H supports A.

Let A be bounded and $u \in \mathbb{R}^n \setminus \{o\}$. Since A is compact, there is a point $x \in A$ satisfying $\langle x, u\rangle = \max\{\langle y, u\rangle : y \in A\}$. Evidently, $\{y \in \mathbb{R}^n : \langle y, u\rangle = \langle x, u\rangle\}$ is a support plane to A with outer normal vector u. □

The existence of support planes through arbitrary boundary points is characteristic for convex sets, in the following precise sense.

Theorem 1.3.3 *Let $A \subset \mathbb{R}^n$ be a closed set such that* $\operatorname{int} A \neq \emptyset$ *and such that through each boundary point of A there is a support plane to A. Then A is convex.*

Proof Suppose that A satisfies the assumptions but is not convex. Then there are points $x, y \in A$ and $z \in [x, y]$ with $z \notin A$. Since $\operatorname{int} A \neq \emptyset$ (and $n \geq 2$, as we may clearly assume), we can choose $a \in \operatorname{int} A$ such that x, y, a are affinely independent. There is a point $b \in \operatorname{bd} A \cap [a, z)$. By assumption, through b there exists a support plane H to A, and $a \notin H$ because $a \in \operatorname{int} A$. Hence, H intersects the plane $\operatorname{aff}\{x, y, a\}$ in a line. The points x, y, a must lie on the same side of this line, which is obviously a contradiction. □

We turn to separation. Let $A, B \subset \mathbb{R}^n$ be sets and $H_{u,\alpha} \subset \mathbb{R}^n$ a hyperplane. The hyperplane $H_{u,\alpha}$ *separates* A and B if $A \subset H^-_{u,\alpha}$ and $B \subset H^+_{u,\alpha}$, or vice versa. This separation is said to be *proper* if A and B do not both lie in $H_{u,\alpha}$. The sets A and B are *strictly separated* by $H_{u,\alpha}$ if $A \subset \operatorname{int} H^-_{u,\alpha}$ and $B \subset \operatorname{int} H^+_{u,\alpha}$, or conversely, and they are *strongly separated* by $H_{u,\alpha}$ if there is a number $\varepsilon > 0$ such that $H_{u,\alpha-\varepsilon}$ and $H_{u,\alpha+\varepsilon}$ both separate A and B. Separation of A and a point x means separation of A and $\{x\}$. We first consider this special case.

Theorem 1.3.4 *Let $A \subset \mathbb{R}^n$ be convex and let $x \in \mathbb{R}^n \setminus A$. Then A and x can be separated. If A is closed, then A and x can be strongly separated.*

Proof If A is closed, the hyperplane through $p(A, x)$ orthogonal to $u(A, x)$ supports A and hence separates A and x. The parallel hyperplane through $(p(A, x) + x)/2$ strongly separates A and x. If A is not closed and $x \notin \operatorname{cl} A$, then a hyperplane separating $\operatorname{cl} A$ and x a fortiori separates A and x. If $x \in \operatorname{cl} A$, then $x \in \operatorname{bd} \operatorname{cl} A$ by Theorem 1.1.15, and by Theorem 1.3.2 there is a support plane to $\operatorname{cl} A$ through x; it separates A and x. $\square$

Corollary 1.3.5 *Every nonempty closed convex set in $\mathbb{R}^n$ is the intersection of its supporting halfspaces.*

Separation of pairs of sets can be reduced to separation of a set and a point:

Lemma 1.3.6 *Let $A, B \subset \mathbb{R}^n$ be nonempty subsets. Then A and B can be separated (strongly separated) if and only if $A - B$ and o can be separated (strongly separated).*

Proof We consider only strong separation; the other case is analogous (or put $\varepsilon = 0$). Suppose that $H_{u,\alpha}$ strongly separates A and B, say $A \subset H^-_{u,\alpha-\varepsilon}$ and $B \subset H^+_{u,\alpha+\varepsilon}$ for some $\varepsilon > 0$. Let $x \in A - B$; thus $x = a - b$ with $a \in A$, $b \in B$. From $\langle a, u\rangle \le \alpha - \varepsilon$ and $\langle b, u\rangle \ge \alpha + \varepsilon$ we get $\langle x, u\rangle \le -2\varepsilon$, so that $A - B$ and o are strongly separated by $H_{u,-\varepsilon}$.

Suppose that $A - B$ and o can be strongly separated. Then there are $u \in \mathbb{R}^n \setminus \{o\}$ and $\varepsilon > 0$ such that $\langle x, u\rangle \le -2\varepsilon$ for all $x \in A - B$. Let

$$\alpha := \sup\{\langle a, u\rangle : a \in A\}, \qquad \beta := \inf\{\langle b, u\rangle : b \in B\}.$$

For $a \in A$, $b \in B$ we have $\langle a, u\rangle - \langle b, u\rangle \le -2\varepsilon$, hence $\beta - \alpha \ge 2\varepsilon$. Thus $H_{u,(\alpha+\beta)/2}$ strongly separates A and B. $\square$

If $A, B \subset \mathbb{R}^n$ are convex, then $A - B$ is convex. If A is compact and B is closed, then $A - B$ is easily seen to be closed. The condition $o \notin A - B$ is equivalent to $A \cap B = \emptyset$. Hence, from Lemma 1.3.6 and Theorem 1.3.4 we deduce the following.

Theorem 1.3.7 *Let $A, B \subset \mathbb{R}^n$ be nonempty convex sets with $A \cap B = \emptyset$. Then A and B can be separated. If A is compact and B is closed, then A and B can be strongly separated.*

The following examples should be kept in mind. Let

$$A := \{(\xi, \eta) \in \mathbb{R}^2 : \xi > 0, \ \eta \geq 1/\xi\},$$
$$B := \{(\xi, \eta) \in \mathbb{R}^2 : \xi > 0, \ \eta \leq -1/\xi\},$$
$$C := \{(\xi, \eta) \in \mathbb{R}^2 : \eta = 0\}.$$

These are pairwise disjoint, closed, convex subsets of $\mathbb{R}^2$. A and B can be strictly separated (by the line C), but not strongly; $A - B$ and o cannot be strictly separated. The sets A and C can be separated, but not strictly.

On the other hand, convex sets may be separable even if they are not disjoint. The exact condition is given by the following theorem.

Theorem 1.3.8 *Let $A, B \subset \mathbb{R}^n$ be nonempty convex sets. Then A and B can be properly separated if and only if*

$$\operatorname{relint} A \cap \operatorname{relint} B = \emptyset. \tag{1.1}$$

Proof Suppose that (1.1) holds. Put $C := \operatorname{relint} A - \operatorname{relint} B$. Then $o \notin C$, and C is convex. By Theorem 1.3.4 there exists a hyperplane $H_{u,0}$ with $C \subset H^-_{u,0}$. Let

$$\beta := \inf\{\langle b, u\rangle : b \in B\}.$$

If $\beta > -\infty$, then $B \subset H^+_{u,\beta}$. Suppose there exists a point $a \in A$ with $\langle a, u\rangle > \beta$ (if $\beta = -\infty$, such a exists). By Theorem 1.1.13 there exists a point $z \in \operatorname{relint} A$, and Lemma 1.1.9 states that $[z, a) \subset \operatorname{relint} A$. Hence, there is a point $\bar{a} \in \operatorname{relint} A$ with $\langle \bar{a}, u\rangle > \beta$. There is a point $b \in B$ with $\langle b, u\rangle < \langle \bar{a}, u\rangle$ and then, by a similar argument as before, a point $\bar{b} \in \operatorname{relint} B$ with $\langle \bar{b}, u\rangle < \langle \bar{a}, u\rangle$. Thus $\bar{a} - \bar{b} \in C$ and $\langle \bar{a} - \bar{b}, u\rangle > 0$, a contradiction. This shows that $\beta > -\infty$ and that $A \subset H^-_{u,\beta}$. Thus A and B are separated by $H_{u,\beta}$. If $A \cup B$ lies in some hyperplane, then this argument yields a hyperplane relative to $\operatorname{aff}(A \cup B)$ separating A and B, and this can be extended to a hyperplane in $\mathbb{R}^n$ that properly separates A and B.

Conversely, let H be a hyperplane properly separating A and B, say with $A \subset H^-$ and $B \subset H^+$. Suppose there exists $x \in \operatorname{relint} A \cap \operatorname{relint} B$. Then $x \in H$. Since $A \subset H^-$ and $x \in \operatorname{relint} A$, we must have $A \subset H$, similarly $B \subset H$, a contradiction. Thus (1.1) holds. □

Occasionally we shall have to use support and separation of convex cones. For these we have:

Theorem 1.3.9 *Let $C \subset \mathbb{R}^n$ be a closed convex cone. Each support plane of C contains o. If $x \in \mathbb{R}^n \setminus C$, then there exists a vector $u \in \mathbb{R}^n$ such that*

$$\langle c, u\rangle \geq 0 \quad \text{for all } c \in C \text{ and } \langle x, u\rangle < 0.$$

Proof Let H be a support plane to C. There is a point $y \in H \cap C$. Then $\lambda y \in C$ for all $\lambda > 0$, which is impossible if $o \notin H$. Hence $o \in H$. The rest is clear from Lemma 1.3.1. □

We shall now prove two more results in the spirit of the theorems of Carathéodory and Helly. They are treated in this section since the first of them needs support planes in its proof and the second one deals with separation.

Theorem 1.3.10 (Steinitz's theorem) *Let $A \subset \mathbb{R}^n$ and $x \in \operatorname{int}\operatorname{conv} A$. Then $x \in \operatorname{int}\operatorname{conv} A'$ for some subset $A' \subset A$ with at most $2n$ points.*

Proof The point x lies in the interior of a simplex with vertices in $\operatorname{conv} A$; hence, by Carathéodory's theorem applied to each vertex, $x \in \operatorname{int}\operatorname{conv} B$ for some subset B of A with at most $(n+1)^2$ points. We can choose a line L through x that does not meet the affine hull of any $n-1$ points of B. Let x_1, x_2 be the endpoints of the segment $L \cap \operatorname{conv} B$. By Theorem 1.3.2, through x_j there is a support plane H_j to $\operatorname{conv} B$ $(j = 1, 2)$. Clearly, $x_j \in \operatorname{conv}(B \cap H_j)$, hence by Carathéodory's theorem there is a representation

$$x_j = \sum_{i=1}^{n} \lambda_{ji} y_{ji} \quad \text{with } y_{ji} \in B \cap H_j,\ \lambda_{ji} \geq 0,\ \sum_{i=1}^{n} \lambda_{ji} = 1$$

$(j = 1, 2)$, and here necessarily $\lambda_{ji} > 0$ by the choice of L. With suitable $\lambda \in (0, 1)$ we have

$$x = (1-\lambda)x_1 + \lambda x_2 = \sum_{i=1}^{n} [(1-\lambda)\lambda_{1i} y_{1i} + \lambda\lambda_{2i} y_{2i}]$$
$$\in \operatorname{relint}\operatorname{conv}\{y_{11}, \dots, y_{1n}, y_{21}, \dots, y_{2n}\},$$

by Theorem 1.1.14. Here relint can be replaced by int, since by the choice of L the points $y_{11}, \dots, y_{1n}$ are affinely independent, and for at least one index k, also $y_{11}, \dots, y_{1n}, y_{2k}$ are affinely independent. $\square$

Theorem 1.3.11 (Kirchberger's theorem) *Let $A, B \subset \mathbb{R}^n$ be compact sets. If for any subset $M \subset A \cup B$ with at most $n+2$ points the sets $M \cap A$ and $M \cap B$ can be strongly separated, then A and B can be strongly separated.*

Proof First we assume that A and B are finite sets. For $x \in \mathbb{R}^n$ define (with $\tau(x) := (x, 1)$)

$$H_x^{\pm} := \{v \in \mathbb{R}^n \times \mathbb{R} : \pm\langle v, \tau(x)\rangle > 0\}.$$

Let $M \subset A \cup B$ and $\operatorname{card} M \leq n+2$. By the assumption there exist $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $\langle u, a\rangle > \alpha$ for $a \in M \cap A$ and $\langle u, b\rangle < \alpha$ for $b \in M \cap B$. Writing $v := (u, -\alpha)$, we see that $\langle v, \tau(a)\rangle = \langle u, a\rangle - \alpha > 0$; thus $v \in H_a^+$ for $a \in M \cap A$. Similarly, $v \in H_b^-$ for $b \in M \cap B$. Thus, the family $\{H_a^+ : a \in A\} \cup \{H_b^- : b \in B\}$ of finitely many convex sets in $\mathbb{R}^n \times \mathbb{R}$ has the property that any $n+2$ or fewer of the sets have nonempty intersection. By Helly's theorem, the intersection of all sets in the family is not empty. Since this intersection is open, it contains an element of the form $v = (u, -\alpha)$ with $u \neq o$. For $a \in A$ we have $v \in H_a^+$, hence $\langle u, a\rangle > \alpha$, and for $b \in B$ similarly $\langle u, b\rangle < \alpha$. Hence A and B, being finite sets, are strongly separated by $H_{u,\alpha}$.

Now let A, B be compact sets satisfying the assumption. Suppose that A and B cannot be strongly separated. Then $\operatorname{conv} A$ and $\operatorname{conv} B$ cannot be strongly separated. By Theorem 1.1.11 these sets are compact and hence by Theorem 1.3.7 they cannot be disjoint. Let $x \in \operatorname{conv} A \cap \operatorname{conv} B$. Then $x \in \operatorname{conv} A' \cap \operatorname{conv} B'$ with finite subsets $A' \subset A$ and $B' \subset B$, which hence cannot be strongly separated. This contradicts the result shown above. ◻

We conclude this section with another application of a separation theorem, which will be useful in the study of Minkowski addition.

Lemma 1.3.12 *Let $A, B \subset \mathbb{R}^n$ be nonempty convex sets. If*

$$x \in \operatorname{relint}(A + B),$$

then x can be represented in the form $x = a + b$ with $a \in \operatorname{relint} A$ and $b \in \operatorname{relint} B$.

Proof There is a representation $x = y+z$ with $y \in A$ and $z \in B$. We may assume that $x = y = z = o$ and also that $\dim(A + B) = n$. Since $o \in \operatorname{int}(A + B)$, the sets $A + B$ and $\{o\}$ cannot be separated. Hence, by Lemma 1.3.6 and Theorem 1.3.8, there is a point

$$a \in \operatorname{relint} A \cap \operatorname{relint}(-B).$$

Then $-a \in \operatorname{relint} B$ and $o = a - a$. ◻

Notes for Section 1.3

1. Separation and support properties of convex sets in finite and infinite dimensions are of fundamental importance in various fields such as functional analysis, optimization, control theory, mathematical economy, and others. For infinite-dimensional separation and support theorems we refer only to Bourbaki [310], Marti [1331], Holmes [985], Bair and Fourneau [112]; see also the survey article by Klee [1116]. A thorough study of several types of separation in $\mathbb{R}^n$ was made by Klee [1114].
2. A stronger version of Theorem 1.3.3 (existence of local support planes) is associated with the name of Tietze; for this, we refer to Valentine [1866], Chapter IV, C. Surveys of characterizations of convexity, also in infinite-dimensional vector spaces, are given by Burago and Zalgaller [356] and by Mani–Levitska [1326].
3. Historical information on the theorems of Steinitz and Kirchberger can be found in the survey article by Danzer, Grünbaum and Klee [464].
4. *Positive bases*. Let $B \subset \mathbb{R}^n$. Using Theorem 1.1.14 one easily sees that $\operatorname{pos} B = \mathbb{R}^n$ holds if and only if $o \in \operatorname{int}\operatorname{conv} B$. The set B is called a *positive basis* of $\mathbb{R}^n$ if $\operatorname{pos} B = \mathbb{R}^n$, but $\operatorname{pos} B' \neq \mathbb{R}^n$ for each proper subset $B' \subset B$. Thus, the theorem of Steinitz implies that a positive basis of $\mathbb{R}^n$ contains at most $2n$ vectors. If B is a linear basis of $\mathbb{R}^n$, then $B \cup (-B)$ is a positive basis, and up to multiplication by positive numbers it is only in this way that the maximal number $2n$ can be achieved. Positive bases were investigated by Davis [468], McKinney [1372], Bonnice and Klee [285], Reay [1561] and Shephard [1786].

1.4 Extremal representations

The purpose of this section is to represent a closed convex set as the convex hull of a smaller set, and here the smallest possible sets will be of particular interest. A first candidate for a smaller set with the same convex hull is the relative boundary. Only the obvious trivial cases must be excluded.

Lemma 1.4.1 *If $A \subset \mathbb{R}^n$ is a closed convex set with $A \neq \operatorname{conv}\operatorname{relbd} A$, then A is either a flat or a half-flat.*

Proof Clearly, we may assume that $\dim A = n$. There is a point $x \in \operatorname{int} A$ with $x \notin \operatorname{conv}\operatorname{bd} A$ (since otherwise $A = \operatorname{int} A \cup \operatorname{bd} A = \operatorname{conv}\operatorname{bd} A$). By the separation theorem, 1.3.4, there is a closed halfspace H^- such that $x \in H^-$ and $\operatorname{conv}\operatorname{bd} A \subset H^+$. Each point $y \in \operatorname{int} H^-$ satisfies $[x, y] \cap \operatorname{bd} A = \emptyset$ and hence $y \in \operatorname{int} A$; thus $H^- \subset A$. By the convexity and closedness of A, each translate of H^- with a point of A in its boundary is contained in A. Thus A is either equal to $\mathbb{R}^n$ or is a halfspace. □

We will exclude the exceptional cases that are the subject of Lemma 1.4.1 by demanding that A be *line-free*, meaning that A does not contain a line. Owing to Lemma 1.4.2 below, this is not a severe restriction. First let $A \subset \mathbb{R}^n$ be a closed convex set. Suppose that A contains a ray $R_{x,u} := \{x + \lambda u : \lambda \geq 0\}$ with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n \setminus \{o\}$. Let $y \in A$. Let $z \in R_{y,u}$ and $w \in [x, z)$. The ray through w with endpoint y meets $R_{x,u}$, hence $w \in A$. Thus $[x, z) \subset A$ and hence $z \in A$. This shows that also $R_{y,u} \subset A$. For this reason, it makes sense to define

$$\operatorname{rec} A := \{u \in \mathbb{R}^n \setminus \{o\} : R_{x,u} \subset A\} \cup \{o\}$$

where $x \in A$; this set does then not depend upon the choice of x. It is evidently a closed convex cone, which is called the *recession cone* of A. One may also write

$$\operatorname{rec} A = \{u \in \mathbb{R}^n : A + u \subset A\}.$$

Lemma 1.4.2 *Every closed convex set $A \subset \mathbb{R}^n$ can be represented in the form $A = \bar{A} \oplus V$, where V is a linear subspace of $\mathbb{R}^n$ and $\bar{A}$ is a line-free closed convex set in a subspace complementary to V.*

Proof Assume that A is not line-free. Then

$$V := \operatorname{rec} A \cap (-\operatorname{rec} A),$$

which is called the *lineality space* of A, is the linear subspace consisting of all vectors that are parallel to some line contained in A. Let U be a linear subspace complementary to V and put $\bar{A} := A \cap U$; then $\bar{A} + V \subset A$. Let $x \in A$. Through x there exists a line contained in A; since it is parallel to V, it meets U in a point y. Then $x = y + (x - y)$ with $y \in \bar{A}$ and $x - y \in V$; hence $x \in \bar{A} + V$. This proves that $A = \bar{A} \oplus V$. Clearly, $\bar{A}$ is closed, convex, and line-free. □

The representation by convex hulls of minimal sets requires some definitions. Let $A \subset \mathbb{R}^n$ be a convex set. A *face* of A is a convex subset $F \subset A$ such that each segment $[x, y] \subset A$ with $F \cap \operatorname{relint}[x, y] \neq \emptyset$ is contained in F or, equivalently, such that $x, y \in A$ and $(x + y)/2 \in F$ implies $x, y \in F$. If $\{z\}$ is a face of A, then z is called an *extreme point* of A. In other words, z is an extreme point of A if and only if it cannot be written in the form $z = (1-\lambda)x + \lambda y$ with $x, y \in A$, $x \neq y$, and $\lambda \in (0, 1)$. The set of all extreme points of A is denoted by $\operatorname{ext} A$. An *extreme ray* of A is a ray that is a face of A. By $\operatorname{extr} A$ we denote the union of the extreme rays of A.

Theorem 1.4.3 *Every line-free closed convex set $A \subset \mathbb{R}^n$ is the convex hull of its extreme points and extreme rays:*

$$A = \operatorname{conv}(\operatorname{ext} A \cup \operatorname{extr} A).$$

Proof The assertion is clear for $n \leq 1$. Suppose that $n \geq 2$, $\dim A = n$ (without loss of generality) and the assertion has been proved for convex sets of smaller dimension. By Lemma 1.4.1 we have $A = \operatorname{conv} \operatorname{bd} A$. By the support theorem, 1.3.2, each point $x \in \operatorname{bd} A$ lies in some support plane H of A. By the induction hypothesis, x lies in the convex hull of the extreme points and extreme rays of $H \cap A$, and it is an immediate consequence of the definition of a face that these are respectively extreme points and extreme rays of A itself. The assertion follows. □

Corollary 1.4.4 *If $A \subset \mathbb{R}^n$ is a line-free closed convex set then*

$$A = \operatorname{conv} \operatorname{ext} A + \operatorname{rec} A.$$

Proof By Theorem 1.4.3, a point $x \in A$ can be written as

$$x = \sum_{i=1}^{k} \lambda_i x_i + \sum_{i=k+1}^{m} \lambda_i v_i \quad \text{with } \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1,$$

with $x_i \in \operatorname{ext} A$ and $v_i \in \operatorname{extr} A$. Then v_i, lying in some extreme ray, can be written as $v_i = y_i + u_i$ where y_i, the endpoint of the ray, is an extreme point of A, and $u_i \in \operatorname{rec} A$. This proves the assertion. □

Corollary 1.4.5 (Minkowski's theorem) *Every convex body $K \in \mathcal{K}^n$ is the convex hull of its extreme points.*

Here the set of extreme points cannot be replaced by a smaller set, since a point $x \in K$ is an extreme point of K if and only if $K \setminus \{x\}$ is convex.

Extreme points can also be characterized in a slightly different way, which is sometimes useful. Let $A \subset \mathbb{R}^n$ be closed and convex and let $x \in A$. A *cap of A around x* is a set of the form $A \cap H^+$, where H^+ is a closed halfspace with $x \in \operatorname{int} H^+$.

Lemma 1.4.6 *Let $A \subset \mathbb{R}^n$ be closed and convex and let $x \in A$. Then x is an extreme point of A if and only if every neighbourhood of x contains a cap of A around x.*

Proof Let $x \in \operatorname{ext} A$. A given neighbourhood of x contains an open ball B_0 with centre x. Let $\bar{A} := \operatorname{conv}(A \setminus B_0)$. Since x is an extreme point of A, it cannot be a convex combination of points from $A \setminus B_0$, thus $x \notin \bar{A}$. Hence, there is a hyperplane H strongly separating $\bar{A}$ and x. If H^+ is the closed halfspace bounded by H and containing x, then $A \cap H^+$ is the required cap.

If $x \notin \operatorname{ext} A$, then $x \in \operatorname{relint}[y,z]$ for suitable $y, z \in A$. Every cap of A around x contains y or z, hence sufficiently small neighbourhoods of x cannot contain a cap around x. □

The notion of an extreme point of a convex set A involves convex combinations of points in A and is thus related to an 'intrinsic' description of A. Looking at a convex set from outside, one is led to a related but different class of special boundary points. The point $x \in A$ is called an *exposed point* of A if there is a support plane H to A with $H \cap A = \{x\}$. The set of exposed points of A is denoted by $\exp A$. Clearly, $\exp A \subset \operatorname{ext} A$, but even for $A \in \mathcal{K}^2$ this inclusion is in general strict. This is shown by the example of a rectangle with a semicircle attached to one of its edges. However, each extreme point of a closed convex set is a limit of exposed points. We formulate the corresponding result for convex bodies only.

Theorem 1.4.7 (Straszewicz's theorem) *For $K \in \mathcal{K}^n$,*

$$\operatorname{ext} K \subset \operatorname{cl} \exp K,$$

hence

$$K = \operatorname{cl} \operatorname{conv} \exp K.$$

Proof Let $x \in \operatorname{ext} K$ and let U be a neighbourhood of x. By Lemma 1.4.6, the set U contains a cap $K \cap H^+$ of K around x. Let G be the ray orthogonal to $H = \operatorname{bd} H^+$ with endpoint x and meeting H. To any point $z \in G$ there is a point $y_z \in K$ with maximal distance from z. Obviously, $y_z \in \exp K$. If $|z - x|$ is sufficiently large, then $y_z \in H^+$, by elementary geometry. Thus $y_z \in K \cap H^+ \subset U$ and hence $\operatorname{ext} K \subset \operatorname{cl} \exp K$.

Using Minkowski's theorem, we get

$$K = \operatorname{conv} \operatorname{ext} K \subset \operatorname{conv} \operatorname{cl} \exp K \subset \operatorname{cl} \operatorname{conv} \exp K \subset \operatorname{cl} \operatorname{conv} \operatorname{ext} K = K,$$

hence $K = \operatorname{cl} \operatorname{conv} \exp K$. □

Notes for Section 1.4

1. *Minkowski's theorem and its aftermath.* Minkowski's theorem was first proved by Minkowski ([1441], §12). Particularly important is its extension to infinite-dimensional spaces. Every compact convex subset of a locally convex Hausdorff linear space is the closed convex hull of its extreme points. This was proved, in a more special case, by Krein and Milman [1147]; see, e.g., Bourbaki [310]. A certain converse is due to Milman [1426]; this states that in the Krein–Milman theorem the set of extreme points cannot be replaced by a set whose closure does not contain the extreme points. We refer the reader to Jacobs [1031] for an elementary introduction to extreme points, including applications, and to Roy [1598] for a survey article on extreme points of convex sets in infinite-dimensional spaces. An extension of the Krein–Milman theorem to locally compact sets, of which Theorem 1.4.3 is the finite-dimensional version, is due to Klee [1107]; see Jongmans [1047] for another extension.

   The Choquet theory of integral representations can be considered as a further extension of the Krein–Milman theorem; see, e.g., Bauer [179], Phelps [1533, 1534], Choquet [423, 424], Alfsen [59] and the survey article by Saškin [1638], which contains many references up to 1972.

2. Straszewicz's theorem was first proved by Straszewicz [1823]; see also Wets [1970] and, for the infinite-dimensional case, Klee [1108] and Bair [107].
3. A good source for the theorems of Minkowski and Straszewicz and their extensions to unbounded closed convex sets is the book of Rockafellar [1583].

   Klee [1109] made an extensive study of the closed sets that are the convex hull of a finite system of points and rays (polyhedral sets).

   If $A \subset \mathbb{R}^n$ is an unbounded line-free closed convex set, then Corollary 1.4.4 states that $A = \operatorname{conv} \operatorname{ext} A + \operatorname{rec} A$. Starting from this representation, Batson [175] found characterizations of the sets A for which $\operatorname{ext} A$ is bounded.
4. The following result is due to Dubins [514]; see also Pranger [1547]. If $K \in \mathcal{K}^n$ and $M \subset \mathbb{R}^n$ is a flat of codimension k, then each extreme point of $K \cap M$ is a convex combination of $k + 1$ (or fewer) extreme points of K. In fact, Dubins proved an infinite-dimensional version. A simplified proof and an extension are due to Klee [1111].
5. *Continuous barycentre functions*. By Minkowski's theorem, each point $x \in K$, where $K \in \mathcal{K}^n$, is a convex combination of extreme points of K. For a polytope P, such a convex combination may involve all the extreme points of P. Let P have the extreme points $x_1, \dots, x_k$. A k-tuple $(\varphi_1, \dots, \varphi_k)$ of real-valued continuous functions $\varphi_1, \dots, \varphi_k$ on P such that

$$x = \sum_{i=1}^{k} \varphi_i(x) x_i \quad \text{with } \varphi_i(x) \geq 0 \ (i = 1, \dots, k), \ \sum_{i=1}^{k} \varphi_i(x) = 1, \tag{1.2}$$

   is called a *continuous barycentre function*. It is called *extreme* if for each $x \in P$ the set $\{x_i : \varphi_i(x) > 0\}$ is affinely independent. The existence of continuous barycentre functions was proved by Kalman [1061] and the existence of extreme continuous barycentre functions by Fuglede [644]. Brøndsted [339] showed that the extreme continuous barycentre functions coincide with the extreme points of the convex set of all continuous barycentre functions on P. Related investigations for general compact convex sets were carried out by Fuglede [644]. Answering a question of Kalman [1061], Wiesler [1977] showed that not all the functions φ_i can be convex.

1.5 Convex functions

The investigation of convex sets is closely tied up with convex functions. We treat convex functions in a slightly more general fashion than would be necessary for our later applications.

For convex functions it is convenient to admit as the range the extended system $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ of real numbers with the usual rules. These are the following conventions. For $\lambda \in \mathbb{R}$, $-\infty < \lambda < +\infty$, $\infty + \infty = \lambda + \infty = \infty + \lambda = \infty$, $-\infty - \infty = -\infty + (-\infty) = \lambda - \infty = -\infty + \lambda = -\infty$, and finally $\lambda\infty = \infty$, 0 or $-\infty$ according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. For a given function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and for $\alpha \in \bar{\mathbb{R}}$ we use the abbreviation

$$\{f = \alpha\} := \{x \in \mathbb{R}^n : f(x) = \alpha\},$$

and $\{f \leq \alpha\}$, $\{f < \alpha\}$ etc. are defined similarly.

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called *convex* if f is *proper*, which means that $\{f = -\infty\} = \emptyset$ and $\{f = \infty\} \neq \mathbb{R}^n$, and if

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$ and for $0 \le \lambda \le 1$. (We warn the reader that this terminology, which is convenient for us, is not unique in the literature: properness is not always included in the definition of a convex function.) A function $f : D \to \bar{\mathbb{R}}$ with $D \subset \mathbb{R}^n$ is called *convex* if its extension $\tilde{f}$ defined by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{for } x \in D, \\ \infty & \text{for } x \in \mathbb{R}^n \setminus D \end{cases}$$

is convex. A function f is *concave* if $-f$ is convex.

Trivial examples of convex functions are the affine functions; these are the functions $f : \mathbb{R}^n \to \mathbb{R}$ of the form $f(x) = \langle u, x \rangle + \alpha$ with $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. A real-valued function on $\mathbb{R}^n$ is affine if and only if it is convex and concave.

The following assertions are immediate consequences of the definition. The supremum of (arbitrarily many) convex functions is convex if it is proper. If f, g are convex functions, then $f + g$ and αf for $\alpha \ge 0$ are convex if they are proper.

Remark 1.5.1 If f is convex, then

$$f(\lambda_1 x_1 + \cdots + \lambda_k x_k) \le \lambda_1 f(x_1) + \cdots + \lambda_k f(x_k)$$

for all $x_1, \ldots, x_k \in \mathbb{R}^n$ and all $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\lambda_1 + \cdots + \lambda_k = 1$. This is called *Jensen's inequality*; it follows by induction.

Convex functions have the important property (important for optimization etc.) that every local minimum is a global minimum. In fact, let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and suppose that $x_0 \in \mathbb{R}^n$ and $\rho > 0$ are such that $f(x_0) < \infty$ and $f(x_0) \le f(x)$ for $|x - x_0| \le \rho$. For $x \in \mathbb{R}^n$ with $|x - x_0| > \rho$, let

$$y := \frac{\rho}{|x - x_0|} x + \left(1 - \frac{\rho}{|x - x_0|}\right) x_0;$$

then $|y - x_0| = \rho$ and hence

$$f(x_0) \le f(y) \le \frac{\rho}{|x - x_0|} f(x) + \left(1 - \frac{\rho}{|x - x_0|}\right) f(x_0),$$

which gives $f(x_0) \le f(x)$.

A convex function determines in a natural way several convex sets. Let the function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex. Then the sets

$$\operatorname{dom} f := \{f < \infty\},$$

the *effective domain* of f, and for $\alpha \in \mathbb{R}$ the *sublevel sets* $\{f < \alpha\}$, $\{f \le \alpha\}$ are convex. The *epigraph* of f,

$$\operatorname{epi} f := \{(x, \zeta) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le \zeta\},$$

is a convex subset of $\mathbb{R}^n \times \mathbb{R}$. The asserted convexity is in each case easy to see. Conversely, a nonempty convex set $A \subset \mathbb{R}^n$ determines a convex function by

$$I_A^\infty(x) := \begin{cases} 0 & \text{for } x \in A, \\ \infty & \text{for } x \in \mathbb{R}^n \setminus A, \end{cases}$$

the *indicator function* of A.

The convexity of the epigraph can be deduced from a local support property, and this leads to the following sufficient criterion for the convexity of a function. It will be used in the proof of Theorem 3.2.3.

Theorem 1.5.2 *Let $D \subset \mathbb{R}^n$ be convex and let $f : D \to \mathbb{R}$ be a continuous function. Suppose that for each point $x_0 \in D$ there are an affine function g on $\mathbb{R}^n$ and a neighbourhood U of x_0 such that $f(x_0) = g(x_0)$ and $f \geq g$ in $U \cap D$. Then f is convex.*

Proof Since a function is convex if its restriction to any segment is convex, it suffices to consider the case where $n = 1$ and D is an interval in $\mathbb{R}$. Suppose that f satisfies the assumption but is not convex. Then there are points $x_1 < z < x_2$ in D and an affine function h on $\mathbb{R}$ such that $f(x_1) = h(x_1)$, $f(x_2) = h(x_2)$ and $f(z) > h(z)$. Since f is continuous, the maximum $b := \max\{f(x) - h(x) : x \in [x_1, x_2]\}$ exists; it is positive. Let $x_0 \in (x_1, x_2)$ be any point where this maximum is attained. By assumption, there exists an affine function g such that $f(x_0) = g(x_0)$ and $f(x) \geq g(x)$ for all x in a neighbourhood of x_0. Since $h + b$ and g are affine functions with $h(x_0) + b = f(x_0) = g(x_0)$ and $h(x) + b \geq f(x) \geq g(x)$ for x in a neighbourhood of x_0, they coincide, hence $f = h + b$ in a neighbourhood of x_0. Thus, the set $\{x \in [x_1, x_2] : f(x) = h(x) + b\}$ is open in $[x_1, x_2]$. Since f is continuous, this set is also closed and thus it is equal to $[x_1, x_2]$. This is a contradiction, since $f(x_1) = h(x_1)$ and $b > 0$. □

We investigate the general analytic properties of convex functions, starting with continuity.

Theorem 1.5.3 *Every convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is continuous on* $\operatorname{int}\operatorname{dom} f$ *and Lipschitzian on any compact subset of* $\operatorname{int}\operatorname{dom} f$.

Proof First we show the continuity. Let $x_0 \in \operatorname{int}\operatorname{dom} f$. We can choose a simplex S with $x_0 \in \operatorname{int} S \subset S \subset \operatorname{int}\operatorname{dom} f$ and a number $\rho > 0$ wit $B(x_0, \rho) \subset S$. For $x \in S$ there is a representation

$$x = \sum_{i=1}^{n+1} \lambda_i x_i \quad \text{with } \lambda_i \geq 0, \ \sum_{i=1}^{n+1} \lambda_i = 1,$$

where $x_1, \dots, x_{n+1}$ are the vertices of S, and we deduce that

$$f(x) \leq \sum_{i=1}^{n+1} \lambda_i f(x_i) \leq c := \max\{f(x_1), \dots, f(x_{n+1})\}.$$

Now let $y = x_0 + \alpha u$ with $\alpha \in [0,1]$ and $|u| = \rho$. From $y = (1-\alpha)x_0 + \alpha(x_0 + u)$ we get

$$f(y) \leq (1-\alpha)f(x_0) + \alpha f(x_0 + u),$$

hence

$$f(y) - f(x_0) \leq \alpha(c - f(x_0))$$

since $x_0 + u \in S$. On the other hand,

$$x_0 = \frac{1}{1+\alpha}y + \frac{\alpha}{1+\alpha}(x_0 - u)$$

and hence

$$f(x_0) \leq \frac{1}{1+\alpha}f(y) + \frac{\alpha}{1+\alpha}f(x_0 - u),$$

which gives

$$f(x_0) - f(y) \leq \alpha(c - f(x_0)).$$

Thus,

$$|f(y) - f(x_0)| \leq \frac{1}{\rho}[c - f(x_0)]|y - x_0|$$

for $y \in B(x_0, \rho)$, which shows the continuity of F at x_0 in a sharpened version. Thus f is continuous on $\operatorname{int}\operatorname{dom} f$.

Let $C \subset \operatorname{int}\operatorname{dom} f$ be a compact subset. By compactness, there exists a number $\rho > 0$ such that $C_\rho := C + B(o,\rho) \subset \operatorname{int}\operatorname{dom} f$. On the compact set C_ρ, the continuous function $|f|$ attains a maximum a. Let $x, y \in C$ be given. Then

$$z := y + \frac{\rho}{|y - x|}(y - x) \in C_\rho$$

and

$$y = (1-\lambda)x + \lambda z \quad \text{with } \lambda = \frac{|y - x|}{\rho + |y - x|},$$

hence $f(y) \leq (1-\lambda)f(x) + \lambda f(z)$ yields

$$f(y) - f(x) \leq \lambda[f(z) - f(x)] \leq \frac{2a}{\rho}|y - x|.$$

Interchanging x and y we get $|f(y) - f(x)| \leq b|y - x|$ with b independent of x and y. Thus f is Lipschitzian on C. □

Differentiability of convex functions will be considered first for the case $n = 1$.

Theorem 1.5.4 *Let $f : \mathbb{R} \to \bar{\mathbb{R}}$ be convex. Then on* $\operatorname{int}\operatorname{dom} f$ *the following holds. The right derivative f_r' and the left derivative f_l' exist and are monotonically increasing functions. The inequality $f_l' \leq f_r'$ is valid, and with the exception of at most countably many points, $f_l' = f_r'$ holds and hence f is differentiable. Further, f_r' is* *continuous from the right and* f_l' *is continuous from the left; in particular, if* f *is differentiable on* $\operatorname{int}\operatorname{dom} f$, *then it is continuously differentiable.*

Proof In the following, all arguments of f are taken from $\operatorname{int}\operatorname{dom} f$. Let $0 < \lambda < \mu$. Then

$$f(x+\lambda) = f\left(\frac{\mu-\lambda}{\mu}x + \frac{\lambda}{\mu}(x+\mu)\right) \le \frac{\mu-\lambda}{\mu}f(x) + \frac{\lambda}{\mu}f(x+\mu),$$

hence

$$\frac{f(x+\lambda)-f(x)}{\lambda} \le \frac{f(x+\mu)-f(x)}{\mu}.$$

Analogously,

$$f(x-\lambda) = f\left(\frac{\mu-\lambda}{\mu}x + \frac{\lambda}{\mu}(x-\mu)\right) \le \frac{\mu-\lambda}{\mu}f(x) + \frac{\lambda}{\mu}f(x-\mu),$$

hence

$$\frac{f(x)-f(x-\mu)}{\mu} \le \frac{f(x)-f(x-\lambda)}{\lambda}.$$

For arbitrary $\lambda, \mu > 0$,

$$\begin{aligned} f(x) &= f\left(\frac{\lambda}{\lambda+\mu}(x-\mu) + \frac{\mu}{\lambda+\mu}(x+\lambda)\right) \\ &\le \frac{\lambda}{\lambda+\mu}f(x-\mu) + \frac{\mu}{\lambda+\mu}f(x+\lambda), \end{aligned}$$

hence

$$\frac{f(x)-f(x-\mu)}{\mu} \le \frac{f(x+\lambda)-f(x)}{\lambda}.$$

From the monotonicity and boundedness properties thus established one deduces the existence of

$$f_r'(x) = \lim_{\lambda\downarrow 0}\frac{f(x+\lambda)-f(x)}{\lambda}, \qquad f_l'(x) = \lim_{\lambda\downarrow 0}\frac{f(x-\lambda)-f(x)}{-\lambda},$$

as well as the inequality $f_l'(x) \le f_r'(x)$.

Thus, for $x < y$,

$$f_l'(x) \le f_r'(x) \le \frac{f(y)-f(x)}{y-x} \le f_l'(y) \le f_r'(y).$$

Hence, f_l' and f_r' are increasing and thus have only countably many discontinuities. At each continuity point of f_l' the above inequalities yield $f_l' = f_r'$ and hence the existence of the derivative f'.

Let $x < y$. We have

$$\frac{f(y)-f(x)}{y-x} = \lim_{z\downarrow x}\frac{f(y)-f(z)}{y-z} \ge \lim_{z\downarrow x} f_r'(z),$$

hence

$$\lim_{z\downarrow x} f_r'(z) \leq f_r'(x) \leq \lim_{z\downarrow x} f_r'(z),$$

by the monotonicity of f_r'. Thus f_r' is continuous from the right. Analogously one obtains that f_l' is continuous from the left. □

Remark 1.5.5 Let $f : \mathbb{R} \to \mathbb{R}$ be convex; let $x_0 \in \mathbb{R}$ and m be a number with $f_l'(x_0) \leq m \leq f_r'(x_0)$. As noted in the preceding proof, one has

$$\frac{f(x) - f(x_0)}{x - x_0} \begin{cases} \geq f_r'(x_0) \geq m, & \text{if } x > x_0, \\ \leq f_l'(x_0) \leq m, & \text{if } x < x_0, \end{cases}$$

thus $f(x) \geq f(x_0) + m(x - x_0)$ for all $x \in \mathbb{R}$. This shows that the line

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = f(x_0) + m(x - x_0)\}$$

supports the epigraph of f at the point $(x_0, f(x_0))$.

Now let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a convex function. Theorem 1.5.4 implies, in particular, for each point $x \in \operatorname{int} \operatorname{dom} f$, the existence of all limits

$$f'(x; u) := \lim_{\lambda\downarrow 0} \frac{f(x + \lambda u) - f(x)}{\lambda},$$

where $u \in \mathbb{R}^n$. If u is a unit vector, this is known as the one-sided directional derivative of f at x in direction u. By a slight abuse of language, we call $f'(x; \cdot)$ the directional derivative function of f at x. We first collect some remarks on this function in connection with sublinearity.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positively homogeneous* if

$$f(\lambda x) = \lambda f(x) \quad \text{for all } \lambda \geq 0 \text{ and all } x \in \mathbb{R}^n,$$

and f is called *subadditive* if

$$f(x + y) \leq f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

A function that is positively homogeneous and subadditive is called *sublinear*. Every sublinear function is convex.

Lemma 1.5.6 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and $x \in \operatorname{int} \operatorname{dom} f$. Then the directional derivative function*

$$f'(x; \cdot) : \mathbb{R}^n \to \mathbb{R}$$

is sublinear.

Proof Let $u \in \mathbb{R}^n \setminus \{o\}$. For $\lambda, \tau > 0$ we may write

$$\frac{f(x + \tau\lambda u) - f(x)}{\tau} = \lambda \frac{f(x + \tau\lambda u) - f(x)}{\tau\lambda},$$

and $\tau \downarrow 0$ gives $f'(x; \lambda u) = \lambda f'(x; u)$. For $u, v \in \mathbb{R}^n$ the convexity of f yields

$$\begin{aligned} f(x + \tau(u + v)) &= f\left(\tfrac{1}{2}(x + 2\tau u) + \tfrac{1}{2}(x + 2\tau v)\right) \\ &\leq \tfrac{1}{2} f(x + 2\tau u) + \tfrac{1}{2} f(x + 2\tau v)), \end{aligned}$$

hence

$$\frac{f(x + \tau(u + v)) - f(x)}{\tau} \leq \frac{f(x + 2\tau u) - f(x)}{2\tau} + \frac{f(x + 2\tau v) - f(x)}{2\tau}$$

for $\tau > 0$, and $\tau \downarrow 0$ gives $f'(x; u + v) \leq f'(x; u) + f'(x; v)$. □

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a sublinear function. The condition $f(-u) = -f(u)$ is necessary and sufficient in order that f be linear on the subspace $\text{lin}\{u\}$, and in this case we say that u is a *linearity direction* of f.

Lemma 1.5.7 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a sublinear function. The set $L[f]$ of all linearity directions of f is a linear subspace. Let $x \in \mathbb{R}^n$. Then $f'(x; \cdot) \leq f$ and*

$$\text{lin}\,(L[f] \cup \{x\}) \subset L[f'(x; \cdot)].$$

Proof Let $u, v \in L[f]$. Then $f(u+v) \leq f(u)+f(v) = -f(-u)-f(-v) \leq -f(-u-v) \leq f(u + v)$, the latter because $0 = f(o) = f(w - w) \leq f(w) + f(-w)$. It follows that $u + v \in L[f]$ and hence that $L[f]$ is a linear subspace.

For $u \in \mathbb{R}^n$ and $\tau > 0$, the sublinearity of f yields

$$\frac{f(x + \tau u) - f(x)}{\tau} \leq f(u),$$

and $\tau \downarrow 0$ gives $f'(x; u) \leq f(u)$.

Let $u \in L[f]$. From

$$2f(x) \leq f(x + \tau u) + f(x - \tau u) \leq 2f(x) + \tau[f(u) + f(-u)] = 2f(x)$$

it follows that

$$\frac{f(x + \tau u) - f(x)}{\tau} + \frac{f(x - \tau u) - f(x)}{\tau} = 0, \tag{1.3}$$

and $\tau \downarrow 0$ gives $f'(x; u) + f'(x; -u) = 0$, hence $u \in L[f'(x; \cdot)]$. By positive homogeneity, equality (1.3) holds also for $u = x$ if $0 < \tau < 1$, hence $\tau \downarrow 0$ gives $f'(x; x) + f'(x; -x) = 0$ and thus $x \in L[f'(x; \cdot)]$. □

We return to general convex functions and consider stronger differentiability properties. For a convex function, the existence of all two-sided directional derivatives (that is, Gâteaux differentiability) implies (total or Fréchet) differentiability. In fact, this is already implied by the existence of the partial derivatives. When talking of partial derivatives, we assume that a fixed orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ is given. The ith partial derivative of f at x, denoted by $\partial_i f(x)$, is the two-sided directional derivative $f'(x; e_i)$.

Theorem 1.5.8 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and $x \in \text{int}\,\text{dom}\, f$. If f has partial derivatives (of first order) at x, then f is differentiable at x.*

Proof If f has partial derivatives at x, we can define the *gradient* of f at x by

$$\nabla f(x) := \sum_{i=1}^{n} \partial_i f(x) e_i$$

and we then have to show that

$$\lim_{|h|\to 0} \frac{g(h)}{|h|} = 0, \tag{1.4}$$

where

$$g(h) := f(x+h) - f(x) - \langle \nabla f(x), h \rangle \quad \text{for } h \in \mathbb{R}^n.$$

The function g is convex and finite in a neighbourhood of o. For $h = \sum_{i=1}^{n} \eta_i e_i$ we have

$$g(h) = g\left(\frac{1}{n} \sum_{i=1}^{n} n\eta_i e_i\right) \le \frac{1}{n} \sum_{i=1}^{n} g(n\eta_i e_i).$$

Using the inequality

$$\langle h, v \rangle \le |h||v| \le |h| \sum_{i=1}^{n} |v_i| \quad \text{for } v = \sum_{i=1}^{n} v_i e_i,$$

we obtain

$$g(h) \le \sum_{i=1}^{n} \eta_i \frac{g(n\eta_i e_i)}{n\eta_i} \le |h| \sum_{i=1}^{n} \left| \frac{g(n\eta_i e_i)}{n\eta_i} \right|$$

(where the summands with $\eta_i = 0$ have to be replaced by 0) and a similar inequality with h replaced by $-h$. Since $g(h) + g(-h) \ge g(o) = 0$ by the convexity of g, we obtain, for $h \ne o$,

$$-\sum_{i=1}^{n} \left| \frac{g(-n\eta_i e_i)}{n\eta_i} \right| \le \frac{-g(-h)}{|h|} \le \frac{g(h)}{|h|} \le \sum_{i=1}^{n} \left| \frac{g(n\eta_i e_i)}{n\eta_i} \right|.$$

Since the partial derivatives of g at o exist and are zero, the left and right sides of this chain of inequalities tend to 0 for $h \to o$, hence (1.4) follows. □

If f is differentiable at x, then the gradient $\nabla f(x)$ of f at x, as defined above, does not depend on the choice of the orthonormal basis.

We note a consequence of the previous theorem, which concerns the distance function $d(K, \cdot)$ of a convex body $K \in \mathcal{K}^n$, as defined in Section 1.2. Recall that also the nearest point map $p(K, \cdot)$ and the vector function $u(K, x) = (x - p(K, x))/d(K, x)$ were defined there.

Lemma 1.5.9 *Let $K \in \mathcal{K}^n$. The distance function $d(K, \cdot)$ is convex. On $\mathbb{R}^n \setminus K$, it is continuously differentiable and satisfies*

$$\nabla d(K, \cdot) = u(K, \cdot). \tag{1.5}$$

Proof Let $x, y \in \mathbb{R}^n$ and $z = (1-\lambda)x + \lambda y$ with $\lambda \in [0,1]$. The point $z' := (1-\lambda)p(K,x) + \lambda p(K,y)$ satisfies $z' \in K$, therefore we obtain

$$d(K,z) \le |z - z'| \le (1-\lambda)|x - p(K,x)| + \lambda|y - p(K,y)| = (1-\lambda)d(K,x) + \lambda d(K,y).$$

Thus, $d(K,\cdot)$ is convex.

Let $x \in \mathbb{R}^n \setminus K$. With respect to an orthonormal basis, one vector of which is a positive multiple of $u(K,x)$, the partial derivatives of $d(K,\cdot)$ at x are easy to compute, and one finds that (1.5) holds. From Theorem 1.5.8 it follows that $d(K,\cdot)$ is differentiable at x. That the partial derivatives are continuous on $\mathbb{R}^n \setminus K$ follows from the continuity of $u(k,\cdot)$. □

For differentiable functions one has some useful criteria for convexity, which we now collect.

Theorem 1.5.10 *Let $D \subset \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ a differentiable function. Then f is convex if and only if f' is increasing.*

Proof If f is convex, then f' is increasing, by Theorem 1.5.4. Let f' be increasing, and let $x, y \in D$, $x < y$, and $\lambda \in (0,1)$. By the mean value theorem, there exist numbers $\theta_1 \in [x, (1-\lambda)x + \lambda y]$ and $\theta_2 \in [(1-\lambda)x + \lambda y, y]$ with

$$f'(\theta_1) = \frac{f((1-\lambda)x + \lambda y) - f(x)}{\lambda(y-x)}, \qquad f'(\theta_2) = \frac{f(y) - f((1-\lambda)x + \lambda y)}{(1-\lambda)(y-x)}.$$

From $f'(\theta_1) \le f'(\theta_2)$ we obtain

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y).$$ □

Corollary 1.5.11 *Let $D \subset \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ a twice differentiable function. Then f is convex if and only if $f'' \ge 0$.*

The extension of these criteria to differentiable functions on $\mathbb{R}^n$ relies on the simple fact, following from the definition of convexity, that a function on $\mathbb{R}^n$ is convex if and only if its restriction to an arbitrary line is convex.

Theorem 1.5.12 *Let $D \subset \mathbb{R}^n$ be convex and open and let $f : D \to \mathbb{R}$ be a differentiable function. Then the following assertions are equivalent.*

(a) *f is convex;*

(b) *$f(y) - f(x) \ge \langle \nabla f(x), y - x \rangle$ for $x, y \in D$;*

(c) *$\langle \nabla f(y) - \nabla f(x), y - x \rangle \ge 0$ for $x, y \in D$.*

Proof Suppose (a) holds. Let $x, y \in D$ and $0 < \lambda < 1$. From

$$f(x + \lambda(y-x)) = f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y)$$

we get

$$\frac{f(x+\lambda(y-x))-f(x)}{\lambda}-\langle\nabla f(x),y-x\rangle\le f(y)-f(x)-\langle\nabla f(x),y-x\rangle.$$

For $\lambda\to 0$ the left side converges to 0, hence (b) follows.

Suppose (b) holds. Interchanging the roles of x and y and adding both inequalities, we get (c).

Suppose (c) holds. Let $x,y\in D$. Let

$$g(\lambda):=f(x+\lambda(y-x))\quad\text{for } 0\le\lambda\le 1.$$

For $0\le\lambda_0<\lambda_1\le 1$ the chain rule gives

$$g'(\lambda_1)-g'(\lambda_0)=\langle\nabla f(x+\lambda_1(y-x)),y-x\rangle-\langle\nabla f(x+\lambda_0(y-x)),y-x\rangle\ge 0.$$

Thus g' is an increasing function and, therefore, g is a convex function. Since $x,y\in D$ were arbitrary, this implies that f is convex. □

Theorem 1.5.13 *Let $D\subset\mathbb{R}^n$ be convex and open and let $f:D\to\mathbb{R}$ be twice differentiable. Then f is convex if and only if its Hessian matrix,*

$$\operatorname{Hess} f(x):=\left(\partial_i\partial_j f(x)\right)_{i,j=1}^n,$$

is positive semi-definite for all $x\in D$.

Proof For $x,y\in D$ define

$$g(\lambda):=f(x+\lambda(y-x))\quad\text{for } x+\lambda(y-x)\in D.$$

Then f is convex if and only if all such functions g are convex, which is equivalent to $g''\ge 0$. We have only to check whether $g''(0)\ge 0$ holds generally. In fact, let $\lambda_1\in(0,1)$. Writing $x_1:=x+\lambda_1(y-x)$ and $h(\tau):=f(x_1+\tau(y-x_1))$, we have $g(\lambda_1+\lambda)=h(\lambda/(1-\lambda_1))$, hence $g''(\lambda_1)\ge 0$ if and only if $h''(0)\ge 0$.

Now for $x=\sum_{i=1}^n\xi_i e_i$, $y=\sum_{i=1}^n\eta_i e_i$ we have

$$\begin{aligned}
g''(0)&=\lim_{\lambda\to 0}\frac{g'(\lambda)-g'(0)}{\lambda}\\
&=\lim_{\lambda\to 0}\frac{1}{\lambda}\langle\nabla f(x+\lambda(y-x))-\nabla f(x),y-x\rangle\\
&=\lim_{\lambda\to 0}\sum_{j=1}^n\frac{1}{\lambda}[\partial_j f(x+\lambda(y-x))-\partial_j f(x)](\eta_j-\xi_j)\\
&=\sum_{i=1}^n\sum_{j=1}^n\partial_i\partial_j f(x)(\eta_i-\xi_i)(\eta_j-\xi_j),
\end{aligned}$$

from which the assertion follows. □

For convex functions that are not necessarily differentiable, the notions of gradient and differential have natural extensions; these have simple geometric interpretations.

If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is convex and differentiable at x, then Theorem 1.5.12 states that

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad \text{for all } y \in \mathbb{R}^n.$$

This gives rise to the following definition. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and let $x \in \operatorname{dom} f$. The set

$$\partial f(x) := \{v \in \mathbb{R}^n : f(y) \geq f(x) + \langle v, y - x \rangle \text{ for all } y \in \mathbb{R}^n\}$$

is called the *subdifferential of f at x*, and each element of $\partial f(x)$ is called a *subgradient of f at x*. Clearly, the subdifferential of f at x is a closed convex subset of $\mathbb{R}^n$. It parametrizes the set of support planes to the epigraph of f above x, in a sense made precise in the following lemma.

Lemma 1.5.14 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and let $x \in \operatorname{dom} f$. The vector v is a subgradient of f at x if and only if the vector $(v, -1) \in \mathbb{R}^n \times \mathbb{R}$ is an outer normal vector to* epi f *at $(x, f(x))$.*

Proof The vector $(v, -1)$ is an outer normal vector to epi f at $(x, f(x))$ if and only if

$$\langle (y, \eta) - (x, f(x)), (v, -1) \rangle \leq 0$$

for all $(y, \eta) \in \operatorname{epi} f$, and this is equivalent to

$$\langle (y, f(y)) - (x, f(x)), (v, -1) \rangle \leq 0$$

for all $y \in \mathbb{R}^n$, hence to

$$f(y) \geq f(x) + \langle v, y - x \rangle$$

for all $y \in \mathbb{R}^n$. □

Differentiability of a convex function is equivalent to the uniqueness of subgradients.

Theorem 1.5.15 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and let $x \in \operatorname{int} \operatorname{dom} f$. Then f is differentiable at x if and only if f has only one subgradient at x.*

Proof Suppose that f is differentiable at x. Let v be a subgradient of f at x, then

$$\frac{f(x + \lambda u) - f(x)}{\lambda} \geq \langle v, u \rangle$$

for $\lambda > 0$, hence $f'(x; u) \geq \langle v, u \rangle$ for all $u \in \mathbb{R}^n$. Differentiability of f at x implies $f'(x; u) = -f'(x; -u) \leq -\langle v, -u \rangle = \langle v, u \rangle$, hence $f'(x; u) = \langle v, u \rangle$. Since this holds for all $u \in \mathbb{R}^n$, the vector v is uniqueley determined.

Conversely, suppose that f has at most one subgradient at x. Let $u \in \mathbb{R}^n \setminus \{o\}$. Put

$$g(\lambda) := f(x + \lambda u) \quad \text{for } \lambda \in \mathbb{R}$$

and choose a number m with $g_l'(0) \leq m \leq g_r'(0)$. By Remark 1.5.5, we have $g(\lambda) \geq g(0) + m\lambda$ and thus $f(x + \lambda u) \geq f(x) + m\lambda$. Hence, the line

$$L := \{(x + \lambda u, f(x) + m\lambda) : \lambda \in \mathbb{R}\} \subset \mathbb{R}^n \times \mathbb{R}$$

satisfies $L \cap \operatorname{int} \operatorname{epi} f = \emptyset$. By Theorem 1.3.7, there exists in $\mathbb{R}^n \times \mathbb{R}$ a hyperplane $H_{(v,-1),\alpha}$ separating L and $\operatorname{int} \operatorname{epi} f$ (the normal vector of this hyperplane can in fact be assumed to be of the form $(v, -1)$ since, due to $x \in \operatorname{int} \operatorname{dom} f$, the hyperplane cannot be 'vertical'). Since $H_{(v,-1),\alpha}$ supports $\operatorname{epi} f$ at $(x, f(x))$, the vector v is a subgradient of f at x and hence is unique. The inclusion $L \subset H_{(v,-1),\alpha}$ implies that

$$\langle (v, -1), (x + \lambda u, f(x) + m\lambda) \rangle = \alpha$$

for all $\lambda \in \mathbb{R}$ and hence that $m = \langle v, u \rangle$. Since v is unique, m is also unique and thus g is differentiable at 0. Since $u \in \mathbb{R}^n \setminus \{o\}$ was arbitrary, f has two-sided directional derivatives at the point x and hence, by Theorem 1.5.8, is differentiable at x. □

Returning to subdifferentials, we call the set

$$\partial f := \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : v \in \partial f(x)\}$$

the *subdifferential* of the convex function f. If $(x_0, v_0) \in \partial f$, then

$$f(x_1) - f(x_0) \geq \langle x_1 - x_0, v_0 \rangle \quad \text{for } x_1 \in \mathbb{R}^n,$$

by the definition of the subdifferential at x_0. By addition we obtain, for any pairs $(x_0, v_0), \dots, (x_k, v_k) \in \partial f$, the inequality

$$\langle x_1 - x_0, v_0 \rangle + \langle x_2 - x_1, v_1 \rangle + \cdots + \langle x_k - x_{k-1}, v_{k-1} \rangle + \langle x_0 - x_k, v_k \rangle \leq 0. \tag{1.6}$$

A subset $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *cyclically monotonic* if (1.6) holds for all $k \in \mathbb{N}$ and all pairs $(x_0, v_0), \dots, (x_k, v_k) \in S$. This property (which extends condition (c) of Theorem 1.5.12) characterizes subsets of the subdifferential of a convex function.

Theorem 1.5.16 (Rockafellar) *Let $S \subset \mathbb{R}^n \times \mathbb{R}^n$. There exists a convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ with $S \subset \partial f$ if and only if S is cyclically monotonic.*

Proof The necessity has already been proved. Suppose that S is cyclically monotonic. Choose $(x_0, v_0) \in S$ (if S is empty, there is nothing to prove), and define

$$f(x) := \sup\{\langle x - x_k, v_k \rangle + \langle x_k - x_{k-1}, v_{k-1} \rangle + \cdots + \langle x_1 - x_0, v_0 \rangle\} \quad \text{for } x \in \mathbb{R}^n,$$

where the supremum is taken over all $k \in \mathbb{N}$ and all choices of $(x_i, v_i) \in S$, $i = 1, \dots, k$. Then $f(x) > -\infty$ and $f(x_0) = 0$, thus f is proper. Being a supremum of affine functions, f is convex.

Now let $(\bar{x}, \bar{v}) \in S$. Choose $\alpha < f(\bar{x})$. By the definition of f, there are a number $k \in \mathbb{N}$ and pairs $(x_i, v_i) \in S$, $i = 1, \dots, k$, such that

$$\alpha < \langle \bar{x} - x_k, v_k \rangle + \langle x_k - x_{k-1}, v_{k-1} \rangle + \cdots + \langle x_1 - x_0, v_0 \rangle. \tag{1.7}$$

With $(x_{k+1}, v_{k+1}) := (\bar{x}, \bar{v})$ we have

$$f(x) \geq \langle x - x_{k+1}, v_{k+1} \rangle + \langle x_{k+1} - x_k, v_n \rangle + \cdots + \langle x_1 - x_0, v_0 \rangle$$

for all $x \in \mathbb{R}^n$, by the definition of f. Together with (1.7) this gives

$$f(x) \geq \langle x - x_{k+1}, v_{k+1} \rangle + \alpha.$$

Since this holds for all $\alpha < f(\bar{x})$, we deduce that

$$f(x) \geq f(\bar{x}) + \langle x - \bar{x}, \bar{v} \rangle$$

and hence that $(\bar{x}, \bar{v}) \in \partial f$. This completes the proof of $S \subset \partial f$. $\square$

Notes for Section 1.5

1. Standard references for convex functions are Rockafellar [1583] and Roberts and Varberg [1581], which we have followed in many respects; see also Marti [1331]. The more recent book by Borwein and Vanderwerff [305] is also recommended.
2. *Differentiability almost everywhere of convex functions.* Let f be a finite convex function on an open convex subset $D \subset \mathbb{R}^n$. By Theorem 1.5.3, f is locally Lipschitzian and hence, by Rademacher's theorem (e.g., Federer [557], p. 216), differentiable almost everywhere on D. This follows also from Theorem 1.5.15 and Theorem 2.2.5, the latter applied to the epigraph of f, or from the one-dimensional case (Theorem 1.5.4) and Fubini's theorem (Roberts and Varberg [1581], p. 116). Rather precise information on the set of non-differentiability is available from work of Zajíček [2027], who also has a related result for continuous convex functions on a separable Banach space.
3. *Twice differentiability almost everywhere.* A convex function f is in fact twice differentiable almost everywhere. Since the first differential need not exist in a full neighhbourhood of the point considered, this assertion must be interpreted carefully. One possible way of defining the twice differentiability of f at x is by the existence of a second-order Taylor expansion. This means that f is differentiable at x and that there exists a symmetric linear map $Af(x) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(y) = f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2}\langle Af(x)(y - x), y - x \rangle + o(|y - x|^2)$$

for all $y \in D$. Another possibility is to say that a map $\vartheta : D \to \mathbb{R}^n$ is a subgradient choice for f if $\vartheta(x) \in \partial f(x)$ for each $x \in D$. In this case, the function f is called twice differentiable at x if the family of subgradient choices for f is uniformly differentiable at x, which means that there are a neighbourhood V of x, a linear map $Af(x) : \mathbb{R}^n \to \mathbb{R}^n$ and a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(\tau) \to 0$ for $\tau \to 0$ such that

$$|\vartheta(y) - \vartheta(x) - Af(x)(y - x)| \leq \psi(|y - x|)|y - x|$$

for $y \in V \cap D$ and each subgradient choice ϑ for f. It follows from the work of Aleksandrov [19] (see also Bangert [136]) that for a convex function twice differentiability in this sense and the existence of a second-order Taylor expansion are equivalent.

With this definition, it is true that a real convex function f on an open set is twice differentiable almost everywhere on its domain. For $n = 1$, this is a consequence of the differentiability almost everywhere of a monotonic function, as was first pointed out by Jessen [1038]. For $n = 2$, it was proved by Busemann and Feller [374], and, by using their result and an induction argument, Aleksandrov [19] obtained the general case. Other proofs, more analytic in character, are due to Rešetnjak [1576], who used techniques from the theory of distributions, and to Bangert [135, 136]. Essential tools of Bangert's proof are the Lipschitz property of the metric projection (Theorem 1.2.1), Rademacher's theorem on the almost everywhere differentiability of Lipschitz maps and a version of Sard's lemma for Lipschitz maps.

The different proofs of Aleksandrov's theorem on the twice differentiability almost everywhere of convex functions are described, and a new proof is presented, by Bianchi, Colesanti and Pucci [225]. Proofs are also found in the books by Evans and Gariepy [538], §6.4, Gruber [834], Subsection 2.2, and Borwein and Vanderwerff [305], Theorem 2.6.4. Extensions to infinite dimensions is the topic of Borwein and Noll [303], who give further references.

A far-reaching extension of Aleksandrov's theorem, to a much wider class of functions defined by certain approximation properties, was proved by Fu [643].

Asplund [96] applied Aleksandrov's theorem to show that a metric projection on any closed (not necessarily convex) subset of $\mathbb{R}^n$ is almost everywhere differentiable.

4. Dudley [518] showed that a Schwartz distribution on $\mathbb{R}^n$ is a convex function if and only if its second derivative is a nonnegative $n \times n$ matrix-valued Radon measure. He also showed the absolute continuity of such a measure with respect to $(n-1)$-dimensional Hausdorff measure, and other related results.
5. Theorem 1.5.16 is due to Rockafellar [1582]; see also [1583]. It plays an important role in the modern theory of measure transport; see McCann [1369].
6. *The second-order subdifferential*. Given a finite convex function f on $\mathbb{R}^n$, Hiriart-Urruty [980] introduced for all $x \in \mathbb{R}^n$ and all $v \in \partial f(x)$ a closed convex set containing o, which he (later) called the second-order subdifferential of f at (x, v), denoting it by $\partial^2 f(x, v)$. Then

$$\partial^2 f(x) := \bigcap_{v \in \partial f(x)} \partial^2 f(x, v)$$

is called the second-order subdifferential of f at x. The set $\partial^2 f(x)$ is compact and convex. If f is twice differentiable at x, then $\partial^2 f(x)$ is the ellipsoid associated with the ordinary second differential of f at x. The second-order subdifferential was further investigated by Hiriart-Urruty and Seeger [981], who also studied it in relation to a notion of generalized Dupin indicatrices for non-smooth convex surfaces.

1.6 Duality

Associated with convex sets, cones and functions, which satisfy some weak conditions, are dual objects of the same kind. This duality permits us, for example, to translate certain results on boundary points of convex sets into results on support planes (or normal vectors), and conversely. But duality is useful in many other respects.

1.6.1 Duality of convex sets

In this section, the position of a convex set with respect to the origin o plays a role. Therefore, we denote by $\mathcal{K}^n_o$ the set of all convex bodies in $\mathcal{K}^n$ containing o, and by $\mathcal{K}^n_{(o)}$ the subset of all convex bodies with o as an interior point.

For subsets $K \subset \mathbb{R}^n$ we define the *polar set* by

$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

(More properly, K° could be defined as the subset

$$\{x^* \in (\mathbb{R}^n)^* : x^*(y) \leq 1 \text{ for all } y \in K\}$$

of the dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$. As, however, we are working with a fixed scalar product in $\mathbb{R}^n$, we always use the isomorphism between $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$ induced by this scalar product to identify both spaces.)

The next theorem shows that for $K \in \mathcal{K}^n_{(o)}$ the set K°, which is called the *polar body* of K, satisfies the same assumptions as K and that this correspondence is a true duality.

Theorem 1.6.1 *Let $K \in \mathcal{K}^n_{(o)}$. Then $K^\circ \in \mathcal{K}^n_{(o)}$ and $K^{\circ\circ} = K$.*

Proof If $x_1, x_2 \in K^\circ$ and $\lambda \in [0,1]$, then $\langle (1-\lambda)x_1 + \lambda x_2, y\rangle \le 1$ for all $y \in K$, hence $(1-\lambda)x_1+\lambda x_2 \in K^\circ$. Thus K° is convex, and it is equally easy to see that K° is closed. For a ball $B(o,\varepsilon)$ one has $B(o,\varepsilon)^\circ = B(o,1/\varepsilon)$ as an immediate consequence of the definition and the Cauchy–Schwarz inequality. Further, $K_1 \subset K_2$ implies $K_1^\circ \supset K_2^\circ$. We can choose $\varepsilon, \rho > 0$ with $B(o,\varepsilon) \subset K \subset B(o,\rho)$; hence $B(o,1/\rho) \subset K^\circ \subset B(o,1/\varepsilon)$, showing that $o \in \operatorname{int} K^\circ$ and that K° is bounded.

Let $y \in K$. For arbitrary $x \in K^\circ$ one has $\langle x,y\rangle \le 1$, hence $y \in K^{\circ\circ}$. Thus $K \subset K^{\circ\circ}$ (observe that up to now neither convexity nor closedness of K have been needed).

Let $z \in \mathbb{R}^n \setminus K$. Since K is convex and closed, K and z can be strongly separated, by Theorem 1.3.4, hence there is a hyperplane $H_{u,\alpha}$ with $K \subset \operatorname{int} H^-_{u,\alpha}$ and $\langle z,u\rangle > \alpha$; here $\alpha > 0$ since $o \in K$. For all $y \in K$ we have $\langle y, u/\alpha\rangle \le 1$, hence $u/\alpha \in K^\circ$. Now $\langle z, u/\alpha\rangle > 1$ shows that $z \notin K^{\circ\circ}$. This finishes the proof of $K^{\circ\circ} = K$. □

We denote by CC^n_o the set of all closed convex subsets of $\mathbb{R}^n$ containing o. By an obvious modification of the previous proof, the following result is obtained.

Theorem 1.6.2 *Let $K \in CC^n_o$. Then $K^\circ \in CC^n_o$ and $K^{\circ\circ} = K$.*

The following theorem describes the extent to which the forming of polar convex sets interchanges the operations of intersection and union.

Theorem 1.6.3 *Let $K_1, K_2 \in \mathcal{K}^n_{(o)}$ or $K_1, K_2 \in CC^n_o$. Then*

$$(K_1 \cap K_2)^\circ = \operatorname{conv}(K_1^\circ \cup K_2^\circ), \qquad [\operatorname{conv}(K_1 \cup K_2)]^\circ = K_1^\circ \cap K_2^\circ.$$

If $K_1 \cup K_2$ is convex, then $K_1^\circ \cup K_2^\circ$ is convex, hence

$$(K_1 \cap K_2)^\circ = K_1^\circ \cup K_2^\circ, \qquad (K_1 \cup K_2)^\circ = K_1^\circ \cap K_2^\circ.$$

Proof From $K_1 \cap K_2 \subset K_i$ we get $K_i^\circ \subset (K_1 \cap K_2)^\circ$, hence $\operatorname{conv}(K_1^\circ \cup K_2^\circ) \subset (K_1 \cap K_2)^\circ$. From $K_i \subset \operatorname{conv}(K_1 \cup K_2)$ we get $[\operatorname{conv}(K_1 \cup K_2)]^\circ \subset K_i^\circ$, hence $[\operatorname{conv}(K_1 \cup K_2)]^\circ \subset K_1^\circ \cap K_2^\circ$. Applying both inclusions to K_i° instead of K_i and using $K^{\circ\circ} = K$, we arrive at the first two equalities of the theorem.

Suppose that $K_1 \cup K_2$ is convex. Let $x \in \mathbb{R}^n \setminus (K_1^\circ \cup K_2^\circ)$. As in the proof of Theorem 1.6.1, there are $u_i \in \mathbb{R}^n \setminus \{o\}$ and $\alpha_i > 0$ with $\langle x, u_i\rangle > \alpha_i$ and $\langle y, u_i\rangle < \alpha_i$ for $y \in K_i^\circ$, which implies $u_i/\alpha_i \in K_i^{\circ\circ} = K_i$ $(i = 1,2)$. Since $K_1 \cup K_2$ is convex, there is a point $z \in [u_1/\alpha_1, u_2/\alpha_2] \cap K_1 \cap K_2$. Since z is a convex combination of u_1/α_1 and u_2/α_2, we get $\langle x,z\rangle > 1$ and hence $x \notin (K_1 \cap K_2)^\circ$. This proves $(K_1 \cap K_2)^\circ \subset K_1^\circ \cup K_2^\circ$. The opposite inclusion being trivial, we obtain $(K_1 \cap K_2)^\circ = K_1^\circ \cup K_2^\circ$, which shows that $K_1^\circ \cup K_2^\circ$ is convex. □

We turn now to the question (first posed by V. Milman) of how far the polarity $K \mapsto K^\circ$ on $\mathcal{K}^n_{(o)}$ is uniquely determined by its essential properties. For this, it is advisable to take a more general viewpoint. We write

$$A \vee B := \operatorname{conv}(A \cup B)$$

for subsets $A, B \subset \mathbb{R}^n$ and observe that $(\mathcal{K}^n_{(o)}, \cap, \vee)$ is a lattice. Theorem 1.6.3 shows that the polarity $K \mapsto K^\circ$ interchanges the lattice operations. More generally, let $\mathcal{M}$ be a system of closed convex subsets of $\mathbb{R}^n$ that is closed under the operations $\cap$ and $\vee$. Then $(\mathcal{M}, \cap, \vee)$ is a lattice. A mapping $T : \mathcal{M} \to \mathcal{M}$ with the properties $T(K \cap L) = T(K) \cap T(L)$ and $T(K \vee L) = T(K) \vee T(L)$ for all $K, L \in \mathcal{M}$ is called a *lattice endomorphism*. The following theorem determines the lattice endomorphisms of $(\mathcal{K}^n_{(o)}, \cap, \vee)$.

Theorem 1.6.4 *Let $n \geq 2$. If T is a lattice endomorphism of $(\mathcal{K}^n_{(o)}, \cap, \vee)$, then either T is a constant mapping or there exists a linear transformation $\Lambda \in \mathrm{GL}(n)$ such that $T(K) = \Lambda K$ for all $K \in \mathcal{K}^n_{(o)}$.*

Note, in particular, that there is no assumption of surjectivity for the mapping T. For the proof, we refer to Böröczky and Schneider [298].

If $(\mathcal{M}, \cap, \vee)$ is a lattice as above, we understand by an *order isomorphism* of this lattice a bijective mapping $T : \mathcal{M} \to \mathcal{M}$ satisfying

$$K \subset L \Leftrightarrow T(K) \subset T(L) \tag{1.8}$$

for all $K, L \in \mathcal{M}$.

Lemma 1.6.5 *Let $\mathcal{M}$ be a system of closed convex sets in $\mathbb{R}^n$ such that $(\mathcal{M}, \cap, \vee)$ is a lattice. Any order isomorphism of this lattice is a lattice endomorphism.*

Proof Let T be an order isomorphism of $(\mathcal{M}, \cap, \vee)$. Let $K, L \in \mathcal{M}$. From $K \cap L \subset K, L$ and (1.8) we get $T(K \cap L) \subset T(K) \cap T(L)$. Since T is bijective, there exists $M \in \mathcal{M}$ such that $T(M) = T(K) \cap T(L)$. From $T(M) \subset T(K), T(L)$ and (1.8) it follows that $M \subset K, L$, thus $M \subset K \cap L$ and hence $T(M) \subset T(K \cap L)$. This yields $T(K \cap L) = T(K) \cap T(L)$. Similarly one obtains that $T(K \vee L) = T(K) \vee T(L)$. □

First we use this to determine the order isomorphisms of $\mathcal{K}^n_{(o)}$. The following corollary follows immediately from Lemma 1.6.5 and Theorem 1.6.4, since the bijectivity excludes a constant mapping.

Corollary 1.6.6 *Let $n \geq 2$. Let $\varphi : \mathcal{K}^n_{(o)} \to \mathcal{K}^n_{(o)}$ be a bijective mapping satisfying*

$$K \subset L \Leftrightarrow \varphi(K) \subset \varphi(L) \tag{1.9}$$

for all $K, L \in \mathcal{K}^n_{(o)}$. Then there exists a linear transformation $\Lambda \in \mathrm{GL}(n)$ such that $\varphi(K) = \Lambda K$ for all $K \in \mathcal{K}^n_{(o)}$.

Now we obtain characterizations of the polarity on $\mathcal{K}^n_{(o)}$, though with some modification, due to the arbitrariness of the scalar product that was used in the definition of polar sets.

Corollary 1.6.7 *Let $n \geq 2$. Let $\psi : \mathcal{K}^n_{(o)} \to \mathcal{K}^n_{(o)}$ be a mapping satisfying*

$$\psi(K \cap L) = \psi(K) \vee \psi(L), \qquad \psi(K \vee L) = \psi(K) \cap \psi(L)$$

for all $K, L \in \mathcal{K}^n_{(o)}$*. Then either* ψ *is a constant mapping or there exists a linear transformation* $\Lambda \in \mathrm{GL}(n)$ *such that* $\psi(K) = \Lambda K^\circ$ *for all* $K \in \mathcal{K}^n_{(o)}$.

Proof Setting $T(K) := \psi(K^\circ)$, we obtain the assertion from Theorems 1.6.3 and 1.6.4. □

Another corollary shows that an order-reversing involution of $\mathcal{K}^n_{(o)}$ is essentially the polarity.

Corollary 1.6.8 *Let* $n \geq 2$*. Let* $\psi : \mathcal{K}^n_{(o)} \to \mathcal{K}^n_{(o)}$ *be a mapping satisfying*

$$\psi(\psi(K)) = K \tag{1.10}$$

and

$$K \subset L \Rightarrow \psi(K) \supset \psi(L) \tag{1.11}$$

for all $K, L \in \mathcal{K}^n_{(o)}$*. Then there exists a selfadjoint linear transformation* $\Lambda \in \mathrm{GL}(n)$ *such that* $\psi(K) = \Lambda K^\circ$ *for all* $K \in \mathcal{K}^n_{(o)}$.

The selfadjointness of the linear transformation Λ in this corollary is a consequence of the involution property (1.10).

If $C \subset \mathbb{R}^n$ is a convex cone, then for $x \in \mathbb{R}^n$ we have $\langle x, y\rangle \leq 1$ for all $y \in C$ if and only if $\langle x, y\rangle \leq 0$ for all $y \in C$, hence the polar convex set is given by

$$C^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 0 \text{ for all } y \in C\}.$$

This is called the *dual cone* of C. If C is a closed convex cone, then C° is again a closed convex cone, and one has $C^{\circ\circ} = C$. The proof is analogous to the corresponding one for polar bodies, except that instead of Theorem 1.3.4, the separation theorem 1.3.9 has to be applied. For the counterpart of Theorem 1.6.3, we observe that for convex cones the convex hull of the union is the same as the sum, and that the sum of closed convex cones is not necessarily closed. Apart from this, the proof of the following theorem is so similar to that of Theorem 1.6.3 that it can be omitted.

Theorem 1.6.9 *Let* $C_1, C_2 \subset \mathbb{R}^n$ *be closed convex cones. Then*

$$(C_1 \cap C_2)^\circ = \mathrm{cl}\,(C_1^\circ + C_2^\circ), \qquad [\mathrm{cl}\,(C_1 + C_2)]^\circ = C_1^\circ \cap C_2^\circ.$$

If $C_1 \cup C_2$ *is convex, then* $C_1^\circ \cup C_2^\circ$ *is convex, hence*

$$(C_1 \cap C_2)^\circ = C_1^\circ \cup C_2^\circ, \qquad (C_1 \cup C_2)^\circ = C_1^\circ \cap C_2^\circ.$$

For the lattice endomorphisms of the lattice $(\mathscr{C}^n, \cap, \vee)$, where $\mathscr{C}^n$ denotes the set of closed convex cones in $\mathbb{R}^n$, there is (for $n \geq 3$) a similar characterization as given in Theorem 1.6.4; see Schneider [1727]. As a consequence, there is essentially only one order-reversing involution for closed convex cones.

Theorem 1.6.10 *Let $n \geq 3$. Let $\psi : \mathscr{C}^n \to \mathscr{C}^n$ be a mapping satisfying*

$$\psi(\psi(C)) = C \quad \text{and} \quad C \subset D \Rightarrow \psi(C) \supset \psi(D)$$

for all $C, D \in \mathscr{C}^n$. Then there exists a selfadjoint linear transformation $\Lambda \in \mathrm{GL}(n)$ such that $\psi(C) = \Lambda C^\circ$ for all $C \in \mathscr{C}^n$.

1.6.2 Duality of convex functions

To treat duality for convex functions, a few preliminaries are necessary. To establish the dualities above, we had to use the fact that a closed convex set is the intersection of its supporting halfspaces. In the case of a convex function, this fact will be applied to the epigraph. One considers, therefore, convex functions whose epigraph is closed; such a function is called, in brief, *closed*. Closedness of a function is evidently equivalent to lower semi-continuity.

We mention a criterion for a convex function to be closed.

Lemma 1.6.11 *The convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is closed if and only if there is a point $z \in \operatorname{relint} \operatorname{dom} f$ such that, for each $y \in \operatorname{cl} \operatorname{dom} f$,*

$$\lim_{\lambda \uparrow 1} f((1-\lambda)z + \lambda y) = f(y). \tag{1.12}$$

Proof If f is closed, then $\operatorname{epi} f$ is closed, hence (1.12) holds for any $z \in \operatorname{relint} \operatorname{dom} f$ (and $\operatorname{dom} f \neq \emptyset$ since f is proper). Conversely, let $z \in \operatorname{relint} \operatorname{dom} f$ be such that (1.12) is satisfied. Let $(y, \eta) \in \operatorname{cl} \operatorname{epi} f$. Since then $y \in \operatorname{cl} \operatorname{dom} f$, it follows from the assumption on z and from Lemma 1.1.9 that $(1-\lambda)(z, f(z)) + \lambda(y, \eta) \in \operatorname{epi} f$ for $0 \leq \lambda < 1$, hence $f((1-\lambda)z + \lambda y) \leq (1-\lambda) f(z) + \lambda\eta$. By (1.12), this implies $f(y) \leq \eta$, hence $(y, \eta) \in \operatorname{epi} f$. Thus $\operatorname{epi} f$ is closed. □

The closed halfspaces containing the epigraph of a convex function are of two kinds. Generally, a closed halfspace in $\mathbb{R}^n \times \mathbb{R}$ containing the epigraph of a proper function is of the form

$$\{(x, \zeta) \in \mathbb{R}^n \times \mathbb{R} : \langle (x, \zeta), (u, \eta) \rangle \leq \alpha\}$$

with $u \in \mathbb{R}^n \setminus \{o\}$, $\eta \leq 0$ and $\alpha \in \mathbb{R}$. If $\eta = 0$, the halfspace is of the form

$$\{(x, \zeta) \in \mathbb{R}^n \times \mathbb{R} : \langle x, u \rangle - \alpha \leq 0\}$$

and is called *vertical*. If it is not vertical, we can assume that $\eta = -1$, so that the halfspace is given by

$$H^-_{(u,-1),\alpha} = \{(x, \zeta) : \langle x, u \rangle - \alpha \leq \zeta\} = \operatorname{epi} h$$

with $h(x) := \langle x, u \rangle - \alpha$. Observing that $\operatorname{epi} f \subset \operatorname{epi} h$ for functions h, f is equivalent to $h \leq f$, we are led to the following lemma.

Lemma 1.6.12 *Let $f := \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and closed. Then*

$$f = \sup\{h : h \text{ affine function on } \mathbb{R}^n, h \leq f\}.$$

Proof The closed convex set epi f is the intersection of the closed halfspaces containing it. We have to show that it is already the intersection of the non-vertical ones among these halfspaces. Let

$$H_1^- := \{(x, \zeta) \in \mathbb{R}^n \times \mathbb{R} : h_1(x) \le 0\}, \quad h_1(x) := \langle x, u_1 \rangle - \alpha_1,$$

be a vertical closed halfspace containing epi f, and let $(x_0, \zeta_0) \in \mathbb{R}^n \times \mathbb{R}$ be a point such that $(x_0, \zeta_0) \notin H_1^-$. There is some non-vertical supporting halfspace to epi f (for instance, at a point $(x, f(x))$ with $x \in \operatorname{relint} \operatorname{dom} f$), say

$$H_2^- := \{(x, \zeta) \in \mathbb{R}^n \times \mathbb{R} : h_2(x) \le \zeta\}, \quad h_2(x) := \langle x, u_2 \rangle - \alpha_2.$$

Define $h := \lambda h_1 + h_2$, with $\lambda > 0$ to be specified below. For $x \in \operatorname{dom} f$ we have $h_1(x) \le 0$ and $h_2(x) \le f(x)$, hence $h(x) \le f(x)$. If $x \in \mathbb{R}^n \setminus \operatorname{dom} f$, then $f(x) = \infty$ and hence $h(x) \le f(x)$ again. Thus

$$\operatorname{epi} f \subset H^- := \{(x, \zeta) \in \mathbb{R}^n \times \mathbb{R} : h(x) \le \zeta\}.$$

Since $(x_0, \zeta_0) \notin H_1^-$, we have $h_1(x_0) > 0$, and hence $\lambda > 0$ can be chosen such that $h(x_0) > \zeta_0$ and thus $(x_0, \zeta_0) \notin H^-$. We have found a non-vertical closed halfspace containing epi f that does not contain (x_0, ζ_0). Thus, epi f is the intersection of all the non-vertical closed halfspaces containing it; equivalently, f is the supremum of all affine functions $\le f$. □

We are now in a position to treat the announced duality for convex functions. For a convex function $f : \mathbb{R}^n \to \bar{R}$ (not necessarily closed) the *conjugate function* is defined by

$$f^*(u) := \sup\{\langle u, x \rangle - f(x) : x \in \mathbb{R}^n\} \quad \text{for } u \in \mathbb{R}^n. \tag{1.13}$$

The following reformulation of the definition sheds some light on the intuitive meaning of the conjugate function. For given $u \in \mathbb{R}^n \setminus \{o\}$, every real number α determines a non-vertical halfspace $H^-_{(u,-1),\alpha}$ in $\mathbb{R}^n \times \mathbb{R}$, and we have

$$(x, f(x)) \in H^-_{(u,-1),\alpha} \Leftrightarrow \langle u, x \rangle - f(x) \le \alpha,$$

where the latter inequality is satisfied for all $x \in \mathbb{R}^n$ if and only if $\alpha \ge f^*(u)$. Hence, we may also write

$$f^*(u) = \inf\{\alpha \in \mathbb{R} : \operatorname{epi} f \subset H^-_{(u,-1),\alpha}\}, \tag{1.14}$$

which can be interpreted as saying that the conjugate function f^* provides a parametrization of the set of supporting halfspaces of the epigraph of f.

Theorem 1.6.13 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a closed convex function. Then f^* is a closed convex function, and $f^{**} = f$.*

Proof From $\{f = \infty\} \neq \mathbb{R}^n$ it follows that $\{f^* = -\infty\} = \emptyset$. As the epigraph epi f has some non-vertical supporting halfspace $\{(x, \zeta) : \langle u, x \rangle - \alpha \le \zeta\}$, we have $\langle u, x \rangle - f(x) \le \alpha$ for $x \in \mathbb{R}^n$ and therefore $f^*(u) \le \alpha$, hence $\{f^* = \infty\} \neq \mathbb{R}^n$. Thus f^* is proper. Being the supremum of a family of affine functions, which are convex and continuous, f^* is convex and lower semi-continuous, hence closed. The definition of f^* can be rewritten as

$$f^*(u) = \sup\{\langle u, x\rangle - \zeta : (x, \zeta) \in \operatorname{epi} f\},$$

thus

$$f^{**}(x) = \sup\{\langle x, u\rangle - \alpha : (u, \alpha) \in \operatorname{epi} f^*\}.$$

Now $(u, \alpha) \in \operatorname{epi} f^*$ is equivalent to $\langle \cdot, u\rangle - \alpha \le f$, hence Lemma 1.6.12 gives

$$\begin{aligned} f(x) &= \sup\{h(x) : h \text{ affine function on } \mathbb{R}^n, h \le f\} \\ &= \sup\{\langle x, u\rangle - \alpha : (u, \alpha) \in \operatorname{epi} f^*\} = f^{**}(x), \end{aligned}$$

which completes the proof. □

Remark 1.6.14 If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a convex function, but not necessarily closed, the previous proof shows that f^* is still a closed convex function, and it shows moreover that $f^{**} = \operatorname{cl} f$, where $\operatorname{cl} f$ is the function with $\operatorname{epi} \operatorname{cl} f = \operatorname{cl} \operatorname{epi} f$. This function is called the *lower semi-continuous hull* of f.

Remark 1.6.15 Let $g : [0, \infty) \to [0, \infty)$ be a convex function satisfying $g(0) = 0$ (and hence non-decreasing). Then the function g^* defined by

$$g^*(u) := \sup_{x \ge 0}\{xu - g(x)\} \qquad \text{for } u \ge 0$$

has the same properties, and we have $g^{**} = g$, by the same proof as above.

The following lemma, which will be needed several times in Section 1.7, shows that under a mild assumption the infimum in (1.14) is attained.

Lemma 1.6.16 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a closed convex function and suppose that $u \in \operatorname{relint} \operatorname{dom} f^*$. Then there exists a point $x \in \operatorname{dom} f$ with $u \in \partial f(x)$.*

Proof First we suppose that $u \in \operatorname{int} \operatorname{dom} f^*$. According to (1.14), we can choose a point $(z, \eta) \in \mathbb{R}^n \times \mathbb{R}$ with $\langle u, z\rangle - \eta > f^*(u)$. Since, by Theorem 1.5.3, f^* is continuous on $\operatorname{int} \operatorname{dom} f^*$, there is a neighbourhood U of u with $\langle v, z\rangle - \eta > f^*(v)$ for all $v \in U$. By (1.14), this means that $\operatorname{epi} f \subset \operatorname{int} H^-_{(v,-1),\langle v,z\rangle-\eta}$ for $v \in U$. For a suitable number β, the set

$$(\operatorname{epi} f) \cap H^+_{(u,-1),\beta} \cap \bigcap_{v \in U} H^-_{(v,-1),\langle v,z\rangle-\eta}$$

is not empty. Since it is compact and convex, the infimum in (1.14) is attained, and the hyperplane $H_{(u,-1),f^*(u)}$ touches $\operatorname{epi} f$ at some point $(x, f(x))$. By Lemma 1.5.14, this means that $u \in \partial f(x)$.

Now suppose that $\dim \operatorname{dom} f^* = k < n$. Since, for any $v \in \mathbb{R}^n$, the function g defined by $g(x) = f(x) + \langle v, x\rangle$ satisfies $g^*(u) = f^*(u - v)$ and $\partial g(x) = \partial f(x) + v$, we may assume that $o \in \operatorname{dom} f^*$. Let $L := \operatorname{lin} \operatorname{dom} f^*$ and $h := f|L$ (f restricted to L). Then $\operatorname{dom} h^* = \operatorname{dom} f^*$.

Let $x \in L$. We assert that

$$(x + L^\perp, f(x)) \subset \operatorname{epi} f. \tag{1.15}$$

Suppose this were false. Then there exists $y \in L^\perp$ such that the point $(x + y, f(x))$ and the closed convex set $\operatorname{epi} f$ can be strongly separated by a hyperplane $H_{(w,-1),\alpha} \subset \mathbb{R}^n \times \mathbb{R}$, such that

$$\operatorname{epi} f \subset H^-_{(w,-1),\alpha}, \qquad (x + y, f(x)) \notin H^-_{(w,-1),\alpha}.$$

The first relation gives $f^*(w) < \infty$, hence $w \in L$. The second relation gives $\langle (x + y, f(x)), (w, -1) \rangle > \alpha$, which together with the inequality $\langle w, x \rangle - f(x) \leq \alpha$, which follows from the first relation, yields $\langle w, y \rangle > 0$ and thus $w \notin L$. This is a contradiction, which proves (1.15).

Relation (1.15) shows that $f(x + y) = f(x)$ for $x \in L$ and $y \in L^\perp$. If now $u \in \operatorname{relint} \operatorname{dom} f^*$, we can apply the first part of the proof to the function $h := f|L$ and obtain a point $x \in L$ with $u \in \partial(f|L)(x)$, and from the special form of f we conclude that $u \in \partial f(x)$. □

To see what conjugation does with sums, we define, for closed convex functions $f_1, f_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$, the *infimal convolution* by

$$(f_1 \square f_2)(x) := \inf_{x_1 + x_2 = x} \{f_1(x_1) + f_2(x_2)\}, \quad x \in \mathbb{R}^n.$$

Writing $\operatorname{epi} f_1 + \operatorname{epi} f_2 =: F$, we have $(x, \eta) \in F$ if and only if there exist $(x_i, \eta_i) \in \mathbb{R}^n \times \mathbb{R}$ with $f_i(x_i) \leq \eta_i$ $(i = 1, 2)$ and $x = x_1 + x_2$, $\eta = \eta_1 + \eta_2$. Therefore, if f denotes the function with $\operatorname{epi} f = F$, we get

$$f(x) = \inf\{\eta : (x, \eta) \in F\} = \inf_{x = x_1 + x_2} \{\eta : f_1(x_1) + f_2(x_2) \leq \eta\} = (f_1 \square f_2)(x).$$

It follows that $f_1 \square f_2$, if it does not attain the value $-\infty$, is a convex function, according to our definition (which includes properness). It need, however, not be closed. For example, if $A := \{(\xi, \eta) \in \mathbb{R}^2 : \xi > 0, \xi\eta \geq 1\}$, $B := \{(\xi, \eta) \in \mathbb{R}^2 : \eta = 0\}$, then $\operatorname{dom} (I_A^\infty \square I_B^\infty) = \{(\xi, \eta) \in \mathbb{R}^2 : \eta > 0\}$, hence $I_A^\infty \square I_B^\infty$ is not closed.

Theorem 1.6.17 *Let $f_1, f_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$ be closed convex functions. Then*

$$\operatorname{epi} (f_1 \square f_2) = \operatorname{epi} f_1 + \operatorname{epi} f_2. \tag{1.16}$$

If $f_1 \square f_2 > -\infty$, then

$$(f_1 \square f_2)^* = f_1^* + f_2^*. \tag{1.17}$$

If $f_1 + f_2 \not\equiv \infty$, then

$$(f_1 + f_2)^* = \operatorname{cl} (f_1^* \square f_2^*). \tag{1.18}$$

Proof The relation (1.16) has already been proved. Suppose that $f_1 \Box f_2 > -\infty$. Then, for $u \in \mathbb{R}^n$,

$$\begin{aligned}(f_1 \Box f_2)^*(u) &= \sup_x \left\{ \langle x, u \rangle - \inf_{x_1+x_2=x} \{f_1(x_1) + f_2(x_2)\} \right\} \\ &= \sup_x \sup_{x_1+x_2=x} \{\langle x, u \rangle - f_1(x_1) - f_2(x_2)\} \\ &= \sup_{x_1,x_2} \{\langle x_1, u \rangle + \langle x_2, u \rangle - f_1(x_1) - f_2(x_2)\} \\ &= f_1^*(u) + f_2^*(u).\end{aligned}$$

Thus (1.17) holds. Suppose that $f_1 + f_2 \not\equiv \infty$. Then the preceding chain of equations also shows that $f_1^* \Box f_2^*$ does not attain the value $-\infty$, and we conclude from (1.17) and Theorem 1.6.13 that $(f_1^* \Box f_2^*)^* = f_1^{**} + f_2^{**} = f_1 + f_2$, hence $(f_1 + f_2)^* = (f_1^* \Box f_2^*)^{**} = \mathrm{cl}\,(f_1^* \Box f_2^*)$, by Remark 1.6.14. This completes the proof. $\Box$

We remark that also

$$(\lambda f)^* = \lambda f^* \left(\frac{\cdot}{\lambda} \right) \tag{1.19}$$

for a convex function f and $\lambda > 0$, as follows immediately from the definition.

The function f^* is often called the *Fenchel conjugate* of f. The conjugation $f \mapsto f^*$ is also known as the *Legendre* (or *Legendre–Fenchel*) *transformation*, and one writes

$$\mathcal{L}f := f^*.$$

Denoting by $\mathrm{Cvx}(\mathbb{R}^n)$ the space of closed convex functions on $\mathbb{R}^n$, we see that $\mathcal{L}$ maps this space bijectively onto itself. The function $\mathcal{L}f$ is called the *Legendre transform* of f. Historically, it was first considered for sufficiently smooth functions.

Remark 1.6.18 Under differentiability assumptions, the Legendre transform can be given a more explicit form. Let f be a convex function with the property that its restriction $f|D$ to some open convex subset $D \subset \mathrm{dom}\, f$ is strictly convex and differentiable. Let U be the image of D under the gradient mapping ∇f of f, restricted to D. Let $u \in U$. Since f is strictly convex on D, the gradient mapping is injective, hence there is a unique $x \in D$ with $u = \nabla f(x)$. By Lemma 1.5.14, the vector $(u, -1) \in \mathbb{R}^n \times \mathbb{R}$ is an outer normal vector to $\mathrm{epi}\, f$ at $(x, f(x))$. Hence, it follows from (1.14) that $f^*(u) = \langle x, u \rangle - f(x)$. Thus, we have

$$(\mathcal{L}f)(u) = \langle (\nabla f)^{-1}(u), u \rangle - f((\nabla f)^{-1}(u)) \quad \text{for } u \in (\nabla f)(D). \tag{1.20}$$

The Legendre transformation is essentially the only order-reversing involution of the space $\mathrm{Cvx}(\mathbb{R}^n)$. The following theorem is due to Artstein-Avidan and Milman [90].

Theorem 1.6.19 (Artstein-Avidan, Milman) *Let $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ be a mapping satisfying*

$$\mathcal{T}\mathcal{T}f = f \quad \text{and} \quad f \le g \Rightarrow \mathcal{T}f \ge \mathcal{T}g$$

for all $f, g \in \mathrm{Cvx}(\mathbb{R}^n)$. Then there are a selfadjoint linear transformation $\Lambda \in \mathrm{GL}(n)$, a vector $t \in \mathbb{R}^n$ and a constant $\gamma \in \mathbb{R}$ such that

$$(\mathcal{T}f)(x) = (\mathcal{L}f)(\Lambda x + t) + \langle x, t\rangle + \gamma$$

for all $f \in \mathrm{Cvx}(\mathbb{R}^n)$.

A distinguished subclass of the set $\mathrm{Cvx}(\mathbb{R}^n)$ of all closed convex functions is the class $\mathrm{Cvx}_0(\mathbb{R}^n)$ of closed convex functions $f : \mathbb{R}^n \to [0, \infty]$ with $f(o) = 0$. Since some geometrically important functions belong to this class, such as norms and support functions of closed convex sets containing the origin (see Section 1.7), it has been called the class of *geometric convex functions*. The Legendre transform $\mathcal{L}$ maps $\mathrm{Cvx}_0(\mathbb{R}^n)$ into itself. But there is also an essentially different order-reversing involution on $\mathrm{Cvx}_0(\mathbb{R}^n)$; see Rockafellar [1583], p. 136 (for an antecedent see the historical remark in Milman [1428]). For $f \in \mathrm{Cvx}_0(\mathbb{R}^n)$ we define

$$f^\bullet(u) := \inf\{\lambda \ge 0 : \langle u, x\rangle - \lambda f(x) \le 1 \text{ for all } x \in \mathbb{R}^n\}, \tag{1.21}$$

with the usual convention that $\inf \emptyset = \infty$. (We use $f^\bullet$ instead of Rockafellar's notation f°, since we reserve the latter for the dual of a log-concave function f; see Section 9.5.) An alternative way to write this is

$$f^\bullet(u) = \begin{cases} \sup\limits_{f(x)>0} \dfrac{\langle u, x\rangle - 1}{f(x)} & \text{if } u \in [f^{-1}(0)]^\circ, \\ \infty & \text{if } u \notin [f^{-1}(0)]^\circ \end{cases}$$

(with the convention that $\sup \emptyset = 0$). Definition (1.21) can be reformulated in terms of epigraphs. For $(u, \eta) \in \mathbb{R}^n \times \mathbb{R}$ we have

$$\begin{aligned} &(u, \eta) \in \operatorname{epi} f^\bullet \Leftrightarrow \eta \ge f^\bullet(u) \Leftrightarrow \langle u, x\rangle \le 1 + \eta f(x)\ \forall\, x \\ &\Leftrightarrow \langle u, x\rangle \le 1 + \eta\zeta\ \forall\, (x, \zeta) \in \operatorname{epi} f \Leftrightarrow \langle (u, -\eta), (x, \zeta)\rangle \le 1\ \forall\, (x, \zeta) \in \operatorname{epi} f \\ &\Leftrightarrow (u, -\eta) \in (\operatorname{epi} f)^\circ, \end{aligned}$$

hence

$$\operatorname{epi} f^\bullet = \{(u, -\eta) \in \mathbb{R}^n \times \mathbb{R} : (u, \eta) \in (\operatorname{epi} f)^\circ\}. \tag{1.22}$$

Theorem 1.6.20 *The mapping $f \mapsto f^\bullet$ is an order-reversing involution of* $\mathrm{Cvx}_0(\mathbb{R}^n)$.

Proof This follows immediately from (1.22) and properties of the polarity of convex sets. □

In analogy to the Legendre transform, a transform $\mathcal{A}$ of $\mathrm{Cvx}_0(\mathbb{R}^n)$ into itself is defined by

$$\mathcal{A}f := f^\bullet.$$

Remarkably, this transform and the Legendre transform are (up to trivial modifications) the only two dualities on $\mathrm{Cvx}_0(\mathbb{R}^n)$. The following theorem was announced by Artstein-Avidan and Milman in [89] and was proved by them in [92].

Theorem 1.6.21 (Artstein-Avidan, Milman) *Let $n \geq 2$. Any order-reversing involution of $\mathrm{Cvx}_0(\mathbb{R}^n)$ is either of the form $f \mapsto (\mathcal{L}f) \circ \Lambda$ or of the form $f \mapsto \gamma(\mathcal{A}f) \circ \Lambda$ for some selfadjoint linear transformation $\Lambda \in \mathrm{GL}(n)$ and some constant $\gamma > 0$.*

One can show that the transforms $\mathcal{L}$ and $\mathcal{A}$ commute on $\mathrm{Cvx}_0(\mathbb{R}^n)$. We write $\mathcal{J} := \mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}$. Rockafellar [1583] (p. 138) calls $\mathcal{J}f$ the *obverse* of f, though the name *gauge transform* might be more appropriate (as explained in Section 1.7, see (1.46)). An explicit representation is given by (*loc. cit.*)

$$(\mathcal{J}f)(x) = f^{*\bullet}(x) = \inf\{\lambda > 0 : \lambda f(\lambda^{-1}x) \leq 1\}.$$

Remarkably, the transformation $\mathcal{J}$ acts 'raywise', that is, $(\mathcal{J}f)(x)$ depends only on the values of f on the ray $\mathbb{R}^+x$. In terms of epigraphs,

$$\mathrm{epi}\,(\mathcal{J}f) = F(\mathrm{epi}\,f),$$

where the map $F : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \times \mathbb{R}^+$ is defined by $F(x, y) := (x/y, 1/y)$.

Notes for Section 1.6

1. *Lattice endomorphisms and characterizations of dualities.* As we have seen, Theorem 1.6.4, determining the lattice endomorphisms of $(\mathcal{K}^n_{(o)}, \cap, \vee)$, and Corollary 1.6.7, essentially characterizing the polarity, are equivalent. The endomorphisms of lattices $(\mathcal{M}, \cap, \vee)$ for other classes $\mathcal{M}$ of convex sets had been studied before the proof of Theorem 1.6.4. In [826], Gruber found all endomorphisms of $(\mathcal{K}^n, \cap, \vee)$, and in [827] he determined the endomorphisms of $(\mathcal{B}^n, \cap, \vee)$; here $\mathcal{B}^n$ denotes the set of all unit balls of norms on $\mathbb{R}^n$, thus the set of all convex bodies with o as centre and interior point. The proof of Theorem 1.6.4 follows Gruber's ideas, but needs additional arguments.

   Some more results of this type have been obtained. First, it is clear from Lemma 1.6.5 that in all cases where for a lattice $(\mathcal{M}, \cap, \vee)$ the lattice endomorphisms are known, also the order isomorphisms are known. Slomka [1793] (where Lemma 1.6.5 appears in a modified form) has determined the endomorphisms of the lattice $(\mathcal{M}, \cap, \vee)$ in the cases where $\mathcal{M}$ is either the set $\mathcal{K}^n_o$ of all convex bodies containing o, or the set CC^n_o of all closed convex sets containing o, or the set CC^n of all nonempty closed convex sets. In the case of CC^n_o, Slomka also proved counterparts to Corollaries 1.6.7, 1.6.8 and 1.6.6. The latter result was first stated in Artstein-Avidan and Milman [87] (Theorem 13) (but there not based on lattice endomorphisms). Segal and Slomka [1767] determine all order-preserving isomorphisms of sets of convex bodies of the type $\{K \in \mathcal{K}^n : A \subset K \subset B\}$, for given convex sets A, B.
2. *Legendre transform.* For a more general treatment of conjugate functions, Legendre transformation and infimal convolution, we refer to the books of Rockafellar [1583] and Hörmander [988].
3. *Dualities and structure-preserving maps of function spaces.* Theorem 1.6.19, which characterizes the Legendre transformation, suitably modified, as the only order-reversing involution of the space $\mathrm{Cvx}(\mathbb{R}^n)$ of closed convex functions, was proved by Artstein-Avidan and Milman [90]. They deduced also results in the style of Corollary 1.6.6, for invertible transformations, for example the following one.

Theorem Let $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ be a bijective mapping satisfying

$$f \le g \Leftrightarrow \mathcal{T} f \le \mathcal{T} g$$

for all $f, g \in \mathrm{Cvx}(\mathbb{R}^n)$. Then there are a linear transformation $\Lambda \in \mathrm{GL}(n)$, two vectors $t_0, t_1 \in \mathbb{R}^n$ and two constants $\gamma_0, \gamma_1 \in \mathbb{R}$ such that

$$(\mathcal{T} f)(x) = \gamma_0 f(\Lambda x + t_0) + \langle x, t_1 \rangle + \gamma_1$$

for all $f \in \mathrm{Cvx}(\mathbb{R}^n)$.

In [87], Artstein-Avidan and Milman proved that an involution of $\mathrm{Cvx}(\mathbb{R}^n)$ that transforms infimal convolution into addition must be a modified Legendre transform.

There are further classes of functions for which previously defined dualities can be characterized in a similar way. This was done by Artstein-Avidan and Milman in [88] for log-concave functions and in [90] for s-concave functions. In the first of these papers, the authors introduced the notion of an *abstract duality*, saying that a transform $\mathcal{T}$ generates a duality transform on a set $\mathcal{S}$ of real functions if it is an order-reversing involution of $\mathcal{S}$, that is, a map $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ with $\mathcal{T}\mathcal{T} f = f$ and $f \le g \Rightarrow \mathcal{T} f \ge \mathcal{T} g$ for $f, g \in \mathcal{S}$. Continuing their search for abstract dualities which are essentially unique, they surprisingly found a situation with essentially two, and only two, dualities; this led to Theorem 1.6.21.

More generally than the spaces $\mathrm{Cvx}(\mathbb{R}^n)$ and $\mathrm{Cvx}_0(\mathbb{R}^n)$, one can choose a closed convex set $K \subset \mathbb{R}^n$ with nonempty interior and consider the space $\mathrm{Cvx}(K)$ of convex functions on K and, if $o \in K$, the space $\mathrm{Cvx}_0(K)$ of nonnegative convex functions on K that vanish at the origin. The bijective transformations of these function spaces which, together with their inverses, preserve the order, have been determined by Artstein-Avidan, Florentin and Milman in [83] (announcement) and [84] (proofs). They can be described with the help of certain permissible projective transformations (called *fractional linear maps*), applied to the epigraphs.

4. *Additive maps.* Besides lattice operations and order, also the preservation of other structures on spaces of convex sets or convex functions may lead to classifications. The following result on additive maps was proved by Artstein-Avidan and Milman [91].

Theorem Let $n \ge 2$. If $\mathcal{T} : \mathcal{CC}^n_o \to \mathcal{CC}^n_o$ is a bijective mapping satisfying

$$\mathcal{T}(K_1 + K_2) = \mathcal{T}(K_1) + \mathcal{T}(K_2)$$

for all $K_1, K_2 \in \mathcal{CC}^n_o$, then there exists a linear transformation $\Lambda \in \mathrm{GL}(n)$ such that $\mathcal{T} K = \Lambda K$ for all $K \in \mathcal{CC}^n_o$.

For the proof, it is essential that the underlying set $\mathcal{CC}^n_o$ contains all closed (not necessarily bounded) convex sets. A corresponding result for bounded convex sets seems to be unknown.

5. *Stability.* There are also stability results for bijective mappings that preserve the considered structures (additivity, order) only approximately, within given bounds; see Artstein-Avidan and Milman [93], Florentin and Segal [619].
6. *Section/projection correspondence.* Another possibility of characterizing essentially the polarity consists in suitably formulating and exploiting the interchange of sections and projections under such an operation. This has been investigated by Milman, Segal and Slomka [1434]. Their result was extended to geometric log-concave functions (i.e., with maximum 1), with suitable notions of section and projection, by Segal and Slomka [1766]. As consequences, they obtained further characterizations of the duality on geometric convex functions and of the support function.
7. *Self-polar functions.* Under the Legendre transform, the only self-polar function is the function $x \mapsto |x|^2/2$. For the transform $\mathcal{A}$ on $\mathrm{Cvx}_0(\mathbb{R}^n)$, there are many self-polar functions. Rotem [1592] determined all the $\mathcal{A}$-self-polar, rotationally invariant functions in $\mathrm{Cvx}_0(\mathbb{R}^n)$.

1.7 Functions representing convex sets

1.7.1 The support function

A convex body can be described by real functions, in different ways. Of such representations, the one by the support function is of fundamental importance in the Brunn–Minkowski theory.

Since a closed convex set is the intersection of its supporting halfspaces, such a set can conveniently be described by specifying the position of its support planes, given their outer normal vectors. This description is provided by the support function.

Let $K \subset \mathbb{R}^n$ be a closed convex set with $\emptyset \neq K \neq \mathbb{R}^n$. The *support function* $h(K, \cdot) = h_K$ of K is defined by

$$h(K, u) := \sup \{\langle x, u\rangle : x \in K\} \quad \text{for } u \in \mathbb{R}^n.$$

If $x \in \mathbb{R}^n \setminus K$, the separation theorem 1.3.7 yields the existence of a vector $u_0 \in \mathbb{R}^n$ with $\langle x, u_0\rangle > h(K, u_0)$, hence for $x \in \mathbb{R}^n$ we have

$$x \in K \Leftrightarrow \langle x, u\rangle \leq h(K, u) \quad \text{for all } u \in \mathbb{R}^n.$$

This can also be expressed by saying that

$$\partial h_K(o) = K; \tag{1.23}$$

that is, K is the subdifferential of its support function at o. Immediately from the definitions of conjugate functions and support functions we see that

$$h_K^*(x) = \sup_u \{\langle x, u\rangle - h_K(u)\} = \begin{cases} 0, & \text{if } x \in K, \\ \infty, & \text{if } x \notin K \end{cases}$$

(the latter since $x \notin K$ yields a vector u_0 with $\langle x, u_0\rangle - h(K, u_0) > 0$, and $\langle x, \lambda u_0\rangle - h(K, \lambda u_0) = \lambda(\langle x, u_0\rangle - h(K, u_0))$ can be made arbitrarily large). Thus,

$$h_K^* = I_K^\infty, \qquad K = \operatorname{dom} h_K^*. \tag{1.24}$$

The support function of a compact convex set $K \subset \mathbb{R}^n$ can also be described as follows. Let

$$C_K := \{\lambda(x, -1) \in \mathbb{R}^n \times \mathbb{R} : x \in K, \ \lambda \geq 0\}.$$

Then C_K is a closed convex cone in $\mathbb{R}^n \times \mathbb{R}$, and its dual cone is the epigraph of the support function of K,

$$C_K^\circ = \operatorname{epi} h_K.$$

In fact, $(y, \eta) \in \operatorname{epi} h_K$ if and only if $\eta \geq \langle x, y\rangle$ for all $x \in K$, and this is equivalent to $\langle (y, \eta), (\lambda x, -\lambda)\rangle \leq 0$ for all $x \in K$ and $\lambda \geq 0$ and hence to $(y, \eta) \in C_K^\circ$.

Associated with the support function are the following important sets. For $u \in \operatorname{dom} h(K,\cdot) \setminus \{o\}$ we put

$$H(K,u) := \{x \in \mathbb{R}^n : \langle x,u\rangle = h(K,u)\},$$
$$H^-(K,u) := \{x \in \mathbb{R}^n : \langle x,u\rangle \le h(K,u)\},$$
$$F(K,u) := H(K,u) \cap K.$$

$H(K,u)$, $H^-(K,u)$ and $F(K,u)$ are, respectively, the *support plane*, *supporting halfspace* and *support set* of K, each with outer normal vector u. Note that these definitions of support plane and supporting halfspace extend the notions of Section 1.3. They obviously coincide with the latter if $F(K,u) \neq \emptyset$, but for an unbounded set K it may happen that $F(K,u) = \emptyset$ although $h(K,u)$ is finite.

The intuitive meaning of the support function is simple. For a unit vector $u \in \mathbb{S}^{n-1} \cap \operatorname{dom} h(K,\cdot)$, the number $h(K,u)$ is the signed distance of the support plane to K with outer normal vector u from the origin; the distance is negative if and only if u points into the open halfspace containing the origin.

From the definition of the support function we see immediately that $h(K,\cdot) = \langle z,\cdot\rangle$ if and only if $K = \{z\}$, that $h(K+t,u) = h(K,u) + \langle t,u\rangle$ for $t \in \mathbb{R}^n$, and further that $h(K,\lambda u) = \lambda h(K,u)$ for $\lambda \ge 0$ and $h(K,u+v) \le h(K,u) + h(K,v)$. Hence, $h(K,\cdot)$ is a convex function if $K \neq \mathbb{R}^n$. It is also clear from the definition that $h_K \le h_L$ if and only if $K \subset L$ and that, for any linear subspace E of $\mathbb{R}^n$, one has $h(K\,|\,E,u) = h(K,u)$ for $u \in E$ (recall that $\cdot\,|\,E$ denotes the orthogonal projection to E). Further, we note that $h(\lambda K,\cdot) = \lambda h(K,\cdot)$ for $\lambda \ge 0$ and $h(-K,u) = h(K,-u)$.

To simplify the exposition, the following considerations will be restricted to compact convex sets, since in later chapters the support function will be used only for these. For a convex body K, the supremum in the definition of $h(K,u)$ is attained and finite for each u, and $h(K,\cdot)$ is a sublinear function.

Theorem 1.7.1 *If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique convex body $K \in \mathcal{K}^n$ with support function f.*

The uniqueness is clear. For the fundamental existence result we give three proofs, each of which has its merits. The first proof is very short (provided the duality theorem for convex functions is taken for granted), the second one is perhaps the simplest, and the third is of historical interest.

First proof By Theorem 1.5.3, the function f is continuous and hence closed. For each real $\lambda > 0$ its conjugate function f^* satisfies

$$\begin{aligned} f^*(x) &= \sup\{\langle x,u\rangle - f(u) : u \in \mathbb{R}^n\} \\ &= \sup\{\langle x,\lambda u\rangle - f(\lambda u) : u \in \mathbb{R}^n\} = \lambda f^*(x), \end{aligned}$$

which is only possible if $f^*(x) \in \{0,\infty\}$ for $x \in \mathbb{R}^n$. By Theorem 1.6.13, f^* is closed and convex, hence it is the indicator function I_K^∞ of a closed convex set K, which is not empty because $\{f^* = \infty\} \neq \mathbb{R}^n$. Then (1.24) gives $h_K^* = I_K^\infty = f^*$, and from Theorem 1.6.13 we deduce that $h_K = f$. $\square$

For the second and third proofs we define

$$K := \{x \in \mathbb{R}^n : \langle x, v\rangle \leq f(v) \text{ for all } v \in \mathbb{R}^n\}.$$

Then K, being the intersection of closed halfspaces, one for each normal direction, is convex and compact, and if $K \neq \emptyset$, we evidently have $h(K,u) \leq f(u)$ for $u \in \mathbb{R}^n$. Therefore, it remains to show that $K \neq \emptyset$ and $h(K,u) \geq f(u)$ for $u \in \mathbb{R}^n$.

Second proof Since f is sublinear and continuous, its epigraph is a closed convex cone in $\mathbb{R}^n \times \mathbb{R}$. Let $u \in \mathbb{R}^n \setminus \{o\}$ be given. Since $(u, f(u)) \in \operatorname{bd} \operatorname{epi} f$, Theorem 1.3.2 implies the existence of a support plane $H_{(y,\eta),\alpha}$ to $\operatorname{epi} f$ through $(u, f(u))$ such that $\operatorname{epi} f \subset H^-_{(y,\eta),\alpha}$. From Theorem 1.3.9 it follows that $\alpha = 0$. Clearly, $\eta \neq 0$ since $\operatorname{dom} f = \mathbb{R}^n$, hence we may assume that $\eta = -1$. Then we have

$$\operatorname{epi} f \subset H^-_{(y,-1),0} = \{(v,\zeta) \in \mathbb{R}^n \times \mathbb{R} : \langle (y,-1),(v,\zeta)\rangle \leq 0\}$$

and thus $\langle y, v\rangle \leq f(v)$ for all $v \in \mathbb{R}^n$. From $(u, f(u)) \in H_{(y,-1),0}$ we get $\langle y, u\rangle = f(u)$. This shows that $y \in K$, hence $K \neq \emptyset$, and $h(K,u) \geq f(u)$. □

Third proof The assertion $K \neq \emptyset$ will be proved by induction with respect to the codimension of the linearity space $L[f]$ (cf. Lemma 1.5.7). If $\dim L[f] = n$, then f is linear, say $f(u) = \langle z, u\rangle$ with $z \in \mathbb{R}^n$, and then $K = \{z\}$. Suppose that $\dim L[f] = k < n$ and that in the case of greater dimension of the linearity space the assertion is true. Let $x \in \mathbb{R}^n \setminus L[f]$. By Lemma 1.5.7, $\dim L[f'(x;\cdot)] \geq k+1$, hence the induction hypothesis applied to the function $f'(x;\cdot)$, which is sublinear by Lemma 1.5.6, yields the existence of a point $y \in \mathbb{R}^n$ satisfying

$$\langle y, v\rangle \leq f'(x;v) \quad \text{for all } v \in \mathbb{R}^n.$$

By Lemma 1.5.7, $f'(x;\cdot) \leq f$, hence $y \in K$. The assertion $K \neq \emptyset$ is proved.

Let $u \in \mathbb{R}^n \setminus \{o\}$. Since $f'(u;\cdot)$ is sublinear, the result already proved yields the existence of a point $y \in \mathbb{R}^n$ with

$$\langle y, v\rangle \leq f'(u;v) \leq f(v) \quad \text{for } v \in \mathbb{R}^n.$$

This shows that $y \in K$, and from $\langle y, -u\rangle \leq f'(u;-u) = -f(u)$ we get $h(K,u) \geq \langle y, u\rangle \geq f(u)$. □

The directional derivatives of support functions appearing implicitly in the last proof have a simple intuitive meaning: they yield the support functions of the corresponding support sets.

Theorem 1.7.2 *For $K \in \mathcal{K}^n$ and $u \in \mathbb{R}^n \setminus \{o\}$,*

$$h'_K(u;x) = h(F(K,u),x) \quad \textit{for } x \in \mathbb{R}^n.$$

Proof By Lemma 1.5.6 and Theorem 1.7.1, $h'_K(u;\cdot)$ is the support function of a convex body K' that satisfies $K' \subset K$ because $h'_K(u;\cdot) \leq h_K$ (Lemma 1.5.7). Let $y \in K'$. Then $\langle y, u\rangle \leq h(K,u)$; from $\langle y, -u\rangle \leq h'_K(u;-u) = -h(K,u)$ we get $\langle y, u\rangle = h(K,u)$ and hence $y \in F(K,u)$. Thus $K' \subset F(K,u)$.

Let $y \in F(K,u)$. Then $\langle y,u\rangle = h(K,u)$ and $\langle y,v\rangle \le h(K,v)$ for $v \in \mathbb{R}^n$. Choosing $v = u + \lambda x$ ($\lambda > 0$, $x \in \mathbb{R}^n$) we get

$$\langle y,x\rangle \le \frac{h(K,u+\lambda x) - h(K,u)}{\lambda}$$

and then $\langle y,x\rangle \le h_K'(u;x)$. This proves $F(K,u) \subset K'$ and thus the theorem. □

Recall that the gradient of a differentiable function f on $\mathbb{R}^n$ is denoted by ∇f.

Corollary 1.7.3 *Let $K \in \mathcal{K}^n$ and $u \in \mathbb{R}^n \setminus \{o\}$. The support function h_K is differentiable at u if and only if the support set $F(K,u)$ contains only one point x. In this case,*

$$x = \nabla h_K(u).$$

Proof By Theorem 1.5.8, differentiability of h_K at u is equivalent to partial differentiability at u, hence to $h_K'(u;e_i) = -h_K'(u;-e_i)$ for the vectors $e_1,\dots,e_n$ of an orthonormal basis. By Theorem 1.7.2 this is equivalent to the fact that $F(K,u)$ is contained in the intersection of n pairwise orthogonal hyperplanes, hence to $F(K,u) = \{x\}$ for some point x.

If $F(K,u) = \{x\}$, then $\langle x,v\rangle = h(F(K,u),v) = h_K'(u;v)$ for $v \in \mathbb{R}^n$, in particular $\langle x,e_i\rangle = h_K'(u;e_i) = \partial_i h_K(u)$ for $i = 1,\dots,n$, thus $x = \nabla h_K(u)$. □

Corollary 1.7.3 also follows from Theorem 1.5.15, but we have preferred to give a more direct proof. The connection to Theorem 1.5.15 is established by the following observation.

Theorem 1.7.4 *If $K \in \mathcal{K}^n$ and $u \in \mathbb{R}^n \setminus \{o\}$, then the subdifferential of the support function h_K at u is the support set of K at u,*

$$\partial h_K(u) = F(K,u). \tag{1.25}$$

Proof By the definition of the subdifferential, $x \in \partial h_K(u)$ if and only if

$$h(K,v) \ge h(K,u) + \langle x, v-u\rangle \quad \text{for } v \in \mathbb{R}^n. \tag{1.26}$$

If $x \in F(K,u)$, then $\langle x,u\rangle = h(K,u)$ and $\langle x,v\rangle \le h(K,v)$ for $v \in \mathbb{R}^n$, hence (1.26) holds. If (1.26) holds, then

$$h(K-x,v) \ge h(K-x,u) \quad \text{for } v \in \mathbb{R}^n.$$

Applying this to $v = o$ and to $v = 2u$, we get $h(K-x,u) = 0$, thus $h(K,u) = \langle x,u\rangle$; then $h(K-x,v) \ge 0$ for $v \in \mathbb{R}^n$ shows that $o \in K - x$, hence $x \in K$. Thus $x \in F(K,u)$. □

We now come to the property of the support function that, for subsequent applications, is of prime importance, namely its additive behaviour (in the first argument) under addition. If K and L are convex bodies, then $K + L$ is also a convex body. The convexity is clear, and compactness follows from the compactness of $K \times L$ and the continuity of addition as a map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$. The addition of convex bodies is usually called *Minkowski addition*. A function f on $\mathcal{K}^n$ with values in some abelian semigroup is called *Minkowski additive* if

$$f(K+L) = f(K) + f(L) \quad \text{for } K, L \in \mathcal{K}^n.$$

Important examples are obtained as follows.

Theorem 1.7.5 *For $K, L \in \mathcal{K}^n$ one has*

(a) $h(K+L,\cdot) = h(K,\cdot) + h(L,\cdot)$,
(b) $H(K+L,\cdot) = H(K,\cdot) + H(L,\cdot)$,
(c) $F(K+L,\cdot) = F(K,\cdot) + F(L,\cdot)$.

Proof Let $u \in \mathbb{R}^n \setminus \{o\}$. There are points $x \in K$ with $h(K,u) = \langle x,u\rangle$ and $y \in L$ with $h(L,u) = \langle y,u\rangle$. This implies $h(K,u) + h(L,u) = \langle x+y,u\rangle \le h(K+L,u)$. Each point $z \in K+L$ has a representation $z = x+y$ with $x \in K$, $y \in L$, and it follows that $\langle z,u\rangle = \langle x,u\rangle + \langle y,u\rangle \le h(K,u) + h(L,u)$. Since $z \in K+L$ was arbitrary, one has $h(K+L,u) \le h(K,u)+h(L,u)$. This proves (a), and (b) is an immediate consequence. Equality (c) follows from Theorem 1.7.2 and (a), or by an easy direct argument. □

A first important consequence of Theorem 1.7.5(a) is the fact that the equality $K+M = L+M$ for convex bodies $K, L, M \in \mathcal{K}^n$ implies $K = L$. Hence, $(\mathcal{K}^n, +)$ is a commutative semigroup with cancellation law. It also has a natural multiplication with nonnegative real numbers, $(\lambda, K) \mapsto \lambda K$, satisfying the rules $\lambda(K+L) = \lambda K+\lambda L$, $(\lambda+\mu)K = \lambda K + \mu K$, $\lambda(\mu K) = (\lambda\mu)K$, $1K = K$. Thus, $\mathcal{K}^n$ together with Minkowski addition and this multiplication is an *abstract convex cone*.

Remark 1.7.6 For arbitrary nonempty subsets $A, B, C \subset \mathbb{R}^n$ with C bounded, the relation

$$A + C \subset B + C \tag{1.27}$$

implies that $\operatorname{cl}\operatorname{conv} A \subset \operatorname{cl}\operatorname{conv} B$. In fact, let $a \in A$ and suppose that $a \notin \operatorname{cl}\operatorname{conv} B$. By Theorem 1.3.4, there are a vector u and a number $\varepsilon > 0$ with $\langle a,u\rangle \ge \langle b,u\rangle + \varepsilon$ for all $b \in \operatorname{cl}\operatorname{conv} B$. This yields

$$\begin{aligned}\sup\{\langle a'+c,u\rangle : a' \in A, c \in C\} &\ge \sup\{\langle a+c,u\rangle : c \in C\}\\ &\ge \sup\{\langle b+c,u\rangle : b \in B, c \in C\} + \varepsilon,\end{aligned}$$

which contradicts (1.27). Therefore, $A \subset \operatorname{cl}\operatorname{conv} B$ and hence $\operatorname{cl}\operatorname{conv} A \subset \operatorname{cl}\operatorname{conv} B$.

Remark 1.7.7 By the map

$$\begin{aligned}\Upsilon:\quad \mathcal{K}^n &\to C(\mathbb{S}^{n-1})\\ K &\mapsto \bar{h}_K\end{aligned} \tag{1.28}$$

(where $\bar{h}_K = h_K|\mathbb{S}^{n-1}$, the restriction of h_K to $\mathbb{S}^{n-1}$) the cone $\mathcal{K}^n$ is embedded isomorphically as a convex cone in the real vector space $C(\mathbb{S}^{n-1})$ of continuous real functions on $\mathbb{S}^{n-1}$.

The following result shows that the vector space generated by the cone just mentioned is dense in $C(\mathbb{S}^{n-1})$ (with respect to the maximum norm).

Lemma 1.7.8 *If $f : \mathbb{S}^{n-1} \to \mathbb{R}$ is a twice continuously differentiable function, there are a convex body $K \in \mathcal{K}^n$ and a number $r > 0$ such that*

$$f = \bar{h}_K - \bar{h}_{rB^n}. \tag{1.29}$$

Hence, the real vector space spanned by the differences of support functions (restricted to $\mathbb{S}^{n-1}$) is dense in the space $C(\mathbb{S}^{n-1})$.

Proof We extend f to $\mathbb{R}^n \setminus \{o\}$ by putting $f(x) := |x| f(x/|x|)$ and we define $g(x) := |x| = h(B^n, x)$ for $x \in \mathbb{R}^n$. Then the function $f + rg$ is positively homogeneous of degree one, for any $r \in \mathbb{R}$. Let $\mathrm{d}^2(f + rg)_x$ denote its second differential at x, considered as a bilinear form on $\mathbb{R}^n$. By homogeneity, $\mathrm{d}^2(f + rg)_x(x, \cdot) = 0$, and for unit vectors x, y with $y \perp x$ we obtain $\mathrm{d}^2(f + rg)_x(y, y) = \mathrm{d}^2 f_x(y, y) + r$. By continuity and compactness, $\mathrm{d}^2 f_x(y, y)$ attains a minimum, hence we can choose $r > 0$ so that $\mathrm{d}^2(f + rg)_x(y, y) \geq 0$ for $x \in \mathbb{S}^{n-1}$ and arbitrary y. By homogeneity, $\mathrm{d}^2(f + rg)_x$ is positive semi-definite for all $x \in \mathbb{R}^n \setminus \{o\}$. Theorem 1.5.13 now shows that $f + rg$ is convex and thus, by Theorem 1.7.1, that it is the support function of a convex body K that satisfies (1.29). The rest of the assertion follows from the fact that the twice continuously differentiable functions on $\mathbb{S}^{n-1}$ are dense in $C(\mathbb{S}^{n-1})$. □

The additivity of the support function (in the first argument) carries over to some important functionals derived from it. Let $K \in \mathcal{K}^n$. The *width function* $w(K, \cdot)$ of K is defined by

$$w(K, u) := h(K, u) + h(K, -u) \quad \text{for } u \in \mathbb{S}^{n-1}.$$

The number $w(K, u)$ is the *width* of K *in the direction* u; this is the distance between the two support planes of K orthogonal to u. The maximum of the width function,

$$D(K) := \max\{w(K, u) : u \in \mathbb{S}^{n-1}\},$$

is at the same time the *diameter* of K, thus $D(K) = \operatorname{diam} K$. In fact, if $x, y \in K$ are points with maximal distance, then the hyperplanes through x and y orthogonal to $x - y$ are support planes to K. On the other hand, to each $u \in \mathbb{S}^{n-1}$ there are points $x, y \in K$ with $w(K, u) \leq |x - y|$.

The *minimal width*

$$\Delta(K) := \min\{w(K, u) : u \in \mathbb{S}^{n-1}\}$$

is called, in brief, the *width* (or the *thickness*) of K. (Of course, diameter and width are not Minkowski additive; one merely has $D(K + L) \leq D(K) + D(L)$ and $\Delta(K + L) \geq \Delta(K) + \Delta(L)$ for $K, L \in \mathcal{K}^n$.)

The mean value of the width function over $\mathbb{S}^{n-1}$ is called the *mean width* and denoted by $w(\cdot)$, thus

$$w(K) := \frac{2}{\omega_n} \int_{\mathbb{S}^{n-1}} h(K,u)\,\mathrm{d}u \tag{1.30}$$

(recall that we write $\mathrm{d}u$ for $\mathrm{d}\mathcal{H}^{n-1}(u)$).

The vector-valued integral

$$s(K) := \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} h(K,u)u\,\mathrm{d}u \tag{1.31}$$

defines the *Steiner point* $s(K)$ of K.

It is clear that mean width w and Steiner point s are Minkowski additive functions. They also have important invariance properties. The mean width is rigid motion invariant; that is, it satisfies $w(gK) = w(K)$ for $K \in \mathcal{K}^n$ and every rigid motion $g : \mathbb{R}^n \to \mathbb{R}^n$. This is obvious from (1.30), since for a rotation g we have

$$h(gK,u) = h(K, g^{-1}u) \tag{1.32}$$

and the spherical Lebesgue measure is rotation invariant; further, for a translation by a vector $t \in \mathbb{R}^n$ we have

$$h(K+t,u) = h(K,u) + \langle t,u\rangle.$$

The invariance of w then follows from

$$\int_{\mathbb{S}^{n-1}} \langle t,u\rangle\,\mathrm{d}u = 0,$$

which holds since the integrand is an odd function on $\mathbb{S}^{n-1}$.

The Steiner point is equivariant under rigid motions; that is to say, $s(gK) = gs(K)$ for every rigid motion g. For a rotation this follows from (1.31) and (1.32), and for translations one has to use that

$$\frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} \langle t,u\rangle u\,\mathrm{d}u = t \quad \text{for } t \in \mathbb{R}^n. \tag{1.33}$$

For the proof, note that

$$\int_{\mathbb{S}^{n-1}} \langle t,u\rangle u\,\mathrm{d}u = \alpha t$$

with a real number α independent of t, since the integral is linear in t and invariant under rotations and reflections fixing t. Choosing $|t| = 1$ we obtain, if $(e_1,\dots,e_n)$ is an orthonormal basis of $\mathbb{R}^n$,

$$\alpha = \int_{\mathbb{S}^{n-1}} \langle t,u\rangle^2\,\mathrm{d}u = \frac{1}{n}\sum_{i=1}^{n} \int_{\mathbb{S}^{n-1}} \langle e_i,u\rangle^2\,\mathrm{d}u = \frac{1}{n}\int_{\mathbb{S}^{n-1}} |u|^2\,\mathrm{d}u = \kappa_n.$$

We mention one further property of the Steiner point, namely that

$$s(K) \in \operatorname{relint} K \quad \text{for } K \in \mathcal{K}^n. \tag{1.34}$$

This will be clear when in Section 5.4 the Steiner point $s(K)$ is interpreted as the centroid of a certain measure concentrated on the relative boundary of K (formula (5.99) for $r = n$).

1.7.2 Further representing functions

The representation of a convex body by its support function can be generalized to representations by a class of functions, which include infinitely differentiable (or even real-analytic) functions. The determination of the convex body K from its support function h_K is condensed in each of the formulae (1.23) and (1.24),

$$K = \partial h_K(o), \qquad K = \operatorname{dom} \mathcal{L} h_K, \tag{1.35}$$

where $\mathcal{L} f = f^*$ denotes the Legendre transform or Fenchel conjugate of the convex function f. Each of these representations can be extended. Generalizing the second one, we say that the convex function $f : \mathbb{R}^n \to \mathbb{R}$ is *associated with* the convex body K if K is the closure of the effective domain of the Legendre transform of f,

$$K = \operatorname{cl} \operatorname{dom} \mathcal{L} f.$$

In particular, the support function of K is associated with K in this sense. For a convex function $f : \mathbb{R}^n \to \mathbb{R}$ we now write

$$K_f := \operatorname{cl} \operatorname{dom} \mathcal{L} f.$$

Lemma 1.7.9 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then K_f is bounded if and only if f is Lipschitz continuous. If this holds, then*

$$f(y) - f(x) \le h(K_f, y - x) \tag{1.36}$$

for all $x, y \in \mathbb{R}^n$.

Proof Suppose there is a constant L with $|f(y) - f(x)| \le L|y - x|$ for all $x, y \in \mathbb{R}^n$. Let $u \in \operatorname{relint} K_f = \operatorname{relint} \operatorname{dom} \mathcal{L} f$ (the latter holds by Theorem 1.1.15(a)). By Lemma 1.6.16, there is a point $x \in \mathbb{R}^n$ with $u \in \partial f(x)$. Therefore, $\langle u, u\rangle \le f(x + u) - f(x) \le L|u|$ and hence $|u| \le L$. Therefore, K_f is bounded.

Conversely, suppose that K_f is bounded. For given $x, y \in \mathbb{R}^n$, choose $v \in \partial f(y)$. Then $v \in K_f$ and $f(y) - f(x) \le \langle v, y - x\rangle \le \sup\{|u| : u \in K_f\}|y - x|$. Thus, f is Lipschitz continuous.

If K_f is bounded, we have shown that for given $x, y \in \mathbb{R}^n$ there exists $v \in K_f$ with $f(y) - f(x) \le \langle v, y - x\rangle$. Since $\langle v, y - x\rangle \le h(K_f, y - x)$, this gives (1.36). □

Lemma 1.7.10 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and Lipschitz continuous. Then, for $z \in \mathbb{R}^n$,*

$$h(K_f, z) = \lim_{\lambda\to\infty} \frac{f(x + \lambda z)}{\lambda}$$

for arbitrary $x \in \mathbb{R}^n$.

Proof Let $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. For $\lambda > 0$,

$$g_y(\lambda) := \frac{f(y + \lambda z) - f(y)}{\lambda} \leq h(K_f, z)$$

by (1.36). The function $\lambda \mapsto f(y + \lambda z)$ is convex, hence g_λ is (weakly) increasing. Therefore, the limit $h(z) = \lim_{\lambda\to\infty} g_y(\lambda)$ exists, and $g_y \leq h(z) \leq h(K_f, z)$. From the Lipschitz continuity of f it follows that $h(z)$ does not depend on y.

Let $u \in \operatorname{relint} K_f$. By Lemma 1.6.16, there is a point x with $u \in \partial f(x)$. Then

$$\langle u, z\rangle \leq f(x + z) - f(x) = g_x(1) \leq h(z) \leq h(K_f, z).$$

Since $\sup\{\langle u, z\rangle : u \in \operatorname{relint} K_f\} = h(K_f, z)$, it follows that $h(K_f, z) = h(z)$. □

Corollary 1.7.11 *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex and Lipschitz continuous, then*

$$K_{\alpha f+\beta g} = \alpha K_f + \beta K_g$$

for $\alpha, \beta > 0$.

Thus, the linearity property of the support function carries over to associated functions.

The following lemma can be seen as a counterpart to the first relation in (1.35).

Lemma 1.7.12 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly convex, Lipschitz continuous, and differentiable. Then* $\operatorname{int} K_f$ *is the image of $\mathbb{R}^n$ under the gradient mapping of f,*

$$\nabla f(\mathbb{R}^n) = \operatorname{int} K_f. \tag{1.37}$$

Proof Under the assumptions, it is clear that $\nabla f(x) \in \operatorname{int} K_f$ for all $x \in \mathbb{R}^n$. That the image of the gradient mapping is all of $\operatorname{int} K_f$ follows from Lemma 1.6.16. □

The following is an example (due to Gromov [805]) of a real-analytic function associated with an arbitrarily given convex body $K \in \mathcal{K}^n_n$. Let

$$f(u) := \log \int_K e^{\langle x,u\rangle}\, \mathrm{d}x, \qquad u \in \mathbb{R}^n \tag{1.38}$$

(recall that we abbreviate $\mathrm{d}\mathcal{H}^n(x)$ by $\mathrm{d}x$). To show the strict convexity of f, we note that Hölder's inequality yields $f((1-\lambda)u + \lambda v) < (1-\lambda)f(u) + \lambda f(v)$ for $u \neq v$ and $0 < \lambda < 1$. From Lemma 1.7.10, with $x = o$, say, it follows that

$$h(K_f, u) = \lim_{\lambda\to\infty} \log\left(\int_K e^{\lambda\langle x,u\rangle}\, \mathrm{d}x\right)^{1/\lambda} = \log \max_{x\in K} e^{\langle x,u\rangle} = h(K, u),$$

for all $u \in \mathbb{R}^n$, thus $K_f = K$.

Further infinitely differentiable convex functions associated with a given convex body can be obtained by applying suitable smoothing procedures to the support function (see Przesławski [1551]).

We turn to further functions that have proved useful to describe convex sets.

Let $K \in \mathcal{CC}^n_o$, thus K is a closed convex set containing the origin o. The function $g(K, \cdot) = \| \cdot \|_K$ defined by

$$g(K, x) = \|x\|_K := \inf \{\lambda \geq 0 : x \in \lambda K\} \quad \text{for } x \in \mathbb{R}^n$$

(with the usual convention that $\inf \emptyset = \infty$) is called the *gauge function* of K. One checks easily that it is convex and positively homogeneous, and that $g(\alpha K, \cdot) = \alpha^{-1} g(K, \cdot)$ for $\alpha > 0$. Clearly,

$$K = \{x \in \mathbb{R}^n : g(K, x) \leq 1\}, \qquad \frac{x}{g(K, x)} \in \operatorname{bd} K \quad \text{for } x \neq o.$$

If $K \in \mathcal{K}^n_{(o)}$ is centrally symmetric with respect to o, then $\| \cdot \|_K$ is the norm induced by K (or 'with unit ball' K). Some authors, but not all, prefer to use the notation $\| \cdot \|_K$ for the gauge function only in the case of a norm. The gauge function is often called the *Minkowski functional*. It is related to the support function by polarity:

Lemma 1.7.13 *For $K \in \mathcal{CC}^n_o$,*

$$g(K, \cdot) = h(K^\circ, \cdot).$$

Proof Writing $\widetilde{K} := \{x \in \mathbb{R}^n : h(K^\circ, x) \leq 1\}$, we have $g(\widetilde{K}, \cdot) = h(K^\circ, \cdot)$ by the definition of the gauge function. Let $x \in \widetilde{K}$. For $u \in K^\circ$ we get $\langle u, x \rangle \leq h(K^\circ, x) \leq 1$, hence $x \in K^{\circ\circ} = K$, thus $\widetilde{K} \subset K$. Let $x \in K$. Then $h(K^\circ, x) = \sup\{\langle x, v \rangle : v \in K^\circ\} \leq 1$, hence $x \in \widetilde{K}$. This shows that $K = \widetilde{K}$. □

Remark 1.7.14 Let $K \in \mathcal{K}^n_{(o)}$ and $x \in \operatorname{bd} K$, and let $u \neq o$ be an outer normal vector of K at x. Then $v := u/g(K^\circ, u) \in \operatorname{bd} K^\circ$ and

$$\langle v, x \rangle = \frac{\langle u, x \rangle}{g(K^\circ, u)} = \frac{\langle u, x \rangle}{h(K, u)} = 1 = g(K, x) = h(K^\circ, x).$$

Thus, x is an outer normal vector to K° at v. (A general version of this remark is contained in Lemma 2.2.3.) In particular, if K is smooth and strictly convex, then the same holds for K°, and we can write the relation between x and v in a convenient form. Anticipating the notation $u_K(x)$ (introduced in Section 2.2) for the outer unit normal vector of K at x, and $N(K, x)$ for the cone of all outer normal vectors of K at x, we have

$$v = h_K(u_K(x))^{-1} u_K(x), \qquad x = g_K(u_{K^\circ}(v))^{-1} u_{K^\circ}(v).$$

For given $x \in \operatorname{bd} K$, the vector v is characterized by

$$v \in N(K, x) \quad \text{and} \quad \langle v, x \rangle = 1.$$

From Corollary 1.7.3 we have $\nabla h_K(u) = x$, hence $\nabla h_K(v) = x$, since ∇h_K is homogeneous of degree zero. Applying this to K° and using Lemma 1.7.13, we obtain $\nabla g_K(x) = v$, or

$$\nabla g_K(x) = h_K(u_K(x))^{-1} u_K(x). \tag{1.39}$$

For $K \in CC_o^n$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} h_{K^\circ}(x) &= \sup\{\langle x, u\rangle : u \in K^\circ\} = \sup_u\{\langle x, u\rangle : h_K(u) \le 1\} \\ &= \sup_u\{\langle x, u\rangle : h_K(u) = 1\}. \end{aligned}$$

If $h_K(u) \neq 0$ for $u \neq o$, this can be written as

$$h_{K^\circ}(x) = \sup_{u \neq o} \frac{\langle x, u\rangle}{h_K(u)} \qquad \text{for } x \in \mathbb{R}^n, \tag{1.40}$$

and, if $\|x\|_K \neq 0$ for $x \neq o$, similarly

$$\|u\|_{K^\circ} = \sup_{x \neq o} \frac{\langle u, x\rangle}{\|x\|_K} \qquad \text{for } u \in \mathbb{R}^n. \tag{1.41}$$

If $K \in \mathcal{K}_{(o)}^n$ is centrally symmetric, then $\|\cdot\|_K$ is a norm on $\mathbb{R}^n$, and, by (1.41),

$$\|u\|_{K^\circ} = \|u\|_K^\star, \tag{1.42}$$

where $\|\cdot\|_K^\star$ denotes the dual norm of $\|\cdot\|_K$ (not to be confused with $\|\cdot\|_K^*$, the conjugate convex function of $\|\cdot\|_K$).

In Section 1.6, we introduced the Legendre transform $\mathcal{L}$, by $\mathcal{L}f := f^*$ for $f \in \mathrm{Cvx}(\mathbb{R}^n)$, where the conjugate convex function f^* is defined by (1.13). We collect here the effects of this transformation on the support, gauge and indicator functions of a closed convex set $K \in CC_o^n$. By (1.24) we have

$$\mathcal{L}h_K = I_K^\infty,$$

hence Theorem 1.6.13 gives

$$\mathcal{L}I_K^\infty = h_K. \tag{1.43}$$

By Lemma 1.7.13, $h_K = \|\cdot\|_{K^\circ}$, hence

$$\mathcal{L}\|\cdot\|_K = I_{K^\circ}^\infty, \qquad \mathcal{L}I_{K^\circ}^\infty = \|\cdot\|_K.$$

If C is a convex cone, then $I_C^\infty = \|\cdot\|_C$, hence for a closed convex cone we have

$$\mathcal{L}\|\cdot\|_C = \|\cdot\|_{C^\circ}.$$

To study the effect of the duality $\mathcal{A} : f \mapsto f^\bullet$ defined on $\mathrm{Cvx}_0(\mathbb{R}^n)$ by (1.21), we note that, for $K \in CC_o^n$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} h_K^\bullet(x) &= \inf_u\{\lambda \ge 0 : \langle x, u\rangle - \lambda h_K(u) \le 1\} \\ &= \inf_u \inf_{\alpha > 0}\{\lambda \ge 0 : \langle x, \alpha u\rangle - \lambda h_K(\alpha u) \le 1\} \\ &= \inf_u\{\lambda \ge 0 : \langle x, u\rangle \le \lambda h_K(u)\} \\ &= \inf_u\{\lambda \ge 0 : \langle x, u\rangle \le \lambda,\ h_K(u) = 1\} \\ &= \sup_u\{\langle x, u\rangle : h_K(u) = 1\}, \end{aligned}$$

thus

$$h_K^\bullet(x) = h_{K^\circ}(x) \qquad \text{for } x \in \mathbb{R}^n. \tag{1.44}$$

Similarly,

$$\|u\|_K^\bullet = \|u\|_{K^\circ} \qquad \text{for } u \in \mathbb{R}^n. \tag{1.45}$$

Formulated in terms of the $\mathcal{A}$-transform, the previous relations read

$$\mathcal{A} h_K = h_{K^\circ}, \qquad \mathcal{A} \|\cdot\|_K = \|\cdot\|_{K^\circ}.$$

Directly from the definition (1.21) we have

$$\mathcal{A} I_K^\infty = I_{K^\circ}^\infty.$$

The transform $\mathcal{J} := \mathcal{L}\mathcal{A}$ satisfies

$$\mathcal{J} I_K^\infty = \|\cdot\|_K. \tag{1.46}$$

Since it associates with the indicator function of a closed convex set containing o its gauge function, it has been called the *gauge transform*.

Let $K \in \mathcal{K}^n_{(o)}$. If $g : [0,\infty) \to [0,\infty)$ is convex and satisfies $g(o) = 0$, then $g \circ h_K \in \mathrm{Cvx}_0(\mathbb{R}^n)$, and we have

$$\mathcal{L}(g \circ h_K) = g^* \circ h_{K^\circ}, \tag{1.47}$$

with g^* according to Remark 1.6.15. For the proof, we use (1.40) and the fact that $\langle x, u\rangle / h_K(u)$ is invariant under multiplication of u with a positive factor. For $x \in \mathbb{R}^n$ this gives

$$\begin{aligned}
g^*(h_{K^\circ}(x)) &= \sup_{a>0}\{h_{K^\circ}(x)a - g(a)\} = \sup_{a>0}\left\{\sup_{u\neq o} \frac{\langle x,u\rangle}{h_K(u)} a - g(a)\right\}\\
&= \sup_{a>0}\sup_{u\neq o}\left\{\frac{\langle x,u\rangle}{h_K(u)} a - g(a) : h_K(u) = a\right\}\\
&= \sup_{u\neq o}\left\{\frac{\langle x,u\rangle}{h_K(u)} a - g(a) : h_K(u) = a\right\}\\
&= \sup_{u\neq o}\{\langle x,u\rangle - g(h_K(u))\} = (g \circ h_K)^*(x)
\end{aligned}$$

and thus (1.47).

The function $g(a) = \frac{1}{2}a^2$ satisfies $g^* = g$, hence a special case of (1.47) is the relation

$$\mathcal{L}\tfrac{1}{2}h_K^2 = \tfrac{1}{2}h_{K^\circ}^2, \tag{1.48}$$

equivalently

$$\mathcal{L}\left(\tfrac{1}{2}\|\cdot\|_K^2\right) = \tfrac{1}{2}\|\cdot\|_{K^\circ}^2. \tag{1.49}$$

This is well known and useful in Finsler geometry, and we elaborate this point a bit. Let $K \in \mathcal{K}^n_{(o)}$ and write $\|\cdot\|_K =: F$ and $\|\cdot\|_{K^o} =: H$. By (1.48), applied to the polar body, we have $\mathcal{L}\frac{1}{2}F^2 = \frac{1}{2}H^2$. Suppose now that K is smooth and strictly convex. Then (1.20) gives, for $u \in \mathbb{R}^n$,

$$\tfrac{1}{2}H^2(u) = \mathcal{L}\tfrac{1}{2}F^2(u) = \langle (\nabla\tfrac{1}{2}F^2)^{-1}(u), u\rangle - \tfrac{1}{2}F^2((\nabla\tfrac{1}{2}F^2)^{-1}(u)). \tag{1.50}$$

Write

$$(\nabla\tfrac{1}{2}F^2)^{-1}(u) = x, \quad \text{thus} \quad u = (\nabla\tfrac{1}{2}F^2)(x). \tag{1.51}$$

Since F^2 is homogeneous of degree 2, Euler's homogeneity relations give

$$\langle (\nabla\tfrac{1}{2}F^2)(x), x\rangle = F^2(x).$$

Together with (1.50) and (1.51) this yields

$$H(u) = F(x).$$

Since $K = \{x : F(x) \le 1\}$ and $K^\circ = \{u : H(u) \le 1\}$, we see that K° *is the image of* K *under the gradient mapping of the function* $\frac{1}{2}F^2$.

From $\frac{1}{2}F^2(x) = \frac{1}{2}H^2(u) = \mathcal{L}\frac{1}{2}F^2(u)$ and the definition (1.13) of the Legendre transform (where the supremum is attained), we have $\frac{1}{2}H^2(u) = \langle x, u\rangle - \frac{1}{2}F^2(x)$ and hence $\langle x, u\rangle = \frac{1}{2}H^2(u) + \frac{1}{2}F^2(x) = F(x)H(u)$. By (1.41),

$$F(x) = \sup_{v \ne o} \frac{\langle v, x\rangle}{H(v)},$$

hence the function $v \mapsto \langle v, x\rangle / H(v)$ attains a maximum at u. The vanishing of its gradient at u yields $x = F(x)\nabla H(u) = H(u)\nabla H(u) = \nabla\frac{1}{2}H^2(u)$. Comparison with (1.51) shows that *the gradient mappings of* $\frac{1}{2}F^2$ *and* $\frac{1}{2}H^2$ *are inverse to each other*.

As usual in classical Finsler geometry, we assume now that $\frac{1}{2}F^2$ is of class C^2 and has positive definite Hessian. We denote the coordinates of x with respect to the standard basis by $x^1, \dots, x^n$ and the coordinates of u by $u_1, \dots, u_n$ (where x and u remain related by (1.51)). Let

$$g_{ij}(x) := \tfrac{1}{2}\frac{\partial^2 F^2}{\partial x^i \partial x^j}(x), \qquad g^{ij}(u) := \tfrac{1}{2}\frac{\partial^2 H^2}{\partial u_i \partial u_j}(u).$$

Since $(\partial/\partial x^i)F^2$ is homogeneous of degree one, the homogeneity relations give

$$\sum_{j=1}^{n} \tfrac{1}{2}\frac{\partial^2 F^2}{\partial x^i \partial x^j}(x)x^j = \tfrac{1}{2}\frac{\partial F^2}{\partial x^i}(x),$$

which is the ith coordinate of $\nabla(\frac{1}{2}F^2)(x)$, hence the gradient mapping of $\frac{1}{2}F^2$ can also be represented by

$$u_i = \sum_{j=1}^{n} g_{ij}(x)x^j.$$

Similarly, the gradient mapping of $\frac{1}{2}H^2$ has the coordinate representation

$$x^i = \sum_{j=1}^{n} g^{ij}(u)u_j.$$

Since the gradient mappings of $\frac{1}{2}F^2$ and $\frac{1}{2}H^2$ are inverse to each other, also their Jacobian matrices are inverse to each other, thus

$$\left((g_{ij}(x))_{i,j=1}^n\right)^{-1} = (g^{ij}(u))_{i,j=1}^n.$$

For more information on the Riemannian metrics defined by (g_{ij}) and (g^{ij}) we refer the reader to Laugwitz [1177].

We turn to a further representing function, particularly useful for bodies containing the origin, namely the radial function. We introduce it for not necessarily convex sets, though not in the greatest possible generality. A set $S \subset \mathbb{R}^n$ is called *starshaped with respect to o* (or briefly 'starshaped') if it is not empty and if $[o, x] \subset S$ for all $x \in S$ (thus, a starshaped set, in our terminology, contains the origin). For a compact starshaped set K, the *radial function* is defined by

$$\rho(K, x) := \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{o\}.$$

It satisfies $\rho(\alpha K, x) = \alpha\rho(K, x)$ and $\rho(K, \alpha x) = \alpha^{-1}\rho(K, x)$ for $\alpha > 0$, so that only its values on unit vectors are needed. Clearly,

$$\rho(K, x)x \in \mathrm{bd}\, K \quad \text{for } x \in \mathbb{R}^n \setminus \{o\}$$

and, for $K \in \mathcal{K}_o^n$,

$$\rho(K, x) = \frac{1}{g(K, x)} = \frac{1}{h(K^\circ, x)} \quad \text{for } x \in \mathbb{R}^n \setminus \{o\}. \tag{1.52}$$

A compact starshaped set with a positive continuous radial function is called a *star body*, and the set of all star bodies in $\mathbb{R}^n$ is denoted by $\mathcal{S}_o^n$.

Since volume is one of the central notions of this book, we will at this point have a brief look at the suitability of the different analytical representations of a convex body or star body for expressing its volume. The *volume functional* V_n on $\mathcal{K}^n$ or $\mathcal{S}_o^n$ is defined as the restriction of the n-dimensional Hausdorff measure $\mathcal{H}^n$ to $\mathcal{K}^n$ or $\mathcal{S}_o^n$, respectively. Most suitable for computing the volume of a star body K is the radial function, since the use of spherical coordinates immediately gives

$$\begin{aligned} V_n(K) &= \int_{\mathbb{R}^n} \mathbf{1}_K(x)\,\mathrm{d}x = \int_0^\infty \int_{\mathbb{S}^{n-1}} \mathbf{1}_K(ru)r^{n-1}\,\mathrm{d}u\,\mathrm{d}r \\ &= \int_{\mathbb{S}^{n-1}} \int_0^{\rho(K,u)} r^{n-1}\,\mathrm{d}r\,\mathrm{d}u, \end{aligned}$$

thus

$$V_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho(K, u)^n\,\mathrm{d}u. \tag{1.53}$$

For $K \in \mathcal{K}_o^n$, disguised versions of (1.53) are the formulae

$$V_n(K) = \frac{1}{n!}\int_{\mathbb{R}^n} e^{-\|x\|_K}\,\mathrm{d}x = \frac{\kappa_n}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\|x\|_K^2/2}\,\mathrm{d}x. \tag{1.54}$$

They follow after using spherical coordinates and observing that $\|ru\|_K = r/\rho(K,u)$ for $u \in \mathbb{S}^{n-1}$ and $r \geq 0$. An extension of (1.54) is

$$V_n(K) = \frac{1}{\Gamma\left(1+\frac{n}{p}\right)}\int_{\mathbb{R}^n} e^{-\|x\|_K^p}\,\mathrm{d}x \tag{1.55}$$

for $1 \leq p < \infty$, proved in the same way.

If f is a twice continuously differentiable, strictly convex function associated with the convex body K, then it follows from Lemma 1.7.12 and the transformation formula for integrals that

$$V_n(K) = \int_{\mathbb{R}^n} |\det \operatorname{Hess} f|\,\mathrm{d}\mathcal{H}^n. \tag{1.56}$$

To compute the volume of a convex body from its support function is less straightforward. If $K \in \mathcal{K}_{(o)}^n$ is smooth and has a support function h_K of class C^2, we may use the fact, established above, that K is the image of K° under the gradient mapping of the function $\frac{1}{2}h_K^2$. Therefore, the transformation formula for integrals gives

$$V_n(K) = \int_{K^\circ}\left(\det \operatorname{Hess} \tfrac{1}{2}h_K^2\right)\mathrm{d}\mathcal{H}^n \tag{1.57}$$

(observe that $\operatorname{Hess} \frac{1}{2}h_K^2$ is positive semi-definite). Since $\operatorname{Hess} \frac{1}{2}h_K^2$ is homogeneous of degree 0, the use of spherical coordinates together with $\rho(K^\circ,\cdot) = h_K^{-1}$ leads to

$$V_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K^{-n}\left(\det \operatorname{Hess} \tfrac{1}{2}h_K^2\right)\mathrm{d}\sigma, \tag{1.58}$$

where σ denotes spherical Lebesgue measure. More often used is the formula (5.3), to be proved later. For a convex body $K \in \mathcal{K}^n$ with a support function of class C^2 it reads

$$V_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K s_{n-1}(K,\cdot)\,\mathrm{d}\sigma.$$

Here $s_{n-1}(K,\cdot)$ denotes the product of the principal radii of curvature of K, as a function of the outer unit normal vector. Corollary 2.5.3 explains how this function is computed from the support function of K.

Notes for Section 1.7

1. *Determination of a convex body by its support function.* The second proof given for Theorem 1.7.1 essentially goes back to Rademacher [1553]. It also appears in Hille and Phillips [977], p. 253, where it is attributed to Fenchel. Separation or support properties are used in several versions of the proof appearing in the literature, at least implicitly, as in our first proof, which can be found in Rockafellar [1583]. These methods have the advantage of being extendable to infinite dimensions. Our third proof is the one of Bonnesen and Fenchel [284]. Yet other proofs are due to Aleksandrov [17], who used the topological

theorem on the invariance of domain, and to McMullen [1381], who found a particularly elementary approach on the basis of Helly's theorem.

2. *Continuous convex sets.* The nonempty closed convex sets $K \subset \mathbb{R}^n$ for which the support function (restricted to the unit sphere)

$$h(K,u) = \sup\{\langle x,u\rangle : x \in K\} \quad \text{for } u \in \mathbb{S}^{n-1}$$

is continuous (with the usual definition of continuity for functions with values in $\mathbb{R} \cup \{\infty\}$) have been called *continuous convex sets* by Gale and Klee [661]. These authors have characterized those sets in several ways and have shown that the system $\mathscr{S}^n$ of continuous convex sets shares some properties with $\mathcal{K}^n$. For instance, $A, B \in \mathscr{S}^n$ implies that $A + B \in \mathscr{S}^n$, that $\operatorname{conv}(A \cup B) \in \mathscr{S}^n$, and that A and B can be strongly separated if they are disjoint.

3. *The support function of an intersection.* Let $(K_i)_{i\in I}$ (where I is an arbitrary index set) be a family of convex bodies in $\mathcal{K}^n$ and suppose that their intersection K is not empty. Then the support function of K can be represented in the form

$$h(K,u) = \inf\left\{\sum_{i\in I} h(K_i,u_i) : \sum_{i\in I} u_i = u\right\},$$

where the infimum is taken over all representations $u = \sum u_i$ with $u_i = o$ for all but finitely many $i \in I$. This was proved by Sandgren [1615], where the result is attributed to F. Riesz, and was rediscovered by Kneser [1121]. It can also be deduced from the more general Theorem 5.6 in Rockafellar [1583].

4. *The semiaxis function.* Let $K \in \mathcal{K}^n_{(o)}$ be a convex body with $o \in \operatorname{int} K$. For $u \in \mathbb{S}^{n-1}$, the value $h(K,u)$ of the support function is the distance of the support plane to K with outer normal vector u from o, while $u/g(K,u)$ is the boundary point of K on the ray with endpoint o and direction u. The polarity $K \mapsto K^\circ$ interchanges hyperplanes and boundary points, hence support function and gauge function interchange their roles under this polarity (Theorem 1.7.13). Leichtweiß [1183] (see also Leichtweiß [1184], §13) suggested a way of describing convex bodies by a class of functions that serve equally well for a body and its polar. Instead of support planes or boundary points one has to consider support elements. A *support element* of K is a pair (x,u) where x is a boundary point of K and u is an outer unit normal vector to K at x. Let $\operatorname{Nor} K$ denote the set of all support elements of K with its natural topology. The map $\sigma : \operatorname{Nor} K \to \mathbb{S}^{n-1}$ defined by

$$\sigma((x,u)) := \frac{x_0 + u}{|x_0 + u|}, \quad x_0 = \frac{x}{|x|},$$

is a homeomorphism. To each $v \in \mathbb{S}^{n-1}$ there exists a unique two-sheeted hyperboloid of revolution, with axis of revolution through o and of direction v, that touches K at $\sigma^{-1}(v)$. Let $a(K,v)$ denote the length of the semiaxis of this hyperboloid. The *semiaxis function* $a(K,\cdot)$ determines K uniquely, and one has $a(K^\circ,\cdot) = a(K,\cdot)^{-1}$. The semiaxis function was further investigated by Baum [181, 182], who also showed how to express the boundary of K explicitly in terms of the semiaxis function and who obtained necessary and sufficient conditions for a function to be a semiaxis function. Later Baum [183] defined, for $K \in \mathcal{K}^n_{(o)}$, a function $F_K : \mathbb{R}^n \to \mathbb{R}$ by

$$F_K(u) := \sup\left\{2\langle u,y\rangle^{\,2} - |y|^2 : y \in K,\ \langle u,y\rangle \geq 0\right\};$$

then the semiaxis function $a(K,\cdot)$ is the restriction of $\sqrt{F_K}$ to $\mathbb{S}^{n-1}$. The supremum in the definition of F_K is attained at a unique point $y \in K$. The function F_K is convex and differentiable. Baum also considered infinite-dimensional generalizations of these concepts. For boundary representations, see Baum [184].

5. *Support flats to convex bodies.* The support function of a convex body $K \in \mathcal{K}^n$ determines the position of its support planes, and the gauge function of K (if $o \in \operatorname{int} K$) fixes its boundary points on rays through the origin. The following common generalization was investigated by Firey [603]. Let $q \in \{0,\dots,n-1\}$. To each q-flat E_q tangent to $\mathbb{S}^{n-1}$,

take a translate E'_q in the subspace spanned by E_q and o. What conditions are necessary and sufficient for the family $\{E'_q\}$ to be the full set of supporting q-flats to some convex body K? (E'_q *supports* K if $E'_q \cap K \neq \emptyset$ and E'_q is contained in a support plane of K.) Firey defined the q-support function H_q of a convex body by means of the distance of the flats E'_q from o, and he answered the question completely by characterizing the q-support functions of convex bodies containing o. He also treated some applications.

6. *A convexity criterion.* In the paper mentioned in the preceding note, Firey [603] also extended Rademacher's plane polar coordinate test for convexity (Bonnesen and Fenchel [284], p. 28) to higher dimensions. If a given function $h : \mathbb{S}^{n-1} \to \mathbb{R}$ is extended to $\mathbb{R}^n$ by positive homogeneity, that is, by $h(\lambda u) = \lambda h(u)$ for$\lambda \geq 0$, when is the extension subadditive and hence a support function? Firey proved that this holds if and only if

$$\det \begin{pmatrix} \langle u_1, e_1\rangle & \cdots & \langle u_1, e_n\rangle & h(u_1) \\ \vdots & & \vdots & \vdots \\ \langle u_n, e_1\rangle & \cdots & \langle u_n, e_n\rangle & h(u_n) \\ \langle v, e_1\rangle & \cdots & \langle v, e_n\rangle & h(v) \end{pmatrix} \det \begin{pmatrix} \langle u_1, e_1\rangle & \cdots & \langle u_1, e_n\rangle \\ \vdots & & \vdots \\ \langle u_n, e_1\rangle & \cdots & \langle u_n, e_n\rangle \end{pmatrix} \leq 0$$

for all choices of $u_1, \dots, u_n, v \in \mathbb{S}^{n-1}$ for which $v \in \operatorname{pos}\{u_1, \dots, u_n\}$; here $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

In the plane, one has another simple criterion. A function $f : \mathbb{R} \to \mathbb{R}$ of period 2π is said to be a sub-sine function if the following holds: If $f(\alpha)$ agrees with $A\cos\alpha + B\sin\alpha$ at $\alpha = \alpha_1, \alpha_2$, where $\alpha_1 < \alpha_2 < \alpha_1 + \pi$, then $f(\alpha) \leq A\cos\alpha + B\sin\alpha$ for $\alpha_1 < \alpha < \alpha_2$. Now there is a support function h on $\mathbb{R}^2$ (with orthonormal basis (e_1, e_2)) for which $f(\alpha) = h(e_1\cos\alpha + e_2\sin\alpha)$ if and only if f is a sub-sine function (see Green [770]). Another formulation is found in Kallay [1057].

7. Lemma 1.7.8 appears, more or less explicitly, at several places in the literature, e.g., Ewald [540], Valette [1867], Schneider [1663]. The last part of the lemma was proved in a different way by Fenchel and Jessen [572].

   Differences of support functions were investigated by Weil [1933] and others.

8. *Power means of support functions.* For $K \in \mathcal{K}^n_n$ and $x \in K$, let $h_K(x)$ be the $(-k)$th power mean of $h(K-x, \cdot)$ over $\mathbb{S}^{n-1}$. Aleksandrov [28] obtained several estimates for the maximum of h_K.

9. *Associated convex functions.* In the presentation of convex functions associated with a convex body, in particular Lemmas 1.7.9 and 1.7.10, we followed Przesławski [1551]. The function (1.38) appears in Gromov [805]. An application of this function, together with proofs of some auxiliary results, is also found in Alesker [30].

10. More general properties than are mentioned here of the transforms $\mathcal{L}$, $\mathcal{A}$ and $\mathcal{J}$ are investigated by Artstein-Avidan and Milman in [92].

11. The volume formula (1.57) appears and is used in Schneider [1658].

1.8 The Hausdorff metric

The set $\mathcal{K}^n$ of convex bodies can be made into a metric space in several geometrically reasonable ways. The Hausdorff metric is particularly convenient and applicable. The natural domain for this metric is the set C^n of nonempty compact subsets of $\mathbb{R}^n$. We shall first treat the Hausdorff metric on this more general class of sets; this will involve no extra effort and is useful in several respects.

The *Hausdorff distance* of the sets $K, L \in C^n$ is defined by

$$\delta(K, L) := \max\left\{\sup_{x\in K}\inf_{y\in L}|x-y|,\ \sup_{x\in L}\inf_{y\in K}|x-y|\right\} \tag{1.59}$$

or, equivalently, by

$$\delta(K, L) = \min\{\lambda \geq 0 : K \subset L + \lambda B^n,\ L \subset K + \lambda B^n\}. \tag{1.60}$$

Then δ is a metric on $\mathcal{C}^n$, the *Hausdorff metric*.

Since we are considering compact sets, sup and inf in (1.59) may be replaced by max and min. Of the two definitions, (1.59) can be used in arbitrary metric spaces if $|x - y|$ is replaced by the distance between x and y in terms of the metric. The equivalence of (1.59) and (1.60) in $\mathbb{R}^n$ is seen as follows. Denote the right-hand side of (1.60) by α. For $x \in K$ one then has $x \in L + \alpha B^n$, hence $x = y + \alpha b$ with suitable $y \in L$ and $b \in B^n$; hence $|x - y| \leq \alpha$ and, therefore, $\inf_{y \in L} |x - y| \leq \alpha$. Since $x \in K$ was arbitrary, $\sup_{x \in K} \inf_{y \in L} |x - y| \leq \alpha$. Interchanging K and L, we get $\delta(K, L) \leq \alpha$. Let $0 < \lambda < \alpha$ and, say, $K \not\subset L + \lambda B^n$. Then $x \notin L + \lambda B^n$ for some $x \in K$, hence $|x - y| \geq \lambda$ for all $y \in L$. This yields $\delta(K, L) \geq \lambda$. Since $\lambda < \alpha$ was arbitrary, we get $\delta(K, L) \geq \alpha$ and thus $\delta(K, L) = \alpha$, as asserted.

That δ is, in fact, a metric is easy to see. To prove, for instance, the triangle inequality, let $K, L, M \in \mathcal{C}^n$ and put $\delta(K, L) = \alpha$, $\delta(L, M) = \beta$. Then $K \subset L + \alpha B^n$ and $L \subset M + \beta B^n$, hence $K \subset M + \beta B^n + \alpha B^n = M + (\alpha + \beta)B^n$; analogously, $M \subset K + (\alpha + \beta)B^n$, hence $\delta(K, M) \leq \alpha + \beta$.

It follows immediately from the definition (1.59) that $\delta(K, L) \leq \delta(\operatorname{bd} K, \operatorname{bd} L)$.

Lemma 1.8.1 *For convex bodies $K, L \in \mathcal{K}^n$,*

$$\delta(K, L) = \delta(\operatorname{bd} K, \operatorname{bd} L).$$

Proof Let $x \in \operatorname{bd} L$. If $x \notin \operatorname{int} K$, then the point $y := p(K, x)$ is a boundary point of K and is the point in K nearest to x, hence $|x - y| \leq \alpha$ and thus $x \in y + \alpha B^n$. If $x \in \operatorname{int} K$, then the ray with endpoint x and direction given by an outer normal vector of L at x meets $\operatorname{bd} K$ in a point z. Since x is the point in K nearest to z, we have $|z - x| \leq \alpha$ and hence $x \in z + \alpha B^n$. Both cases together yield $\operatorname{bd} L \subset \operatorname{bd} K + \alpha B^n$. Similarly, $\operatorname{bd} K \subset \operatorname{bd} L + \alpha B^n$. We conclude that $\delta(\operatorname{bd} K, \operatorname{bd} L) \leq \alpha$, which gives the assertion. □

In the following, all metrical and topological notions occurring in connection with $\mathcal{C}^n$ or $\mathcal{K}^n$ are tacitly understood to refer to the Hausdorff metric and the topology induced by it.

We want to show (in a stronger form) that the metric space $(\mathcal{C}^n, \delta)$ is locally compact. A few preparations are needed.

Lemma 1.8.2 *If $(K_i)_{i \in \mathbb{N}}$ is a decreasing sequence in $\mathcal{C}^n$, that is, if $K_{i+1} \subset K_i$ for $i \in \mathbb{N}$, then*

$$\lim_{i \to \infty} K_i = \bigcap_{j=1}^{\infty} K_j.$$

Proof The set $K := \bigcap_{j=1}^{\infty} K_j$ is compact and not empty. If the assertion is false, then $K_m \not\subset K + \varepsilon B^n$ for all $m \in \mathbb{N}$ with some fixed $\varepsilon > 0$. Let $A_m := K_m \setminus \operatorname{int}(K + \varepsilon B^n)$;

then $(A_m)_{m\in\mathbb{N}}$ is a decreasing sequence of nonempty, compact sets and hence has nonempty intersection A. Clearly, $A \cap K = \emptyset$, but $A_m \subset K_m$ implies $A \subset K$, a contradiction. □

Theorem 1.8.3 *The metric space (C^n, δ) is complete.*

Proof Let $(K_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in C^n. Put $A_m := \operatorname{cl} \bigcup_{i=m}^{\infty} K_i$. Then $(A_m)_{m\in\mathbb{N}}$ is a decreasing sequence of nonempty compact sets (the boundedness of $\bigcup K_i$ is an immediate consequence of the Cauchy property), hence Lemma 1.8.2 yields $A_m \to A := \bigcap_{i\in\mathbb{N}} A_i$ for $m \to \infty$. Thus, for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $A_m \subset A + \varepsilon B^n$ for $m \geq n_0$, hence $K_i \subset A + \varepsilon B^n$ for $i \geq n_0$. Since $(K_i)_{i\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_1 \geq n_0$ with $K_j \subset K_i + \varepsilon B^n$ for $i, j \geq n_1$. Thus, for $i, m \geq n_1$ we have $\bigcup_{j=m}^{\infty} K_j \subset K_i + \varepsilon B^n$ and hence $A_m \subset K_i + \varepsilon B^n$, which implies $A \subset K_i + \varepsilon B^n$. We have proved that $\delta(K_i, A) \leq \varepsilon$ for $i \geq n_1$, from which the assertion follows. □

Theorem 1.8.4 *In (C^n, δ) every closed, bounded subset is compact.*

In particular, the space (C^n, δ) is locally compact. By a known compactness criterion in metric spaces, Theorem 1.8.4 is a consequence of the following result.

Theorem 1.8.5 *From each bounded sequence in C^n one can select a convergent subsequence.*

Proof Let $(K_i^0)_{i\in\mathbb{N}}$ be a sequence in C^n whose elements are contained in some cube C of edge length γ. For each $m \in \mathbb{N}$, the cube C can be written as a union of 2^{mn} cubes of edge length $2^{-m}\gamma$. For $K \in C^n$, let $A_m(K)$ denote the union of all such cubes that meet K. Since (for each m) the number of subcubes is finite, the sequence $(K_i^0)_{i\in\mathbb{N}}$ has a subsequence $(K_i^1)_{i\in\mathbb{N}}$ such that $A_1(K_i^1) =: T_1$ is independent of i. Similarly, there is a union T_2 of subcubes of edge length $2^{-2}\gamma$ and a subsequence $(K_i^2)_{i\in\mathbb{N}}$ of $(K_i^1)_{i\in\mathbb{N}}$ such that $A_2(K_i^2) = T_2$. Continuing in this way, we obtain a sequence $(T_m)_{m\in\mathbb{N}}$ of unions of subcubes (of edge length $2^{-m}\gamma$ for given m) and to each m a sequence $(K_i^m)_{i\in\mathbb{N}}$ such that

$$A_m(K_i^m) = T_m \tag{1.61}$$

and

$$(K_i^m)_{i\in\mathbb{N}} \text{ is a subsequence of } (K_i^k)_{i\in\mathbb{N}} \text{ for } k < m. \tag{1.62}$$

By (1.61) we have $K_i^m \subset K_j^m + \lambda B^n$ with $\lambda = 2^{-m}\sqrt{n}\gamma$, hence $\delta(K_i^m, K_j^m) \leq 2^{-m}\sqrt{n}\gamma$ $(i, j, m \in \mathbb{N})$ and thus, by (1.62),

$$\delta(K_i^m, K_j^k) \leq 2^{-m}\sqrt{n}\gamma \quad \text{for } i, j \in \mathbb{N} \text{ and } k \geq m.$$

For $K_m := K_m^m$ it follows that

$$\delta(K_m, K_k) \leq 2^{-m}\sqrt{n}\gamma \quad \text{for } k \geq m.$$

Thus, $(K_m)_{m\in\mathbb{N}}$ is a Cauchy sequence and hence convergent, by Theorem 1.8.3. This is the subsequence that proves the assertion. □

From now on we again restrict our considerations to the space $\mathcal{K}^n$ of convex bodies.

Theorem 1.8.6 *$\mathcal{K}^n$ is a closed subset of C^n.*

Proof Let $K \in C^n \setminus \mathcal{K}^n$. Then there are points $x, y \in K$ and numbers $\lambda \in (0,1)$, $\varepsilon > 0$ such that $B(z,\varepsilon) \cap K = \emptyset$ for $z = (1-\lambda)x + \lambda y$. Let $K' \in C^n$ satisfy $\delta(K,K') < \varepsilon/2$. There are points $x', y' \in K'$ with $|x' - x| < \varepsilon/2$ and $|y' - y| < \varepsilon/2$, hence the point $z' := (1-\lambda)x' + \lambda y'$ satisfies $|z' - z| < \varepsilon/2$. If $z' \in K'$, then there is a point $w \in K$ with $|w - z'| < \varepsilon/2$, hence with $|w - z| < \varepsilon$, a contradiction. Thus K' is not convex. We have proved that $C^n \setminus \mathcal{K}^n$ is open. □

Theorems 1.8.5 and 1.8.6 together yield the following fundamental result.

Theorem 1.8.7 (Blaschke selection theorem) *Every bounded sequence of convex bodies has a subsequence that converges to a convex body.*

This theorem is a very useful tool in proving the existence of convex bodies with various specific properties.

It is sometimes convenient to have a description of the convergence of convex bodies in terms of convergent sequences of points.

Theorem 1.8.8 *The convergence $\lim_{i\to\infty} K_i = K$ in $\mathcal{K}^n$ is equivalent to the following conditions taken together:*

(a) *each point in K is the limit of a sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in K_i$ for $i \in \mathbb{N}$;*
(b) *the limit of any convergent sequence $(x_{i_j})_{j\in\mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ for $j \in \mathbb{N}$ belongs to K.*

Proof Assume that $K_i \to K$ for $i \to \infty$. Let $x \in K$. Put $x_i := p(K_i, x)$. Then $x_i \in K_i$ and $|x - x_i| \le \delta(K, K_i) \to 0$ for $i \to \infty$. Thus (a) holds. Let $(i_j)_{j\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$ and suppose that $x_{i_j} \in K_{i_j}$ ($j \in \mathbb{N}$) and $x_{i_j} \to x$ for $j \to \infty$, but $x \notin K$. Then $B(x,\rho) \cap (K + \rho B^n) = \emptyset$ for some $\rho > 0$. But for j sufficiently large we have $|x_{i_j} - x| < \rho$ and $x_{i_j} \in K_{i_j} \subset K + \rho B^n$, a contradiction. Thus (b) holds.

Now assume that (a) and (b) are satisfied. Let $\varepsilon > 0$ be given. We have to show that

$$K \subset K_i + \varepsilon B^n \quad \text{for all large } i, \tag{1.63}$$

$$K_i \subset K + \varepsilon B^n \quad \text{for all large } i. \tag{1.64}$$

If (1.63) is false, there exists a sequence $(y_{i_j})_{j\in\mathbb{N}}$ in K, converging to some $y \in K$, with $d(K_{i_j}, y_{i_j}) \ge \varepsilon$ ($j \in \mathbb{N}$). By (a), there exist $x_i \in K_i$ with $x_i \to y$ for $i \to \infty$. Then $|x_{i_j} - y_{i_j}| \to 0$ for $j \to \infty$, a contradiction. Hence (1.63) holds. If (1.64) is false, there exists a sequence $(y_{i_j})_{j\in\mathbb{N}}$ with $y_{i_j} \in K_{i_j}$ and $y_{i_j} \notin K + \varepsilon B^n$ for $j \in \mathbb{N}$. By (a), there exist points $x_{i_j} \in K_{i_j}$ with $x_{i_j} \in K + \varepsilon B^n$ for sufficiently large j. By the convexity of K_{i_j} (only the connectedness is needed), there are points $z_{i_j} \in K_{i_j} \cap \mathrm{bd}\,(K + \varepsilon B^n)$. The sequence $(z_{i_j})_{j\in\mathbb{N}}$ has a convergent subsequence whose limit must belong to K, by (b), a contradiction. Hence (1.64) holds. □

Having endowed $\mathcal{C}^n$ and $\mathcal{K}^n$ with a topology, we should check the continuity of the mappings that we have encountered in previous sections. Some of them are even Lipschitz mappings, for trivial reasons. This holds for the convex hull operator as a map from $\mathcal{C}^n$ to $\mathcal{K}^n$, in fact

$$\delta(\operatorname{conv} K, \operatorname{conv} L) \leq \delta(K, L),$$

and for addition and union as maps from $\mathcal{C}^n \times \mathcal{C}^n$ to $\mathcal{C}^n$, since

$$\delta(K + K', L + L') \leq \delta(K, L) + \delta(K', L'),$$
$$\delta(K \cup K', L \cup L') \leq \max\{\delta(K, L), \delta(K', L')\}.$$

The intersection as a map from $\mathcal{C}^n \times \mathcal{C}^n$ to $\mathcal{C}^n$ is not continuous, as the example after the proof of Theorem 1.8.10 shows.

For convex bodies, the following two observations are sometimes useful.

Lemma 1.8.9 *Let $K, K', L, L' \in \mathcal{K}^n$ and suppose that $K \cap K'$ and $L \cap L'$ contain the same ball of radius $r > 0$ and have diameters at most d. If $\delta(K, L) \leq \varepsilon$ and $\delta(K', L') \leq \varepsilon$, then $\delta(K \cap K', L \cap L') \leq (d/r)\varepsilon$.*

Proof Without loss of generality, we may assume that $B(o, r) \subset K, L, K', L'$. Let $x \in K \cap K'$. There are points $y \in L$ and $y' \in L'$ with $y, y' \in B(x, \varepsilon)$. The convex hull of $B(o, r)$ and any point from $B(x, \varepsilon)$ contains the point $z := (r/(r + \varepsilon))x$ (otherwise, a separation argument would give a contradiction). Thus, $z \in L \cap L'$. We have $|x - z| = (\varepsilon/(r + \varepsilon))|x| \leq (d/r)\varepsilon$. Similarly, to each point $z \in L \cap L'$ we can find a point $x \in K \cap K'$ with $|x - z| \leq (d/r)\varepsilon$. This yields the assertion. □

Theorem 1.8.10 *Let $K, L \in \mathcal{K}^n$ be convex bodies that cannot be separated by a hyperplane. If $K_i, L_i \in \mathcal{K}^n$ ($i \in \mathbb{N}$) are convex bodies with $K_i \to K$ and $L_i \to L$ for $i \to \infty$, then $K_i \cap L_i \neq \emptyset$ for almost all i and $K_i \cap L_i \to K \cap L$ for $i \to \infty$.*

Proof We use the criterion of Theorem 1.8.8. Let $x \in K \cap L$ (note that $K \cap L \neq \emptyset$ since otherwise K and L could be separated). Define $x_i := p(K_i \cap L_i, x)$ for those i for which $K_i \cap L_i \neq \emptyset$. We assert that x_i exists for almost all i and that $x_i \to x$ for $i \to \infty$. If this is false, there exists a ball B with centre x (and positive radius) such that $K_i \cap L_i \cap B = \emptyset$ for infinitely many i. For sufficiently large i we have $K_i \cap B \neq \emptyset$, since $x \in K$ and $K_i \to K$; similarly $L_i \cap B \neq \emptyset$ for large i. Thus, there is a sequence $(H_{i_j})_{j \in \mathbb{N}}$ of hyperplanes H_{i_j} separating $K_{i_j} \cap B$ and $L_{i_j} \cap B$. A subsequence converges to a hyperplane H, and H separates $K \cap B$ and $L \cap B$ (as one easily sees using Theorem 1.8.8). Since $x \in K \cap L \cap B$, necessarily $x \in H$. From the convexity of K and L it now follows that H separates K and L, a contradiction. Hence $x_i \to x$ for $i \to \infty$.

On the other hand, if $(i_j)_{j \in \mathbb{N}}$ is an increasing sequence in $\mathbb{N}$, if $x_{i_j} \in K_{i_j} \cap L_{i_j}$ ($j \in \mathbb{N}$) and $x_{i_j} \to y$ for $j \to \infty$, then clearly $y \in K \cap L$ by Theorem 1.8.8. Now it follows from this theorem that $K_i \cap L_i \to K \cap L$ for $i \to \infty$. □

The following example shows that the non-separability condition in Theorem 1.8.10 cannot be omitted. Let $K = L = L_i = [x, y]$, where $x \neq y$, and let $K_i = [x, y_i]$ be a segment with $[x, y_i] \cap [x, y] = \{x\}$ and such that $y_i \to y$ for $i \to \infty$. Then it follows that $K_i \to K$ and $L_i \to L$, but it does not follow that $K_i \cap L_i \to K \cap L$.

Next we show that the metric projection is continuous in both variables simultaneously, that is, the map

$$\begin{array}{rccc} p: & \mathcal{K}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ & (K, x) & \mapsto & p(K, x) \end{array}$$

is continuous. This is a consequence of the following more precise result.

Lemma 1.8.11 *Let $K, L \in \mathcal{K}^n$, $x, y \in \mathbb{R}^n$, and put $D := \operatorname{diam}(K \cup L \cup \{x, y\})$. Then*

$$|p(K, x) - p(L, y)| \leq |x - y| + \sqrt{5D\delta(K, L)}.$$

Proof Taking Theorem 1.2.1 into account (and the triangle inequality), we only have to show that

$$|p(K, x) - p(L, x)| \leq \sqrt{5D\delta(K, L)}.$$

If $x \in K \cap L$, this is trivial. We assume therefore, say, $x \notin K$. Put $d(K, x) = d$ and $\delta(K, L) = \delta$. The ball $B(p(K, x), \delta)$ contains a point of L, hence $d(L, x) \leq d + \delta$ and thus $p(L, x) \in B(x, d + \delta)$. Let H^- be the supporting halfspace of K with outer normal vector $u(K, x)$. Then L is contained in the translate $H^- + \delta u(K, x)$, hence

$$p(L, x) \in B(x, d + \delta) \cap [H^- + \delta u(K, x)].$$

Elementary geometry now yields that

$$|p(K, x) - p(L, x)| \leq \sqrt{4d\delta + \delta^2} \leq \sqrt{5D\delta},$$

since $d \leq D$ and $\delta \leq D$. □

Also the support function $h : \mathcal{K}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous as a function of two variables; it is in fact locally Lipschitz continuous, as a consequence of the following result.

Lemma 1.8.12 *For $K, L \in \mathcal{K}^n$ with $K, L \subset RB^n$ $(R > 0)$ and for $u, v \in \mathbb{R}^n$,*

$$|h(K, u) - h(L, v)| \leq R\,|u - v| + \max\{|u|, |v|\}\,\delta(K, L).$$

Proof Put $\delta(K, L) = \delta$ and choose a point $x \in K$ with $h(K, u) = \langle x, u\rangle$. From $x \in K \subset L + \delta B^n$ we get $\langle x, v\rangle \leq h(L + \delta B^n, v) = h(L, v) + \delta|v|$, hence

$$h(K, u) - h(L, v) \leq \langle x, u - v\rangle + \delta|v| \leq |x||u - v| + \delta|v| \leq R|u - v| + \delta|v|.$$

The assertion follows after interchanging (K, u) and (L, v). □

Corollary 1.8.13 *The width function $w(K, \cdot)$ of a convex body K satisfies the Lipschitz condition*

$$|w(K, u) - w(K, v)| \leq (\operatorname{diam} K)\,|u - v| \quad \textit{for } u, v \in \mathbb{S}^{n-1}.$$

Proof This follows if Lemma 1.8.12 is applied to the pair $(K - K, K - K)$, since $K - K \subset (\operatorname{diam} K)B^n$ and $h(K - K, \cdot) = w(K, \cdot)$. □

In terms of the support function, the convergence of convex bodies can be treated very conveniently, as a result of the following simple observation. Here $\|\cdot\|$ denotes the maximum norm for real functions on $\mathbb{S}^{n-1}$.

Lemma 1.8.14 *For $K, L \in \mathcal{K}^n$,*

$$\delta(K, L) = \sup_{u \in \mathbb{S}^{n-1}} |h(K, u) - h(L, u)| = \|\bar{h}_K - \bar{h}_L\|.$$

Proof Let $\delta(K, L) \leq \alpha$. Then $K \subset L + \alpha B^n$, hence $h(K, u) \leq h(L + \alpha B^n, u) = h(L, u) + \alpha$ for $u \in \mathbb{S}^{n-1}$. Interchanging K and L, we get $|h(K, u) - h(L, u)| \leq \alpha$ for $u \in \mathbb{S}^{n-1}$ and thus $\|\bar{h}_K - \bar{h}_L\| \leq \alpha$. The argument can be reversed. □

From Lemma 1.8.14 and the definitions (1.30) and (1.31) it follows immediately that the mean width w and the Steiner point s are Lipschitz maps. In both cases, the optimal Lipschitz constants are easily determined. From (1.30) one obtains

$$|w(K) - w(L)| \leq 2\delta(K, L)$$

for $K, L \in \mathcal{K}^n$, with equality if and only if $K = L + \alpha B^n$ or $L = K + \alpha B^n$ with $\alpha \geq 0$. For given $K, L \in \mathcal{K}^n$, take $v \in \mathbb{S}^{n-1}$ so that $|s(K) - s(L)| = \langle s(K) - s(L), v\rangle$; then

$$\begin{aligned} |s(K) - s(L)| &= \left| \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} [h(K, u) - h(L, u)] \langle u, v\rangle \, \mathrm{d}u \right| \\ &\leq \frac{1}{\kappa_n} \delta(K, L) \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}u. \end{aligned}$$

The last integral is twice the $(n-1)$-dimensional measure of the orthogonal projection of $\mathbb{S}^{n-1}$ to a hyperplane orthogonal to v (this is easy to see and is a special case of (5.78)). Thus

$$|s(K) - s(L)| \leq \frac{2\kappa_{n-1}}{\kappa_n} \delta(K, L). \tag{1.65}$$

It is clear that for $K \neq L$ equality in (1.65) cannot be attained, and it is easy to see that the constant cannot be replaced by a smaller one.

Lemma 1.8.14 shows that the map Υ defined by (1.26) maps $\mathcal{K}^n$ not only isomorphically as a convex cone into the vector space $C(\mathbb{S}^{n-1})$, but also isometrically with respect to the Hausdorff metric on $\mathcal{K}^n$ and the metric induced on $C(\mathbb{S}^{n-1})$ by the maximum norm. In particular, convergence of convex bodies is equivalent to uniform convergence on $\mathbb{S}^{n-1}$ of the corresponding support functions. This uniform convergence follows in any case from pointwise convergence.

Theorem 1.8.15 *If a sequence of support functions converges pointwise (on $\mathbb{R}^n$ or, equivalently, on $\mathbb{S}^{n-1}$), then it converges uniformly on $\mathbb{S}^{n-1}$ to a support function.*

Proof Let $(h(K_i, \cdot))_{i\in\mathbb{N}}$ be a pointwise converging sequence of support functions. Clearly, the limit function, say h, is sublinear and, hence, a support function, by Theorem 1.7.1. For each $u \in \mathbb{S}^{n-1}$, $\alpha(u) := \sup_i h(K_i, u)$ is finite. We can choose a number $R > 0$ with

$$\bigcap_{u\in\mathbb{S}^{n-1}} H^-_{u,\alpha(u)} \subset RB^n;$$

then $K_i \subset RB^n$ for $i \in \mathbb{N}$. Let h_i be the restriction of $h(K_i, \cdot)$ to $\mathbb{S}^{n-1}$; then $|h_i(u) - h_i(v)| \leq R|u - v|$ by Lemma 1.8.12. Choose a finite set $S \subset \mathbb{S}^{n-1}$ such that to each $u \in \mathbb{S}^{n-1}$ there is some $v \in S$ with $|u-v| < \varepsilon/3R$. Since S is finite, there exists $n_0 \in \mathbb{N}$ such that $|h_i(v) - h_j(v)| < \varepsilon/3$ for $i, j \geq n_0$ and all $v \in S$. Now let $u \in \mathbb{S}^{n-1}$. Choose $v \in S$ such that $|u - v| < \varepsilon/3R$. Then

$$\begin{aligned}|h_i(u) - h_j(u)| &\leq |h_i(u) - h_i(v)| + |h_i(v) - h_j(v)| + |h_j(v) - h_j(u)| \\ &\leq R|u - v| + \varepsilon/3 + R|u - v| < \varepsilon\end{aligned}$$

for $i, j \geq n_0$ and hence $|h_i(u) - h(u)| \leq \varepsilon$ for $i \geq n_0$. Since $u \in \mathbb{S}^{n-1}$ was arbitrary, the uniform convergence on $\mathbb{S}^{n-1}$ of the sequence $(h_i)_{i\in\mathbb{N}}$ to h is proved. □

Having introduced a metric on the set $\mathcal{K}^n$ of convex bodies, we can now consider approximation. The approximation of general convex bodies by simpler ones such as polytopes or bodies with differentiable boundaries is a useful tool for many investigations. Here we note only the most basic facts on approximation by polytopes. Further approximation results will be treated in Sections 2.4 and 3.4.

Theorem 1.8.16 *Let $K \in \mathcal{K}^n$ and $\varepsilon > 0$. Then there is a polytope $P\in \mathcal{K}^n$ with $P \subset K \subset P + \varepsilon B^n$, hence with $\delta(K, P) \leq \varepsilon$.*

Proof Cover K by finitely many balls with radius ε and centres in K, and let P be the convex hull of their centres. Evidently P satisfies the requirements. □

Remark 1.8.17 In the preceding proof we may impose the condition that the centres of the balls have rational coordinates. Hence, the space $(\mathcal{K}^n, \delta)$ is separable; that is, it has a countable dense subset.

Lemma 1.8.18 *Let $K_1, K_2 \in \mathcal{K}^n$ and $K_2 \subset \operatorname{int} K_1$. Then there is a number $\eta > 0$ such that every convex body $K \in \mathcal{K}^n$ with $\delta(K_1, K) < \eta$ satisfies $K_2 \subset K$.*

Proof Since $K_2 \subset \operatorname{int} K_1$, the function $h(K_1, \cdot) - h(K_2, \cdot)$ is positive on $\mathbb{R}^n \setminus \{o\}$ and hence, being continuous, attains a positive minimum η on $\mathbb{S}^{n-1}$. Let $K \in \mathcal{K}^n$ be a convex body with $\delta(K_1, K) < \eta$. Then $|h(K_1, u) - h(K, u)| < \eta$ and hence $h(K_2, u) \leq h(K_1, u) - \eta < h(K, u)$ for $u \in \mathbb{S}^{n-1}$; thus $K_2 \subset K$. □

Theorem 1.8.19 *Let $K \in \mathcal{K}^n_{(o)}$. For each $\lambda > 1$ there exists a polytope $P \in \mathcal{K}^n$ with $P \subset K \subset \lambda P$.*

Proof Choose a number $\rho > 0$ with $\rho B^n \subset \operatorname{int} K$ and a number ε with $0 < \varepsilon \leq (\lambda - 1)\rho$. By Theorem 1.8.16 and Lemma 1.8.18 we can choose ε small enough so that there is a polytope P with

$$\rho B^n \subset P \subset K \subset P + \varepsilon B^n.$$

For $u \in \mathbb{S}^{n-1}$ we deduce

$$h(\lambda P, u) = h(P, u) + (\lambda - 1)h(P, u) \geq h(P, u) + \varepsilon = h(P + \varepsilon B^n, u) \geq h(K, u),$$

thus $K \subset \lambda P$. □

We prove a further continuity result.

Theorem 1.8.20 *The volume functional V_n is continuous on $\mathcal{K}^n$.*

Proof Let $K \in \mathcal{K}^n$ be given. If $V_n(K) = 0$ and $\bar{K} \in \mathcal{K}^n$ satisfies $\delta(K, \bar{K}) = \alpha \leq 1$ (without loss of generality), then K lies in a hyperplane and $\bar{K} \subset K + \alpha B^n$. Hence, $V_n(\bar{K}) \leq V_n(K + \alpha B^n) \leq c(K)\alpha$ with $c(K)$ independent of α, by an easy estimation using Fubini's theorem. Suppose, therefore, that $V_n(K) > 0$. Then we may assume that $o \in \operatorname{int} K$. Let $\varepsilon > 0$ be given. Choose $\lambda > 1$ with $(\lambda^n - 1)\lambda^n V_n(K) < \varepsilon$ and $\rho > 0$ with $\rho B^n \subset \operatorname{int} K$. By Lemma 1.8.18, there is a number $\alpha > 0$ with $\alpha \leq (\lambda - 1)\rho$ and such that $\rho B^n \subset \bar{K}$ for all $\bar{K} \in \mathcal{K}^n$ satisfying $\delta(K, \bar{K}) < \alpha$. Suppose that the latter holds. Then

$$K \subset \bar{K} + \alpha B^n \subset \bar{K} + (\lambda - 1)\rho B^n \subset \bar{K} + (\lambda - 1)\bar{K} = \lambda \bar{K}$$

and similarly $\bar{K} \subset \lambda K$. It follows that

$$V_n(K) \leq V_n(\lambda \bar{K}) = \lambda^n V_n(\bar{K}),$$

hence

$$V_n(K) - V_n(\bar{K}) \leq (\lambda^n - 1)V_n(\bar{K}) \leq (\lambda^n - 1)\lambda^n V_n(K),$$
$$V_n(\bar{K}) - V_n(K) \leq (\lambda^n - 1)V_n(K) \leq (\lambda^n - 1)\lambda^n V_n(K),$$

thus

$$|V_n(K) - V_n(\bar{K})| \leq (\lambda^n - 1)\lambda^n V_n(K) < \varepsilon.$$ □

Notes for Section 1.8

1. An infinite-dimensional version of Lemma 1.8.1 is found in Wills [1984].
2. *Hausdorff metric and topologies on spaces of subsets*. The Hausdorff metric is usually considered in a more general context. Let (X, d) be a metric space. Then

$$\delta(C, D) := \max \left\{ \sup_{x \in C} \inf_{y \in D} d(x, y),\ \sup_{x \in D} \inf_{y \in C} d(x, y) \right\}$$

defines the Hausdorff metric δ on the set of nonempty closed and bounded subsets of X.

In a special case, this metric was introduced by Pompeiu [1543] (p. 282). A general study of it was made by Hausdorff [941] (chap. VIII, §6), who also described the convergence with respect to this metric in terms of closed limits (see also Hausdorff [942],

p. 149). Both Hausdorff [941] (p. 463) and Kuratowski [1158] (p. 106) refer to Pompeiu for the origin of the definition.

Let $C(X)$ denote the set of nonempty compact subsets of X. Different common topologies introduced on spaces of subsets of a topological space coincide, on $C(X)$, with the topology induced by the Hausdorff metric. This is true for the Vietoris topology and for the topology of closed convergence. The latter, which is usually considered on the set of closed subsets of a locally compact, second countable Hausdorff space, has applications in mathematical economics (Hildenbrand [974]) and stochastic geometry (Matheron [1358]). For properties of the metric space $(C(X), \delta)$ we refer the reader to Kuratowski [1159], §42, and for general investigations on topologies for spaces of subsets to Michael [1424] and the monograph of Klein and Thompson [1118]. See also the survey by McAllister [1368], which gives some information on the early history of the Hausdorff metric.

3. *Selection theorems*. The Blaschke selection theorem for convex bodies was first stated and used by Blaschke [241] (pp. 62 ff). A general theorem, which implies the compactness of $(C(X), \delta)$ for a compact metric space X, appears in Hausdorff [942], p. 147; see also Kuratowski [1159], §42, and Rogers [1585], p. 91. An elegant proof for compact subsets of $\mathbb{R}^n$ was given by Hadwiger [895], [911], §4.3.2; we have followed his approach. A very general form appears in Dierolf [477].

By Lemma 1.8.12, one has $|h(K,u) - h(K,v)| \le R|u - v|$ for $K \in \mathcal{K}^n$ with $K \subset RB^n$ (for given $R > 0$) and for $u, v \in \mathbb{R}^n$. Therefore, the family

$$\{\bar{h}_K : K \in \mathcal{K}^n,\ K \subset RB^n\}$$

is equicontinuous; further, it is uniformly bounded and closed in $C(\mathbb{S}^{n-1})$ (with the maximum norm). Thus, by the Arzelà–Ascoli theorem, it is compact. This is another proof of the Blaschke selection theorem. Two different methods of deducing the Blaschke selection theorem from the Arzelà–Ascoli theorem were described by Heil [949]; he also deduced the selection theorem for compact sets in this way.

Selection theorems for starshaped sets and for more general sets were treated by Hirose [982], Drešević [513], Beer [188] and Spiegel [1801].

4. *Convergence in the sense of the Hausdorff metric*. Theorem 1.8.8 can be generalized and refined in the following way. We consider the space C^n of nonempty compact sets in $\mathbb{R}^n$. Define

$$\lambda(K, L) := \max_{x\in K} \min_{y\in L} |x - y| \quad \text{for } K, L \in C^n,$$

so that $\delta(K, L) = \max\{\lambda(K, L), \lambda(L, K)\}$. For $K \in C^n$ and a sequence $(K_i)_{i\in\mathbb{N}}$ in C^n, the following assertions may hold.

(1) $\lambda(K, K_i) \to 0$ for $i \to \infty$.
(2) Any point $x \in K$ is the limit of a sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in K_i$ ($i \in \mathbb{N}$).
(3) Every open set intersecting K intersects K_i for almost all i.

(1′) $\lambda(K_i, K) \to 0$ for $i \to \infty$.
(2′) The limit of any convergent sequence $(x_{i_j})_{j\in\mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ ($j \in \mathbb{N}$) belongs to K, and the sequence $(K_i)_{i\in\mathbb{N}}$ is bounded.
(3′) Every closed set having empty intersection with K has empty intersection with almost all K_i.

Then the following can be shown: the conditions (1), (2), (3) are equivalent; the conditions (1′), (2′), (3′) are equivalent. Hence

$$\lim_{i\to\infty} K_i = K \Leftrightarrow (2) \text{ and } (2') \Leftrightarrow (3) \text{ and } (3').$$

5. *Convergence of closed convex sets*. On the set of (not necessarily bounded) closed convex subsets of $\mathbb{R}^n$ several different types of convergence have been considered, which coincide on $\mathcal{K}^n$. A comparison of four such types of convergence is made in Salinetti and Wets [1606]; see also Wijsman [1978, 1979] and Beer [189].

6. *Topology of hyperspaces*. For a nonempty subset X of $\mathbb{R}^n$, the space $\mathcal{K}(X)$ (of convex bodies contained in X), metrized by the Hausdorff metric, is also called the *cc-hyperspace* of X. The topology of $\mathcal{K}(X)$, in dependence on the geometry of X, has received some attention. The following results are due to Nadler, Quinn and Stavrakas [1462, 1463, 1464]; see also Chapter XVIII of Nadler [1461]. If $X \in \mathcal{K}^n$ and $\dim X \geq 2$, then $\mathcal{K}(X)$ is homeomorphic to the Hilbert cube $I_\infty = \prod_{i=1}^{\infty}[-1/2^i, 1/2^i]$. For $n \geq 2$, $\mathcal{K}(\mathrm{int}\, B^n)$ and $\mathcal{K}(\mathbb{R}^n)$ are homeomorphic to I_∞ with a point removed. If $X \subset \mathbb{R}^2$ is compact, connected and such that $\mathcal{K}(X)$ is homeomorphic to I_∞, then X is topologically a 2-cell. Curtis, Quinn and Schori [456] found a complete characterization of the polyhedral 2-cells whose cc-hyperspaces are homeomorphic to I_∞. Further results on hyperspaces of convex bodies are due to Montejano [1445] and to Bazylevych and Zarichnyi [186].
7. *Embedding theorems for spaces of convex sets*. The map

$$\begin{array}{rccc} \Upsilon: & \mathcal{K}^n & \to & C(\mathbb{S}^{n-1}) \\ & K & \mapsto & \bar{h}_K \end{array}$$

given by (1.26) maps the cone of convex bodies in $\mathbb{R}^n$ isomorphically, and by Lemma 1.8.14 also isometrically, onto a closed convex cone in a Banach space. For certain spaces of convex subsets of infinite-dimensional vector spaces, similar embedding theorems are known. Rådström [1555] proved the following result.

Theorem Let L be a real normed linear space and M a set of closed bounded convex sets in L with the following properties.

(1) M is closed under addition and multiplication by nonnegative scalars.
(2) If $A \in M$ and S is the unit ball of L, then $A + S$ is closed.
(3) M is metrized by the Hausdorff metric.

Then M can be embedded as a convex cone in a real normed linear space N in such a way that

(a) the embedding is isometric,
(b) addition in N induces addition in M,
(c) multiplication by nonnegative scalars in N induces the corresponding operation in M.

Rådström showed that the conditions on M are satisfied, for instance, by the set of all finite-dimensional compact convex sets and by the set of all compact convex sets.

Rådström's proof was by means of an abstract construction, but it was pointed out later that one can also use support functionals in a similar way to their use in the finite-dimensional case. For locally convex spaces this was done by Hörmander [987], who proved also a generalization of Theorem 1.7.1, characterizing the support functionals of closed convex sets in such a space (essentially by an argument that can be viewed as extending Rademacher's method, which is given as the second proof of Theorem 1.7.1). More general embedding theorems of a similar kind were obtained by Godet-Thobie and The Lai [725], Urbański [1864], Tolstonogov [1849] and Schmidt [1646].

Essential for the existence of such embeddings is the cancellation law, demanding that $A+C = B+C$ implies $A = B$. An algebraic study of the validity of this cancellation law in arbitrary real vector spaces under suitable convexity and other assumptions on the subsets A, B, C was carried out by Jongmans [1048].
8. *Representation of semigroups as systems of compact convex sets.* If the multiplication $(\lambda, K) \mapsto \lambda K = \{\lambda x : x \in K\}$ $(K \in \mathcal{K}^n)$ is considered for all real $\lambda \in \mathbb{R}$, then the rules listed before (1.26) remain the same except that $(\lambda + \mu)K = \lambda K + \mu K$ holds only for $\lambda\mu \geq 0$. The structure of $(\mathcal{K}^n, +)$ with this multiplication is then an example of an abelian $\mathbb{R}$-semigroup with cancellation law. Ratschek and Schröder [1559] have characterized the abstract $\mathbb{R}$-semigroups that can be represented in this way by systems of compact convex sets, with $\mathbb{R}^n$ replaced by a suitable locally convex space.
9. *Different metrics for convex bodies*. In the theory of convex bodies, the Hausdorff metric was first used by Blaschke [241] (§§17, 18). It is, therefore, sometimes called the

Blaschke–Hausdorff metric. In the geometry of convex bodies, other metrics have been proposed. The following are defined on $\mathcal{K}_n^n$ only. The *symmetric difference metric* δ^S is given by

$$\delta^S(K, L) := \mathcal{H}^n(K \Delta L) \quad \text{for } K, L \in \mathcal{K}_n^n,$$

where $K\Delta L = (K \setminus L) \cup (L \setminus K)$ denotes the symmetric difference of K and L. This metric is invariant under volume-preserving affinities of $\mathbb{R}^n$. Two affine-invariant metrics are defined on $\mathcal{K}_n^n$ as follows:

$$\delta^{\mathrm{D}}(K, L) := \log(1 + 2 \inf \{\lambda \geq 0 : K \subset L + \lambda \mathrm{D}L,\ L \subset K + \lambda \mathrm{D}K\}),$$

where $\mathrm{D}K := K - K$ is the difference body of K, and

$$\delta^Q(K, L) := \mathcal{H}^n(K \Delta L)/\mathcal{H}^n(K \cup L).$$

A detailed investigation and comparison of the four metrics $\delta, \delta^S, \delta^{\mathrm{D}}, \delta^Q$ on $\mathcal{K}_n^n$ was carried out by Shephard and Webster [1789]. In particular, they showed that all these metrics induce the same topology on $\mathcal{K}_n^n$, but not the same uniform structures. Further, they studied the completeness and uniform continuity of some functions with respect to these metrics.

Dinghas [492] showed that the symmetric difference metric (for not necessarily convex sets) is decreased under simultaneous symmetrization at an affine subspace.

Another class of metrics is obtained if in Lemma 1.8.14 the maximum norm is replaced by an L_p norm. For $1 \leq p < \infty$ and $K, L \in \mathcal{K}^n$ let

$$\delta_p(K, L) := \left(\int_{\mathbb{S}^{n-1}} |h_K - h_L|^p \, \mathrm{d}\mathcal{H}^{n-1} \right)^{1/p};$$

then δ_p is a metric on $\mathcal{K}^n$, called the L_p metric. These metrics were investigated by Vitale [1885]. One has $\delta_p(K, L) \leq \omega_n^{1/p} \delta(K, L)$ trivially, and Vitale established estimates in the other direction. As a consequence of a sharp, but more complicated, inequality he showed, for instance, that

$$c_p(K, L)[\delta(K, L)]^{(p+n-1)/p} \leq \delta_p(K, L)$$

for $K, L \in \mathcal{K}^n$, where the factor $c_p(K, L)$ depends (for fixed n and p) only on $\operatorname{diam}(K \cup L)$. He deduced that all of the δ_p metrics ($1 \leq p \leq \infty$, with $\delta_\infty = \delta$), induce the same topology on $\mathcal{K}^n$ and yield complete metric spaces in which closed, bounded sets are compact. For δ_1, see also Florian [620].

The *Sobolev distance* $\delta_w(K, L)$ of convex bodies $K, L \in \mathcal{K}_n^n$ is defined by

$$\delta_w(K, L)^2 := \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} |\nabla(h_K - h_L)|^2 \, \mathrm{d}\sigma.$$

For planar convex bodies K, L, Arnold and Wellerding [76] found the optimal constant c for which $\delta(K, L) \leq c\delta_w(K, L)$. For $n \geq 3$, however, they showed that convex bodies of fixed Hausdorff distance can have arbitrarily small Sobolev distance.

10. *Isometries*. Because the Hausdorff metric on $\mathcal{K}^n$ has a simple geometric meaning, the study of the metric space $(\mathcal{K}^n, \delta)$ deserves some attention of its own. A natural first question asks for the isometries that this space permits. Suppose that the map $i : \mathcal{K}^n \to \mathcal{K}^n$ is an isometry with respect to the Hausdorff metric. Under the assumption that i is surjective, it was proved by Schneider [1675] that i is induced by an isometry of $\mathbb{R}^n$, in the sense that there is a rigid motion $g : \mathbb{R}^n \to \mathbb{R}^n$ so that $i(K) = gK$ for all $K \in \mathcal{K}^n$. Without assuming surjectivity, Gruber and Lettl [840] showed that one can find a rigid motion g of $\mathbb{R}^n$ and a convex body L such that $i(K) = gK + L$ for all $K \in \mathcal{K}^n$.

Every isometry of $\mathcal{K}_n^n$ with respect to the symmetric difference metric is induced by an equi-affinity of $\mathbb{R}^n$ (Gruber [813]). For the subspace $\mathcal{K}_n^n(B^n)$ of n-dimensional convex bodies contained in the unit ball, every isometry with respect to the symmetric difference metric is induced by a rotation of the ball (Gruber [816]). The corresponding result for the Hausdorff metric on $\mathcal{K}^n(B^n)$ follows from Bandt [132]. Similar investigations for non-convex sets and for subsets of other spaces were carried out by Gruber [814, 815], Gruber and Lettl [839], Lettl [1203], Gruber and Tichy [844] and Bandt [132]. Very general results on isometries for symmetric difference metrics were obtained by Weisshaupt [1957], in particular, far-reaching extensions of the results by Gruber [813, 816], as well as results for the hyperbolic plane.

11. *Metric convexity and metric segments*. For a metric space it is, furthermore, of interest to study its properties in terms of the geometric notions introduced by Menger [1403] (see also Blumenthal [257]). The space $(\mathcal{K}^n, \delta)$ is a metric segment space, since it follows from Lemma 1.8.14 that

$$\delta(K, (1-\alpha)K + \alpha L) = \alpha\delta(K, L)$$

for $K, L \in \mathcal{K}^n$ and $\alpha \in [0, 1]$. Some generalizations to infinite-dimensional spaces were investigated by Bantegnie [138]. Shephard and Webster [1789] showed that the space $(\mathcal{K}^n_n, \delta^S)$ is metrically convex (in Menger's sense), but not a metric segment space, while $(\mathcal{K}^n_n, \delta^{\mathrm{D}})$ and $(\mathcal{K}^n_n, \delta^Q)$ are not metrically convex.

In the space $(\mathcal{K}^n, \delta)$ there is in general a great variety of metric segments joining two given bodies K and L. A study of their totality was begun by Jongmans [1050]. Answering a question left open by him, Schneider [1696] showed the following. Let $K, L \in \mathcal{K}^n$ be convex bodies joined by only one metric segment (equivalently, $A = \frac{1}{2}(K + L)$ is the only convex body satisfying $\delta(K, A) = \delta(A, L) = \frac{1}{2}\delta(K, L)$). Then either $K = L + \rho B^n$ or $L = K + \rho B^n$ with some $\rho \geq 0$, or else $\dim K < n$ and $L = K + t$, where the vector t is orthogonal to aff K.

12. *Metric entropy*. A characteristic of a totally bounded metric space (X, d) that is designed to describe its 'massivity' is the *metric entropy*. This is, by definition, the function H given by

$$H(\varepsilon) := \log N(\varepsilon) \quad \text{for } \varepsilon > 0,$$

where $N(\varepsilon)$ is the minimal number of closed d-balls of radius ε covering X. For the spaces $(\mathcal{K}^n(A), \delta)$ and $(\mathcal{K}^n_n(A), \delta^S)$, where $\emptyset \neq A \subset \mathbb{R}^n$ is a bounded open set, the asymptotic behaviour of the metric entropy for $\varepsilon \to 0$ was investigated by Dudley [517]. For $(\mathcal{K}^n(B^n), \delta)$, Bronshtein [340] showed more precisely that

$$c_1(n)\varepsilon^{(1-n)/2} \leq H(\varepsilon) \leq c_2(n)\varepsilon^{(1-n)/2} \quad \text{for } \varepsilon \leq c_3(n),$$

where $c_i(n)$ is a positive constant depending only on n. In particular, the *exponent of entropy*, defined by

$$\lim_{\varepsilon\downarrow 0} \inf(\log\log N(\varepsilon)/|\log\varepsilon|),$$

is equal to $(n-1)/2$ for the space $(\mathcal{K}^n(B^n), \delta)$ (see also Bandt [133]).

13. *Diameters of sets of convex bodies*. It is an interesting, and in general difficult, task to determine the diameter, with respect to the Hausdorff metric, of a given bounded subset of $\mathcal{K}^n$ that is defined by some geometric conditions. As an example, McMullen [1388] proved that the space of all convex bodies in $\mathcal{K}^n$ with mean width w equal to that of a line segment of length 2 and with Steiner point s at the origin, has diameter 1. As a corollary, McMullen obtained the sharp estimate

$$\delta(K, L) \leq \frac{\omega_n}{4\kappa_{n-1}} \max\{w(K),\ w(L)\} + |s(K) - s(L)|$$

for $K, L \in \mathcal{K}^n$.

14. *Affine equivalence classes of convex bodies*. Let $\mathcal{AK}^n_n$ denote the set of all affine equivalence classes of n-dimensional convex bodies in $\mathbb{R}^n$. Thus, L belongs to the equivalence class $[K] \in \mathcal{AK}^n_n$ if and only if $K = \alpha L$ for some non-degenerate affine transformation α of $\mathbb{R}^n$. If $\mathcal{AK}^n_n$ is equipped with the quotient topology (induced by the standard topology

on $\mathcal{K}_n^n$), then $\mathcal{AK}_n^n$ is compact. This was proved by Macbeath [1313]. In the course of the proof, Macbeath introduced the function

$$\rho(K, L) := \inf_{\alpha}\{\mathcal{H}^n(\alpha L)/\mathcal{H}^n(K) : K \subset \alpha L\},$$

where α ranges over the affinities of $\mathbb{R}^n$, and showed that

$$\Delta([K], [L]) := \log \rho(K, L) + \log \rho(L, K) \quad \text{for } [K], [L] \in \mathcal{AK}_n^n$$

defines a metric Δ on $\mathcal{AK}_n^n$ that induces the quotient topology.

For the space $(\mathcal{AK}_n^n, \Delta)$, the exponent of entropy (see Note 12 above) was determined by Bronshtein [341]; it is equal to $(n-1)/2$.

15. *Affine equivalence classes of compact sets*. Let $\mathcal{AC}^n$ denote the set of affine equivalence classes of nonempty compact subsets of $\mathbb{R}^n$. Webster [1927] showed how the Hausdorff metric can be used to define a metric on $\mathcal{AC}^n$ that makes $\mathcal{AC}^n$ into a compact space. This metric, however, does not induce the quotient topology on $\mathcal{AC}^n$ (obtained from $(\mathcal{C}^n, \delta)$ and the affine-equivalence relation), since the latter fails to be metrizable. Let $\mathcal{AC}_n^n$ ($\mathcal{AC}_+^n$) be the subset of all affine equivalence classes containing an element C with $\dim \operatorname{conv} C = n$ (respectively, $\mathcal{H}^n(C) > 0$). Then Webster's metric and the quotient topology on $\mathcal{AC}^n$ induce the same topology on $\mathcal{AC}_n^n$, which is compact. On the other hand, there does not exist a metric on $\mathcal{AC}_+^n$ that makes $\mathcal{AC}_+^n$ compact and induces the quotient topology. Webster's results essentially answered a question posed by Grünbaum [847] (p. 263).
16. Corollary 1.8.13 appears in Martini and Wenzel [1354], with a different proof.
17. *Applications of the Steiner point*. The exact Lipschitz constant in (1.65) was determined by Przesławski [1549], Saint Pierre [1604] and Vitale [1885].

    The properties of the Steiner point, in particular Minkowski additivity, the inclusion property (1.34) and the Lipschitz continuity (1.65), make it a useful tool for some applications. For instance, the Steiner point has been used for the construction of continuous selections of multi-valued mappings (Linke [1226], Przesławski [1549]) and in the theory of random sets (Vitale [1884], Giné and Hahn [713, 714]). Motivated by this, Vitale [1886] showed the non-existence of a continuous extension of the Steiner point to all convex bodies in infinite-dimensional Hilbert spaces.
18. *Minkowski additive selections.* Živaljević [2080] studied Minkowski additive functions $f : \mathcal{K}^n \to \mathbb{R}^n$ satisfying $f(K) \in K$ for all $K \in \mathcal{K}^n$ (like the Steiner point map). In particular, he characterized all such Minkowski additive selections f for which $f(K) \in \operatorname{ext} K$ for all $K \in \mathcal{K}^n$.
19. *Invariant measures on $\mathcal{K}^n$*. The following question was posed by McMullen (see Gruber and Schneider [841], p. 268): is there a natural and useful (isometry) invariant measure on the space $\mathcal{K}^n$ with the Hausdorff metric? It appears (depending on the interpretation of 'useful') that the answer is 'no'. Bandt and Baraki [134] showed: for $n > 1$ there is no σ-finite Borel measure on $(\mathcal{K}^n, \delta)$ that is invariant with respect to all isometries from the whole space into itself and is not identically zero. However, there do exist σ-finite Borel measures on $\mathcal{K}^n$ which are positive on nonempty open sets and are invariant under all rigid motions. A large family of such measures was constructed by Hoffmann [983].
20. *Approximation of convex bodies by polytopes.* The approximation result of Theorem 1.8.19, saying that a convex body can be approximated arbitrarily closely by polytopes, is almost trivial. The approximation of convex bodies by polytopes becomes much more interesting if one asks for the quantitative strength of such approximations, under various conditions. This is a wide field, and we can here only refer to the survey articles by Gruber [829, 832] and Bronshtein [346].

2

Boundary structure

2.1 Facial structure

The notions of face, extreme point and exposed point of a convex set were defined in Section 1.4. In the present section we shall study the boundary structure of closed convex sets in relation to these and similar or more specialized notions. We shall assume in the following that $K \subset \mathbb{R}^n$ is a nonempty closed convex set.

An i-dimensional face of K is referred to as an *i-face*. By $\mathcal{F}(K)$ we denote the set of all faces and by $\mathcal{F}_i(K)$ the set of all i-faces of K. A face of dimension $\dim K - 1$ is usually called a *facet*. The empty set $\emptyset$ and K itself are faces of K; the other faces are called *proper*. Conventionally, the empty face has dimension -1. It follows from the definition of a face and from Lemma 1.1.9 that the faces of K are closed. If $F \neq K$ is a face of K, then $F \cap \operatorname{relint} K = \emptyset$. (If $z \in F \cap \operatorname{relint} K$, we choose $y \in K \setminus F$. There is some $x \in K$ with $z \in \operatorname{relint}[x, y]$. Then $[x, y] \subset F$, a contradiction.) In particular, $F \subset \operatorname{relbd} K$ and $\dim F < \dim K$.

Theorem 2.1.1 *If F_i is a face of K for $i \in I$ (a nonempty index set), then $\bigcap_{i \in I} F_i$ is a face of K. If F is a face of K and G is a face of F, then G is a face of K.*

Proof This follows immediately from the definition of a face. □

Theorem 2.1.2 *If $F_1, F_2 \in \mathcal{F}(K)$ are distinct faces, then* $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$. *To each nonempty relatively open convex subset A of K there is a unique face $F \in \mathcal{F}(K)$ with $A \subset$* $\operatorname{relint} F$. *The system*

$$\{\operatorname{relint} F : F \in \mathcal{F}(K)\}$$

is a disjoint decomposition of K.

Proof Assume that $F_1, F_2 \in \mathcal{F}(K)$, $F_1 \neq F_2$ and $z \in \operatorname{relint} F_1 \cap \operatorname{relint} F_2$. There is some $x \in F_1 \setminus F_2$, say. We can choose $y \in F_1$ such that $z \in \operatorname{relint}[x, y]$. From $[x, y] \subset F_1 \subset K$, $z \in F_2$ and $F_2 \in \mathcal{F}(K)$ it follows that $[x, y] \subset F_2$, a contradiction. Thus $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$.

Let $A \neq \emptyset$ be a relatively open convex subset of K. Let F be the intersection of all faces of K containing A; then F is a face of K. Suppose there is some point

$x \in A \setminus \operatorname{relint} F$. Then there is a support plane H to F with $x \in H$ and $H \cap F \neq F$. Since $x \in A \subset F$ and A is relatively open, it follows that $A \subset H$. Now $H \cap F$ is clearly a face of F and hence a face of K. But then $F \subset H \cap F$ by the definition of F, a contradiction. Thus $A \subset \operatorname{relint} F$. It follows from the first part that F is the only face with this property.

In particular, the set A may be one-pointed. Thus, each point $x \in K$ is contained in relint F for a unique face $F \in \mathcal{F}(K)$. □

In the proof above we have used the obvious fact that each support set $H \cap K$, where H is a support plane of K, is a face of K. The support sets are also called *exposed faces* of K. The singleton $\{x\}$ is an exposed face if and only if x is an exposed point. We denote by $\mathcal{E}(K)$ the set of all exposed faces and by $\mathcal{E}_i(K)$ the set of all i-dimensional exposed faces of K. Clearly, not every face is an exposed face: not every extreme point is an exposed point, and taking direct sums yields examples with higher-dimensional faces. However, each proper face F of K is contained in some exposed face. In fact, since $F \cap \operatorname{relint} K = \emptyset$, there exists a hyperplane H separating F and K. From $F \subset K$ it follows that $F \subset H$, hence F is contained in the exposed face $H \cap K$.

Theorem 2.1.3 *If F_i is an exposed face of K for $i \in I \neq \emptyset$, then $\bigcap_{i \in I} F_i$ is either empty or an exposed face of K.*

Proof Let $F := \bigcap_{i \in I} F_i$ and assume that $F \neq \emptyset$. After a translation, we may assume that $o \in F$. For $i \in I$ there exists a vector $u_i \in \mathbb{R}^n \setminus \{o\}$ such that $F_i = K \cap H_{u_i,0}$ and $K \subset H^-_{u_i,0}$. We may assume that $u_1, \dots, u_r$ are linearly independent and that any other u_i linearly depends on these vectors. Then $F = \bigcap_{i=1}^r F_i$. Let $u := \sum_{i=1}^r u_i$. Then $K \subset H^-_{u,0}$ and $o \in K \cap H_{u,0}$, hence $H_{u,0}$ supports K. For $x \in F$ we have $\langle x, u_i \rangle = 0$ for $i \in I$ and hence $\langle x, u \rangle = 0$; thus $F \subset H_{u,0} \cap K$. For $y \in K \setminus F$, we have $\langle y, u_j \rangle < 0$ for some $j \in \{1, \dots, r\}$, hence $\langle y, u \rangle < 0$ and thus $y \notin H_{u,0}$. This proves that $H_{u,0} \cap K = F$ and therefore that $F \in \mathcal{E}(K)$. □

One should observe that the second part of Theorem 2.1.1 has no counterpart for exposed faces: an exposed face of an exposed face of K need not itself be an exposed face of K, but it is, of course, a face of K.

From now on in this section we assume that $K \in \mathcal{K}^n$.

Exposed faces behave well under polarity, as we shall now see. Suppose that $K \in \mathcal{K}^n_{(o)}$. In Section 1.6 the polar body of K was defined as

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

For a subset $F \subset K$ we now define the *conjugate face* of F by

$$\widehat{F} := \{x \in K^\circ : \langle x, y \rangle = 1 \text{ for all } y \in F\}. \tag{2.1}$$

One should keep in mind that $\widehat{F}$ depends also on K and not only on F. For example, in $\widehat{\widehat{F}}$, the upper circumflex refers to K°. This slightly incomplete notation is quite common. Instead of $\widehat{\{y\}}$ we write $\widehat{y}$.

Theorem 2.1.4 *Let $K \in \mathcal{K}^n_{(o)}$ and $\emptyset \neq F \subset K$. Then the following assertions* (a), (b), (c) *are equivalent.*

(a) *F lies in an exposed face of K;*
(b) *$\widehat{F} \neq \emptyset$;*
(c) *$\widehat{F}$ is an exposed face of K°.*

If (a) – (c) *hold, then $\widehat{\widehat{F}}$ is the smallest exposed face of K containing F.*

Proof For $y \in \operatorname{bd} K$, the hyperplane $\{x \in \mathbb{R}^n : \langle x, y\rangle = 1\}$ supports K° (e.g., by Theorem 1.7.13); hence $\widehat{y} = F(K^\circ, y)$ is an exposed face of K°. For $y \in \operatorname{int} K$, $\langle x, y\rangle < 1$ for all $x \in K^\circ$, hence $\widehat{y} = \emptyset$. By definition,

$$\widehat{F} = \bigcap_{y \in F} \{x \in K^\circ : \langle x, y\rangle = 1\} = \bigcap_{y \in F} \widehat{y}.$$

Thus, $\widehat{F}$ is either empty or (by Theorem 2.1.3) an exposed face of K°.

Suppose that F lies in the exposed face $S := K \cap H_{u,1}$ (without loss of generality), where $K \subset H^-_{u,1}$. Then $u \in \widehat{F}$ and hence $\widehat{F} \neq \emptyset$. Conversely, if $u \in \widehat{F}$, then $u \in K^\circ$ and $\langle u, y\rangle = 1$ for all $y \in F$, hence F lies in an exposed face of K. Thus (a), (b) and (c) are equivalent.

Suppose again that F lies in the exposed face $S = K \cap H_{u,1}$. Let $z \in \widehat{\widehat{F}}$. Then $z \in K^{\circ\circ} = K$ and $\langle z, x\rangle = 1$ for all $x \in \widehat{F}$, and in particular $\langle z, u\rangle = 1$, thus $z \in S$. This shows that $\widehat{\widehat{F}} \subset S$ for each exposed face S of K containing F. Since $\widehat{\widehat{F}}$ is itself an exposed face of K containing F, it follows that $\widehat{\widehat{F}}$ is the smallest exposed face containing F. □

Remark 2.1.5 For a proper face F of K that is not an exposed face it follows from Theorem 2.1.4 that $\widehat{\widehat{F}} \neq F$.

By Theorem 2.1.2, each point $x \in K$ belongs to the relative interior of a uniquely determined face F_x of K. This leads to a classification of the points of K according to the dimension of the smallest containing face. The point x is called an *r-extreme* point of K if $\dim F_x \leq r$. Equivalently, x is r-extreme if and only if there is no $(r+1)$-dimensional convex set $A \subset K$ with $x \in \operatorname{relint} A$. Further, the point x is called an *r-exposed* point of K if x is contained in an exposed face $F \in \mathcal{E}(K)$ with $\dim F \leq r$. The set of all the r-extreme (r-exposed) points of K is called its *r-skeleton* (*exposed r-skeleton*) and is denoted by $\operatorname{ext}_r K$ ($\exp_r K$). We have $\operatorname{ext}_0 K = \operatorname{ext} K$, $\exp_0 K = \exp K$; evidently $\exp_r K \subset \operatorname{ext}_r K$ and, trivially, $\operatorname{ext}_r K \subset \operatorname{ext}_s K$ and $\exp_r K \subset \exp_s K$ for $r < s$.

The theorem of Straszewicz, 1.4.7, which states that every extreme point is a limit of exposed points, can be extended to r-extreme and r-exposed points. This extension was obtained by Asplund [95]. His proof, however, is difficult to understand, and we prefer to give a proof due to McMullen [1387]. It requires the following lemma.

Lemma 2.1.6 *Let $G \in \mathcal{F}_r(K)$ (for some $r \in \{0, \ldots, n-1\}$) and $q \in \operatorname{relint} G$. If there is an $(n-r-1)$-flat supporting K precisely at q, then $G \in \mathcal{E}_r(K)$.*

Proof The assumption demands the existence of an $(n-r-1)$-flat L and of a support plane H to K such that $L \subset H$ and $L \cap K = \{q\}$. Let $F := H \cap K$. From $g \in \operatorname{relint} G$ we have $G \subset F$. Suppose that $F \neq G$. Let $x \in F \setminus G$. We may assume that $q = o$. By $L \cap G = \{o\}$ and $o \in$ relint G we have $L \cap \operatorname{lin} G = \{o\}$; from $\dim G = r$ and $\dim L = n-r-1$ we get $L + \operatorname{lin} G = H$. Hence, there is a representation $x = y + z$ with $y \in \operatorname{lin} G$ and $o \neq z \in L$. Since $o \in \operatorname{relint} G$, there is a number $\lambda > 0$ with $-\lambda y \in G$. Then

$$\frac{\lambda}{1+\lambda} z = \frac{\lambda}{1+\lambda} x + \frac{1}{1+\lambda}(-\lambda y) \in F,$$

thus $o \neq [\lambda/(1+\lambda)]z \in L \cap K$, which contradicts the assumption. We conclude that $F = G$ and thus $G \in \mathcal{E}_r(K)$. □

Theorem 2.1.7 *For $r = 0, \dots, n-1$,*

$$\operatorname{ext}_r K \subset \operatorname{cl} \exp_r K.$$

Proof For $r = 0$ this is true by Theorem 1.4.7. Assume that $r \geq 1$ and that the assertion is true for $r-1$ instead of r. Let $p \in \operatorname{ext}_r K$ and a neighbourhood V of p be given. By Theorem 2.1.2 there is a unique face $F \in \mathcal{F}(K)$ such that $p \in \operatorname{relint} F$, and $p \in \operatorname{ext}_r K$ implies $\dim F \leq r$. If $\dim F < r$, then $p \in \operatorname{ext}_{r-1} K$, and by the induction hypothesis we have $p \in \operatorname{cl} \exp_{r-1} K \subset \operatorname{cl} \exp_r K$. Assume, therefore, that $\dim F = r$. Let M be an $(n-r)$-flat through p that is complementary to aff F. Then $p \in \operatorname{ext}(M \cap K)$, hence by Theorem 1.4.7 there is a point $q \in V \cap \exp(M \cap K)$. Let $G \in \mathcal{F}_s(K)$ be the face with $q \in \operatorname{relint} G$. Since $q \in \exp(M \cap K)$ and $q \in \operatorname{relint} G$, we must have $s \leq r$. If $s < r$, then $q \in \operatorname{ext}_{r-1} K$, and the induction hypothesis yields $q \in \operatorname{cl} \exp_{r-1} K \subset \operatorname{cl} \exp_r K$. If $s = r$, let L be an $(n-r-1)$-flat in M that supports $M \cap K$ precisely at q. Lemma 2.1.6 now shows that $G \in \mathcal{E}_r(K)$, hence $q \in \exp_r K$. Since V was an arbitrary neighbourhood of p, the assertion is proved. □

We conclude this section with a simple remark on the topology of the r-skeleton. If $\dim K \leq 2$, it is easy to see that ext K is closed, but for $\dim K \geq 3$ this is not true in general. For example, let K be the convex hull of a circular disc C and of an orthogonal segment S such that there is a point $x \in \operatorname{relbd} C \cap \operatorname{relint} S$. Then x is a limit of extreme points of K but is itself not an extreme point.

It is, however, easy to see that $\operatorname{ext}_r K$ is a G_δ set (an intersection of countably many open sets): for $k \in \mathbb{N}$ let A_k be the set of the centres of all $(r+1)$-dimensional closed balls of radius $1/k$ contained in K. It is not difficult to show that A_k is closed, and obviously

$$\operatorname{ext}_r K = \bigcap_{k \in \mathbb{N}} [(K + B_0(o, 1/k)) \setminus A_k].$$

Notes for Section 2.1

1. *Characterization of sets of extreme points.* If $S = \operatorname{ext} K$ for some $K \in \mathcal{K}^n$, then clearly (a) $\operatorname{cl} S \subset \operatorname{conv} S$, (b) $\operatorname{cl} S$ is compact, (c) $S \cap \operatorname{conv} A \subset A$ for all $A \subset S$. Björck [233] proved that for a nonempty set $S \subset \mathbb{R}^n$ these three conditions are also sufficient for the existence of a convex body $K \subset \mathcal{K}^n$ such that $S = \operatorname{ext} K$, and he considered an infinite-dimensional extension of this result.

2. *A semi-continuity property of skeletons*. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$ converging to $K \in \mathcal{K}^n$. Then, for $r = 0, \ldots, n-1$,

$$\operatorname{cl}\operatorname{ext}_r K \subset \liminf_{i\to\infty} \operatorname{ext}_r K_i.$$

(By definition, $x \in \liminf A_i$ if every neighbourhood of x meets A_i for almost all i.) For a proof, see Schneider [1693], Theorem (2.13) and §8.
3. *The face-function*. Let $K \in \mathcal{K}^n_n$. By Theorem 2.1.2, each point $x \in K$ is contained in the relative interior of a unique face of K; denote this face by F_x. The function $F : x \to F_x$ is called the *face-function* of K; its restriction to $\operatorname{bd} K$ is the face-function of $\operatorname{bd} K$. Klee and Martin [1117] have investigated the semi-continuity properties of the face-function. For $X = K$ or $\operatorname{bd} K$, let X_l (X_u) be the set of all points $x \in X$ at which F is lower (upper) semi-continuous. Klee and Martin studied these sets and showed the following.

Theorem The face-function of $\operatorname{bd} K$ is lower semi-continuous almost everywhere in the sense of category and upper semi-continuous almost everywhere in the sense of measure. However, when $n \geq 3$ it may be lower semi-continuous almost nowhere in the sense of measure and upper semi-continuous almost nowhere in the sense of category.

Actually, upper semi-continuity almost everywhere in the sense of measure for $n > 3$ was only conjectured, but the assertion was completed by Larman [1169], who proved the following.

Theorem The union of the relative boundaries of the proper faces of K of dimension at least one has zero $(n-1)$-dimensional Hausdorff measure.

Lower semi-continuity of face-functions is related to the continuity of so-called envelopes of functions. For a continuous function $f : K \to \mathbb{R}$ one defines the *envelope* f_e by

$$f_e = \sup\{g : g \text{ affine function on } \mathbb{R}^n,\ g \leq f \text{ on } K\}.$$

Let K_e denote the set of all points of K at which every envelope is continuous. Kruskal [1149] gave a three-dimensional example with $K_e \neq K$. Klee and Martin [1117] proved that $K_e = K$ for $n = 2$, and Eifler [536] showed that $K_e = K_l$ (see Note 3) for $K \in \mathcal{K}^n_n$.
4. *Stable convex bodies*. The convex body $K \in \mathcal{K}^n_n$ is called *stable* if the map $(x, y) \mapsto \frac{1}{2}(x+y)$ from $K \times K$ onto K is open. The following assertions are equivalent:

(a) K is stable.
(b) The face-function of K is lower semi-continuous.
(c) All skeletons $\operatorname{ext}_r K$, $r = 0, \ldots, n$, of K are closed.

For the proof, see Papadopoulou [1515] or Debs [473]; the survey by Papadopoulou [1516] explains why stable convex sets are of interest and contains further results. For $n > 2$, the convex bodies K with properties (a), (b), (c) are also characterized by the fact that the metric space consisting of the proper faces of K with the Hausdorff metric is compact. Similarly, the metric space of all exposed faces is compact if and only if all the exposed r-skeletons of K are closed, $0 \leq r \leq n-2$. These results are due to Reiter and Stavrakas [1567].
5. Let $K \in \mathcal{K}^n$, and for $x \in \operatorname{relbd} K$ let S_x be the smallest exposed face of K containing x. In his investigation of the convexity of Chebyshev sets, Brown [348] conjectured the following. If F is an exposed face of K, then there exists an $x \in \operatorname{relbd} F$ such that either $S_x = \{x\}$ or $S_x = F$. He proved this for $n \leq 5$.
6. *Topology of skeletons*. In general, the topological structure of $\operatorname{ext} K$ and $\operatorname{exp} K$ for $K \in \mathcal{K}^n_n$ is not easy to describe. (For the situation in the infinite-dimensional case, see Klee [1105], Bensimon [192].) The following is quoted from Klee [1108] (where $n = 3$): 'Find a useful and simple characterization of the class $\mathcal{X}_n$ of all subsets X of the unit sphere $\mathbb{S}^{n-1}$ such that there is a homeomorphism of $\mathbb{S}^{n-1}$ onto the boundary of a convex body K mapping X into $\operatorname{ext} K$.' The following partial answer was given by Collier [449]. Let X be a subset of a compact zero-dimensional metric space Z, and let $n \geq 3$ be an integer. There is a

homeomorphism of Z into the boundary of a convex body $K \in \mathcal{K}_n^n$ mapping X onto ext K if and only if X is a G_δ set with at least $n + 1$ points. On the other hand, non-trivial examples of subsets of $\mathbb{S}^2$ that do not belong to $\mathcal{X}_3$ can be constructed by means of the following result of Collier [450]. If $K \in \mathcal{K}_3^3$, then each component of $(\mathrm{cl}\,\mathrm{ext}\,K) \setminus \mathrm{ext}\ K$ is a subset of a one-dimensional face of K.

Bronshtein [345] considered the collection $\mathcal{M}_{n,N}$ of sets M in $\mathbb{R}^n$ such that there exist $K \in \mathcal{K}^N$ and a homeomorphism of cl M into K mapping M onto ext K. He showed that $\mathcal{M}_{n,n+1}$ contains all compacta of $\mathbb{R}^n$, and further that each locally compact, bounded set is in $\mathcal{M}_{n,n+2}$ but not necessarily in $\mathcal{M}_{n,n+1}$, and similar results.

The following is known about exp K. For dim $K = 2$, exp K is a G_δ set, but Corson [451] constructed a three-dimensional convex body K for which exp K is not the union of a G_δ set with an F_σ set and contains no dense G_δ set. He also constructed a body $K \in \mathcal{K}_3^3$ for which exp K is the union of a countable number of closed sets each of which has no interior with respect to exp K. Choquet, Corson and Klee [426] proved for $K \in \mathcal{K}_n^n$ that exp K is the union of a G_δ set, an F_σ set and $n - 2$ sets each of which is the intersection of a G_δ set with an F_σ set. For $K \in \mathcal{K}_3^3$ they showed that exp K is the union of a G_δ set and a set that is the intersection of a G_δ and an F_σ set. Answering a question from [426], Holický and Laczkovich [984] constructed a convex body $K \in \mathcal{K}_3^3$ such that exp K is not the intersection of a G_δ and an F_σ set.

Klee [1108] proved for smooth $K \in \mathcal{K}_n^n$ that exp K is a G_δ set and ext $K \setminus$ exp K is a first category F_σ set in ext K. For $n \geq 2$ he showed the existence of a smooth body $K \in \mathcal{K}_n^n$ such that ext K is closed and ext $K \setminus$ exp K is dense in ext K, and he proved the existence of a body $K \in \mathcal{K}_3^3$ such that bd $K \setminus$ ext K, ext $K \setminus$ exp K and exp K are all dense in bd K; see also Edelstein [529].

7. *Paths in the* 1*-skeleton*. The 1-skeleton of a polytope has well-known connectivity properties, which are important, for instance, for the edge-following procedures of the simplex algorithm. Larman and Rogers [1174, 1175] initiated a programme of extending these properties to general convex bodies. In their 1970 paper they proved the following theorem. Let a and b be two distinct exposed points of a convex body $K \in K_n^n$. Then there are n simple arcs $P_1, \dots, P_n$ in the 1-skeleton of K, each joining a to b, such that $P_i \cap P_j = \{a, b\}$ for $1 \leq i < j \leq n$. In their 1971 paper, Larman and Rogers obtained, in two refined versions, the following result on the existence of increasing paths. Let L be a non-constant linear function on $\mathbb{R}^n$ and let $K \in \mathcal{K}_n^n$. Then there is a continuous map s of the closed interval [0,1] to the exposed 1-skeleton of K with
$$L(s(0)) = \inf_{x\in K} L(x), \quad L(s(1)) = \sup_{x\in K} L(x),$$
$$L(s(t_1)) < L(s(t_2)), \text{ when } 0 \leq t_1 < t_2 \leq 1.$$
This line of research was continued by Gallivan [665], Gallivan and Larman [667], and Gallivan and Gardner [666]. Similar investigations for infinite-dimensional compact convex sets were carried out by Larman and Rogers [1176], Larman [1171] and Dalla [460].
8. It was pointed out by Fedotov [561] that a convex body K with $o \in \mathrm{int}\,K$ can have a face F with $\widehat{\widehat{F}} \neq F$ (Remark 2.1.5); this corrects an erroneous statement in Bourbaki [311] (Chapter IV, §1, Exercise 2a).
9. *Hausdorff measures of skeletons*. For a polytope $P \subset \mathbb{R}^n$ and for $r \in \{0, 1, \dots, n\}$, it is natural to consider the total r-dimensional volume, $\eta_r(P)$, of the r-faces of P. Inequalities for these functionals were investigated by Eggleston, Grünbaum and Klee [534] and by Larman and Mani [1173]. The definition
$$\eta_r(K) := \mathcal{H}^r(\mathrm{ext}_r K),$$
where $\mathcal{H}^r$ is the r-dimensional Hausdorff measure, extends the function η_r to arbitrary convex bodies $K \in \mathcal{K}^n$. A different way of extending η_r from polytopes to general convex bodies was proposed by Eggleston, Grünbaum and Klee [534]. They showed that η_r is

lower semi-continuous on the set of n-dimensional polytopes and defined

$$\zeta_r(K) := \liminf_{P \to K} \eta_r(P)$$

for $K \in \mathcal{K}^n$, where the lim inf is taken over all sequences of polytopes converging to K. Then ζ_r is lower semi-continuous on $\mathcal{K}^n$. The comparison of η_r and ζ_r is non-trivial for $1 \leq r \leq n-2$. Burton [361] showed that η_r is lower semicontinuous and that $\zeta_r \geq \eta_r$. It has been asked whether $\zeta_r = \eta_r$ in general (Problem 76 posed by Schneider in [841]). For $r = n-2$ this was proved by Larman [1172], but the other cases remain open.

Eggleston, Grünbaum and Klee [534] also asked for a characterization of those convex bodies $K \in \mathcal{K}^n$ for which $\zeta_r(K) < \infty$. It was observed by Schneider [1693] (p. 19) that a convex body K for which $\zeta_r(K) < \infty$ or $\eta_r(K) < \infty$ has the property that (in the sense of Haar measure) almost every $(n-r)$-flat intersects K in a polytope. This property, however, is not so restrictive as one might expect. Dalla and Larman [462] constructed a convex body $K \in \mathcal{K}^3$ almost all of whose two-dimensional plane sections are polygons, but for which $\operatorname{ext} K$ has Hausdorff dimension one. (This answered in the negative a question posed by Schneider; see Problem 77 in [841].) These authors also showed that a convex body $K \in \mathcal{K}^n$ has $\operatorname{ext}(K \cap E)$ of Hausdorff dimension at most r for almost all k-flats E if and only if the dimension of $\operatorname{ext}_{n-k} K$ is at most $n-k+r$. Burton [361] proved that $\eta_r(K) < \infty$ (where $1 \leq r \leq n-2$) implies $\mathcal{H}^r(\operatorname{ext}_{r-1} K) = 0$. He also obtained results on the facial structure of a convex body K having $\eta_r(K) < \infty$, exhibiting some resemblance to polytopal behaviour.

Burton [362] further found integral-geometric formulae for the functions η_r. The simplest of these says that $\eta_r(K)$ is, up to a factor depending only on n and r, the integral of $\eta_0(K \cap E)$ over all $(n-r)$-flats E, where the integration is with respect to the rigid motion invariant measure on the space of $(n-r)$-flats.

A number of inequalities for the functions η_r are known. Schneider [1688] proved that

$$\eta_r(K) > (n-r+1)V_r(K)$$

for $K \in \mathcal{K}^n_n$ and $r \in \{1, \dots, n-2\}$, where $V_r(K)$ is the intrinsic volume, defined in Section 4.2. This inequality is sharp; that is, the quotient $\eta_r(K)/V_r(K)$ comes arbitrarily close to $n-r+1$ by proper choices of K. A weaker form of the inequality was obtained earlier by Firey and Schneider [611]. For nonnegative integers r and s with $r+s \leq n$, Burton [361] showed that

$$(r+1)(s+1)\eta_{r+s}(K) \leq \eta_r(K)\eta_s(K),$$

which strengthens an inequality obtained by Eggleston, Grünbaum and Klee [534] for polytopes.

A general problem of isoperimetric type can be formulated as follows. For $1 \leq s < r \leq n$ determine the least number $q(n,r,s)$ such that

$$\eta_r(K) \leq q(n,r,s)|\eta_s(K)|^{r/s}$$

for all $K \in \mathcal{K}^n$. For polytopes, this problem was first studied by Eggleston, Grünbaum and Klee [534]. It is not known whether $q(n,r,s) < \infty$ in general. The known results, coming from various sources, are listed in Burton and Larman [363]. The isoperimetric inequality gives the value of $q(n,n,n-1)$. Explicit upper bounds are known for $q(n,r,s)$ when s divides r, and furthermore for $q(n,n-1,s)$, with a better value for $q(3,2,1)$, and for $q(3,3,1)$. Burton and Larman [363] found an upper bound for $q(n,n-2,n-3)$ if $n \geq 4$.

The following inequality of Brunn–Minkowski type was established by Dalla [461]. For $K, L \in \mathcal{K}^n$ and $\lambda \in [0,1]$,

$$\eta_1((1-\lambda)K + \lambda L) \geq (1-\lambda)\eta_1(K) + \lambda\eta_1(L).$$

One may speculate whether $\eta_r((1-\lambda)K+\lambda L)^{1/r}$ is in general a concave function of λ (it is for $r=n$ and $r=n-1$; see Sections 7.1 and 8.2).

Hausdorff measures of finite-dimensional skeletons of infinite-dimensional convex sets were investigated by Dalla [459].

2.2 Singularities

Through each boundary point of a convex body there is a supporting hyperplane, but not necessarily a unique one. Non-uniqueness gives rise to singularities and these will be studied in the present section.

First we introduce some convex cones that describe the behaviour of a convex body at one of its points. Let $K \in \mathcal{K}^n$ be a convex body and let $x \in K$. Define

$$P(K,x) := \{\lambda(y-x) : y \in K,\ \lambda > 0\} = \bigcup_{\lambda>0} \lambda(K-x).$$

Then $P(K,x)$ is a convex cone, and $P(K,x) = \operatorname{aff} K - x$ if and only if $x \in \operatorname{relint} K$. The closed convex cone $S(K,x) := \operatorname{cl} P(K,x)$ can evidently be represented as

$$S(K,x) = \bigcap_{x\in H(K,u)} H^-(K,u) - x,$$

where the intersection extends over all $u \in \mathbb{R}^n \setminus \{o\}$ (or $u \in \mathbb{S}^{n-1}$) for which x lies in the supporting hyperplane $H(K,u)$. (For $x \in \operatorname{int} K$, we adopt the usual convention that the intersection of an empty family of subsets of $\mathbb{R}^n$ is equal to $\mathbb{R}^n$.) The set $S(K,x)$ is called the *support cone* of K at x. (In Bonnesen and Fenchel [284], $S(K,x)+x$ is called the *projection cone* of K at x.) Further, we define the *normal cone* of K at x by

$$N(K,x) := \{u \in \mathbb{R}^n \setminus \{o\} : x \in H(K,u)\} \cup \{o\}.$$

If $x \in K \cap H(K,u)$, then u is called an *outer* or *exterior normal vector* of K at x. Thus $N(K,x)$ consists of all outer normal vectors of K at x together with the zero vector. Clearly, $N(K,x)$ is a closed convex cone and

$$N(K,x) = S(K,x)^\circ; \tag{2.2}$$

that is, $N(K,x)$ and $S(K,x)$ are a pair of dual convex cones. The normal cone is closely related to the metric projection; in fact, it follows from Lemma 1.3.1 that

$$N(K,x) = p_K^{-1}(x) - x \tag{2.3}$$

for $x \in K$. We also remark that

$$\partial I_K^\infty(x) = N(K,x) \qquad \text{for } x \in K, \tag{2.4}$$

as follows immediately from the definitions of the normal cone and of the subdifferential. We may see (2.4) as a counterpart to (1.25).

If K lies in a proper linear subspace E of $\mathbb{R}^n$ and if $S_E(K, x)$ and $N_E(K, x)$ are, respectively, the support cone and normal cone of K at x, taken with respect to the subspace E, then obviously

$$S(K, x) = S_E(K, x),$$

while

$$N(K, x) = N_E(K, x) + E^{\perp}, \tag{2.5}$$

where $E^{\perp}$ is the orthogonal complement of E.

The following theorem describes the behaviour of the support cone and the normal cone under addition and intersection.

Theorem 2.2.1 *Let $K, L \in \mathcal{K}^n$.*

(a) *If $x \in K$ and $y \in L$, then*

$$S(K + L, x + y) = S(K, x) + S(L, y),$$
$$N(K + L, x + y) = N(K, x) \cap N(L, y).$$

(b) *If $x \in K \cap L$ and* $\operatorname{relint} K \cap \operatorname{relint} L \neq \emptyset$*, then*

$$S(K \cap L, x) = S(K, x) \cap S(L, x),$$
$$N(K \cap L, x) = N(K, x) + N(L, x).$$

Proof (a) Performing translations if necessary, we may clearly assume that $x = y = o$. Let $u \in N(K + L, o) \setminus \{o\}$. Then $K + L \subset H^-_{u,0}$. From $K = K + \{o\} \subset H^-_{u,0}$ and $o \in K$ it follows that $u \in N(K, o)$. Similarly we get $u \in N(L, o)$.

Let $u \in N(K, o) \cap N(L, o) \setminus \{o\}$. From $u \in N(K, o)$ we have $K \subset H^-_{u,0}$; similarly $L \subset H^-_{u,0}$ and thus $K + L \subset H^-_{u,0}$. Since $o \in K + L$, we get $u \in N(K + L, o)$. This proves $N(K + L, o) = N(K, o) \cap N(L, o)$.

From (2.2) and Theorem 1.6.9 we now get $S(K + L, o) = S(K, o) + S(L, o)$.

(b) Let $x \in K \cap L$ and $y \in \operatorname{relint} K \cap \operatorname{relint} L$. The first equality of (b) is clear if $x = y$; hence we assume that $x \neq y$. Let $z \in S(K, x) \cap S(L, x)$. Since $y - x \in \operatorname{relint} K - x \subset \operatorname{relint} P(K, x)$ and $z \in \operatorname{cl} P(K, x)$, it follows from Lemma 1.1.9 that $[y - x, z) \subset \operatorname{relint} P(K, x)$. Hence, for $w \in [y - x, z)$ there is a number $\lambda > 0$ such that $[x, x + \lambda w] \subset K$. Similarly, there is a number $\mu > 0$ such that $[x, x + \mu w] \subset L$. It follows that $w \in P(K \cap L, x)$. Since $w \in [y - x, z)$ was arbitrary, we deduce that $z \in \operatorname{cl} P(K \cap L, x) = S(K \cap L, x)$. Thus $S(K, x) \cap S(L, x) \subset S(K \cap L, x)$. The opposite inclusion is trivial. This proves the first equality of (b). The second equality is again obtained by applying (2.2) and Theorem 1.6.9. □

For a point $x \in \operatorname{bd} K$ we denote by S_x the smallest exposed face of K containing x.

Lemma 2.2.2 *Let $K \in \mathcal{K}^n$ and $x, y \in \operatorname{bd} K$. Then $N(K, x) = N(K, y)$ if and only if $S_x = S_y$ (in particular, if $y \in \operatorname{relint} S_x$).*

Proof If $N(K,x) = N(K,y)$, then each exposed face containing x contains y also, and each exposed face containing y contains x. It follows that $S_x = S_y$. Suppose that $S_x = S_y$ (which is clearly satisfied if $y \in \operatorname{relint} S_x$). Let $u \in N(K,x)$. Then $S_x \subset K \cap H(K,u)$, hence $y \in H(K,u)$ and thus $u \in N(K,y)$. The assertion follows. □

We can use this to extend the definition of the normal cone. Let F be a nonempty convex subset of K. Then we define $N(K,F) := N(K,x)$, where $x \in \operatorname{relint} F$. This does not depend on the choice of x: if $F \cap \operatorname{int} K \neq \emptyset$, then $N(K,x) = \{o\}$. Otherwise, let $x,y \in \operatorname{relint} F$. Then S_x is the smallest exposed face of K containing F, hence $S_x = S_y$ and Lemma 2.2.2 gives $N(K,x) = N(K,y)$.

The normal cone $N(K,F)$ is closely related to the conjugate face $\widehat{F}$ (introduced in (2.1)), as shown in the following lemma.

Lemma 2.2.3 *Suppose that $K \in \mathcal{K}^n_{(o)}$ and let F be a nonempty convex subset of K. Then*

$$N(K,F) = \operatorname{pos} \widehat{F} \cup \{o\}.$$

Proof Let $u \in N(K,F) \setminus \{o\}$. Then $H_{v,1}$ with $v = h(K,u)^{-1}u$ is a support plane of K satisfying $x \in K \cap H_{v,1}$ for some point $x \in \operatorname{relint} F$. It follows that $F \subset H_{v,1}$; hence $\langle y, v\rangle \leq 1$ for $y \in K$ and $\langle y, v\rangle = 1$ for $y \in F$. This shows that $v \in \widehat{F}$ and hence $u \in \operatorname{pos} \widehat{F}$. Thus $N(K,F) \subset \operatorname{pos} \widehat{F} \cup \{o\}$. The argument can be reversed. □

It follows from Lemma 2.2.3, (2.5) and Theorem 2.1.2 that distinct normal cones of a convex body have disjoint relative interiors. In particular, we have

$$\operatorname{int} N(K,x) \cap \operatorname{int} N(K,y) = \emptyset \quad \text{for } N(K,x) \neq N(K,y). \tag{2.6}$$

We note that

$$\bigcup_{x \in \operatorname{ext} K} N(K,x) = \mathbb{R}^n, \tag{2.7}$$

since each support set of K contains an extreme point of K.

The normal cones lead, in a natural way, to a classification of the singular boundary points of a convex body. Let $K \in \mathcal{K}^n_n$ be a convex body with interior points and let $x \in \operatorname{bd} K$. We say that x is an *r-singular* point of K if $\dim N(K,x) \geq n - r$. (Bonnesen and Fenchel [284] (p. 14) call x a p-Kantenpunkt (ridge point of order p) if $\dim N(K,x) = n - p$. Thus, x is r-singular if and only if x is a p-Kantenpunkt for some $p \leq r$.) An $(n-2)$-singular point is briefly called *singular*. If the supporting hyperplane to K at x is unique, i.e., if x is not singular, then x is called a *regular* or *smooth* point of K. The convex body K (of dimension n) is called *regular* or *smooth* if all its boundary points are regular. We denote the set of regular boundary points of a convex body $K \in \mathcal{K}^n_n$ by $\operatorname{reg} K$. At each point $x \in \operatorname{reg} K$, there is a unique outer unit normal vector to K, which we denote by $u_K(x)$.

Theorem 2.2.4 *If the convex body $K \in \mathcal{K}^n_n$ has only regular boundary points, then its boundary is a C^1 submanifold of $\mathbb{R}^n$.*

Proof Locally, the boundary bd K can be represented as the graph of a convex function, defined on some open $(n-1)$-dimensional convex domain. It follows from Theorem 1.5.15 that this function is differentiable. The continuity of its partial derivatives follows from Theorem 1.5.4. This yields the assertion. □

The following theorem shows that a convex body cannot have a large set of r-singular boundary points.

Theorem 2.2.5 *Let $K \in \mathcal{K}_n^n$ and $r \in \{0, \dots, n-1\}$. The set of r-singular points of K can be covered by countably many compact sets of finite r-dimensional Hausdorff measure.*

Proof We use the metric projection p_K. Let M_r denote the intersection of a fixed closed ball, containing K in its interior, with the union of all r-dimensional affine subspaces of $\mathbb{R}^n$ that are spanned by points with rational coordinates. Let x be an r-singular point of K. Then the translated normal cone

$$N(K, x) + x = p_K^{-1}(x)$$

is of dimension at least $n-r$. It is, therefore, easy to see that this set meets M_r; hence $x \in p_K(M_r)$. As M_r is a union of countably many compact sets of finite r-dimensional Hausdorff measure and as p_K is a Lipschitz map by Theorem 1.2.1, the assertion follows. □

Theorem 2.2.5 implies that the set of r-singular points of a convex body K has $(r+1)$-dimensional Hausdorff measure zero. In particular, K has at most countably many 0-singular points and the set of singular points has $(n-1)$-dimensional Hausdorff measure zero. In particular, the normal vector $u_K(x)$ exists for $\mathcal{H}^{n-1}$-almost all $x \in \text{bd}\, K$, if $K \in \mathcal{K}_n^n$.

The following consequence of the latter assertion is often useful. A supporting halfspace of K is called *regular* if its boundary contains some regular (smooth) boundary point of K.

Theorem 2.2.6 *A convex body with interior points is the intersection of its regular supporting halfspaces.*

Proof Let $K \in \mathcal{K}_n^n$, and let $x \in \mathbb{R}^n \setminus K$. The body K contains a ball B with positive radius, and $\text{conv}\,(B \cup \{x\})$ intersects bd K in a set of positive $(n-1)$-dimensional Hausdorff measure. This set contains a regular boundary point y of K, and the support plane to K at y separates K and x. The assertion follows. □

The assertion about the countability of the set of 0-singular points can be generalized as follows. By a *perfect* face of $K \in \mathcal{K}_n^n$ one understands a face F of K for which

$$\dim F + \dim N(K, F) = n.$$

Theorem 2.2.7 *A convex body $K \in \mathcal{K}_n^n$ has at most countably many perfect faces.*

Proof If F is a perfect face of K, then the set $p_K^{-1}(\operatorname{relint} F)$ is of dimension n and hence contains a point z with rational coordinates in its interior. Since $p_K(z) \in \operatorname{relint} F$ and different faces have disjoint relative interiors by Theorem 2.1.2, the assertion follows. □

The classification of support planes of a convex body, which we now describe, will in later chapters turn out to be of greater importance than the classification of boundary points. Let $K \in \mathcal{K}^n$ be a convex body. Let $u \in \mathbb{R}^n \setminus \{o\}$. Then u is an outer normal vector of K at each point of the support set $F(K,u) = K \cap H(K,u)$. The normal cone $N(K, F(K,u))$ (which is equal to $N(K,x)$ for any point $x \in \operatorname{relint} F(K,u)$) has a unique face, denoted by $T(K,u)$, that contains u in its relative interior (cf. Theorem 2.1.2). We call $T(K,u)$ the *touching cone* of K at u, since for every vector $v \in \operatorname{relint} T(K,u)$ the support plane $H(K,v)$ touches K at the same set: choose $x \in \operatorname{relint} F(K,u)$ and let D be the flat through x totally orthogonal to $\operatorname{lin} T(K,u)$; then $F(K,v) = D \cap K$ for each $v \in \operatorname{relint} T(K,u)$. In fact, $D \cap K \subset F(K,v)$ is clear. On the other hand, let $y \in F(K,v)$; then $\langle y - x, v\rangle = 0$ and $\langle y - x, t\rangle \le 0$ for each $t \in T(K,u)$. Let $t \in T(K,u)$. Since $v \in \operatorname{relint} T(K,u)$, there are a vector $w \in T(K,u)$ and a number $\lambda > 0$ such that $\lambda v = t + w$. We get $0 = \langle y - x, \lambda v\rangle = \langle y - x, t + w\rangle$ and hence $\langle y - x, t\rangle = 0$. This shows that $y \in D$.

The vector u will be called an *r-extreme normal vector* of K, and $H(K,u)$ an *r-extreme support plane* of K, if $\dim T(K,u) \le r + 1$ ($r = 0, \dots, n-1$). Thus u is an r-extreme normal vector of K if and only if there do not exist $r + 2$ linearly independent normal vectors $u_1, \dots, u_{r+2}$ at one and the same boundary point of K such that $u = u_1 + \cdots + u_{r+2}$. For 0-extreme we say *extreme*.

Remark 2.2.8 The set D is the affine hull of the set which in Bonnesen and Fenchel [284] (p. 16) is denoted by $\mathcal{D}_m$ and is found in a different way. (The construction described there can be interpreted as a dual way of finding the minimal face $T(K,u)$ of $N(K, F(K,u))$ containing u by successively determining the minimal exposed face containing u.) Bonnesen and Fenchel call $H(K,u)$ a p−Kantenstützebene if $\dim D = p$. Thus, $H(K,u)$ is a p−Kantenstützebene if and only if it is an $(n-p-1)$-extreme, but not an $(n-p-2)$-extreme, support plane of K.

Further, the vector u is called an *r-exposed normal* vector of K, and $H(K,u)$ an *r-exposed support plane* of K, if $\dim N(K, F(K,u)) \le r + 1$.

The choice of notation suggests an underlying duality. In fact, let $K \in \mathcal{K}^n_{(o)}$. A given support plane H of K can be written as $H = H_{u,1}$ with $u \in \mathbb{R}^n \setminus \{o\}$. Writing $F(K,u) = F$, we have $N(K,F) = \operatorname{pos} \widehat{F}$ by Lemma 2.2.3 and $u \in \widehat{F}$. Hence, the touching cone $T(K,u)$ is the smallest face of $\operatorname{pos} \widehat{F}$ containing u, and this is the positive hull of the smallest face of $\widehat{F}$ containing u. It follows that u is an r-extreme point of K° if and only if u is an r-extreme normal vector of K. If u is an r-exposed normal vector of K, then $\dim \operatorname{pos} \widehat{F} = \dim N(K,F) \le r + 1$, hence $\dim \widehat{F} \le r$. Thus, u is contained in the exposed face $\widehat{F}$ of dimension at most r and is, therefore, an r-exposed point of K°. Vice versa, let u be an r-exposed point of

K°. Since $\widehat{u} = F(K,u) = F$ (where the circumflex refers to K°), $\widehat{F} = \widehat{\widehat{u}}$ is, by Theorem 2.1.4, the smallest exposed face of K° containing u. Hence, $\dim \widehat{F} \leq r$ and thus $\dim N(K,F) \leq r+1$; that is, u is an r-exposed normal vector of K.

The preceding observation leads us to a dual version of Theorem 2.1.7.

Theorem 2.2.9 *Let $K \in \mathcal{K}^n$ and $r \in \{0,\dots,n-1\}$. Then each r-extreme support plane of K is a limit of r-exposed support planes of K.*

Proof If $\dim K = n$, we may assume that $o \in \operatorname{int} K$ and then apply Theorem 2.1.7 to K° to get the assertion. Suppose that $\dim K = k < n$. Let $u \in \mathbb{R}^n \setminus \{o\}$ be a given vector. We may assume that $o \in K$; let $E = \operatorname{lin} K$ and write $u = u_0 + u_1$ with $u_0 \in E$ and $u_1 \in E^\perp$. If u is an r-extreme normal vector of K and $u_0 = o$, then u is an r-exposed normal vector of K, since $T(K,u) = N(K,F(K,u)) = E^\perp$ in this case. Suppose that $u_0 \neq o$. Then $F(K,u) = F(K,u_0) = F$, say. Let $N_E(K,F) \subset E$ be the normal cone of K at F and $T_E(K,u_0) \subset E$ the touching cone of K at u_0, both with respect to E. Then

$$N(K,F) = N_E(K,F) + E^\perp, \quad T(K,u) = T_E(K,u_0) + E^\perp.$$

It is now clear that the vector u is an r-extreme (r-exposed) normal vector of K if and only if u_0 is an $(r-n+k)$-extreme ($(r-n+k)$-exposed) normal vector of K. Thus, the assertion of the theorem is obtained in an obvious way if the result already proved is applied in E. $\square$

The classification of support planes plays a role since certain problems in later chapters lead to pairs of convex bodies that have some, but not all, support planes in common. In particular, the convex body $K \in \mathcal{K}^n$ containing the convex body $L \in \mathcal{K}^n$ is called a *p-tangential body* of L if each $(n-p-1)$-extreme support plane of K is a support plane of L ($p \in \{0,\dots,n-1\}$). Thus, a 0-tangential body of L is the body L itself, and each p-tangential body of L is also a q-tangential body for $p < q \leq n-1$. An $(n-1)$-tangential body is briefly called a *tangential body*. The following theorem characterizes p-tangential bodies in a slightly different way.

Theorem 2.2.10 *Let $K, L \in \mathcal{K}^n$, $L \subset K$, and $p \in \{0,\dots,n-1\}$. Then K is a p-tangential body of L if and only if the following condition holds.*

(*) *Each support plane of K that is not a support plane of L contains only $(p-1)$-singular points of K.*

Proof Suppose that K is a p-tangential body of L. Let H be a support plane of K and assume that H contains a point x of K that is not $(p-1)$-singular, which means that $\dim N(K,x) \leq n-p$. If u is the outer normal vector of H, then $N(K,F(K,u)) \subset N(K,x)$, hence $\dim T(K,u) \leq n-p$, so that u is an $(n-p-1)$-extreme normal vector of K. Therefore, H is a support plane of L. Thus condition $(*)$ holds.

Assume that $(*)$ holds. Let H be an $(n-p-1)$-extreme support plane of K. By Theorem 2.2.9, H is the limit of a sequence $(H_i)_{i\in\mathbb{N}}$ of $(n-p-1)$-exposed support planes of K. Each H_i contains a point x_i of K with $\dim N(K,x_i) \leq n-p$; that is, x_i is not $(p-1)$-singular. By condition $(*)$, H_i is a support plane of L. Since this is true for all i, the hyperplane H supports L. This proves that K is a p-tangential body of L. $\square$

If K is a 1-tangential body of L, then each point of K in a support plane of K that does not support L is a 0-singular point of K. There are at most countably many such points, and K is clearly the convex hull of L and these points. If x and y are any two of these 0-singular points, then $[x, y]$ meets L, since otherwise one finds a support plane of K not supporting L but containing a point of K that is not 0-singular. For intuitive reasons that are now evident, 1-tangential bodies are also called *cap-bodies*.

The notion of p-extreme normal vector of a convex body admits a generalization that will be useful in Section 7.6. Let $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ be $n-1$ convex bodies. We say that the vector $u \in \mathbb{R}^n \setminus \{o\}$ is $(K_1, \dots, K_{n-1})$-*extreme* if there exist $(n-1)$-dimensional linear subspaces $E_1, \dots, E_{n-1}$ of $\mathbb{R}^n$ such that

$$T(K_i, u) \subset E_i \quad \text{for } i = 1, \dots, n-1$$

and

$$\dim (E_1 \cap \dots \cap E_{n-1}) = 1.$$

In particular, u is a p-extreme normal vector of the convex body K if and only if u is

$$(\underbrace{K, \dots, K}_{n-1-p}, \underbrace{B^n, \dots, B^n}_{p})\text{–extreme.}$$

Here, of course, the p-tuple of unit balls B^n may be replaced by any other p-tuple of regular convex bodies.

We could also dualize the notion of r-singular boundary points in a similar way to that in which we dualized the notions of p-extreme and p-exposed boundary points. We need only the following case. Let $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$. The vector u is called a *regular normal vector* of K if $\dim F(K, u) = 0$; otherwise u is called *singular*. Thus, $u \in \mathbb{S}^{n-1}$ is a singular normal vector of K if and only if $H(K, u) \cap K$ contains a segment. A convex body is called *strictly convex* if its boundary does not contain a segment. We denote the set of regular normal vectors of K by $\operatorname{regn} K$.

Theorem 2.2.11 *The set of singular unit normal vectors of a convex body $K \in \mathcal{K}^n$ has $(n-1)$-dimensional Hausdorff measure zero.*

Proof Obviously we may assume that $\dim K = n$ and $o \in \operatorname{int} K$. Let u be a singular unit normal vector of K. Then $H(K, u)$ contains a segment F. The point $x := h(K, u)^{-1} u$ is a boundary point of the polar body K°, and $F \subset \widehat{x}$ (where the circumflex refers to K°). From Lemma 2.2.3 it follows that $N(K^\circ, x) \supset F$, hence x is a singular boundary point of K°. Thus u is contained in the image M of the set of singular boundary points of K° under the radial projection from $\operatorname{bd} K^\circ$ to $\mathbb{S}^{n-1}$. Since this radial projection is obviously a Lipschitz map, it follows from Theorem 2.2.5 that M has $(n-1)$-dimensional Hausdorff measure zero. $\square$

It is now in order to define spherical images and reverse spherical images. Let $K \in \mathcal{K}^n$. For a subset $\beta \subset \mathbb{R}^n$ we denote by $\sigma(K,\beta)$ the set of all outer unit normal vectors of K at points of β, thus

$$\sigma(K,\beta) := \bigcup_{x\in K\cap\beta} N(K,x) \cap \mathbb{S}^{n-1},$$

and we call this the *spherical image of* K at β. Similarly, for a subset $\omega \subset \mathbb{S}^{n-1}$ we denote by $\tau(K,\omega)$ the set of all boundary points of K at which there exists a normal vector of K belonging to ω, thus

$$\tau(K,\omega) := \bigcup_{u\in\omega} F(K,u).$$

We may call this the *reverse spherical image of* K at ω. The *spherical image map of* K is the map

$$u_K : \operatorname{reg} K \to \mathbb{S}^{n-1}$$

defined by letting $u_K(x)$, for $x \in \operatorname{reg} K$, be the unique outer unit normal vector of K at x. The *reverse spherical image map of* K (not the inverse of u_K, which does not exist in general) is the map

$$x_K : \operatorname{regn} K \to \operatorname{bd} K$$

for which $x_K(u)$, for $u \in \operatorname{regn} K$, is the unique point in $F(K,u)$.

Lemma 2.2.12 *The spherical image map u_K and the reverse spherical image map x_K of a convex body K are continuous. If K is smooth and strictly convex, then u_K is a homeomorphism from* bd K *to $\mathbb{S}^{n-1}$, and x_K is its inverse.*

Proof Let $x_j, x \in \operatorname{reg} K$ and $x_j \to x$ for $j \to \infty$. There exists a convergent subsequence $(u_K(x_{j_k}))_{k\in\mathbb{N}}$; let $u_k := u_K(x_{j_k})$ and $u := \lim_{k\to\infty} u_K(x_{j_k})$. Then $\langle x_{j_k}, u_k\rangle = 0$, hence $\langle x, u\rangle = 0$, and $\langle y, u_k\rangle \le 0$, thus $\langle y, u\rangle \le 0$, for each $y \in K$, which means that $u = u_K(x)$. Since every convergent subsequence of the sequence of $(u_K(x_j))_{j\in\mathbb{N}}$ converges to $u_K(x)$, the map u_K is continuous. In a similar way one proves that the map x_K is continuous. The remaining assertion is clear. □

Since we want to be able to speak of the Lebesgue measure of the spherical image $\sigma(K,\beta)$, we show that it exists for a sufficiently large family of sets β.

Lemma 2.2.13 *Let $K \in \mathcal{K}^n$ and let $\beta \subset \mathbb{R}^n$ be a Borel set. Then the spherical image $\sigma(K,\beta)$ of K at β is Lebesgue measurable on $\mathbb{S}^{n-1}$.*

Proof Let $\mathcal{M}$ be the family of all subsets $\beta \subset \mathbb{R}^n$ for which the set $\sigma(K,\beta)$ is Lebesgue measurable. First let β be closed. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $\sigma(K,\beta)$ for which $u_j \to u$ for $j \to \infty$. For $j \in \mathbb{N}$, we can choose $x_j \in K \cap \beta$ such that $u_j \in N(K, x_j)$. Since $K \cap \beta$ is compact, a subsequence of $(x_j)_{j\in N}$ converges to a point $x \in K \bigcap \beta$, and from $u_j \to u$ it follows that $u \in N(K, x)$, hence $u \in \sigma(K,\beta)$. Thus $\sigma(K,\beta)$ is closed and hence $\beta \in \mathcal{M}$. Next let $\beta \in \mathcal{M}$ and $\beta^c := \mathbb{R}^n \setminus \beta$. Suppose that

$u \in \sigma(K,\beta) \cap \sigma(K,\beta^c)$. Then $F(K,u)$ contains a point of β and a point of β^c, hence u is a singular normal vector of K. From Theorem 2.2.11 we deduce that

$$\mathcal{H}^{n-1}(\sigma(K,\beta) \cap \sigma(K,\beta^c)) = 0.$$

Hence, the set $\sigma(K,\beta^c)$ differs from the set $\mathbb{S}^{n-1} \setminus \sigma(K,\beta)$, which is Lebesgue measurable, only by a set of (spherical Lebesgue) measure zero. It follows that $\sigma(K,\beta^c)$ is Lebesgue measurable and thus $\beta^c \in \mathcal{M}$. Finally, it is clear that $\sigma(K,\bigcup_i \beta_i) = \bigcup_i \sigma(K,\beta_i)$ for any family (β_i). We have shown that $\mathcal{M}$ is a σ-algebra containing the closed sets. Hence, $\mathcal{M}$ contains all Borel subsets of $\mathbb{R}^n$, as we set out to prove. □

To formulate an analogous result for the reverse spherical image $\tau(K,\omega)$, we define the *radial map*

$$f : \mathbb{R}^n \setminus \{o\} \to \mathbb{S}^{n-1} \quad \text{by} \quad f(x) := \frac{x}{|x|}.$$

Lemma 2.2.14 *Let $K \in \mathcal{K}^n_{(o)}$, and let $\omega \subset \mathbb{S}^{n-1}$ be a Borel set. Then $f(\tau(K,\omega))$ is Lebesgue measurable on $\mathbb{S}^{n-1}$.*

The proof, which uses Theorem 2.2.5, is so similar to that of Lemma 2.2.13 that we can omit it.

Notes for Section 2.2

1. Some considerations using the first part of Theorem 2.2.1 can be found in Jongmans [1051], for instance the remark that the sum of a polytope and a non-smooth convex body cannot be smooth, and some related results.
2. Theorem 2.2.5 is essentially due to Anderson and Klee [69]. For $n = 3$, a different proof was given by Besicovitch [214]. A generalization to certain non-convex sets can be found in Federer [556] (4.15(3) on p. 447). A weaker form of Theorem 2.2.5 was claimed without proof by Favard [553], p. 228. That the set of singular boundary points of a convex body $K \in \mathcal{K}^n_n$ is of $(n-1)$-dimensional Hausdorff measure zero was first proved by Reidemeister [1562]; see also Aleksandrov [23] (chap. V, §2) for an equivalent result.

   Theorem 2.2.5 implies, in particular, that the set of regular points of a closed convex set with interior points is dense in the boundary of the set. This is also true in arbitrary separable Banach spaces (Mazur [1367]; see also Marti [1331], p. 112).
3. *Refined information on sets of singular points.* A stronger, and in a sense optimal, version of Theorem 2.2.5 was obtained by Zajíček [2027], who showed that the countably many sets covering the r-singular points can be chosen to be k-dimensional surfaces which have a special representation in terms of differences of convex functions. Further results of this kind, with stronger regularity properties for the covering sets, and partially concerning the analogous notions for convex functions, are due to Alberti, Ambrosio and Cannarsa [10], Alberti [9], Colesanti and Pucci [446].

   By taking into account the strength of the singularities, quantitative improvements of Theorem 2.2.5 are possible. For $K \in \mathcal{K}^n_n$, $r \in \{0,\dots,n-1\}$, and $\tau > 0$, let $\Sigma^r(K,\tau)$ denote the set of all r-singular points x of K for which $\mathcal{H}^{n-1-r}(N(K,x) \cap \mathbb{S}^{n-1}) \geq \tau$. Extending a three-dimensional result of Colesanti and Pucci [446], Hug [1004] showed that

$$\mathcal{H}^r(\Sigma^r(K,\tau)) \leq \frac{n}{\tau}\binom{n-1}{r} W_{n-r}(K), \tag{2.8}$$

   where $W_0(K),\dots,W_n(K)$ are the quermassintegrals of K (see Section 4.2). This estimate is sharp; equality holds, for example, for regular polytopes and suitable τ. For a more general result, from which (2.8) is deduced, see Note 4 in Section 4.5. Hug also obtained an analogous estimate for sets of singular normal vectors.

4. The special case of Theorem 2.2.7 for one-dimensional faces was first proved by Fujiwara [649]. The general case appears (with different proofs) in Bourbaki [311] (chap. IV, §1, Exercise 2e) and in Brøndsted [336]. The latter author introduced the term 'perfect' face, while Bourbaki speaks of 'regular' faces.

   Fedotov [561] called a face F of $K \in \mathcal{K}^n$ *isolated at* a point $x \in F$ if there exists a neighbourhood U of x for which
   $$U \cap \operatorname{ext}_{\dim F} K = U \cap F.$$
   If this holds, then he showed that x is an internal point of F, that F is isolated at each of its internal points, and that F is a perfect face.

5. Fedotov [559] made a study of faces and of extreme and exposed points in relation to normal vectors and obtained, for example, the following results. Let $K \in \mathcal{K}^n_n$ and $\beta \subset \mathbb{R}^n$. If $x \in \exp_i K$ and β is a neighbourhood of x, then the spherical image $\sigma(K,\beta)$ of K at β contains an $(n-i-1)$-dimensional submanifold. Conversely, if β is open and the spherical image of β contains an $(n-i-1)$-dimensional Lipschitz submanifold, then $\beta \cap \exp_i K \neq \emptyset$.

6. The touching cone $T(K,u)$ of a convex body K at u was introduced here as the smallest face of the normal cone $N(K, F(K,u))$ containing u. Defining the touching cones of K as the (nonempty) faces of the normal cones of K, the cone $T(K,u)$ is thus the unique touching cone of K containing u in its relative interior. Weis [1952] made a systematic study of the lattices of normal cones and touching cones and their relations with the lattices of exposed faces and faces of a (not necessarily closed) convex set in $\mathbb{R}^n$.

7. Lemma 2.2.13 appears (essentially) in Aleksandrov [23] (chap. V, §2) and Lemma 2.2.14 in Aleksandrov [12], §2. A more detailed study of spherical images can be found in Aleksandrov [19], §5.

8. *Minkowski's characterization of extreme support planes*. Let $K \in \mathcal{K}^n_n$ and $u \in \mathbb{R}^n \setminus \{o\}$. For small $\delta > 0$, let $r(\delta)$ be the radius of the largest $(n-1)$-dimensional ball contained in $K \cap H_{u,h(K,u)-\delta}$. Then u is an extreme normal vector of K if and only if $r(\delta)/\delta \to \infty$ for $\delta \to 0$. This was proved by Minkowski [1441] for $n = 3$ and extended to higher dimensions by Favard [553].

9. *Tangential bodies*. The definition of a p-tangential body given above is essentially the one in Bonnesen and Fenchel [284], p. 14. Favard [553], p. 273, used property $(*)$ of Theorem 2.2.10 as a definition (and talked of a tangential body of order $n-p$). The equivalence expressed by Theorem 2.2.10 was proved in Schneider [1689].

10. The convex cone $P(K,x)$ defined at the beginning of Section 2.2 is often denoted by cone (x,K) and called the *cone of K at x*. Above, we were only interested in its closure, but of course the cone $P(K,x)$ itself carries more information about the behaviour of K at its point x. Let $C \subset \mathbb{R}^n$ be a convex cone. There exist a convex body $K \in \mathcal{K}^n$ and a point $x \in K$ such that $P(K,x) = C$ if and only if C is an F_σ-set. This was proved by Sung and Tam [1826]. Their work is also related to earlier work by Larman [1170] and Brøndsted [337] on certain cones (inner aperture, barrier cone) associated with an unbounded convex set, and to a certain classification of boundary points of convex bodies by Waksman and Epelman [1902].

2.3 Segments in the boundary

We have called a convex body *strictly convex* if its boundary does not contain a segment. As the example of a polytope shows, the boundary of a convex body can contain many segments. The directions of these segments, however, make up a relatively small set; in the example it is confined to finitely many great spheres (where directions are represented by elements of the unit sphere $\mathbb{S}^{n-1}$). It is easy to construct convex bodies where the set of directions of segments in the boundary is dense in $\mathbb{S}^{n-1}$, but still of $\mathcal{H}^{n-1}$-measure zero. The latter holds true in general, but no easy proof seems to be known. With some effort, however, the following stronger result can be shown.

Theorem 2.3.1 (Ewald, Larman, Rogers) *Let $K \in \mathcal{K}^n$ be a convex body and let $U(K) \subset \mathbb{S}^{n-1}$ be the set of all unit vectors that are parallel to a segment in the boundary of K. Then $U(K)$ has σ-finite $(n-2)$-dimensional Hausdorff measure.*

Since the set of unit vectors parallel to a segment in a facet of K has positive $(n-2)$-dimensional Hausdorff measure and since a convex body can have infinitely many facets, the result of Theorem 2.3.1 is best possible.

Theorem 2.3.1 can be considered as one of the most remarkable results on the boundary structure of general convex bodies. For this reason, and since a generalization to be treated below will be applied in Chapter 4, we shall give the full proof, essentially following Ewald, Larman and Rogers [542], with a variant at the end of the proof that was proposed by Zalgaller [2028].

An important tool, which is useful in other contexts too, is the following cap covering theorem due to Ewald, Larman and Rogers. Recall that a cap of a convex body $K \in \mathcal{K}^n_n$ is an n-dimensional set of the form $K \cap H^+$ where H^+ is a closed halfspace. For given $r, R > 0$, we denote by $\mathcal{K}^n(r, R)$ the set of all convex bodies $K \in \mathcal{K}^n$ satisfying

$$rB^n + x \subset K \subset RB^n + x$$

for some $x \in K$.

Theorem 2.3.2 (Ewald, Larman, Rogers) *For $K \in \mathcal{K}^n(r, R)$ there exist positive constants $c_1, \dots, c_4$ depending only on n, r, R such that the following holds. For $0 < \varepsilon < c_1$, there exist caps $K_1, \dots, K_m$ of K whose union covers the boundary of K and whose widths $\Delta(K_i)$ and volumes $V_n(K_i)$ satisfy*

$$c_2\varepsilon < \Delta(K_i) < c_3\varepsilon \quad \text{for } i = 1, \dots, m,$$

$$\sum_{i=1}^{m} V_n(K_i) < c_4\varepsilon.$$

The proof will be split into a sequence of lemmas. The first of these is well known (Bonnesen and Fenchel [284], p. 53) and quoted here without proof.

Lemma 2.3.3 *If $K \in \mathcal{K}^n_n$ and o is the centroid of K, then $-K \subset nK$.*

For $K \in \mathcal{K}^n$, $x \in K$, and $\lambda > 0$ the convex body

$$M(x, \lambda) := x + \lambda[(K - x) \cap (x - K)]$$

is called a *Macbeath region*.

We assume in the following that a convex body $K \in \mathcal{K}^n(r, R)$ is given and, without loss of generality, that

$$rB^n \subset K \subset RB^n.$$

In particular, the origin o is a point of K.

Lemma 2.3.4 *If $x, y \in K$ and $M\left(x, \frac{1}{2}\right) \cap M\left(y, \frac{1}{2}\right) \neq \emptyset$, then $M(y, 1) \subset M(x, 5)$.*

Proof A point $z \in M\left(x, \frac{1}{2}\right) \cap M\left(y, \frac{1}{2}\right)$ is of the form

$$z = x + \tfrac{1}{2}(x - k_1) = y + \tfrac{1}{2}(k_2 - y) \tag{2.9}$$

with $k_1, k_2 \in K$. Let $w \in M(y, 1)$. Then

$$w = y + (k_3 - y) = y + (y - k_4)$$

with suitable $k_3, k_4 \in K$; hence

$$w = k_3 = x + 5\left[\left(\tfrac{4}{5}x + \tfrac{1}{5}k_3\right) - x\right] \in x + 5(K - x)$$

and, using (2.9),

$$\begin{aligned} w = y + (y - k_4) &= 6x - 2k_1 - 2k_2 - k_4 \\ &= x + 5\left[x - \left(\tfrac{2}{5}k_1 + \tfrac{2}{5}k_2 + \tfrac{1}{5}k_4\right)\right] \\ &\in x + 5(x - K), \end{aligned}$$

hence $w \in M(x, 5)$. □

Lemma 2.3.5 *Let $H^+ = H^+_{u,\alpha}$ be a closed halfspace meeting K but not rB^n and such that the distance between its bounding hyperplane H and the parallel support plane $H(K, u)$ is at most $\frac{1}{2}r$. Let c be the centre of gravity (with respect to H) of $K \cap H$. Then $K \cap H^+ \subset M(c, 3n)$.*

Proof We suppose that the assertion is false. Since $K \subset c + 3n(K - c)$, there must exist a point $k \in K \cap H^+$ with $k \notin c + 3n(c - K)$. Writing $\eta = 1/(3n)$, we have

$$c - \eta(k - c) \notin K. \tag{2.10}$$

We may assume that $|u| = 1$. The point $k \in K \cap H^+$ can be written in the form $k = h + tu$ with $h \in H$ and $0 \leq t \leq \frac{1}{2}r$. We can also write $c = su + z$ with $\langle z, u\rangle = 0$. Since $rB^n \subset \mathbb{R}^n \setminus H^+$, we have $s \geq r \geq 2t$. Define

$$v := \frac{s}{s + t}(h - c) = \frac{t}{s + t}z,$$

then $\langle v, u\rangle = 0$ and

$$c + v = \frac{s}{s + t}k \in K \cap H. \tag{2.11}$$

On the other hand, using (2.11),

$$c - \frac{\eta(s+t)}{s-\eta t} v = \frac{s}{s-\eta t}[c - \eta(k-c)]$$

belongs to H but not to K, by (2.10) and $\eta t < \frac{1}{2} r < s$. Since c is the centre of gravity of $K \cap H$, Lemma 2.3.3 now implies that

$$\frac{\eta(s+t)}{s-\eta t} > \frac{1}{n-1} > \frac{1}{n}.$$

Using $\eta t \leq t \leq \frac{1}{2} r \leq \frac{1}{2} s$, we arrive at

$$\frac{1}{3n} = \eta > \frac{1}{n} \frac{s-\eta t}{s+t} \geq \frac{1}{n} \frac{s - \frac{1}{2}s}{s + \frac{1}{2}s} = \frac{1}{3n},$$

a contradiction. □

For $0 < \varepsilon < r$ we put

$$K_{-\varepsilon} := \bigcap_{u \in \mathbb{S}^{n-1}} [H^-(K,u) - \varepsilon u]$$

(this is the inner parallel body of K at distance ε, cf. Section 3.1).

Lemma 2.3.6 *If* $0 < \varepsilon < \frac{1}{4} r^2 R^{-1}$, *then*

$$(1 - 4r^{-2} R\varepsilon) K \subset K_{-\varepsilon} \subset K$$

and

$$V_n(K \setminus K_{-\varepsilon}) < 4nr^{-2} R V_n(K) \varepsilon.$$

Proof Let $x \in \operatorname{bd} K_{-\varepsilon}$. The nonnegative function $h(K - x, \cdot) - \varepsilon$ attains a minimum on $\mathbb{S}^{n-1}$, say at u. This minimum is equal to 0, since otherwise $x \in \operatorname{int} K_{-\varepsilon}$. Hence, $x \in H(K,u) - \varepsilon u$ and, therefore, $|x| \geq \langle x, u \rangle = h(K,u) - \varepsilon \geq r - \varepsilon > \frac{1}{2} r$ (observe that $\varepsilon < \frac{1}{4} r$ since $r < R$). Since $|x| < R$, the angle θ between u and the vector x satisfies $\cos\theta \geq r/(2R)$. Let $x' = \xi x$ with $\xi > 0$ and $x' \in \operatorname{bd} K$. Then $|x' - x| \cos\theta \leq \varepsilon$ and hence $|x' - x|/|x| \leq 4r^{-2} R\varepsilon$. Thus

$$(1 - 4r^{-2} R\varepsilon) K \subset K_{-\varepsilon} \subset K$$

and

$$\begin{aligned} V_n(K \setminus K_{-\varepsilon}) &\leq V_n(K \setminus (1 - 4r^{-2} R\varepsilon) K) = [1 - (1 - 4r^{-2} R\varepsilon)^n] V_n(K) \\ &< 4nr^{-2} R\varepsilon V_n(K). \end{aligned}$$

□

Lemma 2.3.7 *Let* $0 < \varepsilon < r$ *and* $x \in \operatorname{bd} K_{-\varepsilon}$. *Then*

$$B(x, \varepsilon) \subset M(x, 1) \quad \textit{and} \quad \Delta(M(x,1)) = 2\varepsilon.$$

If $\varepsilon < (R/n)[r/(2R)]^{2n}$, there is a cap C^0 of K, bounded by a hyperplane through x, so that

$$C^0 \subset M(x, 3n).$$

Proof From the definition of $K_{-\varepsilon}$ it is clear that $B(x, \varepsilon) \subset K$ and thus $B(x, \varepsilon) \subset M(x, 1)$. This implies $\Delta(M(x, 1)) \geq 2\varepsilon$. With u chosen as in the proof of the preceding lemma, we have

$$M(x, 1) \subset H^-(K, u) \cap [x - (H^-(K, u) - x)].$$

The right-hand side is a strip bounded by two parallel hyperplanes at a distance 2ε apart, hence $\Delta(M(x, 1)) = 2\varepsilon$.

Among all caps of K bounded by hyperplanes through x, there is one with minimal volume, say C^0. The special cap $K \cap [H^+(K, u) - \varepsilon u]$, with u as above, can be covered by a cylinder of height ε whose base is an $(n-1)$-ball of radius R, hence

$$V_n(C^0) \leq \kappa_{n-1} R^{n-1} \varepsilon. \tag{2.12}$$

There is a point $y \in C^0$ lying in a support hyperplane $\bar{H}$ of K parallel to the bounding hyperplane H of C^0 through x. Let t be the distance of x from $\bar{H}$ and τ the distance of o from $\bar{H}$. The convex body K contains the convex hull of y and the intersection of rB^n with the hyperplane through o parallel to $\bar{H}$. The cap C^0 contains a set similar to this convex hull, with similarity ratio t/τ. Hence,

$$V_n(C^0) \geq \left(\frac{t}{\tau}\right)^n \frac{1}{n} \kappa_{n-1} r^{n-1} \tau,$$

which together with (2.12) and $\tau \leq R$ yields

$$t^n \leq \frac{nR^{2n-2}}{r^{n-1}} \varepsilon.$$

If now $\varepsilon < (R/n)[r/(2R)]^{2n}$, this yields $t < \frac{1}{4}r$. In particular, the hyperplane H does not meet the ball $\frac{1}{2}rB^n$. From the minimum property of C^0 it follows that x is the centre of gravity of $K \cap H$; otherwise a suitable small rotation of H around x would yield a cap with smaller volume. We can now apply Lemma 2.3.5, with r replaced by $\frac{1}{2}r$, to conclude that $C^0 \subset M(x, 3n)$. □

We are now in a position to prove the cap covering theorem of Ewald, Larman and Rogers.

Proof of Theorem 2.3.2 Put $c_1 = (R/n)[r/(2R)]^{2n}$ (which is the constant appearing in Lemma 2.3.7), and let $0 < \varepsilon < c_1$. Choose points $x_1, \ldots, x_m \in \operatorname{bd} K_{-\varepsilon}$ with

$$M\left(x_i, \tfrac{1}{2}\right) \cap M\left(x_j, \tfrac{1}{2}\right) = \emptyset \quad \text{for } i \neq j \tag{2.13}$$

and such that m is maximal (clearly a finite m exists, since $V_n\left(M\left(x_i, \frac{1}{2}\right)\right) \geq V_n\left(B\left(x_i, \frac{1}{2}\varepsilon\right)\right)$ by Lemma 2.3.7).

Let $z \in \operatorname{bd} K$, and let $y \in [o, z] \cap \operatorname{bd} K_{-\varepsilon}$. As in the proof of Lemma 2.3.6 and by the choice of ε,

$$|z - y| < 4r^{-2}R\varepsilon|y| < 4r^{-2}R^2\varepsilon < \tfrac{1}{4}r,$$

which implies that $y-(z-y) \in K$ and hence $z \in M(y, 1)$. By the maximality property of $x_1, \dots, x_m$, there is an index $i \in \{1, \dots, m\}$ for which

$$M\left(x_i, \tfrac{1}{2}\right) \cap M\left(y, \tfrac{1}{2}\right) \neq \emptyset.$$

Lemma 2.3.4 now shows that $z \in M(y, 1) \subset M(x_i, 5)$. Since $z \in \operatorname{bd} K$ was arbitrary, we have proved that

$$\operatorname{bd} K \subset \bigcup_{i=1}^{m} M(x_i, 5). \tag{2.14}$$

Let $i \in \{1, \dots, m\}$. The point x_i lies in the boundary of a supporting halfspace H_i^- of $K_{-\varepsilon}$. Clearly,

$$M\left(x_i, \tfrac{1}{2}\right) \setminus H_i^- \subset K \setminus K_{-\varepsilon}$$

and hence, by (2.13),

$$\sum_{i=1}^{m} V_n\left(M\left(x_i, \tfrac{1}{2}\right) \setminus H_i^-\right) \leq V_n(K \setminus K_{-\varepsilon}).$$

Since $M\left(x_i, \tfrac{1}{2}\right)$ is symmetric with respect to the point x_i, we have

$$V_n\left(M\left(x_i, \tfrac{1}{2}\right) \setminus H_i^-\right) = \tfrac{1}{2}V_n\left(M\left(x_i, \tfrac{1}{2}\right)\right)$$

and we conclude from Lemma 2.3.6 that

$$\sum_{i=1}^{m} V_n\left(M\left(x_i, \tfrac{1}{2}\right)\right) < 8nr^{-2}RV_n(K)\varepsilon. \tag{2.15}$$

By Lemma 2.3.7, for each $i \in \{1, \dots, m\}$ a cap C_i^0 of K, bounded by a hyperplane through x_i, can be found such that

$$C_i^0 \subset M(x_i, 3n). \tag{2.16}$$

There is a supporting halfspace $H^-_{u_i,\alpha_i}$ of K so that

$$C_i^0 = K \cap H^+_{u_i,\alpha_i - t_i}$$

with suitable $t_i > 0$. We define the cap C_i of K by

$$C_i := K \cap H^+_{u_i,\alpha_i - 6t_i}.$$

From

$$x_i - 5(K - x_i) \subset x_i - 5(H^-_{u_i,\alpha_i} - x_i) = H^+_{u_i,\alpha_i - 6t_i}$$

it follows that $K \cap M(x_i, 5) \subset C_i$ and thus

$$\mathrm{bd}\, K \subset \bigcup_{i=1}^{m} C_i,$$

by (2.14).

Choose a point $y_i \in K \cap H_{u_i,\alpha_i}$. From the convexity of K and from (2.16) we infer that

$$C_i \subset y_i + 6(C_i^0 - y_i) \subset y_i + 6[M(x_i, 3n) - y_i] = M(x_i, 18n) + 5x_i - 5y_i.$$

From Lemma 2.3.7 it follows that $\Delta(C_i) \leq 36n\varepsilon$, and $B(x_i, \varepsilon) \subset M(x_i, 1) \subset C_i$ gives $\Delta(C_i) \geq 2\varepsilon$. Finally,

$$\begin{aligned}\sum_{i=1}^{m} V_n(C_i) &\leq \sum_{i=1}^{m} V_n(M(x_i, 18n)) = (36n)^n \sum_{i=1}^{m} V_n\left(M\left(x_i, \tfrac{1}{2}\right)\right) \\ &< (36n)^n 8nr^{-2} R V_n(K)\varepsilon\end{aligned}$$

by (2.15), which completes the proof of Theorem 2.3.2. □

The proof of Theorem 2.3.1 requires two more lemmas. For the first one, a rectangular parallelepiped is referred to as a *box*.

Lemma 2.3.8 *A convex body $K \in \mathcal{K}_n^n$ can be covered by a box P with smallest edge length $\Delta(K)$ and with volume $V_n(P) \leq n! V_n(K)$.*

Proof We use induction on n. The case $n = 1$ being trivial, assume that $n \geq 2$ and the assertion has been proved in dimension $n-1$. The convex body of width $\Delta(K)$ has two parallel supporting hyperplanes H_0, H_1 of mutual distance $\Delta(K)$. Let π denote the orthogonal projection from $\mathbb{R}^n$ onto H_0. There must be a point $x_0 \in K \cap H_0 \cap \pi(K \cap H_1)$, since otherwise $K \cap H_0$ and $\pi(K \cap H_1)$ could be strongly separated by an $(n-2)$-plane E in H_0, and small rotations of H_0 and H_1 around E, respectively $H_1 \cap \pi^{-1}(E)$, would yield a parallel strip containing K, of width strictly less than $\Delta(K)$. Let $x_1 \in K \cap H_1$ be the point for which $\pi x_1 = x_0$ and let C be the convex hull of πK and x_1. For $y \in \pi K$, let $y_1 \in \mathrm{relbd}\, \pi K$ be such that $y \in [x_0, y_1]$. If $z \in \pi^{-1}(y_1) \cap K$, then $\mathrm{conv}\{z, x_0, x_1\} \subset K$, and the intersections of the line $\pi^{-1}(y)$ with this triangle and with C have the same length. By Cavalieri's principle, $V_n(K) \geq V_n(C) = \frac{1}{n}\Delta(K) V_{n-1}(\pi K)$, where V_{n-1} denotes $(n-1)$-dimensional volume. By the induction hypothesis, πK can be covered by an $(n-1)$-dimensional box P' of smallest edge length $\Delta(\pi K_1) \geq \Delta(K)$ and $(n-1)$-volume $V_{n-1}(P') \leq (n-1)! V_{n-1}(\pi K)$. Hence, K can be covered by a box P of smallest edge length $\Delta(K)$ and volume $V_n(P) = \Delta(K) V_{n-1}(P') \leq \Delta(K)(n-1)! V_{n-1}(\pi K) \leq n! V_n(K)$. □

Lemma 2.3.9 *Let $K_0, K_1 \in \mathcal{K}_n^n$. Let $\Lambda(K_0, K_1)$ be the set of all differences $x_1 - x_0$ of boundary points x_0 of K_0 and x_1 of K_1 lying in parallel support planes, thus*

$$\Lambda(K_0, K_1) = \bigcup_{u \in \mathbb{S}^{n-1}} [F(K_1, u) - F(K_0, u)].$$

If $C_1, \ldots, C_m$ are caps of $K_0 + K_1$ covering the boundary of $K_0 + K_1$, then

$$\Lambda(K_0, K_1) \subset \bigcup_{i=1}^{m} (\mathrm{D}C_i + a_i),$$

where $\mathrm{D}C_i = C_i - C_i$ is the difference body of C_i and a_i is a suitable translation vector.

Proof For $i \in \{1, \ldots, m\}$ we can write $C_i = (K_0 + K_1) \cap H^+_{u_i,\alpha_i-\tau_i}$, where H_{u_i,α_i} is a support plane of $K_0 + K_1$, with outer unit normal vector u_i, and $\tau_i > 0$. Let $\alpha_i^\nu := h(K_\nu, u_i)$ and $C_i^\nu = K_\nu \cap H^+_{u_i,\alpha_i^\nu-\tau_i}$ for $\nu = 0, 1$. A given element of $\Lambda(K_0, K_1)$ is of the form $x_1 - x_0$ with $x_\nu \in F(K_\nu, v)$ for suitable $v \in \mathbb{S}^{n-1}$. Then $x_0 + x_1 \in F(K_0 + K_1, v) \subset \mathrm{bd}\,(K_0 + K_1)$, hence there is an index $j \in \{1, \ldots, m\}$ for which $x_0 + x_1 \in C_j$. We have

$$\alpha_j - \tau_j \le \langle x_0 + x_1, u_j\rangle \le \alpha_j, \qquad \langle x_\nu, u_j\rangle \le \alpha_j^\nu \quad \text{for } \nu = 0, 1.$$

From $\langle x_\nu, u_j\rangle < \alpha_j^\nu - \tau_j$ for $\nu = 0$ or 1 it would follow that

$$\langle x_0 + x_1, u_j\rangle < \alpha_j^0 + \alpha_j^1 - \tau_j = \alpha_j - \tau_j,$$

a contradiction. Thus $\alpha_j^\nu - \tau_j \le \langle x_\nu, u_j\rangle$ and, therefore, $x_\nu \in C_j^\nu$ for $\nu = 0, 1$. This shows that

$$x_1 - x_0 \in \bigcup_{i=1}^{m} (C_i^1 - C_i^0). \tag{2.17}$$

If $b_i^\nu \in F(K_\nu, u_i)$, then $C_i^0 + b_i^1 \subset C_i$ and $C_i^1 + b_i^0 \subset C_i$, hence

$$C_i^1 - C_i^0 \subset \mathrm{D}C_i + b_i^1 - b_i^0,$$

which together with (2.17) proves the assertion. □

Proof of Theorem 2.3.1 Let $K \in \mathcal{K}^n$ be given. Since each boundary segment of K is contained in a support set $F(K, u)$ and since $F(K + B^n, u) = F(K, u) + u$, we may assume from the beginning that $K = K' + B^n$ with a convex body K'; thus K is 1-smooth. (A convex body $L \in \mathcal{K}^n$ is called *η-smooth* if to each $x \in \mathrm{bd}\,L$ there is a vector t such that $x \in \eta B^n + t \subset L$.)

Let H_0, H_1 be a pair of parallel hyperplanes in $\mathbb{R}^n$. We say that a vector $u \in \mathbb{S}^{n-1}$ is *associated with* (K, H_0, H_1) if the boundary of K contains a segment parallel to u that intersects H_0 and H_1. Let $\mathcal{R}$ denote the set of all hyperplanes in $\mathbb{R}^n$ that are spanned by points with rational coordinates. We fix a pair H_0, H_1 of parallel hyperplanes in $\mathcal{R}$ meeting the interior of K. Let U^* be the set of all $u \in \mathbb{S}^{n-1}$ that are associated with (K, H_0, H_1). Since there are only countably many possible choices for H_0, H_1 and since the union of the sets U^* corresponding to these different choices covers the set $U(K)$ of Theorem 2.3.1, it suffices to show that U^* has finite $(n-2)$-dimensional Hausdorff measure.

Since directions of boundary segments remain unchanged under homotheties, we may assume that $H_0 = H_{w,-1}$ and $H_1 = H_{w,1}$ with $w \in \mathbb{S}^{n-1}$. Let $K_0 = K \cap H_0$

and $K_1 = K \cap H_1$. In the following, $c_1, c_2, \dots$ denote constants depending only on n, K_0, K_1 but not on ε. By Λ we denote the set of all differences $x_1 - x_0$ where $x_0 \in K_0$, $x_1 \in K_1$ and x_0 and x_1 lie in a support plane of K. Then $U^* = \{y/|y| : y \in \Lambda \cup (-\Lambda)\}$. Since U^* is the image of $\Lambda \cup (-\Lambda)$ under a Lipschitz map, it suffices to show that Λ has finite $(n-2)$-dimensional Hausdorff measure.

We apply Theorem 2.3.2 to the $(n-1)$-dimensional convex body $K_0 + K_1$, with $\mathbb{R}^n$ replaced by $\operatorname{aff}(K_0 + K_1) = H_{w,0}$. Let $c_1, \dots, c_4$ be the constants appearing in that theorem (they now depend only on n, K_0, K_1). Let $0 < \varepsilon < \sqrt{c_1}$ be given. There exist $(n-1)$-dimensional caps $C_1, \dots, C_m$ of $K_0 + K_1$ covering the boundary of $K_0 + K_1$ and such that the widths $\Delta'(C_i)$ and volumes $V_{n-1}(C_i)$, computed in $H_{w,0}$, satisfy

$$c_2\varepsilon^2 < \Delta'(C_i) < c_3\varepsilon^2, \tag{2.18}$$

$$\sum_{i=1}^{m} V_{n-1}(C_i) < c_4\varepsilon^2. \tag{2.19}$$

Let $x_1 - x_0 \in \Lambda$, with $x_0 \in K_0$ and $x_1 \in K_1$ lying in a support plane of K. Then $x_1 - w$ and $x_0 + w$ lie in parallel support planes (with respect to $H_{w,0}$) of $K_1 - w$ and $K_0 + w$, respectively. From Lemma 2.3.9 we now infer that

$$\Lambda \subset \bigcup_{i=1}^{m} (\mathrm{D}C_i + t_i) \tag{2.20}$$

with suitable translation vectors t_i.

Let $i \in \{1, \dots, m\}$. The difference body $\mathrm{D}C_i$ has width $\Delta'(\mathrm{D}C_i) = 2\Delta'(C_i)$ and $(n-1)$-volume

$$V_{n-1}(\mathrm{D}C_i) \le n^{n-1} V_{n-1}(C_i),$$

since $\mathrm{D}C_i = C_i - C_i$ is contained in a translate of $C_i + (n-1)C_i$, according to Lemma 2.3.3. By Lemma 2.3.8, $\mathrm{D}C_i$ can be covered by an $(n-1)$-dimensional box P_i with smallest edge length $w_1 = \Delta'(\mathrm{D}C_i)$ and with volume $V_{n-1}(P_i) \le (n-1)!\, V_{n-1}(\mathrm{D}C_i)$.

We have assumed that K is 1-smooth. Since H_0, H_1 meet the interior of K, it follows that K_0 and K_1, and thus also $K_0 + K_1$, are η-smooth for some $\eta > 0$. Hence, each cap C_i contains a cap, of the same width, cut from an $(n-1)$-ball of radius η. Now it follows from (2.18) that the edge lengths $w_1 \le w_2 \le \dots \le w_{n-1}$ of the box P_i satisfy $w_2 > c_5\varepsilon$. The box P_i can be covered by N_i cubes of dimension $n-1$ and edge length ε, where

$$\begin{aligned} N_i &\le \prod_{k=1}^{n-1} \left(\frac{w_k}{\varepsilon} + 1\right) \le \varepsilon^{-n} w_1 \cdots w_{n-1} \left(\varepsilon + \frac{\varepsilon^2}{\Delta'(\mathrm{D}C_i)}\right)\left(1 + \frac{1}{c_5}\right)^{n-2} \\ &\le c_6\varepsilon^{-n} V_{n-1}(P_i) \le c_7\varepsilon^{-n} V_{n-1}(\mathrm{D}C_i) \le c_8\varepsilon^{-n} V_{n-1}(C_i). \end{aligned}$$

By (2.20), the set Λ can be covered by

$$N \le c_8 \varepsilon^{-n} \sum_{i=1}^{m} V_{n-1}(C_i) \le c_9 \varepsilon^{2-n}$$

cubes of edge length ε, using (2.19). Since this holds for all sufficiently small numbers $\varepsilon > 0$, the set Λ has finite $(n-2)$-dimensional Hausdorff measure. This completes the proof of Theorem 2.3.1. □

Theorem 2.3.1 can be considered as a special case of a more general theorem, referring to corresponding segments in the boundaries of a pair of convex bodies. Recall that $\mathrm{SO}(n)$ is the group of proper rotations of $\mathbb{R}^n$, equipped with its usual topology and differentiable structure. The normalized Haar measure on $\mathrm{SO}(n)$ is denoted by ν. On the compact Lie group $\mathrm{SO}(n)$ we can choose a bi-invariant Riemannian metric. This induces, in the usual way, a metric (distance function), which we denote by d_1. Hausdorff measures on $\mathrm{SO}(n)$ refer to this distance function. (The σ-finiteness of Hausdorff measures, in which we are interested, does not depend on the choice of the Riemannian metric, since any two continuous Riemannian metrics on $\mathrm{SO}(n)$ induce equivalent distance functions.)

Theorem 2.3.10 *Let $K, K' \in \mathcal{K}^n$ be convex bodies. The set $U(K, K')$ of all rotations $\rho \in \mathrm{SO}(n)$ for which K and $\rho K'$ contain parallel segments lying in parallel supporting hyperplanes has σ-finite $[\frac{1}{2}n(n-1)-1]$-dimensional Hausdorff measure.*

It is not difficult to obtain Theorem 2.3.1 from this result by taking K' equal to a segment. Theorem 2.3.1 is interesting in itself as a strong result on the boundary structure of convex bodies. The following corollary of Theorem 2.3.10 will be needed in Section 4.4.

Corollary 2.3.11 *Let $K \in \mathcal{K}^n$. For $K' \in \mathcal{K}^n$, the set $U(K, K')$ of Theorem 2.3.10 has ν-measure zero.*

Let E be a k-dimensional linear subspace of $\mathbb{R}^n$, where $k \in \{1, \dots, n-1\}$. The set of all rotations $\rho \in \mathrm{SO}(n)$ for which ρE is parallel to some supporting k-flat of K that contains more than one point of K is of ν-measure zero.

Recall that a supporting flat of K is an affine subspace of $\mathbb{R}^n$ contained in a supporting hyperplane of K and having nonempty intersection with K. The first part of Corollary 2.3.11 is true since Theorem 2.3.10 implies that $U(K, K')$ has $\frac{1}{2}n(n-1)$-dimensional Hausdorff measure zero and hence ν-measure zero, because $\frac{1}{2}n(n-1)$ is the dimension of $\mathrm{SO}(n)$. The second part is obtained by taking $K' = E \cap B^n$.

Although Corollary 2.3.11 is weaker than Theorem 2.3.10, no simpler proof of the former seems to be known. The proof of Theorem 2.3.10 requires some preliminaries.

A pair of orthogonal unit vectors in $\mathbb{R}^n$ will be called a 2-*frame*. We denote by $V_{n,2}$ the set of all 2-frames in $\mathbb{R}^n$. For $(u, v) \in V_{n,2}$ and $\rho \in \mathrm{SO}(n)$ we write $\rho(u, v) = (\rho u, \rho v)$. On $V_{n,2}$ a metric τ is introduced by

$$\tau((u_1, v_1), (u_2, v_2)) := |u_1 - u_2| + |v_1 - v_2|.$$

In the following, $c_{10}, c_{11}, \ldots$ denote positive constants independent of ε (but c_k may depend on those c_j with $j < k$). The assertions below are understood to hold for all sufficiently small $\varepsilon > 0$.

Lemma 2.3.12 *Let $A, B \subset V_{n,2}$ be sets of diameter $c_{10}\varepsilon$. The set $X(A,B)$ of all rotations $\rho \in \mathrm{SO}(n)$ for which $A \cap \rho B \neq \emptyset$ can be covered by less than $c_{14}\varepsilon^{-(n-2)(n-3)/2}$ balls, in $(\mathrm{SO}(n), d_1)$, of diameter ε.*

Proof First we assume that $a, b \in V_{n,2}$ are 2-frames such that $\tau(a,b) \leq c_{10}\varepsilon$, and we assert that there exists a rotation $\rho \in \mathrm{SO}(n)$ in a $c_{12}\varepsilon$-neighbourhood of the neutral element e such that $b = \rho a$.

We embed $\mathrm{SO}(n)$ into $\mathbb{R}^{n^2}$ by associating with each $\rho \in \mathrm{SO}(n)$ the entries, in the usual order, of its matrix with respect to a given orthonormal basis $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$. The standard Euclidean metric on $\mathbb{R}^{n^2}$ induces a second metric d_2 on $\mathrm{SO}(n)$. It also induces a Riemannian metric on the smooth embedded submanifold $\mathrm{SO}(n)$, and this induces another metric d_3 on $\mathrm{SO}(n)$. Since the submanifold $\mathrm{SO}(n)$ is compact, the metrics d_2 and d_3 are equivalent; that is, there exist constants γ_1, γ_2 such that $\gamma_1 d_2(\rho, \sigma) \leq d_3(\rho, \sigma) \leq \gamma_2 d_2(\rho, \sigma)$ for $\rho, \sigma \in \mathrm{SO}(n)$. Also, the metrics d_1 and d_3 are equivalent, since both are induced by continuous Riemannian metrics.

Since the metric τ is invariant under the operation of $\mathrm{SO}(n)$ and since d_1 is bi-invariant, we may, without loss of generality, assume that $b = (b_1, b_2)$. Let $a = (a_1, a_2)$. Applying the Gram–Schmidt orthonormalization process to the n-tuple $(a_1, a_2, b_3, \ldots, b_n)$ (which is linearly independent if ε is sufficiently small) and using $|a_1 - b_1| + |a_2 - b_2| \leq c_{10}\varepsilon$, we get an orthonormal n-tuple $(a_1, \ldots, a_n)$ satisfying $|a_i - b_i| \leq c_{11}\varepsilon$ for $i = 1, \ldots, n$. If we take the coordinate vectors (with respect to the basis $b_1, \ldots, b_n$) of $a_1, \ldots, a_n$ as the row vectors of a matrix, then this matrix represents a rotation $\rho \in \mathrm{SO}(n)$ for which $\rho a = b$. By the definition of the metric d_2 we have

$$d_2(\rho, e) = (|a_1 - b_1|^2 + \cdots + |a_n - b_n|^2)^{1/2} \leq \sqrt{n} c_{11}\varepsilon$$

and hence $d_1(\rho, e) \leq c_{12}\varepsilon$.

To prove the lemma, let $a \in A$, $b \in B$. The set M of all $\sigma \in \mathrm{SO}(n)$ for which $a = \sigma b$ is a left coset of the isotropy subgroup of b, which can be identified with $\mathrm{SO}(n-2)$, hence M is a submanifold of dimension $\frac{1}{2}(n-2)(n-3)$. Let $\rho \in X(A,B)$. Then there are elements $a' \in A$ and $b' \in B$ for which $a' = \rho b'$. By assumption we have $\tau(a, a') \leq c_{10}\varepsilon$ and $\tau(b, b') \leq c_{10}\varepsilon$. As shown above, there exist rotations ρ_1, ρ_2 in a $c_{12}\varepsilon$-neighbourhood of e such that $a' = \rho_1 a$ and $b' = \rho_2 b$. It follows that $a = \rho_1^{-1}\rho\rho_2 b$, hence $\mu := \rho_1^{-1}\rho\rho_2 \in M$. Since the metric d_1 is bi-invariant, we get

$$\begin{aligned} d_1(\rho, \mu) &= d_1(\rho_1\mu\rho_2^{-1}, \mu) = d_1(\rho_1\mu, \mu\rho_2) \\ &\leq d_1(\rho_1\mu, \mu) + d_1(\mu, \mu\rho_2) = d_1(\rho_1, e) + d_1(e, \rho_2) \\ &\leq 2c_{12}\varepsilon. \end{aligned}$$

Thus, $X(A,B)$ is contained in a $2c_{12}\varepsilon$-neighbourhood of M. The smooth, compact submanifold M of dimension $\frac{1}{2}(n-2)(n-3)$ can be covered by less than $c_{13}\varepsilon^{-(n-2)(n-3)/2}$

balls of diameter ε (for a proof, see Flatto and Newman [617], Theorem 2.1). Then $X(A, B)$ can be covered by less than $c_{14}\varepsilon^{-(n-2)(n-3)/2}$ balls of diameter ε. $\square$

The proof of Theorem 2.3.10 is an extension of the one given for Theorem 2.3.1, and we use the same notation, as well as some results established in the course of that proof.

Proof of Theorem 2.3.10 Let $K, K' \in \mathcal{K}^n$ be given. Both bodies can be assumed to be 1-smooth. Let H_0, H_1 be a pair of parallel hyperplanes in $\mathbb{R}^n$. The 2-frame (u, v) is said to be *associated with* (K, H_0, H_1) if the support set $F(K, v)$ contains a segment parallel to u that intersects H_0 and H_1 and if $\langle u, w\rangle \geq \frac{1}{2}$, where w is the unit vector orthogonal to H_0 and pointing from H_0 to H_1. Let $\rho \in U(K, K')$. Then there exists a 2-frame (u, v) such that $F(K, v)$ and $F(\rho K', v)$ each contain a segment parallel to u. Hence, there exist four hyperplanes H_0, H_1, H_0', H_1' in $\mathcal{R}$ such that H_0 and H_1 are parallel, say at distance a, H_0' and H_1' are parallel and at the same distance a, (u, v) is associated with (K, H_0, H_1), and $(\rho^{-1}u, \rho^{-1}v)$ is associated with (K', H_0', H_1').

We fix a pair H_0, H_1 of parallel hyperplanes in $\mathcal{R}$ that meet the interior of K, and a pair H_0', H_1' of parallel hyperplanes in $\mathcal{R}$ at the same distance that meet the interior of K'. Let $U^*(K, K')$ be the set of all rotations $\rho \in \mathrm{SO}(n)$ for which there exists a 2-frame (u, v) associated with (K, H_0, H_1) and such that $(\rho^{-1}u, \rho^{-1}v)$ is associated with (K', H_0', H_1'). Clearly, it suffices to show that $U^*(K, K')$ has finite $[\frac{1}{2}n(n-1)-1]$-dimensional Hausdorff measure.

We may assume that $H_0 = H_{w,-1}$ and $H_1 = H_{w,1}$. Below, $c_{15}, c_{16}, \ldots$ denote positive constants depending only on n, on $K_\nu = K \cap H_\nu$, and on $K_\nu' = K' \cap H_\nu'$ $(\nu = 0, 1)$. Let V^* be the set of all 2-frames associated with (K, H_0, H_1). If $(u, v) \in V^*$, then there are points x_0, x_1 satisfying

$$
\begin{gathered}
x_0 \in K_0, \quad x_1 \in K_1, \\
x_0, x_1 \in F(K, v), \\
u = \frac{x_1 - x_0}{|x_1 - x_0|}.
\end{gathered}
\tag{2.21}
$$

The point $x_0 + x_1$ lies in one of the caps $C_1, \ldots, C_m$ used in the proof of Theorem 2.3.1, say $x_0 + x_1 \in C_i$. Then $x_1 - x_0 \in \mathrm{D}C_i + t_i$ with some vector t_i (compare the proof of Lemma 2.3.9). The set $\mathrm{D}C_i$ was covered by N_i cubes of dimension $n-1$ and of edge length ε, say by $W_1^{(i)}, \ldots, W_{N_i}^{(i)}$. We denote by V_{ij}^* the set of all $(u, v) \in V^*$ for which points x_0, x_1 satisfying (2.21) exist such that $x_0 + x_1 \in C_i$ and $x_1 - x_0 \in W_j^{(i)} + t_i$. Then V^* is the union of all these V_{ij}^*.

For $x \in \mathbb{R}^n$, we denote by $\bar{x}$ the orthogonal projection of x onto the linear subspace $H_{w,0}$. Let $u_1, u_2 \in \mathbb{S}^{n-1}$, $\langle u_\nu, w\rangle \geq \frac{1}{2}$ and $x_\nu \in H_{u_\nu,0}$ for $\nu = 1, 2$. Then

$$
x_\nu = \bar{x}_\nu - \frac{\langle \bar{x}_\nu, u_\nu\rangle}{\langle w, u_\nu\rangle} w
$$

and hence

$$|x_1 - x_2| \le |\bar{x}_1 - \bar{x}_2| + \left|\frac{\langle \bar{x}_1, u_1 - u_2\rangle}{\langle w, u_1\rangle}\right| + \left|\frac{\langle \bar{x}_1, u_2\rangle}{\langle w, u_1\rangle} - \frac{\langle \bar{x}_1, u_2\rangle}{\langle w, u_2\rangle}\right| + \left|\frac{\langle \bar{x}_1 - \bar{x}_2, u_2\rangle}{\langle w, u_2\rangle}\right|$$
$$\le 3|\bar{x}_1 - \bar{x}_2| + 6|\bar{x}_1|\,|u_1 - u_2|. \tag{2.22}$$

Now let $(u_1, v_1), (u_2, v_2) \in V^*_{ij}$. Then

$$|u_1 - u_2| \le \operatorname{diam} W_j^{(i)} \le \sqrt{(n-1)}\varepsilon.$$

The vectors $\bar{v}_1, \bar{v}_2$ are exterior normal vectors of $K_0 + K_1$ at points belonging to the same cap C_i. As in the proof of Theorem 2.3.1, this cap contains a cap, of the same width $\Delta'(C_i) < c_3\varepsilon^2$, that is cut, by the same $(n-2)$-plane, from an $(n-1)$-ball of some positive radius η. It follows that the angle between $\bar{v}_1$ and $\bar{v}_2$ is less than $c_{15}\varepsilon$. Let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1\bar{v}_1, \lambda_2\bar{v}_2$ are unit vectors, then

$$|\lambda_1\bar{v}_1 - \lambda_2\bar{v}_2| \le c_{15}\varepsilon.$$

Since $\lambda_1, \lambda_2 \ge 1$ and $v_1,\ v_2$ are unit vectors, we have

$$|v_1 - v_2| \le |\lambda_1 v_1 - \lambda_2 v_2|.$$

From (2.22) we obtain

$$|\lambda_1 v_1 - \lambda_2 v_2| \le 3|\lambda_1\bar{v}_1 - \lambda_2\bar{v}_2| + 6|u_1 - u_2|$$

and thus

$$|v_1 - v_2| \le c_{16}\varepsilon.$$

It follows that, in terms of the metric τ introduced on $V_{n,2}$,

$$\operatorname{diam} V^*_{ij} \le c_{17}\varepsilon.$$

By the proof of Theorem 2.3.1, the total number of the sets V^*_{ij} is $N \le c_{18}\varepsilon^{2-n}$. Thus, V^* can be covered by $N \le c_{18}\varepsilon^{2-n}$ subsets $A_1, \dots, A_N$ of $V_{n,2}$ of diameter $c_{17}\varepsilon$.

Similarly, the set of all 2-frames associated with (K', H_0', H_1') can be covered by N subsets $A_1', \dots, A_N'$ of $V_{n,2}$ of diameter $c_{17}\varepsilon$.

Now let $\rho \in U^*(K, K')$. Then there exists a 2-frame (u, v) associated with (K, H_0, H_1) such that $\rho^{-1}(u, v)$ is associated with (K', H_0', H_1'). For suitable i, j we have $(u, v) \in A_i$ and $\rho^{-1}(u, v) \in A_j'$, hence $(u, v) \in A_i \cap \rho A_j'$ and therefore $\rho \in X(A_i, A_j')$ (as defined in Lemma 2.3.12). Thus

$$U^*(K, K') \subset \bigcup_{i,j=1}^{N} X(A_i, A_j').$$

According to Lemma 2.3.12, each set $X(A_i, A_j')$ can be covered by less than $c_{19}\varepsilon^{-(n-2)(n-3)/2}$ balls of diameter ε, hence $U^*(K, K')$ can be covered by less than

$$(c_{18}\varepsilon^{2-n})^2 c_{19}\varepsilon^{-(n-2)(n-3)/2} = c_{20}\varepsilon^{-[n(n-1)/2-1]}$$

balls of diameter ε. Since this is true for all sufficiently small $\varepsilon > 0$, the set $U^*(K, K')$ has finite $[n(n-1)/2-1]$-dimensional Hausdorff measure. This completes the proof of Theorem 2.3.10. □

Without proof, we mention two similar results, now referring to the group of rigid motions of $\mathbb{R}^n$ and its Haar measure. Let $K, K' \in \mathcal{K}^n$. By a *common support plane* of K and K' we understand a hyperplane that supports both bodies, leaving each of them in the same halfspace. Such a common support plane, H, is said to be *exceptional* if the affine hulls of the support sets $H \cap K$ and $H \cap K'$ have a nonempty intersection or contain parallel lines. Further, we say that a common boundary point x of K and K' is *exceptional* if the linear hulls of the normal cones $N(K, x)$ and $N(K', x)$ have an intersection different from $\{o\}$. For more information about the following result, we refer to Note 5 below.

Theorem 2.3.13 *Let $K, K' \in \mathcal{K}^n$ be convex bodies.*

(a) *The set of all rigid motions g for which K and gK' have some exceptional common support plane is of Haar measure zero.*
(b) *The set of all rigid motions g for which K and gK' have some exceptional common boundary point is of Haar measure zero.*

Notes for Section 2.3

1. The question of whether the set $U(K)$ in Theorem 2.3.1 has $(n-1)$-dimensional measure zero was first asked by Klee [1106]. (Ewald, Larman and Rogers [542] mention an example due to Bing and Klee of a topological 2-sphere in $\mathbb{R}^3$ containing segments of all directions; thus some assumption such as convexity is necessary.) For $n = 3$ (the two-dimensional case is trivial), McMinn [1373] showed that $U(K)$ can be covered by the ranges of countably many Lipschitz mappings from $[-1, 1]$ to $\mathbb{S}^2$. This implies that $U(K)$ is of σ-finite one-dimensional Hausdorff measure. A shorter proof was given by Besicovitch [215]. For $n \geq 4$, Klee [1115] posed the problem of showing that $U(K) \neq \mathbb{S}^{n-1}$. This was proved in the stronger form given in Theorem 2.3.1 by Ewald, Larman and Rogers [542]. In fact, these authors obtained a more general result. To formulate it, let $G(n, r)$ be the Grassmann manifold of all r-dimensional linear subspaces of $\mathbb{R}^n$. It is a compact differentiable manifold of dimension $r(n-r)$ and can be equipped with a rotation invariant Riemannian metric. This induces a distance function and thus Hausdorff measures. The following general result can be stated.

   Theorem Let $K \in \mathcal{K}^n$ and $1 \leq s \leq r \leq n-1$. The set of all r-dimensional linear subspaces of $\mathbb{R}^n$ parallel to some supporting r-flat of K that contains an s-dimensional convex subset of K has σ-finite $[r(n-r)-s]$-dimensional Hausdorff measure.

   The case $s = r$ of this theorem was proved by Ewald, Larman and Rogers [542]. By extending their methods, Zalgaller [2028] treated the case $s = 1$, and briefly sketched a proof of the general theorem above. Several partial refinements of the preceding theorem were obtained by Pavlica and Zajíček [1520].

   Some of these results can be interpreted in terms of the existence of r-dimensional Chebyshev subspaces of n-dimensional Banach spaces; see Klee [1115] and Zalgaller [2028]. The case $s = 1$ implies, in particular, that for almost all r-dimensional linear subspaces E, the shadow boundary of K in direction E is sharp. This was stated, without a complete proof, by Ewald [539].

By further extending the methods of Zalgaller, Schneider [1690] obtained Theorem 2.3.10.

As mentioned after Theorem 2.3.1, the possible presence of facets is an obstruction to improving the result, and a similar remark concerns the generalizations mentioned above. However, Ewald, Larman and Rogers [542] showed the following. If $2 \le r \le n-2$ and $K \in \mathcal{K}^n_n$ has no facets, then the set of all r-dimensional linear subspaces of $\mathbb{R}^n$ parallel to some r-dimensional convex subset in the boundary of K has $r(n-r-1)$-dimensional Hausdorff measure zero. Further, the following was proved by Larman and Rogers [1175]. Let H be a hyperplane in $\mathbb{R}^n$. The set of unit vectors parallel to segments in the boundary of a convex body K that are parallel to H but not lying in the support planes of K parallel to H has $(n-2)$-dimensional Hausdorff measure zero.

2. Using the result of Ewald, Larman and Rogers, Ivanov [1028] deduced the following result on segments in the boundary. The union of all lines in $\mathbb{R}^n$ that meet the boundary of the convex body $K \in \mathcal{K}^n$ in a segment has σ-finite $(n-1)$-dimensional Hausdorff measure. Consequently, almost all points $x \in \mathbb{R}^n \setminus K$ have the property that the shadow boundary of K under central projection from x is sharp.
3. For convex subsets in the boundary of a convex body cut out by concurrent sections, Ivanov [1029] showed the following. Let $K \in \mathcal{K}^n_{(o)}$ and let $2 \le r \le n-1$. The set of all r-dimensional linear subspaces of $\mathbb{R}^n$ containing an $(r-1)$-dimensional convex subset of $\mathrm{bd}\, K$ that does not meet the relative interior of a facet of K has Haar measure zero in $G(n,r)$.
4. *The economic cap covering theorem.* Variants of the cap covering theorem of Ewald, Larman and Rogers, which is Theorem 2.3.2 above, have subsequently become very important in stochastic geometry, in connection with the asymptotic behaviour of the convex hull of independent random points in a given convex body. Beginning with work of Bárány and Larman [147] and of Bárány [140], an economic cap covering theorem for certain parts of convex bodies close to the boundary has become a useful tool for delicate estimates. We refer to later work by Bárány and Reitzner [151], Bárány and Steiger [153] and the surveys by Bárány [144, 145] and Reitzner [1569].
5. Theorem 2.3.13 arose from some questions of integral geometry and was later applied accordingly. It was conjectured by Glasauer [717], who proved it for $n \le 3$ (in a way not extending to higher dimensions) or if one of the bodies is a polytope (see Glasauer [717, 718, 719]), as well as a counterpart in spherical space. The general result was proved by Schneider [1721]. In the special case where K' is a singleton, part (a) of the theorem reduces to the result of Ivanov mentioned in Note 2.

2.4 Polytopes

The boundary structure of polytopes is particularly simple as far as the facial structure and the classification of boundary points are concerned (not, of course, from the combinatorial or metrical viewpoint). In this book, polytopes are mainly used as tools for obtaining results on general convex bodies. We treat, therefore, only the most basic properties of polytopes.

A polytope in $\mathbb{R}^n$ is, by definition, the convex hull of a finite (possibly empty) subset of $\mathbb{R}^n$. We denote by $\mathcal{P}^n$ the set of nonempty polytopes in $\mathbb{R}^n$ and by $\mathcal{P}^n_n$ the subset of n-dimensional polytopes. If $P = \mathrm{conv}\,\{x_1, \dots, x_k\}$, then clearly the extreme points of P are among the $x_1, \dots, x_k$; hence, by Minkowski's theorem, the nonempty polytopes are precisely the convex bodies with finitely many extreme points. If H is a support plane of P, then

$$H \cap P = \mathrm{conv}\,(H \cap \{x_1, \dots, x_k\}),$$

hence each support set of a polytope is itself a polytope. Since each extreme point of a Minkowski sum $P_1 + P_2$ is the sum of an extreme point of P_1 and an extreme point of P_2, the sum of two polytopes is a polytope.

First we show that for polytopes the distinction between faces and exposed faces (support sets) is unnecessary.

Theorem 2.4.1 *Let $P \subset \mathbb{R}^n$ be a polytope, F_1 a support set of P and F a support set of F_1. Then F is a support set of P.*

Proof We may assume that $o \in F$. There is a support plane $H_{u,0}$ to P with $H_{u,0} \cap P = F_1$ and $P \subset H^-_{u,0}$. In $H_{u,0}$ there is a support plane H to F_1 with $H \cap F_1 = F$, say

$$H = \{x \in H_{u,0} : \langle x, v\rangle = 0\}, \qquad F_1 \subset \{x \in H_{u,0} : \langle x, v\rangle \leq 0\}$$

with $v \in u^\perp \cup \mathbb{S}^{n-1}$. Define

$$\eta_0 := \max\{-\langle x, v\rangle/\langle x, u\rangle : x \in \operatorname{ext} P \setminus \operatorname{ext} F_1\}$$

and $H(\eta) := H_{\eta u+v,0}$ with $\eta > \eta_0$. We have $\langle x, u\rangle < 0$ for $x \in \operatorname{ext} P \setminus \operatorname{ext} F_1$, hence

$$\langle x, \eta u + v\rangle < \eta_0\langle x, u\rangle + \langle x, v\rangle \leq 0$$

by the definition of η_0. For $x \in \operatorname{ext} F_1 \setminus \operatorname{ext} F$ we get

$$\langle x, \eta u + v\rangle = \langle x, v\rangle < 0,$$

and for $x \in \operatorname{ext} F$ we have $\langle x, \eta u + v\rangle = 0$. Thus $\operatorname{ext} F \subset H(\eta)$, whereas $\operatorname{ext} P \setminus \operatorname{ext} F \subset \operatorname{int} H^-_{\eta u+v,0}$. We see that $H(\eta)$ is a support plane to P with $H(\eta) \cap P = F$ and therefore F is a support set of P. □

Corollary 2.4.2 *Each proper face of a polytope is a support set.*

Proof Assume that P is a polytope of dimension n and that the assertion has been proved for polytopes of smaller dimension (for zero-dimensional polytopes there is nothing to prove). Let F be a proper face of P. As remarked earlier (before Theorem 2.1.3), F is contained in a support set F_1 of P and it is a face of F_1. Now either $F = F_1$ and then F is a support set of P, or F is a proper face of F_1 and then F is a support set of F_1, by the induction hypothesis. By Theorem 2.4.1, F is also a support set of P. □

In particular, each extreme point of a polytope P is an exposed point of P. The extreme points of a polytope are called its *vertices*; the set of vertices of P is often denoted by $\operatorname{vert} P$. The one-dimensional faces of a polytope P are its *edges*, and the $(\dim P - 1)$-dimensional faces are its *facets*. If $x_1, \dots, x_k$ are the vertices of the polytope P and F is a proper face of P, then there is a hyperplane H with

$$F = H \cap P = \operatorname{conv}(H \cap \{x_1, \dots, x_k\}).$$

Thus each face is the convex hull of a subset of the vertices, and $\mathcal{F}(P)$ is finite.

A polytope was defined as the convex hull of finitely many points. Alternatively, it can be represented as the intersection of finitely many halfspaces.

Theorem 2.4.3 *Every polytope is the intersection of finitely many closed halfspaces.*

Proof Let $P \in \mathcal{P}^n$. Since flats and half-flats can be represented as intersections of finitely many closed halfspaces, it suffices to assume that $\dim P = n$. Let $F_1, \dots, F_k$ be the facets of P; then $F_i = H_i \cap P$ where H_i is a unique support plane of P. Let H_i^- be the closed halfspace bounded by H_i and containing P $(i = 1, \dots, k)$. We assert that

$$P = H_1^- \cap \cdots \cap H_k^-. \tag{2.23}$$

The inclusion $P \subset H_1^- \cap \cdots \cap H_k^-$ is trivial. Let $x \in \mathbb{R}^n \setminus P$. Let A be the union of the affine hulls of x and any $n-1$ vertices of P. We can choose a point $y \in (\operatorname{int} P) \setminus A$. There is a point $z \in \operatorname{bd} P \cap [x, y]$; it lies in some support plane to P and hence in some face F of P. Suppose that $\dim F =: j \le n-2$. By Carathéodory's theorem, z belongs to the convex hull of some $j+1 \le n-1$ vertices of P and hence to A. But then $y \in A$, a contradiction. This shows that F is a facet, hence $F = F_i$ for suitable $i \in \{1, \dots, k\}$. From $y \in \operatorname{int} P \subset \operatorname{int} H_i^-$ we deduce that $x \notin H_i^-$. This proves (2.23). □

Let $P \in \mathcal{P}_n^n$ and let F be a facet of P. Then $F = P \cap H_{u,\alpha}$ and $P \subset H_{u,\alpha}^-$ for suitable u and α. We may assume that $u \in \mathbb{S}^{n-1}$; then u is called the *outer unit normal vector* of the facet F. The outer unit normal vectors of the facets of P are called, in brief, the *normal vectors* of P.

Corollary 2.4.4 *If $P \in \mathcal{P}_n^n$ is a polytope and $u_1, \dots, u_k$ are its normal vectors, then*

$$P = \bigcap_{i=1}^{k} H^-_{u_i, h(P,u_i)}.$$

In particular, the numbers $h(P, u_1), \dots, h(P, u_k)$ determine P uniquely.

Proof If F_i is the facet of P with normal vector u_i, then $F_i = P \cap H_{u_i, h(P,u_i)}$. The assertion is now clear from the proof of Theorem 2.4.3. □

At this point, it is convenient to use polarity, and this requires the following lemma. It shows, in particular, that the polar body of a polytope (with respect to an interior point) is again a polytope.

Lemma 2.4.5 *Let $u_1, \dots, u_k \in \mathbb{R}^n \setminus \{o\}$. If*

$$P := \bigcap_{i=1}^{k} H^-_{u_i,1}$$

is bounded, then

$$P^\circ = \operatorname{conv}\{u_1, \dots, u_k\}.$$

Proof We remark that $o \in \operatorname{int} P$, hence P° is well defined.

Let $Q := \operatorname{conv}\{u_1, \dots, u_k\}$. For $x \in P$ we have $\langle x, u_i \rangle \le 1$, hence $u_i \in P^\circ$. Thus $Q \subset P^\circ$.

Let $v \in \mathbb{R}^n \setminus Q$. Then there is a hyperplane $H_{z,\alpha}$ strongly separating Q and v, and by an appropriate choice of signs we may assume that $\langle z, u_i \rangle < \alpha$ for $i = 1, \dots, k$ and $\langle z, v \rangle > \alpha$. Since P is bounded, clearly $\operatorname{pos}\{u_1, \dots, u_k\} = \mathbb{R}^n$ (otherwise the closed convex cone $\operatorname{pos}\{u_1, \dots, u_k\}$ has a boundary point and hence a supporting hyperplane, necessarily through o, each outer normal vector of which is an element of P); hence there is a representation

$$o = \sum_{i=1}^{k} \lambda_i u_i \quad \text{with } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i > 0.$$

This yields $0 < \sum \lambda_i \alpha$ and thus $\alpha > 0$; without loss of generality, $\alpha = 1$. But then $z \in P$, hence $v \notin P^\circ$. This proves the equality $Q = P^\circ$. □

Theorem 2.4.6 *Any bounded intersection of finitely many closed halfspaces is a polytope.*

Proof Let $P = \bigcap_{i=1}^{k} H_i^-$ be bounded, where $H_i^- \subset \mathbb{R}^n$ is a closed halfspace. We may assume that $\dim P = n$ (otherwise we work in $\operatorname{aff} P$) and, after a translation, that $o \in \operatorname{int} P$. Then $o \in \operatorname{int} H_i^-$, hence H_i^- can be represented in the form $H_i^- = H_{u_i,1}^-$. Lemma 2.4.5 shows that P° is a polytope. By Theorem 2.4.3, P° is the intersection of finitely many closed halfspaces (and $o \in \operatorname{int} P^\circ$, as always). Hence $P^{\circ\circ} = P$ is a polytope. □

Theorem 2.4.6 implies, in particular, that the intersection of finitely many polytopes is a polytope and that the intersection of a polytope with a flat is a polytope.

More information on the faces of a polytope is now easy to obtain.

Theorem 2.4.7 *Each proper face of a polytope P is contained in some facet of P.*

Proof We may assume that $\dim P = n$. In the proof of Theorem 2.4.3 it has been shown that P has a representation

$$P = \bigcap_{i=1}^{k} H_i^-,$$

where H_i^- is a closed halfspace and $F_i := P \cap H_i$ is a facet of P ($i = 1, \dots, k$). Since $\operatorname{bd} P \subset \bigcup_{i=1}^{k} (P \cap H_i)$, each boundary point of P is contained in some facet F_j.

Now let F be a proper face of P and choose $x \in \operatorname{relint} F$. Then $x \in F_j = P \cap H_j$ for some $j \in \{1, \dots, k\}$. Since $x \in \operatorname{relint} F$ and H_j is a support plane of P, we deduce $F \subset H_j$, hence $F \subset F_j$. □

Corollary 2.4.8 *Let P be a polytope and let $F^j \in \mathcal{F}_j(P)$ and $F^k \in \mathcal{F}_k(P)$ be faces such that $F^j \subset F^k$. Then there are faces $F^i \in \mathcal{F}_i(P)$ $(i = j+1, \dots, k-1)$ such that $F^j \subset F^{j+1} \subset \cdots \subset F^{k-1} \subset F^k$.*

Proof Clearly, F^j is a face of F^k (being a support set of P and hence of F^k). If F^j is a proper face of F^k, then F^j is contained in a facet F^{k-1} of F^k, and by Theorem 2.1.1, F^{k-1} is a face of P. By repeating this argument, the assertion is obtained. □

By Lemma 2.2.3, the normal cones of an n-dimensional polytope can be related to the faces of its polar polytope (with respect to an interior point). This yields the following.

Theorem 2.4.9 *Let $P \in \mathcal{P}^n_n$, let $F_1, \dots, F_k$ be its facets and let u_i be the outer unit normal vector of F_i ($i = 1, \dots, k$). If F is a proper face of P, then*

$$N(P, F) = \operatorname{pos}\{u_i : F \subset F_i\}.$$

Proof We may assume that $o \in \operatorname{int} P$. From Lemma 2.2.3 we know that $N(P, F) = \operatorname{pos} \widehat{F}$. Writing $v_i = u_i/h(P, u_i)$, we have

$$P = \bigcap_{i=1}^{k} H^-_{v_i,1}$$

by Corollary 2.4.4, hence Lemma 2.4.5 yields

$$P^\circ = \operatorname{conv}\{v_1, \dots, v_k\}.$$

The face $\widehat{F}$ of P° is the convex hull of those v_i that it contains, and $\operatorname{pos} \widehat{F}$ is the positive hull of these v_i and hence of the corresponding u_i. Now $v_i \in \widehat{F}$ is equivalent to $F \subset H_{v_i,1}$, which is equivalent to $F \subset F_i$. $\square$

We note some consequences of Theorem 2.4.9. Let $P \in \mathcal{P}^n_n$ be a polytope and F a proper face of P. Then F is the intersection of the facets of P containing F; this follows from Theorem 2.4.7 and the fact that the intersection of all facets containing F is a face of the same dimension as F. Let G be another proper face of P. From Theorem 2.4.9 it follows that

$$F \subset G \Leftrightarrow N(P, F) \supset N(P, G) \tag{2.24}$$

and that

$$\dim N(P, F) = n - \dim F. \tag{2.25}$$

As a consequence, a point x of P is r-singular if and only if it belongs to some face of dimension $\le r$. Thus for n-polytopes, the notions of r-extreme, r-exposed, and r-singular boundary point all coincide. In particular, the extreme points (vertices) of P are precisely the points x with $\dim N(P, x) = n$.

From (2.24) one further deduces for $v \in \mathbb{R}^n \setminus \{o\}$ that

$$F = F(P, v) \Leftrightarrow v \in \operatorname{relint} N(P, F) \tag{2.26}$$

and that

$$N(P, F) = \bigcap_{x \in \operatorname{ext} F} N(P, x). \tag{2.27}$$

If $o \in \operatorname{int} P$, so that the polar polytope P° is defined, then $F \subset G$ for proper faces F and G of P implies $\widehat{F} \supset \widehat{G}$ by (2.24) and Lemma 2.2.3. Thus, the map $F \mapsto \widehat{F}$ from the face lattice $\mathcal{F}(P)$ to the face lattice $\mathcal{F}(P^\circ)$ (observe that $\widehat{\emptyset} = P^\circ$ and $\widehat{P} = \emptyset$) is an antimorphism, that is, a bijection that reverses the inclusion relation; it satisfies

$$\dim \widehat{F} = n - 1 - \dim F \tag{2.28}$$

for $F \in \mathcal{F}(P)$.

A polytope is called *simplicial* if all its proper faces (equivalently, all its facets) are simplices. An n-polytope P is called *simple* if each of its vertices is contained in exactly n facets. Clearly, this holds if and only if P° (taken in aff P and with respect to some point in relint P as origin) is simplicial.

The normal cones of a polytope give rise to a series of important metric invariants. Let $P \in \mathcal{P}^n$ be a polytope and $F \in \mathcal{F}_k(P)$ be a k-face of P. Then the number

$$\gamma(F, P) := \mathcal{H}^{n-k-1}(N(P, F) \cap \mathbb{S}^{n-1})/\omega_{n-k}$$

(with the convention $\gamma(F, P) = 1$ if $k = n$) is called the *external angle* of P at its face F. Thus $\gamma(F, P)$ is the $(n - k - 1)$-dimensional measure of the spherical image $\sigma(P, \operatorname{relint} F)$ of P at relint F, divided by the total measure of the $(n - k - 1)$-dimensional unit sphere. If $\dim P < n$, the number $\gamma(F, P)$ remains the same if computed in any affine subspace containing P; this follows from (2.5) and Fubini's theorem. From (2.6) and (2.7) one deduces that

$$\sum_{F \in \mathcal{F}_0(P)} \gamma(F, P) = 1. \tag{2.29}$$

We shall now establish a special result dealing with approximation by polytopes, which will be useful in later chapters. Some preparations are required.

Two polytopes P_1, P_2 are called *strongly isomorphic* if

$$\dim F(P_1, u) = \dim F(P_2, u)$$

for all $u \in \mathbb{S}^{n-1}$. Obviously, this is an equivalence relation; the corresponding equivalence class of a polytope P will be called its *a-type*.

Lemma 2.4.10 *If P_1, P_2 are strongly isomorphic polytopes then, for each $u \in \mathbb{S}^{n-1}$, the support sets $F(P_1, u)$, $F(P_2, u)$ are strongly isomorphic.*

Proof Let $u \in \mathbb{S}^{n-1}$ be given. Let $v \in \mathbb{S}^{n-1}$ and put $F(F(P_i, u), v) =: F_i'$ for $i = 1, 2$. We may assume that $o \in F_1' \cap F_2'$. As shown in the proof of Theorem 2.4.1, for η sufficiently large the hyperplane $H_{\eta u+v,0}$ supports P_i and $H_{\eta u+v,0} \cap P_i = F_i'$ for $i = 1, 2$. Since P_1, P_2 are strongly isomorphic, we get $\dim F_1' = \dim F_2'$ and thus the assertion. □

Lemma 2.4.11 *The polytopes $P_1, P_2 \in \mathcal{P}^n$ are strongly isomorphic if and only if*

$$\{N(P_1, x) : x \in \operatorname{ext} P_1\} = \{N(P_2, y) : y \in \operatorname{ext} P_2\}. \tag{2.30}$$

Proof Suppose that P_1, P_2 are strongly isomorphic. Let $x \in \operatorname{ext} P_1$. Choose $u \in \operatorname{int} N(P_1, x)$ and $y \in F(P_2, u)$; then $y \in \operatorname{ext} P_2$. Let $v \in N(P_1, x)$ and $\lambda \in [0, 1)$; then $v_\lambda := (1 - \lambda)u + \lambda v \in \operatorname{int} N(P_1, x)$, hence $\dim F(P_2, v_\lambda) = \dim F(P_1, v_\lambda) =$

$\dim\{x\} = 0$ and, therefore, $F(P_2, v_\lambda) = \{y_\lambda\}$ for some $y_\lambda \in \operatorname{ext} P_2$, which implies $v_\lambda \in \operatorname{int} N(P_2, y_\lambda)$. Since this holds for all $\lambda \in [0, 1)$, we conclude that $y_\lambda = y$. It follows that $v \in N(P_2, y)$, thus $N(P_1, x) \subset N(P_2, y)$. Since $x \in \operatorname{ext} P_1$ was arbitrary and the roles of P_1 and P_2 can be interchanged, it follows that $N(P_1, x) = N(P_2, y)$. Thus (2.30) holds.

Suppose, conversely, that (2.30) is true. Let $u \in \mathbb{S}^{n-1}$ and choose a vertex x_1 of $F(P_1, u)$. There is a vertex x_2 of P_2 for which $N(P_2, x_2) = N(P_1, x_1)$. The vector u lies in the relative interior of a face of $N(P_i, x_i)$ of dimension $n - \dim F(P_i, u)$ (as follows from (2.25) and (2.26)), hence $\dim F(P_1, u) = \dim F(P_2, u)$. Since u was arbitrary, P_1 and P_2 are strongly isomorphic. □

Corollary 2.4.12 *For $P_1, P_2 \in \mathcal{P}^n$, all polytopes $\lambda_1 P_1 + \lambda_2 P_2$ with $\lambda_1, \lambda_2 > 0$ are strongly isomorphic. If P_1, P_2 are strongly isomorphic, then all polytopes $\lambda_1 P_1 + \lambda_2 P_2$ with $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 > 0$ are strongly isomorphic.*

Proof Let $u \in \mathbb{S}^{n-1}$ and write $F_i := F(P_i, u)$, $i = 1, 2$. By Theorem 1.7.5(c),

$$\dim F(\lambda_1 P_1 + \lambda_2 P_2, u) = \dim(\lambda_1 F_1 + \lambda_2 F_2),$$

which is independent of λ_1, λ_2 if both numbers are positive. This proves the first part. If P_1, P_2 are strongly isomorphic, it follows from (2.27) and Lemma 2.4.11 that $N(P_1, F_1) = N(P_2, F_2)$. Therefore, the affine hulls of F_1 and F_2 are translates of each other, hence $\dim(\lambda_1 F_1 + \lambda_2 F_2)$ is independent of $\lambda_1, \lambda_2 \geq 0$ if $\lambda_1 + \lambda_2 > 0$. □

Let P_1, P_2 be strongly isomorphic polytopes. Let F be a proper face of P_1. The normal cone $N(P_1, F)$ is a face of $N(P_1, x)$ for some vertex x of F, hence by Lemma 2.4.11 a face of $N(P_2, y)$ for some vertex y of P_2 and thus the normal cone of a unique face $\phi(F)$ of P_2. It is clear (using (2.24)) that ϕ (extended to improper faces) is a bijective map from $\mathcal{F}(P_1)$ to $\mathcal{F}(P_2)$ that preserves inclusions, that is, a combinatorial isomorphism. This explains the term *strongly isomorphic*. In the literature, strongly isomorphic polytopes have also been called *analogous* (for this reason, we speak of the *a*-type of a polytope), *strongly combinatorially isomorphic*, *strongly combinatorially equivalent* or *locally similar*.

Lemma 2.4.13 *Let P be a simple n-polytope with normal vectors $u_1, \dots, u_k$. Then there is a number $\beta > 0$ such that every polytope of the form*

$$P' := \bigcap_{i=1}^{k} H^-_{u_i, h(P,u_i)+\alpha_i} \tag{2.31}$$

with $|\alpha_i| \leq \beta$ is simple and strongly isomorphic to P.

Proof We can choose pairwise disjoint balls $B(x, \rho)$ for $x \in \operatorname{ext} P$ such that each ball $B(x, \rho)$ meets only those facets of P that contain x. Next, we can choose a number $\beta > 0$ with the following property: if x is a vertex of P and if, say, $u_1, \dots, u_n$ are the outer normal vectors of the facets containing x, then for $|\alpha_i| \leq \beta$ the intersection point of the hyperplanes $H_{u_i, h(P,u_i)+\alpha_i}$ $(i = 1, \dots, n)$ lies in $B(x, \rho)$, and $H_{u_j, h(P,u_j)+\alpha_j}$ does not meet $B(x, \rho)$ for $j \notin \{1, \ldots, n\}$. It is then clear that P' defined by (2.31) with $|\alpha_i| \le \beta$ is a polytope having a vertex x' in each of the balls $B(x, \rho)$ ($x \in \operatorname{ext} P$) such that $N(P', x') = N(P, x)$. We deduce that P' has no further vertices and that it is simple and strongly isomorphic to P, by Lemma 2.4.11. □

Lemma 2.4.14 *Let P be an n-polytope with normal vectors $u_1, \ldots, u_k$. Then to each $\beta > 0$, numbers $\alpha_1, \ldots, \alpha_k$ with $|\alpha_i| \le \beta$ can be chosen such that*

$$P' =: \bigcap_{i=1}^{k} H^-_{u_i, h(P, u_i) + \alpha_i}$$

is a simple n-polytope with normal vectors $u_1, \ldots, u_k$ and the property that each normal cone of a vertex of P' is contained in some normal cone of P.

Proof We choose balls $B(x, \rho)$ ($x \in \operatorname{ext} P$) as in the proof of Lemma 2.4.13. If $|\alpha_1|, \ldots, |\alpha_k|$ are sufficiently small, then P' is an n-polytope with normal vectors $u_1, \ldots, u_k$ and such that any vertex x' of P' is contained in some ball $B(x, \rho)$. By Theorem 2.4.9, $N(P', x')$ is then the positive hull of only those normal vectors that belong to facets of P containing x; hence $N(P', x') \subset N(P, x)$. By choosing the α_i successively in an appropriate way, we can clearly achieve that the resulting polytope P' is simple. □

We are now in a position to prove the following result on simultaneous approximation of convex bodies by strongly isomorphic polytopes.

Theorem 2.4.15 *Let $K_1, \ldots, K_m \in \mathcal{K}^n$ be convex bodies. To each $\varepsilon > 0$ there exist simple strongly isomorphic polytopes $P_1, \ldots, P_m \in \mathcal{P}^n$ satisfying $\delta(K_i, P_i) < \varepsilon$ for $i = 1, \ldots, m$.*

Proof Let $\varepsilon > 0$ be given. Theorem 1.8.16 yields the existence of polytopes $Q_1, \ldots, Q_m \in \mathcal{P}^n$ with $\delta(K_i, Q_i) < \varepsilon/2$ for $i = 1, \ldots, m$. Let $P := Q_1 + \cdots + Q_m$ (which is also a polytope) and let $u_1, \ldots, u_k$ be the normal vectors of P. Starting with P, we construct P' as in Lemma 2.4.14 and put $P_i := Q_i + \alpha P'$, where $\alpha > 0$ is sufficiently small so that $\delta(P_i, Q_i) < \varepsilon/2$ and hence $\delta(K_i, P_i) < \varepsilon$ for $i = 1, \ldots, m$. Let x be a vertex of some P_i. Then $x = q + \alpha p$ with suitable $q \in \operatorname{ext} Q_i$ and $p \in \operatorname{ext} P'$. From Theorem 2.2.1 we have

$$N(P_i, x) = N(Q_i, q) \cap N(P', p).$$

By the construction of P', the normal cone $N(P', p)$ is contained in some normal cone of P, and by Theorem 2.2.1 the latter is contained in some normal cone of Q_i, necessarily $N(Q_i, q)$, because $N(P', p) \cap N(Q_i, q)$ is n-dimensional. Thus $N(P_i, x) = N(P', p)$. Since x was an arbitrary vertex of P_i, we deduce from Lemma 2.4.11 that P_i is strongly isomorphic to P'. This implies also that P_i is simple. □

Notes for Section 2.4

1. Standard references on convex polytopes, from which we have taken some of the basic arguments, are the books by Grünbaum [849] and Ziegler [2079]; see also McMullen and Shephard [1398] and Brøndsted [338].
2. *Strongly isomorphic polytopes*. The approximation theorem 2.4.15 is due to Aleksandrov [13]. He defined strongly isomorphic polytopes in a different but equivalent way (and called them *analogous*). Our definition appears in Shephard [1776], where the term 'locally similar' is used. Still another term was introduced by Grünbaum [848], p. 50.

   Lemma 2.4.11 can also be found, in essence, in Meyer [1419] (Corollary 2.7) and in Kallay [1059].

   Corollary 2.4.12, showing the convexity of the a-types, can be refined considerably. McMullen [1376], using the technique of diagrams and representations of polyhedra, showed how, for given normal vectors, the set of the closures of the a-types of translation classes can be viewed as a polyhedral complex.
3. *Monotypic polytopes*. McMullen, Schneider and Shephard [1397] studied the n-polytopes P, called *monotypic*, with the property that the a-type of P is uniquely determined by the system of normal vectors of P.
4. *Approximation by polytopes*. While in this book the approximation of convex bodies by polytopes is only used as a tool, it is an interesting topic in its own right and can be studied under various aspects. We refer the reader to the survey articles by Gruber [818, 828], which contain many references.

   Many asymptotic results about convex hulls of random points or intersections of random halfspaces concern the approximation of convex bodies by random polytopes. We refer to the following surveys, which contain many references: Schneider [1709], Bárány [144], Subsection 8.2.4 in the book by Schneider and Weil [1740], Reitzner [1569].

2.5 Higher regularity and curvature

Another special class of convex bodies, besides the polytopes, with an accessible boundary structure are the convex bodies whose boundaries are regular submanifolds of $\mathbb{R}^n$, in the sense of differential geometry, and which satisfy suitable differentiability properties. The present section is mainly concerned with the second-order boundary structure of such convex bodies. The central notions are those of curvatures and radii of curvature.

For the purpose of introduction, we begin with a general notion of curvature in a direction. Let $K \in \mathcal{K}^n_n$ be a convex body with interior points. We assume that x is a smooth boundary point of K. Let $u = u_K(x)$ be the unique outer unit normal vector to K at x. The linear subspace $H(K,u) - x$ of $\mathbb{R}^n$ is called the *tangent space* of K at x and is denoted by T_xK. We can choose a number $\varepsilon > 0$ and a neighbourhood U of x such that $U \cap \operatorname{bd} K$ can be described in the form

$$U \cap \operatorname{bd} K = \{x + t - f(t)u : t \in T_xK \cap B(o,\varepsilon)\}, \tag{2.32}$$

where $f : T_xK \cap B(o,\varepsilon) \to \mathbb{R}$ is a convex function which satisfies $f \geq 0$ and $f(o) = 0$.

Let $t \in T_xK \cap \mathbb{S}^{n-1}$ be a given unit tangent vector of K at x. For $0 < \tau < \varepsilon$, let $c(\tau)$ be the centre of the circle lying in the plane through x spanned by u and t, going through the point $x + \tau t - f(\tau t)u$ and touching $H(K,u)$ at x. If $(\tau_j)_{j\in\mathbb{N}}$ is a sequence such that $\tau_j \to 0$ for $j \to \infty$ and $(c(\tau_j))_{j\in\mathbb{N}}$ converges to some point $x - \rho u$, and thus

$$\rho = \lim_{j\to\infty} \frac{\tau_j^2}{2f(\tau_j t)},$$

then ρ is called a *radius of curvature* of K at x in direction t. The set of all numbers ρ (∞ admitted) arising in this way is an interval $[\rho_i(x,t), \rho_s(x,t)]$ with $0 \le \rho_i(x,t) \le \rho_s(x,t) \le \infty$. We call

$$\rho_i(x,t) = \liminf_{\tau\downarrow 0} \frac{\tau^2}{2f(\tau t)}$$

the *lower radius of curvature* and

$$\rho_s(x,t) = \limsup_{\tau\downarrow 0} \frac{\tau^2}{2f(\tau t)}$$

the *upper radius of curvature* of K at x in direction t. The reciprocal values (possibly ∞)

$$\kappa_i(x,t) := 1/\rho_s(x,t), \quad \kappa_s(x,t) := 1/\rho_i(x,t)$$

are, respectively, the *lower curvature* and *upper curvature* of K at x in direction t. If both values coincide and are finite, their common value $\kappa(x,t)$ is the *curvature* of K at x in direction t.

Now we introduce differentiability assumptions. A convex body $K \in \mathcal{K}_n^n$ is said to be of class C^k, for some $k \in \mathbb{N}$, if its boundary hypersurface is a regular submanifold of $\mathbb{R}^n$, in the sense of differential geometry, that is k-times continuously differentiable. The body K is of class C^∞ if it is of class C^k for each $k \in \mathbb{N}$. It was already stated in Theorem 2.2.4 that a smooth convex body (that is, a convex body with only regular boundary points) is of class C^1.

We assume that the reader is familiar with the elementary differential geometry of hypersurfaces of class C^2. We recall some of the basic definitions and facts, restricting ourselves to boundary hypersurfaces of convex bodies.

Let $K \in \mathcal{K}_n^n$ be of class C^2. We identify, in the canonical way, the abstract tangent spaces of $\mathbb{R}^n$ with the vector space $\mathbb{R}^n$ itself and thus also the abstract tangent space of the submanifold $\mathrm{bd}\, K$ at x with the linear subspace $T_x K$, as introduced above. For $x \in \mathrm{bd}\, K$, the vector $u_K(x)$ is the outer unit normal vector of K at x. The map $u_K : \mathrm{bd}\, K \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$ thus defined is the *spherical image map (or Gauss map)* of K; it is of class C^1. Its differential at x, $\mathrm{d}(u_K)_x =: W_x$ maps $T_x K$ into itself. The linear map $W_x : T_x K \to T_x K$ is called the *Weingarten map*. The bilinear form defined on $T_x K$ by

$$\mathrm{II}_x(v,w) := \langle W_x v, w\rangle \quad \text{for } v, w \in T_x K$$

is the *second fundamental form* of bd K at x. The *first fundamental form* is just the restriction of the scalar product to T_xK,

$$\mathrm{I}_x(v,w) := \langle v,w\rangle \quad \text{for } v,w \in T_xK.$$

In local coordinates, the foregoing can be expressed as follows. Let $X : M \to \mathbb{R}^n$, where $M \subset \mathbb{R}^{n-1}$ is open, be a parametrization of class C^2 of a neighbourhood of x on bd K where, say, $x = X(y)$ with $y \in M$. Put $N := u_K \circ X$, so that $N : M \to \mathbb{S}^{n-1}$ is a map of class C^1. We have $\mathrm{d}N_y = \mathrm{d}(u_K)_x \circ \mathrm{d}X_y$ and hence $W_x = \mathrm{d}N_y \circ \mathrm{d}X_y^{-1}$. For a differentiable function φ on M we denote by φ_i the ith partial derivative with respect to a given orthonormal basis $(e_1,\dots,e_{n-1})$ of $\mathbb{R}^{n-1}$. Then $\mathrm{d}X(e_i) = X_i$, $\mathrm{d}N(e_i) = N_i$ and hence $W_xX_i = N_i$ (as usual, we omit the argument y). The matrix $(l_{ij})_{i,j=1}^{n-1}$ of the bilinear form II_x with respect to the standard basis $(X_1,\dots,X_{n-1})$ of T_xK has entries

$$l_{ij} = \mathrm{II}_x(X_i,X_j) = \langle W_xX_i,X_j\rangle = \langle N_i,X_j\rangle = -\langle N,X_{ij}\rangle, \tag{2.33}$$

the latter because of $\langle N,X_j\rangle = 0$. Thus, II_x is symmetric, and the Weingarten map is selfadjoint (with respect to $\langle\cdot,\cdot\rangle$). The matrix $(g_{ij})_{i,j=1}^{n-1}$ of the first fundamental form is given by $g_{ij} = \langle X_i,X_j\rangle$. For the matrix $(l_i^j)_{i,j=1}^{n-1}$ of the Weingarten map, which is defined by

$$N_i = \sum_{j=1}^{n-1} l_i^j X_j, \quad i = 1,\dots,n-1, \tag{2.34}$$

one obtains $l_{ik} = \langle N_i,X_k\rangle = \sum_j l_i^j\langle X_j,X_k\rangle = \sum_j l_i^j g_{jk}$, hence

$$l_i^j = \sum_{k=1}^{n-1} l_{ik}g^{kj}, \tag{2.35}$$

where $(g^{ij})_{i,j}$ denotes the inverse matrix of $(g_{ij})_{i,j}$.

The eigenvalues of the Weingarten map W_x are, by definition, the *principal curvatures* of K at x. The equality $W_xv = \lambda v$ for $v \in T_xK \setminus \{o\}$ and $\lambda \in \mathbb{R}$ is equivalent to $\mathrm{II}_x(v,w) = \lambda\mathrm{I}_x(v,w)$ for all $w \in T_xK$. Thus, the principal curvatures are also the eigenvalues of the second fundamental form relative to the first fundamental form. In local coordinates as above, $v = \sum_{i=1}^{n-1} v^iX_i$ is an eigenvector of the Weingarten map and λ is a corresponding principal curvature if and only if

$$\sum_{i=1}^{n-1}\left(l_i^j - \lambda\delta_i^j\right)v^i = 0,$$

or equivalently

$$\sum_{i=1}^{n-1}\left(l_{ij} - \lambda g_{ij}\right)v^i = 0,$$

for $j = 1,\dots,n-1$.

For the special representation given by (2.32), that is, $X(t) = x + t - f(t)u$ for $t \in T_xK \cap B_0(o,\varepsilon)$, one obtains (at the point x) that $l_{ij} = f_{ij}$ and $g_{ij} = \delta_{ij}$ and hence that the principal curvatures at x are the eigenvalues of the Hessian matrix of f. This provides a simple interpretation of the principal curvatures and shows that they are nonnegative, by Theorem 1.5.13.

By H_j we denote the jth normalized elementary symmetric function of the principal curvatures; thus $H_0 = 1$ and

$$H_j = \binom{n-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le n-1} k_{i_1} \cdots k_{i_j} \tag{2.36}$$

for $j = 1, \dots, n-1$. We refer to H_1 as the *mean curvature* and to H_{n-1} as the *Gauss–Kronecker curvature*.

As is well known from differential geometry, under our present assumptions the curvature $\kappa(x,t)$ defined initially exists and is given by

$$\kappa(x,t) = \frac{\mathrm{II}_x(t,t)}{\langle t,t\rangle} \quad \text{for } t \in T_xK \setminus \{o\}. \tag{2.37}$$

Let $(v_1, \dots, v_{n-1})$ be an orthonormal basis of T_xK consisting of eigenvectors of the Weingarten map W_x, and let k_i be the principal curvature corresponding to v_i. Then

$$\kappa(x,t) = k_1\langle t, v_1\rangle^2 + \cdots + k_{n-1}\langle t, v_{n-1}\rangle^2 \tag{2.38}$$

for each unit vector $t \in T_xK$. This equation is known as *Euler's theorem*.

The following assumption, stronger than C^k, will often be important. We say that K is of class C^k_+ (for $k \ge 2$) if K is of class C^k and the spherical image map $u_K : \mathrm{bd}\, K \to \mathbb{S}^{n-1}$ is a diffeomorphism (of class C^1, and hence C^{k-1}). This is equivalent to the assumption that its differential, the Weingarten map, is everywhere of maximal rank, and thus to the assumption that all principal curvatures are non-zero, or equivalently that $H_{n-1} \neq 0$.

Let K be of class C^2_+. Then the map u_K has an inverse u_K^{-1} of class C^1, and for the support function h_K of K we get

$$h_K(u) = \langle u_K^{-1}(u), u\rangle;$$

hence h_K is differentiable on $\mathbb{S}^{n-1}$ and thus on $\mathbb{R}^n \setminus \{o\}$. From Corollary 1.7.3 we have

$$\nabla h_K(u) = u_K^{-1}\left(\frac{u}{|u|}\right)$$

for $u \in \mathbb{R}^n \setminus \{o\}$. It follows that the support function h_K is, in fact, of class C^2.

For several investigations on convex bodies, particularly in connection with Minkowski addition, it appears more natural to impose differentiability assumptions on the support function than on the boundary. Let $K \in \mathcal{K}^n_n$. By Corollary 1.7.3, the support function h_K is differentiable on $\mathbb{R}^n \setminus \{o\}$ if and only if K is strictly convex, and if this is satisfied, then h_K is continuously differentiable. Let h_K be of class C^1 (note that K may well have singularities, so that $\mathrm{bd}\, K$ need not be of class C^1). For $u \in \mathbb{R}^n \setminus \{o\}$, let $\xi_K(u) = x_K(u/|u|)$ be the unique point of $\mathrm{bd}\, K$ at which u is attained as an outer normal vector. The map ξ_K thus defined on $\mathbb{R}^n \setminus \{o\}$ is positively homogeneous of degree zero. By Corollary 1.7.3,

$$\xi_K(u) = \nabla h_K(u). \tag{2.39}$$

Now we assume that h_K is of class C^2. Then ξ_K is of class C^1. Its restriction to $\mathbb{S}^{n-1}$, the map $x_K : \mathbb{S}^{n-1} \to \operatorname{bd} K$, is the reverse spherical image map, introduced in Section 2.2. For $u \in \mathbb{S}^{n-1}$, let T_u be the $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ that is orthogonal to u; thus T_u is the tangent space of $\mathbb{S}^{n-1}$ at u. The differential $\mathrm{d}(x_K)_u =: \overline{W}_u$ maps T_u into T_u. We call the linear map $\overline{W}_u : T_u \to T_u$ the *reverse Weingarten map* and the bilinear form

$$\overline{\mathrm{II}}_u(v, w) := \langle \overline{W}_u v, w \rangle, \qquad v, w \in T_u,$$

the *reverse second fundamental form* of K at u.

Let $N : M \to \mathbb{R}^n$, where $M \subset \mathbb{R}^{n-1}$ is open, be a parametrization of class C^2 of a neighbourhood of u on $\mathbb{S}^{n-1}$ and let $u = N(y)$. Put $X := x_K \circ N$. We have $\mathrm{d}X_y = \mathrm{d}(x_K)_u \circ \mathrm{d}N_y$, hence $\overline{W}_u = \mathrm{d}X_y \circ \mathrm{d}N_y^{-1}$ and thus $\overline{W}_u N_i = X_i$. The matrix $(b_{ij})_{i,j=1}^{n-1}$ of $\overline{\mathrm{II}}_u$ with respect to the standard basis $(N_1, \dots, N_{n-1})$ of T_u is given by

$$\begin{aligned} b_{ij} &= \overline{\mathrm{II}}_u(N_i, N_j) = \langle \overline{W}_u N_i, N_j \rangle = \langle X_i, N_j \rangle \\ &= \sum_{k=1}^{n} \sum_{m=1}^{n} (\partial_k \partial_m h_K) \circ N\, \eta_i^k \eta_j^m \end{aligned} \tag{2.40}$$

by (2.39), where $N = (\eta^1, \dots, \eta^n)$. Thus, $\overline{\mathrm{II}}_u$ is symmetric and $\overline{W}_u$ is selfadjoint. The first fundamental form $\bar{\mathrm{I}}_u$ of the sphere $\mathbb{S}^{n-1}$ at u is given by $\bar{\mathrm{I}}_u(v, w) := \langle v, w \rangle$ for $v, w \in T_u$. Its matrix with respect to $(N_1, \dots, N_{n-1})$ is denoted by $(e_{ij})_{i,j=1}^{n-1}$; thus $e_{ij} = \bar{\mathrm{I}}_u(N_i, N_j) = \langle N_i, N_j \rangle$. Introducing the matrix $(b_i^j)_{i,j=1}^{n-1}$ of the reverse Weingarten map by

$$X_i = \sum_{j=1}^{n-1} b_i^j N_j, \qquad i = 1, \dots, n-1, \tag{2.41}$$

we find that $b_{ik} = \langle X_i, N_k \rangle = \sum_j b_i^j \langle N_j, N_k \rangle = \sum_j b_i^j e_{jk}$ and hence

$$b_i^j = \sum_{k=1}^{n-1} b_{ik} e^{kj}, \tag{2.42}$$

where $(e^{ij})_{i,j}$ is the matrix inverse to $(e_{ij})_{i,j}$.

The eigenvalues $r_1, \dots, r_{n-1}$ of the reverse Weingarten map are called the *principal radii of curvature* of K at u. It should be kept in mind that, while the principal curvatures are functions on the boundary of K, the principal radii of curvature are considered as functions of the outer unit normal vector, in other words, as functions on the spherical image. We denote by s_j the jth normalized elementary symmetric function of the principal radii of curvature:

$$s_j = \binom{n-1}{j}^{-1} \sum_{1\le i_1<\cdots<i_j\le n-1} r_{i_1}\cdots r_{i_j}. \tag{2.43}$$

The vector $v \in T_u \setminus \{o\}$ is an eigenvector of the reverse Weingarten map and λ is a corresponding radius of curvature if $\overline{W}_u v = \lambda v$, or, equivalently, $\overline{\mathrm{II}}_u(v,w) = \lambda \overline{\mathrm{I}}_u(v,w)$ for all $w \in T_u$. In terms of the local parametrization N of $\mathbb{S}^{n-1}$ used above and with $v = \sum_{i=1}^{n-1} v^i N_i$, this is equivalent to

$$\sum_{i=1}^{n-1}\left(b_i^j - \lambda\delta_i^j\right)v^i = 0, \quad j = 1,\dots,n-1 \tag{2.44}$$

and to

$$\sum_{i=1}^{n-1}\left(b_{ij} - \lambda e_{ij}\right)v^i = 0, \quad j = 1,\dots,n-1. \tag{2.45}$$

In particular, the product of the principal radii of curvature is given by

$$s_{n-1}\circ N = \det\left(b_i^j\right) = \frac{\det(b_{ij})}{\det(e_{ij})}. \tag{2.46}$$

(Here and in the following, we commit a misuse of notation, writing $\det(a_{ij})$ for the determinant of the matrix $(a_{ij})_{i,j=1}^n$.)

The following remark will later play a role. Since the principal radii of curvature, denoted by $r_1,\dots,r_{n-1}$, are the eigenvalues of the matrix $\left(b_i^j\right)_{i,j}$, the matrix $\left(b_i^j + \rho\delta_i^j\right)_{i,j}$ with $\rho \in \mathbb{R}$ has the eigenvalues $r_1+\rho,\dots,r_{n-1}+\rho$, hence

$$\det\left(b_i^j + \rho\delta_i^j\right) = \sum_{k=0}^{n-1}\rho^k\binom{n-1}{k}s_{n-1-k}\circ N. \tag{2.47}$$

In the special case where $K \in \mathcal{K}_n^n$ is of class C_+^2, we have seen above that its support function h_K is of class C^2, and we have $x_K(u) = u_K^{-1}(u)$ for $u \in \mathbb{S}^{n-1}$. This yields

$$\overline{W}_u = \mathrm{d}(x_K)_u = (\mathrm{d}(u_K)_{x_K(u)})^{-1} = W_{x_K(u)}^{-1}, \tag{2.48}$$

so that the reverse Weingarten map at u coincides with the inverse Weingarten map taken at $x_K(u)$. In particular, the eigenvectors of $W_{x_K(u)}$ and $\overline{W}_u$ are the same, all principal radii of curvature are positive and

$$r_i(u) = \frac{1}{k_i(x_K(u))} \tag{2.49}$$

if the ordering is chosen properly. This implies that

$$s_j(u) = \frac{H_{n-1-j}}{H_{n-1}}(x_K(u)), \tag{2.50}$$

$$H_j(x) = \frac{s_{n-1-j}}{s_{n-1}}(u_K(x)), \tag{2.51}$$

for $j = 1, \dots, n-1$.

It may sometimes be desirable to express the reverse Weingarten map more directly in terms of the support function. This can be achieved by means of the following lemma. Let $\mathrm{d}^2 h_u$ be the second differential of $h = h_K$ at $u \in \mathbb{R}^n \setminus \{o\}$, considered as a bilinear form on $\mathbb{R}^n$. For $u \in \mathbb{R}^n \setminus \{o\}$, let π_u be the orthogonal projection onto the orthogonal subspace T_u.

Lemma 2.5.1 *For $u \in \mathbb{R}^n \setminus \{o\}$,*

$$\mathrm{d}^2 h_u(a,b) = |u|^{-1} \langle \overline{W}_{u/|u|} \pi_u a, \pi_u b \rangle \tag{2.52}$$

for all $a, b \in \mathbb{R}^n$.

Proof Let $u \in \mathbb{R}^n \setminus \{o\}$ be given. By homogeneity, we may assume that $|u| = 1$. Let $(e_1, \dots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$ with $e_n = u$. We write $(y^1, \dots, y^n)$ for the coordinates of $y \in \mathbb{R}^n$ and h_{ij} for the second partial derivatives of h with respect to this basis. The homogeneity relation

$$\sum_{j=1}^{n} h_{ij}(y) y^j = 0 \quad \text{for } i = 1, \dots, n \tag{2.53}$$

yields $h_{in}(u) = 0$ and hence

$$\mathrm{d}^2 h_u(a,b) = \sum_{i,j=1}^{n-1} h_{ij}(u) a^i b^j \quad \text{for } a, b \in \mathbb{R}^n.$$

From (2.39) we get, for $i, j \le n-1$,

$$h_{ij}(u) = \langle (\xi_K)_i(u), e_j \rangle = \langle \mathrm{d}(\xi_K)_u(e_i), e_j \rangle = \langle \mathrm{d}(x_K)_u(e_i), e_j \rangle = \langle \overline{W}_u e_i, e_j \rangle,$$

hence

$$\langle \overline{W}_u \pi_u a, \pi_u b \rangle = \sum_{i,j=1}^{n-1} h_{ij}(u) a^i b^j,$$

which completes the proof of the lemma. □

Immediately from (2.53) and (2.52) we see the following.

Corollary 2.5.2 *The eigenvectors (with respecto to $\langle \cdot, \cdot \rangle$) of the second differential of h_K at $u \in \mathbb{S}^{n-1}$ are u, with corresponding eigenvalue 0, and the eigenvectors of the reverse Weingarten map at u, with corresponding eigenvalues $r_1, \dots, r_{n-1}$, the principal radii of curvature.*

It now follows from Theorem 1.5.13 that the principal radii of curvature are non-negative.

Corollary 2.5.3 *Let $K \in \mathcal{K}^n_n$ have a support function h of class C^2. For $j \in \{1, \dots, n-1\}$, the number $\binom{n-1}{j} s_j(u)$, the jth elementary symmetric function of the principal radii of curvature of K at $u \in \mathbb{S}^{n-1}$, is equal to the sum of the principal minors of order j of the Hessian matrix (with respect to any orthonormal basis of $\mathbb{R}^n$) of h at u.*

In particular,

$$s_1(u) = \frac{1}{n-1} \Delta h(u), \tag{2.54}$$

where Δ denotes the Laplace operator on $\mathbb{R}^n$.

If the orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ is chosen such that $e_n = u$, then

$$s_{n-1}(u) = \det \left(h_{ij}(u) \right)_{i,j=1}^{n-1}. \tag{2.55}$$

We mention briefly which form (2.54) and (2.55) take if we work with the restriction $\overline{h}$ of the support function of K to the unit sphere $\mathbb{S}^{n-1}$. The assumptions and notation are as in the paragraphs before (2.39); in particular, $N : M \to \mathbb{R}^n$ is a local C^2 parametrization of $\mathbb{S}^{n-1}$ around u, and $X = x_K \circ N$. Then $\overline{h} \circ N = \langle X, N \rangle$. Using classical tensor notation, with covariant derivatives on $\mathbb{S}^{n-1}$ denoted by $_{\|j}$, we obtain $\overline{h}_i = \langle X, N_i \rangle$ (since $\langle X_i, N \rangle = 0$) and

$$\overline{h}_{i\|j} = \langle X_j, N_i \rangle + \langle X, N_{i\|j} \rangle = b_{ij} - \langle X, e_{ij} N \rangle = b_{ij} - \overline{h} e_{ij},$$

because, on the sphere, $N_{i\|j} = -e_{ij} N$. Thus,

$$b_{ij} = \overline{h}_{i\|j} + \overline{h} e_{ij}.$$

Therefore, at u,

$$s_1 = \overline{h} + \frac{1}{n-1} \Delta_S \overline{h}, \tag{2.56}$$

where Δ_S denotes the spherical Laplace operator, and

$$s_{n-1} = \frac{\det \left(\overline{h}_{i\|j} + \overline{h} e_{ij} \right)}{\det \left(e_{ij} \right)}. \tag{2.57}$$

A different approach to (2.56) and the spherical Laplace operator (see, e.g., Seeley [1764]) is as follows. If $f : \mathbb{S}^{n-1} \to \mathbb{R}$ is a function of class C^2, one extends it to $\mathbb{R}^n \setminus \{o\}$ as a positively homogeneous function of degree zero; then $\Delta_S f$ is the restriction of Δf to $\mathbb{S}^{n-1}$. If $f : \mathbb{R}^n \setminus \{o\} \to R$ is positively homogeneous of degree k, then

$$\Delta f = \Delta_S f + k(k+n-2) f \quad \text{on } \mathbb{S}^{n-1}.$$

Hence, (2.54) gives (2.56).

To derive a counterpart to Euler's theorem (2.38), let $K \in \mathcal{K}^n_n$ have a support function of class C^2. Let $u \in \mathbb{S}^{n-1}$ and a unit vector $t \in T_u$ be given. We denote by $r(u, t)$ the radius of curvature, at u, of the image $\mathrm{proj}_E K$ of K under orthogonal projection to the linear subspace E spanned by u and t. To compute $r(u,t)$, we can choose an orthonormal basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$ such that $e_1 = t$ and $e_n = u$. The support function of $\mathrm{proj}_E K$ is just the restriction of h_K to E, hence (2.54) gives

$$r(u,t) = h_{11}(u) + h_{nn}(u) = h_{11}(u),$$

where the notation is as in the proof of Lemma 2.5.1. By that lemma,

$$\overline{\mathrm{II}}_u(t,t) = \langle \overline{W}_u t, t\rangle = \mathrm{d}^2 h_u(t,t) = \sum_{i,j=1}^{n-1} h_{ij}(u)\delta_1^i\delta_1^j = h_{11}(u).$$

Thus,

$$r(u,t) = \frac{\overline{\mathrm{II}}_u(t,t)}{\langle t,t\rangle} \quad \text{for } t \in T_u \setminus \{o\}, \tag{2.58}$$

in analogy to (2.37), and in analogy to Euler's theorem (2.38) we have

$$r(u,t) = r_1\langle t, v_1\rangle^2 + \cdots + r_{n-1}\langle t, v_{n-1}\rangle^2 \tag{2.59}$$

for each unit vector $t \in T_u$, where v_i is the unit eigenvector of $\overline{W}_u$ corresponding to r_i.

For $n = 2$, there is only one principal radius of curvature, and we call it the radius of curvature, denoted by r. In the planar case, it is often more convenient not to use the homogeneous support function h_K, but the function $h : [0, 2\pi] \to \mathbb{R}$ defined by $h(\alpha) = h_K(u_\alpha)$, where $u_\alpha = e_1 \cos\alpha + e_2 \sin\alpha$ and (e_1, e_2) is an orthonormal basis of $\mathbb{R}^2$. Using the homogeneity relations, one then computes $\Delta h_K(u_\alpha) = (h'' + h)(\alpha)$; hence the radius of curvature is given by

$$r(u_\alpha) = (h'' + h)(\alpha). \tag{2.60}$$

Above, we have considered convex bodies with either a regular boundary hypersurface of class C^2 or a support function of class C^2, and we have shown that a convex body of class C^2_+ has a support function of class C^2. To have complete symmetry, it remains to show that a convex body $K \in \mathcal{K}^n_n$ with a support function of class C^2 and with everywhere positive radii of curvature is necessarily of class C^2_+. For the proof, we may assume that $o \in \mathrm{int}\, K$. By (1.52), the radial function of the polar body K° is given by $\rho(K^\circ,\cdot) = h(K,\cdot)^{-1}$ on $\mathbb{S}^{n-1}$, hence K° is of class C^2. As before, we use a local parametrization $N : M \to \mathbb{R}^n$ of class C^2 of the sphere $\mathbb{S}^{n-1}$ around u. We put $X = x_K \circ N$. From $\overline{W}_u N_i = X_i$ and the assumption that all eigenvalues of $\overline{W}_u$ are non-zero, it follows that $X_1,\dots,X_{n-1}$ are linearly independent. Hence, X is a local parametrization of class C^1 of $\mathrm{bd}\, K$. In particular, it follows that K is of class C^1 and that x_K is a diffeomorphism of class C^1 from $\mathbb{S}^{n-1}$ onto $\mathrm{bd}\, K$. The spherical image map u_{K° of K° can be described by

$$u_{K^\circ} : \rho(K^\circ,u)u \mapsto \frac{x_K(u)}{|x_K(u)|}, \qquad u \in \mathbb{S}^{n-1}$$

(see Remark 1.7.14), and thus is a composition of C^1-diffeomorphisms. Since u_{K° is a diffeomorphism, K° is of class C^2_+ and hence has a support function of class C^2 and with nowhere vanishing radii of curvature. Repetition of the argument, with the roles of K and K° interchanged, shows that K is of class C^2_+.

The elementary symmetric functions s_j and H_j appear in some formulae related to surface area and volume computations. Let $K \in \mathcal{K}^n_n$ be a convex body of class C^2_+. Let $X : M \to \mathbb{R}^n$, where $M \subset \mathbb{R}^{n-1}$ is open, be a local C^2-parametrization of $\operatorname{bd} K$. As is well known from differential geometry, the $\mathcal{H}^{n-1}$-measure of $X(M)$ can be computed from

$$\mathcal{H}^{n-1}(X(M)) = \int_M \mathrm{d}A,$$

with the differential geometric surface area element given by

$$\mathrm{d}A = \sqrt{\det(g_{ij})}\, \mathrm{d}y^1 \cdots \mathrm{d}y^{n-1},$$

where $y^1, \dots, y^{n-1}$ are Cartesian coordinates in M. Since K is of class C^2_+, the map $N = u_K \circ X$ is a local C^1-parametrization of $\mathbb{S}^{n-1}$, hence

$$\mathcal{H}^{n-1}(N(M)) = \int_M \mathrm{d}\sigma$$

with

$$\mathrm{d}\sigma = \sqrt{\det(e_{ij})}\, \mathrm{d}y^1 \cdots \mathrm{d}y^{n-1}.$$

Now (2.41) gives

$$g_{ij} = \langle X_i, X_j \rangle = \left\langle \sum_r b^r_i N_r, \sum_s b^s_j N_s \right\rangle = \sum_{r,s} b^r_i b^s_j e_{rs},$$

hence $\det(g_{ij}) = \left[\det\left(b^j_i\right)\right]^2 \det(e_{ij})$ and thus

$$\mathrm{d}A = \left|\det\left(b^j_i\right)\right| \mathrm{d}\sigma = s_{n-1} \circ N\, \mathrm{d}\sigma$$

by (2.46). We deduce that for any integrable real function f on $\operatorname{bd} K$ we have

$$\int_{\operatorname{bd} K} f(x)\, \mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\mathbb{S}^{n-1}} f(x_K(u)) s_{n-1}(u)\, \mathrm{d}\mathcal{H}^{n-1}(u). \tag{2.61}$$

Similarly, for any integrable real function g on $\mathbb{S}^{n-1}$ we have

$$\int_{\mathbb{S}^{n-1}} g(u)\, \mathrm{d}\mathcal{H}^{n-1}(u) = \int_{\operatorname{bd} K} g(u_K(x)) H_{n-1}(x)\, \mathrm{d}\mathcal{H}^{n-1}(x). \tag{2.62}$$

Next, we compute a 'local parallel volume', which will play an essential role in Chapter 4. For a relatively open subset β of $\operatorname{bd} K$, let

$$A_\rho(K, \beta) := \{x \in \mathbb{R}^n : 0 < d(K, x) \le \rho \text{ and } p(K, x) \in \beta\}$$

for $\rho > 0$. Thus $A_\rho(K,\beta)$ is the set of all points in $\mathbb{R}^n$ at positive distance at most ρ from K and with the nearest point in K falling in β. Similarly, for an open subset ω of $\mathbb{S}^{n-1}$ we define

$$B_\rho(K,\omega) := \{x \in \mathbb{R}^n : 0 < d(K,x) \le \rho \text{ and } u(K,x) \in \omega\}$$

for $\rho > 0$. If K is smooth and strictly convex, we clearly have $A_\rho(K,\beta) = B_\rho(K,u_K(\beta))$. For K of class C^2_+, the measure of this set can be computed as follows. It suffices to assume that $\beta = X(M)$ for some local parametrization X as above. Then, obviously,

$$A_\rho(K,\beta) = \{X(y) + \lambda N(y) : y \in M,\ 0 < \lambda \le \rho\}.$$

Thus, $A_\rho(K,\beta)$ is the image of the region $M \times (0,\rho] \subset \mathbb{R}^{n-1} \times \mathbb{R}$ under the injective map $(y,\lambda) \mapsto X(y) + \lambda N(y)$, which has Jacobian

$$|X_1 + \lambda N_1, \dots, X_{n-1} + \lambda N_{n-1}, N|,$$

where $|\cdot,\dots,\cdot|$ denotes the determinant. For this Jacobian, we obtain from (2.41) and (2.47) that it is equal to

$$\begin{aligned}
&\left|\sum_j \left(b_i^j + \lambda\delta_1^j\right) N_j, \dots, \sum_j \left(b_{n-1}^j + \lambda\delta_{n-1}^j\right) N_j, N\right| \\
&= \det\left(b_i^j + \lambda\delta_i^j\right) \left|N_1, \dots, N_{n-1}, N\right| \\
&= \pm \sum_{k=0}^{n-1} \lambda^k \binom{n-1}{k} s_{n-1-k} \circ N \sqrt{\det(e_{ij})}.
\end{aligned}$$

Here we have used the fact that $|N_1,\dots,N_{n-1},N|^2$ is equal to the square of the $(n-1)$-volume of the parallelepiped spanned by $N_1,\dots,N_{n-1}$, which is equal to the Gram determinant $\det(\langle N_i, N_j\rangle)$. Integration of the absolute value now yields, with $\omega = u_K(\beta)$, or $\beta = x_K(\omega)$,

$$\begin{aligned}
\mathcal{H}^n(A_\rho(K,\beta)) &= \mathcal{H}^n(B_\rho(K,\omega)) \\
&= \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m}\int_\omega s_m \,\mathrm{d}\mathcal{H}^{n-1} \\
&= \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m}\int_\beta H_{n-1-m} \,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned} \tag{2.63}$$

In Chapter 4, this relation will be generalized considerably.

So far, we have considered only one convex body K and have, therefore, not always indicated that the notions introduced depend on K. In the following, we shall write $\kappa(K,x,t)$, $\mathrm{II}_x(K,v,w)$, $\mathrm{II}_x(K)$, $r(K,u,t)$, respectively, for $\kappa(x,t)$, $\mathrm{II}_x(v,w)$, II_x, $r(u,t)$, and so on, where necessary.

The following theorem, on pairs of convex bodies, allows us to deduce a global inclusion result (up to translations) from local curvature comparisons. Here $\mathrm{II}_x \ge \mathrm{II}'_y$ means $\mathrm{II}_x(t,t) \ge \mathrm{II}'_y(t,t)$ for all t.

Theorem 2.5.4 *Let $K, L \in \mathcal{K}^n$ be convex bodies of class C^2_+. Then the following conditions are equivalent.*

(a) $\mathrm{II}_x(L) \geq \mathrm{II}_y(K)$ *for all pairs of points $x \in \mathrm{bd}\, L$ and $y \in \mathrm{bd}\, K$ at which the outer unit normal vectors are the same;*

(b) $r(L, u, t) \leq r(K, u, t)$ *for each orthonormal pair of vectors u, t;*

(c) $h_K - h_L$ *is a support function.*

Proof Of course, (a) is also equivalent to

(a') $\kappa(L, x, t) \geq \kappa(K, y, t)$ for x, y as in (a) and all $t \in T_x L = T_y K$,

and (b) is equivalent to

(b') $\overline{\mathrm{II}}_u(L) \leq \overline{\mathrm{II}}_u(K)$ for $u \in \mathbb{S}^{n-1}$.

By Theorems 1.5.13 and 1.7.1, $h_k - h_L$ is a support function if and only if $\mathrm{d}^2(h_K - h_L)$ is positive semi-definite. By Lemma 2.5.1, this is equivalent to (b′) and thus, by (2.58), to (b). Using (2.48), we have, for $t \in T_u \cap \mathbb{S}^{n-1}$,

$$\overline{\mathrm{II}}_u(t, t) = \langle \overline{W}_u t, t \rangle = \langle W^{-1}_{x_K(u)} t, t \rangle, \qquad \mathrm{II}_{x_K(u)}(t, t) = \langle W_{x_K(u)} t, t \rangle.$$

The equivalence of (a) and (b′) now follows, since $A \geq B$ for selfadjoint positive endomorphisms A, B of T_u implies $B^{-1} \geq A^{-1}$. ($A \geq B$ means $\langle At, t \rangle \geq \langle Bt, t \rangle$ for all $t \in T_u$ and hence is equivalent to the fact that each eigenvalue of A relative to B is ≥ 1. But $Ae_i = \lambda_i Be_i$ is equivalent to $B^{-1}(Be_i) = \lambda_i A^{-1}(Be_i)$; thus the eigenvalues of A relative to B coincide with the eigenvalues of B^{-1} relative to A^{-1}.) □

The assertion (c) in Theorem 2.5.4 implies that for suitable $z \in \mathbb{R}^n$ we have $h_K - h_{L+z} \geq 0$ and thus $L + z \subset K$ (see also Theorem 3.2.12).

The reverse Weingarten map and the notions derived from it have the advantage that they show simple behaviour under Minkowski addition. For given convex bodies $K_1, \ldots, K_m \in \mathcal{K}^n_n$ with support functions of class C^2, let

$$K = \lambda_1 K_1 + \cdots + \lambda_m K_m$$

with $\lambda_1, \ldots, \lambda_m \geq 0$. Then (2.39) implies that

$$x_K(u) = \sum_{r=1}^{m} \lambda_r x_{K_r}(u),$$

which in turn yields

$$\overline{W}_u(K) = \sum_{r=1}^{m} \lambda_r \overline{W}_u(K_r), \qquad \overline{\mathrm{II}}_u(K) = \sum_{r=1}^{m} \lambda_r \overline{\mathrm{II}}_u(K_r).$$

If $N : M \to \mathbb{R}^n$ is a local C^2-parametrization of $\mathbb{S}^{n-1}$ and if we write $X^{(r)} := x_{K_r} \circ N$, then the matrix of $\overline{\mathrm{II}}_u(K_r)$ with respect to the basis $(N_1, \dots, N_{n-1})$ is given by $b_{ij}(K_r) = \langle X_i^{(r)}, N_j \rangle$, hence

$$b_{ij}(K) = \sum_{r=1}^{m} \lambda_r b_{ij}(K_r).$$

In particular, for the product of the principal radii of curvature of the Minkowski linear combination K we get, by (2.46),

$$s_{n-1}(K, \cdot) \circ N = [\det(e_{ij})]^{-1} \det\left(\lambda_1 b_{ij}(K_1) + \cdots + \lambda_m b_{ij}(K_m)\right).$$

Generally, if $A_1, \dots, A_m$ are symmetric real $k \times k$ matrices, the determinant of $\lambda_1 A_1 + \cdots + \lambda_m A_m$ is a homogeneous polynomial of degree k in $\lambda_1, \dots, \lambda_m$. It can be written as

$$\det(\lambda_1 A_1 + \cdots + \lambda_m A_m) = \sum_{i_1, \dots, i_k = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_k} D(A_{i_1}, \dots, A_{i_k}), \tag{2.64}$$

since the coefficient of $\lambda_{i_1}, \dots, \lambda_{i_k}$ does, in fact, depend only on $A_{i_1}, \dots, A_{i_k}$. One may assume that the coefficients are symmetric in their arguments and then they are uniquely determined. $D(A_1, \dots, A_k)$ is called the *mixed discriminant* of $A_1, \dots, A_k$ (see Section 5.5).

We deduce that we have an identity

$$s_{n-1}(\lambda_1 K_1 + \cdots + \lambda_m K_m, u) = \sum_{i_1, \dots, i_{n-1} = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n-1}} s(K_{i_1}, \dots, K_{i_{n-1}}, u) \tag{2.65}$$

for $u \in \mathbb{S}^{n-1}$, where s is symmetric in its first $n-1$ arguments. We call $s(K_1, \dots, K_{n-1}, \cdot)$, which is defined by (2.65), the *mixed curvature function* of $K_1, \dots, K_{n-1}$. In local coordinates as above,

$$s(K_1, \dots, K_{n-1}, \cdot) \circ N = [\det(e_{ij})]^{-1} D\left(\left(b_{ij}(K_1)\right), \dots, \left(b_{ij}(K_{n-1})\right)\right). \tag{2.66}$$

In particular, we see from this and (2.47) that

$$s_j(K, \cdot) = s(\underbrace{K, \dots, K}_{j}, \underbrace{B^n, \dots, B^n}_{n-1-j}, \cdot) \tag{2.67}$$

for $j = 0, \dots, n-1$ and any convex body K with a support function of class C^2.

If we want to express $s(K_1, \dots, K_{n-1}, u)$ by means of support functions, we may, for fixed $u \in \mathbb{S}^{n-1}$, choose an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ with $e_n = u$. If $K \in \mathcal{K}^n$ has a support function h of class C^2, then Corollary 2.5.3 gives

$$s_{n-1}(K, u) = \det\left(h_{ij}(u)\right)_{i,j=1}^{n-1}.$$

We deduce that

$$s(K_1, \dots, K_{n-1}, u) = D\left(\left(h_{ij}(K_1, u)\right)_{i,j=1}^{n-1}, \dots, \left(h_{ij}(K_{n-1}, u)\right)_{i,j=1}^{n-1}\right). \tag{2.68}$$

Notes for Section 2.5

1. *Normals.* Before considering curvatures, which describe second-order properties, we should have a look at first-order properties, for example, the behaviour of normals. A *normal* of the convex body $K \in \mathcal{K}_n^n$ is a line $x + \mathbb{R}u$, where $x \in \operatorname{bd} K$ and u is an outer normal vector of K at x.

   Extending a result of Hann [936], Hug [1001] proved the following. For $K \in \mathcal{K}_n^n$ let $n(K)$ denote the average number of normals that pass through an interior point of K. Then $2 \le n(K) \le V_n(K + \mathrm{D}K)/V_n(K) - 1$, where $\mathrm{D}K$ denotes the difference body of K (see Section 10.1). In particular, $2 \le n(K) \le 3^n - 1$ if K is centrally symmetric. Both of the latter inequalities are sharp.

   Balls, bodies of constant width and centrally symmetric convex bodies can all be characterized by a very specific behaviour of their normals. This motivated Groemer [802] to introduce corresponding deviation measures for normals and to prove estimates for the deviation of a convex body from one of the special types in terms of deviations of the normals.
2. *Jessen's radii of curvature*. The definitions of lower and upper radii of curvature given at the beginning of Section 2.5 are not the only natural ones. To describe a different possibility, we assume that $n = 2$ and that $K \in \mathcal{K}_2^2$ is represented in a neighbourhood of the smooth boundary point x in the way given by (2.32). Let $t \in T_x K \cap \mathbb{S}^1$. For $0 < \tau_j < \varepsilon$ let v_j be an outer unit normal vector of K at $x(\tau_j) := x + \tau_j t - f(\tau_j t)u$, and let $z(\tau_j, v_j)$ be the intersection point of the normals $\{x - \lambda u : \lambda \ge 0\}$ and $\{x(\tau_j) - \lambda v_j : \lambda \ge 0\}$. If $\tau_j \to 0$ for $j \to \infty$ and $z(\tau_j, v_j)$ converges to some point $x - ru$, thus

$$r = \lim_{j\to\infty} \frac{\tau_j}{\sqrt{1 - \langle u, v_j\rangle^2}},$$

   then r is called a *Jessen radius of curvature* of K at x in the direction t. The set of all numbers r (∞ admitted) obtainable in this way is an interval $[r_i, r_s]$ with $0 \le r_i \le r_s \le \infty$. It contains the interval $[\rho_i, \rho_s]$. Jessen [1038] showed that each Jessen radius of curvature, say r, satisfies

$$\rho_s - \sqrt{\rho_s(\rho_s - \rho_i)} \le r \le \rho_s + \sqrt{\rho_s(\rho_s - \rho_i)},$$

   and that these inequalities comprise the only general restrictions for ρ_i, ρ_s, r_i, r_s. In particular, $\rho_i = \rho_s$ implies $r_i = r_s$. Connections with the second-order differentiability properties of f at o are described in Busemann [370], §2.
3. *Complete Riemannian submanifolds with nonnegative sectional curvatures*. If $K \in \mathcal{K}_n^n$ is of class C^∞ (say), then $\operatorname{bd} K$ with the induced Riemannian metric is a Riemannian manifold with nonnegative sectional curvatures. This assertion has an inverse that is not restricted to the compact case.

   Theorem (Sacksteder) If M is a C^∞ $(n-1)$-dimensional complete orientable Riemannian manifold of nonnegative sectional curvatures that is not identically zero, and if $X : M \to \mathbb{R}^n$ is an isometric immersion, then X is an embedding and $X(M)$ is the boundary of a convex set.

   This theorem is due to Sacksteder [1602], with special cases proved before by Hadamard [884], Stoker [1820], van Heijenoort [1868], Chern and Lashof [420]. The Gauss map of such submanifolds was investigated by Wu [1991]. More generally, let $K \subset \mathbb{R}^n$ be a closed convex set with interior points, $K \ne \mathbb{R}^n$, and let $\sigma(K)$ be the set of all outer unit normal vectors at boundary points of K. Then Wu showed that there exist a k-dimensional great subsphere $\mathbb{S}^k$ of $\mathbb{S}^{n-1}$ $(0 \le k \le n-1)$ and a unique subset u_0 of $\mathbb{S}^k$ that is relatively open and (geodesically) convex such that $u_0 \subset \sigma(K) \subset \operatorname{cl} u_0$. Wu also showed by an example that $\sigma(K)$ itself need not be convex.
4. *Boundaries and support functions of class C^2*. That the support function of a convex surface of class C^2 with positive curvatures is of class C^2 was noted by Wintner [1988],

Appendix. Here the assumption of positive curvatures cannot be omitted. Hartman and Wintner [939] gave an example showing that the support function of a convex body need not be of class C^2 even if the boundary is real-analytic, and they also treated some related questions.

5. *Differentiability properties of projections*. Let $K \in \mathcal{K}^n_n$ be a convex body of class C^2_+ and let K' be its projection onto a k-dimensional linear subspace E. The support function h_K of K is of class C^2 and the support function of K' is obtained from h_K by restriction; hence it is also of class C^2. Since all radii of curvature of K are positive, the same holds for K', as can be deduced from (2.58) in connection with Lemma 2.5.1. Thus K' is of class C^2_+.

   Without the assumption of positive curvatures, no such argument is possible, and strange things can happen. The case $n = 3,\ k = 2$ was investigated by Kiselman [1079]. He showed that K' need not be of class C^2 even if K is of class C^∞. Further, he proved: if the boundary of K is of class C^2 with Lipschitz continuous second derivatives, then K' has a twice differentiable boundary. If bd K is real-analytic, then the boundary of K' is of Hölder class $C^{2+\varepsilon}$ for some $\varepsilon > 0$ (but it may be exactly of class $C^{2+2/q}$ for any odd integer $q \geq 3$).
6. *Differentiability properties of Minkowski sums*. Even more unexpected than the results of the previous note are the results that Kiselman [1080] obtained for Minkowski sums. He posed the question whether it is true that $K + L$ is of class C^k if $K, L \in \mathcal{K}^n_n$ are of class C^k. He showed that for $n = 2$ the answer is in the affirmative for $k = 1, 2, 3, 4$ but not for $k = 7$. Further, the boundary of $K + L$, where $K, L \in \mathcal{K}^2_2$ have real-analytic boundaries, is of Hölder class $C^{20/3}$, and this is best possible. He further mentioned an example of convex bodies $K, L \in \mathcal{K}^3_3$ with C^∞ boundaries such that $K + L$ is not of class C^2.

   These investigations were continued by Boman, who showed in [278] the existence of two strictly convex sets $A, B \in \mathcal{K}^2$ with C^∞ boundaries such that the boundary of $A + B$ is not of class C^5. In [279] he proved that there exist convex bodies $A, B \in \mathcal{K}^4$ with real-analytic boundaries such that the boundary of $A + B$ is not of class C^2.

   On the other hand, Krantz and Parks [1146] showed that the sum $A + B$ of two closed convex sets $A, B \subset \mathbb{R}^n$ of which one is bounded and of class $C^{1,\alpha}$, $0 < \alpha \leq 1$, is itself of class $C^{1,\alpha}$.
7. A weaker form of Theorem 2.5.4 is due to Rauch [1560]. He proved, in a different way, that condition (a) of Theorem 2.5.4 implies that some translate of L is contained in K. For the proof given above, see Schneider [1708]; compare also Note 6 in Section 3.2.
8. *Tangential radii of curvature*. The number $r(u, t)$ defined by (2.58) is sometimes called the *tangential radius of curvature* of K at $x_K(u)$ in the direction t. Formula (2.59) was proved by Blaschke [241], p. 117, for $n = 3$; see also Firey [593], p. 12, for the general case.

2.6 Generalized curvatures

In this section we present, mainly without proofs, different approaches to describe the curvature behaviour of general convex bodies. First we consider curvatures that exist almost everywhere on the boundary of a convex body. Let $K \in \mathcal{K}^n_n$, let x be a smooth boundary point of K and let u be the outer unit normal vector of K at x. We may assume that bd K is represented around x by a convex function f, as in (2.32). For $h > 0$, define the set

$$\mathcal{D}(h) := (2h)^{-1/2}\{t \in T_x K \cap B(o, \varepsilon) : f(t) \leq h\}.$$

If the Hausdorff closed limit $\lim_{h\downarrow 0} \mathcal{D}(h) := \mathcal{D}$ exists, this set is called the (Dupin) *indicatrix* of K at x. If K is of class C^2, then differential geometry tells us that $\mathcal{D}$ exists and is given by

$$\mathcal{D} = \left\{ \sum_{i=1}^{n-1} y^i e_i : k_1(y^1)^2 + \cdots + k_{n-1}(y^{n-1})^2 \le 1 \right\} \tag{2.69}$$

if $(e_1, \ldots, e_{n-1})$ is an orthonormal basis of eigenvectors of the Weingarten map at x and k_i is the principal curvature corresponding to e_i. For a general convex body K, the (smooth) point x is called a *normal* point or *Euler point* of K if at x the indicatrix exists and its boundary (if any) is a quadric in $T_x K$ with centre x. If x is a normal point, then the indicatrix at x can be represented in the form (2.69), with a suitable orthonormal basis $(e_1, \ldots, e_{n-1})$, and this defines the principal curvatures $k_1, \ldots, k_{n-1}$. They are nonnegative, since the indicatrix is always a convex set. Furthermore, it follows from the definition of $\mathcal{D}$ that the curvature $\kappa(x, t)$ exists for each unit vector $t \in T_x$ and is given by

$$\kappa(x, t) = k_1 \langle t, e_1 \rangle^2 + \cdots + k_{n-1} \langle t, e_{n-1} \rangle^2;$$

thus Euler's theorem holds.

It follows from work of Aleksandrov [19] that x is a normal point of K if and only if the representing function f is twice differentiable at o, in the sense explained in Section 1.5, Note 3. The almost everywhere twice differentiability of convex functions then yields the following theorem, which is due to Busemann and Feller [374] for $n \le 3$ and to Aleksandrov [19] for arbitrary n.

Theorem 2.6.1 *For every convex body $K \in \mathcal{K}_n^n$, $\mathcal{H}^{n-1}$-almost all boundary points are normal.*

Although this is a strong theorem, it should be kept in mind that essential information on the shape of a general convex body may be carried by those points that are not normal. This is shown clearly by the example of a polytope.

An occasionally more useful notion of curvatures for general convex bodies is obtained if these are not defined on the boundary of the body, but on its normal bundle, as we now explain. A generalized second fundamental form on the normal bundle was introduced by Walter [1904] (more generally, for closed locally convex sets in a Riemannian manifold), and generalized curvatures on the normal bundle were introduced and used by Zähle [2020]. For proofs we refer to these papers; we follow also the presentation given by Hug [1001].

Let $K \in \mathcal{K}^n$. Recall from Section 1.2 that $p(K, \cdot) : \mathbb{R}^n \to K$ is the nearest-point map of K, the distance of x from K is denoted by $d(K, x)$, and for $x \in \mathbb{R}^n \setminus K$, the unit vector $u(K, \cdot)$ is defined by $u(K, x) := (x - p(K, x))/d(K, x)$.

The product space $\Sigma = \mathbb{R}^n \times \mathbb{S}^{n-1}$ gets its metric from the Euclidean metric of $\mathbb{R}^n \times \mathbb{R}^n$. A pair $(x, u) \in \Sigma$ is a *support element* of K if x is a boundary point of K and u is an outer unit normal vector of K at x. In particular, if $x \in \mathbb{R}^n \setminus K$, then the pair $(p(K, x), u(K, x))$ is a support element of K. The set $\operatorname{Nor} K \subset \Sigma$ of all support elements of K is the *normal bundle* of K.

Now let $K \in \mathcal{K}^n$ and let $\rho > 0$ be given. The set

$$K_\rho := K + \rho B^n = \{x \in \mathbb{R}^n : d(K, x) \leq \rho\}$$

is the outer parallel body of K at distance ρ. Since K_ρ has only regular boundary points, $\operatorname{bd} K_\rho$ is a C^1 submanifold of $\mathbb{R}^n$, by Theorem 2.2.4. For $y \in \operatorname{bd} K_\rho$ we have

$$y = p(K, y) + \rho u(K, y), \qquad u_{K_\rho}(y) = u(K, y).$$

The map

$$F_K : \mathbb{R}^n \setminus K \to \Sigma, \qquad F_K := (p(K, \cdot), u(K, \cdot))$$

has the property that its restriction $F_K | \operatorname{bd} K_\rho : \operatorname{bd} K_\rho \to \operatorname{Nor} K$ is a bi-Lipschitz homeomorphism. This follows from the facts that the nearest-point map $p(K, \cdot)$ is contracting, by Theorem 1.2.1, and that the inverse of $F_K | \operatorname{bd} K_\rho$ is given by $(F_K | \operatorname{bd} K_\rho)^{-1}(x, u) = x + \rho u$.

Let $\mathcal{D}_K \subset \mathbb{R}^n \setminus K$ be the set of all points at which $p(K, \cdot)$ is differentiable. At the points of $\mathcal{D}_K$, also $u(K, \cdot)$ and F_K are differentiable. Since $p(K, \cdot)$ is a Lipschitz map, it is differentiable $\mathcal{H}^n$-almost everywhere on $\mathbb{R}^n \setminus K$, by Rademacher's theorem. It can be deduced from Lemma 1.2.2 that together with $y \in \mathcal{D}_K$ the whole ray $\{p(K, y) + \lambda u(K, y) : \lambda > 0\}$ belongs to $\mathcal{D}_K$; for a proof, see Walter [1904], Theorem 3.2. We conclude that there exists a set $\mathcal{D}_K^* \subset \operatorname{Nor} K$ with $\mathcal{H}^{n-1}(\operatorname{Nor} K \setminus \mathcal{D}_K^*) = 0$ such that $u(K, \cdot)$ is differentiable at $x + \rho u$ for all $(x, u) \in \mathcal{D}_K^*$ and all $\rho > 0$ (see also [1904], Theorem 3.3).

In the following, we fix $(x, u) \in \mathcal{D}_K^*$. For given $\rho > 0$, we set $y := x + \rho u$, denote by $\mathrm{d}u(K, \cdot)_y$ the differential of $u(K, \cdot)$ at y and define

$$\mathrm{II}_y(v, w) := \langle \mathrm{d}u(K, \cdot)_y \, v, w \rangle \qquad \text{for } v, w \in T_y K_\rho.$$

Then II_y is a bilinear form on $T_y K_\rho$. By Lemma 1.5.9, the distance function $d(K, \cdot)$ is continuously differentiable on $\mathbb{R}^n \setminus K$ and satisfies $\nabla d(K, \cdot) = u(K, \cdot)$. It follows that

$$\mathrm{II}_y(v, w) = \mathrm{d}^2 d(K, \cdot)_y(v, w) \qquad \text{for } v, w \in T_y K_\rho.$$

Hence, II_y is symmetric (see [557], 3.1.11). Since $d(K, \cdot)$ is convex, II_y is positive semi-definite (compare also Walter [1904], Section 5).

For all $\tau > -\rho$ we have

$$u(K, y + \tau u(K, y)) = u(K, y).$$

Differentiation gives

$$\mathrm{d}u(K, \cdot)_{y+\tau u} \circ [\mathrm{id} + \tau \mathrm{d}u(K, \cdot)_y] = \mathrm{d}u(K, \cdot)_y, \tag{2.70}$$

where id denotes the identity map on $\mathbb{R}^n$.

Let b be an eigenvector of $\mathrm{d}u(K, \cdot)_{x+\rho u}$ with eigenvalue k^ρ (ρ is an upper index). Let $\sigma > \rho$. Then (2.70) with $\tau = \sigma - \rho$ gives

$$\mathrm{d}u(K, \cdot)_{x+\sigma u} \, (b + (\sigma - \rho) k^\rho b) = k^\rho b$$

and hence

$$\mathrm{d}u(K,\cdot)_{x+\sigma u}\, b = \frac{k^\rho}{1+(\sigma-\rho)k^\rho}\, b.$$

Thus, b is also an eigenvector of $\mathrm{d}u(K,\cdot)_{x+\sigma u}$, with eigenvalue

$$k^\sigma = \frac{k^\rho}{1+(\sigma-\rho)k^\rho}. \tag{2.71}$$

This holds for all $\sigma > \rho$ and yields $k^\sigma \le 1/(\sigma-\rho)$ and hence $k^\sigma \le 1/\sigma$. Since $\rho > 0$ was arbitrary, this holds for all $\sigma > 0$. Writing (2.71) in the form

$$k^\sigma(1-\rho k^\rho) = k^\rho(1-\sigma k^\sigma),$$

we see that $k^\rho = 1/\rho$ for one $\rho > 0$ implies $k^\sigma = 1/\sigma$ for all $\sigma > 0$, and $k^\rho = 0$ for one $\rho > 0$ implies $k^\sigma = 0$ for all $\sigma > 0$, and further that

$$\frac{k^\sigma}{1-\sigma k^\sigma} = \frac{k^\rho}{1-\rho k^\rho}.$$

Therefore,

$$\frac{k^\rho}{1-\rho k^\rho} =: k \in [0,\infty] \tag{2.72}$$

does not depend on $\rho > 0$ and is equal to ∞ if and only if $k^\rho = 1/\rho$. From (2.72) we now have

$$k^\rho = \begin{cases} \dfrac{k}{1+\rho k} & \text{if } k < \infty, \\ 1/\rho & \text{if } k = \infty. \end{cases} \tag{2.73}$$

Since k^ρ was defined as an eigenvalue of the differential of $u(K,\cdot)$ at $x+\rho u$ (where $(x,u) \in \mathcal{D}_K^*$), we write $k^\rho = k^\rho(x+\rho u)$ in the following. Since k is independent of ρ, we may write $k = k(x,u)$.

The differential $\mathrm{d}u(K,\cdot)_{x+\rho u}$ always has the eigenvector $u(K,x+\rho u)$, with eigenvalue 0. Let $b_1,\dots,b_{n-1}$ be an orthonormal basis of $u^\perp$ consisting of eigenvectors of $\mathrm{d}u(K,\cdot)_{x+\rho u}$ (as seen, this basis can be chosen independent of ρ). Let $k_i^\rho(x+\rho u)$ be the eigenvalue corresponding to b_i, and let

$$k_i(x,u) := \frac{k_i^\rho(x+\rho u)}{1-\rho k_i^\rho(x+\rho u)} \qquad (= \infty \text{ if } k_i^\rho(x+\rho u) = 1/\rho),$$

corresponding to (2.72). This is independent of ρ. We call $k_1(x,u),\dots,k_{n-1}(x,u)$ the *generalized principal curvatures* of K at $(x,u) \in \mathcal{D}_K^*$. According to (2.73) we have

$$k_i^\rho(x+\rho u) = \frac{k_i(x,u)}{1+\rho k_i(x,u)}. \tag{2.74}$$

The relation of these generalized curvatures on the normal bundle to the principal curvatures on the boundary, as introduced at the beginning of this section, is as follows. Let $K \in \mathcal{K}_n^n$ and let x be a normal boundary point of K. Then, for all $\rho > 0$, the point $x+\rho u$, where $u = u_K(x)$, is a normal boundary point of K_ρ. Hug [1005] deduced this from a result of Noll [1475] and concluded that $k_1(x,u),\dots,k_{n-1}(x,u)$ are the principal curvatures of bd K, as defined by (2.69). Hug also showed the following. For $\mathcal{H}^{n-1}$-almost all $u \in \mathbb{S}^{n-1}$, the values $k_1(x_K(u),u)^{-1},\dots,k_{n-1}(x_K(u),u)^{-1}$ are the principal radii of curvature of K at u. They are defined as the eigenvalues of the second differential $\mathrm{d}^2 h_K(u)|u^\perp$ at the points u where h_K is second-order differentiable.

Now again let $K \in \mathcal{K}^n$. It is useful to construct from the preceding a special orthonormal basis of the tangent space of Nor K at $(x,u) \in \mathcal{D}_K^*$, which we denote by Tan(Nor $K,(x,u)$). Define the mapping $f : \mathrm{bd}\, K_\rho \to \mathrm{Nor}\, K_\rho$ by $f(z) := (z, u(K,z))$ for $z \in \mathrm{bd}\, K_\rho$. Its differential at $y = x+\rho u$ maps the basis vector b_i of $T_y K_\rho$ to $(b_i, k_i^\rho(y) b_i)$. By (2.74), this is equal to

$$\left(b_i, \frac{k_i(x,u)}{1+\rho k_i(x,u)}\, b_i\right). \tag{2.75}$$

Define, further, the mapping $g : \mathrm{Nor}\, K_\rho \to \mathrm{Nor}\, K$ by $g(z,v) := (z-\rho v, v)$ for $(z,v) \in \mathrm{Nor}\, K_\rho$. Its differential at (y,u) maps the vector (2.75) to

$$\left(\left(1-\frac{\rho k_i(x,u)}{1+\rho k_i(x,u)}\right) b_i, \frac{k_i(x,u)}{1+\rho k_i(x,u)}\, b_i\right).$$

We normalize this vector to a unit vector and then let ρ tend to zero, which is legitimate because of the independence of ρ established above. The result is that the vectors

$$a_i(x,u) := \left(\frac{1}{\sqrt{1+k_i(x,u)^2}}\, b_i, \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}\, b_i\right), \tag{2.76}$$

$i = 1,\dots,n-1$, constitute an orthonormal basis of Tan(Nor $K,(x,u)$).

The generalized principal curvatures on the normal bundle can be used to generalize the expressions (2.63) for the local parallel volume in two directions, namely to general convex bodies and to a more general notion of local parallel set. For $\eta \in \mathcal{B}(\Sigma)$ and for $\rho > 0$, we define

$$M_\rho(K,\eta) := \{x \in \mathbb{R}^n : 0 < d(K,x) \le \rho,\ (p(K,x), u(K,x)) \in \eta\}, \tag{2.77}$$

so that the sets appearing in (2.63) are given by $A_\rho(K,\beta) = M_\rho(K, \beta \times \mathbb{S}^{n-1})$ and $B_\rho(K,\omega) = M_\rho(K, \mathbb{R}^n \times \omega)$. For the computation of the measure $\mathcal{H}^n(M_\rho(K,\eta))$ we refer to Zähle [2020]. This computation makes essential use of the orthonormal basis constructed above and uses the coarea formula. Combining [2020] and formulae (10), (9) and (11), we get

$$\mathcal{H}^n(M_\rho(K,\eta)) \tag{2.78}$$
$$= \sum_{m=0}^{n-1} \rho^{n-m} \frac{1}{n-m} \int_{\eta\cap \mathrm{Nor}\, K} \sum_{\substack{I\subset\{1,\dots,n-1\}\\ |I|=n-m-1}} \frac{\prod_{i\in I} k_i(x,u)}{\prod_{i=1}^{n-1} \sqrt{1+k_i(x,u)^2}}\, \mathcal{H}^{n-1}(\mathrm{d}(x,u)).$$

Since $k_i(x,u) = \infty$ is possible, the summands in the integrand in (2.78)) have to be read as

$$\prod_{i\in I} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} \prod_{i\notin I} \frac{1}{\sqrt{1+k_i(x,u)^2}},$$

together with

$$\frac{k_i}{\sqrt{1+k_i^2}} = 1 \quad \text{and} \quad \frac{1}{\sqrt{1+k_i^2}} = 0 \quad \text{if } k_i = \infty.$$

Notes for Section 2.6

1. *Differential-geometric properties of general convex surfaces*. Theorem 2.6.1 is, of course, closely related to second-order differentiability properties of convex functions, and we refer to Section 1.5, Note 2, for relevant references. A detailed investigation of the differential-geometric properties of convex surfaces without differentiability assumptions was undertaken by Busemann and Feller [374, 372, 373, 375]. Among the topics that they treated, for convex surfaces for which no regularity was assumed, were versions of the theorems of Meusnier, Olinde Rodrigues and Euler, umbilics, shortest curves, upper and lower indicatrices, Gauss curvature and spherical images; see also Busemann [370]. The extension of Theorem 2.6.1 from $n = 3$ (Busemann and Feller [374]) to general n by Aleksandrov [19] is not straightforward. Among several other related results, Aleksandrov also investigated consequences of twice differentiability almost everywhere of support functions.
2. *ε-smooth points*. If $K \in \mathcal{K}^n_n$ and $x \in \operatorname{bd} K$ is a normal point of K, then x is an ε-smooth point of K for some $\varepsilon > 0$; that is, there exists a ball B of radius ε such that $x \in B \subset K$. Hence, it follows from Theorem 2.6.1 that the set of all boundary points of K that are not ε-smooth for any $\varepsilon > 0$ is of $\mathcal{H}^{n-1}$-measure zero. For this result, McMullen [1377] gave a simple direct proof, thus answering a question posed by Sallee [1611].
3. *Approximation with converging radii of curvature*. From the counterpart of Theorem 2.6.1 for support functions (Aleksandrov [19]) it follows that a convex body $K \in \mathcal{K}^n_n$ has principal radii of curvature $r_i(u)$, where

$$0 \le r_1(u) \le r_2(u) \le \cdots \le r_{n-1}(u) \le \infty$$

for σ-almost all $u \in \mathbb{S}^{n-1}$; here σ denotes the spherical Lebesgue measure on $\mathbb{S}^{n-1}$. The numbers $r_i(u)$ can be defined as the eigenvalues, corresponding to eigenvectors orthogonal to u, of the second differential of the support function h_K at $u \in \mathbb{S}^{n-1}$ (cf. Corollary 2.5.2); they are defined σ-almost everywhere on $\mathbb{S}^{n-1}$ and are measurable and σ-integrable. The following approximation result was proved and applied by Weil [1932]; for an extension, see Weil [1933].

Theorem For any convex body $K \in \mathcal{K}^n_n$ there exists a sequence $(K_i)_{i\in\mathbb{N}}$ of convex bodies of class C^2_+ converging to K such that the radii of curvature of K_i and K, denoted by $r_1^{(i)}, \dots, r_{n-1}^{(i)}$ and $r_1, \dots, r_{n-1}$, respectively, satisfy the following assertions.

(a) $r_j^{(i)} \to r_j$ for $i \to \infty$ holds σ-almost everywhere, $j = 1, \dots, n-1$;

(b) $r_j^{(i)} \to r_j$ for $i \to \infty$ in the $L^1(\mathbb{S}^{n-1})$ norm, $j = 1, \dots, n-1$;

(c) The measures $\int r_j^{(i)}\, d\sigma$ converge weakly to the measure $\int r_j\, d\sigma$ for $i \to \infty$, $j = 1, \dots, n-2$.

4. *Generalized second fundamental form and generalized curvatures on the normal bundle.* As already mentioned, a generalized second fundamental form on the normal bundle (in

a Riemannian manifold) was introduced by Walter [1904]. A generalized second fundamental form for Lipschitzian hypersurfaces was investigated by Noll [1475]. In the convex case, he gave a geometric characterization in terms of outer parallel surfaces. For sets of positive reach, generalized principal curvatures on the normal bundle were introduced by Zähle [2020]. The application of these generalized curvatures in spaces of constant curvature was further developed by Kohlmann [1128, 1129]. Extensive use of generalized curvatures for the investigation of local properties of convex bodies and of more general sets was made by Hug [1001, 1002, 1003, 1004, 1005, 1006, 1007].

5. *A different notion of normal bundle.* The name 'normal bundle' has also been used for a different object. Let $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$. If the supporting hyperplane to K with outer normal vector u touches K at a single point (as is the case for $\mathcal{H}^{n-1}$-almost all $u \in \mathbb{S}^{n-1}$), then Groemer [803] called the directed line through that point with direction given by u a *normal* of K. He further called the set $\mathbf{N}(K)$ of all such normals the 'normal bundle' of K. He proved that $\mathbf{N}(K) = \mathbf{N}(L)$ for $K, L \in \mathcal{K}^n$ implies that one of K, L is a parallel body of the other; in fact, he proved a stability version of this result.

2.7 Generic boundary structure

Due to their simple boundary structure, polytopes as well as C^2_+-regular convex bodies are favourite objects of investigation. On the other hand, one may argue that these are very special convex bodies, and one may ask whether there exists something like the boundary behaviour of a 'general' convex body. This question can be made precise if the notion of Baire category is used, and then several affirmative answers, often surprising ones, are possible.

The space $\mathcal{K}^n$ of convex bodies with the Hausdorff metric is a complete metric space, by Theorems 1.8.3 and 1.8.6, and thus a Baire space. This means that in $\mathcal{K}^n$ the intersection of countably many dense open sets is dense (such an intersection is called a dense G_δ set). In particular, such a set is not empty. A subset of $\mathcal{K}^n$ is called *meagre* or *of first category* if it is a countable union of nowhere dense subsets. The complement of a meagre set is called *residual*. In particular, a dense open subset is residual, and an intersection of countably many residual sets is residual. Meagre subsets of a Baire space can be considered as 'small'. We say, therefore, that a certain property E for elements of a Baire space is *generic* if the subset of elements not having property E is meagre. For the space $\mathcal{K}^n$ it has become customary to say instead that *most* convex bodies have the property E, or that a *typical* convex body has this property.

Category arguments can be used to give (non-constructive) existence proofs for objects with certain 'irregularity' properties, which otherwise may be hard to obtain. When such an argument works it shows that, in fact, most elements of the space have the property in question. For the boundary structure of convex bodies, which is our concern in the present chapter, there are several results showing that a typical convex body has properties that, at first sight, one would be inclined to consider as 'pathological'. This holds, at least, for second-order properties, whereas the typical first-order boundary behaviour is not very exciting. The following theorem shows, in a slightly more precise form, that most convex bodies are smooth and strictly convex.

Theorem 2.7.1 *In $\mathcal{K}^n$, the set of smooth convex bodies and the set of strictly convex bodies are dense G_δ-sets; hence, most convex bodies are smooth and strictly convex.*

Proof For $k \in \mathbb{N}$, let $\mathcal{A}_k$ be the set of all $K \in \mathcal{K}^n$ having a boundary point at which there exist two outer normal vectors forming an angle $\geq 1/k$, and let $\mathcal{B}_k$ be the set of all $K \in \mathcal{K}^n$ for which $\operatorname{bd} K$ contains a segment of length $\geq 1/k$. Since $\mathcal{K}^n \setminus \bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ is the set of all smooth convex bodies in $\mathcal{K}^n$ and $\mathcal{K}^n \setminus \bigcup_{k\in\mathbb{N}} \mathcal{B}_k$ is the set of all strictly convex bodies, it suffices to show that $\mathcal{A}_k$ and $\mathcal{B}_k$ are closed and nowhere dense. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{A}_k$ converging to $K \in \mathcal{K}^n$. For each $i \in \mathbb{N}$, we choose a point $x_i \in \operatorname{bd} K_i$ at which there exist two outer unit normal vectors u_i, v_i of K_i, forming an angle $\geq 1/k$. After selecting subsequences and changing the notation, we may assume that $x_i \to x$, $u_i \to u$ and $v_i \to v$ for $i \to \infty$. Then $x \in K$ (by Theorem 1.8.8), and from $\langle x_i, u_i\rangle = h(K_i, u_i)$, $\langle x_i, v_i\rangle = h(K_i, v_i)$ and Lemma 1.8.12 it follows that $\langle x, u\rangle = h(K, u)$ and $\langle x, v\rangle = h(K, v)$; thus $x \in \operatorname{bd} K$ and $u, v \in N(K, x)$. Since u and v form an angle $\geq 1/k$, we conclude that $K \in \mathcal{A}_k$. Thus $\mathcal{A}_k$ is closed. In a similar way, one easily shows that $\mathcal{B}_k$ is closed. Since the set of smooth and strictly convex bodies is dense in $\mathcal{K}^n$ (this follows, e.g., from Section 3.4), the closed sets $\mathcal{A}_k$ and $\mathcal{B}_k$ have empty interior and thus are nowhere dense. □

While Theorem 2.7.1 shows that the boundary and the support function of a typical convex body are differentiable, in general no stronger differentiability properties are satisfied. This is shown by the behaviour of the curvatures described in the theorems below. These results were discovered by Zamfirescu [2038, 2039].

By $\mathcal{K}^n_r \subset \mathcal{K}^n$ we denote the subset of smooth (regular) convex bodies. For these bodies, the lower and upper radii of curvature were defined in Section 2.5. By Theorem 2.7.1, $\mathcal{K}^n_r$ is a dense G_δ set in $\mathcal{K}^n$. In particular, it is itself a Baire space, and every residual subset of $\mathcal{K}^n_r$ is residual in $\mathcal{K}^n$. Hence, it causes no loss of generality if we restrict the following considerations to the space $\mathcal{K}^n_r$.

Theorem 2.7.2 *For most convex bodies K in $\mathcal{K}^n_r$, at every boundary point $x \in \operatorname{bd} K$ and for every tangent direction t at x,*

$$\rho_i(x, t) = 0 \quad or \quad \rho_s(x, t) = \infty.$$

Thus the typical convex body is smooth and strictly convex, and its curvatures $\kappa(x, t)$ are zero wherever they exist and are finite. By Aleksandrov's theorem, 2.6.1, the latter holds for (in the measure sense) almost all boundary points x of a convex body and for every tangent direction t at x.

Corollary 2.7.3 *For most convex bodies K in $\mathcal{K}^n_r$, the curvatures satisfy $\kappa(x, t) = 0$ for $\mathcal{H}^{n-1}$-almost all $x \in \operatorname{bd} K$ and every tangent direction t at x.*

The preceding results do not exclude the possibility that simultaneously $\rho_i(x, t) = 0$ and $\rho_s(x, t) = \infty$ for the points x in a boundary set of K of $\mathcal{H}^{n-1}$-measure zero. Typically, in fact, this holds in a residual set:

Theorem 2.7.4 *For most convex bodies K in $\mathcal{K}_r^n$, at most boundary points x of K and for every tangent direction t at x,*

$$\rho_i(x,t) = 0 \quad \textit{and} \quad \rho_s(x,t) = \infty.$$

The ideas involved in the proofs given below for Theorems 2.7.2 and 2.7.4 are taken from Zamfirescu [2038, 2044]. We start with some preparations. Let $K \in \mathcal{K}_r^n$, $x \in \operatorname{bd} K$, and let u be the unique outer unit normal vector of K at x; let $t \in T_x K$ be a unit tangent vector. We define a two-dimensional half-plane by

$$D(x,u,t) := \{x + \lambda u + \mu t : \lambda \in \mathbb{R},\ \mu \geq 0\}$$

and a convex arc by

$$S(K,x,t) := D(x,u,t) \cap \operatorname{bd} K.$$

For $\alpha > 0$, we say that K is *α-flat at* (x,t) if

$$S(K,x,t) \cap \operatorname{int} B(x - \alpha u, \alpha) = \emptyset,$$

and K is *α-curved at* (x,t) if

$$S(K,x,t) \subset B(x - \alpha u, \alpha).$$

It follows from the definitions of the lower and upper radii of curvature that $\rho_i(x,t) > 0$ if and only if K is α-flat at (x,t) for some $\alpha > 0$, and that $\rho_s(x,t) < \infty$ if and only if K is α-curved at (x,t) for some $\alpha > 0$.

For $k \in \mathbb{N}$ we define the following sets.

$$\begin{aligned}\mathcal{A}_k := \{K \in \mathcal{K}_r^n :\ & \text{There exist } x \in \operatorname{bd} K \text{ and } t \in T_x K \text{ such that}\\ & K \text{ is } k^{-1}\text{-flat at } (x,t) \text{ and } k\text{-curved at } (x,t)\}.\end{aligned}$$

For given $K \in \mathcal{K}_r^n$, let

$$\begin{aligned}A_k^1(K) := \{x \in \operatorname{bd} K :\ & \text{There exists } t \in T_x K \text{ such that}\\ & K \text{ is } k^{-1}\text{-flat at } (x,t)\},\\ A_k^2(K) := \{x \in \operatorname{bd} K :\ & \text{There exists } t \in T_x K \text{ such that}\\ & K \text{ is } k\text{-curved at } (x,t)\}.\end{aligned}$$

For $m \in \mathbb{N}$ and $i = 1,2$ we put

$$\begin{aligned}C_{k,m}^i := \{K \in \mathcal{K}_r^n :\ & \text{There exists } x \in \operatorname{bd} K \text{ such that}\\ & B_0(x, m^{-1}) \cap \operatorname{bd} K \subset A_k^i(K)\}.\end{aligned}$$

Lemma 2.7.5 *The sets $A_k^1(K)$, $A_k^2(K)$ are closed in* bd K. *The sets $\mathcal{A}_k$, $C_{k,m}^1$, $C_{k,m}^2$ are closed in $\mathcal{K}_r^n$ $(k,m \in \mathbb{N})$.*

Proof We start with $\mathcal{A}_k$. Let $(K_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{A}_k$ converging to a convex body $K \in \mathcal{K}^n_r$. For each $j \in \mathbb{N}$, we choose a point $x_j \in \operatorname{bd} K_j$ and a unit tangent vector $t_j \in T_{x_j} K_j$ such that

$$S(K_j, x_j, t_j) \cap \operatorname{int} B(x_j - k^{-1}u_j, k^{-1}) = \emptyset,$$
$$S(K_j, x_j, t_j) \subset B(x_j - ku_j, k),$$

where u_j is the outer unit normal vector of K_j at x_j. After selecting suitable subsequences and changing the notation, we may assume that $x_j \to x$ and $t_j \to t$ for $j \to \infty$. Then it is clear that $x \in \operatorname{bd} K$, the sequence $(u_j)_{j\in\mathbb{N}}$ converges to the unique outer unit normal vector u of K at x, and t is a unit tangent vector of K at x. We assert that

$$S(K, x, t) \cap \operatorname{int} B(x - k^{-1}u, k^{-1}) = \emptyset, \tag{2.79}$$
$$S(K, x, t) \subset B(x - ku, k). \tag{2.80}$$

Suppose that (2.79) is false. Then there are a point $z \in S(K, x, t)$ and a ball B with centre z such that $B \subset \operatorname{int} B(x - k^{-1}u, k^{-1})$. Now

$$K_j \cap D(x_j, u_j, t_j) \to K \cap D(x, u, t) \quad \text{for } j \to \infty.$$

This follows from Theorem 1.8.10, applied to K_j and the intersection L_j of $D(x_j, u_j, t_j)$ with some ball containing all the bodies $K_1, K_2, \dots$, and from the fact that K and $D(x, u, t)$ cannot be separated by a hyperplane since K is smooth. It follows that

$$S(K_j, x_j, t_j) \cap \operatorname{int} B(x_j - k^{-1}u_j, k^{-1}) \neq \emptyset$$

for sufficiently large j, a contradiction. Thus (2.79) holds. Suppose that (2.80) is false. Then there are a point $z' \in S(K, x, t)$ and a ball B' with centre z' such that $B' \cap B(x - ku, k) = \emptyset$. As above, we conclude that

$$S(K_j, x_j, t_j) \not\subset B(x_j - ku_j, k)$$

for sufficiently large j, again a contradiction. Thus (2.80) holds. Hence, K is k^{-1}-flat and k-curved at (x, t). This proves that $\mathcal{A}_k$ is closed.

Let $(x_j)_{j\in\mathbb{N}}$ be a sequence in $A^1_k(K)$ converging to a point x. For each j, we choose a unit tangent vector $t_j \in T_{x_j} K$ such that

$$S(K, x_j, t_j) \cap \operatorname{int} B(x_j - k^{-1}u_j, k^{-1}) = \emptyset,$$

where u_j is the outer unit normal vector of K at x_j. Precisely as above, with K_j replaced by K, we find that K is k^{-1}-flat at (x, t) for suitable $t \in T_x(K)$. Thus $A^1_k(K)$ is closed. In a similar way, we may show that $A^2_k(K)$ is closed.

Let $(K_j)_{j\in\mathbb{N}}$ be a sequence in $C^1_{k,m}$ converging to some $K \in \mathcal{K}^n_r$. For each $j \in \mathbb{N}$, we choose $x_j \in \operatorname{bd} K_j$ such that

$$B_0(x_j, m^{-1}) \cap \operatorname{bd} K_j \subset A^1_k(K_j).$$

After selecting a subsequence and changing the notation, we may assume that $x_j \to x \in \operatorname{bd} K$ for $j \to \infty$. Let $y \in B_0(x, m^{-1}) \cap \operatorname{bd} K$. For each $j \in \mathbb{N}$, we can choose

$y_j \in \mathrm{bd}\, K_j$ in such a way that $y_j \to y$ for $j \to \infty$. For j sufficiently large, we have $|x_j - y_j| < m^{-1}$ and thus $y_j \in A_k^1(K_j)$. Hence, there is a unit tangent vector $t_j \in T_{y_j}K_j$ such that K_j is k^{-1}-flat at (y_j, t_j). The argument used before shows that K is k^{-1}-flat at (y, t) for suitable $t \in T_y K$. Thus, $y \in A_k^1(K)$ and hence $B_0(x, m^{-1}) \cap \mathrm{bd}\, K \subset A_k^1(K)$, which shows that $C_{k,m}^1$ is closed. Similarly one shows that $C_{k,m}^2$ is closed. □

Proof of Theorem 2.7.2 Let $\mathcal{A} \subset \mathcal{K}_r^n$ be the subset of smooth convex bodies having a boundary point x and a tangent direction t at x such that $\rho_i(x, t) \neq 0$ and $\rho_s(x, t) \neq \infty$. Then

$$\mathcal{A} = \bigcup_{k \in \mathbb{N}} \mathcal{A}_k.$$

Let $k \in \mathbb{N}$ be given. If $P \in \mathcal{K}^n$ is a polytope and $0 < \varepsilon < k^{-1}$, then $P + \varepsilon B^n \in \mathcal{K}_r^n \setminus \mathcal{A}_k$. Since the set of bodies of this type is dense in $\mathcal{K}_r^n$ and $\mathcal{A}_k$ is closed in $\mathcal{K}_r^n$ by Lemma 2.7.5, $\mathcal{A}_k$ is nowhere dense in $\mathcal{K}_r^n$. Thus $\mathcal{A}$ is of first category in $\mathcal{K}_r^n$. □

Proof of Theorem 2.7.4 Let $\mathcal{K}^* \subset \mathcal{K}_r^n$ be the set of smooth convex bodies K for which the set of all boundary points x with some tangent direction t at x satisfying

$$\rho_i(x, t) > 0 \quad \text{or} \quad \rho_s(x, t) < \infty$$

is not meagre in $\mathrm{bd}\, K$. We have to show that $\mathcal{K}^*$ is meagre in $\mathcal{K}_r^n$. Clearly,

$$\mathcal{K}^* = \left\{ K \in \mathcal{K}_r^n : \bigcup_{k \in \mathbb{N}} A_k^1(K) \cup \bigcup_{k \in \mathbb{N}} A_k^2(K) \text{ is not meagre} \right\}.$$

Let $K \in \mathcal{K}^*$. Then there are a number $i \in \{1, 2\}$ and a number $k \in \mathbb{N}$ such that $A_k^i(K)$ is not nowhere dense. Since the set $A_k^i(K)$ is closed by Lemma 2.7.5, it has nonempty interior relative to $\mathrm{bd}\, K$. Hence, $K \in C_{k,m}^i$ for some $m \in \mathbb{N}$ and we conclude that

$$\mathcal{K}^* \subset \bigcup_{k,m \in \mathbb{N}} \left(C_{k,m}^1 \cup C_{k,m}^2 \right).$$

Let $k, m \in \mathbb{N}$ and $i \in \{1, 2\}$ be given. Let $P \in \mathcal{K}^n$ be a polytope with faces of diameter less than m^{-1} and choose $0 < \varepsilon < (2km)^{-1}$. Then $P + \varepsilon B^n \in \mathcal{K}_r^n \setminus C_{k,m}^i$. Since the bodies of this type are dense in $\mathcal{K}_r^n$ and $C_{k,m}^i$ is closed in $\mathcal{K}_r^n$ by Lemma 2.7.5, the set $C_{k,m}^i$ is nowhere dense in $\mathcal{K}_r^n$. Thus K^* is meagre in $\mathcal{K}_r^n$. □

Notes for Section 2.7

1. Theorem 2.7.1 was first proved, in an infinite-dimensional version, by Klee [1110]; it was rediscovered by Gruber [812]. Klee also proved that in each infinite-dimensional Banach space most compact convex sets are the closures of their sets of extreme points.

   Theorem 2.7.1 can be strengthened as follows. In a metric space (X, ρ), a set M is called *porous* if there is a number $\alpha > 0$ such that, for every $x \in X$ and any ball $B(x, \varepsilon)$ with centre x and radius ε, there exists a point $y \in B(x, \varepsilon)$ such that

$$B(y, \alpha\rho(x, y)) \cap M = \emptyset.$$

A countable union of porous sets is called σ-porous. One says that nearly all elements of a metric Baire space have a property E if the set of elements not having property E is a σ-porous set. Zamfirescu [2045] proved that nearly all convex bodies in $\mathcal{K}^n$ are smooth and strictly convex.

2. *Typical curvature properties*. Gruber [812] proved that the set of convex bodies in $\mathcal{K}^n$ that have a boundary of class C^2 or which are, in a certain sense, uniformly strictly convex, is of first category. Theorem 2.7.2 and Corollary 2.7.3 (as well as some other corollaries) are due to Zamfirescu [2038]. In connection with Corollary 2.7.3, the following simple example in $\mathbb{R}^2$ is of interest. Take a convex polygon $P \in \mathcal{P}_2^2$ and let $\mathrm{T}P$ be the convex hull of all the points dividing each side of P into three equal parts. Then $\lim_{k\to\infty} \mathrm{T}^k P$ is a smooth, strictly convex body with curvature vanishing almost everywhere on the boundary. This example is due to de Rham [476].

   A weaker form of Theorem 2.7.4, in which the set of boundary points x of K at which $\rho_i(x,t) = 0$ and $\rho_s(x,t) = \infty$ for every tangent direction t is merely dense in $\mathrm{bd}\, K$, was proved by Schneider [1692]. The general form of Theorem 2.7.4 was obtained by Zamfirescu [2039], and the simpler proof given above appears in Zamfirescu [2044]. The curvature properties of typical convex bodies were further investigated by Zamfirescu [2047]. He proved, among other results, that on the boundary of a typical convex body K, the set of points x with $\kappa(x,t) = \infty$ for some $t \in T_x(K)$ and that of points in which the lower curvature in some direction equals the upper curvature in the opposite direction, are dense. Further results on the boundary structure of typical convex bodies can be found in Zamfirescu [2044, 2047].

   Answering a question of Zamfirescu, Adiprasito [6] proved that a typical convex body in $\mathcal{K}^n$, $n \geq 2$, has a boundary point at which the curvature is infinite in every tangent direction.

3. *Limit sections*. The concept of the Dupin indicatrix can be generalized as follows, say, for $n = 3$. Let $K \in \mathcal{K}_3^3$, $x \in \mathrm{bd}\, K$, H be a support plane to K at x, and $(H_i)_{i\in\mathbb{N}}$ be a sequence of planes meeting K and converging to H. If there is a sequence $(\eta_i)_{i\in\mathbb{N}}$ of homotheties such that the sequence $(\eta_i(H_i \cap K))_{i\in\mathbb{N}}$ converges to a two-dimensional convex body C, then C is called a *limit section* of K at x. The point x is called *universal* if every convex body in H is a limit section of K at x, for suitable choices of H_i and η_i; it is called *p-universal* if an analogous statement is true where the planes H_i have to be parallel to H. Melzak [1402] proved the existence of a convex body with a p-universal point (and otherwise of class C^∞), the existence of a convex body for which every boundary point is universal, and related results. The following theorem was proved by Bárány and Schneider [152]. Here 'most', in either case, is understood in the sense of Baire category.

   Theorem For most convex bodies in $\mathcal{K}^3$, most boundary points are p-universal.

4. *Typical properties of normals*. The behaviour of the normals of typical convex bodies was investigated by Zamfirescu [2040, 2043]. For instance, he proved that for most convex bodies K in $\mathcal{K}^n$, most points of $\mathbb{R}^n$ lie on infinitely many normals of K.

5. Other topics treated under category aspects include geodesics (Zamfirescu [2041, 2052], Gruber [824]), convex curves on convex surfaces (Zamfirescu [2046]), diameters (Zamfirescu [2042], Bárány and Zamfirescu [154]), approximation (Schneider and Wieacker [1741], Gruber and Kenderov [838], Gruber [817]), shadow boundaries (Zamfirescu [2048, 2050], Gruber and Sorger [843]), properties of non-differentiability of the metric projection (Zamfirescu [2049]); see also Theorem 3.2.18.

   There are several further category results in convexity, partly concerning properties that are farther away from the main themes of this book. One can find references in the survey articles by Zamfirescu [2044], Gruber [820, 828], Zamfirescu [2051] and in Zamfirescu's book [2053].

6. *Convex functions*. Smoothness properties of typical convex functions were studied by Gruber [812] and Klima and Netuka [1119]; see also Howe [993].

7. Schwarz and Zamfirescu [1762] considered typical convex sets of convex sets.

8. *Typical compact sets*. By Theorem 1.8.3, the space $\mathcal{C}^n$ of nonempty compact subsets of $\mathbb{R}^n$ with the Hausdorff metric is also a Baire space. Some category results are known for this space. Most compact sets in $\mathcal{C}^n$ have Hausdorff measure zero, if any Hausdorff measure is given (see Gruber [817, 823] for more general results), and they are homeomorphic to Cantor's ternary set (see Wieacker [1975] for references). Wieacker found some surprising results on the (first-order) boundary structure of the convex hull of a typical compact set. In the following theorem, which collects the main results of Wieacker [1975], $\exp^* K$ denotes the set of farthest points of the convex body K; thus $x \in \exp^* K$ if and only if there is a ball B such that $K \subset B$ and $x \in K \cap \operatorname{bd} B$.

   Theorem (Wieacker) For most compact sets $C \in \mathcal{C}^n$, the convex hull $K = \operatorname{conv} C$ has the following properties:

   (a) $\operatorname{bd} K$ is of class C^1 but not of class C^2;
   (b) $\exp^* K$ and $\operatorname{ext} K \setminus \exp K$ are dense in $\operatorname{ext} K$;
   (c) $\operatorname{ext} K$ is homeomorphic to Cantor's ternary set;
   (d) $\exp K$ is homeomorphic to the space of irrational numbers;
   (e) $\exp^* K$ is homeomorphic to the topological product of the space of rational numbers and Cantor's ternary set;
   (f) The k-skeleton $\operatorname{ext}_k K$ of K is of (Hausdorff and topological) dimension k ($k = 0, \dots, n-1$).

3

Minkowski addition

3.1 Minkowski addition and subtraction

The aim of this chapter is a more systematic study of Minkowski addition. In the present section we first collect some elementary formal properties of this addition and then investigate its convexifying effect. After that, we introduce a certain counterpart to Minkowski addition, a useful operation that has been called Minkowski subtraction.

The sum of two subsets $A, B \subset \mathbb{R}^n$ was defined by

$$A + B := \{a + b : a \in A,\ b \in B\},$$

and the scalar multiple by

$$\lambda A := \{\lambda a : a \in A\}$$

for real numbers λ. One may rewrite the definition of the sum in the form

$$A + B = \bigcup_{b \in B} (A + b) = \bigcup_{a \in A} (a + B).$$

This gives a 'kinematic' interpretation that may support the intuition: $A + B$ is the set that is covered if A undergoes all translations by vectors of B, or conversely.

We recall some of the information on Minkowski sums that is scattered in former sections. If A and B are convex, compact or polytopes, then $A + B$ is, respectively, convex, compact or a polytope. As a map from $C^n \times C^n$ into C^n, Minkowski addition is continuous. The sets C^n and $\mathcal{K}^n$ with Minkowski addition are commutative semigroups with unit element $\{o\}$. $\mathcal{K}^n$ satisfies the cancellation law (Section 1.7); that is, if $K, L, M \in \mathcal{K}^n$ and $K + M = L + M$, then $K = L$. A stronger fact is true: $K + M \subset L + M$ implies $K \subset L$ (known as the 'order cancellation law'), as follows by considering the support functions (or from Remark 1.7.6).

The convex body K is called a *summand* of the convex body M if there exists a convex body L such that $K + L = M$. If this is the case, we also say that M is an *antisummand* of K.

There are some trivial rules valid for arbitrary subsets A, B, C of $\mathbb{R}^n$ and non-negative real numbers λ, μ, namely

$$(A \cup B) + C = (A + C) \cup (B + C), \tag{3.1}$$

$$(A \cap B) + C \subset (A + C) \cap (B + C), \tag{3.2}$$

$$\lambda A + \lambda B = \lambda(A + B),$$

$$\lambda A + \mu A \supset (\lambda + \mu)A.$$

If A is convex, then $\lambda A + \mu A = (\lambda + \mu)A$ (Remark 1.1.1). Special relations hold for convex bodies with convex union.

Lemma 3.1.1 *Let $K, L \in \mathcal{K}^n$ be convex bodies such that $K \cup L$ is convex. Then*

$$(K \cap L) + C = (K + C) \cap (L + C) \tag{3.3}$$

for any convex set $C \in \mathbb{R}^n$, and

$$(K \cup L) + (K \cap L) = K + L. \tag{3.4}$$

Proof Let $x \in (K+C)\cap(L+C)$; then $x = y+c = z+d$ with $y \in K$, $z \in L$ and $c, d \in C$. Since $K \cup L$ is convex, there is a number $\lambda \in [0, 1]$ such that $(1 - \lambda)y + \lambda z \in K \cap L$ and hence

$$\begin{aligned} x &= (1 - \lambda)(y + c) + \lambda(z + d) \\ &= (1 - \lambda)y + \lambda z + (1 - \lambda)c + \lambda d \\ &\in (K \cap L) + C. \end{aligned}$$

Thus $(K + C) \cap (L + C) \subset (K \cap L) + C$, which together with (3.2) yields (3.3).

For the proof of (3.4), we note that $(K \cup L) + (K \cap L) \subset K + L$ holds trivially. Let $x \in K$, $y \in L$. Since $K \cup L$ is convex, the segment $[x, y]$ contains a point $x' = (1 - \lambda)x + \lambda y \in K \cap L$ where $\lambda \in [0, 1]$. Then $y' := \lambda x + (1 - \lambda)y \in K \cup L$ and $x + y = x' + y'$. This shows that $K + L \subset (K \cap L) + (K \cup L)$ and thus proves (3.4). □

We remind the reader that $A - B$ for $A, B \subset \mathbb{R}^n$ is the set

$$A - B = \{a - b : a \in A,\ b \in B\},$$

and $-A = (-1)A = \{-a : a \in A\}$ is the image of A under reflection in the origin.

If $K \in \mathcal{K}^n$ is a convex body, the set

$$\mathrm{D}K := K - K = \{x - y : x, y \in K\}$$

is a convex body which is centrally symmetric with respect to the origin; it is called the *difference body* of K. Since $h(-K, u) = h(K, -u)$, we have

$$h(\mathrm{D}K, u) = h(K, u) + h(K, -u) = w(K, u)$$

for $u \in \mathbb{S}^{n-1}$; thus the support function of $\mathrm{D}K$ is the width function of K. If the difference body of K is a ball, then $w(K, \cdot)$ is constant, and K is called a *body of constant width*.

Minkowski addition of non-convex sets is, in more than one sense, a convexifying operation. Roughly speaking, vector sums of many bounded sets are approximately convex. This observation can be made quantitively precise and can be connected with several possible ways of measuring the non-convexity of a compact set. The following results of this kind have proved useful in mathematical economics and in stochastic geometry. The first result is often called the Shapley–Folkman lemma.

Theorem 3.1.2 (Shapley–Folkman) *Let $A_1, \dots, A_k \subset \mathbb{R}^n$ and let*

$$x \in \sum_{i=1}^{k} \operatorname{conv} A_i.$$

Then there is an index set $I \subset \{1, \dots, k\}$ with $\operatorname{card} I \le n$ *such that*

$$x \in \sum_{i \in I} \operatorname{conv} A_i + \sum_{i \notin I} A_i.$$

Proof If

$$x \in \sum_{i=1}^{k} \operatorname{conv} A_i = \operatorname{conv}(A_1 + \cdots + A_k)$$

(by Theorem 1.1.2), then $x \in \operatorname{conv}(A_1' + \cdots + A_k')$ with suitable finite sets $A_i' \subset A_i$. Hence, it suffices to prove the result under the assumption that the sets $A_1, \dots, A_k$ are finite.

Define $f : (\mathbb{R}^n)^k \to \mathbb{R}^n$ by $f(y_1, \dots, y_k) := y_1 + \cdots + y_k$ and let $P \subset (\mathbb{R}^n)^k$ be the polytope defined by

$$P := \operatorname{conv} A_1 \times \cdots \times \operatorname{conv} A_k,$$

then $x \in f(P)$. Let z be an extreme point of $f^{-1}(x) \cap P$, and let F be the face of P with $z \in \operatorname{relint} F$. Since $f^{-1}(x)$ is of codimension n and z is extreme, it follows that $\dim F \le n$. The face F is of the form $F = F_1 \times \cdots \times F_k$, where F_i is a face of $\operatorname{conv} A_i$. Writing $J := \{i : \dim F_i > 0\}$, we deduce that $\operatorname{card} J \le n$. For $i \in \{1, \dots, k\} \setminus J$ we have $F_i = \{a_i\}$ with an extreme point a_i of $\operatorname{conv} A_i$, hence with $a_i \in A_i$. This yields

$$x = f(z) \in f(F_1 \times \cdots \times F_k) = F_1 + \cdots + F_k \subset \sum_{i \in J} \operatorname{conv} A_i + \sum_{i \notin J} A_i$$

and thus the assertion. □

For convenience, we formulate the following results only for compact subsets (although some of them could be generalized).

Corollary 3.1.3 *If $A_1, \dots, A_k \in \mathcal{C}^n$ and $A_i \subset RB^n$ for $i = 1, \dots, k$ with some real number R, then*

$$\delta\left(\sum_{i=1}^{k} A_i, \operatorname{conv} \sum_{i=1}^{k} A_i\right) \le nR.$$

Proof Let $x \in \operatorname{conv}(A_1 + \cdots + A_k)$. By Theorem 3.1.2 we can assume, after renumbering, that $x = a + a_{m+1} + \cdots + a_k$ with $m \leq n$, $a \in \sum_{i=1}^{m} \operatorname{conv} A_i$ and $a_i \in A_i$ for $i = m+1, \ldots, k$. By the definition of the Hausdorff distance, there is a point $a' \in A_1 + \cdots + A_m$ such that

$$|a - a'| \leq \delta(A_1 + \cdots + A_m,\ \operatorname{conv}(A_1 + \cdots + A_m)) \leq mR,$$

the latter because $A_1 + \cdots + A_m \subset mRB^n$. Hence, the point $\overline{a} := a' + a_{m+1} + \cdots + a_k \in A_1 + \cdots + A_k$ satisfies $|x - \overline{a}| = |a - a'| \leq mR \leq nR$. The assertion follows. □

Remark 3.1.4 The essential feature of the estimate of Corollary 3.1.3 is that the right-hand side is independent of k. In particular, if $(A_i)_{i \in \mathbb{N}}$ is a sequence of uniformly bounded compact sets in $\mathbb{R}^n$, it follows that

$$\delta\left(\frac{1}{k}(A_1 + \cdots + A_k),\ \frac{1}{k}\operatorname{conv}(A_1 + \cdots + A_k)\right) \to 0 \quad \text{for } k \to \infty.$$

Thus, averaging of compact sets in the Minkowski sense is 'asymptotically convexifying'.

There are sharper estimates that demonstrate the convexifying effect of Minkowski addition. They can, for instance, be deduced from the following lemma. For a compact set $A \in \mathcal{C}^n$ there is a unique ball of smallest radius containing A, the *circumball* of A. Its radius, $R(A)$, is called the *circumradius* of A.

Lemma 3.1.5 *Let $A_1, \ldots, A_m \in \mathcal{C}^n$ and $x \in \operatorname{conv}(A_1 + \cdots + A_m)$. Then there is a point $a \in A_1 + \cdots + A_m$ such that*

$$|x - a|^2 \leq \sum_{i=1}^{m} R(A_i)^2.$$

Proof We use induction with respect to m. If $m = 1$, let z be the centre of the circumball of A_1. Since $x \in \operatorname{conv} A_1$, we can choose $a \in A_1$ with $\langle z - x, a - x\rangle \leq 0$ and then obtain

$$|x - a|^2 \leq |x - a|^2 + |z - x|^2 + 2\langle z - x, x - a\rangle = |z - a|^2 \leq R(A_1)^2.$$

Thus the assumption holds for $m = 1$. Suppose it is true for some $m \geq 1$. Let $x \in \operatorname{conv}(A_1 + \cdots + A_{m+1})$. Then $x = y + z$ with $y \in \operatorname{conv}(A_1 + \cdots + A_m)$ and $z \in \operatorname{conv} A_{m+1}$. By the inductive assumption, there exists a point $a \in A_1 + \cdots + A_m$ such that

$$|y - a|^2 \leq \sum_{i=1}^{m} R(A_i)^2.$$

Let $w := p(\operatorname{conv} A_{m+1}, x - a)$; then

$$|x - a - w|^2 \leq |x - a - z|^2 = |y - a|^2 \leq \sum_{i=1}^{m} R(A_i)^2.$$

Suppose for a moment that $x - a - w \neq o$. The hyperplane H through w orthogonal to $x - a - w$ separates $x - a$ and $\operatorname{conv} A_{m+1}$ (Section 1.2), and $w \in H \cap \operatorname{conv} A_{m+1} = \operatorname{conv}(H \cap A_{m+1})$. By the case $m = 1$ of the lemma, there is some $a_{m+1} \in H \cap A_{m+1}$ such that

$$|w - a_{m+1}|^2 \leq R(H \cap A_{m+1})^2 \leq R(A_{m+1})^2.$$

We have

$$\langle x - a - w, a_{m+1} - w \rangle = 0. \tag{3.5}$$

If $x-a-w = o$, we choose $a_{m+1} \in A_{m+1}$ with $|w-a_{m+1}|^2 \leq R(A_{m+1})^2$, and (3.5) holds trivially. Now $a + a_{m+1} \in A_1 + \cdots + A_{m+1}$ and

$$\begin{aligned} |x - (a + a_{m+1})|^2 &= |(x - a - w) - (a_{m+1} - w)|^2 \\ &= |x - a - w|^2 + |a_{m+1} - w|^2 \leq \sum_{i=1}^{m+1} R(A_i)^2, \end{aligned}$$

which completes the proof. □

To formulate a consequence, we define, for a compact set $A \in \mathcal{C}^n$, the *inner radius* of A by

$$\rho(A) := \sup_{x \in \operatorname{conv} A} \inf \{\lambda \leq 0 : x \in \operatorname{conv}(A \cap B(x, \lambda))\}.$$

Then $\rho(A) = 0$ if and only if A is convex, and trivially $\rho(A) \leq \operatorname{diam} A$, with equality if A is affinely independent.

Theorem 3.1.6 (Shapley–Folkman–Starr) *Let $A_1, \dots, A_k \in \mathcal{C}^n$, and suppose that $x \in \operatorname{conv}(A_1 + \cdots + A_k)$. Then there exists a point $a \in A_1 + \cdots + A_k$ such that*

$$|x - a| \leq \sqrt{n} \max_{1 \leq i \leq k} \rho(A_i), \tag{3.6}$$

hence

$$\delta\left(\sum_{i=1}^{k} A_i, \operatorname{conv} \sum_{i=1}^{k} A_i\right) \leq \sqrt{n} \max_{1 \leq i \leq k} \rho(A_i). \tag{3.7}$$

Proof Let $x \in \operatorname{conv}(A_1 + \cdots + A_k)$. By Theorem 3.1.2, there is an index set $I \subset \{1, \dots, k\}$ with $\operatorname{card} I \leq n$ such that $x = \sum_{i=1}^{k} a_i$ with $a_i \in \operatorname{conv} A_i$ for $i \in I$ and $a_i \in A_i$ for $i \notin I$. By Lemma 3.1.5, there exists a point $b \in \sum_{i \in I} A_i$ such that

$$\left|\sum_{i \in I} a_i - b\right|^2 \leq \sum_{i \in I} R(A_i)^2.$$

Hence, the point $a := b + \sum_{i \notin I} a_i$ satisfies $a \in \sum_{i=1}^{k} A_i$ and

$$|x - a|^2 \leq \sum_{i \in I} R(A_i)^2 \leq n \max_{1 \leq i \leq k} R(A_i)^2. \tag{3.8}$$

We can write

$$x = \sum_{i=1}^{k} x_i \quad \text{with } x_i \in \operatorname{conv} A_i.$$

Put $A_i' := A_i \cap B(x_i, \rho(A_i))$; then $x_i \in \operatorname{conv} A_i'$ by the definition of $\rho(A_i)$ (and by compactness), hence $x \in \operatorname{conv}(A_1' + \cdots + A_k')$. By the result proved above, there is a point $a \in \sum_{i=1}^{k} A_i' \subset \sum_{i=1}^{k} A_i$ such that

$$|x - a| \leq \sqrt{n} \max_{1 \leq i \leq k} R(A_i') \leq \sqrt{n} \max_{1 \leq i \leq k} \rho(A_i),$$

which proves (3.6). The estimate (3.7) is an immediate consequence. □

Remark 3.1.7 Inequality (3.8) can be slightly improved by noting that the sum $\sum_{i \in I} R(A_i)^2$ can be estimated from above by the sum of the $\min\{n, k\}$ largest $R(A_i)^2$. With this improvement, the assertion up to (3.8) is often quoted as the Shapley–Folkman theorem, and the rest of the assertion as Starr's corollary.

Remark 3.1.8 Applying Theorem 3.1.2 with $k = n + 1$ and $A_i = A \subset \mathbb{R}^n$, we get

$$A + n \operatorname{conv} A = \operatorname{conv} A + n \operatorname{conv} A. \tag{3.9}$$

This implies that 'cancellation characterizes convexity' (Borwein and O'Brien [304]), in the following sense. Let $A \subset \mathbb{R}^n$ be a compact set and suppose that the cancellation law

$$\forall\, B, C \in \mathcal{K}^n : A + C = B + C \Rightarrow A = B \tag{3.10}$$

holds. By (3.9),

$$A + n \operatorname{conv} A = \operatorname{conv} A + n \operatorname{conv} A,$$

and (3.10) yields $A = \operatorname{conv} A$.

Both functions appearing in Theorem 3.1.6, namely $\delta(A, \operatorname{conv} A)$ and $\rho(A)$, are measures for the deviation of a compact set A from its convex hull. As such, they also measure the non-convexity of the set, but in a way that depends on the Euclidean metric and that is not invariant under similarities. Since convexity is an affine notion, a geometrically significant measure for the non-convexity of sets should be invariant under affine transformations. Such a function can also be derived from the relation (3.9). If we define

$$c(A) := \inf\{\lambda \geq 0 : A + \lambda \operatorname{conv} A \text{ is convex}\},$$

then $c(A) \leq n$. It is clear that the function c is invariant under non-singular affine maps and that $c(A) = 0$ for convex sets A. The following theorem (Schneider [1678]) shows that, for compact sets, c can serve as a measure of non-convexity, and it describes the sets that are least convex, in this sense.

Theorem 3.1.9 *For every set $A \subset \mathbb{R}^n$,*

$$0 \le c(A) \le n.$$

For compact sets A, equality holds on the left if and only if A is convex, and on the right if and only if A consists of $n + 1$ affinely independent points.

Proof It is convenient to write

$$A_\lambda := (1 + \lambda)^{-1}(A + \lambda \operatorname{conv} A)$$

for $\lambda \ge 0$; then $A_\lambda \subset \operatorname{conv} A \subset \operatorname{conv} A_\lambda$, hence

$$c(A) = \inf\{\lambda \ge 0 : A_\lambda = \operatorname{conv} A\}.$$

The inequalities $0 \le c(A) \le n$ are clear. Now let $A \in \mathcal{C}^n$ be compact. Assume that $c(A) = 0$. Let $x \in \operatorname{conv} A$, and let $\lambda > 0$. Then $x \in A_\lambda$, hence $x = (1 + \lambda)^{-1}(a_\lambda + \lambda b_\lambda)$ with $a_\lambda \in A$ and $b_\lambda \in \operatorname{conv} A$. This implies $x - a_\lambda = \lambda(b_\lambda - x)$ and thus $\delta(A, x) \le 2\lambda \operatorname{diam} A$. Since $\lambda > 0$ was arbitrary, we see that $x \in A$, hence A is convex.

Now suppose that $c(A) = n$. Then A cannot be contained in a hyperplane (since otherwise $c(A) \le n - 1$), hence $\operatorname{conv} A$ has interior points.

Let $x \in \operatorname{bd} \operatorname{conv} A$. Then x lies in a supporting hyperplane H of $\operatorname{conv} A$, hence

$$x \in H \cap \operatorname{conv} A = \operatorname{conv}(H \cap A) = (H \cap A)_{n-1} \subset A_{n-1}.$$

Let $\lambda > n - 1$ and put $\alpha := (\lambda - n + 1)(1 + \lambda)^{-1}$; then

$$x + \alpha(\operatorname{conv} A - x) \subset (1 - \alpha)A_{n-1} + \alpha \operatorname{conv} A = A_\lambda.$$

Since $\alpha > 0$ and this holds for all $x \in \operatorname{bd} \operatorname{conv} A$, we see that the set

$$R_\lambda := (\operatorname{conv} A) \setminus A_\lambda$$

satisfies

$$\operatorname{cl} R_\lambda \subset \operatorname{int} \operatorname{conv} A \quad \text{for } \lambda > n - 1. \tag{3.11}$$

If $\mu > \lambda$, then

$$\begin{aligned} A_\mu &= (1 + \mu)^{-1}A + (\mu - \lambda)(1 + \mu)^{-1}(1 + \lambda)^{-1}\operatorname{conv} A + \lambda(1 + \lambda)^{-1}\operatorname{conv} A \\ &\supset (1 + \mu)^{-1}A + (\mu - \lambda)(1 + \mu)^{-1}(1 + \lambda)^{-1}A + \lambda(1 + \lambda)^{-1}\operatorname{conv} A \\ &= A_\lambda; \end{aligned}$$

hence $\mu > \lambda$ implies $R_\mu \subset R_\lambda$. Since the sets $\operatorname{cl} R_\lambda$ are compact and nonempty for $\lambda < n$, there exists a point

$$z \in \bigcap_{0<\lambda<n} \operatorname{cl} R_\lambda.$$

By (3.11), $z \in \operatorname{int} \operatorname{conv} A$. On the other hand,

$$z \notin \operatorname{int}(1 + n)^{-1}(a + n \operatorname{conv} A) \quad \text{for } a \in A, \tag{3.12}$$

since otherwise, for sufficiently large $\lambda < n$,

$$z \in \operatorname{int}(1+\lambda)^{-1}(a + \lambda \operatorname{conv} A) \subset \operatorname{int} A_\lambda$$

and thus $z \notin \operatorname{cl} R_\lambda$, a contradiction.

By Carathéodory's theorem, there exists an affinely independent set $Y \subset A$ such that $z \in \operatorname{conv} Y$, and some subset $X \subset Y$ satisfies $z \in \operatorname{relint}\operatorname{conv} X$. Suppose that $\dim\operatorname{aff} X < n$. Then

$$z \in \operatorname{conv} X = (1+\lambda)^{-1}(X + \lambda \operatorname{conv} X) \quad \text{for } \lambda \geq n-1$$

and hence $z \in \operatorname{relint}(1+n)^{-1}(x + n \operatorname{conv} X)$ for suitable $x \in X$. Since $z \in \operatorname{int}\operatorname{conv} A$, we must have $\operatorname{relint}\operatorname{conv} X \subset \operatorname{int}\operatorname{conv} A$ and hence $z \in \operatorname{int}(1+n)^{-1}(x+n \operatorname{conv} A)$. This contradicts (3.12). Hence $\dim\operatorname{aff} X = n$ and Y is the set of vertices of an n-simplex S. From (3.12) it follows that z is the centroid of S, since this is the only point not contained in $\bigcup_{x \in Y} \operatorname{int}(1+n)^{-1}(x + nS)$. If A contains a point $a \notin Y$, we can replace an appropriate point of Y by a and still obtain an affinely independent set $\overline{Y} \subset A$ satisfying $z \in \operatorname{conv} \overline{Y}$. By the same argument, $\overline{Y}$ is the set of vertices of an n-simplex of which z is the centroid. But as Y and $\overline{Y}$ differ in precisely one point, this is impossible. Hence $A = Y$; that is, A consists of $n+1$ affinely independent points. Conversely, if A is of this kind and if $A + \lambda \operatorname{conv} A$ is convex for some $\lambda \geq 0$, then the barycentre of A belongs to A_λ, which implies $\lambda \geq n$. This completes the proof. □

Having studied some properties of Minkowski addition, we now introduce a complementary operation called *Minkowski subtraction* (although it was not introduced by Minkowski). While the sum of two sets $A, B \subset \mathbb{R}^n$ can be defined by

$$A + B = \bigcup_{b \in B} (A + b),$$

the *Minkowski difference* of A and B is, by definition, the set

$$A \div B := \bigcap_{b \in B} (A - b).$$

(If B is empty, $A \div B$ is, by convention, equal to $\mathbb{R}^n$.) Evidently, we may also write

$$A \div B = \{x \in \mathbb{R}^n : B + x \subset A\}.$$

Lemma 3.1.10 *For $A, B \subset \mathbb{R}^n$,*

$$A \div (A \div B) = \bigcap_{B \subset A - x} (A - x), \tag{3.13}$$

thus $A \div (A \div B)$ is the intersection of all translates of A that contain B. In particular,

$$A \div (A \div B) = B$$

holds if and only if B is an intersection of translates of A.

Proof Since $A \div (A \div B) = \bigcap_{x \in A \div B}(A - x)$ and $x \in A \div B \Leftrightarrow B + x \subset A \Leftrightarrow B \subset A - x$, relation (3.13) is true. If $A \div (A \div B) = B$, then B is an intersection of translates of A, by (3.13). The latter implies that always $B \subset A \div (A \div B)$. If B is an intersection of translates of A, then it contains the intersection of all translates of A containing B. □

There are some trivial rules connecting addition and subtraction, namely

$$(A + B) \div B \supset A, \tag{3.14}$$

$$(A \div B) + B \subset A \ (B \neq \emptyset), \tag{3.15}$$

$$(A \div B) + C \subset (A + C) \div B, \tag{3.16}$$

$$(A \div B) \div C = A \div (B + C), \tag{3.17}$$

$$A + B \subset C \Leftrightarrow A \subset C \div B. \tag{3.18}$$

The verification is immediate.

Under convexity assumptions, more is true. If A is convex, then $A \div B$ is an intersection of convex sets and hence is convex. For convex bodies $K, L \in \mathcal{K}^n$, recall that L is a *summand* of K if there exists a convex body M such that $K = L + M$. A closer look at summands is the aim of the next section. Here we observe the following.

Lemma 3.1.11 *Let $K, L \in \mathcal{K}^n$ be convex bodies. Then*

$$(K + L) \div L = K.$$

The relation

$$(K \div L) + L = K$$

holds if and only if L is a summand of K.

Proof Let $x \in (K + L) \div L$; hence $L + x \subset K + L$ and, therefore, $h_L + h_{\{x\}} \leq h_K + h_L$. Subtracting h_L, we get $x \in K$. Thus $(K + L) \div L \subset K$, which together with (3.14) proves the first assertion.

If $(K \div L) + L = K$, then L is a summand of K. Conversely, suppose that $K = M + L$ for some $M \in \mathcal{K}^n$. Then $K \div L = (M + L) \div L = M$, which proves the second assertion. □

For convex bodies $K, L \in \mathcal{K}^n$, the Minkowski difference can also be represented as an intersection of halfspaces, in the following form.

$$\begin{aligned} K \div L &= \bigcap_{u \in \mathbb{S}^{n-1}} [H^-(K, u) - h(L, u)u] \\ &= \bigcap_{u \in \mathbb{S}^{n-1}} H^-_{u, h(K,u) - h(L,u)}. \end{aligned} \tag{3.19}$$

In fact, $x \in K \div L$ is equivalent to $L + x \subset K$, which is equivalent to $h(L+x, u) \le h(K, u)$ and hence to $\langle x, u\rangle \le h(K, u) - h(L, u)$ for all $u \in \mathbb{S}^{n-1}$.

In particular, (3.19) implies that

$$h(K \div L, u) \le h(K, u) - h(L, u) \quad \text{for } u \in \mathbb{S}^{n-1}.$$

Let $x \in \operatorname{bd}(K \div L)$. Then $L + x$ must contain a boundary point y of K (otherwise, $L + x + v \subset K$ for all v in some neighbourhood of o and thus $x \in \operatorname{int}(K \div L)$). If u is an outer unit normal vector of K at y, then $h(K, u) = h(L + x, u)$, hence $h(K, u) - h(L, u) = \langle x, u\rangle \le h(K \div L, u)$. Thus, to each boundary point x of $K \div L$ there is a normal vector u of $K \div L$ at x such that $h(K \div L, u) = h(K, u) - h(L, u)$.

Remark 3.1.12 It should be observed that Minkowski subtraction is not continuous. For example, let K be a rectangle, not a square, in the plane, and let L be one of its incircles touching three edges of K. There are quadrangles K_i, obtained from K by tilting one of its longer edges, such that L is the unique incircle of K_i, for each $i \in \mathbb{N}$, and $K_i \to K$ for $i \to \infty$. Then $K_i \div L$ is one-pointed for all i, whereas $K \div L$ is a segment of positive length. Hence, $K_i \div L \not\to K \div L$.

The following notion of parallel bodies is useful in several respects. Let $K, B \in \mathcal{K}^n$ be convex bodies and let $\lambda \ge 0$ be a real number. Then the convex body $K + \lambda B$ is called an *outer parallel body* of K, and $K \div \lambda B$ is an *inner parallel body*, both *relative to* B. The greatest number λ for which $K \div \lambda B$ is not empty is called the *inradius* of K *relative* to B and denoted by $r(K, B)$, thus

$$r(K, B) := \max\{\lambda \ge 0 : \lambda B + t \subset K \ \text{ for some } t \in \mathbb{R}^n\}.$$

For fixed B, one defines the full system of relative parallel bodies of K by

$$K_\rho := \begin{cases} K + \rho B & \text{for } 0 \le \rho < \infty, \\ K \div (-\rho)B & \text{for } -r(K, B) \le \rho \le 0. \end{cases}$$

Then for $K, L \in \mathcal{K}^n$ and arbitrary $\rho, \sigma \ge -r(K, B)$ the rule

$$K_\rho + L_\sigma \subset (K + L)_{\rho+\sigma} \tag{3.20}$$

is valid.

Proof Let $\rho, \sigma \ge 0$. We have

$$K_\rho + L_\sigma = K + \rho B + L + \sigma B = K + L + (\rho + \sigma)B = (K + L)_{\rho+\sigma}.$$

Next,

$$K_{-\rho} + L_{-\sigma} + (\rho + \sigma)B = (K \div \rho B) + \rho B + (L \div \sigma B) + \sigma B \subset K + L,$$

hence $K_{-\rho} + L_{-\sigma} \subset (K + L) \div (\rho + \sigma)B = (K + L)_{-\rho-\sigma}$.

If $\rho \ge \sigma$, then

$$\begin{aligned} K_\rho + L_{-\sigma} &= K + (\rho - \sigma)B + (L \div \sigma B) + \sigma B \subset K + (\rho - \sigma)B + L \\ &= (K + L)_{\rho-\sigma}. \end{aligned}$$

If $\rho \le \sigma$, then

$$K_\rho + L_{-\sigma} + (\sigma - \rho)B = K + \rho B + (\sigma - \rho)B + (L \div \sigma B)$$
$$= K + (L \div \sigma B) + \sigma B \subset K + L,$$

hence $K_\rho + L_{-\sigma} \subset (K + L) \div (\sigma - \rho)B = (K + L)_{\rho - \sigma}$. □

From (3.20) we deduce the following lemma, which later will turn out to be very useful.

Lemma 3.1.13 *If $K, B \in \mathcal{K}^n$, then the full system $\rho \mapsto K_\rho$ of parallel sets of K relative to B is concave with respect to inclusion; that is,*

$$(1 - \lambda)K_\rho + \lambda K_\sigma \subset K_{(1-\lambda)\rho + \lambda\sigma}$$

for $\lambda \in [0, 1]$ and $\rho, \sigma \in [-r(K, B), \infty)$.

Proof By (3.20), $(1 - \lambda)K_\rho + \lambda K_\sigma = [(1 - \lambda)K]_{(1-\lambda)\rho} + [\lambda K]_{\lambda\sigma} \subset K_{(1-\lambda)\rho+\lambda\sigma}$. □

In some applications of the preceding lemma, it is important to know in which cases the inner parallel bodies of K are homothetic to K. The following lemma is a slightly improved version of a result of Bol [272]. Recall (Section 2.2) that the convex body K is a tangential body of the body $B \in \mathcal{K}^n$ if each extreme support plane of K is a support plane of B. For the following, an equivalent description is convenient. Let K be a tangential body of B. If x is a boundary point of K, then through x there is an extreme support plane of K, which is hence a support plane of B. Conversely, suppose that through each boundary point of K there is a support plane of B. Then a support plane of K that is not a support plane of B contains only singular points of K. By Theorem 2.2.10, K is a tangential body of B. Thus, a convex body K is a tangential body of B if and only if through each boundary point of K there exists a support plane to K that also supports B.

Lemma 3.1.14 *Let $K, B \in \mathcal{K}^n_n$ be convex bodies, and let $\rho \in (0, r(K, B))$. Then $K \div \rho B$ is homothetic to K if and only if K is homothetic to a tangential body of B.*

Proof Suppose that $K \div \rho B$ is homothetic to K. After a suitable choice of the origin we may assume that $\lambda K = K \div \rho B$, where $0 < \lambda < 1$. Then $\lambda^2 K = \lambda(\lambda K) = \lambda(K \div \rho B) = \lambda K \div \lambda\rho B = (K \div \rho B) \div \lambda\rho B = K \div (1 + \lambda)\rho B$ by (3.17). By induction we get

$$\lambda^k K = K \div (1 + \lambda + \cdots + \lambda^{k-1})\rho B$$

for $k \in \mathbb{N}$, hence

$$\{o\} = K \div \tau B \quad \text{with } \tau = \frac{\rho}{1 - \lambda},$$

because $\rho_k \uparrow \tau$ obviously implies $K \div \rho_k B \to K \div \tau B$. Let $x \in \operatorname{bd} K$. Then $\lambda x \in \lambda K = K \div \rho B$, hence $\rho B + \lambda x \subset K$. Since λx is a boundary point of $K \div \rho B$, the body $\rho B + \lambda x$ contains a boundary point y of K, say $y = \rho b + \lambda x$ with $b \in B$. Then

$$v := \frac{y - \lambda x}{1-\lambda} = \frac{\rho b}{1-\lambda} \in \tau B \subset K.$$

The point $y \in \operatorname{bd} K$ is a convex combination, with positive coefficients, of the points $x \in \operatorname{bd} K$ and $v \in K$, hence any support plane to K through y contains x and v and thus supports K at x and τB at v. Thus K is a tangential body of τB (and hence is homothetic to a tangential body of B).

Conversely, suppose that K is a tangential body of τB, for some $\tau > 0$. Then

$$K = \bigcap_{u\in\omega} H^-(\tau B, u) = \bigcap_{u\in\omega} H^-(K, u),$$

where $\omega \subset \mathbb{S}^{n-1}$ is the set of those normal vectors u for which the support plane $H(K,u)$ also supports τB. (Note that, by Theorem 2.2.6, K is the intersection of its supporting halfspaces that contain some regular point of K in their boundaries.) Now

$$K \div \rho B = \bigcap_{u\in\mathbb{S}^{n-1}} H^-_{u,h(K,u)-\rho h(B,u)} \subset \bigcap_{u\in\omega} H^-_{u,h(K,u)-\rho h(B,u)}.$$

For $u \in \omega$ we have

$$h(K,u) - \rho h(B,u) = \frac{\tau-\rho}{\tau} h(K,u),$$

hence $K \div \rho B \subset (1-\rho/\tau)K$. From Lemma 3.1.13 we get

$$\frac{\rho}{\tau} K_{-\tau} + \left(1 - \frac{\rho}{\tau}\right) K_0 \subset K_{-\rho},$$

and since $K_{-\tau} = \{o\}$, this gives $K \div \rho B = (1-\rho/\tau)K$. □

Notes for Section 3.1

1. *The theorems of Shapley, Folkman and Starr.* Theorem 3.1.2, Lemma 3.1.5 and Theorem 3.1.6 are all from Starr [1814], where credit for most of the results (namely, up to (3.8)) is given to unpublished work of Folkman and Shapley; see also Arrow and Hahn [78], Appendix B, Ekeland and Temam [537], Appendix I, Starr [1815], and Molchanov [1444], Theorem C.13. Our proof of Theorem 3.1.2 essentially follows Artstein [80], Theorem 5.1. Simple proofs of Theorems 3.1.2 and 3.1.6 were also given by Cassels [395]. He introduced, for compact sets $A \in \mathcal{C}^n$, the function

$$c_3(A) := \sup_{x\in\operatorname{conv} A} \inf \left(\sum_{i=1}^m \alpha_i |x - a_i|^2\right)^{1/2},$$

where the infimum is taken over all representations $x = \sum_{i=1}^m \alpha_i a_i$ with $m \in \mathbb{N}$, $a_i \in A$, $\alpha_i \geq 0$ and $\sum \alpha_i = 1$, and he showed that

$$c_3\left(\sum_{i=1}^k A_i\right) \leq \sqrt{n} \max_{1\leq i\leq k} c_3(A_i)$$

for $A_1, \ldots, A_k \in \mathcal{C}^n$. Like the functions defined by $c_1(A) := \delta(A, \operatorname{conv} A)$ and by $c_4(A) := \rho(A)$, also c_3 measures the deviation of A from its convex hull. Since $c_1(A) \leq c_3(A) \leq c_4(A)$, the result of Cassels implies the inequality (3.7), which can be written as

$$c_1\left(\sum_i A_i\right) \leq \sqrt{n} \max_i c_4(A_i).$$

Wegmann [1929] investigated relations between c_1, c_3, c_4 and a further function c_2 defined in a similar way. In particular, he showed that $c_3 = c_4$, while simple examples $A \in \mathcal{C}^n$ exist with $c_1(A) < c_3(A)$.

The theorem of Shapley, Folkman and Starr is applied in mathematical economics; see, for example, Starr [1814], Arrow and Hahn [78]. Hildenbrand [974].

2. The main part of Theorem 3.1.9 was formulated and proved above for compact sets only. Some more general and additional results were proved in Schneider [1678]. Let $A \subset \mathbb{R}^n$ be an arbitrary set. If A is bounded and $c(A) = 0$, then $\operatorname{cl} A$ is convex; if $c(A) = n$, then A consists of $n+1$ affinely independent points. If A is either unbounded or connected, then $c(A) \leq n-1$; in both cases the bound $n-1$ is sharp.

   A question posed by the author (part of Problem 68 in [841]), whether there exists an affine-invariant, continuous measure of convexity, has a simple negative answer. This was pointed out by D'Agata [458]. It suffices to consider a sequence of two-pointed sets converging to a one-pointed set.

   The relation (3.9),

$$A + n \operatorname{conv} A = (n+1) \operatorname{conv} A$$

   for $A \subset \mathbb{R}^n$, appears also in Borwein and O'Brien [304]. These authors prove: a Banach space X is finite-dimensional if and only if for any compact set $A \subset X$ there exists $\alpha > 0$ with

$$A + \alpha \operatorname{cl} \operatorname{conv} A = (\alpha + 1) \operatorname{cl} \operatorname{conv} A.$$

   The least such α that works for all A is the dimension of the space.

3. *Minkowski addition and subtraction.* The systematic treatment of Minkowski addition, after some special appearances in the work of Steiner and Brunn, began with the work of Minkowski [1438] (§17), [1441].

   The origins of Minkowski subtraction are not so clear, except that it was not introduced by Minkowski (contrary to some statements in the literature). The earliest appearance I could find was in the form of inner parallel bodies, in the work of Bol [269, 270, 272], where credit for the idea is given to unpublished work of Kaluza from 1938 (see [269], p. 219, and [270], p. 254). It seems that the term 'Minkowski subtraction' was introduced by Hadwiger [899], for the reason of analogy to Minkowski addition. In part of the literature, the term 'Minkowski–Pontryagin difference' is used, since Pontryagin [1545] also introduced this difference and used it in his work on linear differential games.

   All the formal rules for Minkowski addition and subtraction that we have mentioned, and several more, can be found in the book [911], §4.2.1, of Hadwiger (who uses the notation $A \times B$ and A/B instead of $A + B$ and $A \div B$), with one exception. The exception is equation (3.4), which presumably first appeared in Sallee [1608]. Further rules, more generally for closed, bounded, convex sets in Hausdorff topological vector spaces, are found in Grzybowski, Przybycień and Urbański [854].

   Addition and subtraction (for not necessarily convex sets) have found applications in mathematical morphology and image processing; see Matheron [1358], Serra [1770, 1771] and Stoyan, Kendall and Mecke [1822]; in the latter book, note in particular the remark in §1.4 on quantitative image processing. A useful survey is given by Haralick, Sternberg and Zhuang [937]. In these and related references, different notation is used, namely

$$A \oplus B := A + B,$$
$$A \ominus B := A \div -B.$$

These operations are complementary to each other, in the sense that

$$A \ominus B = (A^c \oplus B)^c,$$
$$A \oplus B = (A^c \ominus B)^c,$$

where A^c is the complement of the set A. In applications, B is usually a fixed convex body, for instance in the plane a disc or a hexagon, which is then called the *structuring element*. It is used to transform a given set A in various ways, in particular by means of the following operations.

$$A \oplus -B, \text{ the } \textit{dilation} \text{ of } A \text{ by } B;$$
$$A \ominus -B, \text{ the } \textit{erosion} \text{ of } A \text{ by } B;$$
$$A_B := (A \ominus -B) \oplus B, \text{ the } \textit{opening} \text{ of } A \text{ by } B,$$
$$A^B := (A \oplus -B) \ominus B, \text{ the } \textit{closing} \text{ of } A \text{ by } B.$$

The Minkowski difference also plays a role in robot motion planning and corresponding algorithms.

4. *Axiomatic characterization of Minkowski addition.* The following characterization was proved by Gardner, Hug and Weil [679] (Corollary 9.11).

   Theorem Minkowski addition is the unique operation $* : (\mathcal{K}^n)^2 \to \mathcal{K}^n$ ($n \geq 2$) which is continuous, $\mathrm{GL}(n)$ covariant and satisfies $K * \{o\} = K = \{o\} * K$ for all $K \in \mathcal{K}^n$.

5. *Analysis of $\mathcal{K}^n$-valued functions.* It is not surprising that some parts of the elementary analysis of real-valued functions can be carried over to $\mathcal{K}^n$-valued functions, with Minkowski addition replacing the addition of real numbers. Integrals of functions from an interval into $\mathcal{K}^n$ were constructed by Dinghas [491, 494]. For later developments see, e.g., Artstein [79] and the references given there. Artstein emphasizes the usefulness of the support function. It appears that a consequent use of the support function, together with Theorem 1.8.15 and some elementary estimates, would also permit one to deduce many of the results in Ahrens [7, 8] and in Chapter IV of Meschkowski and Ahrens [1404] directly from the corresponding results for real-valued functions.

   Vitale [1883] has studied a Bernstein type of approximation for functions from [0,1] into $\mathcal{K}^n$, where Minkowski addition is used.

6. *Vector spaces consisting of classes of convex bodies.* The set $\mathcal{K}^n$ of convex bodies, equipped with Minkowski addition and multiplication by nonnegative real numbers, forms a commutative semigroup, satisfying the cancellation law, with scalar operator. More algebraic structure can be introduced if suitable equivalence classes of convex bodies are considered. Vector spaces of classes of convex bodies, in addition endowed with norms or scalar products, were investigated by Ewald and Shephard [543], Ewald [540], Shephard [1781], Schmitt [1647], Lewis [1208], Geivaerts [692, 693]; see also Sorokin [1800] and Zamfirescu [2037].

   In the background of part of these investigations is, roughly speaking, the idea of decomposing a support function into its even and odd parts, and of concentrating attention on one of these parts. Valette [1867] treats a related topic in this sense also.

7. *Prolongation of linear series.* Let $K_0, K_1 \in \mathcal{K}^n$ and $K_\lambda := (1-\lambda)K_0 + \lambda K_1$ for $\lambda \in [0,1]$. We say that the linear series $(K_\lambda)_{\lambda \in [0,1]}$ can be *prolonged* via K_1 if, for some $\lambda > 1$, the function $(1-\lambda)h(K_0,\cdot) + \lambda h(K_1,\cdot)$ is still a support function; in other words, if there exists a convex body K_λ such that $K_1 = (1-\lambda^{-1})K_0 + \lambda^{-1}K_\lambda$. Evidently, this is equivalent to the condition that K_0 is homothetic to a summand of K_1. Known criteria for summands (see Section 3.2) can therefore be used to study this prolongation. It was investigated by

Vincensini [1872, 1873]. He applied it to the 'construction' of convex bodies with given difference body (domaine vectoriel). With his notation, but using the support function with advantage, we can describe this procedure as follows. Let $D \in \mathcal{K}^n$ be a given convex body, centrally symmetric with respect to the origin. Let Γ be an arbitrary convex body with the property that $C := \mathrm{D}\Gamma = \Gamma - \Gamma$ is homothetic to a summand of D (this is always the case if D and Γ are of class C^2_+); say, αC is a summand of D where $\alpha \in (0, 1)$. Then

$$h(\Delta, u) := \frac{\alpha}{2}[h(\Gamma, u) - h(\Gamma, -u)] + \frac{1}{2} h(D, u)$$

defines a support function, and the body Δ evidently satisfies $\mathrm{D}\Delta = D$. In a series of papers, Vincensini [1874, 1875, 1876, 1877, 1878, 1879, 1880] has studied related questions and applications; see also Vincensini and Zamfirescu [1881].

8. *Hammer's associated bodies and reducibility*. For $K \in \mathcal{K}^n$, Hammer [934] defined and studied the associated bodies

$$K(\rho) := \begin{cases} \bigcap_{b \in \mathrm{bd}\, K} [\rho(K - b) + b] & \text{for } \rho \in (0, 1), \\ \bigcup_{b \in \mathrm{bd}\, K} [\rho(K - b) + b] & \text{for } \rho \in [1, \infty). \end{cases}$$

This construction can also be described in terms of addition and subtraction. In fact, it is not difficult to show (see also Voiculescu [1896]) that

$$K(\rho) = K + (\rho - 1)\mathrm{D}K = \rho K + (\rho - 1)(-K) \quad \text{for } \rho \geq 1,$$
$$K(\rho) = K \div (1 - \rho)\mathrm{D}K = \rho K \div (1 - \rho)(-K) \quad \text{for } \rho \leq 1.$$

Hammer further proposed a notion of reducibility, as follows. There is a number $r_K \leq 1$ such that

$$K = K(\rho)(\rho/(2\rho - 1)) \quad \text{for } \rho \geq r_K,$$
$$K \neq K(\rho)(\rho/(2\rho - 1)) \quad \text{for } \rho < r_K.$$

Then K is called *reducible* if $r_K < 1$, otherwise *irreducible*. This notion of reducibility was studied in a series of papers by Zamfirescu [2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037]. It appears that part of the results can be obtained more easily using the following reformulation. Write

$$\alpha_K := \sup\{\alpha \geq 0 : -\alpha K \text{ is a summand of } K\}$$

or, equivalently, using support functions,

$$\alpha_K = \sup\{\alpha \geq 0 : h_K - \alpha h_{(-K)} \text{ is convex}\}.$$

Then it follows from the representation of $K(\rho)$ above that $r_K = (1 + \alpha_K)^{-1}$. Denoting by h_K^+ the even part and by h_K^- the odd part of h_K, we may also write

$$r_K = \inf\{\rho \geq 0 : (2\rho - 1)h_K^+ + h_K^- \text{ is convex}\}.$$

In particular, K is reducible if and only if it has a summand that is positively homothetic to $-K$ (Zamfirescu [2036]). Now, elementary properties of Minkowski addition and criteria for convex functions (Sections 1.5, 2.5) and for summands (Section 3.2) can be used to obtain results on reducibility.

9. *Extremal and facial structure of sums of convex sets*. The following was observed by Klee [1110], Proposition 6.5. Let $A, B \subset \mathbb{R}^n$ be convex sets, A closed and B compact. Then for each point $z \in \mathrm{ext}\,(A + B)$ there are unique points $x_z \in A$ and $y_z \in B$ (necessarily extreme points) such that $z = x_z + y_z$. Further, for each point $x \in \mathrm{ext}\, A$ there is a point $y \in B$ such that $x + y \in \mathrm{ext}\,(A + B)$. However, there may be points $y \in \mathrm{ext}\, B$ such that $x + y \notin \mathrm{ext}\,(A + B)$ for arbitrary $x \in A$. This was further studied by Bair [111].

The extremal and facial structure of sums of closed convex sets in certain infinite-dimensional spaces was investigated by Husain and Tweddle [1025], Roy [1597], Edelstein and Fesmire [530], Jongmans [1049], Bair, Fourneau and Jongmans [113].

Some observations on smoothness properties of Minkowski sums are found in Jongmans [1050, 1051, 1052]. See also Section 2.5, Note 5.

10. The behaviour of certain cones associated with convex sets in real vector spaces under Minkowski addition of the convex sets was studied by Dubois and Jongmans [515].
11. *Minkowski addition in the theory of random sets*. The theorem of Shapley, Folkman and Starr has also found applications in stochastic geometry. For these applications, the crude estimate of Corollary 3.1.3 is sufficient, since one only needs Remark 3.1.4. The latter is used to deduce results on sequences of independent random compact sets and their partial Minkowski sums from results on the corresponding convex hulls; for convex sets, the embedding into a Banach space via the support function can be utilized. For such applications, see Artstein [81], Artstein and Vitale [82], Ljašenko [1230] (who gives independent proofs for the required estimates), Weil [1948], Giné, Hahn and Zinn [716].

    Further applications of Minkowski addition in the theory of random sets are found in Cressie [455], Mase [1355], Schürger [1745], Giné and Hahn [713, 714]; for infinite dimensions see Giné and Hahn [715] and the references given there.

    Chapter three of the book by Molchanov [1444] is devoted to the role of Minkowski addition in the theory of random sets.
12. *Combinatorial and computational aspects of Minkowski addition of polytopes*. Minkowski sums of polytopes raise many questions of combinatorial and computational nature.

    Gritzmann and Sturmfels [779] determined the computational complexity of computing the Minkowski sum of r convex polytopes in $\mathbb{R}^n$, which are presented either in terms of vertices or in terms of facets. For fixed n, they obtained a polynomial time algorithm for adding r polytopes with up to N vertices. Among other combinatorial results, they derived bounds for the number of vertices of the sum in terms of the vertex numbers of the given polytopes.

    Tiwary [1847] showed, among similar results, that enumerating the facets of the sum $P_1 + P_2$ of two polytopes is NP-hard if P_1 and P_2 are specified by facets.

    Tight bounds for the number of k-faces, $k = 0, 1, 2$, of the sum of two three-polytopes with given numbers of vertices were obtained by Fukuda and Weibel [652]. Complementing one of their further results, Sanyal [1632] showed the following. If $r \geq n$ and if $P_1, \dots, P_r \in \mathcal{P}^n$ are polytopes each of which has at least $n + 1$ vertices, then

$$f_0(P_1 + \cdots + P_r) \leq (1 - (n + 1)^{-n}) \prod_{i=1}^{r} f_0(P),$$

    where f_0 is the number of vertices. Thus, the maximal number $\prod_i f_0(P_i)$ cannot be attained if $r \geq n$.

    The following was shown by Fogel, Halperin and Weibel [624]. Let $P_1, \dots, P_r$ be polytopes in $\mathbb{R}^3$, and let m_i be the number of facets of P_i, $i = 1, \dots, r$. Then the sum $P_1 + \cdots + P_r$ has at most

$$\sum_{1 \leq i < j \leq r} (2m_i - 5)(2m_j - 5) + \sum_{i=1}^{r} m_i + \binom{r}{2}$$

    facets, and this bound is tight.

    Karavelas and Tzanaki [1066] derived tight bounds for the maximum number of k-faces, $k = 0, \dots, n-1$, of the sum of two n-dimensional convex polytopes. Their proof uses ideas from McMullen's proof of the Upper bound theorem.

    Fukuda and Weibel [653] proved the following Euler-type relation. Let $P_1, \dots, P_r$ be n-dimensional polytopes in $\mathbb{R}^n$ which are in relatively general position. Denoting, as usual, by $f_k(Q)$ the number of k-faces of a polytope Q, put $P := P_1 + \cdots + P_r$ and $f_k^\delta(P) := f_k(P) - (f_k(P_1) + \cdots + f_k(P_r))$. Then

$$\sum_{k=0}^{d-1}(-1)^k k f_k^\delta(P) = 0.$$

13. *The Minkowski measure of symmetry.* The number $r(K, -K)$, the inradius of K relative to $-K$, is an affine invariant of K which, under different names, has often been discussed in the literature. By definition,

$$r(K, -K) = \max\{\lambda \geq 0 : -\lambda K + t \subset K \text{ for some } t \in \mathbb{R}^n\}.$$

Grünbaum [847], Secton 6.1, calls this the *Minkowski measure of symmetry* and denotes it by $F_1(K)$. Clearly, $F_1(K) = 1$ if and only if K is centrally symmetric. If $K \in \mathcal{K}_n^n$ has its centroid at o, then

$$h(K, u) \geq \frac{1}{n} h(K, -u) = \frac{1}{n} h(-K, u) \quad \text{for } u \in \mathbb{S}^{n-1},$$

which for special cases goes back to Minkowski [1435] (see also Bonnesen and Fenchel [284], §34); hence $-K \subset nK$ and thus $F_1(K) \geq 1/n$. The equality $F_1(K) = 1/n$ holds if and only if K is a simplex; see the references in Grünbaum [847], §6.1. For stability versions of this extremal property of the simplex, see Schneider [1728] and the references given there. The set

$$\begin{aligned} C(K) &:= \{a \in \mathbb{R}^n : -F_1(K)(K - a) + a \subset K\} \\ &= \frac{K \div F_1(K)(-K)}{1 + F_1(K)} = K \div \frac{F_1(K)}{1 + F_1(K)} \mathrm{D}K \end{aligned}$$

is called the *critical set* of K. Klee [1104] proved that

$$\frac{1}{F_1(K)} + \dim C(K) \leq n,$$

which implies that $\dim C(K) \leq n - 2$ for $K \in \mathcal{K}^n$. Dziechcińska-Halamoda and Szwiec [525] for polytopes and Laget [1166] in general proved that every convex body in $\mathbb{R}^n$ of dimension at most $n - 2$ is the critical set of some convex body $K \in \mathcal{K}^n$.

Jin and Guo [1042] showed that on the space of convex bodies of constant width, the Minkowski measure of symmetry attains one extremum precisely at the balls, and the other extremum is attained by the diametric completions of regular simplices.

The Minkowski measure of symmetry was generalized by Guo [870], who thus established another extremal property of simplices.

14. *Asymmetry in mathematical morphology*. Inner parallel bodies of a convex body K with respect to its difference body $\mathrm{D}K$ and the Minkowski measure of symmetry have also been discussed in mathematical morphology; see Jourlin and Laget [1054]. Jourlin and Laget [1053] asked for a characterization of convex bodies K with the property that the inner parallel body $K \div \tau \mathrm{D}K$, for some $\tau > 0$, is positively homothetic to K. For the case of the plane, an answer was given by Schneider [1710]: if $K \in \mathcal{K}_2^2$ has this property and is not centrally symmetric, then K is a polygon with the property that some homothet Q of $-K$ is properly contained in K and is such that each edge of K contains a vertex of Q. Examples of such polygons are the affine images of regular polygons with an odd number of edges, but there are many others. The corresponding problem in higher dimensions is open.
15. *Modifications of Minkowski addition.* Minkowski addition can be generalized in different ways, and some of these generalizations find increasing interest and are gaining in importance. We refer to Section 9.1.
16. In a different direction, Minkowski addition was modified by McMullen [1394]. He considered a fixed decomposition of a finite-dimensional real vector space V into complementary linear subspaces L and M. If $x \in V$, $x = y + z$ with $y \in L$ and $z \in M$, write $x = (y, z)$. The *fibre sum* of convex sets $K_0, K_1 \subset V$ relative to (L, M) is defined by

$$K_0 \uplus K_1 := \{(y_0 + y_1, z) : (y_i, z) \in K_i \text{ for } i = 0, 1\}.$$

Thus, $\uplus$ interpolates between Minkowski sum (case $L = V$) and intersection (case $M = V$). The set $K_0 \uplus K_1$ is again convex. Various properties of this operation are investigated in [1394], and a related operation for convex functions is also introduced.

3.2 Summands and decomposition

Let $K, L \in \mathcal{K}^n$ be convex bodies. As defined in Section 3.1, the body L is a *summand* of K if there exists a convex body $M \in \mathcal{K}^n$ such that $K = L + M$. In this section, we shall study different aspects of the summand relation. After a remark on direct summands, we collect some criteria for summands; then we investigate convex bodies having only trivial summands.

We start with a result concerning direct summands. We write $K = K_1 \oplus \cdots \oplus K_m$ if $K = K_1 + \cdots + K_m$ for suitable K_i lying in linear subspaces E_i of $\mathbb{R}^n$ such that $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_m$. The convex body L is called a *direct summand* of K if there exists a convex body M such that $K = L \oplus M$. If a representation $K = L \oplus M$ is only possible with $\dim L = 0$ or $\dim M = 0$, then K is called *directly indecomposable*.

Theorem 3.2.1 *Every convex body $K \in \mathcal{K}^n$ with* $\dim K \geq 1$ *has a representation*

$$K = K_1 \oplus \cdots \oplus K_m$$

with $m \in \{1, \dots, n\}$ and directly indecomposable bodies $K_1, \dots, K_m$ with positive dimension. The representation is unique up to the order of the summands.

Proof The existence of the representation is obvious. To show the essential uniqueness, suppose that

$$K = K_1 \oplus \cdots \oplus K_m = L_1 \oplus \cdots \oplus L_r$$

with $m \geq 2$, where the bodies K_i, L_j are directly indecomposable. Then also $r \geq 2$. Put $\overline{K} := K_2 \oplus \cdots \oplus K_m$. We can choose a vector $u \neq 0$ orthogonal to K_1 such that the support set $F(\overline{K}, u)$ contains only one point, say x. Then $F(K, u) = F(K_1, u) + F(\overline{K}, u) = K_1 + x$ by Theorem 1.7.5(c); thus

$$K_1 + x = F(K, u) = F(L_1, u) \oplus \cdots \oplus F(L_r, u).$$

Since K_1 is directly indecomposable, the decomposition on the right-hand side must be trivial, that is, at most one of the summands has positive dimension. Thus K_1 is a translate of a subset of some L_j. By a similar argument, L_j is a translate of a subset of some K_i. Since $\dim K_1 \geq 1$ and K_1, K_i for $1 \neq i$ are in complementary subspaces, we must have $i = 1$ and thus $K_1 = L_j$. Repeating the argument for $K_2 \oplus \cdots \oplus K_m$, and so on, we arrive at the assertion. $\square$

We turn to summands in the general case and first prove a simple intuitive criterion. For convex bodies $K, L \in \mathcal{K}^n$ one says that L *slides freely inside* K if to each boundary point x of K there exists a translation vector $t \in \mathbb{R}^n$ such that $x \in L + t \subset K$. The following theorem gives a necessary and sufficient criterion for summands and then strengthens the sufficiency part.

Theorem 3.2.2 *Let $K, L \in \mathcal{K}^n$. Then L is a summand of K if and only if L slides freely inside K.*

Suppose that for each support plane H of K there are a point $x \in H$ and a vector $t \in \mathbb{R}^n$ such that $x \in L + t \subset K$. Then L is a summand of K.

Proof If $K = L + M$ and $x \in K$, there exist $y \in L$ and $t \in M$ such that $x = y + t$, hence $x \in L + t \subset L + M = K$.

Conversely, assume that the weaker conditon above is satisfied. Then $K \div L \neq \emptyset$, and from (3.15) it follows that $(K \div L) + L \subset K$. Suppose this inclusion is strict. Then K has a support plane H such that $H \cap ((K \div L) + L) = \emptyset$. By assumption, there exist a point $x \in H$ and a vector $t \in \mathbb{R}^n$ such that $x \in L + t \subset K$. This means that $t \in K \div L$ and hence that $x \in L + (K \div L)$, a contradiction. Thus $(K \div L) + L = K$, which shows that L is a summand of K. □

For strictly convex bodies, it is sufficient that the sliding condition of Theorem 3.2.2 is satisfied locally. We say that the convex body L is *locally embeddable* in the convex body K if to each point $x \in \operatorname{bd} K$ there are a point $y \in L$ and a neighbourhood U of y such that

$$(L \cap U) + x - y \subset K.$$

For example, if P is a polytope, then $2P$ is locally embeddable in P. This shows that in the following theorem the assumption of strict convexity cannot be omitted.

Theorem 3.2.3 *Let $K, L \in \mathcal{K}^n$ and let L be strictly convex. If L is locally embeddable in K, then L is a summand of K.*

Proof Let $u_0 \in \mathbb{S}^{n-1}$ and let $x \in \operatorname{bd} K$ be a point where u_0 is attained as an outer normal vector to K. By assumption, there are a point $y \in L$ and a neighbourhood U of y such that $(L \cap U) + x - y \subset K$. Since the reverse spherical image map of the strictly convex body L is defined on the whole sphere $\mathbb{S}^{n-1}$ and is continuous (Lemma 2.2.12), there exists a neighbourhood V of u_0 in $\mathbb{S}^{n-1}$ such that the unique boundary point of L where a vector $u \in V$ is attained as outer normal vector of L, lies in U. It follows that $h(L+x-y, u_0) = h(K, u_0)$ and $h(L+x-y, u) \leq h(K, u)$ for all $u \in V$. Thus, the function $g := h(K, \cdot) - h(L, \cdot)$ and the linear function $l = \langle x - y, \cdot\rangle$ satisfy $g(u_0) = l(u_0)$ and $g \geq l$ in a neighbourhood of u_0. Now it follows from Theorem 1.5.2 that g is convex and hence a support function. This yields the assertion. □

As a consequence of Theorem 3.2.2, we can show that every summand L of a polytope P is itself a polytope. To a given vertex x of P there is a unique vector t_x such that $x \in L + t_x \subset P$. Clearly, $N(L, x - t_x) \supset N(P, x)$. Thus L has finitely many boundary points $y_1, \dots, y_k$ such that $\bigcup_{i=1}^k N(L, y_i) = \mathbb{R}^n$. If u is a normal vector of L at some exposed point a, then $u \in N(L, y_i)$ for some i and hence $a = y_i$. Consequently, L has only finitely many exposed points and thus is a polytope.

The possible summands of a polytope have a structure that is strongly related to the polytope itself. Let $P, Q_1, Q_2 \in \mathcal{P}^n$ and $P = Q_1 + Q_2$. Let F be a face of P. Choose

$u \in \mathbb{S}^{n-1}$ such that $F(P, u) = F$ and put $G_i := F(Q_i, u)$ for $i = 1, 2$. Then $F = G_1 + G_2$, hence G_1 is a summand of F. In particular, the normal vectors (of the facets) of Q_1 are among the normal vectors of P. Choose $x \in \operatorname{relint} F$. Then x has a representation $x = y_1 + y_2$ with $y_i \in \operatorname{relint} G_i$ (Lemma 1.3.12). By Theorem 2.2.1(a) we have

$$N(P, F) = N(P, x) = N(Q_1, y_1) \cap N(Q_2, y_2) = N(Q_1, G_1) \cap N(Q_2, G_2),$$

hence $N(P, F) \subset N(Q_1, G_1)$. Since $u \in \operatorname{relint} N(Q_1, G_1)$, the face G_1 does not depend on the choice of u. We say that G_1 is the face of Q_1 *corresponding to* F. If F' is a face of P containing F and G_1' is the face of Q_1 corresponding to F', then a vector v with $F' = F(P, v)$ satisfies $v \in N(P, F) \subset N(Q_1, G_1)$ and $v \in \operatorname{relint} N(Q_1, G_1')$, hence $N(Q_1, G_1') \subset N(Q_1, G_1)$. Thus G_1 is contained in G_1'.

Particularly easy to describe are the summands of a polygon $P \in \mathcal{P}^2$. Choose an orthonormal basis (e_1, e_2) of $\mathbb{R}^2$ and let $x_0 = x_0(P)$ be the vertex of P with smallest e_2-coordinate and, if there are two vertices with this property, also with smallest e_1-coordinate. Let $x_0, x_1, \dots, x_k, x_{k+1} = x_0$ be the vertices of P in cyclic order, in such a way that the vectors

$$v_i := x_i - x_{i-1}, \quad i = 1, \dots, k+1,$$

satisfy

$$v_i = L_i(e_1 \cos \alpha_i + e_2 \sin \alpha_i) \tag{3.21}$$

with $L_i > 0$ and

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{k+1} < 2\pi. \tag{3.22}$$

Then

$$\sum_{i=1}^{k+1} v_i = o. \tag{3.23}$$

Conversely, if vectors $v_1, \dots, v_{k+1}$ satisfying (3.21), (3.22) and (3.23) are given, then

$$x_0 \text{ (arbitrary)}, \quad x_j := x_0 + \sum_{i=1}^{j} v_i, \quad j = 1, \dots, k,$$

are the vertices of a convex polygon P. In fact, if $j \in \{1, \dots, k+1\}$ is given, we can choose $u \in \mathbb{S}^1$ with

$$\begin{aligned} \langle v_i, u \rangle &> 0 \quad \text{for } i = j, j-1, \dots, m, \\ \langle v_i, u \rangle &< 0 \quad \text{for } i = j+1, \dots, m-1, \end{aligned}$$

where the indices i have to be understood as residue classes modulo $(k+1)$. For $r \in \{j-1, j-2, \dots, m-1\}$ we then have

$$\langle x_j - x_r, u \rangle = \langle v_{r+1} + v_{r+2} + \cdots + v_j, u \rangle > 0$$

and for $r \in \{j+1, \dots, m-1\}$,

$$\langle x_j - x_r, u \rangle = -\langle v_{j+1} + v_{j+2} + \cdots + v_r, u \rangle > 0.$$

Thus each x_j is a vertex of $P := \operatorname{conv}\{x_0, x_1, \dots, x_k\}$.

Now suppose that $P = Q + Q'$. Since each edge of Q is a summand of an edge of P, the polygon Q has the vertices

$$x_0(Q) \text{ and } y_i = \sum_{i=1}^{j} \lambda_i v_i, \quad j = 1, \dots, k \tag{3.24}$$

(not necessarily distinct), where $0 \le \lambda_i \le 1$. Clearly, Q' then has the vertices

$$x_0(P) - x_0(Q) \text{ and } y'_j = \sum_{i=1}^{j} (1 - \lambda_i) v_i, \quad j = 1, \dots, k. \tag{3.25}$$

Conversely, if Q is a polygon whose vertices can be given by (3.24), then Q is a summand of P, since (3.25) can be used to define the vertices of the other summand. Thus we see that Q is a summand of the polygon P if and only if for each edge $F(Q, u)$ of Q the support set $F(P, u)$ is an edge of P of at least the same length. This remark will later be extended (Theorem 3.2.11).

We return to summands of general convex bodies and first observe that there is a close relation to intersections of translates. Let $K, L \in \mathcal{K}^n$ be convex bodies such that L is a summand of K. Then $K = L + M$ for some $M \in \mathcal{K}^n$, and Lemma 3.1.11 gives

$$L = K \div M = \bigcap_{t \in M} (K - t).$$

Thus, each summand of K is an intersection of a family of translates of K. For $n = 2$, this necessary condition for a summand is also sufficient:

Theorem 3.2.4 *For $K \in \mathcal{K}^2$, every nonempty intersection of translates of K is a summand of K.*

Proof Suppose that $\emptyset \neq L = \bigcap_{t \in T} (K + t)$ with some set $\emptyset \neq T \subset \mathbb{R}^2$. First let K be a polygon, say

$$K = \bigcap_{i=1}^{k} H^-_{u_i, \alpha_i}.$$

Then

$$L = \bigcap_{t \in T} (K + t) = \bigcap_{t \in T} \bigcap_{i=1}^{k} H^-_{u_i, \alpha_i + \langle u_i, t \rangle} = \bigcap_{i=1}^{k} \bigcap_{t \in T} H^-_{u_i, \alpha_i + \langle u_i, t \rangle} = \bigcap_{i=1}^{k} H^-_{u_i, \alpha_i + \beta_i},$$

where $\beta_i := \inf_{t \in T} \langle u_i, t \rangle$. It is now easy to see that the polygon L satisfies the sufficient assumption stated above for summands of a polygon K and hence that L is a summand of K.

Now let $K \in \mathcal{K}^2$ be arbitrary. We can choose a sequence $(P_i)_{i\in\mathbb{N}}$ of polygons such that

$$P_1 \supset P_2 \supset P_3 \dots \quad \text{and} \quad \bigcap_{i\in\mathbb{N}} P_i = K,$$

hence $\lim_{i\to\infty} P_i = K$ (cf. Lemma 1.8.2). We have

$$L = \bigcap_{t\in T}(K+t) = \bigcap_{t\in T}\bigcap_{i\in\mathbb{N}}(P_i+t) = \bigcap_{i\in\mathbb{N}} P_i'$$

with $P_i' := \bigcap_{t\in T}(P_i+t)$. By the first part, $P_i = P_i' + Q_i$ with $Q_i \in \mathcal{K}^2$. From $P_1' \supset P_2' \supset \dots$ we infer that $\lim P_i' = L$, and since $\lim P_i = K$, there exists $\lim Q_i =: M \in \mathcal{K}^2$, and $K = L + M$. Thus L is a summand of K. □

For $K, L \in \mathcal{K}^2$ we deduce from Theorem 3.2.4 that $K \div L$, if not empty, is always a summand of K, and that L is a summand of K if and only if $K \div (K \div L) = L$.

Adopting terminology introduced by Polovinkin, we say that a convex body $K \in \mathcal{K}^n$ is a *generating set* if any nonempty intersection of translates of K is a summand of K. Thus, two-dimensional convex bodies are generating sets. In $\mathbb{R}^n$ with $n \geq 3$, an intersection of two translates of a convex body $K \in \mathcal{K}^n$ is in general not a summand of K. This is shown by the example of a square pyramid in $\mathbb{R}^3$. However, the Euclidean ball is a generating set. This was first proved by Maehara [1317].

Theorem 3.2.5 (Maehara) *Any nonempty intersection of translates of the ball B^n is a summand of B^n.*

Proof Let L be a nonempty intersection of translates of B^n; then Lemma 3.1.10 shows that $L = B^n \div (B^n \div L)$. Let $x \in \mathrm{bd}\, B^n$ and let H be the supporting hyperplane to B^n at x. We can choose $t \in \mathbb{R}^n$ such that H supports $L + t$ at x. Suppose there exists a point $y \in (L+t) \setminus B^n$. Then $|x - y| \leq 2$; in fact, $|x - y| < 2$, since $L + t$ is an intersection of unit balls and would coincide with B^n if it contained two points at distance 2. Let E be the two-dimensional linear subspace spanned by x and y. In E there exist two minor circular arcs of radius 1 with endpoints x and y, and precisely one of them, say α, crosses the line $H \cap E$. Every unit ball containing $L+t$ contains x and y and hence α. It follows from (3.13) that $\alpha \subset B^n \div (B^n \div (L+t)) = L+t$. This is a contradiction, since H is a supporting hyperplane of $L+t$. Thus $L+t \subset B^n$. Since x was an arbitrary boundary point of B, we have proved that L slides freely inside B^n. By Theorem 3.2.2, L is a summand of B^n. □

Some interest in generating sets comes from the following result, which is due to Maehara (*loc. cit.*) for $S = B^n$ and to Sallee [1614] in the general case.

Theorem 3.2.6 (Maehara, Sallee) *Suppose $S \in \mathcal{K}^n$ is a generating set with $S = -S$. Let $A \in \mathcal{K}^n$ be a convex body with*

$$A - A \subset S.$$

If $\eta(A) := S \div (-A)$ *and* $\theta(A) := S \div (-\eta(A))$*, then* $K := \frac{1}{2}[\eta(A) + \theta(A)]$ *is a convex body that satisfies*

$$A \subset K \quad \textit{and} \quad K - K = S.$$

Proof Since $\eta(A) = \bigcap_{x \in A}(S + x)$, it follows from $A - A \subset S$ that $A \subset \eta(A)$. Further, $\theta(A) = S \div (S \div A)$ (where $S = -S$ is used), hence (3.13) gives $A \subset \theta(A)$. Therefore, $A \subset K$.

We have $\eta(A) - \theta(A) = \eta(A) - (S \div (-\eta(A))) = \eta(A) + (S \div \eta(A))$ (again using $S = -S$). Since $\eta(A)$ is an intersection of translates of S, and S is a generating set, $\eta(A)$ is a summand of S, hence it follows from Lemma 3.1.11 that $\eta(A) + (S \div \eta(A)) = S$, thus $\eta(A) - \theta(A) = S$. This yields

$$K - K = \tfrac{1}{2}[\eta(A) + \theta(A)] - \tfrac{1}{2}[\eta(A) + \theta(A)] = \tfrac{1}{2}[\eta(A) - \theta(A)] + \tfrac{1}{2}[\eta(A) - \theta(A)] = S$$

and thus the assertion. □

Together with Theorem 3.2.5, this yields the following.

Corollary 3.2.7 *If* $A \in \mathcal{K}^n$ *is a convex body with* $\operatorname{diam} A \leq 1$*, then the body* K *constructed in Theorem* 3.2.6 *with* $S = B^n$ *is a body of constant width* 1 *that contains* A*.*

Thus, every convex body of diameter d is contained in a convex body of constant width d. This fact is often deduced with the aid of an infinite iteration procedure, with many free choices, or even with Zorn's lemma. Maehara's approach provides a remarkably explicit construction.

Returning to Theorem 3.2.4 we remark that, although it does not extend beyond dimension two, criteria for summands involving intersections of translates can still be formulated in higher dimensions. Lemma 3.2.8 and Theorem 3.2.10 below are due, in more general forms, to Wieacker [1974].

Lemma 3.2.8 *Let* $K, L \in \mathcal{K}^n$*. Then* L *is a summand of* K *if and only if, for all* $x \in K$ *and* $y_1, \dots, y_{n+1} \in L$*,*

$$(x - L) \cap (K - y_1) \cap \dots \cap (K - y_{n+1}) \neq \emptyset.$$

Proof By Lemma 3.1.11, L is a summand of K if and only if $(K \div L) + L = K$, hence if and only if, for each $x \in K$, there is $z \in L$ such that

$$x - z \in K \div L = \bigcap_{y \in L}(K - y)$$

or, equivalently,

$$(x - L) \cap \bigcap_{y \in L}(K - y) \neq \emptyset.$$

Now Helly's theorem, 1.1.6, applied to the finite subfamilies of

$$\{(x - L) \cap (K - y) : y \in L\},$$

together with the compactness of L and K, yields the assertion. □

Lemma 3.2.8 is improved by Theorem 3.2.10. Its proof uses the following lemma. Recall that $K|E = \mathrm{proj}_E K$ denotes the image of K under orthogonal projection to the subspace E.

Lemma 3.2.9 *Let $K, L \in \mathcal{K}^n$. If $L|E$ is a summand of $K|E$, for all two-dimensional linear subspaces E in some dense subset of $G(n, 2)$, then L is a summand of K.*

Proof Define $g(u) := h(K, u) - h(L, u)$ for $u \in \mathbb{R}^n$. Let $E \in G(n, 2)$ be a two-dimensional linear subspace for which $K|E = L|E + M_E$ with a convex body $M_E \subset E$. For $u, v \in E$ we have $u + v \in E$ and hence

$$\begin{aligned} g(u + v) &= h(K, u + v) - h(L, u + v) = h(K|E, u + v) - h(L|E, u + v) \\ &= h(M_E, u + v) \leq h(M_E, u) + h(M_E, v) = g(u) + g(v). \end{aligned}$$

The set of all pairs (u, v) for which the inequality $g(u+v) \leq g(u)+g(v)$ has thus been established is dense in $\mathbb{R}^n \times \mathbb{R}^n$; therefore, the continuous, positively homogeneous function g is sublinear and hence, by Theorem 1.7.1, it is the support function of a convex body $M \in \mathcal{K}^n$. Then $K = L + M$. □

Theorem 3.2.10 (Wieacker) *Let $K, L \in \mathcal{K}^n$. Then L is a summand of K if and only if, for all $x \in K$ and $y_1, y_2 \in L$,*

$$(x - L) \cap (K - y_1) \cap (K - y_2) \neq \emptyset. \tag{3.26}$$

Proof Only the sufficiency of condition (3.26) has to be proved. Suppose first that $n = 2$. We assert that

$$(K - y_1) \cap (K - y_2) \cap (K - y_3) \neq \emptyset \tag{3.27}$$

for $y_1, y_2, y_3 \in L$. Suppose this were false for some $y_1, y_2, y_3 \in L$. Then there is a line H that strongly separates $(K - y_1) \cap (K - y_2)$ (which is not empty, by (3.26)) and $K - y_3$. For $x \in K$, the set $x - L$ meets $(K - y_1) \cap (K - y_2)$, $(K - y_2) \cap (K - y_3)$ and $(K-y_1)\cap(K-y_3)$ and hence also the disjoint segments $H\cap(K-y_1)$ and $H\cap(K-y_2)$. Thus $x - L$ contains the open segment S between them. From

$$S \subset \bigcap_{x \in K} (L - x) = L \div K$$

we deduce that K is strictly contained in a translate of L, say in L. Then $h(K, \cdot) \leq h(L, \cdot)$, while $h(K, u) < h(L, u)$ for some $u \in \mathbb{S}^1$. For $y, z \in L$ with $\langle y, u\rangle = h(L, u)$ and $\langle z, u\rangle = h(L, -u)$ we get

$$\begin{aligned} h(K - y, u) &= h(K, u) - \langle y, u\rangle < 0, \\ h(K - z, -u) &= h(K, -u) - \langle z, -u\rangle \leq 0 \end{aligned}$$

and therefore $(K - y) \cap (K - z) = \emptyset$. This contradicts (3.26); thus (3.27) must hold.

From (3.26), (3.27) and Helly's theorem we now get

$$(x - L) \cap (K - y_1) \cap (K - y_2) \cap (K - y_3) \neq \emptyset$$

whenever $x \in K$ and $y_1, y_2, y_3 \in L$. Lemma 3.2.8 then shows that L is a summand of K. Thus the theorem is proved for $n = 2$.

Now let $n \geq 3$. Let $E \in G(n, 2)$ and $\pi = \mathrm{proj}_E$. For $x \in \pi K$ and $y_1, y_2 \in \pi L$ we choose $\bar{x} \in K$ and $\bar{y}_1, \bar{y}_2 \in L$ with $x = \pi\bar{x}$, $y_i = \pi\bar{y}_i$ and find that

$$(\bar{x} - L) \cap (K - \bar{y}_1) \cap (K - \bar{y}_2) \neq \emptyset,$$

hence

$$(x - \pi L) \cap (\pi K - y_1) \cap (\pi K - y_2) \neq \emptyset.$$

By the first part of the proof, πL is a summand of πK. By Lemma 3.2.9, L is a summand of K. □

For special convex bodies in $\mathcal{K}^n$, such as polytopes or sufficiently smooth bodies, more effective criteria for summands can be proved. The case of polytopal summands is easy to check by means of the following result.

Theorem 3.2.11 *Let $P, K \in \mathcal{K}^n$, where P is a polytope. Then P is a summand of K if and only if the support set $F(K, u)$ contains a translate of $F(P, u)$ whenever $F(P, u)$ is an edge of P ($u \in \mathbb{S}^{n-1}$).*

Proof If $K = P + M$ with $M \in \mathcal{K}^n$, then $F(K, u) = F(P, u) + F(M, u) \supset F(P, u) + t$ for some $t \in M$, so that the condition is necessary. Suppose now that the condition is satisfied. First let $n = 2$. We can easily construct a sequence $(P_i)_{i\in\mathbb{N}}$ of polygons such that $P_i \to K$ for $i \to \infty$ and $F(K, u_j) \subset F(P_i, u_j)$ for each normal vector u_j of an edge of P. By the criterion for summands of polygons described earlier, we have $P_i = P + Q_i$ with some polygon Q_i. From $P_i \to K$ we get $Q_i \to Q$ for some $Q \in \mathcal{K}^2$ and $K = P + Q$, thus P is a summand of K.

Now let $n \geq 3$. Let $E \in G(n, 2)$ be a two-dimensional linear subspace with the property that $\dim F(P, w) \leq 1$ for each $w \in E$. Let $\pi = \mathrm{proj}_E$. If $w \in E \setminus \{o\}$ is such that $F(\pi P, w)$ is an edge of the polygon πP, then, by the choice of E, the set $\pi^{-1}F(\pi P, w) \cap P = F(P, w)$ is an edge of P. By the assumption of the theorem, the support set $F(K, w)$ contains a translate of the edge $F(P, w)$, hence $F(\pi K, w)$ contains a translate of the edge $F(\pi P, w)$. We have proved that πP and πK satisfy the assumptions of the theorem. By the first part of the proof, πP is a summand of πK. Since the set of all 2-subspaces E that could have been chosen above is dense in $G(n, 2)$, it follows from Lemma 3.2.9 that P is a summand of K. □

For a convex body $K \in \mathcal{K}^n$ we denote by $\mathcal{S}(K)$ the set of all convex bodies that are homothetic to a summand of K. Thus $\mathcal{S}(K)$ is a convex cone in $\mathcal{K}^n$, in the sense that $L, M \in \mathcal{S}(K)$ and $\lambda \geq 0$ imply $L + M \in \mathcal{S}(K)$ and $\lambda L \in \mathcal{S}(K)$. Further, for convex bodies $L, K \in \mathcal{K}^n$ we write $L \leq K$ if $\dim F(L, u) \leq \dim F(K, u)$ for all $u \in \mathbb{S}^{n-1}$. Then the special case of Theorem 3.2.11 where K is also a polytope (this is due to

Shephard [1776]) evidently implies: if $P, K \in \mathcal{P}^n$ are polytopes, then $P \in \mathcal{S}(K)$ if and only if $P \leq K$.

Another class of convex bodies for which criteria for summands are easily formulated are the bodies of class C^2_+. The following result is just a reformulation of Theorem 2.5.4.

Theorem 3.2.12 *Let $K, L \in \mathcal{K}^n$ be convex bodies of class C^2_+. Then L is a summand of K if and only if the radii of curvature satisfy*

$$r(L, u, v) \leq r(K, u, v)$$

for each orthogonal pair of vectors $u, v \in \mathbb{S}^{n-1}$; equivalently, if the curvatures satisfy

$$\kappa(L, x, t) \geq \kappa(K, y, t)$$

for $x \in F(L, u)$, $y \in F(K, u)$, $u \in \mathbb{S}^{n-1}$ and $t \perp u$.

Consequently, if K is a convex body of class C^2_+, then there are positive numbers r, R such that rK is a summand of the unit ball B^n and B^n is a summand of RK.

Let $K, L \in \mathcal{K}^n$. One says that the convex body L *rolls freely inside* K if for each rotation $\rho \in \mathrm{SO}(n)$ and each boundary point x of K there is a translation vector $t \in \mathbb{R}^n$ such that $x \in \rho L + t \subset K$. By Theorem 3.2.2, this is equivalent to the condition that each congruent image of L is a summand of K. Hence, Theorem 3.2.12 yields the following.

Corollary 3.2.13 *Let $K, L \in \mathcal{K}^n$ be convex bodies of class C^2_+. Then L rolls freely inside K if and only if*

$$\max_{u,v} r(L, u, v) \leq \min_{u,v} r(K, u, v).$$

This is a generalization of Blaschke's 'rolling theorem'. Blaschke [241], §24, considered the case where $n \leq 3$ and L is a ball.

We turn now to the investigation of convex bodies having only trivial summands. The convex body $K \in \mathcal{K}^n$ of positive dimension is called *indecomposable* if a representation $K = K_1 + K_2$ with $K_1, K_2 \in \mathcal{K}^n$ is only possible with K_1, K_2 homothetic to K; otherwise K is called *decomposable*. Thus, K is indecomposable precisely if $\mathcal{S}(K) = \{\lambda K + t : \lambda \geq 0,\ t \in \mathbb{R}^n\}$.

In dealing with indecomposable convex bodies (and elsewhere too) it is convenient to introduce the classes

$$\mathcal{K}^n_s := \{K \in \mathcal{K}^n : s(K) = o\}$$

and

$$\mathcal{K}^n_{s,1} := \{K \in \mathcal{K}^n : s(K) = o,\ w(K) = 1\},$$

where s is the Steiner point and w is the mean width (see Section 1.7). We call $\mathcal{K}^n_{s,1}$ the class of *normalized convex bodies*. Evidently it contains precisely one element from each homothety class of convex bodies with more than one point. From the linearity properties of Steiner point and mean width it follows that $\mathcal{K}^n_{s,1}$ is a convex subset of $\mathcal{K}^n$, by which we mean, of course, that $K, L \in \mathcal{K}^n_{s,1}$ and $\lambda \in [0,1]$ implies that $(1-\lambda)K + \lambda L \in \mathcal{K}^n_{s,1}$. By the continuity of s and w, the set $\mathcal{K}^n_{s,1}$ is closed. If $K \in \mathcal{K}^n_{s,1}$ and $x \in K$, then $[o, x] \subset K$ and hence $(2\kappa_{n-1}/n\kappa_n)|x| = w([o, x]) \leq w(K) = 1$. Thus $\mathcal{K}^n_{s,1}$ is bounded, and by the Blaschke selection theorem it is compact. We see that the map $\Upsilon : K \mapsto h_K$ (see (1.26) and Lemma 1.8.14) maps $\mathcal{K}^n_{s,1}$ bijectively onto a compact convex set in the Banach space $C(\mathbb{S}^{n-1})$. Under this map, the indecomposable bodies in $\mathcal{K}^n_{s,1}$ correspond precisely to the extreme points of $\Upsilon(\mathcal{K}^n_{s,1})$. (The definition of an extreme point is the same as in Section 1.4: $x \in \Upsilon(\mathcal{K}^n_{s,1})$ is an extreme point of $\Upsilon(\mathcal{K}^n_{s,1})$ if and only if it cannot be represented in the form $x = (1-\lambda)y + \lambda z$ with $y, z \in \Upsilon(\mathcal{K}^n_{s,1})$, $y \neq z$ and $\lambda \in (0, 1)$.) In fact, let $K \in \mathcal{K}_s$ be decomposable. Then $K = L + M$ with suitable $L, M \in \mathcal{K}^n$ that are not homothetic to K. We have $w(L) + w(M) = w(L + M) = w(K) = 1$ and similarly $s(L) + s(M) = o$. Writing $\overline{L} := [L - s(L)]/w(L)$, $\overline{M} := [M - s(M)]/w(M)$ and $\lambda = w(M)$, we get $\overline{L}, \overline{M} \in \mathcal{K}^n_{s,1}$ and $(1-\lambda)\overline{L} + \lambda\overline{M} = K$, hence $\Upsilon(K) = (1-\lambda)\Upsilon(\overline{L}) + \lambda\Upsilon(\overline{M})$ with $\Upsilon(\overline{L}) \neq \Upsilon(K)$, so that $\Upsilon(K)$ is not an extreme point of $\Upsilon(\mathcal{K}^n_{s,1})$. The converse conclusion is obvious.

In the plane, the indecomposable convex bodies are easily determined.

Theorem 3.2.14 *The indecomposable bodies in $\mathcal{K}^2$ are the segments and the triangles.*

Proof That segments and triangles are indecomposable is clear (a summand of a triangle must be a polygon with the same normal vectors). Conversely, let $K \in \mathcal{K}^2$ be indecomposable and not a segment. By Theorem 3.2.4, each nonempty intersection $K \cap (K + t)$ must be homothetic to K. Let U denote the set of all exterior unit normal vectors at regular boundary points of K. Choose $u, v \in U$, $u \neq \pm v$, and let x, y be regular points of $\operatorname{bd} K$ where these normal vectors are attained. Then $\overline{K} := K \cap (K + x - y)$ has interior points. Since $\overline{K}$ is homothetic to K, the body K must have a boundary point where both u and v are normal vectors. This implies that no vector $w \in \operatorname{int} \operatorname{pos}\{u, v\}$ can be a normal vector at a regular point of K. Thus, U must consist either of three vectors or of four vectors in opposite pairs. Since the support lines at regular points (which are dense in $\operatorname{bd} K$) determine K uniquely (Theorem 2.2.6), K is either a triangle or a parallelogram, but the latter is a sum of two non-parallel segments and thus decomposable. □

Remark 3.2.15 A regular hexagon in the plane can be written as the Minkowski sum of two triangles and also as the sum of three segments. Thus a representation of a convex body as a finite sum of indecomposable bodies, if possible, is in general not unique.

The assertion of Theorem 3.2.14 is typical for the plane; in higher dimensions the situation changes drastically. In fact, most convex bodies in $\mathbb{R}^n$, for $n \geq 3$, are indecomposable. We first show that there are many indecomposable polytopes.

There are several criteria from which the indecomposability of a polytope can be deduced; we choose one due to McMullen [1389]. Let $P \in \mathcal{P}^n$ be a polytope.

A *strong chain of faces* of P is a sequence $F_0, F_1, \ldots, F_k$ of faces of P such that $\dim(F_{j-1} \cap F_j) \geq 1$ for $j = 1, \ldots, k$. A family $\mathcal{F}$ of faces of P is called *strongly connected* if for all $F, G \in \mathcal{F}$ there exists a strong chain $F = F_0, F_1 \ldots, F_k = G$ with $F_j \in \mathcal{F}$ for $j = 1, \ldots, k$. The family $\mathcal{F}$ *touches* the face F if $F \cap G \neq \emptyset$ for some $G \in \mathcal{F}$.

Theorem 3.2.16 *Let $P \in \mathcal{P}^n$ be a polytope having a strongly connected family $\mathcal{F}$ of indecomposable faces that touches each facet of P. Then P is indecomposable.*

Proof We may assume that $\dim P = n$. Let Q be a summand of P. Let $F_0, F_1, \ldots, F_k \in \mathcal{F}$ form a strong chain. Let G_j be the face of Q corresponding to F_j (in the sense explained after Theorem 3.2.2). Then G_j is a summand of F_j and F_j is indecomposable, hence $G_j = \lambda_j F_j + t_j$ with suitable $\lambda_j \geq 0$ and $t_j \in \mathbb{R}^n$. Because dim $(F_{j-1} \cap F_j) \geq 1$ for each j, we conclude that $\lambda_{j-1} = \lambda_j$ and $t_{j-1} = t_j$ for $j = 1, \ldots, k$. Since the family $\mathcal{F}$ is strongly connected, there are a number $\lambda \geq 0$ and a vector $t \in \mathbb{R}^n$ such that $F(Q, v) = \lambda F(P, v) + t$ whenever $F(P, v) \in \mathcal{F}$.

Let $u_1, \ldots, u_m$ be the outer unit normal vectors of the facets of P; let $i \in \{1, \ldots, m\}$. By assumption, $F(P, u_i)$ has a vertex x lying in some face $F(P, v) \in \mathcal{F}$. The corresponding vertex y of Q lies in $F(Q, v)$, hence $y = \lambda x + t$. This yields $h(Q, u_i) = \langle y, u_i \rangle = \langle \lambda x + t, u_i \rangle = \lambda h(P, u_i) + \langle t, u_i \rangle = h(\lambda P + t, u_i)$ for $i = 1, \ldots, m$ and hence $Q = \lambda P + t$, by Corollary 2.4.4. Since Q was an arbitrary summand of P, the polytope P is indecomposable. □

Corollary 3.2.17 *If all two-dimensional faces of the polytope P are triangles, then P is indecomposable.*

This can be used to show that, for $n \geq 3$, most convex bodies in $\mathcal{K}^n$ are indecomposable.

Theorem 3.2.18 *For $n \geq 3$, the set of indecomposable bodies in $\mathcal{K}^n$ is a dense G_δ set.*

Proof Since the simplicial polytopes are dense in $\mathcal{K}^n$ if $n \geq 3$, it follows from Corollary 3.2.17 that the set $\mathcal{A}$ of indecomposable polytopes is a dense subset of $\mathcal{K}^n$. Let $K \in \mathcal{K}^n$ be decomposable. Then one easily checks that K can be written in the form $K = (L + M)/2$ with convex bodies $L, M \in \mathcal{K}^n$ such that $s(K) = s(L) = s(M)$, $w(K) = w(L) = w(M)$ and $L \neq M$. For $k \in \mathbb{N}$, let $\mathcal{B}_k$ be the set of all $K \in \mathcal{K}^n$ having such a representation with $\delta(L, M) \geq 1/k$. Using the Blaschke selection theorem, one shows that $\mathcal{B}_k$ is closed. Since $\mathcal{A} = \mathcal{K}^n \setminus \bigcup_{k \in \mathbb{N}} \mathcal{B}_k$, the assertion follows. □

Since in a Baire space the intersection of two dense G_δ sets is a dense G_δ set, the last result together with Theorem 2.7.1 shows that most convex bodies in $\mathcal{K}^n$, $n \geq 3$, are smooth, strictly convex and indecomposable. It appears that no concrete example of such a body is explicitly known. This is not too surprising, since it is hard to imagine how such a body should be described. By a result of Sallee [1611], to be indecomposable it cannot be too smooth. More precisely, a neighbourhood U

(of some point) in the boundary of a convex body $K \in \mathcal{K}^n$ is called *ε-smooth*, for some $\varepsilon > 0$, if for each $x \in U$ there exist $t \in \mathbb{R}^n$ such that $x \in \varepsilon B^n + t \subset K$. Sallee proved: if the boundary of a strictly convex body K contains a neighbourhood U that is ε-smooth for some $\varepsilon > 0$, then K is decomposable.

Notes for Section 3.2

1. *Direct decompositions.* The proof of Theorem 3.2.1 can be extended to the case of line-free closed convex sets. In this form, the theorem was proved by Gruber [809], in a less direct way. For an approach involving matroids, see Kincses [1078], Theorem 2.5. From a general and abstract viewpoint, such decompositions were studied by Gale and Klee [662].
2. The first part of Theorem 3.2.2 is found at several places in the literature (sometimes described as 'obvious' or 'folklore'); see Geivaerts [690], Sallee [1612], Weil [1934], Firey [608]. The proof of the sharpening formulated in the second part of the theorem is taken from Moreno and Schneider [1448]. The result follows also from Theorem 2.3 in Montejano [1446]. This paper contains further characterizations of summands and has several results about Minkowski differences.

   Theorem 3.2.3 is taken from Karasëv [1065].

   Some more criteria for summands, more generally for closed, bounded, convex sets in Hausdorff topological vector spaces, are found in Grzybowski, Przybycień and Urbański [854].
3. *Intersections of translates.* Theorem 3.2.4 was proved in a special case (intersections of two translates) by Meyer [1422]. His proof can be extended to the general case, which appears in Geivaerts [690].

   A general study of relations between summands and intersections of translates was presented by Wieacker [1974], from which Theorem 3.2.10 is taken. He has other, more general, results in this spirit, also for unbounded convex sets. Restricted to convex bodies, one of his results reads as follows.

   Theorem Let $K, L_i \in \mathcal{K}^n$ for $i \in I$ (an index set) and suppose that $\bigcap_{i\in I} L_i \neq \emptyset$. If for each subset J of I with $n-1$ or fewer elements there is a subset $J' \subset I$ such that $\bigcap_{j\in J\cup J'} L_j$ is a summand of K, then $\bigcap_{i\in I} L_i$ is a summand of K.

   This result can be considered as a higher-dimensional extension of Theorem 3.2.4, to which it reduces if $n = 2$ and each L_i is a translate of K.
4. *Generating sets.* Generating sets play a role in the work of Polovinkin; see Balashov and Polovinkin [115] and the citations there. Karasëv [1065] proved that a convex body K is already a generating set if any nonempty intersection of two translates of K is a summand of K. Theorem 3.2.6 above, due to Sallee, was later rediscovered by Polovinkin [1541] (more generally, in reflexive Banach spaces). Maehara's result, Theorem 3.2.5, that the Euclidean ball is a generating set, was rediscovered by Polovinkin [1540] (Theorem 2); a different proof appears in Karasëv [1065], and an extension to Hilbert spaces in Balashov and Polovinkin [115]. That two-dimensional convex bodies are generating sets was also rediscovered (Balashov and Polovinkin [115], Corollary 2.6). Sallee observed that the system of generating sets is closed under linear transformations and direct sums. From the work of McMullen, Schneider and Shephard [1397] on monotypic polytopes, the n-dimensional convex polytopes ($n \geq 3$) which are generating sets are known in the following cases (see assertions (36), (37), (43) and the supplement by Borowska and Grzybowski [301]): in dimension three, in dimension n if they have at most $n+3$ facets and in any dimension n if they are centrally symmetric. In fact, a centrally symmetric polytope is a generating set if and only if it is a direct sum of polygons. Further examples of generating sets (under the name of 'Sallee sets') were constructed by Borowska and Grzybowski [302], but none of them is centrally symmetric, or smooth, or strictly convex.

Borowska and Grzybowski [300] have also extended the notion of Sallee set to ordered commutative semigroups satisfying the order cancellation law.

For convex bodies with a twice continuously differentiable support function, Ivanov [1030] has derived a criterion for generating sets. He deduced that by sufficiently smooth perturbations of a ball one can obtain centrally symmetric generating sets which are not ellipsoids. For example, the outer parallel bodies of an ellipsoid at a sufficiently small positive distance are generating sets.

5. Geivaerts [690, 691] has studied the class $\mathcal{S}(K)$ of all convex bodies homothetic to a summand of the body $K \in \mathcal{K}^n$. For $n = 2$ he proved that $\mathcal{S}(K)$ is closed if and only if K is a polygon, and that $\mathcal{S}(K)$ is dense in $\mathcal{K}^2$ if and only if K is smooth. He also investigated the class $\mathcal{S}'(K) \subset \mathcal{K}^n$ of all bodies L with the property that for each $x \in \operatorname{bd} K$ there exist $\lambda_x > 0$ and $t_x \in \mathbb{R}^n$ such that $x \in \lambda_x L + t_x \subset K$.
6. *Criteria for summands.* Theorem 3.2.11 was proved by Shephard [1776] in the case where K is a polytope and by Weil [1934] in the general case, though in a more indirect way using deeper results. The proof given here is taken from Schneider [1671]. A strengthening of the theorem appears in Grzybowski, Urbański and Wiernowolski [855].

   Weil [1944] proved the following theorem and explained that it can be considered as a generalization of Theorem 3.2.11 to general convex bodies.

   Theorem Let $K, L \in \mathcal{K}^n$. Then L is a summand of K if and only if, for each translate L' of L not contained in K, the set of all points of L' farthest from K is a support set of L'.

   If L is strictly convex, this simplifies as follows: L is a summand of K if and only if each translate L' of L not contained in K has a unique point farthest from K.

   Wieacker [1976] found a way to generalize Weil's theorem to unbounded closed convex sets. Since for such sets the vector sum need not be closed, and sets of farthest points may be empty, this requires a careful reformulation.

   Weil [1944] and Firey (oral communication, Oberwolfach 1980) proposed to characterize summands L of convex bodies K by imposing topological conditions on $(L + t) \cap \operatorname{bd} K$. Such a result was proved by Goodey [733]:

   Theorem Let $K, L \in \mathcal{K}^n_n$. If $\operatorname{bd} K \cap \operatorname{int} L'$ is acyclic (in particular, if it is a topological ball) for every translate L' of L, then L is a summand of K.

   In fact, Goodey did not speak of summands but of freely sliding bodies, but this is equivalent by Theorem 3.2.2.

   Burton [360] obtained the following sufficient criterion for summands. Let $K, L \in \mathcal{K}^n$ be convex bodies with $L \subset K$ whose nearest-point maps commute, that is, $p_L \circ p_K = p_L$ $(= p_K \circ p_L)$. Then L is a summand of K. He also proved the following. Suppose that L is smooth; then $p_L \circ p_K = p_L$ if and only if K is an outer parallel body of L.

   Criteria for summands involving mixed volumes or mixed area measures were proved by Weil [1934] and McMullen [1391]; see also Matheron [1361] (cf. Section 4.2, Note 2, and Section 5.1, Note 9).
7. *Rolling theorems.* Extensions of Blaschke's rolling theorem, also to non-compact convex sets, were treated by Koutroufiotis [1145], Rauch [1560], Delgado [475] and very thoroughly and in great generality by Brooks and Strantzen [347]. Analogues for convex curves in spherical or hyperbolic two-dimensional space are due to Karcher [1067]. In a question involving geometric probabilities (see §8.5 of the book by Schneider and Weil [1740]), freely rolling convex bodies were used, and some characterizations proved, by Firey [608] and Weil [1947]. In particular, these authors (Firey, §5, for the planar case; Weil, Theorem 1, in $\mathbb{R}^n$) give versions of Theorem 3.2.12 that hold for general convex bodies, without differentiability assumptions. Brooks and Strantzen [347], Theorem 4.3.2, have a very general result of this kind where, however, the conclusion is not in terms of the summand relation but in terms of the inclusion relation.
8. Apparently the first to consider indecomposable convex bodies was Gale [660]. Without proof he gave some examples of decomposable and indecomposable convex bodies, and

stated the assertion of Theorem 3.2.14. For this theorem, an analytic proof was given by Silverman [1791] and a geometric one by Meyer [1419, 1422]. Meyer proved a weaker form of Theorem 3.2.4 and then used a characterization of the simplex due to Rogers and Shephard [1586], which holds in n-dimensional space (see Section 10.1, Note 4, for further results related to this theorem).

Simple examples of indecomposable convex bodies in $\mathbb{R}^n$ ($n \geq 3$) that are not polytopes can be constructed in the following way. A *general frustum* over A and B in $\mathbb{R}^n$ is, by definition, the convex hull of two convex bodies A and B that lie in different parallel hyperplanes. Sallee [1612] proved the following. Suppose that K is a general frustum over A and B, where A is indecomposable and no homothet of A is a summand of B. Then K is indecomposable.

9. *Summands and decomposability of polytopes.* The first results on summands of polytopes and on indecomposability are due to Shephard [1776] (see also Grünbaum [848], Chapter 15). Shephard proved the polytope case of Theorem 3.2.11 and deduced the following sufficient condition for indecomposability.

   Theorem A polytope $P \in \mathcal{P}^n$ is indecomposable if it has an edge to which each vertex is connected by some strong chain of indecomposable faces.

   Among the consequences that Shephard deduced are Corollary 3.2.17 and the fact (in a more general version) that every simple n-polytope, except the simplex, is decomposable.

   Meyer [1419, 1421, 1423] has some weaker sufficient indecomposability conditions for 3-polytopes. He further studied the cone $\mathcal{S}(P)$ of all polytopes homothetic to a summand of the polytope P and deduced a necessary and sufficient condition for indecomposability in terms of the rank of a certain system of linear equalities and inequalities determined by the facets of the polytope. A clear picture of the situation and new proofs of these results can be obtained if one uses the diagram technique of polyhedral representations; see McMullen [1376] and also McMullen [1384], §5. Fourneau [625] used McMullen's representations to define a certain metric related to decomposability.

   Kallay [1059] studied the decomposability of a polytope in connection with the decomposability of its (geometric) edge graph. Smilansky [1794] gave an example of an indecomposable 4-polytope all of whose facets are decomposable.

   Although some sufficient criteria for the indecomposability of a polytope are of a combinatorial nature, such as Corollary 3.2.17, decomposability is not a combinatorial property. Meyer [1419, 1421] and Kallay [1059] both gave examples of two combinatorially equivalent 3-polytopes, one decomposable and the other not. Decomposability of polytopes is, however, a projective property: if T is a projective transformation of $\mathbb{R}^n$ that is permissible for the polytope P, then the cones $\mathcal{S}(P)$ and $\mathcal{S}(TP)$ have the same dimension. This was shown by Kallay [1060]; see also Smilansky [1795], pp. 40–41.

   Smilansky [1795] studied the decomposability of polytopes (and polyhedral sets) by investigating the space of affine dependences of the vertices of the polar polytope. He obtained new proofs for results of Meyer [1423] and McMullen [1376] and several new results, in particular on combinatorial decomposability, for instance: if a 3-polytope has more vertices than facets, then it is decomposable; if a 3-polytope has at most three triangular facets, then it is decomposable.

10. *Summands and symmetry.* Shephard [1780] called a convex body K with centre o *irreducible* if it is not the difference body of some body that is not homothetic to K; otherwise K is called *reducible* (this notion of reducibility is different from the one in Section 3.1, Note 13). He proved that K is reducible if and only if it has a summand without a centre of symmetry, and gave examples of reducible and of irreducible convex bodies. An application of Shephard's investigation to Banach space geometry can be found in Payá and Yost [1521]. A thorough study of irreducible polytopes, including various necessary and sufficient criteria, is due to Yost [2003]. One of his results states that every n-dimensional convex polytope, for $n \geq 3$, with less than $4n$ vertices is irreducible; the bound $4n$ is best possible.

For $K \in \mathcal{K}_n^n$, Tennison [1843] defined $f(K) := \sup V(L)/V(K)$, where the supremum is taken over the centrally symmetric summands L ($\{o\}$ admitted) of K. He showed that f has many of the properties usually associated with a measure of symmetry, except possibly continuity. In fact, it follows from Corollary 3.2.17 and the denseness of simplicial polytopes that f is not continuous.

11. *Asymmetry classes.* For $K_1, K_2 \in \mathcal{K}^n$ we write $K_1 \approx K_2$ if there exist centrally symmetric convex bodies $S_1, S_2 \in \mathcal{K}^n$ such that $K_1 + S_1 = K_2 + S_2$. Thus, $K_1 \approx K_2$ if and only if $K_1 - K_2$ is centrally symmetric. Defining the *asymmetry function* of K by $a(K, u) := h(K - s(K), u) - h(K - s(K), -u)$ (where s denotes the Steiner point), it is also true that $K_1 \approx K_2$ if and only if K_1 and K_2 have the same asymmetry function. The relation $\approx$ is an equivalence relation on $\mathcal{K}^n$, and the corresponding classes are called *asymmetry classes*. Let $[K]$ denote the asymmetry class of K. In [543], Ewald and Shephard posed the question of whether every asymmetry class $[K]$ has a *strongly minimal* member M, in the sense that each other element $L \in [K]$ can be expressed in the form $L = M + S$ with a centrally symmetric body $S \in \mathcal{K}^n$. (We say here 'strongly minimal', instead of 'minimal' as in [543], to avoid confusion with the subsequent note.) A strongly minimal member of an asymmetry class is necessarily unique up to translation. Schneider [1674] proved that for $n = 2$ the answer to the question of Ewald and Shephard is in the affirmative (for a different proof, see §7.3 of the book by Pallaschke and Urbański [1498]). For $n \geq 3$, the answer is negative, in general. This follows from the result in [1674], that a polytope P is a strongly minimal member of its asymmetry class if and only if P and $-P$ do not have a pair of equiparallel edges, that is, parallel edges of the form $F(P, u)$ and $F(-P, u)$. (See also Section 8.3, Note 1.)
12. *Minimal pairs of convex bodies.* Motivated by quasidifferential calculus, Pallaschke, Scholtes and Urbański [1497] have introduced, for pairs $(A, B) \in \mathcal{K}^n \times \mathcal{K}^n$ (in fact, more generally in Hausdorff topological vector spaces), an equivalence relation $\sim$ by

$$(A, B) \sim (C, D) \Leftrightarrow A + D = B + C$$

and an order relation $\leq$ by

$$(A, B) \leq (C, D) \Leftrightarrow A \subset C \text{ and } B \subset D.$$

A pair (A, B) is called a *minimal pair* if it is a minimal element, with respect to the order $\leq$, in its $\sim$ equivalence class $[A, B]$. Recall that a convex body K is an *antisummand* of the convex body A if A is a summand of K. The following holds (Lemma 3.1 in Scholtes [1742]). The pair (C, D) is a minimal member of the equivalence class $[A, B]$ if and only if $A + D$ $(= B + C)$ is (with respect to inclusion) a minimal common antisummand of A and B (in other words, if A and B are summands of the convex body C and $C \subset A + D$, then $C = A + D$). Every equivalence class $[A, B]$ contains a minimal pair, which is in general not unique up to translations. Pallaschke and Urbański [1498] have devoted a book to the topic of minimal pairs in topological vector spaces; to this book we refer the reader for details. Here we mention only a few results in the finite-dimensional case. In the plane, minimal pairs in a given equivalence class are uniquely determined up to translations. This was proved independently by Scholtes [1743] and Grzybowski [851]. A third proof was given by Bauer [177]; it has the advantage that it yields a characterization of the minimal pairs in the plane (see Section 8.3, Note 1). Bauer has also proved that most pairs $(A, B) \in \mathcal{K}^n \times \mathcal{K}^n$, in the Baire category sense, are the unique minimal element (up to translations) in their equivalence class $[A, B]$.

In their book [1498], Pallaschke and Urbański asked the following question (Q1 on page 279). If (A, B) is a pair of polytopes, does the equivalence class $[A, B]$ contain a minimal pair consisting of polytopes? For $n = 2$, this follows from known results. An affirmative answer in dimension 3 was given by Grzybowski [853]. The higher-dimensional case seems to be open.
13. *Reduced pairs of convex bodies.* Related to minimal pairs is the following notion introduced in Bauer [177]. The pair $(K, M) \in \mathcal{K}^n \times \mathcal{K}^n$ is called *reduced* if $K + M$ is a

summand of each common antisummand of K and M. Equivalently, (K, M) is reduced if and only if each element of the equivalence class $[K, M]$ (see the previous note) is of the form $(K + L, M + L)$ with $L \in \mathcal{K}^n$. Every reduced pair is minimal. Bauer characterized the reduced pairs in the plane (see Section 8.3, Note 1) and showed that a pair (P, Q) of n-dimensional polytopes is reduced if and only if P and Q do not have a pair of equiparallel edges, that is, parallel edges of the form $F(P, u)$ and $F(Q, u)$.

14. *Maximal common summands.* A *maximal common summand* of the convex bodies $A, B \in \mathcal{K}^n$ is a maximal element, with respect to the inclusion order, of the set of all common summands of A and B. Maximal common summands exist, and in the plane they are uniquely determined up to translation. This was proved, in a stronger form, by Grzybowski [852]. A different proof is given in the first part of Theorem 8.3.5. Grzybowski further showed that the uniqueness result does not extend to higher dimensions (cf. also [1498], Example 8.2.3).
15. *Indecomposable pairs of convex bodies.* As the example of a triangle T and its reflected image $-T$ in the plane shows, it may well happen that the sum $K + M$ of two convex bodies K, M has summands that are not homothetic to a combination of K and M. We say that the pair (K, M) of convex bodies is *indecomposable* if K and M are indecomposable and every summand of $K + M$ is a translate of $\alpha K + \beta M$ with some $\alpha, \beta \geq 0$. The indecomposable pairs of convex bodies in the plane are easily determined; see Schneider [1719]. In that paper, a sufficient criterion for the indecomposability of a pair of polytopes in $\mathbb{R}^n$, $n \geq 3$, is given. It is deduced that, for $n \geq 3$, the set of indecomposable pairs of convex bodies in $\mathcal{K}^n \times \mathcal{K}^n$ is a dense G_δ set.
16. *Relative indecomposability.* As explained in Section 3.2, indecomposable convex bodies correspond to extreme points of the compact convex set $\mathcal{K}^n_{s,1}$. More generally, if $\mathcal{M}$ is a nonempty, compact, convex subset of $\mathcal{K}^n$, we say that $K \in \mathcal{M}$ is *extreme in* $\mathcal{M}$ if $K = (1 - \lambda)K_1 + \lambda K_2$ with $K_1, K_2 \in \mathcal{M}$ and $\lambda \in (0, 1)$ implies that $K_1 = K_2$. For a closed convex subset $\mathcal{M} \subset \mathcal{K}^n$ we also say that $K \in \mathcal{M}$ is *indecomposable relative to* $\mathcal{M}$ if $K = (1 - \lambda)K_1 + \lambda K_2$ with $K_1, K_2 \in \mathcal{M}$ and $\lambda \in (0, 1)$ implies that K_1 and K_2 are homothetic to K. The following cases have been considered. Kallay [1057, 1058] gave a complete characterization, in the plane, of all convex bodies that are extreme in the class of bodies with a given width function. In particular, a convex body $K \in \mathcal{K}^2$ of constant width 1 is indecomposable relative to the class of bodies of constant width if and only if its radius of curvature function (which exists almost everywhere on $\mathbb{S}^1$) is almost everywhere either 0 or 1. In higher dimensions, it seems unknown whether the relatively indecomposable bodies of constant width have a similar characterization, and whether they are dense in the set of all bodies of constant width.

    McMullen [1388] determined (and used for an extremum problem) the convex bodies that are indecomposable relative to the set of bodies of revolution.

    Grzaślewicz [850] determined the convex bodies that are extreme in $\mathcal{K}(C)$ (the set of convex bodies contained in C) for a strictly convex set $C \in \mathcal{K}^2_0$. He showed that K is extreme in $\mathcal{K}(C)$ if and only if either (1) K is a line segment with endpoints on $\operatorname{bd} C$ or (2) $\operatorname{int} K \neq \emptyset$ and each connected component of $\operatorname{bd} K \setminus \operatorname{bd} C$ contains at most one extreme point of K.
17. Most of the preceding treatment of decompositions $K = L + M$ was restricted to convex bodies K, L, M in $\mathbb{R}^n$. For unbounded convex sets and in infinite-dimensional vector spaces, many new problems arise. A detailed study was begun by Bair [106, 108, 109, 110], Bair and Jongmans [114].
18. *Decomposition of non-convex sets*. In this note only, 'indecomposable' will have a different meaning. The set $Z \subset \mathbb{R}^n$ (not necessarily convex) is called *indecomposable* if it cannot be represented as a vector sum $Z = Z_1 + Z_2$ with $Z_1, Z_2 \in \mathbb{R}^n$, where each Z_i contains more than one point. Milka [1425] investigated the indecomposability of convex surfaces (i.e., boundaries of convex sets with interior points). Milka showed that every (relative) neighbourhood of the set of extreme points of a closed convex surface in $\mathbb{R}^3$ is indecomposable and that a closed strictly convex hypersurface in $\mathbb{R}^n$ is indecomposable. Among further results, he proved: if $K \subset \mathbb{R}^3$ is closed and convex, with interior points

and with recession cone $\operatorname{rec} K$ a ray, and if $\operatorname{bd} K$ is decomposable in two different ways, then $\operatorname{bd} K$ is a paraboloid.

19. *Extremal convex functions* are, in some sense, analogous to indecomposable convex bodies. They were investigated by Bronshtein [342] and others.

3.3 Additive maps

Since Minkowski addition is a basic structure on the set $\mathcal{K}^n$ of convex bodies, the maps defined on $\mathcal{K}^n$ and compatible with this addition deserve special interest. Let φ be a map from $\mathcal{K}^n$ into some abelian group. (If the natural range is an abelian semigroup with cancellation law, we embed it into an abelian group.) According to Section 1.7, the map φ is called *Minkowski additive* if

$$\varphi(K + L) = \varphi(K) + \varphi(L) \quad \text{for } K, L \in \mathcal{K}^n.$$

At this point, we remark that nothing would be gained by studying additive maps on the larger semigroup C^n of nonempty, compact subsets of $\mathbb{R}^n$ with Minkowski addition. In fact, let φ be an additive map from C^n to an abelian group. Applying φ to (3.9), we obtain

$$\varphi(A) + \varphi(n \operatorname{conv} A) = \varphi(\operatorname{conv} A) + \varphi(n \operatorname{conv} A)$$

for $A \in C^n$ and hence $\varphi(A) = \varphi(\operatorname{conv} A)$. Thus, φ is completely determined by its values on $\mathcal{K}^n$.

Remark 3.3.1 For the commutative semigroup $\mathcal{K}^n$, the Grothendieck group can be taken as the group of differences of support functions, with corresponding homomorphism $K \mapsto h_K$. The preceding argument shows that the commutative semigroup C^n has the same Grothendieck group, with corresponding homomorphism $C \mapsto h_{\operatorname{conv} C}$.

Returning to $\mathcal{K}^n$, we notice that it follows immediately from (3.4) that every Minkowski additive map φ also satisfies the relation

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \quad \text{if } K, L, K \cup L \in \mathcal{K}^n. \tag{3.28}$$

A map φ with this property is called *additive* or a *valuation*. Chapter 6 is devoted to a more detailed study of valuations.

If φ takes its values in a real vector space or in $\mathcal{K}^n$, we say that φ is *Minkowski linear* if it is Minkowski additive and satisfies

$$\varphi(\lambda K) = \lambda \varphi(K) \quad \text{for } K \in \mathcal{K}^n \text{ and } \lambda \geq 0.$$

Examples of Minkowski linear maps are provided by the mean width, the Steiner point, the map $K \mapsto F(K, u)$ for fixed $u \in \mathbb{S}^{n-1}$ and the map $K \mapsto h_K$ that associates with a convex body its support function.

From a geometric point of view, the Minkowski additive maps enjoying an invariance property are of particular interest. Under additional continuity assumptions, one can obtain uniqueness results.

Theorem 3.3.2 *Let $n \geq 2$. If the map $\varphi : \mathcal{K}^n \to \mathbb{R}$ is Minkowski additive, invariant under proper rotations and continuous at the unit ball B^n, then φ is a constant multiple of the mean width.*

A map φ from $\mathcal{K}^n$ into $\mathbb{R}^n$ or into $\mathcal{K}^n$ is called equivariant under a group G of transformations of $\mathbb{R}^n$ if $\varphi(gK) = g\varphi(K)$ for all $g \in G$.

Theorem 3.3.3 *Let $n \geq 2$. If the map $\varphi : \mathcal{K}^n \to \mathbb{R}^n$ is Minkowski linear, equivariant under proper rigid motions and continuous at the unit ball B^n, then φ is the Steiner point map.*

Remark 3.3.4 Let φ be a Minkowski additive map from $\mathcal{K}^n$ into $\mathbb{R}$ or $\mathbb{R}^n$. For $K \in \mathcal{K}^n$ we have $2K = K + K$, hence $\varphi(2K) = 2\varphi(K)$, and induction gives $\varphi(kK) = k\varphi(K)$ for $k \in \mathbb{N}$. For $k, m \in \mathbb{N}$ one then obtains $k\varphi(K) = \varphi(kK) = \varphi(m(k/m)K) = m\varphi((k/m)K)$, thus $\varphi(qK) = q\varphi(K)$ for rational $q > 0$. If now φ is assumed to be continuous on $\mathcal{K}^n$, then we deduce that $\varphi(\lambda K) = \lambda\varphi(K)$ for real $\lambda \geq 0$. Thus, the assumption 'Minkowski linear and continuous at B^n' in Theorem 3.3.3 is weaker than the assumption 'Minkowski additive and continuous'.

Theorem 3.3.2 and its proof are essentially due to Hadwiger (e.g., [911], p. 213). We present a slightly modified version, also based on the method of rotation averaging.

If $K \in \mathcal{K}^n$ is a convex body, we say that K' is a *rotation mean* of K if there are a number $m \in \mathbb{N}$ and rotations $\rho_1, \dots, \rho_m \in \mathrm{SO}(n)$ such that

$$K' = \frac{1}{m}(\rho_1 K + \dots + \rho_m K).$$

Theorem 3.3.5 (Hadwiger) *For every convex body $K \in \mathcal{K}^n$ with* $\dim K > 0$ *there is a sequence of rotation means of K converging to a ball.*

Proof For $L \in \mathcal{K}^n$ let $d(L) := \min\{\lambda \geq 0 : L \subset \lambda B^n\}$. Clearly, the function d is continuous. A ball λB^n containing K contains all rotation means of K, hence the family $\mathcal{R}(K)$ of rotation means of K is bounded. The function d attains a minimum $d_0 > 0$ on the compact set $\mathrm{cl}\,\mathcal{R}(K)$, say at L. Assume that $L \neq d_0 B^n$. Then there is a vector $u_0 \in \mathbb{S}^{n-1}$ with $h(L, u_0) < d_0$, hence $h(L, u) < d_0$ for all u in a suitable neighbourhood U of u_0 on $\mathbb{S}^{n-1}$. Since $\mathbb{S}^{n-1}$ is compact, we can find finitely many rotations $\rho_1, \dots, \rho_m \in \mathrm{SO}(n)$ such that $\bigcup_{i=1}^m \rho_i U = \mathbb{S}^{n-1}$. Put

$$\overline{L} := \frac{1}{m}(\rho_1 L + \dots + \rho_m L).$$

Let $u \in \mathbb{S}^{n-1}$. There is a number $i \in \{1, \dots, m\}$ with $u \in \rho_i U$, hence $\rho_i^{-1} u \in U$ and thus $h(L, \rho_i^{-1} u) < d_0$, which implies

$$h(\overline{L}, u) = \frac{1}{m} \sum_{j=1}^{m} h(\rho_j L, u) = \frac{1}{m} \sum_{j=1}^{m} h(L, \rho_j^{-1} u) < d_0.$$

By the continuity of $h(\overline{L}, \cdot)$, this yields $d(\overline{L}) < d_0$. There is a sequence $(K_j)_{j \in \mathbb{N}}$ in $\mathcal{R}(K)$ converging to L, and we deduce that

$$\overline{K}_j := \frac{1}{m}(\rho_1 K_j + \cdots + \rho_m K_j) \to \overline{L}$$

for $j \to \infty$, hence $d(\overline{K}_j) < d_0$ for large j. Since $\overline{K}_j \in \mathcal{R}(K)$, this contradicts the minimality of d_0. Thus L is a ball, which proves the theorem. □

Proof of Theorem 3.3.2 For $x \in \mathbb{R}^n$ we can choose $\rho \in \mathrm{SO}(n)$ with $\rho x = -x$, hence $\varphi(\{o\}) = \varphi(\{x\} + \rho\{x\}) = 2\varphi(\{x\})$. This yields $\varphi(\{o\}) = 0$ and then $\varphi(\{x\}) = 0$ in general. Let $K \in \mathcal{K}^n$ and $\dim K > 0$. If

$$K' = \lambda_1 \rho_1 K + \cdots + \lambda_m \rho_m K$$

with positive rational numbers $\lambda_1, \dots, \lambda_m$ and rotations $\rho_1, \dots, \rho_m \in \mathrm{SO}(n)$, then $\varphi(K') = (\lambda_1 + \cdots + \lambda_m)\varphi(K)$ by the properties of φ and Remark 3.3.4. Since the mean width has the same properties as φ, we deduce that $\varphi(K)/w(K) = \varphi(K')/w(K')$. It follows from Theorem 3.3.5 that the integer m, the rotations ρ_i and the rational numbers λ_i can be chosen so that $\delta(K', B^n)$ is smaller than a given number $\varepsilon > 0$. Since φ is continuous at B^n, we deduce that $\varphi(K)/w(K) = \varphi(B^n)/2$. □

The following elegant proof of Theorem 3.3.3 is due to Positsel'skii [1546], except that he assumed equivariance under improper rigid motions also.

Proof of Theorem 3.3.3 For given $\varepsilon > 0$, we decompose the compact group $\mathrm{SO}(n)$ into finitely many nonempty Borel sets $\Delta_{1,\varepsilon}, \dots, \Delta_{m(\varepsilon),\varepsilon}$ of diameter less than ε. We choose $\rho_{k,\varepsilon} \in \Delta_{k,\varepsilon}$ and write $\nu_{k,\varepsilon} := \nu(\Delta_{k,\varepsilon})$, where ν is the normalized Haar measure on $\mathrm{SO}(n)$. Then the usual estimate shows that

$$\lim_{\varepsilon \to 0} \sum_{k=1}^{m(\varepsilon)} f(\rho_{k,\varepsilon}) \nu_{k,\varepsilon} = \int_{\mathrm{SO}(n)} f \, \mathrm{d}\nu$$

for any continuous real function f on $\mathrm{SO}(n)$.

Let $K \in \mathcal{K}^n$. If $K = \{x\}$, then $\varphi(K) = x = s(K)$ by the rigid motion equivariance of φ. Assume, therefore, that $\dim K > 0$. We choose vectors $v \in \mathbb{R}^n$, $x \in \mathbb{S}^{n-1}$ and a number $c > |v|$ and define for $\varepsilon > 0$ a convex body K_ε by

$$K_\varepsilon := \sum_{k=1}^{m(\varepsilon)} [c + \langle v, \rho_{k,\varepsilon} x \rangle] \nu_{k,\varepsilon} \rho_{k,\varepsilon}^{-1} K.$$

Applying the map φ to this body and using the properties of φ we get, for an arbitrary vector $y \in \mathbb{S}^{n-1}$,

$$\langle \varphi(K_\varepsilon), y\rangle = \sum_{k=1}^{m(\varepsilon)} [c + \langle v, \rho_{k,\varepsilon} x\rangle]\langle \varphi(K), \rho_{k,\varepsilon} y\rangle \nu_{k,\varepsilon}$$

and hence

$$\begin{aligned} \lim_{\varepsilon\to 0}\langle \varphi(K_\varepsilon), y\rangle &= \int_{\mathrm{SO}(n)} [c + \langle v, \rho x\rangle]\langle \varphi(K), \rho y\rangle \,\mathrm{d}\nu(\rho) \\ &= \int_{\mathrm{SO}(n)} \langle v, \rho x\rangle\langle \varphi(K), \rho y\rangle \,\mathrm{d}\nu(\rho), \end{aligned}$$

since $\int_{\mathrm{SO}(n)}\langle \varphi(K), \rho y\rangle \,\mathrm{d}\nu(\rho)$, as a function of y, is odd and rotation invariant, and hence zero.

On the other hand, the support function of K_ε is given by

$$h(K_\varepsilon, y) = \sum_{k=1}^{m(\varepsilon)} [c + \langle v, \rho_{k,\varepsilon} x\rangle] h(K, \rho_{k,\varepsilon} y)\nu_{k,\varepsilon}$$

and hence satisfies

$$\lim_{\varepsilon\to 0} h(K_\varepsilon, y) = \int_{\mathrm{SO}(n)} [c + \langle v, \rho x\rangle] h(K, \rho y) \,\mathrm{d}\nu(\rho).$$

Now

$$\int_{\mathrm{SO}(n)} ch(K, \rho y) \,\mathrm{d}\nu(\rho) =: r$$

is a positive number (the integral is invariant under translations of K and clearly positive if $o \in \operatorname{relint} K$) that depends only on K and v. The integral

$$I(x, y) := \int_{\mathrm{SO}(n)} \langle v, \rho x\rangle h(K, \rho y) \,\mathrm{d}\nu(\rho)$$

satisfies $I(x, y) = I(y, x)$ if $n \geq 3$, since $I(\tau x, \tau y) = I(x, y)$ for each rotation $\tau \in \mathrm{SO}(n)$, and for $n \geq 3$ we can choose τ such that $\tau x = y$ and $\tau y = x$. Writing

$$\int_{\mathrm{SO}(n)} \rho^{-1} v h(K, \rho x) \,\mathrm{d}\nu(\rho) =: z$$

for $n \geq 3$, we thus have $I(x, y) = \langle z, y\rangle$. For $n = 2$, $I(x, y)$ is not symmetric in x and y, but in this case we can write

$$I(x, y) = \frac{1}{2\pi}\int_0^{2\pi} \langle v, u(\xi + \alpha)\rangle h(K, u(\eta + \alpha)) \,\mathrm{d}\alpha$$

with $u(\alpha) := e_1 \cos\alpha + e_2 \sin\alpha$, where (e_1, e_2) is an orthonormal basis of $\mathbb{R}^2$, and $x = u(\xi),\ y = u(\eta)$. An elementary computation now yields

$$I(x, y) = (A_1 \cos\xi + A_2 \sin\xi)\cos\eta + (A_1 \sin\xi - A_2 \cos\xi)\sin\eta,$$

where the A_i depend only on K and v; hence again we have $I(x, y) = \langle z, y\rangle$ with some vector z depending only on K, v and x. Thus, in both cases we arrive at

$$\lim_{\varepsilon\to 0} h(K_\varepsilon, y) = r + \langle z, y\rangle = h(B(z, r), y),$$

where $B(z, r)$ is the ball with centre z and radius r. This holds for each $y \in \mathbb{S}^{n-1}$, hence $\lim_{\varepsilon\to 0} K_\varepsilon = B(z, r)$ in the Hausdorff metric (observe that pointwise convergence of support functions implies uniform convergence on $\mathbb{S}^{n-1}$, by Theorem 1.8.15). From this we get $r^{-1}(K_\varepsilon - z) \to B^n$ for $\varepsilon \to 0$ and hence $\varphi(r^{-1}(K_\varepsilon - z)) \to \varphi(B^n)$ by the assumed continuity of φ at B^n. Since $\varphi(B^n) = o$ by the rotation equivariance of φ, we obtain $\varphi(K_\varepsilon) \to z$ and thus

$$\lim_{\varepsilon\to 0}\langle\varphi(K_\varepsilon), y\rangle = \langle z, y\rangle$$

for $y \in \mathbb{S}^{n-1}$. We have proved that

$$\int_{\mathrm{SO}(n)} \langle v, \rho x\rangle\langle\varphi(K), \rho y\rangle \, \mathrm{d}\nu(\rho) = \langle z, y\rangle,$$

which depends only on K, v, x, y and not on φ. Since the Steiner point map has all the properties of φ, this yields

$$\int_{\mathrm{SO}(n)} \langle v, \rho x\rangle\langle\varphi(K) - s(K), \rho y\rangle \, \mathrm{d}\nu(\rho) = 0.$$

The choices $v := \varphi(K) - s(K)$ and $x = y$ now yield $\varphi(K) = s(K)$. □

Remark 3.3.6 Inspection of the proofs of Theorems 3.3.2 and 3.3.3 shows that in neither case is it necessary to assume that φ is defined on all of $\mathcal{K}^n$. In fact, the methods work if the domain of φ is a closed subset of $\mathcal{K}^n$ that is closed under Minkowski addition and under similarities, and consists not only of singletons.

It is, however, not known whether similar characterization theorems hold for maps that are only defined on polytopes. Here, of course, the continuity at the ball must be replaced by some other assumption. To formulate a definite problem, suppose that the map $\varphi : \mathcal{P}^n \to \mathbb{R}$ is Minkowski additive, invariant under rigid motions and continuous. Is it then true that $\varphi(P) = cw(P)$ for all $P \in \mathcal{P}^n$, with some constant c? A similar question can be posed for the Steiner point.

Although the continuity assumptions in Theorems 3.3.2 and 3.3.3 are rather weak, they cannot be omitted. Examples showing this can be constructed in different ways. Here we refer to the fact that in Chapter 4 (see also Section 8.3) we shall associate with each convex body $K \in \mathcal{K}^n$ a certain finite positive Borel measure $S_1(K, \cdot)$ on the unit sphere $\mathbb{S}^{n-1}$, its area measure of order one. The dependence on K is Minkowski linear, that is,

$$S_1(K + L, \cdot) = S_1(K, \cdot) + S_1(L, \cdot), \qquad S_1(\lambda K, \cdot) = \lambda S_1(K, \cdot)$$

for $K, L \in \mathcal{K}^n$ and $\lambda \geq 0$. If $S_1^s(K, \cdot)$ denotes the singular part (with respect to spherical Lebesgue measure) of $S_1(K, \cdot)$, then it follows from the uniqueness of the

Lebesgue decomposition of a measure into its singular and its absolutely continuous part that the map $K \mapsto S_1^s(K,\cdot)$ is also Minkowski linear. Furthermore, the map $K \mapsto S_1(K,\cdot)$ is translation invariant and rotation equivariant, that is,

$$S_1(K+t,\cdot) = S_1(K,\cdot), \qquad S_1(\rho K, \rho\omega) = S_1(K,\omega)$$

for any translation vector $t \in \mathbb{R}^n$, any rotation $\rho \in \mathrm{SO}(n)$ and any Borel set $\omega \subset \mathbb{S}^{n-1}$. Again, these properties carry over to $S_1^s(K,\cdot)$. Now we can define

$$w'(K) := S_1^s(K, \mathbb{S}^{n-1}), \qquad s'(K) := \int_{\mathbb{S}^{n-1}} u\, S_1^s(K, \mathrm{d}u) + s(K)$$

for $K \in \mathcal{K}^n$. Then w' and s' are Minkowski linear functions, respectively invariant and equivariant under rigid motions, but w' is not a multiple of w and s' is not equal to s. This can be shown by constructing suitable convex bodies (as will become clear in Chapter 4), and it implies that w' and s' are not continuous.

Having found that real-valued and vector-valued Minkowski additive maps satisfying certain natural invariance and continuity properties are essentially unique, we may go a step further and consider body-valued Minkowski additive maps (but without proofs, essentially). We call a map $T : \mathcal{K}^n \to \mathcal{K}^n$ a *Minkowski endomorphism* of $\mathcal{K}^n$ if it is Minkowski additive, rotation equivariant and continuous. Thus, such a map is compatible with the most basic geometric structures on $\mathcal{K}^n$. Clearly, the assumptions imply that also $T(\lambda K) = \lambda T K$ for $K \in \mathcal{K}^n$ and $\lambda \geq 0$. We do not assume (in contrast to [1675]) that T is equivariant under translations, since that can always be achieved by a simple modification: if T has the properties above, then the map T' defined by $T'(K) := T(K - s(K)) + s(K)$ for $K \in \mathcal{K}^n$ has the same properties and satisfies, in addition, $T'(K+t) = T'(K) + t$ for all $t \in \mathbb{R}^n$. Synonymously with 'rotation equivariant' one also says 'rotation intertwining', or just 'intertwining'.

To construct examples of Minkowski endomorphisms, we need a notion of convolution on the sphere. By $C(\mathbb{S}^{n-1})$ we denote the real vector space of continuous real functions on $\mathbb{S}^{n-1}$ with the maximum norm $\|\cdot\|$, and by $\mathcal{M}(\mathbb{S}^{n-1})$ the real vector space of finite signed Borel measures on $\mathbb{S}^{n-1}$ with the total variation norm $\|\cdot\|_{TV}$. The rotation group $\mathrm{SO}(n)$ operates on these spaces by means of $(\vartheta f)(u) := f(\vartheta^{-1}u)$ for $f \in C(\mathbb{S}^{n-1})$, $u \in \mathbb{S}^{n-1}$, $\vartheta \in \mathrm{SO}(n)$, respectively $(\vartheta\mu)(A) := \mu(\vartheta^{-1}A)$ for $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$, $A \in \mathcal{B}(\mathbb{S}^{n-1})$, $\vartheta \in \mathrm{SO}(n)$. We fix a point $p \in \mathbb{S}^{n-1}$ and denote by S_p the subgroup of $\mathrm{SO}(n)$ fixing p. Let $\mathcal{M}(\mathbb{S}^{n-1}, p) \subset \mathcal{M}(\mathbb{S}^{n-1})$ be the subset of signed measures that are invariant under S_p. The elements of $\mathcal{M}(\mathbb{S}^{n-1}, p)$ are called *zonal signed measures* (with pole p). We define the *convolution* of $f \in C(\mathbb{S}^{n-1})$ and $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ by

$$(f * \mu)(\vartheta p) := \int_{\mathbb{S}^{n-1}} f(\vartheta v)\, \mathrm{d}\mu(v), \qquad \vartheta \in \mathrm{SO}(n). \tag{3.29}$$

This defines $(f*\mu)(u)$ for each $u \in \mathbb{S}^{n-1}$, since $u = \vartheta p$ for suitable $\vartheta \in \mathrm{SO}(n)$, and if also $u = \rho p$ with $\rho \in \mathrm{SO}(n)$, then $\rho^{-1}\vartheta \in S_p$ and hence $\int f(\vartheta v)\, \mathrm{d}\mu(v) = \int f(\vartheta v)\, \mathrm{d}\mu(\rho^{-1}\vartheta v) = \int f(\rho u)\, \mathrm{d}\mu(u)$. It is easy to check that the mapping $f \mapsto f * \mu$ is a linear operator from $C(\mathbb{S}^{n-1})$ into itself, satisfying

$$(\rho f) * \mu = \rho(f * \mu) \quad \text{for } \rho \in \mathrm{SO}(n), \qquad \|f * \mu\| \le \|f\| \cdot \|\mu\|_{TV}. \tag{3.30}$$

If μ has a density with respect to the spherical Lebesgue measure σ, then there is an integrable function $g : [-1, 1] \to \mathbb{R}$ such that $\mu(A) = \int_A g(\langle p, v\rangle)\, \mathrm{d}\sigma(v)$ for $A \in \mathcal{B}(\mathbb{S}^{n-1})$ and

$$(f * \mu)(u) := \int_{\mathbb{S}^{n-1}} f(v) g(\langle u, v\rangle)\, \mathrm{d}\sigma(v).$$

An equivalent description of the convolution is useful. For this, we associate with $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ the signed measure $\tilde{\mu}$ on $[-1, 1]$ defined by

$$\tilde{\mu}(A) := \mu(\{u \in \mathbb{S}^{n-1} : \langle u, p\rangle \in A\}), \qquad A \in \mathcal{B}([-1, 1]).$$

For $f \in C(\mathbb{S}^{n-1})$ and $t \in [-1, 1]$ we define

$$(R_t f)(u) := \int_{\mathbb{S}^{n-1}} f\, \mathrm{d}\sigma_{u,t} \qquad \text{for } u \in \mathbb{S}^{n-1},$$

where $\sigma_{u,t}$ denotes the unique Borel probability measure on $\mathbb{S}^{n-1}$ that is concentrated on the $(n-2)$-sphere $\{x \in \mathbb{S}^{n-1} : \langle x, u\rangle = t\}$ and is invariant under the rotations of this sphere. Then we have

$$(f * \mu)(u) = \int_{-1}^{1} (R_t f)(u)\, \mathrm{d}\tilde{\mu}(t), \qquad u \in \mathbb{S}^{n-1}, \tag{3.31}$$

for $f \in C(\mathbb{S}^{n-1})$ and $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$. Denoting the right-hand side of (3.31) for the moment by $\Phi(f, \mu)(u)$, we note that $\Phi(\rho f, \mu) = \rho\Phi(f, \mu)$ for $\rho \in \mathrm{SO}(n)$. Since also $(\rho f) * \mu = \rho(f * \mu)$, it suffices to prove (3.31) for $u = p$. For this, we denote by ν_p the Haar probability measure on the group S_p. Then we get

$$\begin{aligned}
(f * \mu)(p) &= \int_{\mathbb{S}^{n-1}} f(v)\, \mathrm{d}\mu(v) = \int_{\mathbb{S}^{n-1}} f(\vartheta v)\, \mathrm{d}\mu(v) \quad \text{for } \vartheta \in S_p \\
&= \int_{S_p} \int_{\mathbb{S}^{n-1}} f(\vartheta v)\, \mathrm{d}\mu(v)\, \mathrm{d}\nu_p(\vartheta) = \int_{\mathbb{S}^{n-1}} \int_{S_p} f(\vartheta v)\, \mathrm{d}\nu_p(\vartheta)\, \mathrm{d}\mu(v) \\
&= \int_{\mathbb{S}^{n-1}} (R_{\langle p,v\rangle} f)(p)\, \mathrm{d}\mu(v) = \int_{-1}^{1} (R_t f)(p)\, \mathrm{d}\tilde{\mu}(t) = \Phi(f, \mu)(p).
\end{aligned}$$

This completes the proof of (3.31).

Now let $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ and assume that, for each $K \in \mathcal{K}^n$, the function $\bar{h}_K * \mu$ is a support function, restricted to $\mathbb{S}^{n-1}$. Then there exists a convex body $TK \in \mathcal{K}^n$ with $\bar{h}_{TK} = \bar{h}_K * \mu$. It follows from the mentioned properties of the convolution that this defines a Minkowski endomorphism T of $\mathcal{K}^n$. This Minkowski endomorphism is not only continuous, but Lipschitz continuous, since $\delta(TK, TM) \le \|\mu\|_{TV}\delta(K, M)$ by (3.30) and Lemma 1.8.14. Conversely, the following holds.

Theorem 3.3.7 *If T is a uniformly continuous Minkowski endomorphism of $\mathcal{K}^n$, then there exists a zonal signed measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ such that $\bar{h}_{TK} = \bar{h}_K * \mu$ for $K \in \mathcal{K}^n$.*

Proof Let T be given. If $f \in C(\mathbb{S}^{n-1})$ is a difference of support functions, say $f = \bar{h}_K - \bar{h}_M$, we define $T'f := \bar{h}_{TK} - \bar{h}_{TM}$. It follows from the Minkowski additivity of T that this definition does not depend on the special representation of f and that T', thus defined, is a linear operator from the vector space of differences of support functions into itself. Since this vector space is dense in $C(\mathbb{S}^{n-1})$, by Lemma 1.7.8, it follows from the uniform continuity of T that T' has a continuous extension to $C(\mathbb{S}^{n-1})$; we denote it also by T'. From the equivariance property of T we deduce that $T'\rho f = \rho T' f$ for $f \in C(\mathbb{S}^{n-1})$ and $\rho \in \mathrm{SO}(n)$. The rest of the proof follows Dunkl [520], Theorem 8. The mapping $f \mapsto (T'f)(p)$ is a continuous linear functional on $C(\mathbb{S}^{n-1})$, hence by the Riesz representation theorem there is a signed measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ with $(T'f)(p) = \int_{\mathbb{S}^{n-1}} f \,\mathrm{d}\mu$. For $\vartheta \in \mathrm{SO}(n)$ this gives

$$(T'f)(\vartheta p) = (\vartheta^{-1} T' f)(p) = (T' \vartheta^{-1} f)(p) = \int_{\mathbb{S}^{n-1}} f(\vartheta v)\,\mathrm{d}\mu(v) = (f * \mu)(\vartheta p)$$

by (3.29), which completes the proof. □

Theorem 3.3.7 is not very satisfactory, for two reasons. First, the assumption of uniform continuity is rather strong. With only continuity, a similar result can be obtained using distributions instead of signed measures (Kiderlen [1075]). Second, it is not known for which zonal signed measures μ the convolution $\bar{h}_K * \mu$ always yields a support function. This does hold for positive zonal measures, as proved by Kiderlen [1075], using formula (3.31) and showing that each $R_t h_K$ yields a support function. Positivity, however, is not necessary. Suppose that μ has the property that $\bar{h}_K * \mu$ is a support function for $K \in \mathcal{K}^n$. Then $\mu + \eta$ also has this property if $\eta \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ is a *linear measure*, that is, of the form $\eta = \alpha \int_{(\cdot)} \langle p, y \rangle \,\mathrm{d}\sigma(y)$ with $\alpha \in \mathbb{R}$. In fact, we have $\bar{h}_K * (\mu + \eta) = \bar{h}_K * \mu + \beta \bar{h}_{\{s(K)\}}$ with a constant β. A zonal signed measure $\eta \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ is called a *weakly positive measure* if the sum of μ and a suitable linear measure is positive. Whether $\bar{h}_K * \mu$ is a support function for all K only in the case where μ is a weakly positive measure (as conjectured by F. E. Schuster) is open.

If $\bar{h}_{TK} = \bar{h}_K * \mu$ with a weakly positive measure μ, then the Minkowski endomorphism T has an additional property: it is *weakly monotonic*. This means that $s(K) = s(M) = o$ together with $K \subset M$ implies $TK \subset TM$, for all $K, M \in \mathcal{K}^n$. The following result is due to Kiderlen [1075].

Theorem 3.3.8 *The mapping $T : \mathcal{K}^n \to \mathcal{K}^n$ is a weakly monotonic Minkowski endomorphism if and only if $\bar{h}_{TK} = \bar{h}_K * \mu$ with a weakly positive zonal measure μ.*

A satisfactory description of all Minkowski endomorphisms is known in the two-dimensional case, which is simpler, due to the commutativity of the rotation group of $\mathbb{R}^2$. For a convenient formulation, let $\vartheta_\alpha \in \mathrm{SO}(2)$ for $\alpha \in [0, 2\pi)$ denote the rotation by the angle α. A signed measure μ on $\mathcal{B}([0, 2\pi))$ is called *weakly positive* if it is positive up to the addition of a linear measure, which is now one of the form $\int_{(\cdot)} (c_1 \cos\alpha + c_2 \sin\alpha)\,\mathrm{d}\alpha$. The following was proved by Schneider [1672].

Theorem 3.3.9 *A map $T : \mathcal{K}^2 \to \mathcal{K}^2$ is a Minkowski endomorphism if and only if there exists a weakly positive measure μ on $\mathcal{B}([0, 2\pi))$ such that*

$$h(TK, \cdot) = \int_0^{2\pi} h(\vartheta_\alpha[K - s(K)], \cdot)\, \mathrm{d}\mu(\alpha) + \langle s(K), \cdot \rangle \tag{3.32}$$

for all $K \in \mathcal{K}^2$. The measure μ is unique up to the addition of a linear measure.

In higher dimensions, it is possible to characterize special Minkowski endomorphisms by means of simple additional assumptions. Theorem 3.3.10 below collects some results obtained in Schneider [1670]. Their proof uses a combination of elementary facts from harmonic analysis for the rotation group with convexity arguments. It is a crucial fact that a Minkowski endomorphism is a *multiplier transformation*, in the following sense. As explained in the Appendix, we denote by π_m the orthogonal projection from $C(\mathbb{S}^{n-1})$ to the space $\mathscr{S}^m$ of spherical harmonics of degree m on $\mathbb{S}^{n-1}$. Then, for a Minkowski endomorphism T of $\mathcal{K}^n$, the following holds. There is a sequence $(\gamma_m)_{m\in\mathbb{N}_0}$ of real numbers, and for $n = 2$ an additional sequence $(\rho_m)_{m\in\mathbb{N}_0}$ of rotations $\rho_m \in \mathrm{SO}(2)$, such that

$$\pi_m \bar{h}_{TK} = \gamma_m \pi_m \bar{h}_K \qquad \text{for } K \in \mathcal{K}^n$$

if $n \geq 3$, and

$$\pi_m \bar{h}_{TK} = \gamma_m \rho_m \pi_m \bar{h}_K \qquad \text{for } K \in \mathcal{K}^2$$

if $n = 2$ ($m = 0, 1, 2, \dots$). The Minkowski endomorphism T is uniquely determined by the sequence $(\gamma_m)_{m\in\mathbb{N}_0}$, together with the sequence $(\rho_m)_{m\in\mathbb{N}_0}$ if $n = 2$. This can be used to obtain the following results which, without loss of generality, we formulate for translation equivariant Minkowski endomorphisms.

Theorem 3.3.10 *Let T be a translation equivariant Minkowski endomorphism of $\mathcal{K}^n$.*

(a) *T is uniquely determined by the image of one suitably chosen convex body, for example, a triangle with at least one irrational angle.*
(b) *If the image under T of some convex body of positive dimension is zero-dimensional, then $TK = s(K)$ for $K \in \mathcal{K}^n$.*
(c) *If the image under T of some convex body is a segment, then*

$$TK = \lambda[K - s(K)] + \mu[-K + s(K)] + s(K) \quad \text{for } K \in \mathcal{K}^n$$

with real numbers $\lambda, \mu \geq 0$, $\lambda + \mu > 0$.

(d) *Let T be surjective. If $n \geq 3$, then*

$$TK = \lambda[K - s(K)] + s(K) \qquad \text{for } K \in \mathcal{K}^n$$

with $\lambda \neq 0$. If $n = 2$, then

$$TK = \lambda\rho[K - s(K)] + s(K) \qquad \text{for } K \in \mathcal{K}^2$$

with $\lambda \neq 0$ and $\rho \in \mathrm{SO}(2)$.

Notes for Section 3.3

1. *Characterizations of the Steiner point*. For the early history of the Steiner point we refer to the notes for Section 5.4. The problem of characterizing the Steiner point by some of its properties was first posed by Grünbaum [847], p. 239, who asked whether Minkowski additivity and similarity equivariance are sufficient to characterize the Steiner point. Sallee [1610] constructed an example showing that one needs a continuity assumption. The example given above after the proof of Theorem 3.3.3 is taken from Schneider [1670]. By the properties of Minkowski additivity, rigid motion equivariance and continuity, the Steiner point was characterized by Shephard [1784] for $n = 2$ and by Schneider [1663] for $n \geq 3$. Before that, Meyer [1420] had proved a slightly weaker version of the characterization, by assuming uniform continuity. Two earlier attempts ([1648], [917]) contained errors. For dimensions 2 and 3, interesting elementary proofs were given by Hadwiger [920] and Berg [199].

   Positsel'skii [1546], whose proof we presented above in essence, assumed equivariance under improper motions also; that proper motions suffice was pointed out (for $n \geq 3$) by Saint Pierre [1604].
2. *Abstract Steiner points of polytopes*. If $P \in \mathcal{P}^n$ is a convex polytope, then its Steiner point can be represented by

$$s(P) = \sum_{i=1}^{f_0(P)} \omega_n^{-1} \mathcal{H}^{n-1}(N(P, v_i) \cap \mathbb{S}^{n-1}) v_i,$$

   where $v_1, \dots, v_{f_0(P)}$ are the vertices of P. This is a special case of (5.99). Motivated by this, Berg [199] defined

$$s_\psi(P) := \sum_{i=1}^{f_0(P)} \psi(N(P, v_i) \cap \mathbb{S}^{n-1}) v_i,$$

   where ψ is a real-valued simple valuation on spherical polytopes, that is,

$$\psi(Q_1 \cup Q_2) = \psi(Q_1) + \psi(Q_2)$$

   if Q_1, Q_2 are intersections of $\mathbb{S}^{n-1}$ with convex polyhedral cones having convex union and no common interior points. It is not difficult to show that the map s_ψ is Minkowski linear, and that it is translation equivariant if $\psi(\mathbb{S}^{n-1}) = 1$. Berg [199] calls a map $\varphi : \mathcal{P}^n \to \mathbb{R}^n$ an *abstract Steiner point* if it is Minkowski additive and equivariant under (proper and improper) similarities. The definition of s_ψ above yields an abstract Steiner point if ψ is invariant under rotations and reflections and satisfies $\psi(\mathbb{S}^{n-1}) = 1$. It is not known whether every abstract Steiner point is of this form, but Berg [199] proved this for $n = 2$ and $n = 3$. As a consequence, every abstract Steiner point on $\mathcal{P}^n$, for $n = 2$ and $n = 3$, that is bounded on bounded sets of polytopes, is the usual Steiner point.
3. *Rotation means*. Apparently, the first proof of Theorem 3.3.5 appeared (for $n = 3$) in Hadwiger [908], p. 27; see also Hadwiger [911], §4.5.3. The basic idea of the proof seems to be older. For instance, it is very similar to an idea in the usual construction of an invariant mean on a compact topological group; compare Pontryagin [1544].

   Interesting problems arise if one asks for quantitative improvements in Theorem 3.3.5: how many rotations are required to reach a prescribed degree of accuracy of the approximation? Results of this type are due to Bourgain, Lindenstrauss and Milman [320, 321]. We mention a result on random Minkowski symmetrizations. For $u \in \mathbb{S}^{n-1}$, let π_u denote the orthogonal reflection at the hyperplane through zero with normal vector u. For $K \in \mathcal{K}^n$, the convex body $\frac{1}{2}(K + \pi_u K)$ is said to be obtained from K by a *Minkowski symmetrization* (or Blaschke symmetrization; see Blaschke [241], §22, VII). This Minkowski symmetrization is said to be random if u is chosen randomly on $\mathbb{S}^{n-1}$ with rotation invariant distribution. Now Theorem 14 of Bourgain, Lindenstrauss and Milman [320] says the following. Let K be a centrally symmetric convex body in $\mathbb{R}^n$ and let $\varepsilon > 0$. If $n > n_0(\varepsilon)$ and if we perform

$N = cn \log n + c(\varepsilon)n$ independent random Minkowski symmetrizations on K, we obtain with probability $1 - \exp[-\tilde{c}(\varepsilon)n]$ a body $\tilde{K}$ that satisfies

$$(1-\varepsilon)w(K)B^n \subset \tilde{K} \subset (1+\varepsilon)w(K)B^n,$$

where $w(K)$ denotes the mean width of K and $c, c(\varepsilon), \tilde{c}(\varepsilon)$ are suitable constants. This investigation was continued by Klartag [1094].

Strong results on deterministic Minkowski symmetrizations were obtained by Klartag [1095, 1097]. We mention one result from [1097]. For $K \in \mathcal{K}^n$ and $0 < \varepsilon < 1/2$, there exist $cn \log 1/\varepsilon$ Minkowski symmetrizations that transform K into a body $\tilde{K}$ that satisfies

$$(1-\varepsilon)w(K)B^n \subset \tilde{K} \subset (1+\varepsilon)w(K)B^n,$$

where c is some numerical constant.

4. Theorem 3.3.5 can be used, in a similar way to other symmetrization procedures, to prove extremal properties of the ball; see, for example, Hadwiger [907] and Schneider [1655].
5. *Minkowski endomorphisms.* Minkowski endomorphisms were first introduced and studied (under the name of 'endomorphisms') in Schneider [1670]. The corresponding purely analytical question, of determining all rotation equivariant linear operators of $C(\mathbb{S}^{n-1})$ into itself, was solved before by Dunkl [520]. The Minkowski endomorphisms in dimension 2 were completely determined in Schneider [1672]. Further results in the style of Theorem 3.3.10 can be obtained if one considers only Minkowski endomorphisms preserving some functional, such as volume or surface area. For example, the following theorem was proved in Schneider [1670] (the quermassintegrals W_k are defined in Section 4.2 below).

Theorem If φ is a translation equivariant Minkowski endomorphism of $\mathcal{K}^n$ satisfying $W_k(\varphi(K)) = W_k(K)$ for some $k \in \{0, 1, \dots, n-2\}$ and all $K \in \mathcal{K}^n$, then

$$\varphi(K) = \lambda[K - s(K)] + s(K) \quad \text{for } K \in \mathcal{K}^n,$$

where $\lambda \in \{1, -1\}$.

The following result of Schneider [1673] requires no continuity assumption. Here V_n denotes the volume.

Theorem Let $n \geq 2$. If $\varphi : \mathcal{K}^n \to \mathcal{K}^n$ is a Minkowski additive map satisfying $V_n(\varphi(K)) = V_n(K)$ for $K \in \mathcal{K}^n$, then there exists a volume-preserving affine map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(K)$ is a translate of αK for each $K \in \mathcal{K}^n$.

Minkowski endomorphisms were further studied by Kiderlen [1074, 1075], some of whose results we have quoted above. He and later Schuster [1747] investigated a similar notion, where Minkowski addition is replaced by a different composition of convex bodies called Blaschke addition; see Subsection 8.2.2. For Minkowski endomorphisms, the following questions remain open for $n \geq 3$: is every Minkowski endomorphism weakly monotonic? If not, is every Minkowski endomorphism uniformly continuous?

6. *Bijective additive maps.* Under the assumption of bijectivity, additive mappings of systems of unbounded convex sets can show a very rigid behaviour. Recall that CC^n denotes the system of all nonempty closed convex sets in $\mathbb{R}^n$, and CC^n_o is the subsystem of sets containing the origin. Artstein-Avidan and Milman [91] proved that a bijective map $T : CC^n \to CC^n$ satisfying $T(K_1 + K_2) = TK_1 + TK_2$ for all $K_1, K_2 \in CC^n$ must be of the form $TK = gK$ with a linear transformation $g \in \mathrm{GL}(n)$. For dimensions $n \geq 2$, the same result holds with CC^n replaced by CC^n_o. Even the following stability version was proved by Artstein-Avidan and Milman [92].

Theorem Let $n \geq 2$, and let $T : CC^n_o \to CC^n_o$ be a bijective mapping such that, with suitable constants $0 < c_1 < c_2$,

$$c_1(TK_1 + TK_2) \subset T(K_1 + K_2) \subset c_2(TK_1 + TK_2)$$

for all $K_1, K_2 \in CC^n_o$. Then there exists $g \in \mathrm{GL}(n)$ such that

$$gK \subset TK \subset (c_2/c_1)^3 gK$$

for all $K \in CC^n_o$.

It is essential for the proofs that unbounded convex sets are admitted.

3.4 Approximation and addition

Our objective in this section is a treatment of approximation of convex bodies with a special view to Minkowski addition. First we describe an approximation procedure by smooth bodies that has particularly useful properties with regard to addition. Then we study briefly the possibility of approximating a convex body by Minkowski sums of bodies from a given class.

It is often useful to approximate a convex body by sufficiently smooth bodies. We can do this by applying to its support function a suitable regularization process (the usual convolution, with a minor modification to retain homogeneity).

Theorem 3.4.1 *Let $\varepsilon > 0$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a function of class C^∞ with support in $[\varepsilon/2, \varepsilon]$ and with*

$$\int_{\mathbb{R}^n} \varphi(|z|)\,\mathrm{d}z = 1.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is a support function, then the function $\widetilde{f}$ defined by

$$\widetilde{f}(x) := \int_{\mathbb{R}^n} f(x + |x|z)\,\varphi(|z|)\,\mathrm{d}z \quad \textit{for } x \in \mathbb{R}^n$$

is a support function of class C^∞ on $\mathbb{R}^n \setminus \{o\}$. The map $T : \mathcal{K}^n \to \mathcal{K}^n$ defined by $h_{TK} = \widetilde{h}_K$ is a Minkowski endomorphism; more precisely, it has the following properties.

(a) $T(K+L) = TK + TL$ *and* $T(\lambda K) = \lambda TK$ *for* $K, L \in \mathcal{K}^n$ *and* $\lambda \geq 0$*;*
(b) $T(gK) = gTK$ *for* $K \in \mathcal{K}^n$ *and every rigid motion* g *of* $\mathbb{R}^n$*, further* $T(-K) = -T(K)$*;*
(c) $\delta(K, TK) \leq R\varepsilon$ *for all convex bodies* $K \subset RB^n$ *(where* $R > 0$*);*
(d) $\delta(TK, TL) \leq (1+\varepsilon)\delta(K, L)$ *for* $K, L \in \mathcal{K}^n$*.*

Proof Let φ be as in the theorem and let f be a support function. It follows from the definition that $\widetilde{f}$ is positively homogeneous. Let $z \in \mathbb{R}^n$ and put

$$g_z(x) := f(x + |x|z) + f(x - |x|z) \quad \text{for } x \in \mathbb{R}^n.$$

For $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$ we have

$$\begin{aligned} g_z(x+y) &= f(x+y+|x+y|z) + f(x+y-|x+y|z) \\ &\leq f(x + \alpha|x+y|z) + f(y + (1-\alpha)|x+y|z) \\ &\quad + f(x - \alpha|x+y|z) + f(y - (1-\alpha)|x+y|z). \end{aligned}$$

Without loss of generality we may assume that x and y are linearly independent. Put

$$\alpha := \frac{|x|}{|x|+|y|}, \quad \beta := \frac{|x+y|}{|x|}, \quad \gamma := \frac{|x+y|}{|y|};$$

then $1 - \alpha\beta > 0$ and $1 - (1-\alpha)\gamma > 0$. From

$$\begin{aligned}2(x + \alpha|x+y|z) &= (1+\alpha\beta)(x + |x|z) + (1-\alpha\beta)(x - |x|z),\\ 2(x - \alpha|x+y|z) &= (1-\alpha\beta)(x + |x|z) + (1+\alpha\beta)(x - |x|z)\end{aligned}$$

we infer that

$$f(x + \alpha|x+y|z) + f(x - \alpha|x+y|z) \leq f(x + |x|z) + f(x - |x|z) = g_z(x).$$

Similarly we obtain

$$f(y + (1-\alpha)|x+y|z) + f(y - (1-\alpha)|x+y|z) \leq g_z(y).$$

Together with the inequality for $g_z(x+y)$ as obtained above this yields

$$g_z(x+y) \leq g_z(x) + g_z(y).$$

Since

$$\widetilde{f}(x) = \frac{1}{2}\int_{\mathbb{R}^n} g_z(x)\,\varphi(|z|)\,\mathrm{d}z$$

and $\varphi \geq 0$, the convexity of $\widetilde{f}$ follows. By Theorem 1.7.1, $\widetilde{f}$ is a support function.

For $u \in \mathbb{S}^{n-1}$ we have

$$\widetilde{f}(u) = \int_{\mathbb{R}^n} f(u+z)\,\varphi(|z|)\,\mathrm{d}z = \int_{\mathbb{R}^n} f(y)\,\varphi(|y-u|)\,\mathrm{d}y$$

and hence, for $x \in \mathbb{R}^n \setminus \{o\}$,

$$\widetilde{f}(x) = |x|\int_{\mathbb{R}^n} f(y)\,\varphi\left(\left|y - \frac{x}{|x|}\right|\right)\,\mathrm{d}y,$$

from which we see that $\widetilde{f}$ is of class C^∞ on $\mathbb{R}^n \setminus \{o\}$.

The map $T : \mathcal{K}^n \to \mathcal{K}^n$ defined by $h_{TK} = \widetilde{h_K}$ clearly has property (a) of the theorem. For $t \in \mathbb{R}^n$ and $u \in \mathbb{S}^{n-1}$ we have

$$h_{T\{t\}}(u) = \int_{\mathbb{R}^n} \langle u+z, t\rangle\,\varphi(|z|)\,\mathrm{d}z = \langle u, t\rangle = h_{\{t\}}(u),$$

hence $T\{t\} = \{t\}$. Together with (a) this yields $T(K+t) = TK + T\{t\} = TK + t$, thus T commutes with translations. Let ρ be a rotation of $\mathbb{R}^n$, proper or improper. Defining ρf by $(\rho f)(x) := f(\rho^{-1}x)$ for $x \in \mathbb{R}^n$, we obtain

$$\begin{aligned}(\rho\widetilde{f})(x) &= \int_{\mathbb{R}^n} f(\rho^{-1}x + |\rho^{-1}x|z)\,\varphi(|z|)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^n} f(\rho^{-1}(x + |x|\rho z))\,\varphi(|\rho z|)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^n} (\rho f)(x + |x|z)\,\varphi(|z|)\,\mathrm{d}z,\end{aligned}$$

hence $(\widetilde{\rho f}) = \rho \tilde{f}$. This shows that $T(\rho K) = \rho T K$ for $K \in \mathcal{K}^n$; in particular (since ρ was allowed to be improper) $T(-K) = -T(K)$. This completes the proof of part (b).

For $K \in \mathcal{K}^n$ with $K \subset RB^n$ and for $u \in \mathbb{S}^{n-1}$ we obtain

$$\begin{aligned}
|h_{TK}(u) - h_K(u)| &= \left| \int_{\mathbb{R}^n} h_K(u+z)\, \varphi(|z|)\, \mathrm{d}z - h_K(u) \right| \\
&\leq \int_{\mathbb{R}^n} |h_K(u+z) - h_K(u)|\, \varphi(|z|)\, \mathrm{d}z \\
&\leq \int_{\mathbb{R}^n} R|z|\, \varphi(|z|)\, \mathrm{d}z \leq R\varepsilon,
\end{aligned}$$

where Lemma 1.8.12 together with $\varphi(|z|) = 0$ for $|z| > \varepsilon$ was used. Since $u \in \mathbb{S}^{n-1}$ was arbitrary, this proves (c). In a similar way assertion (d) is obtained, again using Lemma 1.8.12. □

Although the bodies TK obtained in Theorem 3.4.1 have support functions of class C^∞, they are not necessarily smooth since they can have singular points. This can easily be remedied by defining $T'K := TK + \varepsilon B^n$ for $K \in \mathcal{K}^n$. Then $T'K$ is of class C^∞_+ and the map T' has properties that are easily derived from those of the map T.

If a convex body K is approximated by the use of Theorem 3.4.1, then the approximating bodies inherit many properties from K. For instance, TK has (at least) the same symmetries as K, and if K is of constant width w, then $TK - TK = T(K - K) = T(wB^n)$ is a ball by (b); hence TK is of constant width.

The approximation procedure of Theorem 3.4.1 can be refined further, as we now briefly describe. For this, we use spherical harmonics (see the Appendix for the necessary explanations and definitions). By π_m we denote the orthogonal projection from the space $C(\mathbb{S}^{n-1})$ of the real continuous functions on $\mathbb{S}^{n-1}$ onto the subspace of spherical harmonics of degree m. Let $K \in \mathcal{K}^n$ be a convex body of class C^∞_+. Since $\overline{h}_K$ (the restriction of the support function of K to $\mathbb{S}^{n-1}$) is of class C^∞, we have

$$\overline{h}_K = \sum_{m=0}^{\infty} \pi_m \overline{h}_K$$

with convergence in the maximum norm. If, for some integer $k \geq 2$, the partial sum $\sum_{m=0}^{k} \pi_m \overline{h}_K$ happens to be the restriction of a support function, we denote the convex body which it determines by $S_k K$. For given k, the map S_k is defined only on a subset of $\mathcal{K}^n$, but if K and L are in its domain, then it follows from the properties of spherical harmonics that

$$S_k(K + L) = S_k K + S_k L, \qquad S_k(gK) = g S_k K$$

for each rigid motion of $\mathbb{R}^n$. Now, for K of class C^∞_+, the body $S_k K$ does, in fact, exist for all sufficiently large k (see the Appendix).

Now let $K \in \mathcal{K}^n$ and $\varepsilon > 0$ be given. We can choose a map T according to Theorem 3.4.1 so that $\delta(K, TK) < \varepsilon$. Then the body $TK + \varepsilon B^n$ is of class C^∞_+, hence k can be chosen so that $T_k K := S_k(TK + \varepsilon B^n)$ exists and satisfies $\delta(TK + \varepsilon B^n, T_k K) < \varepsilon$.

Thus the body T_kK satisfies $\delta(K, T_kK) < 3\varepsilon$ and is of class C^∞_+ (for sufficiently large k). Finally, the support function of T_kK is, in fact, algebraic, since $\pi_m h$ (for any $h \in C(\mathbb{S}^{n-1})$) is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous polynomial on $\mathbb{R}^n$. Hence, $h(T_kK, x) = |x|P(x/|x|)$, where $P(y)$ is a polynomial in the Cartesian coordinates of y. The map T_k commutes with rigid motions, and if K and L are in its domain, then $T_k(K + L) = T_kK + T_kL$ for $k > 0$.

Our second topic in this section is the approximation of convex bodies by Minkowski sums of bodies from a given class. This vaguely formulated programme gives rise to many interesting questions. Here we treat only one special aspect.

We say that a convex body K is approximable by the class $\mathcal{A} \subset \mathcal{K}^n$ if K is the limit of a sequence of bodies of the form

$$\lambda_1 K_1 + \cdots + \lambda_r K_r \quad \text{with } K_i \in \mathcal{A},\ \lambda_i \geq 0\ (i = 1, \ldots, r),\ r \in \mathbb{N}.$$

The following theorem shows that for many convex bodies, namely the indecomposable ones, such an approximation is only possible in an essentially trivial way.

Theorem 3.4.2 *If the indecomposable body $K \in \mathcal{K}^n$ is approximable by the class $\mathcal{A} \subset \mathcal{K}^n$, then the closure of $\mathcal{A}$ contains a positive homothetic copy of K.*

To formulate a consequence, call a subset $\mathcal{A} \subset \mathcal{K}^n$ a *universal approximating class* if any convex body in $\mathcal{K}^n$ is approximable by $\mathcal{A}$. For $n = 2$, the set of all triangles and all line segments is such a universal approximating class, since every convex polygon is a sum of triangles and line segments. For $n \geq 3$, however, the set of indecomposable polytopes is dense in $\mathcal{K}^n$, by Theorem 3.2.18. Hence, for $n \geq 3$, the only closed, homothety invariant, universal approximating class in $\mathcal{K}^n$ is $\mathcal{K}^n$ itself.

Before the proof of the theorem, a few remarks are in order. The theorem was proved for the case of indecomposable polytopes by Shephard [1777]; this proof is reproduced in Grünbaum [848] (Section 15.2). Shephard's proof seems to be restricted to polytopes. In its general form, Theorem 3.4.2 is due to Berg [197], who observed that it is a consequence of D. Milman's converse to the Krein–Milman theorem (see, e.g., Phelps [1533], p. 9), applied to the set $\Upsilon(\mathcal{K}^n_{s,1})$ (see Section 3.2), which is a compact convex set in $C(\mathbb{S}^{n-1})$. Since we have not treated this theorem about convexity in topological vector spaces, we present here an elementary version of the proof, adapted to our particular situation.

Proof of Theorem 3.4.2 Let $K \in \mathcal{K}^n$ be an indecomposable body of positive dimension that is approximable by a class $\mathcal{A}$. We can assume that $\mathcal{A}$ is closed.

By assumption, there is a sequence $(K_i)_{i \in \mathbb{N}}$ converging to K with

$$K_i = \sum_{j=1}^{m_i} \lambda_{ij} K_{ij}, \quad K_{ij} \in \mathcal{A}, \quad \lambda_{ij} \geq 0. \tag{3.33}$$

For $L \in \mathcal{K}^n$ with $\dim L > 0$, we denote its normalized homothet by $\overline{L}$, that is, $\overline{L} = w(L)^{-1}[L - s(L)]$. Then $K_i \to K$ implies $\overline{K}_i \to \overline{K}$ for $i \to \infty$, and we have

$$\overline{K}_i = \sum_j \overline{\lambda}_{ij}\overline{K}_{ij} \quad \text{with } \overline{\lambda}_{ij} = \frac{\lambda_{ij} w(K_{ij})}{w(K_i)},$$

so that

$$\sum_j \overline{\lambda}_{ij} = 1.$$

Since indecomposability is invariant under homothety, we may, therefore, assume from the beginning that $\mathcal{A} \subset \mathcal{K}^n_{s,1}$, $K \in \mathcal{K}^n_{s,1}$ and that (3.33) holds with

$$\sum_j \lambda_{ij} = 1.$$

We assert now that $K \in \mathcal{A}$. Suppose this were false. Then for each $A \in \mathcal{A}$ we have $\delta(A, K) > 0$, hence the set

$$\mathcal{U}(A) := \left\{L \in \mathcal{K}^n : \delta(A, L) \leq \tfrac{1}{2}\delta(A, K)\right\}$$

has interior points. For $i \in \mathbb{N}$, let $\mu_i(\mathcal{U}(A))$ be the sum of those coefficients λ_{ij} for which $K_{ij} \in \mathcal{U}(A)$. We assert that

$$\limsup_{i\to\infty} \mu_i(\mathcal{U}(A)) > 0$$

for at least one $A \in \mathcal{A}$. Suppose this were false. The compact set $\mathcal{A}$ can be covered by finitely many sets $\mathcal{U}(A_1), \dots, \mathcal{U}(A_p)$. Then

$$\sum_{r=1}^{p} \mu_i(\mathcal{U}(A_r)) \geq \sum_{j=1}^{m_i} \lambda_{ij} = 1,$$

but on the other hand, $\mu_i(\mathcal{U}(A_r)) \to 0$ for $i \to \infty$, a contradiction. Hence, there is some $A_0 \in \mathcal{A}$ for which $\limsup \mu_i(\mathcal{U}(A_0)) > 0$, and taking a subsequence and changing the notation, we may assume that

$$\lim_{i\to\infty} \mu_i(\mathcal{U}(A_0)) > 0. \tag{3.34}$$

Now for $i \in \mathbb{N}$ we define

$$K_i' := \frac{\sum' \lambda_{ij} K_{ij}}{\sum' \lambda_{ij}}, \quad K_i'' := \frac{\sum'' \lambda_{ij} K_{ij}}{\sum'' \lambda_{ij}},$$

where the sums $\sum'$ extend over those j for which $K_{ij} \in \mathcal{U}(A_0)$ and the sums $\sum''$ extend over the remaining j (with the convention that $K_i' = \{o\}$ if $\sum' \lambda_{ij} = 0$, and similarly for K_i''). Then we have

$$K_i = \tau_i K_i' + (1 - \tau_i) K_i''$$

with

$$\tau_i = \sum{}' \lambda_{ij} = \mu_i(\mathcal{U}(A_0)). \tag{3.35}$$

Since K_i', K_i'' belong to the compact set $\mathcal{K}^n_{s,1} \cup \{\{o\}\}$ and τ_i belongs to the compact interval [0,1], we may assume, after selecting subsequences and changing the notation, that $K_i' \to K'$, $K_i'' \to K''$, with $K', K'' \in \mathcal{K}^n_{s,1} \cup \{\{o\}\}$ and $\tau_i \to \tau \in [0,1]$, and thus

$$K = \tau K' + (1 - \tau)K''.$$

From (3.34) and (3.35) we have $\tau > 0$, in particular $K_i' \neq \{o\}$ for large i. For these i, the body K_i' is a convex combination of elements from $\mathcal{U}(A_0)$, hence $K_i' \in \mathcal{U}(A_0)$ and thus $K' \in \mathcal{U}(A_0)$, which implies $K' \neq K$. Since $K', K'' \in \mathcal{K}^n_{s,1}$, the bodies K' and K'' are not homothetic to K, thus K is decomposable. This contradiction shows that $K \in \mathcal{A}$, which proves the theorem. □

Notes for Section 3.4

1. *Regularization and approximation*. A regularization process very similar to that of Theorem 3.4.1 was first proposed by Radon [1554], but it appears that his paper remained unnoticed for a long time. Radon considered (in a more general form) the transformation defined by

$$\widetilde{f}(x) := \int_{\mathbb{R}^n} [f(x - |x|z) + |x| f(z)]\, \varphi(|z|)\, \mathrm{d}z$$

   and showed that it yields a support function if applied to such a function. He used this to show that a convex body can be approximated by convex bodies with real-analytic support functions.

   The map T of Theorem 3.4.1 appears in essence in Schneider [1670], where it was considered for a different reason, and where the convexity of f was proved on the basis of a communication from W. Weil. As a special approximation procedure for convex bodies, the method of Theorem 3.4.1 was used by Weil [1936]. He investigated for which pairs of polytopes $P, Q \in \mathcal{P}^n_n$ with $P \subset Q$ it is possible to find a convex body K of class C^∞_+ with $P \subset K \subset Q$. He proved that this is possible if and only if each point of $\mathrm{bd}\, P \cap \mathrm{bd}\, Q$ is a vertex of P and an internal point of some facet of Q.

   Schneider [1700] extended the method, in the way sketched above, by using spherical harmonics. A regularization process close to that of Theorem 3.4.1 was independently proposed by Saint Pierre [1604].

   As mentioned, the approximation process of Theorem 3.4.1 automatically yields bodies of constant width if applied to such a body. This does not hold for other procedures, described in the literature, that involve approximation by bodies with very smooth boundaries or support functions; see Minkowski [1438], §2, Bonnesen and Fenchel [284], p. 36, Hammer [935] and Firey [604]. Special approximations for plane ovals of constant width were constructed by Tanno [1839] and Wegner [1930].

   The approximation theorem 3.4.1 uses convolution in $\mathbb{R}^n$. Grinberg and Zhang [775] (Theorem 5.4) describe a similar approximation process which uses convolution in the rotation group $\mathrm{SO}(n)$. They point out that in addition to preserving the property of constant width, this process produces zonoids if applied to zonoids. We remark that the process T of Theorem 3.4.1 has the same property, since TS is a zonoid if S is a segment.
2. A thorough study of the approximation of centrally symmetric bodies (mainly in the plane) by polynomial bodies was made by Faro Rivas [548].

3.5 Minkowski classes and additive generation

By a *Minkowski class* in $\mathbb{R}^n$ we understand a nonempty subset $\mathcal{M}$ of $\mathcal{K}^n$ that is closed in the Hausdorff metric and closed under Minkowski addition and dilatation. Given a subgroup G of the group of affine transformations of $\mathbb{R}^n$, a Minkowski class $\mathcal{M}$ is called *G-invariant* if $K \in \mathcal{M}$ implies $gK \in \mathcal{M}$ for all $g \in G$. For 'G-invariant' in the cases $G = \mathrm{SO}(n)$ or $G = \mathrm{G}_n$ we say *rotation invariant* and *motion invariant*, respectively, and *translation invariant* and *affine invariant* have the analogous meanings. The set of all singletons (one-pointed convex bodies) will be called the *trivial* Minkowski class.

Perhaps the best known motion invariant Minkowski classes, besides the trivial class and the class of balls, are provided by the centrally symmetric convex bodies and by the of bodies of constant width. They are, in a sense, complementary to each other: a convex body is centrally symmetric if and only if its support function is essentially (that is, up to a linear function) an even function, and a body is of constant width if and only if its support function is essentially (meaning, here, up to the support function of a ball) an odd function. If S is a given centrally symmetric convex body, then the set of all convex bodies of constant S-width (that is, the set of all $K \in \mathcal{K}^n$ with $K - K = \lambda S$ for some $\lambda \geq 0$) is an example of a translation invariant Minkowski class.

Before studying more general Minkowski classes, we have a brief look at special convex bodies which, together with their dilatates, form Minkowski classes of particular geometric appeal. There is a way of looking at bodies of constant width that readily suggests a generalization. A body of constant width can be freely turned inside a suitable cube while always touching the facets of the cube. More generally, let $P \subset \mathbb{R}^n$ be a polyhedral set, that is, an n-dimensional intersection of finitely many closed halfspaces. A convex body $K \in \mathcal{K}^n_n$ is said to be a *rotor* of P if for every rotation ρ of $\mathbb{R}^n$ there exists a translation vector $t \in \mathbb{R}^n$ such that $\rho K + t$ is contained in P and touches each facet of P. Clearly, the set of all rotors of a given polyhedral set, together with their dilatates, is a motion invariant Minkowski class, if it is not empty. By Theorem 3.3.5, such a class contains a ball with positive radius. Thus, P can have a rotor only if it has an inscribed ball touching all its facets. If the exterior normal vectors of the facets of P are linearly independent, then obviously every convex body is a rotor of P. Leaving this trivial case aside, we ask for the polyhedral sets admitting non-spherical rotors. A complete classification is possible. Of course, if K is a rotor of the polyhedral set

$$P = \bigcap_{i \in I} H^-(P, u_i),$$

where the u_i for $i \in I$ (a finite index set) are the exterior normal vectors of the facets of P, then K is also a rotor of

$$P' := \bigcap_{i \in I'} H^-(P, u_i)$$

whenever I' is a subset of I. Such a polyhedral set P' is said to be *derived from* P. The following theorem (Schneider [1664]), which we quote without proof, lists all the non-trivial pairs of polyhedral sets and corresponding rotors. In the formulation of this theorem, support functions are restricted to $\mathbb{S}^{n-1}$.

Theorem 3.5.1 *Let $P \subset \mathbb{R}^n$ be a polyhedral set whose facets have linearly dependent normal vectors. Suppose that K is a non-spherical rotor of P. Then one of the following assertions holds.*

(a) *P is derived from a parallelepiped with equal distances between parallel facets; K is a body of constant width.*
(b) *n=2; P is derived from a regular k-gon with $k \in \{3, 4, \dots\}$; the Fourier coefficients a_i, b_i of the support function of K are zero except for $i = 0$ and $i \equiv \pm 1$ $(\mathrm{mod}\ k)$.*
(c) *$n = 3$; P is a regular tetrahedron; the support function of K is a sum of spherical harmonics of orders* $0, 1, 2, 5$.
(d) *$n = 3$; P is derived from a regular octahedron, but is not a tetrahedron; the support function of K is a sum of spherical harmonics of orders* $0, 1, 5$.
(e) *$n = 3$; P is congruent to the cone*

$$\{x \in \mathbb{R}^3 : \langle x, v_i \rangle \leq 0 \ \textit{for}\ i = 1, 2, 3, 4\},$$

where

$$v_1 = \frac{1}{\sqrt{7}}\left(\sqrt{6}, 0, 1\right), \quad v_2 = \frac{1}{\sqrt{7}}\left(-\sqrt{6}, 0, 1\right),$$

$$v_3 = \frac{1}{\sqrt{7}}\left(0, \sqrt{6}, 1\right), \quad v_4 = \frac{1}{\sqrt{7}}\left(0, -\sqrt{6}, 1\right)$$

with respect to some orthonormal basis; the support function of K is a sum of spherical harmonics of orders $0, 1, 4$.
(f) *$n \geq 4$; P is a regular simplex; the support function of K is a sum of spherical harmonics of orders* $0, 1, 2$.

Conversely, each of the cases listed here does really occur, so that no further reduction is possible. We wish to point out that cases (c) to (f), unlike (a) and (b), describe rotors that depend only on finitely many real parameters.

Minkowski classes enter the scene naturally if one asks for the extent to which it is possible to generate convex bodies by adding simpler bodies, taken from a small supply. If $\mathcal{A} \subset \mathcal{K}^n$ is any set of convex bodies and G is a subgroup of the affine transformation group of $\mathbb{R}^n$, let us consider the smallest Minkowski class (smallest G-invariant Minkowski class) containing $\mathcal{A}$; we call this the Minkowski class (G-invariant Minkowski class) *generated by* $\mathcal{A}$. Thus, the G-invariant Minkowski class generated by $\mathcal{A}$ consists of all convex bodies of the form

$$\lambda_1 g_1 K_1 + \cdots + \lambda_m g_m K_m \quad \text{with } m \in \mathbb{N},\ \lambda_i \geq 0,\ g_i \in G,\ K_i \in \mathcal{A}\,(i = 1, \dots, m)$$

and their limits.

Theorem 3.4.2 implies, for $n \geq 3$, that the Minkowski class generated by a homothety invariant subset $\mathcal{A} \subset \mathcal{K}^n$ is equal to $\mathcal{K}^n$ only in the trivial case where $\mathcal{A}$ is dense in $\mathcal{K}^n$. Therefore, Minkowski classes generated by essentially smaller sets will always have particular properties.

The perhaps disappointing picture, that one cannot generate $\mathcal{K}^n$ (for $n \geq 3$) by additions and limits from an essentially smaller set, changes if also differences (of support functions) are allowed. If $\mathcal{M}$ is a Minkowski class, we call a convex body $K \in \mathcal{K}^n$ a *generalized $\mathcal{M}$-body* if there exist bodies $M_1, M_2 \in \mathcal{M}$ with $K + M_1 = M_2$. We shall see that the set of generalized $\mathcal{M}$-bodies is considerably (and usefully) larger than $\mathcal{M}$.

With good reason, the simplest non-trivial Minkowski class is that generated by the segments; it is also the smallest non-trivial affine-invariant Minkowski class. Its elements are called *zonoids*. Thus, a zonoid in $\mathbb{R}^n$ is a convex body that can be approximated by finite sums of line segments. From the viewpoint of Minkowski addition, the zonoids are generated in a particularly simple way and deserve, therefore, special attention. It turns out that zonoids appear in several different contexts. We collect now some basic facts about zonoids.

A Minkowski sum of finitely many segments is called a *zonotope*. Thus, a zonotope is a polytope of the form $Z = S_1 + \cdots + S_k$ with $k \in \mathbb{N}$ and segments $S_1, \ldots, S_k$. After applying suitable translations to the S_i (which results in one translation applied to Z) we may assume that each S_i has its centre at the origin; hence Z is centrally symmetric. Let $u \in \mathbb{S}^{n-1}$. By Theorem 1.7.5,

$$F(Z, u) = F(S_1, u) + \cdots + F(S_k, u),$$

hence each face of a zonotope is itself a zonotope and, in particular, is centrally symmetric. This fact characterizes zonotopes.

Theorem 3.5.2 *A convex polytope is a zonotope if and only if all its two-dimensional faces are centrally symmetric.*

Proof Only the sufficiency has to be proved. Let $P \in \mathcal{P}^n$ be a polytope of dimension at least two all of whose 2-faces are centrally symmetric. Let S be an edge of P, and let $u \in \mathbb{S}^{n-1}$ be orthogonal to S. By π we denote the orthogonal projection from P onto a hyperplane orthogonal to S. Let p be a vertex of $\pi F(P, u)$. In the polytope πP we can find an edge path $(E_1, \ldots, E_m)$ connecting the vertex πS with p, such that πS is an endpoint of E_1 and p is an endpoint of E_m. The set $\pi^{-1}(E_1)$ is a 2-face of P and hence centrally symmetric and has, therefore, an edge $S' \neq S$ that is a translate of S. The projection of S' under π is the other endpoint of E_1. Applying the same argument to E_2, and so on, we deduce that $\pi^{-1}(p)$ is a translate of S. Thus the face $F(P, u)$ contains a translate of S. Since $u \perp S$ was arbitrary, it follows from Theorem 3.2.11 that S is a summand of P, hence $P = S + P'$ with a polytope P'. Since $F(P, u) = F(S, u) + F(P', u)$ for $u \in \mathbb{S}^{n-1}$, it is clear that each 2-face of P' is centrally symmetric and that P' has no edge parallel to S. Repeating the argument, we conclude after finitely many steps that P is a sum of segments. $\square$

If $P = S + P'$ with a segment S, then each support set $F(P, u)$ with $u \in \mathbb{S}^{n-1}$ orthogonal to S contains a translate of S. The union of all these sets $F(P, u)$ makes up a ‘zone’ in the boundary of P; this explains the name ‘zonotope’.

If $Z = S_1 + \cdots + S_k$ with $S_i = \operatorname{conv}\{\alpha_i v_i, -\alpha_i v_i\}$ where $v_i \in \mathbb{S}^{n-1}$ and $\alpha_i > 0$ for $i = 1, \ldots, k$, then the support function of the zonotope Z is given by

$$h(Z, \cdot) = \sum_{i=1}^{k} \alpha_i |\langle \cdot, v_i \rangle|. \tag{3.36}$$

Conversely, a convex body Z with such a support function is a zonotope with centre at the origin.

By definition, a *zonoid* in $\mathbb{R}^n$ is a convex body that can be approximated, in the Hausdorff metric, by a sequence of zonotopes. Thus, each zonoid has a centre of symmetry. The representation (3.36) is generalized as follows. By a signed measure (a measure) on $\mathbb{S}^{n-1}$ we understand a real-valued σ-additive (and nonnegative) function on the σ-algebra $\mathcal{B}(\mathbb{S}^{n-1})$ of Borel subsets of $\mathbb{S}^{n-1}$. A signed measure on $\mathbb{S}^{n-1}$ and a function on $\mathbb{S}^{n-1}$ are called *even* if they are invariant under reflection in the origin.

Theorem 3.5.3 *A convex body $K \in \mathcal{K}^n$ is a zonoid with centre at o if and only if its support function can be represented in the form*

$$h(K, x) = \int_{\mathbb{S}^{n-1}} |\langle x, v \rangle| \, \mathrm{d}\rho(v) \quad \textit{for } x \in \mathbb{R}^n \tag{3.37}$$

with some even measure ρ on $\mathbb{S}^{n-1}$.

Proof Suppose that (3.37) holds. For $k \in \mathbb{N}$ we decompose $\mathbb{S}^{n-1}$ into finitely many nonempty Borel sets $\Delta_1^k, \ldots, \Delta_{m(k)}^k$ of diameter less than $1/k$ and choose $v_i^k \in \Delta_i^k$. Then

$$\lim_{k\to\infty} \sum_{i=1}^{m(k)} |\langle x, v_i^k \rangle| \rho(\Delta_i^k) = \int_{\mathbb{S}^{n-1}} |\langle x, v \rangle| \, \mathrm{d}\rho(v),$$

uniformly for $x \in \mathbb{S}^{n-1}$. Hence, the zonotopes Z_k defined by

$$Z_k := \sum_{i=1}^{m(k)} \rho(\Delta_i^k) \operatorname{conv}\{v_i^k, -v_i^k\}$$

satisfy $h(Z_k, \cdot) \to h(K, \cdot)$ and thus $Z_k \to K$ for $k \to \infty$. By definition, K is a zonoid.

For the converse, observe that (3.36) can also be written in the form (3.37), with an even measure ρ concentrated in finitely many points. Hence, we may assume that

$$h(Z_k, \cdot) = \int_{\mathbb{S}^{n-1}} |\langle \cdot, v \rangle| \, \mathrm{d}\rho_k(v) \tag{3.38}$$

with an even measure ρ_k on $\mathbb{S}^{n-1}$ ($k \in \mathbb{N}$) and that $Z_k \to K$ for $k \to \infty$. Integrating (3.38) over $\mathbb{S}^{n-1}$ and using Fubini's theorem, we obtain

$$\int_{\mathbb{S}^{n-1}} h(Z_k, u)\,\mathrm{d}u = c(n)\rho_k(\mathbb{S}^{n-1})$$

with a constant $c(n)$ depending only on n. Since the left-hand side converges for $k \to \infty$, the sequence $(\rho_k(\mathbb{S}^{n-1}))_{k\in\mathbb{N}}$ is bounded. In the space of signed measures on $\mathbb{S}^{n-1}$ with the total variation norm, which can be identified with the dual space of $C(\mathbb{S}^{n-1})$, bounded sets are relatively weak*-compact. Hence, some subsequence $(\rho_{k_i})_{i\in\mathbb{N}}$ is weak*-convergent to a signed measure ρ, necessarily an even measure, and we get

$$h(K, x) = \lim_{i\to\infty} h(Z_{k_i}, x) = \lim_{i\to\infty} \int_{\mathbb{S}^{n-1}} |\langle x, v\rangle|\,\mathrm{d}\rho_{k_i}(v) = \int_{\mathbb{S}^{n-1}} |\langle x, v\rangle|\,\mathrm{d}\rho(v)$$

for each $x \in \mathbb{R}^n$. □

If the support function of a zonoid K with centre at the origin is represented in the form (3.37) with an even measure ρ, we say that ρ is the *generating measure* of K. The next theorem shows that in fact this generating measure is unique. It also shows that a representation of the form (3.37) with a signed measure ρ is always possible if K is centrally symmetric with respect to o and has a support function that is sufficiently often differentiable.

Theorem 3.5.4 *If ρ is an even signed measure on $\mathbb{S}^{n-1}$ with*

$$\int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\,\mathrm{d}\rho(v) = 0 \quad \textit{for } u \in \mathbb{S}^{n-1}, \tag{3.39}$$

then $\rho = 0$.

If G is an even real function on $\mathbb{S}^{n-1}$ of differentiability class C^k, where $k \geq n+2$ is even, then there exists an even continuous function g on $\mathbb{S}^{n-1}$ such that

$$G(u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, g(v)\,\mathrm{d}v \tag{3.40}$$

for $u \in \mathbb{S}^{n-1}$.

Proof We use spherical harmonics and refer to the Appendix for some of the details. In the following, all integrations without specified domain are over $\mathbb{S}^{n-1}$. Let Y_m be a spherical harmonic of order m on $\mathbb{S}^{n-1}$. The Funk–Hecke theorem yields

$$\int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| Y_m(v)\,\mathrm{d}v = \lambda_m Y_m(u) \tag{3.41}$$

with

$$\begin{aligned}\lambda_m &= \frac{(-1)^m \pi^{(n-1)/2}}{2^{m-1}\Gamma(m+(n-1)/2)} \int_{-1}^{1} |t| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (1-t^2)^{m+(n-3)/2}\,\mathrm{d}t \\ &= \frac{(-1)^{(m-2)/2}\pi^{(n-1)/2}\Gamma(m-1)}{2^{m-2}\Gamma(m/2)\Gamma((m+n+1)/2)} \quad \text{for even } m\end{aligned} \tag{3.42}$$

and $\lambda_m = 0$ for odd m. If now (3.39) holds, we have

$$0 = \int\int |\langle u, v\rangle| \, \mathrm{d}\rho(v) Y_m(u) \, \mathrm{d}u = \lambda_m \int Y_m(v) \, \mathrm{d}\rho(v)$$

and hence

$$\int Y_m(v) \, \mathrm{d}\rho(v) = 0.$$

For even m this holds since $\lambda_m \neq 0$, and for odd m it holds since Y_m is odd and ρ is even. From the completeness of the system of spherical harmonics it now follows that $\rho = 0$.

Now let G satisfy the assumptions of the theorem. To solve the integral equation (3.40), we develop G into a series of spherical harmonics:

$$G \sim \sum_{m=0}^{\infty} Y_m \quad \text{with } Y_m := \pi_m G.$$

If the series

$$g(v) := \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \lambda_m^{-1} Y_m(v) \tag{3.43}$$

converges uniformly on $\mathbb{S}^{n-1}$, then the continuous function g solves (3.40), by (3.41). For the uniform convergence of (3.43), it is sufficient to prove that

$$\|\lambda_m^{-1} Y_m\| = \mathrm{O}\left(m^{-2}\right). \tag{3.44}$$

Now from (3.42) we get, using Stirling's formula, that

$$|\lambda_m^{-1}| = \mathrm{O}\left(m^{(n+2)/2}\right) \tag{3.45}$$

for even $m \to \infty$. By Appendix relation (A.12), to which we refer for the notation,

$$\|Y_m\| \leq \left[\frac{N(n,m)}{\omega_n}\right]^{1/2} \|Y_m\|_2. \tag{3.46}$$

From Appendix equation (A.1),

$$\left[\frac{N(n,m)}{\omega_n}\right]^{1/2} = \mathrm{O}\left(m^{(n-2)/2}\right). \tag{3.47}$$

It remains to estimate $\|Y_m\|_2$. Let $(Y_{mj})_{j=1}^{N(n,m)}$ be an orthonormal basis of $\mathscr{S}^m$ ($m \in \mathbb{N}_0$), and put $a_{mj} := (G, Y_{mj})$, so that

$$Y_m = \pi_m G = \sum_{j=1}^{N(n,m)} a_{mj} Y_{mj},$$

$$\|Y_m\|_2^2 = (G, Y_m) = \sum_{j=1}^{N(n,m)} a_{mj}^2 =: a_m^2.$$

Using $\Delta_S Y_m = -m(m+n-2)Y_m$ and applying Green's formula $(f, \Delta_S g) = (\Delta_S f, g)$ on the sphere $k/2$ times, we get

$$a_m^2 = (G, Y_m) = \left[\frac{-1}{m(m+n-2)}\right]^{k/2} (\Delta_S^{k/2} G, Y_m)$$

for $m > 0$. Writing

$$b_{mj} := (\Delta_S^{k/2} G, Y_{mj}), \qquad b_m^2 := \sum_{j=1}^{N(n,m)} b_{mj}^2 = \left\|\pi_m \Delta_S^{k/2} G\right\|_2^2,$$

we deduce, using the Cauchy–Schwarz inequality, that

$$|a_m| \leq m^{-k} |b_m|.$$

By the Parseval relation,

$$\sum_{m=0}^{\infty} b_m^2 = \left\|\Delta_S^{k/2} G\right\|_2^2 < \infty,$$

hence $|a_m| = \mathrm{o}\left(m^{-k}\right)$. This, together with (3.45), (3.46) and (3.47) yields (3.44), which completes the proof. □

Remark 3.5.5 For $n = 2$, the solution of the integral equation (3.40) can easily be given explicitly. It is convenient to write the argument of G as an angle. If $G : [0, 2\pi] \to \mathbb{R}$ is a periodic function of class C^2, then

$$G(\varphi) = \frac{1}{4} \int_0^{2\pi} |\cos(\varphi - \psi)| \left[G\left(\psi - \frac{\pi}{2}\right) + G''\left(\psi - \frac{\pi}{2}\right)\right] \mathrm{d}\psi,$$

as one may check by partial integration.

A convex body $K \in \mathcal{K}^n$ whose support function can be represented in the form

$$h(K, u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}\rho(v)$$

with an even signed measure ρ, and any of its translates, is called a *generalized zonoid*. Since the signed measure ρ can be written as the difference of two measures (Hahn–Jordan decomposition), we see that there exist two zonoids M_1, M_2 such that $K + M_1 = M_2$. Conversely, if this holds, then K is a generalized zonoid. Thus, if $\mathcal{Z}$ denotes the Minkowski class of zonoids, then the generalized zonoids are precisely the generalized $\mathcal{Z}$-bodies.

The following lemma shows that each support set of a generalized zonoid is itself a generalized zonoid. For $e \in \mathbb{S}^{n-1}$ we write

$$\Omega_e := \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle > 0\},$$
$$\omega_e := \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle = 0\},$$

and for $K \in \mathcal{K}^n$ we abbreviate the support set $F(K, e)$ by K_e.

Lemma 3.5.6 *If the support function of the convex body $K \in \mathcal{K}^n$ is given by*

$$h(K, u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \,\mathrm{d}\rho(v) \quad \text{for } u \in \mathbb{R}^n$$

with an even signed measure ρ, then, for $e \in \mathbb{S}^{n-1}$,

$$h(K_e, u) = \int_{\omega_e} |\langle u, v\rangle| \,\mathrm{d}\rho(v) + \langle v_e, u\rangle \quad \text{for } u \in \mathbb{R}^n,$$

where

$$v_e = 2\int_{\Omega_e} v \,\mathrm{d}\rho(v).$$

Proof Theorem 1.7.2 tells us that

$$h(K_e, u) = h'_K(e; u) = \lim_{\lambda \downarrow 0} \lambda^{-1}[h(K, e + \lambda u) - h(K, e)].$$

The assertion of the lemma is true for $u = \pm e$, because $h(K_e, \pm e) = \pm h(K, e)$; hence we may assume that u and e are linearly independent.

Put

$$\begin{aligned} A &:= \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle > 0,\ \langle e + \lambda u, v\rangle > 0\},\\ B &:= \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle \le 0,\ \langle e + \lambda u, v\rangle > 0\},\\ C &:= \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle > 0,\ \langle e + \lambda u, v\rangle \le 0\}. \end{aligned}$$

Then $\Omega_{e+\lambda u} = A \cup B$ and $\Omega_e = A \cup C$ and we obtain

$$\begin{aligned} h(K_e, u) &= \lim \lambda^{-1}\left[\int_{\mathbb{S}^{n-1}} |\langle e + \lambda u, v\rangle| \,\mathrm{d}\rho(v) - \int_{\mathbb{S}^{n-1}} |\langle e, v\rangle| \,\mathrm{d}\rho(v)\right]\\ &= 2\lim \lambda^{-1}\left[\int_{A\cup B} \langle e + \lambda u, v\rangle \,\mathrm{d}\rho(v) - \int_{A\cup C} \langle e, v\rangle \,\mathrm{d}\rho(v)\right]\\ &= 2\lim \lambda^{-1}\left[\int_B \langle e, v\rangle \,\mathrm{d}\rho(v) - \int_C \langle e, v\rangle \,\mathrm{d}\rho(v)\right] + 2\lim \int_{A\cup B} \langle u, v\rangle \,\mathrm{d}\rho(v), \end{aligned}$$

where, as also in the following, all limits refer to $\lambda \downarrow 0$. For $v \in B$ we have $|\langle e, v\rangle| \le c\lambda$ with a constant c independent of λ. Writing

$$B' := \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle < 0,\ \langle e + \lambda u, v\rangle > 0\},$$

we obtain

$$\left|\lambda^{-1}\int_B \langle e, v\rangle \,\mathrm{d}\rho(v)\right| = \left|\lambda^{-1}\int_{B'} \langle e, v\rangle \,\mathrm{d}\rho(v)\right| \le c\,|\rho|\,(B'),$$

where $|\rho|$ denotes the total variation measure of ρ. Since (in the set-theoretic sense) $\lim B' = \emptyset$, we have

$$\lim |\rho|(B') = 0,$$

hence

$$\lim \lambda^{-1} \int_B \langle e, v\rangle \, \mathrm{d}\rho(v) = 0.$$

From $\lim C = \emptyset$ we similarly find

$$\lim \lambda^{-1} \int_C \langle e, v\rangle \, \mathrm{d}\rho(v) = 0.$$

Further, $\lim A = \Omega_e$ and

$$\lim B = D := \{v \in \mathbb{S}^{n-1} : \langle e, v\rangle = 0,\ \langle u, v\rangle > 0\}.$$

This yields

$$\begin{aligned} h(K_e, u) &= 2 \int_{\Omega_e} \langle u, v\rangle \, \mathrm{d}\rho(v) + 2 \int_D \langle u, v\rangle \, \mathrm{d}\rho(v) \\ &= 2 \left\langle u, \int_{\Omega_e} v \, \mathrm{d}\rho(v) \right\rangle + \int_{\omega_e} |\langle u, v\rangle| \, \mathrm{d}\rho(v), \end{aligned}$$

which completes the proof of the lemma. □

We collect some consequences of these and former observations.

Corollary 3.5.7 *Every support set of a generalized zonoid is itself a generalized zonoid and is, in particular, centrally symmetric. Every support set of a zonoid is a summand of the zonoid. A polytope is a generalized zonoid only if it is a zonotope.*

For $n = 2$, the zonoids coincide with the centrally symmetric convex bodies. In the set of centrally symmetric convex bodies in $\mathbb{R}^n$ ($n \geq 3$), the class of zonoids is closed and nowhere dense, and the class of generalized zonoids is dense but not closed.

Proof If K is a generalized zonoid, then

$$h(K, u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}\rho(v), \quad u \in \mathbb{S}^{n-1},$$

with an even signed measure ρ. For $e \in \mathbb{S}^{n-1}$, Lemma 3.5.6 shows that

$$h(K_e - v_e, u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}\rho_e(v),$$

where ρ_e is the restriction of ρ to ω_e; hence K_e is a generalized zonoid. If K is a zonoid and hence ρ is nonnegative, then

$$h(K_e - v_e, u) + \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}(\rho - \rho_e)(v) = h(K, u).$$

Since $\rho - \rho_e \geq 0$, the integral defines a support function, hence K_e is a summand of K. If the polytope P is a generalized zonoid, then each of its faces is centrally symmetric, hence P is a zonotope, by Theorem 3.5.2.

That the class of generalized zonoids is dense in the class of centrally symmetric convex bodies follows from Theorem 3.5.4, since the set of convex bodies with support function of class C^∞ is dense, by Theorem 3.4.1. A convex body with a non-symmetric support set cannot be a generalized zonoid; hence for $n \geq 3$ the set of generalized zonoids is not closed, and the set of zonoids, which is closed by definition, is nowhere dense in the set of centrally symmetric convex bodies. Every centrally symmetric polygon is a sum of segments and hence a zonotope. □

The statement that the zonoids are nowhere dense in the space of centrally symmetric convex bodies if $n \geq 3$ can be strengthened in a quantitative way by estimating how far away some centrally symmetric bodies are from the nearest zonoid. We show this (following Schneider [1724]) for the crosspolytope $Q = \operatorname{conv}\{\pm e_1, \dots, \pm e_n\}$, where $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

Theorem 3.5.8 *Let* $0 < \lambda < 2^{-n} n\gamma(n)$ *with*

$$\gamma(n) = 2\binom{n-1}{\left[\frac{n-1}{2}\right]}.$$

Then there exists no zonoid Z *with* $Q \subset Z \subset \lambda Q$.

Proof We assume that λ is as in the theorem, but there exists a zonoid Z with $Q \subset Z \subset \lambda Q$. Since $-Z$ satisfies the same inclusions, so does $\frac{1}{2}(Z - Z)$, hence we may assume that Z has centre o. Then the support function of Z has a representation

$$h(Z, x) = \int_{\mathbb{S}^{n-1}} |\langle u, x\rangle| \,\mathrm{d}\rho(u) \quad \text{for } x \in \mathbb{R}^n$$

with an even finite measure ρ on $\mathbb{S}^{n-1}$. This gives

$$\begin{aligned} &\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} h(Z, \varepsilon_1 e_1 + \dots + \varepsilon_n e_n) \\ &= \int_{\mathbb{S}^{n-1}} \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} |\varepsilon_1\langle u, e_1\rangle + \dots + \varepsilon_n\langle u, e_n\rangle| \,\mathrm{d}\rho(u). \end{aligned}$$

We assert that

$$F(a_1, \dots, a_n) := \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} |\varepsilon_1 a_1 + \dots + \varepsilon_n a_n| \geq \gamma(n) \sum_{i=1}^n |a_i| \tag{3.48}$$

for $a_1, \dots, a_n \in \mathbb{R}$. For the proof, it suffices, for reasons of symmetry and homogeneity, to restrict $(a_1, \dots, a_n)$ to the simplex

$$\Delta := \left\{(a_1, \dots, a_n) \in \mathbb{R}^n : a_i \geq 0,\ \sum a_i = 1\right\}.$$

Since the function F is convex and the restriction $F|\Delta$ is invariant under the affine symmetry group of the simplex Δ, the function $F|\Delta$ attains its minimum at a nonempty compact convex set containing the centroid of Δ. This gives

$$F(a_1,\dots,a_n) \geq F\left(\frac{1}{n},\dots,\frac{1}{n}\right) = \frac{1}{n} \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} |\varepsilon_1 + \cdots + \varepsilon_n|$$
$$= \frac{1}{n}\sum_{j=0}^{n} \binom{n}{j} |n-2j| = \gamma(n),$$

where the last equation follows by induction. This proves (3.48).

From (3.48) and $Q \subset Z$, we conclude that

$$\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} h(Z, \varepsilon_1 e_1 + \cdots + \varepsilon_n e_n)$$
$$\geq \gamma(n) \int_{\mathbb{S}^{n-1}} \sum_{j=1}^{n} |\langle u, e_j\rangle| \,\mathrm{d}\rho(u) = \gamma(n) \sum_{j=1}^{n} h(Z, e_j) \geq \gamma(n) \sum_{j=1}^{n} h(Q, e_j)$$
$$= n\gamma(n).$$

On the other hand, the inclusion $Z \subset \lambda Q$ implies

$$\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} h(Z, \varepsilon_1 e_1 + \cdots + \varepsilon_n e_n) \leq \lambda \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} h(Q, \varepsilon_1 e_1 + \cdots + \varepsilon_n e_n) = 2^n \lambda.$$

Both estimates together give $2^n \lambda \geq n\gamma(n)$, a contradiction. □

Remark 3.5.9 We note that the number $2^{-n} n\gamma(n)$ appearing in Theorem 3.5.8 is equal to $3/2$ for $n = 3$ and tends to infinity for $n \to \infty$.

Now we turn to extensions of the observation that the set of generalized zonoids is dense in the space of centrally symmetric convex bodies. We want to obtain a similar result for general (not necessarily symmetric) convex bodies. A zonoid is a limit of Minkowski sums of the form

$$\lambda_1 \vartheta_1 S + \cdots + \lambda_k \vartheta_k S \quad \text{with } k \in \mathbb{N},\ \lambda_i \geq 0,\ \vartheta_i \in \mathrm{SO}(n)\ (i = 1,\dots,k)$$

and thus is additively generated by a single convex body, a segment S. A generalized zonoid is a convex body whose support function is the difference of the support functions of two zonoids. We want to investigate whether, by replacing the segment S by a non-symmetric convex body, we can in a similar way achieve a dense subset of the set of all convex bodies.

For this, we denote by $\mathcal{M}_A$ the rotation invariant Minkowski class generated by the convex body $A \in \mathcal{K}^n$. By definition, the convex body K is a generalized $\mathcal{M}_A$-body if there exist convex bodies $M_1, M_2 \in \mathcal{M}_A$ with $K + M_1 = M_2$. Thus, $M \in \mathcal{M}_A$ if and only if M can be approximated by bodies of the form

$$\lambda_1 \vartheta_1 A + \cdots + \lambda_k \vartheta_k A \quad \text{with } k \in \mathbb{N},\ \lambda_i \geq 0,\ \vartheta_i \in \mathrm{SO}(n)\ (i = 1,\dots,k),$$

and K is a generalized $\mathcal{M}_A$-body if $h_K = h_{M_2} - h_{M_1}$ with $M_1, M_2 \in \mathcal{M}_A$. To decide whether the set of generalized $\mathcal{M}_A$-bodies is dense in $\mathcal{K}^n$ is a matter of harmonic analysis. As explained in the Appendix, we denote by π_m the orthogonal projection from the function space $C(\mathbb{S}^{n-1})$ (with scalar product $(\cdot,\cdot)$; see the Appendix) to the space $\mathcal{S}^m$ of spherical harmonics of degree m, $m = 0, 1, 2, \ldots$ The convex body A is called *universal* if $\pi_m h_A \neq 0$ for all $m \in \mathbb{N}_0$. The following result appears in Schneider and Schuster [1734].

Theorem 3.5.10 *The set of generalized $\mathcal{M}_A$-bodies is dense in $\mathcal{K}^n$ if and only if the body A is universal.*

Proof Suppose, first, that A is not universal; then $\pi_m h_A = 0$ for some $m \in \mathbb{N}_0$. If $m = 0$, then A is one-pointed, by (A.6). Hence, we can assume that $m \geq 1$. Let $0 \neq Y_m \in \mathcal{S}^m$. By Lemma 1.7.8, there is a constant $c > 0$ such that $Y_m + c = \bar{h}_M$ for some convex body M (where $\bar{h}_M = h_M | \mathbb{S}^{n-1}$). If the set of generalized $\mathcal{M}_A$-bodies is dense in $\mathcal{K}^n$, then $\bar{h}_M$ can be approximated uniformly (on $\mathbb{S}^{n-1}$) by linear combinations of functions of the form $h_{\vartheta A}$ with $\vartheta \in \mathrm{SO}(n)$. But this implies that $\pi_m \bar{h}_M = 0$, a contradiction.

Now suppose that A is universal. Let $K \in \mathcal{K}^n$. First we assume that the support function $\bar{h}_K$ is a finite sum of spherical harmonics,

$$\bar{h}_K = \sum_{m=0}^{k} \sum_{j=1}^{N(n,m)} a_{mj} Y_{mj} \tag{3.49}$$

with real coefficients a_{mj}. Define $c_{mj} := (\bar{h}_A, Y_{mj})$. Since A is universal, for each $m \in \mathbb{N}_0$ we can choose an index $j(m)$ with $c_{mj(m)} \neq 0$. We define

$$b_{ij}^m := \begin{cases} a_{mi}/c_{mj(m)} & \text{for } j = j(m), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g := N(n,m) \sum_{m=0}^{k} \sum_{i,j=1}^{N(n,m)} b_{ij}^m t_{ij}^m,$$

with the functions t_{ij}^m on $\mathrm{SO}(n)$ defined in the Appendix by (A.17). Using formula (A.18) (and observing that $\bar{h}_{\vartheta A} = \vartheta \bar{h}_A$), we obtain, for $u \in \mathbb{S}^{n-1}$,

$$\int_{\mathrm{SO}(n)} \bar{h}_{\vartheta A}(u) g(\vartheta) \, \mathrm{d}\nu(\vartheta) = \sum_{m=0}^{k} \sum_{i,j=1}^{N(n,m)} b_{ij}^m N(n,m) \int_{\mathrm{SO}(n)} \bar{h}_{\vartheta A}(u) t_{ij}^m(\vartheta) \, \mathrm{d}\nu(\vartheta)$$
$$= \sum_{m=0}^{k} \sum_{i,j=1}^{N(n,m)} b_{ij}^m (\bar{h}_A, Y_{mj}) Y_{mi}(u) = \sum_{m=0}^{k} \sum_{i=1}^{N(n,m)} a_{mi} Y_{mi}(u) = \bar{h}_K(u).$$

Writing the function g as the difference of two nonnegative functions and approximating the integrals by finite sums, we see that K is a generalized $\mathcal{M}_A$-body.

As proved in Section 3.4, the set of convex bodies whose support function (restricted to $\mathbb{S}^{n-1}$) is a finite sum of spherical harmonics is dense in $\mathcal{K}^n$. Therefore, the set of generalized $\mathcal{M}_A$-bodies is dense in $\mathcal{K}^n$. □

Theorem 3.5.10 is only useful with additional information about universal convex bodies. Here we mention the following result. For proofs, we refer to the article of

Schneider and Schuster [1734], which strengthened an earlier result of Alesker [37], and to Schneider [1670], §5.

Theorem 3.5.11 *For every non-symmetric (that is, not centrally symmetric) convex body $K \in \mathcal{K}^n$ there exists a linear transformation $g \in GL(n)$, which can be chosen arbitrarily close to the identity, such that gK is universal.*

Any triangle in $\mathbb{R}^n$ $(n \geq 2)$ for which at least one angle is an irrational multiple of π, is universal.

A convex body $K \in \mathbb{R}^n$ $(n \geq 2)$ is called a *triangle body* if it can be approximated by finite sums of triangles, and a *generalized triangle body* if there are triangle bodies T_1, T_2 such that $K + T_1 = T_2$. The following is clear from Theorem 3.5.10 and 3.5.11.

Corollary 3.5.12 *The set of generalized triangle bodies is dense in $\mathcal{K}^n$.*

Notes for Section 3.5

1. *Bodies of constant width.* Much material about convex bodies of constant width can be found in the survey articles by Chakerian and Groemer [403] and by Heil and Martini [953]. More recent papers on bodies of constant width are by Bayen, Lachand-Robert and Oudet [185], Lachand-Robert and Oudet [1163], Guilfoyle and Klingenberg [868].
2. *Rotors.* Theorem 3.5.1 includes some special cases which were known for a longer time; the general classification was completed by Schneider [1664]. The proof is reproduced in §5.7 of the book by Groemer [800].

   Special rotors in regular polygons have been considered, often in connection with kinematic considerations, by Reuleaux [1577], Meissner [1400], Wunderlich [1993], Goldberg [727, 728], Schaal [1640] and others. All the rotors of the regular polygons were determined by Meissner [1400]. Case (b) of Theorem 3.5.1 is due to Fujiwara [648]; simpler proofs were given by Kameneckii [1062] and Schaal [1639].

   Meissner [1401] determined the rotors of the three-dimensional regular polyhedra. Models of such rotors are shown in the survey article by Goldberg [729], who also discussed the problem of the existence of rotors in higher dimensions and for non-regular polytopes.

   Since the class of rotors of a regular k-gon depends on infinitely many parameters, it is sufficiently interesting to be the subject of extremal problems. In particular, one may ask which rotors of a given regular k-gon have the least area. The case $k = 4$, where the Reuleaux triangle is the solution, was solved independently by Lebesgue [1179] and Blaschke [238]. The survey by Chakerian and Groemer [403] (p. 67) lists several later proofs, of which the one by Chakerian [397] is especially elegant. The case $k = 3$ was settled by Fujiwara and Kakeya [651]; alternative proofs were given by Weissbach [1954, 1955]; see also Jaglom and Boltjanski [1032]. For arbitrary $k \geq 3$, the problem was treated by Focke [622] under special symmetry assumptions and solved completely by Klötzler [1120].

   Some other extremal problems for rotors of regular polygons were investigated by Fujiwara and Kakeya [651], Fujiwara [650], Schaal [1640], Focke [622], Focke and Gensel [623], Weissbach [1956].

   Other Minkowski classes in the plane, more general than rotors, are made up by the so-called U_k-curves, for which all circumscribed equiangular k-gons have the same perimeter, or by the P_k-curves, for which all circumscribed equiangular k-gons are regular. These classes were studied by Meissner [1400], Jaglom and Boltjanski [1032], Goldberg [730], Weissbach [1955].
3. Surveys on zonoids and related topics were given by Bolker [274], Schneider and Weil [1738], Goodey and Weil [749].

4. *Zonotopes and generalizations*. The characterization of zonotopes contained in Theorem 3.5.2 must have been folklore for a long time, at least for $n = 3$; compare Blaschke [248], p. 250, and Coxeter [454], §2.8. Proofs for n-dimensional space were given by Bolker [274] and Schneider [1660].

   Some results have been proved for polytopes with centrally symmetric faces of dimension greater than two. Aleksandrov [11] (see also Burckhardt [358]) proved that a polytope of dimension at least three all of whose facets have a centre of symmetry is itself centrally symmetric. As a generalization of the zonotopes, one may consider the class $\mathcal{P}^n_{(k)}$ of n-polytopes all of whose k-faces are centrally symmetric, where $k \in \{2, \dots, n\}$. Aleksandrov's theorem implies that $\mathcal{P}^n_{(k)} \subset \mathcal{P}^n_{(k+1)} \subset \cdots \subset \mathcal{P}^n_{(n)}$ for $k \geq 2$; see also Shephard [1782] for further results, and McMullen [1382]. McMullen [1374] showed that $\mathcal{P}^n_{(n-2)} \neq \mathcal{P}^n_{(n-1)}$ for $n \geq 4$ and proved that $\mathcal{P}^n_{(k)}$ for $2 \leq k \leq n-2$ is equal to the class of n-dimensional zonotopes.

   Various combinatorial and geometric results on zonotopes have been obtained by McMullen [1375] and Shephard [1787]; for a survey and further references see McMullen [1384], §7.

   Zonotopes are also related to space-filling polytopes, that is, polytopes that tile space by translations. Every space-filling polytope in $\mathbb{R}^3$ is a zonotope, with the additional property that every zone contains four or six facets. Coxeter [453] used this fact for a simple classification of the space-filling polytopes in $\mathbb{R}^3$. In higher dimensions, the space-filling zonotopes were investigated by Shephard [1788] and McMullen [1379].

   Hadwiger [902] proved for $n = 3$ and Mürner [1456] proved for general n that every space-filling n-polytope is T-equidecomposable (in the sense of scissors congruence) to a cube, where T denotes the group of translations. In three-space, this property leads to a characterization of zonotopes, and, more generally: an n-polytope is T-equidecomposable to a cube if and only if it belongs to the class $\mathcal{P}^n_{(n-1)}$ defined above. This was conjectured by Hadwiger [921] and proved by Mürner [1457].

   For lattice zonotopes in $\mathbb{R}^n$ (i.e., zonotopes whose vertices belong to the integer lattice $\mathbb{Z}^n$), Betke and Gritzmann [217] proved two interesting inequalities of discrete geometry, which for general polytopes are respectively not true and unknown.

   Several aspects of the geometry of zonotopes are discussed in Martini [1350]. Some extremum problems for zonotopes were treated by Linhart [1223, 1224] and Filliman [580].

5. *The sum of a zonotope and an ellipsoid.* For a convex body $K \subset \mathbb{R}^n$, consider the following property. (*) For each $u \in \mathbb{S}^{n-1}$, there exists $\varepsilon(u) > 0$ such that the section $[H(K,u) - \alpha u] \cap K$ is centrally symmetric for $0 < \alpha < \varepsilon(u)$. Generalizing a classical characterization of the ellipsoid, Burton [359] proved the following remarkable result. An n-dimensional convex body K ($n \geq 3$) has property (*) if and only if K is the Minkowski sum of a (not necessarily n-dimensional) zonotope and an (n-dimensional) ellipsoid.

6. *The integral equation for zonoids*. Proofs of Theorem 3.5.3 may be found in Bolker [274], p. 336, Schneider [1660], Lindquist [1221], Matheron [1358], p. 94.

   Theorem 3.5.4 goes back (at least) to Blaschke [241]. For the uniqueness part, two proofs for $n = 3$ and special measures appear on pp. 152 and 154–155 of that book. The general uniqueness theorem was first proved by Aleksandrov [13], §8, and rediscovered by Petty [1525], Rickert [1579, 1580] and Matheron [1356] (reproduced in Matheron [1358], §4.5). Aleksandrov, Petty and Rickert use spherical harmonics for the uniqueness proof, while Matheron's proof is motivated by probability theory. Clearly, the uniqueness assertion of Theorem 3.5.4 is equivalent to the fact that the real vector space spanned by the functions $u \mapsto |\langle u, v\rangle|$, $v \in \mathbb{S}^{n-1}$, is dense in the space $C_e(\mathbb{S}^{n-1})$ of even continuous real functions on $\mathbb{S}^{n-1}$ with the maximum norm. A simple elegant proof of this fact was given by Choquet ([424], p. 53; [425], p. 1971).

   Matheron [1358], p. 73, had conjectured a generalization of the uniqueness assertion of Theorem 3.5.4, with the sphere replaced by Grassmannians. This was disproved by Goodey and Howard [739].

The fact that the integral equation (also known as the *zonoid equation*)

$$G(u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, g(v)\, \mathrm{d}v, \tag{3.50}$$

where G is a given even function on $\mathbb{S}^{n-1}$, can be solved by means of spherical harmonics was mentioned by Blaschke [241], p. 154 (for $n = 3$ and without specifying the necessary smoothness assumptions). The general argument given above was carried out in Schneider [1656]. Geometric applications of a solution of (3.50) appear in Chakerian [399], Firey [592], Petty [1527], Schneider [1656].

For special functions G, an explicit solution of the integral equation (3.50) may be found. In this way (and using the uniqueness result) one can obtain explicit examples of centrally symmetric convex bodies of class C^∞_+ that are not zonoids; for instance, the body $K \in \mathcal{K}^3$ with support function given by

$$h(K, u) = 1 + \alpha P_2(\langle e, u\rangle) \quad \text{for } u \in \mathbb{S}^2,$$

where $e \in \mathbb{S}^2$ is fixed, $P_2(t) = (3t^2 - 1)/2$ denotes the second Legendre polynomial and $\alpha \in [-2/5, -1/4]$ is a real constant (see Schneider [1660], p. 69).

7. *Zonoids whose polars are zonoids.* Answering a question of Bolker [274], Schneider [1677] constructed (in any $\mathbb{R}^n$) zonoids whose polars (with respect to the centre) are zonoids and which are not ellipsoids. The proof used perturbations of the ball, together with a (weak) stability result for the solutions of (3.50), namely $\|g\| \le \|G\|_{2k}$, where $\|\cdot\|$ denotes the maximum norm and $\|\cdot\|_{2k}$ is a certain norm involving derivatives up to order $2k$. Since the constructed zonoids were very close to ellipsoids and smooth, this raised the following questions.

   In $\mathbb{R}^n$, let d_n denote the maximal Banach–Mazur distance of a ball from all zonoids whose polars are zonoids. Is it true that $\lim_{n\to\infty} d_n = 1$? This was asked by the author at an Oberwolfach conference in 1976; see problem 72 in [841] (where λE should read $(1 + \lambda)E$). The question remains unanswered.

   That zonoids whose polars are zonoids need not be smooth was shown by Lonke [1231]. He proved that the sum of an n-ball and an $(n - 1)$-ball in $\mathbb{R}^n$, $n = 3$ or 4, is a zonoid whose polar is a zonoid. On the other hand, he showed that if a convex body K in $\mathbb{R}^n$ is of the form $K = B + C$ with centrally symmetric convex bodies B, C and $1 \le \dim C \le n - 2$, then the polar of K is not a zonoid. In particular, an n-dimensional zonoid whose polar is a zonoid cannot have a proper face of dimension different from zero or $n - 1$.

8. *Approximation problems for zonoids*. By definition, a zonoid can be approximated arbitrarily closely by finite sums of segments. Interesting problems arise if one asks for the minimal number of segments necessary to reach a given degree of approximation. For a zonoid Z in $\mathbb{R}^n$ (with centre at the origin) and for $\varepsilon > 0$ (sufficiently small, in dependence on the cases considered below), let $N(Z, \varepsilon)$ be the smallest number N such that there is a sum P_N of N segments satisfying

$$Z \subset P_N \subset (1 + \varepsilon)Z.$$

   Improving a slightly weaker result of Figiel, Lindenstrauss and Milman [578], Gordon [760] showed for the ball B^n that

$$N(B^n, \varepsilon) \le c\varepsilon^{-2} n,$$

   where c is an absolute constant. The remarkable fact here is the linear dependence of the upper bound on n. For general zonoids Z, Bourgain, Lindenstrauss and Milman [321] (announced in [319]) obtained results that are only slightly weaker. For $\tau > 0$ they showed that

$$N(Z, \varepsilon) \le c(\tau)\varepsilon^{-(2+\tau)}(\log n)^3 n$$

with a constant $c(\tau)$ depending only on τ, and

$$N(Z,\varepsilon) \le c(\tau,\delta)\varepsilon^{-(2+\tau)}n$$

if the zonoid Z is the unit ball of a uniformly convex norm, δ being the degree of uniform convexity. For given n, the above bounds are not optimal in ε, but here for the ball sharper estimates are known. There are constants $c_i(n)$, depending only on n, such that

$$c_1(n)\varepsilon^{-2(n-1)/(n+2)} \le N(B^n,\varepsilon) \le c_2(n)\left(\varepsilon^{-2}|\log\varepsilon|\right)^{(n-1)/(n+2)}.$$

The left-hand inequality is due to Bourgain, Lindenstrauss and Milman [321], Theorem 6.5 and the right-hand one to Bourgain and Lindenstrauss [317]. The latter authors also have similar, but in general weaker, results for arbitrary zonoids. Weaker inequalities of the above type were obtained earlier by Betke and McMullen [218] and by Linhart [1225].

9. *Characterizations of zonoids*. Theorem 3.5.2 characterizes zonotopes in a very simple and intuitive way. An equally simple characterization of zonoids apparently does not exist. In particular, it is not possible to characterize zonoids by a strictly local criterion. The question of such a characterization has been posed repeatedly; see Blaschke [248], p. 250, Blaschke and Reidemeister [255], pp. 81–82, and Bolker [275]. However, the following was shown by Weil [1939]. There exists a convex body $K \in \mathcal{K}^n$ ($n \ge 3$), arbitrarily smooth, that is not a zonoid but has the following property. For each $u \in \mathbb{S}^{n-1}$ there exist a zonoid Z with centre at the origin and a neighbourhood U of u in $\mathbb{S}^{n-1}$ such that the boundaries of K and Z coincide at all points where the exterior unit normal vector belongs to U. Thus, no characterization of zonoids is possible that involves only arbitrarily small neighbourhoods of boundary points. Instead, Weil [1939] proposed the following conjecture. Let $K \in \mathcal{K}^n$ be a convex body such that to any great sphere $\sigma \subset \mathbb{S}^{n-1}$ there exist a zonoid Z and a neighbourhood U of σ in $\mathbb{S}^{n-1}$ such that the boundaries of K and Z coincide at all points where the exterior unit vector belongs to U; then K is a zonoid. Affirmative answers for even dimensions were given independently by Panina [1500, 1501, 1502] and Goodey and Weil [746]. In odd dimensions, Nazarov, Ryabogin and Zvavitch [1470] found that the answer is negative. An extension of the problem of local equatorial determination to bodies that embed in L_p, for $p > 0$, was treated by Schlaerth [1644]. Goodey and Weil [752] have investigated such questions within a more general theory for intertwining operators on the sphere.

It is also not possible to characterize zonoids by means of their projections. Weil [1949] constructed a convex body $K \in \mathcal{K}^n$ ($n \ge 3$) that is not a zonoid but has the property that all its projections onto hyperplanes are zonoids.

An approach to characterizations of zonoids by means of a different integral representation was proposed by Ambartzumian [67].

Non-trivial characterizations of zonoids are possible if systems of inequalities satisfied by support functions are taken into account. Some definitions are needed. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if

$$\sum_{i,j=1}^{k} f(x_i - x_j)\,\alpha_i\alpha_j \ge 0$$

holds for all $k \in \mathbb{N}$, $x_1,\dots,x_k \in \mathbb{R}^n$, $\alpha_1,\dots,\alpha_k \in \mathbb{R}$, and it is called *conditionally positive definite* if these inequalities are only assumed for $\sum \alpha_i = 0$. The function f is *of negative type* if for every $t > 0$ the function e^{-tf} is positive definite. A real normed vector space $(V, \|\cdot\|)$ is called *hypermetric* if

$$\sum_{i,j=1}^{k} \|(x_i - x_j)\|\,\alpha_i\alpha_j \le 0$$

for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in V$ and all integers $\alpha_1, \dots, \alpha_k$ satisfying $\sum \alpha_i = 1$. With these definitions, the following properties of an n-dimensional convex body $Z \in \mathcal{K}^n$ with centre at the origin are equivalent:

(a) Z is a zonoid;
(b) $h(Z, \cdot)$ is of negative type;
(c) $e^{-h(Z,\cdot)}$ is positive definite;
(d) $\mathbb{R}^n$ with norm $\|\cdot\|_Z = h(Z, \cdot)$ is hypermetric;
(e) $-h(Z, \cdot)$ is conditionally positive definite.

Here the equivalence of (a) and (b) is a classical theorem due to Lévy [1205], §63; see also Choquet [425], p. 173. The equivalence of (b) and (c) is trivial; for the equivalence of (c), (d) and (e) see Witsenhausen [1989]. Witsenhausen [1990] and Assouad [97] also obtained characterizations of zonotopes among the polytopes by means of norm inequalities.

10. *Relations of zonoids to other fields.* Zonoids appear, and are useful, in various contexts. We mention briefly some of their occurrences; for more information we refer to the survey articles mentioned in Note 3 above; we have already quoted from them.

A well-known theorem of Liapounoff [1216] says that the range of a non-atomic $\mathbb{R}^n$-valued measure is compact and convex; shorter proofs are due to Halmos [933], Blackwell [234], Lindenstrauss [1219] and others. It is not difficult to show that the convex bodies occurring as such ranges are precisely the zonoids; see Rickert [1580] and Bolker [274].

In Banach space theory, zonoids occur as follows. Let $Z \in \mathbb{R}^n$ be a convex body with o as centre and interior point. Then $\|\cdot\|_Z = h(Z, \cdot)$ defines a norm on $\mathbb{R}^n$, for which the polar body Z° is the unit ball. The body Z is a zonoid if and only if the space $\mathbb{R}^n$ with norm $\|\cdot\|_Z$ is isometric to a subspace of $L_1 = L_1([0, 1])$ (a proof and further discussion can be found in Bolker [274]). Thus, questions about the isometric embedding of finite-dimensional Banach spaces in L_1 reduce to problems on zonoids. For instance, Bolker [274] pointed out that the space l_p^n (which is $\mathbb{R}^n$ endowed with the norm defined by $\|(\xi_1, \dots, \xi_n)\| = (\sum_{i=1}^n |\xi_i|^p)^{1/p}$) embeds in L_1 isometrically if $1 \le p \le 2$, and he conjectured that this is not the case if $n \ge 3$ and $p > 2$. This was proved for $p > 2.7$ by Witsenhausen [1989] and for all $p > 2$ by Dor [511]. Bolker [274, 275] also formulated the following question. If E is an n-dimensional Banach space ($n \ge 3$) such that E and its dual space both embed isometrically in L_1, is E isometric to l_2^n? The result of Schneider [1677] mentioned in Note 6 provides counterexamples.

In stochastic geometry, zonoids have proved useful as auxiliary bodies. If ρ is an even measure on $\mathbb{S}^{n-1}$, which can be, e.g., the orientation distribution of a random hyperplane or a measure associated in some way with a stochastic process of other geometric objects, one defines an *associated zonoid* K by means of equation (3.37). Some parameters describing the stochastic situation can then be expressed in terms of functionals of K, so that, for instance, inequalities for convex bodies can be used to solve extremal problems of stochastic geometry. This method was initiated by Matheron [1357, 1358], who called the associated zonoid a ‘Steiner compact’. For the subsequent development, we refer to the book by Schneider and Weil [1740], in particular §4.6. Later applications of the associated zonoid appear in Schneider [1729, 1730], Hug and Schneider [1019].

Alexander [56] made an interesting contribution to Hilbert’s fourth problem, using zonoids and generalized zonoids. In particular, the equivalence of (a) and (d) in Note 5 plays an essential role in his argument.

An application of zonoids to mathematical economics may be found in Hildenbrand [975].

11. *Lift zonoids.* Let μ be a Borel measure on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |x| \, d\mu(x) < \infty$. Denoting by $a_+ := \max\{0, a\}$ the positive part of a, we can define a zonoid $Z(\mu)$ by its support function,

$$h(Z(\mu), x) = \int_{\mathbb{R}^n} \langle x, y \rangle_+ \, d\mu(y) \quad \text{for } x \in \mathbb{R}^n$$

(it has centre $\frac{1}{2}\int_{\mathbb{R}^n} x\,\mathrm{d}\mu(x)$). This zonoid does not determine the measure μ uniquely, but this can be remedied by the following lifting procedure. Define the product measure $\widehat{\mu} := \delta_1 \otimes \mu$ on $\mathbb{R}\times\mathbb{R}^n$, where δ_1 denotes the Dirac measure at 1. The zonoid $\widehat{Z}(\mu) := Z(\widehat{\mu})$ in $\mathbb{R}\times\mathbb{R}^n$ is called the *lift zonoid* of μ. It has the support function

$$h(\widehat{Z}(\mu), (\xi, x)) = \int_{\mathbb{R}^n} (\xi + \langle x, y\rangle)_+ \,\mathrm{d}\mu(y), \qquad (\xi, x) \in \mathbb{R}\times\mathbb{R}^n.$$

Remarkably, the lift zonoid $\widehat{Z}(\mu)$ determines the measure μ uniquely.

This lift zonoid has applications in multivariate statistics and mathematical finance. We refer to the book by Mosler [1450] and to the articles by Cascos and by Molchanov in the book [1071].

12. *Generalized zonoids*. Lemma 3.5.6 is taken from Schneider [1660].

The existence of generalized zonoids that are not zonoids motivates two questions: to characterize the even signed measures ρ on $\mathbb{S}^{n-1}$ for which

$$h(K, \cdot) = \int_{\mathbb{S}^{n-1}} |\langle \cdot, v\rangle| \,\mathrm{d}\rho(v) \tag{3.51}$$

yields a support function, and to characterize the convex bodies K obtainable in this way. Aiming at answering these questions, Weil [1937] developed formulae for mixed area measures of generalized zonoids (Section 5.3). With their aid and using results from Weil [1933], he obtained the following result. For $u_1, \dots, u_j \in \mathbb{S}^{n-1}$ let $D_j(u_1, \dots, u_j)$ denote the j-dimensional volume of the parallelepiped spanned by $u_1, \dots, u_j$. Further, define a partial map $T : (\mathbb{S}^{n-1})^{n-1} \to \mathbb{S}^{n-1}$ by the normalized vector product. Then (3.51) defines a support function if and only if

$$T\left[\int_{(\cdot)} D_{n-1}\,\mathrm{d}(\rho^j \times \sigma^{n-j-1})\right] \geq 0$$

for $j = 1, \dots, n-1$ (where σ denotes the normalized rotation invariant measure on $\mathbb{S}^{n-1}$). A different characterization of the signed measures ρ that generate convex bodies by means of (3.51) can be obtained by considering projections. For a signed measure ρ on $\mathbb{S}^{n-1}$ and for a j-dimensional great subsphere S_j of $\mathbb{S}^{n-1}$, Weil [1949] defined a projection of ρ onto S_j such that, for any generalized zonoid K with generating measure ρ and any $(j+1)$-dimensional linear subspace E_{j+1} of $\mathbb{R}^n$, the orthogonal projection of K onto E_{j+1} has the projection of ρ on $E_{j+1} \cap \mathbb{S}^{n-1}$ as its generating measure.

He deduced, for instance, that ρ generates a convex body if and only if all its projections onto great circles are nonnegative. Another application generalizes a convexity criterion of Lindquist [1222].

Several characterizations of zonoids, generalized zonoids and centrally symmetric convex bodies, expressed in terms of inequalities involving mixed volumes, were investigated by Weil [1938, 1940] and Goodey [731, 734, 735] (some of the assertions in Goodey [734, 735] have to be modified; see the corrections in Goodey and Weil [746]). See also Section 5.1, Note 8.

13. Generalized zonoids were obtained by extending the integral representation (3.37) from measures to signed measures. The fact that the generalized zonoids are dense in the class of centrally symmetric convex bodies but do not exhaust this class is a reason for a further extension, namely to distributions. It was shown by Weil [1938] that for each convex body $K \in \mathcal{K}^n$ with centre at the origin there exists an even distribution T_K on $\mathbb{S}^{n-1}$, whose domain can be extended to include the functions $|\langle u, \cdot\rangle|$, $u \in \mathbb{S}^{n-1}$, such that $T_K(|\langle u, \cdot\rangle|) = h(K, u)$. The properties of the correspondence $K \longmapsto T_K$ and some applications were studied further by Weil [1938, 1940].

14. *Universal convex bodies.* Theorems 3.5.10 and 3.5.11 (first part) are taken from Schneider and Schuster [1734]. They strengthen an earlier result by Alesker [37], who proved that the generalized $\mathcal{M}$-bodies are dense in $\mathcal{K}^n$ if $\mathcal{M}$ is the $GL(n)$ invariant Minkowski class

generated by a non-symmetric convex body. Both papers, [37] and [1734], also contain analogous results for centrally symmetric convex bodies.

15. *Minkowski classes generated by lower-dimensional bodies*. Corollary 3.5.12 can also be deduced from a result of Goodey and Weil [748]. They introduce a certain mean section body $M_2(K)$ and show a uniqueness result for a certain integral equation. By a Hahn–Banach argument, the uniqueness is equivalent to the denseness of differences of support functions of mean section bodies in the space of all support functions. Since a mean section body $M_2(K)$ is a mean of two-dimensional sections of K, it is a triangle body; hence Corollary 3.5.12 follows.

    Let $k \in \{1, \dots, n\}$ and let $\mathcal{S}_k$ be the set of all simplices in $\mathbb{R}^n$ of dimension at most k. The Minkowski class generated by $\mathcal{S}_k$ was investigated by Ricker [1578]. He found an integral representation for its elements in terms of $\mathbb{R}^n$-valued measures.

    Let $k \in \{2, \dots, n-1\}$ and let $\mathcal{K}_k$ be the set of all convex bodies in $\mathbb{R}^n$ of dimension at most k. Hildenbrand and Neyman [976] proved a result that implies that the Minkowski class generated by $\mathcal{K}_k$ is nowhere dense in $\mathcal{K}^n$. This follows also from Theorem 3.4.2 and the denseness of the indecomposable bodies for $n \geq 3$.

4

Support measures and intrinsic volumes

With this chapter, we start a detailed study of functionals on convex bodies, such as volume, surface area, functionals derived from them and measures related to them. The investigation of inequalities for such functionals will begin with Chapter 7.

4.1 Local parallel sets

Recall that for a convex body $K \in \mathcal{K}^n$ and a number $\rho \geq 0$, the *outer parallel body* of K at distance ρ is the set

$$K_\rho := K + \rho B^n = \{x \in \mathbb{R}^n : d(K, x) \leq \rho\}.$$

The *Steiner formula* (to be proved in Section 4.2) says that its volume can be expressed as a polynomial of degree at most n in the parameter ρ. For the coefficients, two different normalizations are in use, so that the Steiner formula appears in the two versions,

$$V_n(K + \rho B^n) = \sum_{i=0}^{n} \rho^i \binom{n}{i} W_i(K) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K). \tag{4.1}$$

The polynomial on the right side is known as the *Steiner polynomial*. In the classical terminology, the functionals $W_0, \dots, W_n$ are called the *quermassintegrals*, and $V_0, \dots, V_n$, which came into use only later, have been coined the *intrinsic volumes* (it appears that the German name 'Quermaßintegral' was first used in Bonnesen and Fenchel [284]; the name 'intrinsic volume' was proposed by McMullen [1378]). Also the name *Minkowski functionals* (possibly with a third normalization) can be encountered. (Note that the term 'Minkowski functional' is also used for the gauge function.)

The coefficients defined by the Steiner formula turn out to be important quantities, carrying basic information on the convex body under investigation. Formula (4.1) is only a very special case of a similar expansion, valid for a general Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$; this leads to the notion of the mixed volume and will be extensively investigated in later chapters. The particular case of the outer parallel body deserves a separate study, because its strong ties with the Euclidean metric result in a theory of different character and lead to a number of notions and results related to the Euclidean geometry of convex bodies. Moreover, these functionals admit useful local versions, which replace curvature functions in the case of general convex bodies.

The Minkowski sum $K + \rho B^n$ is at the same time the set of all points of $\mathbb{R}^n$ having distance from K at most ρ. This point of view leads naturally to the more general notion of a local parallel set. Given a subset $\beta \subset K$, we may consider the set of all points $x \in \mathbb{R}^n$ for which $0 < d(K, x) \le \rho$ and for which the nearest point $p(K, x)$ belongs to β. Alternatively, we may prescribe a set $\beta \subset \mathbb{S}^{n-1}$ of unit vectors and then consider the set of all $x \in \mathbb{R}^n$ for which $0 < d(K, x) \le \rho$ and for which the unit vector $u(K, x)$ pointing from $p(K, x)$ to x belongs to β. Restricting β to Borel sets, we find in either case that the Lebesgue measure of the local parallel set thus defined is again a polynomial in the parameter ρ. The coefficients now depend on K and β, and as functions of the latter argument they are measures. These are the curvature or area measures studied in the present chapter. Their basic properties will be derived from corresponding properties of the local parallel sets, as established in this section. Instead of treating the two types of local parallel sets separately, we consider a common generalization, which leads to the support measures.

We recall some notation from Section 2.6. We denote the product space $\mathbb{R}^n \times \mathbb{S}^{n-1}$ by Σ. If $K \in \mathcal{K}^n$, a pair $(x, u) \in \Sigma$ is called a *support element* of K if $x \in \operatorname{bd} K$ and u is an outer unit normal vector of K at x. For each $y \in \mathbb{R}^n \setminus K$, the pair $(p(K, y), u(K, y))$ is a support element of K. The set of all support elements of K, denoted by $\operatorname{Nor} K$, is the normal bundle of K. For a topological space X, the σ-algebra of Borel subsets of X is denoted by $\mathcal{B}(X)$. 'Measurable' without further comment means Borel measurable.

Now let a convex body $K \in \mathcal{K}^n$ and a number $\rho > 0$ be given. The map

$$\begin{array}{rccc} f_\rho : & K_\rho \setminus K & \to & \Sigma \\ & x & \mapsto & (p(K, x), u(K, x)) \end{array}$$

is continuous and hence measurable. We may, therefore, consider the image measure $\mu_\rho(K, \cdot)$ under f_ρ of the Lebesgue measure. Thus, $\mu_\rho(K, \cdot)$ is a finite measure on $\mathcal{B}(\Sigma)$, and, for a Borel set $\eta \in \mathcal{B}(\Sigma)$, the value $\mu_\rho(K, \eta)$ is the Lebesgue measure of the *local parallel set*

$$M_\rho(K, \eta) := f_\rho^{-1}(\eta) = \{x \in \mathbb{R}^n : 0 < d(K, x) \le \rho \text{ and } (p(K, x), u(K, x)) \in \eta\}.$$

We establish some basic properties of the map $\mu_\rho : \mathcal{K}^n \times \mathcal{B}(\Sigma) \to \mathbb{R}$. For the measure-theoretic notions and results to be used we refer, e.g., to Ash [94] and Bauer [180]. By $\xrightarrow{w}$ we denote weak convergence of Borel measures.

Theorem 4.1.1 *Let $(K_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{K}^n$ with $K_j \to K$ for $j \to \infty$. Then $\mu_\rho(K_j, \cdot) \xrightarrow{w} \mu_\rho(K, \cdot)$ for $j \to \infty$.*

Proof Let $\eta \subset \Sigma$ be open, and let $x \in M_\rho(K,\eta)$ be a point with $d(K,x) < \rho$. From Lemma 1.8.11 it follows that $d(K_j,x) \to d(K,x)$ and

$$(p(K_j,x),u(K_j,x)) \to (p(K,x),u(K,x)) \quad \text{for } j \to \infty.$$

Hence, for almost all j we have $d(K_j,x) < \rho$ and $(p(K_j,x),u(K_j,x)) \in \eta$, thus $x \in M_\rho(K_j,\eta)$. This shows that

$$M_\rho(K,\eta) \setminus \operatorname{bd} K_\rho \subset \liminf_{j\to\infty} M_\rho(K_j,\eta)$$

and hence that

$$\begin{aligned}\mu_\rho(K,\eta) = \mathcal{H}^n(M_\rho(K,\eta) \setminus \operatorname{bd} K_\rho) &\le \mathcal{H}^n(\liminf_{j\to\infty} M_\rho(K_j,\eta))\\ &\le \liminf_{j\to\infty} \mathcal{H}^n(M_\rho(K_j,\eta)) = \liminf_{j\to\infty} \mu_\rho(K_j,\eta).\end{aligned}$$

This holds for all open sets $\eta \subset \Sigma$. Further, we have

$$\mu_\rho(K,\Sigma) = \mathcal{H}^n(K_\rho \setminus K) = \mathcal{H}^n(K_\rho) - \mathcal{H}^n(K).$$

The continuity of the volume on $\mathcal{K}^n$ (Theorem 1.8.20) together with the continuity of Minkowski addition shows that

$$\mu_\rho(K_j,\Sigma) \to \mu_\rho(K,\Sigma) \quad \text{for } j \to \infty.$$

The assertion follows (see, e.g., Ash [94], p. 196). $\square$

Theorem 4.1.2 *For each $\eta \in \mathcal{B}(\Sigma)$, the function $\mu_\rho(\cdot,\eta) : \mathcal{K}^n \to \mathbb{R}$ is measurable.*

Proof In the preceding proof it was shown that $K_j \to K$ implies

$$\liminf_{j\to\infty} \mu_\rho(K_j,\eta) \ge \mu_\rho(K,\eta)$$

for open η, hence for such η the function $\mu_\rho(\cdot,\eta)$ is lower semi-continuous and thus measurable. Let $\mathcal{D}$ be the family of all sets $\eta \in \mathcal{B}(\Sigma)$ for which $\mu_\rho(\cdot,\eta)$ is measurable. We show that $\mathcal{D}$ is a Dynkin system. For $\eta_1,\eta_2 \in \mathcal{D}$ with $\eta_2 \subset \eta_1$ we have $M_\rho(K,\eta_2) \subset M_\rho(K,\eta_1)$ and

$$M_\rho(K,\eta_1 \setminus \eta_2) = M_\rho(K,\eta_1) \setminus M_\rho(K,\eta_2),$$

hence

$$\mu_\rho(K,\eta_1 \setminus \eta_2) = \mu_\rho(K,\eta_1) - \mu_\rho(K,\eta_2)$$

for $K \in \mathcal{K}^n$, thus $\eta_1 \setminus \eta_2 \in \mathcal{D}$. For a sequence $(\eta_j)_{j\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{D}$ we have

$$\mu_\rho\left(K, \bigcup_{j=1}^\infty \eta_j\right) = \sum_{j=1}^\infty \mu_\rho(K,\eta_j)$$

for $K \in \mathcal{K}^n$, since $\mu_\rho(K,\cdot)$ is a measure. It follows that $\bigcup_{j=1}^\infty \eta_j \in \mathcal{D}$. Thus $\mathcal{D}$ is a Dynkin system that contains the open sets and, therefore, the σ-algebra generated by them (Bauer [180], p. 20). Hence, $\mathcal{B}(\Sigma) \subset \mathcal{D}$, which was the assertion. $\square$

Theorem 4.1.3 *For each $\eta \in \mathcal{B}(\Sigma)$, the function $\mu_\rho(\cdot, \eta)$ is additive, that is,*

$$\mu_\rho(K_1 \cup K_2, \eta) + \mu_\rho(K_1 \cap K_2, \eta) = \mu_\rho(K_1, \eta) + \mu_\rho(K_2, \eta)$$

if $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n$.

Proof Let $K_1, K_2 \in \mathcal{K}^n$ be convex bodies such that $K_1 \cup K_2 \in \mathcal{K}^n$. By $I_\rho(K, \eta, \cdot)$ we denote the characteristic function of the set $M_\rho(K, \eta)$. Let $x \in \mathbb{R}^n$ be given and suppose that $y := p(K_1 \cup K_2, x) \in K_1$, without loss of generality. Then clearly

$$p(K_1 \cup K_2, x) = p(K_1, x).$$

Let $p(K_2, x) =: z$. Since $K_1 \cup K_2$ is convex, there is a point $a \in [z, y]$ for which $a \in K_1 \cap K_2$. Since $y = p(K_1 \cup K_2, x)$, we have $|y - x| \le |z - x|$ and hence $|a - x| \le |z - x|$. Here strict inequality is impossible; hence $a = z$ and thus $z \in K_1 \cap K_2$. It follows that

$$p(K_1 \cap K_2, x) = p(K_2, x).$$

We deduce that

$$d(K_1 \cup K_2, x) = d(K_1, x), \qquad d(K_1 \cap K_2, x) = d(K_2, x),$$
$$u(K_1 \cup K_2, x) = u(K_1, x), \qquad u(K_1 \cap K_2, x) = u(K_2, x),$$

from which we get

$$I_\rho(K_1 \cup K_2, \eta, x) = I_\rho(K_1, \eta, x), \qquad I_\rho(K_1 \cap K_2, \eta, x) = I_\rho(K_2, \eta, x).$$

As x was arbitrary, we conclude that

$$I_\rho(K_1 \cup K_2, \eta, \cdot) + I_\rho(K_1 \cap K_2, \eta, \cdot) = I_\rho(K_1, \eta, \cdot) + I_\rho(K_2, \eta, \cdot).$$

Integration of this equality with respect to $\mathcal{H}^n$ yields the assertion. □

4.2 Steiner formula and support measures

We shall now show that the measure $\mu_\rho(K, \eta)$ of the local parallel set $M_\rho(K, \eta)$ can be expressed as a polynomial in the parameter ρ (the 'local Steiner formula'). First let $P \in \mathcal{K}^n$ be a polytope. For $x \in \mathbb{R}^n \setminus P$, the nearest point $p(P, x)$ belongs to the relative interior of a unique face of P. For a given face F of P, the Lebesgue measure of the set

$$M_\rho(P, \eta) \cap p(P, \ \cdot)^{-1}(\mathrm{relint}\, F),$$

where $\eta \in \mathcal{B}(\Sigma)$ and $\rho > 0$, can be computed by means of Fubini's theorem. For this, we observe that

$$(P_\rho \setminus P) \cap p(P, \ \cdot)^{-1}(\mathrm{relint}\, F)$$

is, up to a set of measure zero, equal to the direct sum

$$F \oplus (N(P, F) \cap \rho B^n),$$

where $N(P, F)$ is the normal cone of P at F. If $\dim F =: m \in \{0, \dots, n-1\}$, we obtain

$$
\begin{aligned}
&\mathcal{H}^n(M_\rho(P,\eta) \cap p(P,\cdot)^{-1}(\operatorname{relint} F)) \\
&= \int_F \mathcal{H}^{n-m}\,\{z \in N(P,F) : 0 < |z| \le \rho,\ (y, z/|z|) \in \eta\}\, \mathrm{d}\mathcal{H}^m(y) \\
&= \int_F \mathcal{H}^{n-m}\left\{\lambda u : 0 < \lambda \le \rho,\ u \in N(P,F) \cap \mathbb{S}^{n-1},\ (y,u) \in \eta\right\} \mathrm{d}\mathcal{H}^m(y) \\
&= \frac{\rho^{n-m}}{n-m} \int_F \int_{N(P,F)\cap\mathbb{S}^{n-1}} \mathbf{1}_\eta(y,u)\, \mathrm{d}\mathcal{H}^{n-m-1}(u)\, \mathrm{d}\mathcal{H}^m(y),
\end{aligned}
$$

where $\mathbf{1}_\eta$ denotes the characteristic function of η. Now the disjoint decomposition

$$
\mathbb{R}^n \setminus P = \bigcup_{m=0}^{n-1} \bigcup_{F \in \mathcal{F}_m(P)} \{x \in \mathbb{R}^n \setminus P : p(P,x) \in \operatorname{relint} F\}
$$

gives

$$
\mu_\rho(P,\eta) = \sum_{m=0}^{n-1} \frac{\rho^{n-m}}{n-m} \sum_{F \in \mathcal{F}_m(P)} \int_F \int_{N(P,F)\cap\mathbb{S}^{n-1}} \mathbf{1}_\eta(y,u)\, \mathrm{d}\mathcal{H}^{n-m-1}(u)\, \mathrm{d}\mathcal{H}^m(y). \tag{4.2}
$$

We define

$$
\binom{n-1}{m} \Theta_m(P,\eta) := \sum_{F \in \mathcal{F}_m(P)} \int_F \int_{N(P,F)\cap\mathbb{S}^{n-1}} \mathbf{1}_\eta(y,u)\, \mathrm{d}\mathcal{H}^{n-m-1}(u)\, \mathrm{d}\mathcal{H}^m(y) \tag{4.3}
$$

for $m = 0, \dots, n-1$, so that

$$
\mu_\rho(P,\eta) = \frac{1}{n} \sum_{m=0}^{n-1} \rho^{n-m} \binom{n}{m} \Theta_m(P,\eta). \tag{4.4}
$$

This representation can be extended to arbitrary convex bodies.

Theorem 4.2.1 *For every convex body $K \in \mathcal{K}^n$ there exist finite positive measures $\Theta_0(K,\cdot), \dots, \Theta_{n-1}(K,\cdot)$ on $\mathcal{B}(\Sigma)$ such that, for every $\eta \in \mathcal{B}(\Sigma)$ and every $\rho > 0$, the measure $\mu_\rho(K,\eta)$ of the local parallel set $M_\rho(K,\eta)$ is given by*

$$
\mu_\rho(K,\eta) = \frac{1}{n} \sum_{m=0}^{n-1} \rho^{n-m} \binom{n}{m} \Theta_m(K,\eta). \tag{4.5}
$$

The mapping $K \mapsto \Theta_m(K,\cdot)$ (from $\mathcal{K}^n$ into the space of Borel measures on Σ) is weakly continuous and additive, that is,

$$
K_j \to K \quad \textit{implies} \quad \Theta_m(K_j,\cdot) \xrightarrow{w} \Theta_m(K,\cdot),
$$

and $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n$ implies

$$
\Theta_m(K_1 \cup K_2,\cdot) + \Theta_m(K_1 \cap K_2,\cdot) = \Theta_m(K_1,\cdot) + \Theta_m(K_2,\cdot).
$$

For each $\eta \in \mathcal{B}(\Sigma)$, the function $\Theta_m(\cdot,\eta)$ (from $\mathcal{K}^n$ to $\mathbb{R}$) is measurable.

Proof First let $P \in \mathcal{K}^n$ be a polytope. Let $\eta \in \mathcal{B}(\Sigma)$ and $\rho > 0$ be given. Writing down equation (4.4) successively for $\rho = 1, \dots, n$, we obtain a system of linear equations that can be solved for $\Theta_0(P, \eta), \dots, \Theta_{n-1}(P, \eta)$ (its determinant is a Vandermonde determinant, different from zero). We get representations of the form

$$\Theta_m(P, \eta) = \sum_{k=1}^{n} a_{mk} \mu_k(P, \eta)$$

with certain constants a_{mk}. With these constants we define

$$\Theta_m(K, \cdot) := \sum_{k=1}^{n} a_{mk} \mu_k(K, \cdot) \tag{4.6}$$

for arbitrary convex bodies $K \in \mathcal{K}^n$. Then $\Theta_m(K, \cdot)$ is a finite measure on $\mathcal{B}(\Sigma)$, for the moment possibly a signed one. From Theorem 4.1.1 it follows that $K_j \to K$ (for $K_j, K \in \mathcal{K}^n$) implies $\Theta_m(K_j, \cdot) \xrightarrow{w} \Theta_m(K, \cdot)$. Applying this to a sequence of polytopes converging to K, we see that $\Theta_m(K, \cdot)$ is a positive measure and that (4.4) can be extended to give (4.5). The final assertions of the theorem follow from (4.6) and Theorems 4.1.3 and 4.1.2. $\square$

Formula (4.5) is the *local Steiner formula*. The measure $\Theta_m(K, \cdot)$ is called the *support measure* or *generalized curvature measure* of order m of the convex body K. Although we have defined the measure $\Theta_m(K, \cdot)$ on the whole space Σ, it is clearly concentrated on the set $\operatorname{Nor} K$ of support elements of K: if $x \in M_\rho(K, \eta)$, then $(p(K, x), u(K, x)) \in \eta \cap \operatorname{Nor} K$, hence $\mu_\rho(K, \eta) = \mu_\rho(K, \eta \cap \operatorname{Nor} K)$ and, therefore, by (4.6),

$$\Theta_m(K, \eta) = \Theta_m(K, \eta \cap \operatorname{Nor} K) \tag{4.7}$$

for $\eta \in \mathcal{B}(\Sigma)$ and $m = 0, \dots, n-1$. This explains the name 'support measure'. The reason for the term 'curvature measure' will become clear below.

Taking $\eta = \Sigma$ in (4.5) and adding the volume of K on both sides, we obtain the classical Steiner formula, which we write in the two forms

$$V_n(K + \rho B^n) = \sum_{i=0}^{n} \rho^i \binom{n}{i} W_i(K) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K). \tag{4.8}$$

Here

$$\kappa_{n-m} V_m(K) = \binom{n}{m} W_{n-m}(K) = \binom{n}{m} \frac{1}{n} \Theta_m(K, \Sigma) \tag{4.9}$$

for $m = 0, \dots, n-1$ are the total support measures in two different normalizations, and

$$V_n(K) = W_0(K)$$

is the volume of K. The functions $W_0, \dots, W_n$ are called the *quermassintegrals*. This term comes from the German 'Quermaß', which can be the measure of either a cross-section or a projection. The reason for this terminology will be clear when certain integral-geometric interpretations of the functions W_i have been obtained (formulae (4.60) and (5.72), for $k = j$). The functions $V_0, \dots, V_n$ are called the *intrinsic volumes*. This name can be explained by the facts that $V_m(K)$ does not depend on the dimension of the ambient space (as will be clear from (4.22)) and that, for an m-dimensional convex body, $V_m(K)$ is its m-dimensional volume.

The functional V_0 is identically 1 on nonempty convex bodies. It is usually denoted by χ and called the *Euler characteristic*. This may seem strange at the present stage, but the reason becomes clear when this functional has been additively extended to finite unions of convex bodies (see Theorem 4.3.1).

From the support measures $\Theta_m(K, \cdot)$, which are defined on sets of support elements, we derive the two series of marginal measures, which are defined either on sets of boundary points or on sets of normal vectors. For $m = 0, \dots, n-1$ we put

$$C_m(K, \beta) := \Theta_m(K, \beta \times \mathbb{S}^{n-1}) \quad \text{for } \beta \in \mathcal{B}(\mathbb{R}^n), \tag{4.10}$$

$$S_m(K, \omega) := \Theta_m(K, \mathbb{R}^n \times \omega) \quad \text{for } \omega \in \mathcal{B}(\mathbb{S}^{n-1}). \tag{4.11}$$

These measures can replace the support measures if K is either smooth or strictly convex.

Lemma 4.2.2 *Let $K \in \mathcal{K}^n$, $m \in \{0, \dots, n-1\}$, and let $f : \Sigma \to \mathbb{R}$ be a nonnegative measurable function.*

If K is smooth, then

$$\int_\Sigma f(x, u)\, \Theta_m(K, \mathrm{d}(x, u)) = \int_{\mathbb{R}^n} f(x, u_K(x))\, C_m(K, \mathrm{d}x),$$

where $u_K(x)$ is the unique outer unit normal vector of K at x.

If K is strictly convex, then

$$\int_\Sigma f(x, u)\, \Theta_m(K, \mathrm{d}(x, u)) = \int_{\mathbb{S}^{n-1}} f(x_K(u), u)\, S_m(K, \mathrm{d}u),$$

where $x_K(u)$ is the unique boundary point of K at which u is attained as outer normal vector.

Proof If f is the characteristic function of a set $\eta \in \mathcal{B}(\Sigma)$, this follows immediately from the definitions and (4.7). The extension to nonnegative measurable functions is a standard procedure of integration theory. □

The measures $C_0(K, \cdot), \dots, C_{n-1}(K, \cdot)$ are called the *curvature measures* of K and $S_0(K, \cdot), \dots, S_{n-1}(K, \cdot)$ are called the *area measures* of K. When distinction is necessary, we call $C_m(K, \cdot)$ the curvature measure of order m, or the mth curvature measure, and similarly for the area measures. The area measure of order $n-1$ is also known as the *surface area measure*, because of its intuitive meaning expressed by (4.32) below. Observe that the curvature measures are Borel measures on $\mathbb{R}^n$, while the area measures are Borel measures on the unit sphere $\mathbb{S}^{n-1}$.

For the reader's convenience, we repeat the definition of these measures by specializing formula (4.5) to these cases. Writing

$$A_\rho(K,\beta) := \{x \in \mathbb{R}^n : 0 < d(K,x) \le \rho \text{ and } p(K,x) \in \beta\}$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $\rho > 0$, we have

$$\mathcal{H}^n(A_\rho(K,\beta)) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m} C_m(K,\beta), \tag{4.12}$$

and with

$$B_\rho(K,\omega) := \{x \in \mathbb{R}^n : 0 < d(K,x) \le \rho \text{ and } u(K,x) \in \omega\}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ we obtain

$$\mathcal{H}^n(B_\rho(K,\omega)) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m} S_m(K,\omega). \tag{4.13}$$

It is clear that the properties established for the support measures carry over to curvature measures and area measures. The mappings $K \mapsto C_m(K,\cdot)$ and $K \mapsto S_m(K,\cdot)$ are weakly continuous and additive; the function $C_m(\cdot,\beta)$ is measurable for each $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $S_m(\cdot,\omega)$ is measurable for each $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. The following invariance and homogeneity properties are easily obtained from (4.12) and (4.13). If g is a rigid motion of $\mathbb{R}^n$ and g_0 denotes the corresponding rotation, then

$$C_m(gK, g\beta) = C_m(K,\beta), \tag{4.14}$$

$$S_m(gK, g_0\omega) = S_m(K,\omega). \tag{4.15}$$

If $\lambda > 0$, then

$$C_m(\lambda K, \lambda\beta) = \lambda^m C_m(K,\beta), \tag{4.16}$$

$$S_m(\lambda K, \omega) = \lambda^m S_m(K,\omega). \tag{4.17}$$

The measure $C_m(K,\cdot)$ is concentrated on the boundary of K, since $A_\rho(K,\beta) = \emptyset$ if $\beta \cap \operatorname{bd} K = \emptyset$. It is *locally determined*, in the following sense. If $\beta \subset \mathbb{R}^n$ is open and if $K_1, K_2 \in \mathcal{K}^n$ are such that $K_1 \cap \beta = K_2 \cap \beta$, then $C_m(K_1,\beta') = C_m(K_2,\beta')$ for every Borel set $\beta' \subset \beta$. This follows from the observation that $p(K_i,x) \in \beta$ ($i = 1$ or 2) for an open set β, together with $K_1 \cap \beta = K_2 \cap \beta$, implies $p(K_1,x) = p(K_2,x)$, hence $A_\rho(K_1,\beta') = A_\rho(K_2,\beta')$ for $\beta' \subset \beta$. The assertion then follows from (4.12). Also, the measure $S_m(K,\cdot)$ is *locally determined*, which is now meant in the following sense. If $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ (not necessarily open in this case) and if $K_1, K_2 \in \mathcal{K}^n$ are such that $\tau(K_1,\omega) = \tau(K_2,\omega)$, then $S_m(K_1,\omega) = S_m(K_2,\omega)$. In fact, the local parallel set $B_\rho(K_i,\omega)$ depends only on the reverse spherical image $\tau(K_i,\omega)$.

Also for the support measures, curvature measures and area measures, different normalizations are in use. One defines $\Lambda_m(K,\cdot)$, $\Phi_m(K,\cdot)$ and $\Psi_m(K,\cdot)$ by

$$n\kappa_{n-m}\Lambda_m(K,\cdot) = \binom{n}{m}\Theta_m(K,\cdot), \tag{4.18}$$

$$n\kappa_{n-m}\Phi_m(K,\cdot) = \binom{n}{m}C_m(K,\cdot), \tag{4.19}$$

$$n\kappa_{n-m}\Psi_m(K,\cdot) = \binom{n}{m}S_m(K,\cdot) \tag{4.20}$$

for $m = 0,\dots,n-1$. It is reasonable to supplement the second definition by

$$\Phi_n(K,\beta) := \mathcal{H}^n(K\cap\beta) \quad \text{for } \beta\in\mathcal{B}(\mathbb{R}^n).$$

Neither of the two nomenclatures deserves unique preference over the other. In studies related to differential geometry, the C_m, S_m appear more natural, while in integral geometry, the Φ_m, Ψ_m yield slightly less clumsy formulae. As for the intrinsic volumes, one advantage of the second normalization is that $\Phi_m(K,\beta)$ depends only on K and β and not on the dimension of the ambient space in which it is computed. This follows from (4.22) below.

We consider special cases where the curvature and area measures can be given a more direct and intuitive interpretation. First let $P\in\mathcal{K}^n$ be a polytope. From (4.3) and (4.18), we get

$$\Lambda_m(P,\eta) = \frac{1}{\omega_{n-m}}\sum_{F\in\mathcal{F}_m(P)}\int_F\int_{N(P,F)\cap\mathbb{S}^{n-1}}\mathbf{1}_\eta(x,u)\,\mathrm{d}\mathcal{H}^{n-m-1}(u)\,\mathrm{d}\mathcal{H}^m(x) \tag{4.21}$$

for $\eta\in\mathcal{B}(\Sigma)$ and $m=0,\dots,n-1$. Using the definition of the external angle $\gamma(F,P)$ (see Section 2.4), we obtain

$$\Phi_m(P,\beta) = \sum_{F\in\mathcal{F}_m(P)}\gamma(F,P)\mathcal{H}^m(F\cap\beta) \tag{4.22}$$

for $\beta\in\mathcal{B}(\mathbb{R}^n)$ and $m=0,\dots,n$, in particular

$$V_m(P) = \sum_{F\in\mathcal{F}_m(P)}\gamma(F,P)\mathcal{H}^m(F). \tag{4.23}$$

Since $\gamma(F,P)$ depends only on P and F, as remarked in Section 2.4, we see that $\Phi_m(P,\beta)$ does not depend on the dimension of the ambient space; that is, it can be computed in any Euclidean space containing P. Similarly we get

$$\Psi_m(P,\omega) = \sum_{F\in\mathcal{F}_m(P)}\frac{\mathcal{H}^{n-1-m}(N(P,F)\cap\omega)}{\omega_{n-m}}\mathcal{H}^m(F) \tag{4.24}$$

for $\omega\in\mathcal{B}(\mathbb{S}^{n-1})$ and $m=0,\dots,n-1$.

If K is of class C^2_+, then a comparison of (2.63) with (4.12) and (4.13) shows that

$$C_m(K,\beta) = \int_{\beta\cap \mathrm{bd}\, K} H_{n-1-m}\, \mathrm{d}\mathcal{H}^{n-1}, \tag{4.25}$$

$$S_m(K,\omega) = \int_\omega s_m\, \mathrm{d}\mathcal{H}^{n-1} \tag{4.26}$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$, $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, $m = 0,\dots,n-1$ (first for open sets β,ω, but then for Borel sets, since both sides of (4.25) and (4.26) are measures in respectively β and ω).

Formula (4.25) is, of course, the reason for the name 'curvature measure'. These measures replace, for general convex bodies, the elementary symmetric functions of the principal curvatures that can be defined in the C^2 case. Similarly, the measures $S_m(K,\cdot)$ replace the elementary symmetric functions of the principal radii of curvature (as functions on the spherical image). The name 'area measures' for this series of measures comes from the facts that $S_{n-1}(K,\omega)$ is the area of the reverse spherical image of K at ω (see (4.32) below for the general case) and that $S_m(K,\omega)$ can be obtained from this by means of the Steiner-type formula

$$S_{n-1}(K_\rho,\cdot) = \sum_{m=0}^{n-1} \rho^{n-1-m}\binom{n-1}{m} S_m(K,\cdot), \tag{4.27}$$

which is a special case of (4.36).

We return to general convex bodies and note that (4.25) and (4.26) have a generalization in terms of elementary symmetric functions of the generalized principal curvatures on the normal bundle. A comparison of (4.5), (4.18) and (2.78) gives

$$\Lambda_m(K,\eta) = \frac{1}{\omega_{n-m}} \int_{\eta\cap \mathrm{Nor}\, K} \sum_{\substack{I\subset\{1,\dots,n-1\}\\ |I|=n-m-1}} \frac{\prod_{i\in I} k_i(x,u)}{\prod_{i=1}^{n-1} \sqrt{1+k_i(x,u)^2}}\, \mathcal{H}^{n-1}(\mathrm{d}(x,u)) \tag{4.28}$$

for $K \in \mathcal{K}^n$, $\eta \in \mathcal{B}(\Sigma)$ and $m = 0,\dots,n-1$. This representation is due to Zähle [2020]. Moreover, Zähle has defined $(n-1)$-forms φ_m (of class C^∞) on $\mathbb{R}^n \times \mathbb{R}^n$, $m = 0,\dots,n-1$, such that, for each $K \in \mathcal{K}^n$ and each $\eta \in \mathcal{B}(\Sigma)$, the value $\Lambda_m(K,\eta)$ is obtained by integrating the $(n-1)$-form $\mathbf{1}_\eta\varphi_m$ over the Lipschitz manifold Nor K. This representation of the support measures is known as the *current representation*.

Now we establish the intuitive meaning of the measures C_m, S_m in the extreme cases $m = 0$ and $n-1$, whereby we shall show that they are, in fact, Hausdorff measures of suitable sets.

Theorem 4.2.3 *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. Then*

$$C_0(K,\beta) = \mathcal{H}^{n-1}(\sigma(K,\beta)), \tag{4.29}$$

$$S_0(K,\omega) = \mathcal{H}^{n-1}(\omega). \tag{4.30}$$

If K is n-dimensional, then

$$C_{n-1}(K,\beta) = \mathcal{H}^{n-1}(\beta \cap \operatorname{bd} K), \tag{4.31}$$

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}(\tau(K,\omega)). \tag{4.32}$$

Proof We remark that (4.29) and (4.31) could be deduced from (4.28), using (2.76) to determine the approximate Jacobians of the mappings $(x,u) \mapsto x$ and $(x,u) \mapsto u$ on $\operatorname{Nor} K$. However, we give an independent and more elementary proof.

First we remark that the formulae are valid if K is a polytope, as follows from (4.22) and (4.24).

For $K \in \mathcal{K}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$ we define

$$\kappa(K,\beta) := \mathcal{H}^{n-1}(\sigma(K,\beta)).$$

By Lemma 2.2.13, $\sigma(K,\beta)$ is a Lebesgue measurable subset of $\mathbb{S}^{n-1}$. In the proof of that lemma it was shown that $\beta_1 \cap \beta_2 = \emptyset$ implies that

$$\mathcal{H}^{n-1}(\sigma(K,\beta_1) \cap \sigma(K,\beta_2)) = 0.$$

Since the restriction of $\mathcal{H}^{n-1}$ to the Lebesgue measurable subsets of $\mathbb{S}^{n-1}$ is σ-additive, it follows that $\kappa(K,\cdot)$ is a measure on $\mathcal{B}(\mathbb{R}^n)$.

Let $(K_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$ converging to a convex body K. We take an open set $\beta \subset \mathbb{R}^n$ and assume that $u \in \sigma(K,\beta) \cap \operatorname{regn} K$. There is a unique point $x \in F(K,u)$; it belongs to β. For $j \in \mathbb{N}$ choose $x_j \in F(K_j,u)$. Any accumulation point y of the sequence $(x_j)_{j\in\mathbb{N}}$ belongs to K and lies in $H(K,u)$, hence $y = x$. Thus $x_j \to x$ for $j \to \infty$ and, therefore, $x_j \in \beta$ and $u \in \sigma(K_j,\beta)$ for almost all j. This shows that

$$\sigma(K,\beta) \cap \operatorname{regn} K \subset \liminf_{j\to\infty} \sigma(K_j,\beta)$$

and in view of Theorem 2.2.11 we deduce that

$$\begin{aligned}\kappa(K,\beta) &= \mathcal{H}^{n-1}\left(\sigma(K,\beta) \cap \operatorname{regn} K\right) \le \mathcal{H}^{n-1}\left(\liminf_{j\to\infty} \sigma(K_j,\beta)\right)\\ &\le \liminf_{j\to\infty} \mathcal{H}^{n-1}\left(\sigma(K_j,\beta)\right) = \liminf_{j\to\infty} \kappa(K_j,\beta).\end{aligned}$$

Since this holds for all open sets $\beta \subset \mathbb{R}^n$ and since $\kappa(K_j,\mathbb{R}^n) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \kappa(K,\mathbb{R}^n)$, we have proved the weak convergence $\kappa(K_j,\cdot) \xrightarrow{w} \kappa(K,\cdot)$. Now the equality $\kappa(K,\cdot) = C_0(K,\cdot)$ follows immediately if K is approximated by a sequence of polytopes, since it is true for polytopes and both sides are weakly continuous. This proves (4.29).

Equality (4.30) holds since it is true for polytopes and $S_0(K,\cdot)$ depends weakly continuously on K.

For the proof of (4.31) we put

$$\eta(K,\beta) := \mathcal{H}^{n-1}(\beta \cap \operatorname{bd} K)$$

for $K \in \mathcal{K}^n_n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$. Then $\eta(K,\cdot)$ is a measure, and as above it suffices to prove that $K_j \to K$ (for $K_j, K \in \mathcal{K}^n_n$) implies $\eta(K_j,\cdot) \xrightarrow{w} \eta(K,\cdot)$.

We may suppose that $o \in \operatorname{int} K$. Define

$$\varphi : \mathbb{R}^n \setminus \{o\} \to \mathbb{S}^{n-1} \quad \text{by} \quad \varphi(x) := \frac{x}{|x|}.$$

Let $\rho(K, \cdot)$ be the radial function of K (see Section 1.7) so that, for $y \in \mathbb{S}^{n-1}$, the point $\rho(K, y)y$ is in the boundary of K. By $n(K, y)$ we denote an arbitrary outer unit normal vector of K at this point. Theorem 2.2.5 tells us that $n(K, y)$ is unique for $\mathcal{H}^{n-1}$-almost all $y \in \mathbb{S}^{n-1}$. We can now write $\eta(K, \beta)$ as the integral

$$\eta(K, \beta) = \int_{\varphi(\beta \cap \operatorname{bd} K)} \frac{\rho(K, y)^{n-1}}{\langle y, n(K, y)\rangle} \, d\mathcal{H}^{n-1}(y). \tag{4.33}$$

This can be proved with the aid of Theorem 3.2.3 in Federer [557]. If now the sequence $(K_j)_{j\in\mathbb{N}}$ of convex bodies converges to K, we may assume that $o \in \operatorname{int} K_j$ for all j. We have $\rho(K_j, \cdot) \to \rho(K, \cdot)$, $n(K_j, \cdot) \to n(K, \cdot)$ almost everywhere on $\mathbb{S}^{n-1}$, and the functions $\langle \cdot, n(K_j, \cdot)\rangle$ are bounded from below by a positive constant. Using Fatou's lemma and the bounded convergence theorem, it is now easy to see that $\eta(K_j, \cdot) \xrightarrow{w} \eta(K, \cdot)$ for $j \to \infty$. This completes the proof of (4.31).

For the proof of (4.32) we put

$$F(K, \omega) := \mathcal{H}^{n-1}(\tau(K, \omega))$$

for $K \in \mathcal{K}^n_n$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. Again we may assume that $o \in \operatorname{int} K$. From Lemma 2.2.14 we know that $\varphi(\tau(K, \omega))$ is Lebesgue measurable on $\mathbb{S}^{n-1}$. If $\omega_1 \cap \omega_2 = \emptyset$, then

$$\mathcal{H}^{n-1}(\tau(K, \omega_1) \cap \tau(K, \omega_2)) = 0,$$

as a consequence of Theorem 2.2.4. We deduce that $F(K, \cdot)$ is a measure on $\mathcal{B}(\mathbb{S}^{n-1})$, and from (4.33) we have

$$F(K, \omega) = \int_{\varphi(\tau(K,\omega))} \frac{\rho(K, y)^{n-1}}{\langle y, n(K, y)\rangle} \, d\mathcal{H}^{n-1}(y).$$

As above, this can be used to show that $K_j \to K$ (with $K \in \mathcal{K}^n_n$) implies $F(K_j, \cdot) \xrightarrow{w} F(K, \cdot)$ for $j \to \infty$. Using this, (4.32) is extended from polytopes to general convex bodies. $\square$

It is evident that the need to introduce two distinct series of measures is due to the fact that a convex body can have singular boundary points and singular normal vectors. The following lemma expresses one aspect under which both types of measures are related to each other.

Lemma 4.2.4 *Let $m \in \{0, \ldots, n-1\}$, let $\omega \subset \mathbb{S}^{n-1}$ and $\beta \subset \mathbb{R}^n$ be closed. Then*

$$C_m(K, \tau(K, \omega) \cap \operatorname{reg} K) \le S_m(K, \omega) \le C_m(K, \tau(K, \omega)),$$
$$S_m(K, \sigma(K, \beta) \cap \operatorname{regn} K) \le C_m(K, \beta) \le S_m(K, \sigma(K, \beta)).$$

Proof Let $(x, u) \in \operatorname{Nor} K$ be a support element of K for which $x \in \tau(K, \omega) \cap \operatorname{reg} K$; then $u \in N(K, x)$. Since $x \in \tau(K, \omega)$, there is a normal vector of K at x belonging to ω, and this is equal to u since $x \in \operatorname{reg} K$. Thus $u \in \omega$. This shows that

$$([\tau(K, \omega) \cap \operatorname{reg}\ K] \times \mathbb{S}^{n-1}) \cap \operatorname{Nor} K \subset (\mathbb{R}^n \times \omega) \cap \operatorname{Nor} K.$$

The sets occurring here are Borel sets, since $\tau(K, \omega)$ and $\operatorname{Nor} K$ are closed and bd $K \setminus \operatorname{reg} K$ is an F_σ set. We deduce that

$$\Theta_m(K, ([\tau(K, \omega) \cap \operatorname{reg} K] \times \mathbb{S}^{n-1}) \cap \operatorname{Nor} K) \leq \Theta_m(K, (\mathbb{R}^n \times \omega) \cap \operatorname{Nor} K)$$

and hence, using (4.7), that

$$\begin{aligned} C_m(K, \tau(K, \omega) \cap \operatorname{reg} K) &= \Theta_m(K, ([\tau(K, \omega) \cap \operatorname{reg} K] \times \mathbb{S}^{n-1}) \cap \operatorname{Nor} K) \\ &\leq \Theta_m(K, (\mathbb{R}^n \times \omega) \cap \operatorname{Nor} K) = S_m(K, \omega). \end{aligned}$$

Let $(x, u) \in \operatorname{Nor} K$ be a support element of K for which $u \in \omega$; then $x \in \tau(K, \omega)$. Thus,

$$(\mathbb{R}^n \times \omega) \cap \operatorname{Nor} K \subset [\tau(K, \omega) \times \mathbb{S}^{n-1}] \cap \operatorname{Nor} K$$

and hence

$$\begin{aligned} S_m(K, \omega) &= \Theta_m(K, (\mathbb{R}^n \times \omega) \cap \operatorname{Nor} K) \\ &\leq \Theta_m(K, [\tau(K, \omega) \times \mathbb{S}^{n-1}] \cap \operatorname{Nor} K) = C_m(K, \tau(K, \omega)). \end{aligned}$$

This proves the first two inequalities of the lemma. The proof of the last two is completely analogous. □

If $K \in \mathcal{K}^n$ is smooth, then the spherical image map σ_K is defined on all of bd K, and $\tau(K, \omega) = \sigma_K^{-1}(\omega)$ for $\omega \subset \mathbb{S}^{n-1}$. If ω is a Borel set, this shows that $\tau(K, \omega)$ is a Borel set, since σ_K is continuous. From Lemma 4.2.4 we obtain, first for closed sets and then for general sets $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$,

$$C_m(K, \sigma_K^{-1}(\omega)) = C_m(K, \tau(K, \omega)) = S_m(K, \omega).$$

An analogous result holds for strictly convex bodies, hence we can state the following.

Theorem 4.2.5 *If $K \in \mathcal{K}^n$ is smooth, then $S_m(K, \cdot)$ is the image measure of $C_m(K, \cdot)$ under the spherical image map u_K. If K is strictly convex, then $C_m(K, \cdot)$ is the image measure of $S_m(K, \cdot)$ under the reverse spherical image map x_K* $(m = 0, \ldots, n-1)$.

Thus, for convex bodies K that are both smooth and strictly convex there is in fact no essential difference between the two types of measures, since

$$C_m(K, \beta) = S_m(K, u_K(\beta \cap \operatorname{bd}\ K)) \quad \text{for } \beta \in \mathcal{B}(\mathbb{R}^n).$$

If the body $K \in \mathcal{K}^n$ is of dimension less than n, it is easy to see (e.g., from (4.12), (4.13)) how the equations (4.31), (4.32) have to be modified. If $\dim K = n-1$, then

$$C_{n-1}(K, \beta) = \mathcal{H}^{n-1}(\beta \cap K)$$

and

$$S_{n-1}(K,\omega) = 2\mathcal{H}^{n-1}(K),\ \ \mathcal{H}^{n-1}(K),\ \text{or } 0,$$

according to whether both, one or none of the unit normal vectors of the affine hull of K belong to ω. If $\dim K < n-1$, then $C_{n-1}(K,\beta) = 0 = S_{n-1}(K,\omega)$ (so that (4.31), (4.32) are true again).

In Chapters 9 and 10, for bodies $K \in \mathcal{K}^n_{(o)}$, measures μ of the form

$$\mu(K,\cdot)(\omega) = \int_\omega G \circ h_K \, \mathrm{d}S_{n-1}(K,\cdot), \qquad \omega \in \mathcal{B}(\mathbb{S}^{n-1}), \tag{4.34}$$

with a continuous function $G : (0,\infty) \to \mathbb{R}$ will occur. We show that they define valuations.

Lemma 4.2.6 *If $\mu(K,\cdot)$ is defined by (4.34) and if $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n_{(o)}$, then*

$$\mu(K_1 \cup K_2,\cdot) + \mu(K_1 \cap K_2,\cdot) = \mu(K_1,\cdot) + \mu(K_2,\cdot).$$

Proof Let $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n_{(o)}$. Using intersections of supporting halfspaces, it is not difficult to construct polytopes $P_{1,k}, P_{2,k} \in \mathcal{K}^n_{(o)}$ such that $P_{1,k} \to K_1$, $P_{2,k} \to K_2$ for $k \to \infty$ and such that $P_{1,k} \cup P_{2,k}$ is convex for each k. Since the surface area measure is weakly continuous, it therefore suffices to prove the assertion for the case where K_1, K_2 are polytopes, what we now assume. Let $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, and let $u_1, \dots, u_m$ be the outer unit normal vectors of K_1 and K_2 that are contained in ω. Then

$$\mu(K,\omega) = \sum_{i=1}^m G(h(K,u_i))V_{n-1}(F(K,u_i))$$

for $K \in \{K_1, K_2, K_1 \cup K_2, K_1 \cap K_2\}$. Therefore, the assertion follows if we prove that

$$\begin{aligned} &G(h(K_1 \cup K_2,u_i))V_{n-1}(F(K_1 \cup K_2,u_i)) \\ &+ G(h(K_1 \cap K_2,u_i))V_{n-1}((F(K_1 \cap K_2,u_i)) \\ &= G(h(K_1,u_i))V_{n-1}(F(K_1,u_i)) + G(h(K_2,u_i))V_{n-1}(F(K_2,u_i)) \end{aligned}$$

for each $i \in \{1,\dots,m\}$. If $h(K_1,u_i) = h(K_2,u_i) =: h_i$, then

$$h(K_1 \cup K_2,u_i) = h(K_1 \cap K_2,u_i) = h_i$$

and

$$V_{n-1}(F(K_1 \cup K_2,u_i)) + V_{n-1}(F(K_1 \cap K_2,u_i)) = V_{n-1}(F(K_1,u_i)) + V_{n-1}(F(K_2,u_i)).$$

If, say, $h(K_1,u_i) < h(K_2,u_i)$, then

$$h(K_1 \cup K_2,u_i) = h(K_2,u_i), \quad h(K_1 \cap K_2,u_i) = h(K_1,u_i)$$

and

$$F(K_1 \cup K_2,u_i) = F(K_2,u_i), \quad F(K_1 \cap K_2,u_i) = F(K_1,u_i).$$

The assertion follows. $\square$

Finally, we return to our starting point, polynomial expansions of Steiner type, and remark that the existence of a polynomial expansion of type (4.5) carries over to the support measures. For $\rho > 0$ we define a map $t_\rho : \Sigma \to \Sigma$ by $t_\rho(x, u) := (x + \rho u, u)$. Then we have:

Theorem 4.2.7 *Let $K \in \mathcal{K}^n$, $\eta \in \mathcal{B}(\Sigma)$, $\rho > 0$ and $m \in \{0, \dots, n-1\}$. Then*

$$\Theta_m(K_\rho, t_\rho \eta) = \sum_{j=0}^{m} \rho^j \binom{m}{j} \Theta_{m-j}(K, \eta). \tag{4.35}$$

Proof Let $x \in \mathbb{R}^n \setminus K_\rho$. There is a unique point $y \in [x, p(K, x)] \cap \operatorname{bd} K_\rho$. Assume that $p(K_\rho, x) \neq y$. Then $|x - p(K_\rho, x)| < |x - y|$. Let $p(K, p(K_\rho, x)) =: z$. From $p(K_\rho, x) \in \operatorname{bd} K_\rho$ and $y \in \operatorname{bd} K_\rho$ it follows that

$$|p(K_\rho, x) - z| = \rho \leq |y - p(K, x)|$$

and hence

$$|x - z| \leq |x - p(K_\rho, x)| + |p(K_\rho, x) - z| < |x - y| + |y - p(K, x)| = |x - p(K, x)|,$$

a contradiction. Hence $p(K_\rho, x) = y$. This implies that $u(K_\rho, x) = u(K, x)$, $d(K_\rho, x) = d(K, x) - \rho$ and $p(K_\rho, x) = p(K, x) + \rho u(K, x)$. Now we obtain the disjoint decomposition

$$M_{\rho+\lambda}(K, \eta) = M_\rho(K, \eta) \cup M_\lambda(K_\rho, t_\rho \eta)$$

for $\lambda > 0$ and hence the equality

$$\mu_{\rho+\lambda}(K, \eta) = \mu_\rho(K, \eta) + \mu_\lambda(K_\rho, t_\rho \eta).$$

Here we insert (4.5) and compare the coefficients of equal powers of λ to obtain the assertion (4.35). □

For area measures, formula (4.35) takes the simple form

$$S_m(K_\rho, \omega) = \sum_{j=0}^{m} \rho^j \binom{m}{j} S_{m-j}(K, \omega) \tag{4.36}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and $\rho > 0$.

Using (4.18), we write the case $m = n - 1$ of (4.35) in the form

$$2\Lambda_{n-1}(K_\rho, \cdot) = \sum_{k=0}^{n-1} \rho^{n-k-1} \omega_{n-k}\, t_\rho \Lambda_k(K, \cdot) \tag{4.37}$$

(where $t_\rho \Lambda_k(K, \cdot)$ denotes the image measure of $\Lambda_k(K, \cdot)$ under t_ρ), to derive the following general version of the local Steiner formula.

Theorem 4.2.8 *Let $K \in \mathcal{K}^n$, and let $f : \mathbb{R}^n \setminus K \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^n \setminus K} f(x)\, \mathrm{d}x = \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\infty \rho^{n-j-1} \int_\Sigma f(x + \rho u)\, \Lambda_j(K, \mathrm{d}(x, u))\, \mathrm{d}\rho. \tag{4.38}$$

Proof First we consider a polytope P and assume that f is continuous with compact support. The disjoint decomposition

$$\mathbb{R}^n \setminus P = \bigcup_{j=0}^{n-1} \bigcup_{F \in \mathcal{F}_j(P)} \{x \in \mathbb{R}^n \setminus P : p(P, x) \in \operatorname{relint} F\},$$

together with $A(\rho, F) := \{x \in \operatorname{bd} P_\rho : p(P, x) \in \operatorname{relint} F\}$, gives

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus P} f(x)\,\mathrm{d}x &= \int_0^\infty \sum_{j=0}^{n-1} \sum_{F \in \mathcal{F}_j(P)} \int_{A(\rho,F)} f(y)\,\mathcal{H}^{n-1}(\mathrm{d}y)\,\mathrm{d}\rho \\
&= 2 \int_0^\infty \int_\Sigma f(y)\,\Lambda_{n-1}(P_\rho, d(y, u))\,\mathrm{d}\rho \\
&= \int_0^\infty \int_\Sigma \sum_{j=0}^{n-1} \rho^{n-j-1} \omega_{n-j} f(x + \rho u)\,\Lambda_j(P, \mathrm{d}(x, u))\,\mathrm{d}\rho,
\end{aligned}$$

where (4.37) and the transformation formula for integrals were used. By the weak continuity of the support measures, the result extends to general convex bodies K, and by standard measure-theoretic arguments, it further extends to nonnegative measurable functions f. □

Notes for Section 4.2

1. *The Steiner formula.* The Steiner formula goes back to Steiner [1818], who derived it in $\mathbb{R}^2$ and $\mathbb{R}^3$, for polytopes and surfaces of class C^2_+. Nowadays, this formula is just a special case of the more general expansion formula leading to the mixed volumes (see Chapter 5).

   For the volume of the 'tube' around a smooth submanifold of Euclidean or spherical space, Weyl [1972] derived a polynomial expansion, now called 'Weyl's tube formula'. The essential point here is that the coefficients can be expressed in terms of the Riemann curvature tensor of the submanifold and thus depend only on its intrinsic metric. The extension of Weyl's tube formula to general Riemannian spaces was studied by Gray and Vanhecke [769]; see also Gray [768].

   Counterparts to the original Steiner formula in spaces of constant curvature were treated by Herglotz [972], Vidal Abascal [1870], Allendoerfer [60], Santaló [1627]. Hadwiger [887, 889, 890, 891] used integral-geometric methods to obtain Steiner-type formulae for certain non-convex domains also. Other analogues, partly in the form of inequalities, are due to Hadwiger [892], Ohmann [1481], v. Sz.-Nagy [1835], Makai [1320], Meyer [1418].

   Under the name of 'weighted parallel volumes', Kampf [1063] studied functionals of the form $K \mapsto \int V_n(K + \lambda B^n)\,\mathrm{d}\rho(\lambda)$, where ρ is a suitable signed measure with support in $\mathbb{R}_{\geq 0}$.

   A certain generalization of outer parallel bodies was also studied by Sangwine-Yager [1624].
2. *Local parallel sets and sets of positive reach.* Local parallel sets, of the type denoted above by $B_\rho(K, \omega)$, appeared first in the beautiful paper by Fenchel and Jessen [572]. There (§7) they were called brush sets ('Bürstenmengen') and used to give an intuitive interpretation of the surface area measure, after that had been introduced via the theory of mixed volumes and approximation. Also the local Steiner formula (4.13) appears there.

   A natural common generalization of convex sets and of smooth submanifolds, for which local parallel sets can be defined in such a way that their measure has a polynomial expansion, yielding a Steiner formula, are the sets of positive reach. A compact set $A \subset \mathbb{R}^n$ is said to be *of positive reach* if there exists a number $\rho > 0$ such that, for each point $x \in \mathbb{R}^n$ whose distance from A is less than ρ, there is a unique point in A that is nearest to x. For sets A of positive reach, Federer [556] introduced the curvature measures $\Phi_m(A, \cdot)$. He made an extensive study and proved many of the relevant results, which in the present book are considered only for convex bodies.

The simpler introduction of the curvature measures for the case of convex bodies and the derivation of their properties, as presented here, is taken from Schneider [1687]. The generalized curvature measures, defined on sets of support elements and later called support measures, appear in Schneider [1691].

3. *A local Steiner-type formula for closed sets.* Stachó [1804] constructed generalizations of Federer's curvature measures to arbitrary sets (with weaker properties). This was continued by Hug, Last and Weil [1010], who used a local Steiner-type formula to introduce generalizations of the support measures for arbitrary closed sets in $\mathbb{R}^n$. They gave explicit descriptions of these support measures in terms of generalized principal curvatures on a normal bundle and the $(n-1)$-dimensional Hausdorff measure on this normal bundle. They were able to prove enough analytical properties of these support measures to make them applicable to the investigation of contact distributions in stochastic geometry.
4. *Polynomial parallel volume and convexity.* Heveling, Hug and Last [971] reversed Steiner's observation and asked whether polynomial parallel volume characterizes convexity. More precisely, let $\emptyset \neq A \subset \mathbb{R}^n$ be a compact set and assume that $V_n(A + rB^d)$ is a polynomial in $r \geq 0$. Must A be convex? The authors gave a positive answer for $n = 2$ and found counterexamples for $n \geq 3$. However, they found a way to characterize convexity in all dimensions by a suitable version of polynomiality for local parallel volumes. Applications to stochastic geometry appear in Hug, Last and Weil [1011]. For compact sets $\emptyset \neq A \subset \mathbb{R}^2$, Kampf [1064] showed that

$$\lim_{r\to\infty}[V_2((\operatorname{conv} A) + rB^2) - V_2(A + rB^2)] = 0$$

and used this to give a new proof of the two-dimensional result of Heveling, Hug and Last mentioned above.

We mention that the surface area (in the Minkowski sense) of the parallel sets of nonempty compact subsets of $\mathbb{R}^n$ was studied by Kneser [1123].
5. *Roots of the Steiner polynomial.* In view of the importance of the Steiner polynomial, it is a natural question to ask whether the complex roots of the equation

$$\sum_{i=0}^{n} \binom{n}{i} W_i(K)\lambda^i = 0 \tag{4.39}$$

bear geometric significance on the convex body K. If they were all real (as is the case for the cube; see Katsnelson [1068]), this would give a simple proof of the quadratic inequalities for the quermassintegrals; but this does not hold in general. Motivated by an analogy in algebraic geometry, Teissier [1842] (p. 87) asked whether inradius and circumradius of a convex body can be estimated in terms of its quermassintegrals, and Oda [1476] (p. 188) asked whether the roots of (4.39) can be used in such an estimate. For the analogue for algebraic varieties, Teissier posed a more specific question (p. 101), which was later formulated by Sangwine-Yager [1619, 1622] as a conjecture about convex bodies: do the real parts $a_1 \leq \cdots \leq a_n$ of the roots of (4.39) and the inradius $r(K)$ and circumradius $R(K)$ of K satisfy $a_1 \leq -R(K) \leq -r(K) \leq a_n \leq 0$? (In fact, the question was posed more generally, with the unit ball replaced by some other convex gauge body.) This is true for $n = 2$, as a consequence of an inequality due to Bonnesen (see Section 7.2, Note 4). However, Henk and Hernández Cifre [956] found a convex body K for which (4.39) has a root with positive real part, and another convex body K for which the real parts of all the roots of (4.39) are less than $-r(K)$. The paper [956] contains some estimates for inradius and circumradius in terms of the roots of the Steiner polynomial. Before that, Hernández Cifre and Saorín [965] had made a systematic study of the roots of

(4.39) for three-dimensional convex bodies and their relation to the Blaschke diagram. A general study of the location of the roots of the Steiner polynomial is made in Henk and Hernández Cifre [957], again in a relative setting, with a general gauge body.

For convex bodies of class C^2, results on the location of the roots of the Steiner polynomial in relation to the minimal and maximal radii of curvature were found by Green and Osher [772] for $n = 2$ and by Jetter [1039] in some higher dimensions.

Root location problems for Steiner and Weyl polynomials (coming from Weyl's tube formula) are further systematically investigated by Katsnelson [1068].

6. *Alternating Steiner polynomial and inner parallel bodies.* A special case of (4.36) is the system of Steiner formulae for the quermassintegrals, that is,

$$W_q(K + \rho B^n) = \sum_{i=0}^{n-q} \binom{n-q}{i} W_{q+i}(K)\rho^i$$

for $K \in \mathcal{K}^n$, $\rho \geq 0$ and $q = 0, \ldots, n$. If B^n is a summand of K, then it is not difficult to see that

$$W_q(K \div \rho B^n) = \sum_{i=0}^{n-q} \binom{n-q}{i} W_{q+i}(K)(-\rho)^i \tag{4.40}$$

for $0 \leq \rho \leq 1$ and for $q = 0, \ldots, n$. This was pointed out by Matheron [1361] (in a more general form, with B^n replaced by an arbitrary convex gauge body E and with the quermassintegrals replaced by the corresponding mixed volumes). He showed, conversely, that the validity of (4.40) for $\rho \in (0, 1)$ and $q = 0, \ldots, n$ implies that B^n is a summand of K. He conjectured that this is already true if (4.40) holds for $\rho \in (0, 1)$ and $q = 0$ (the case of the volume). More generally, he conjectured (with a general gauge body instead of B^n) that, for $K \in \mathcal{K}^n$ and $0 \leq \rho < r(K)$ (the inradius of K),

$$V_n(K \div \rho B^n) \geq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i,$$

with equality if and only if B^n is a summand of K. He proved this for $n = 2$. However, Sangwine-Yager [1618] gave a three-dimensional counterexample. More general results were obtained by Hernández Cifre and Saorín [967]. They defined classes $\mathcal{R}_p$ of convex bodies ($p = 0, \ldots, n-1$) according to the differentiability properties of the quermassintegrals with respect to the parameter of the inner parallel bodies, and showed, for instance, the following. Let n be odd. If $K \in \mathcal{K}^n$ belongs to the class $\mathcal{R}_{n-2}$, then, for $0 \leq \rho < r(K)$,

$$V_n(K \div \rho B^n) \leq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.$$

Equality holds if and only if $K \in \mathcal{R}_{n-1}$. For even-dimensional 1-tangential bodies of a ball, the latter inequality holds with the reverse inequality sign.

7. *An Euler-type version of the local Steiner formula.* For the local Steiner formula in the form of Theorem 4.2.8, Glasauer [720] has proved the following counterpart of Euler type.

$$\begin{aligned}&(1-(-1)^n)\int_{\mathbb{R}^n} f\,\mathrm{d}\mathcal{H}^n\\ &= \sum_{m=0}^{n-1}(-1)^m\binom{n-1}{m}\int_0^\infty t^{n-m-1}\int_\Sigma f(x-tu)\,\Theta_m(K,\mathrm{d}(x,u))\,\mathrm{d}t,\end{aligned}$$

for all integrable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which each term in the sum is finite. Special cases had been treated before, by Hann [936] and Hug [1001], with more elaborate methods.

8. *A volume representation.* The following formula shares with the previous one that it involves an alternating sum of integrals with respect to curvature measures. For a convex body $K \in \mathcal{K}_n^n$, define the *interior reach* of K at its boundary point x by

$$r(K, x) := \sup\{\lambda \geq 0 : x \in B(z, \lambda) \subset K \text{ for some } z \in K\}.$$

Sangwine-Yager [1623] proved that

$$V_n(K) = \frac{1}{n} \sum_{m=0}^{n-1} \binom{n}{m} (-1)^{n-m+1} \int_{\mathrm{bd}\, K} r(K, x)^{n-1}\, C_m(K, \mathrm{d}x).$$

9. *Representation of curvature measures by generalized curvatures.* The integral representation (4.28) of the support measures in terms of generalized curvatures on the normal bundle was established by Zähle [2020] in a more general form, namely for sets of positive reach. Also the current representation was obtained in this generality. This work was continued in Zähle [2021].
10. Arguments leading to special cases of Theorem 4.2.3 can be found at several places in the work of Aleksandrov [12], §1, [18], [19], §6, [23], Chapter V, §2.
11. *Axiomatic characterizations of curvature measures and area measures*. Linear combinations of curvature measures or of area measures can be characterized by some of their properties, in a manner analogous to Hadwiger's characterization theorem for the intrinsic volumes (Theorem 6.4.14). Although Hadwiger's theorem was not proved in this section, we mention the characterizations of curvature measures here, since their proofs are of a different character.

Theorem Let φ be a map from $\mathcal{K}^n$ into the set of finite (nonnegative) Borel measures on $\mathbb{R}^n$ satisfying the following conditions (where $\varphi(K)(\beta)$ is written as $\varphi(K, \beta)$).

(a) φ is rigid motion equivariant, that is, $\varphi(gK, g\beta) = \varphi(K, \beta)$ for $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and every rigid motion g of $\mathbb{R}^n$;
(b) φ is additive;
(c) φ is weakly continuous;
(d) φ is locally determined, which means: if $\beta \subset \mathbb{R}^n$ is open and $K \cap \beta = L \cap \beta$, then $\varphi(K, \beta') = \varphi(L, \beta')$ for every Borel set $\beta' \subset \beta$.

Under these assumptions, there are nonnegative real constants $c_0, \dots, c_n$ such that

$$\varphi(K, \beta) = \sum_{i=0}^{n} c_i \Phi_i(K, \beta)$$

for $K \in \mathcal{K}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$.

This theorem was proved by Schneider [1687]. It gives a partial answer to a question of Federer [556], Remark 5.17, who had asked for a characterization, similar to Hadwiger's characterization theorem for intrinsic volumes, of his curvature measures on sets of positive reach. A similar result for area measures was obtained by Schneider [1679].

Theorem Let ψ be a map from $\mathcal{K}^n$ into the set of finite signed Borel measures on $\mathbb{S}^{n-1}$ satisfying the following conditions (where $\psi(K)(\omega)$ is written as $\psi(K, \omega)$).

(a) ψ is rigid motion equivariant, that is $\psi(gK, g_0\omega) = \psi(K, \omega)$ for $K \in \mathcal{K}^n$, $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and every rigid motion g of $\mathbb{R}^n$, where g_0 denotes the rotation corresponding to g;
(b) ψ is additive;
(c) ψ is weakly continuous;
(d) ψ is locally determined, which means: if $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and $\tau(K, \omega) = \tau(L, \omega)$, then $\psi(K, \omega) = \psi(L, \omega)$.

Under these assumptions, there are real constants $c_0, \dots, c_{n-1}$ such that

$$\psi(K, \omega) = \sum_{i=0}^{n-1} c_i \Psi_i(K, \omega)$$

for $K \in \mathcal{K}^n$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$.

Generalizations of these theorems to certain classes of non-convex sets were proved by Zähle [2026].

12. *Axiomatic characterizations of support measures*. For the support measures, a characterization is possible which does not assume the valuation property. The following was proved by Glasauer [718].

Theorem Let ψ be a map from $\mathcal{P}^n$ into the set of finite signed Borel measures on Σ satisfying the following conditions (where $\psi(P)(\eta)$ is written as $\psi(P, \eta)$).

(a) $\psi(gP, g\eta) = \psi(P, \eta)$ for $P \in \mathcal{P}^n$, $\eta \in \mathcal{B}(\Sigma)$ and every rigid motion g of $\mathbb{R}^n$, where $g\eta := \{(gx, g_0 u) : (x, u) \in \eta\}$ and g_0 denotes the rotation corresponding to g;
(b) ψ is locally determined, which means: if $\eta \in \mathcal{B}(\Sigma)$ and $P, P' \in \mathcal{P}^n$ satisfy $\eta \cap \operatorname{Nor} P = \eta \cap \operatorname{Nor} P'$, then $\psi(P, \eta) = \psi(P', \eta)$.

Under these assumptions, there are real constants $c_0, \dots, c_{n-1}$ such that

$$\psi(P, \eta) = \sum_{j=0}^{n-1} c_j \Lambda_j(P, \eta)$$

for $P \in \mathcal{P}^n$ and $\eta \in \mathcal{B}(\Sigma)$.

Under additional continuity assumptions, $\mathcal{P}^n$ can be replaced by $\mathcal{K}^n$ in this characterization.

13. *Flag measures.* A natural generalization of the support measures are the flag measures, which take touching flats into account. Let $K \in \mathcal{K}^n$ and $k \in \{0, \dots, n-1\}$. Let $A(n, k, K)$ be the (Borel) set of flats $E \in A(n, k)$ with $E \cap K = \emptyset$ for which the pair $p(K, E) \in K$ and $l(K, E) \in E$ of nearest points is unique. Define

$$u(K, E) := \frac{l(K, E) - p(K, E)}{d(K, E)},$$

where $d(K, E) := |l(K, E) - p(K, E)|$. If $E \in A(n, k)$, we denote by $L(E) \in G(n, k)$ the parallel linear subspace. For $\rho > 0$ and $\eta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1} \times G(n, k))$, the local parallel set

$$M_\rho^{(k)}(K, \eta) := \{E \in A(n, k, K) : 0 < d(K, E) \le \rho,\ (p(K, E), u(K, E), L(E)) \in \eta\}$$

is a Borel set in $A(n, k)$, and for its Haar measure a local Steiner formula

$$\mu_k(M_\rho^{(k)}(K, \eta)) = \sum_{m=0}^{n-k-1} \rho^{n-k-m} \kappa_{n-k-m}\, \Xi_m^{(k)}(K, \cdot)$$

holds. This defines finite Borel measures $\Xi_0^{(k)}(K, \cdot), \dots, \Xi_{n-k-1}^{(k)}(K, \cdot)$ on $\mathbb{R}^n \times \mathbb{S}^{n-1} \times G(n, k)$, which are the *flag measures* of order k of the convex body K.

The flag measures were first systematically investigated by Kropp [1148] and Hinderer [978]. A thorough presentation is given by Hug, Türk and Weil [1022]; to this we refer for further historical hints. Applications to translation invariant valuations appear in Hinderer, Hug and Weil [979], and to projection functions in Goodey, Hinderer, Hug, Rataj and Weil [738].

14. *Hermitian curvature and area measures.* Within the modern theory of valuations and its applications to integral geometry in Hermitian and other spaces (which are far outside the scope of this book), Hermitian versions of curvature measures appear in the work of

Bernig, Fu and Solanes [212], and Hermitian versions of area measures in the work of Wannerer [1924].

15. *Hessian measures of convex functions.* In analogy to support measures of convex bodies, one can introduce Hessian measures of convex functions. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, convex set and $f : \Omega \to \mathbb{R}$ a convex Lipschitz function. Let

$$\Gamma(f) := \{(x, v) \in \Omega \times \mathbb{R}^n : v \in \partial f(x)\},$$

where $\partial f(x)$ is the subdifferential of f at x. For $\rho \geq 0$ and $\eta \in \mathcal{B}(\Omega \times \mathbb{R}^n)$, define a local parallel set by

$$P_\rho(f, \eta) := \{x + \rho v : (x, v) \in \eta \cap \Gamma(f)\}.$$

In analogy to (4.5), one has a polynomial expansion

$$\mathcal{H}^n(P_\rho(f, \eta)) = \sum_{m=0}^{n} \rho^m \binom{n}{m} \Theta_m(f, \eta),$$

and this defines finite Borel measures $\Theta_0(f, \cdot), \dots, \Theta_n(f, \cdot)$ on $\Omega \times \mathbb{R}^n$, the *Hessian measures* of f. The special cases $F_m(f, \alpha) := \Theta_m(f, \alpha \times \mathbb{R}^n)$ with $\alpha \in \mathcal{B}(\Omega)$ were introduced in this way by Colesanti [434] (and in a different way before by Trudinger and Wang [1853]). In particular, $F_0(f, \alpha) = \mathcal{H}^n(\alpha)$, and $F_n(f, \alpha)$ is the Lebesgue measure of the image of α under the subgradient map of f. If f is smooth of class C^2, then $F_m(f, \alpha \times \mathbb{R}^n)$ is the integral over α of the mth normalized elementary symmetric function of the eigenvalues of the Hessian matrix of f. Colesanti proved an upper estimate for the value of $F_m(f, \cdot)$ at a sublevel set of f in terms of a quermassintegral of the sublevel set and the Lipschitz constant of f.

This investigation was continued and generalized by Colesanti and Hug [442, 443, 444]. In [442], the Hessian measures $\Theta(f, \cdot)$ are introduced and an analytic representation for them is obtained in the case of a general convex function. An application of the Hessian measures estimates the singularities of convex functions. For a convex Lipschitz function f on a bounded open convex set Ω, let $\Sigma^r(f) := \{x \in \Omega : \dim \partial f(x) \geq n - r\}$ for $r \in \{0, \dots, n\}$. It is proved that

$$\int_{\Sigma^r(f)} \mathcal{H}^{n-r}(\partial f(x))\, \mathrm{d}\mathcal{H}^r(x) \leq \binom{n}{r} L^{n-r} W_{n-r}(\mathrm{cl}\, \Omega),$$

where L is the Lipschitz constant of f. The bound is sharp.

The paper [443] studies Hessian measures for semi-convex functions and treats, in particular, a Crofton theorem, absolute continuity properties and Radon–Nikodym derivatives, a duality theorem, and relations to curvature and area measures of convex bodies. An application is, for example, Theorem 4.5.6 below. In [444], geometric characterizations of the support of the Hessian measures are provided, and the Radon–Nikodym derivative and absolute continuity of Hessian measures with respect to Lebesgue measure are investigated in greater detail.

16. For some surprising uses of intrinsic volumes in connection with Gaussian processes, including certain infinite-dimensional versions of intrinsic volumes, we refer to Vitale [1891, 1893], Gao and Vitale [669] and the references given there.

4.3 Extensions of support measures

It may be desirable from various points of view to extend the support measures or the curvature measures from convex bodies to more general sets. In this book, we want to stay close to convexity and treat, therefore, only one such extension, namely to finite unions of convex bodies. The extension will be made in such a way that the additivity of the mapping $K \mapsto \Theta_m(K, \cdot)$ is preserved; for that, we have to sacrifice positivity. We prepare the extension with some remarks about additive maps, or valuations (which are more thoroughly dealt with in Chapter 6).

A function φ defined on a family $\mathscr{S}$ of sets and with values in an abelian group is called *additive* or a *valuation* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

whenever $K, L, K \cup L, K \cap L \in \mathscr{S}$. If $\emptyset \in \mathscr{S}$, we moreover assume that $\varphi(\emptyset) = 0$.

The family $\mathscr{S}$ is called *intersectional* if $K, L \in \mathscr{S}$ implies $K \cap L \in \mathscr{S}$. If $\mathscr{S}$ is intersectional, we denote by $U(\mathscr{S})$ the lattice consisting of all finite unions of elements of $\mathscr{S}$. It is a question of general interest whether a valuation φ on an intersectional family $\mathscr{S}$ can be extended, as a valuation, to the lattice $U(\mathscr{S})$. If this is the case (and if the extension is denoted by the same symbol φ), then the *inclusion–exclusion principle*

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \left(\sum_{i_1 < \cdots < i_r} \varphi(K_{i_1} \cap \cdots \cap K_{i_r}) \right)$$

holds for $K_1, \dots, K_m \in U(\mathscr{S})$. This follows by an obvious induction argument, using the additivity property. To write this equality in a more concise form, we let $S(m)$ denote the set of nonempty subsets of $\{1, \dots, m\}$, and for $v \in S(m)$ we write $|v| :=$ card v. If $K_1, \dots, K_m$ are given, then we use the abbreviation

$$K_v := K_{i_1} \cap \cdots \cap K_{i_r} \quad \text{for } v = \{i_1, \dots, i_r\} \in S(m).$$

The inclusion–exclusion principle now takes the form

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{v \in S(m)} (-1)^{|v|-1} \varphi(K_v). \tag{4.41}$$

In general, of course, this formula cannot be used to define an additive extension of φ from $\mathscr{S}$ to $U(\mathscr{S})$, since the representation of an element of $U(\mathscr{S})$ in the form $K_1 \cup \cdots \cup K_m$ with $K_i \in \mathscr{S}$ need not be unique. But if such an extension exists, then (4.41) gives its values, so that the extension is unique.

Of particular interest for us is the lattice $U(\mathcal{K}^n)$ generated by the intersectional family $\mathcal{K}^n \cup \{\emptyset\}$. Thus, $U(\mathcal{K}^n)$ consists of all subsets of $\mathbb{R}^n$ that can be represented as finite unions of convex bodies. $U(\mathcal{K}^n)$ was much studied by Hadwiger, who called it the 'Konvexring'. We shall use the disputable translation *convex ring*. The following result is fundamental in a study of extensions of valuations to the convex ring.

Theorem 4.3.1 *There is a unique valuation χ on the convex ring $U(\mathcal{K}^n)$ that satisfies*

$$\chi(K) = 1 \quad \textit{for } K \in \mathcal{K}^n. \tag{4.42}$$

That (4.42) defines a valuation on $\mathcal{K}^n$ is clear, since $K_1 \cap K_2 \neq \emptyset$ if $K_1, K_2 \in \mathcal{K}^n$ and $K_1 \cup K_2 \in \mathcal{K}^n$. The uniqueness of an additive extension to $U(\mathcal{K}^n)$ follows from

(4.41). The existence is generally known, since the Euler characteristic as defined, for example, in singular homology theory yields a valuation on $U(\mathcal{K}^n)$. For this reason, the function χ on $U(\mathcal{K}^n)$ given by Theorem 4.3.1 is, of course, called the *Euler characteristic*. We present Hadwiger's elementary existence proof (Hadwiger [906]; see also [911], p. 239).

Proof of Theorem 4.3.1 The proof proceeds by induction with respect to the dimension. The existence for $n = 0$ being trivial, suppose that the existence of χ has been proved in dimension $n-1$. Choose $u \in \mathbb{S}^{n-1}$ and define

$$\chi(K) := \sum_{\lambda \in \mathbb{R}} \left[\chi(K \cap H_{u,\lambda}) - \lim_{\mu \downarrow \lambda} \chi(K \cap H_{u,\mu}) \right] \tag{4.43}$$

for $K \in U(\mathcal{K}^n)$, where on the right-hand side χ denotes the (unique) Euler characteristic which, by the inductive assumption, exists in each hyperplane $H_{u,\lambda}$ ($\lambda \in \mathbb{R}$). Obviously, $\chi(K) = 1$ for $K \in \mathcal{K}^n$. For $K = K_1 \cup \cdots \cup K_m$ with $K_i \in \mathcal{K}^n$ we have from (4.41)

$$\chi(K \cap H_{u,\lambda}) = \sum_{v \in S(m)} (-1)^{|v|-1} \chi(K_v \cap H_{u,\lambda}).$$

Since $\lambda \mapsto \chi(K_v \cap H_{u,\lambda})$ is the characteristic function of a compact interval, it is clear that in (4.43) the limits exist and the sum is finite. It is also clear (using the inductive assumption) that χ, thus defined on $U(\mathcal{K}^n)$, is additive. □

The support measure $\Theta_m(K, \cdot)$ is an additive function of K, by Theorem 4.2.1 (thus, a valuation on $\mathcal{K}^n$ with values in the abelian group of finite signed Borel measures on Σ). We now construct the additive extension of this valuation to the convex ring. While the existence of such an extension could be deduced from a general extension theorem (see Chapter 6), we prefer to give an explicit construction that provides additional insight and at the same time extends the Steiner formula to local 'parallel sets with multiplicity'. The availability of curvature measures for sets more general than convex bodies is useful for applications, and the additivity of the extension permits the almost automatic generalization of some integral-geometric formulae, to be proved in the next section.

First let $K \in \mathcal{K}^n$ be a convex body, let $\rho > 0$ and $\eta \in \mathcal{B}(\Sigma)$. In Section 4.1 we have defined the local parallel set $M_\rho(K, \eta)$, and equation (4.5) states that

$$\mathcal{H}^n(M_\rho(K, \eta)) = \frac{1}{n} \sum_{m=0}^{n-1} \rho^{n-m} \binom{n}{m} \Theta_m(K, \eta).$$

In order to obtain a Steiner formula such as the above for non-convex sets $K \in U(\mathcal{K}^n)$ as well, we consider, instead of the parallel set $M_\rho(K, \eta)$, its characteristic function, denoted by $c_\rho(K, \eta, \cdot)$, and extend this additively to $U(\mathcal{K}^n)$. The value $c_\rho(K, \eta, x)$ is then interpreted as counting the multiplicity with which the point x belongs to the parallel set of K with respect to η.

For $K \in U(\mathcal{K}^n)$ and points $q, x \in \mathbb{R}^n$ we define the *index of K at q with respect to x* by

$$j(K, q, x) := \begin{cases} 1 - \lim\limits_{\delta \downarrow 0} \lim\limits_{\varepsilon \downarrow 0} \chi(K \cap B(x, |x - q| - \varepsilon) \cap B(q, \delta)) & \text{if } q \in K, \\ 0 & \text{if } q \notin K. \end{cases}$$

If K is convex, then clearly

$$j(K, q, x) = \begin{cases} 1 & \text{if } q = p(K, x), \\ 0 & \text{otherwise.} \end{cases} \tag{4.44}$$

To prove the existence of the limits, we choose a representation $K = \bigcup_{i=1}^{r} K_i$ with $K_i \in \mathcal{K}^n$. Without loss of generality, we may assume that $q \in K_i$ for $i = 1, \dots, m$ and $q \notin K_i$ for $i > m$, where $1 \le m \le r$. We can choose $\delta_0 > 0$ such that $K_i \cap B(q, \delta) = \emptyset$ for $i \in \{m + 1, \dots, r\}$ and $0 < \delta < \delta_0$. If $\delta < \delta_0$ is fixed, then for all sufficiently small $\varepsilon > 0$ the inequality

$$K_v \cap B(x, |x - q| - \varepsilon) \cap B(q, \delta) \neq \emptyset$$

holds for all $v \in S(m)$ with $j(K_v, q, x) = 0$. In fact, for $v \in S(m)$ we have $q \in K_v$, and if $j(K_v, q, x) = 0$, then (4.44) implies that K_v and hence $K_v \cap B(q, \delta)$ contains a point whose distance from x is smaller than that of q. For sufficiently small $\varepsilon > 0$ we thus have

$$j(K_v, q, x) = 1 - \chi(K_v \cap B(x, |x - q| - \varepsilon) \cap B(q, \delta)).$$

Using the additivity of χ, we deduce that

$$\begin{aligned} & 1 - \chi(K \cap B(x, |x - q| - \varepsilon) \cap B(q, \delta)) \\ &= \sum_{v \in S(m)} (-1)^{|v|-1} \left[1 - \chi(K_v \cap B(x, |x - q| - \varepsilon) \cap B(q, \delta))\right] \\ &= \sum_{v \in S(m)} (-1)^{|v|-1} j(K_v, q, x) = \sum_{v \in S(r)} (-1)^{|v|-1} j(K_v, q, x). \end{aligned}$$

The right-hand side does not depend on ε or δ, which shows the existence of the limits.

The index function j thus defined is additive in its first argument; that is, for fixed $q, x \in \mathbb{R}^n$ and for $K, L \in U(\mathcal{K}^n)$ we have

$$j(K \cup L, q, x) + j(K \cap L, q, x) = j(K, q, x) + j(L, q, x).$$

This follows from the definition; in fact, for $q \notin K \cup L$ both sides are zero, and for $q \in K \cap L$ it follows from the additivity of χ. If $q \in K \setminus L$ (and similarly for $q \in L \setminus K$) one has to observe that $(K \cup L) \cap B(q, \delta) = K \cap B(q, \delta)$ for all sufficiently small $\delta > 0$.

Now for $K \in U(\mathcal{K}^n)$, $\eta \in \mathcal{B}(\Sigma)$, $\rho > 0$ and $x \in \mathbb{R}^n$ we define

$$c_\rho(K, \eta, x) := \sum_{\substack{q \in \mathbb{R}^n \setminus \{x\} \\ (q, \overline{x - q}) \in \eta}} j(K \cap B(x, \rho), q, x),$$

where $\overline{x-q} := (x-q)/|x-q|$. Actually, the sum is finite, since $j(K,q,x) \neq 0$ for $K = \bigcup_{i=1}^r K_i$ with $K_i \in \mathcal{K}^n$ implies, by the additivity of $j(\cdot,q,x)$, that there is some $v \in S(r)$ for which $j(K_v,q,x) \neq 0$ and hence $q = p(K_v,x)$.

If K is convex, then (4.44) obviously implies that

$$c_\rho(K,\eta,x) := \begin{cases} 1 & \text{if } 0 < d(K,x) \text{ and } (p(K,x),u(K,x)) \in \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $c_\rho(K,\eta,\cdot)$ is the characteristic function of the local parallel set $M_\rho(K,\eta)$. From the additivity of $j(\cdot,q,x)$ it follows that $c_\rho(\cdot,q,x)$ is additive on $U(\mathcal{K}^n)$. Hence, for $K = \bigcup_{i=1}^r K_i$ with $K_i \in \mathcal{K}^n$ we have

$$c_\rho(K,\eta,\cdot) = \sum_{v\in S(r)} (-1)^{|v|-1} c_\rho(K_v,\eta,\cdot).$$

Since the right-hand side is a finite sum of integrable functions, we can define

$$\mu_\rho(K,\eta) := \int_{\mathbb{R}^n} c_\rho(K,\eta,x)\,\mathrm{d}\mathcal{H}^n(x)$$

for $K \in U(\mathcal{K}^n)$ and $\eta \in \mathcal{B}(\Sigma)$. The notation is consistent with that of Section 4.1, and $\mu_\rho(\cdot,\eta)$ is an additive function on $U(\mathcal{K}^n)$. Hence, for $K = \bigcup_{i=1}^r K_i$ with $K_i \in \mathcal{K}^n$ we have

$$\begin{aligned} \mu_\rho(K,\eta) &= \sum_{v\in S(r)} (-1)^{|v|-1} \mu_\rho(K_v,\eta) = \sum_{v\in S(r)} (-1)^{|v|-1} \frac{1}{n} \sum_{m=0}^{n-1} \rho^{n-m} \binom{n}{m} \Theta_m(K_v,\eta) \\ &= \frac{1}{n} \sum_{m=0}^{n-1} \rho^{n-m} \binom{n}{m} \sum_{v\in S(r)} (-1)^{|v|-1} \Theta_m(K_v,\eta). \end{aligned}$$

Since the left-hand side depends only on the point set K and not on its chosen representation as a finite union of convex bodies, the same is true for the coefficients of the polynomial in ρ on the right-hand side. Hence, we are now in a position to define

$$\Theta_m(K,\eta) := \sum_{v\in S(r)} (-1)^{|v|-1} \Theta_m(K_v,\eta)$$

for $m = 0,\dots,n-1$. Thus $\Theta_m(K,\cdot)$ is a finite signed measure on $\mathcal{B}(\Sigma)$. From this representation (or from the additivity of $\mu_\rho(\cdot,\eta)$) we deduce that $\Theta_m(\cdot,\eta)$ is additive on $U(\mathcal{K}^n)$. In this way, we obtain the (unique) additive extension of the support measures to the convex ring, and for the generalized parallel volume $\mu_\rho(K,\eta)$ we arrive at the Steiner formula

$$\mu_\rho(K,\eta) = \frac{1}{n} \sum_{m=0}^{n-1} \rho^{n-m} \binom{n}{m} \Theta_m(K,\eta).$$

As in Section 4.2, we specialize the support measures by putting

$$C_m(K,\beta) = \frac{n\kappa_{n-m}}{\binom{n}{m}} \Phi_m(K,\beta) = \Theta_m(K,\beta\times\mathbb{S}^{n-1}),$$

$$S_m(K,\omega) = \frac{n\kappa_{n-m}}{\binom{n}{m}} \Psi_m(K,\omega) = \Theta_m(K,\mathbb{R}^n\times\omega)$$

for $K \in U(\mathcal{K}^n)$, Borel sets $\beta \in \mathcal{B}(\mathbb{R}^n)$, $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and for $m = 0,\dots,n-1$; further,

$$\Phi_n(K,\beta) := \mathcal{H}^n(K\cap\beta).$$

Of these additive extensions of the curvature measures and area measures to the convex ring, it will mainly be the signed measures C_m (or Φ_m) that will be used in the sequel. We shall first extend the interpretation of C_{n-1} given by (4.31).

Theorem 4.3.2 *If $K \in U(\mathcal{K}^n)$ is the closure of its interior, then*

$$C_{n-1}(K,\beta) = \mathcal{H}^{n-1}(\beta\cap\operatorname{bd}K)$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$.

Proof Let $K = \bigcup_{i=1}^r K_i$ with $K \in \mathcal{K}^n$. If among the convex bodies $K_1,\dots,K_r$ there were one of dimension less than n not covered by the union of the others, then K would not be the closure of its interior. Hence, lower-dimensional bodies in the representation may be omitted, and we can assume from the beginning that $\dim K_i = n$ for $i = 1,\dots,r$.

For $L \in \mathcal{K}^n$, let $\varphi(L,\cdot)$ denote the characteristic function of the boundary of L. Let $x \in \operatorname{bd}K$. If, say, $x \in K_i$ precisely for $i = 1,\dots,m$, then $x \in \operatorname{bd}K_v$ for $v \in S(m)$ and hence

$$\sum_{v\in S(r)} (-1)^{|v|-1}\varphi(K_v,x) = \sum_{v\in S(m)} (-1)^{|v|-1} = 1.$$

Let $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $v \in S(r)$. Theorem 4.2.3 together with the remark after its proof tells us that $C_{n-1}(K_v,\beta) = \mathcal{H}^{n-1}(\beta\cap\operatorname{bd}K_v)$, provided that $\dim K_v \neq n-1$. If $\dim K_v = n-1$, then $\operatorname{relint}K_v \subset \operatorname{int}K$, hence $C_{n-1}(K_v,\beta\cap\operatorname{bd}K) = 0 = \mathcal{H}^{n-1}(\beta\cap\operatorname{bd}K\cap\operatorname{bd}K_v)$. Further, $C_{n-1}(K,\cdot)$ is concentrated on the boundary of K, since $c_\rho(K,\beta\times\mathbb{S}^{n-1},\cdot) = c_\rho(K,(\beta\cap\operatorname{bd}K)\times\mathbb{S}^{n-1},\cdot)$ by the definition of c_ρ. For $\beta \in \mathcal{B}(\mathbb{R}^n)$ we conclude that

$$\begin{aligned} C_{n-1}(K,\beta) = C_{n-1}(K,\beta\cap\operatorname{bd}K) &= \sum_{v\in S(r)} (-1)^{|v|-1} C_{n-1}(K_v,\beta\cap\operatorname{bd}K) \\ &= \sum_{v\in S(r)} (-1)^{|v|-1}\mathcal{H}^{n-1}(\beta\cap\operatorname{bd}K\cap\operatorname{bd}K_v) \\ &= \sum_{v\in S(r)} (-1)^{|v|-1}\int_{\beta\cap\operatorname{bd}K}\varphi(K_v,x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \mathcal{H}^{n-1}(\beta\cap\operatorname{bd}K), \end{aligned}$$

which completes the proof. □

For convex bodies $K \in \mathcal{K}^n$, formula (4.29) interprets the curvature measure C_0 as the content of the spherical image, namely

$$C_0(K,\beta) = \mathcal{H}^{n-1}(\sigma(K,\beta)).$$

If we want to extend this representation to the elements of the convex ring, we have to count the points of the spherical image with a suitable multiplicity. For this, we use a similar approach to the above and introduce another index function. For $K \in U(\mathcal{K}^n)$, a point $q \in \mathbb{R}^n$ and a vector $u \in \mathbb{S}^{n-1}$ we define

$$i(K,q,u) := \begin{cases} 1 - \lim\limits_{\delta\downarrow 0}\lim\limits_{\varepsilon\downarrow 0}\chi(K \cap B(q + (\delta+\varepsilon)u, \delta)) & \text{if } q \in K, \\ 0 & \text{if } q \notin K. \end{cases}$$

If K is convex, then clearly

$$i(K,q,u) = \begin{cases} 1 & \text{if } (q,u) \in \operatorname{Nor} K, \\ 0 & \text{otherwise.} \end{cases} \tag{4.45}$$

As for the index function $j(K,q,\cdot)$, one sees that $i(K,q,u)$ is well defined and that $i(\cdot,q,u)$ is an additive function on the convex ring $U(\mathcal{K}^n)$. For $K \in U(\mathcal{K}^n), \beta \in \mathcal{B}(\mathbb{R}^n)$ and $u \in \mathbb{S}^{n-1}$ we define

$$c(K,\beta,u) := \sum_{q\in\beta} i(K,q,u).$$

Here we have to allow infinite values, but these can be neglected. In fact, let $K = \bigcup_{i=1}^r K_i$ with $K_i \in \mathcal{K}^n$. If $u \in \bigcap_{i=1}^r \operatorname{regn} K_i$ and $v \in S(r)$, there is at most one point q for which $(q,u) \in \operatorname{Nor} K_v$ and thus $i(K_v,q,u) \neq 0$. It follows from Theorem 2.2.11 that $c(K,\beta,\cdot)$ is finite $\mathcal{H}^{n-1}$-almost everywhere on $\mathbb{S}^{n-1}$. If K is convex and $u \in \operatorname{regn} K$, then (4.45) implies that $c(K,\beta,\cdot)$ is the characteristic function of the spherical image $\sigma(K,\beta)$, hence (4.29) can be written as

$$C_0(K,\beta) = \int_{\mathbb{S}^{n-1}} c(K,\beta,u)\,\mathrm{d}\mathcal{H}^{n-1}(u). \tag{4.46}$$

Since both $C_0(\cdot,\beta)$ and $c(\cdot,\beta,u)$ are additive on $U(\mathcal{K}^n)$, the equality extends immediately to sets K of the convex ring. This is the desired interpretation of the curvature measure $C_0(K,\cdot)$ as the $\mathcal{H}^{n-1}$-integral of a multiplicity function for the spherical image.

Finally we remark that

$$\frac{1}{\omega_n} C_0(K,\mathbb{R}^n) = V_0(K) = \chi(K) \tag{4.47}$$

for $K \in U(\mathcal{K}^n)$. This is true for convex bodies, and the general case follows by additivity. In the same way, a corresponding result for the index functions is obtained. This is in analogy to well-known index sum formulae for smooth submanifolds. For $K \in U(\mathcal{K}^n)$ and $x \in \mathbb{R}^n \setminus K$ we have

$$\sum_{q\in\mathbb{R}^n} j(K,q,x) = \chi(K), \tag{4.48}$$

and if $u \in \mathbb{S}^{n-1}$ is regular for K, which means that $u \in \bigcap_{i=1}^r \operatorname{regn} K_i$ for some representation $K = \bigcup_{i=1}^r K_i$ with $K_i \in \mathcal{K}^n$, then we have

$$\sum_{q \in \mathbb{R}^n} i(K,q,u) = \chi(K). \tag{4.49}$$

Notes for Section 4.3

1. For comments concerning the introduction of the Euler characteristic in Theorem 4.3.1, see Note 2 in Section 6.2.
2. *Index functions*. The additive extension of the support measures to the convex ring by means of an index function was carried out by Schneider [1694]. This had previously been done for the curvature measure C_0 in Schneider [1685], extending some work of Hadwiger [919] and Banchoff [129] for polyhedra and cell complexes. The index $i(K,q,u)$ is also used in Zähle [2022].

   As mentioned at the end of Section 4.3, the index functions that were introduced are analogues of index functions known in differential geometry. Calling the point q a critical point of $K \in U(\mathcal{K}^n)$ with respect to the height function $\langle \cdot, u\rangle$ if $i(K,q,u) \neq 0$, one may consider equality (4.49) as an analogue of the critical point theorem for smooth submanifolds. Similarly, (4.47) is an analogue of the Gauss–Bonnet theorem.

   Suppose that $K \in U(\mathcal{K}^n)$ is the point set of a polyhedral cell complex of which Δ^k is the set of k-dimensional cells. Then, the index $i(K,q,u)$ satisfies

$$i(K,q,u) = \sum_{k=0}^{n} (-1)^k \sum_{Z \in \Delta^k} i(Z,q,-u). \tag{4.50}$$

   This was proved by Shephard [1785] for the special case of the boundary complex of a convex polytope, and it can be extended to the general case by means of an argument due to Perles and Sallee [1523]. A definition equivalent to (4.50) was used by Banchoff [129, 130] in his investigation of critical point theory, curvature and the Gauss–Bonnet theorem for polyhedra.
3. *Absolute curvature measures on the convex ring*. If the additivity assumption is dropped, other extensions of the curvature measures and support measures to the convex ring are possible. Of particular interest are nonnegative, or absolute, curvature measures. A nonnegative extension of Federer's curvature measures to the convex ring was proposed by Matheron [1358], pp. 119 ff. His construction was extended in Schneider [1694], as follows. For given $K \in U(\mathcal{K}^n)$ and $x \in \mathbb{R}^n$, a point $q \in \mathbb{R}^n$ is called a *projection* of x in K if $q \in K$ and there exists a neighbourhood N of q such that $|x-y| > |x-q|$ for all $y \in K \cap N$, $y \neq q$. The set $\Pi(K,x)$ of all projections of x in K is finite. For $\eta \in \mathcal{B}(\Sigma)$ and $\rho > 0$ let

$$\bar{c}_\rho(K,\eta,x) := \operatorname{card}\{q \in \Pi(K,x) : 0 < |x-q| \le \rho \text{ and } (q, \overline{x-q}) \in \eta\}$$

   and

$$\bar{\mu}_\rho(K,\eta) := \int_{\mathbb{R}^n} \bar{c}_\rho(K,\eta,x)\, \mathrm{d}\mathcal{H}^n(x).$$

   One can show that $\bar{\mu}_\rho(K,\eta)$ satisfies a Steiner formula, that is, a polynomial expansion

$$\bar{\mu}_\rho(K,\eta) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m}\bar{\Theta}_m(K,\eta),$$

   and that this defines (positive) measures $\bar{\Theta}_0(K,\cdot),\dots,\bar{\Theta}_{n-1}(K,\cdot)$ on $\mathcal{B}(\Sigma)$, which for $K \in \mathcal{K}^n$ coincide with respectively $\Theta_0(K,\cdot),\dots,\Theta_{n-1}(K,\cdot)$. The specialization

$$\bar{C}_m(K,\beta) := \bar{\Theta}_m(K,\beta \times \mathbb{S}^{n-1}), \quad \beta \in \mathcal{B}(\mathbb{R}^n)$$

   yields the measures introduced by Matheron.

The measure $\bar{C}_0(K,\cdot)$ can also be interpreted as follows. For $K \in U(\mathcal{K}^n)$ and $x \in \operatorname{bd} K$, a unit vector $u \in \mathbb{S}^{n-1}$ is called a *normal vector* of K at x if there exists a neighbourhood N of x such that $\langle x,u\rangle \geq \langle y,u\rangle$ for all $y \in K \cap N$. For $\beta \in \mathcal{B}(\mathbb{R}^n)$, let $\bar{c}(K,\beta,u)$ be the (possibly infinite) number of points $x \in K \cap \beta$ for which u is a normal vector. Then $\bar{C}_0(K,\beta) = \int_{\mathbb{S}^{n-1}} \bar{c}(K,\beta,u)\,\mathrm{d}\mathcal{H}^{n-1}(u)$. Thus, $\bar{C}_0(K,\cdot)$ can be interpreted as the measure of the 'spherical image with multiplicity'. The normalized total measure $\bar{C}_0(K,\mathbb{R}^n)/\omega_n$ was called the *convexity number* of K by Matheron [1358]. It is equal to one for convex bodies, but not only for these.

4. *Other generalizations of curvature measures*. The treatment of support and curvature measures given in this section was restricted to convex bodies and the sets of the convex ring. Its extension to more general classes of sets is possible, but requires in general deeper methods from geometric measure theory. We give only some brief hints to the literature. Federer's [556] curvature measures for sets of positive reach were further extended and investigated by Zähle [2018, 2019, 2020, 2021, 2022]. The extension is to finite unions of sets of positive reach and later to so-called second-order rectifiable sets. Applications to cell complexes and mosaics appear in Zähle [2023, 2024], Weiss [1953] and Weiss and Zähle [2024]. Fu [636] relates curvature measures to generalized Morse theory.

   The approach to curvature measures via so-called normal cycles, as begun by Wintgen [1987] and Zähle [2020, 2021, 2022], was further extended in work of Fu [638, 639]. Other extensions of curvature measures to different singular objects were investigated by Bröcker and Kuppe [335], Bernig and Bröcker [207, 208], Rataj and Zähle [1557, 1558].

   General versions of absolute curvature measures were introduced by Baddeley [104] and Zähle [2025].

   Kuiper [1155] has used singular homology to define a sequence of curvature measures, which includes a generalization of C_0.

   Curvature measures for non-convex polyhedra were considered by Flaherty [614]. A thorough study of curvatures for piecewise linear spaces was made by Cheeger, Müller and Schrader [409, 410]. The latter paper is also related to some (more general) work of Zähle quoted above. Part of the work of Cheeger, Müller and Schrader can be simplified; see Budach [353] and also the remark at the end of §2 in Schneider [1715].

   Approximation results, of different kinds, for curvature measures appear in Brehm and Kühnel [333], Cheeger, Müller and Schrader [409], Lafontaine [1164], Zähle [2026].

5. *Curvature measures in spaces of constant curvature*. Kohlmann [1128, 1129, 1130] introduced curvature measures for sets of positive reach in Riemannian space forms. Among other results, he obtained integral representations, Minkowski-type integral formulae and, as applications in his further work, characterizations of balls and stability results (see Section 8.5, Note 4).

4.4 Integral-geometric formulae

In this section, we prove integral-geometric formulae for curvature measures, area measures and support measures. Generally speaking, integral geometry is concerned with the computation and application of mean values for geometrically defined functions with respect to invariant measures. Here we shall treat local versions of some intersection and projection formulae, which in their global versions, namely for the intrinsic volumes, are central results of classical integral geometry. We shall also consider mean value formulae for Minkowski sums, and counterparts to the intersection formulae in the form of kinematic formulae involving the distances between non-intersecting bodies, or between bodies and flats. This yields, among other results, an integral-geometric interpretation of the support measures.

We assume that the reader is familiar with the topological groups $\mathrm{SO}(n)$ of proper rotations of $\mathbb{R}^n$ and G_n of rigid motions of $\mathbb{R}^n$, and with their Haar measures. We normalize the Haar measure ν on $\mathrm{SO}(n)$ by imposing that $\nu(\mathrm{SO}(n)) = 1$. Every rigid motion $g \in \mathrm{G}_n$ determines uniquely a rotation $\rho \in \mathrm{SO}(n)$ and a translation vector $t \in \mathbb{R}^n$ so that $gx = \rho x + t$ for $x \in \mathbb{R}^n$, and we write $g = g_{t,\rho}$. The map

$$\begin{array}{rccc} \gamma: & \mathbb{R}^n \times \mathrm{SO}(n) & \to & \mathrm{G}_n \\ & (t,\rho) & \mapsto & g_{t,\rho} \end{array}$$

is a homeomorphism (products of topological spaces are always equipped with the product topology). We can define a Haar measure μ on G_n by the image measure

$$\mu := \gamma(\lambda_n \otimes \nu),$$

where λ_n denotes the restriction of $\mathcal{H}^n$ to the Borel sets of $\mathbb{R}^n$; this implies a definite normalization of μ. The measure μ, defined on the σ-algebra $\mathcal{B}(\mathrm{G}_n)$ of Borel subsets of G_n, is left invariant and right invariant, thus the motion group is unimodular.

Let $k \in \{0,\dots,n\}$. The rotation group $\mathrm{SO}(n)$ operates in the natural way on $G(n,k)$, the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$, and the motion group G_n operates in the natural way on $A(n,k)$, the space of k-dimensional affine subspaces of $\mathbb{R}^n$. To facilitate the handling of the invariant measures on $G(n,k)$ and $A(n,k)$, we choose a k-dimensional linear subspace L_k of $\mathbb{R}^n$ and denote its orthogonal complement by $L_k^\perp$. The mappings

$$\begin{array}{rccc} \beta_k: & \mathrm{SO}(n) & \to & G(n,k) \\ & \rho & \mapsto & \rho L_k \end{array}$$

and

$$\begin{array}{rccc} \gamma_k: & L_k^\perp \times \mathrm{SO}(n) & \to & A(n,k) \\ & (t,\rho) & \mapsto & \rho(L_k + t) \end{array}$$

are surjective. The usual topologies on $G(n,k)$ and $A(n,k)$ are the finest topologies for which β_k and γ_k, respectively, are continuous. With these topologies, $G(n,k)$ and $A(n,K)$ are second countable Hausdorff spaces, $G(n,k)$ is compact and $A(n,k)$ is locally compact, the natural operations are continuous, $G(n,k)$ is a homogenoeus $\mathrm{SO}(n)$-space and $A(n,k)$ is a homogeneous G_n-space. The image measure of ν under β_k, denoted by ν_k, is the invariant probability measure on $G(n,k)$. The image measure

$$\mu_k := \gamma_k(\lambda_{n-k} \otimes \nu)$$

of the product measure $\lambda_{n-k} \otimes \nu$, where λ_{n-k} is the restriction of $\mathcal{H}^{n-k}$ to the Borel sets of $L_k^\perp$, is a G_n-invariant measure on $A(n,k)$ with a suitable normalization. Neither measure depends on the special choice of the subspace L_k.

Below, we shall sometimes need to know that certain sets of rotations, rigid motions or flats are of measure zero. Some of these facts follow from the results of Section 2.3; a more elementary one is contained in the following lemma.

Two linear subspaces E, F of $\mathbb{R}^n$ are said to be in *special position* if

$$\operatorname{lin}(E \cup F) \neq \mathbb{R}^n \quad \text{and} \quad E \cap F \neq \{o\}.$$

Lemma 4.4.1 *Let $E \in G(n, p)$, $F \in G(n, q)$, $0 \leq p, q \leq n-1$. The set of all rotations $\rho \in \mathrm{SO}(n)$ for which E and ρF are in special position is of ν-measure zero.*

Proof To avoid computations requiring an explicit description of the Haar measure ν, we give a proof that uses only its invariance and finiteness.

For given $E \in G(n, p)$, $F \in G(n, q)$, let X be the set of all $\rho \in \mathrm{SO}(n)$ for which E and ρF are in special position. We prove the assertion $\nu(X) = 0$ by induction on $p = \dim E$. The case $p = 0$ is trivial; we assume that $p \geq 1$ and the assertion has been proved for $\dim E < p$. Let $\dim E = p$. We choose a linear subspace $U \in G(n, n-p+1)$ and denote its orthogonal complement by $U^\perp$. Further, we choose a number $k \in \mathbb{N}$ and k vectors $u_1, \dots, u_k \in U$ of which any $n - p + 1$ are linearly independent. Put $E_i := \operatorname{lin}(U^\perp \cup \{u_i\})$; then $\dim E_i = p$. Let X_i be the set of all $\rho \in \mathrm{SO}(n)$ for which E_i and ρF are in special position. As $E_i = \rho_i E$ for suitable $\rho_i \in \mathrm{SO}(n)$, we have $X_i = \rho_i X$ and hence $\nu(X_i) = \nu(X)$. Let Y be the set of all $\rho \in \mathrm{SO}(n)$ for which $U^\perp$ and ρF are in special position. Since $\dim U^\perp = p-1$, we have $\nu(Y) = 0$ by the inductive hypothesis. We assert that each $\rho \in \mathrm{SO}(n) \setminus Y$ lies in at most $n - p$ of the sets $X_1, \dots, X_k$. Let $\rho \in \mathrm{SO}(n) \setminus Y$ and, say, $\rho \in X_i$ for $i = 1, \dots, r$. Thus E_i and ρF are in special position for $i = 1, \dots, r$. In particular, $\operatorname{lin}(E_i \cup \rho F) \neq \mathbb{R}^n$ and hence $\operatorname{proj}_U \rho F \neq U$, since $U^\perp \subset E_i$. The subspaces $U^\perp$ and ρF are not in special position since $\rho \notin Y$; however, E_i and ρF are in special position. This is only possible if $u_i \in \operatorname{proj}_U \rho F$. Since the vectors $u_1, \dots, u_{n-p+1}$ are linearly independent and $\dim \operatorname{proj}_U \rho F \leq n - p$, we deduce that $r \leq n - p$. Let ζ_i be the characteristic function of X_i on $\mathrm{SO}(n)$. We have proved that $\sum_{i=1}^k \zeta_i(\rho) \leq n - p$ for $\rho \in \mathrm{SO}(n) \setminus Y$. Since $\nu(Y) = 0$, integration yields

$$k\nu(X) = \sum_{i=1}^{k} \nu(X_i) = \int_{\mathrm{SO}(n)} \sum_{i=1}^{k} \zeta_i(\rho)\, \mathrm{d}\nu(\rho) \leq n - p.$$

Letting $k \to \infty$, we conclude that $\nu(X) = 0$. $\square$

Our first major goal in this section is the proof of the following theorem.

Theorem 4.4.2 *Let $K, K' \in U(\mathcal{K}^n)$ be sets of the convex ring, let $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ and $j \in \{0, \dots, n\}$. Then*

$$\int_{G_n} \Phi_j(K \cap gK', \beta \cap g\beta')\, \mathrm{d}\mu(g) = \sum_{k=j}^{n} \alpha_{njk} \Phi_k(K, \beta) \Phi_{n+j-k}(K', \beta') \tag{4.51}$$

with

$$\alpha_{njk} = \frac{\binom{k}{j} \kappa_k \kappa_{n+j-k}}{\binom{n}{k-j} \kappa_j \kappa_n} = \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n+j-k+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}.$$

The special case $\beta = \beta' = \mathbb{R}^n$ reduces to what Hadwiger [911] (with different notation) called the 'complete system of kinematic formulae', namely

$$\int_{G_n} V_j(K \cap gK')\,\mathrm{d}\mu(g) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) V_{n+j-k}(K'). \tag{4.52}$$

The further specialization $j = 0$ can be written in the form

$$\int_{G_n} \chi(K \cap gK')\,\mathrm{d}\mu(g) = \frac{1}{\kappa_n} \sum_{k=0}^{n} \frac{\kappa_k \kappa_{n-k}}{\binom{n}{k}} V_k(K) V_{n-k}(K') \tag{4.53}$$

$$= \frac{1}{\kappa_n} \sum_{k=0}^{n} \binom{n}{k} W_k(K) W_{n-k}(K'). \tag{4.54}$$

This result is known as the 'principal kinematic formula' (for convex bodies). It expresses the total measure of the set of all rigid motions g for which the moving body gK' meets K, in terms of the intrinsic volumes of K and K'.

The proof of Theorem 4.4.2 is divided into several steps. As a preliminary, we remark that it suffices to prove the theorem for convex bodies K, K'. In fact, if (4.51) is true in this case, then one can utilize the fact that both sides of (4.51), as functions of K, are additive on $U(\mathcal{K}^n)$. By additivity, (4.51) remains true if K is replaced by a finite union of convex bodies. In a second step, K' can similarly be replaced by a finite union of convex bodies. Hence, from now on we assume that $K, K' \in \mathcal{K}^n$.

First we have to show that the integrand in (4.51) is, in fact, a measurable function of g. For this, let $K, K' \in \mathcal{K}^n$ and $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ be given. Let $X(K, K')$ be the set of all rigid motions $g \in G_n$ for which K and gK' cannot be separated by a hyperplane. For given $\varepsilon > 0$, we define a function $h : G_n \to \mathbb{R}$ by

$$h(g) := U_\varepsilon(K \cap gK', \beta \cap g\beta'),$$

where $U_\varepsilon(L, \alpha) := \mu_\varepsilon(L, \alpha \times \mathbb{S}^{n-1}) = \mathcal{H}^n(A_\varepsilon(L, \alpha))$ for $L \in \mathcal{K}^n$, $\alpha \in \mathcal{B}(\mathbb{R}^n)$ (the measure μ_ε was defined in Section 4.1, and the local parallel set $A_\varepsilon(L, \alpha)$ in 4.2), and we define a function $f : X(K, K') \times \mathbb{R}^n \to \mathbb{R}$ by

$$f(g, x) := \zeta_\beta(x) \zeta_{\beta'}(g^{-1}x) U_\varepsilon(K \cap gK', \mathbb{R}^n),$$

where ζ_α is the characteristic function of α on $\mathbb{R}^n$. The function $(g, x) \mapsto g^{-1}x$ is continuous on $G_n \times \mathbb{R}^n$ and the function $g \mapsto U_\varepsilon(K \cap gK', \mathbb{R}^n)$ is continuous on $X(K, K')$, by Theorem 1.8.10 and by the continuity of $U_\varepsilon(\cdot, \mathbb{R}^n)$ (which follows from Theorem 1.8.20). Since $\zeta_\beta, \zeta_{\beta'}$ are measurable on $\mathbb{R}^n$, the function f is measurable. Let

$$P(g, \alpha) := \frac{U_\varepsilon(K \cap gK', \alpha)}{U_\varepsilon(K \cap gK', \mathbb{R}^n)}$$

for $g \in X(K, K')$ and $\alpha \in \mathcal{B}(\mathbb{R}^n)$. Since the function $g \mapsto K \cap gK'$ is continuous on $X(K, K')$ and $U_\varepsilon(\cdot, \alpha)$ is measurable on $\mathcal{K}^n$ by Theorem 4.1.2, it follows that $P(\cdot, \alpha)$

is measurable on $X(K, K')$. For fixed g, $P(g, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R}^n)$. Since, for $g \in X(K, K')$,

$$h(g) = \int_{\mathbb{R}^n} \zeta_{\beta\cap g\beta'}(x)\, U_\varepsilon(K \cap gK', \mathrm{d}x) = \int_{\mathbb{R}^n} f(g, x)\, P(g, \mathrm{d}x),$$

it follows (e.g., Neveu [1471], p. 95) that h is measurable on $X(K, K')$. Now $h(g) = 0$ if $K \cap gK' = \emptyset$, and the (closed) set M of all $g \in \mathrm{G}_n \setminus X(K, K')$ for which $K \cap gK' \neq \emptyset$ satisfies

$$\mu(M) = \int_{\mathrm{SO}(n)} \int_{T(\rho)} \mathrm{d}\mathcal{H}^n \, \mathrm{d}\nu(\rho), \tag{4.55}$$

where $T(\rho)$ is the set of all $t \in \mathbb{R}^n$ for which $\rho K'$ and K meet but can be separated by a hyperplane. It is easy to see that $T(\rho) = \mathrm{bd}\,(K - \rho K')$ and hence $\mu(M) = 0$. Thus h is measurable. Since $\varepsilon > 0$ was arbitrary, we deduce from (4.6) that the function $g \mapsto \Phi_j(K \cap gK', \beta \cap g\beta')$ is measurable if $j \in \{0, \dots, n-1\}$; for $j = n$ this is easily seen directly.

Now we enter into the essential part of the proof of Theorem 4.4.2, by writing the integral of (4.51) in the form

$$\begin{aligned} &\int_{\mathrm{G}_n} \Phi_j(K \cap gK', \beta \cap g\beta')\, \mathrm{d}\mu(g) \\ &= \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \Phi_j(K \cap (\rho K' + t), \beta \cap (\rho\beta' + t))\, \mathrm{d}t\, \mathrm{d}\nu(\rho). \end{aligned} \tag{4.56}$$

This is possible by the description of the invariant measure μ given initially and by Fubini's theorem (this argument was already used in (4.55)). We first consider the inner integral for the case of polytopes, where an explicit computation is possible.

Some preparatory definitions are needed. First, for $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we often abbreviate $A + x$ by A_x. For linear subspaces L, L' of $\mathbb{R}^n$ we define a number $[L, L']$ as follows. We choose an orthonormal basis of $L \cap L'$ and extend it to an orthonormal basis of L as well as to an orthonormal basis of L'. The resulting vectors span a parallelepiped of dimension $\dim \mathrm{lin}\,(L \cup L')$; let $[L, L']$ be its n-dimensional volume. This definition does not depend on the choice of the bases. Next, let $P, P' \in \mathcal{P}^n$ be convex polytopes, and let F be a face of P and F' a face of P'. By $L(F) := \mathrm{aff}\, F - \mathrm{aff}\, F$ we denote the linear subspace parallel to aff F. Then we define $[F, F'] := [L(F), L(F')]$. We also define a common external angle

$$\gamma(F, F', P, P') := \gamma(F \cap F'_x, P \cap P'_x),$$

where $x \in \mathbb{R}^n$ is chosen such that $\mathrm{relint}\, F \cap \mathrm{relint}\, F'_x \neq \emptyset$; this definition does not depend on the choice of x. Finally, we write

$$\lambda_F(\beta) := \mathcal{H}^{\dim F}(F \cap \beta) \quad \text{for } \beta \in \mathcal{B}(\mathbb{R}^n),$$

so that λ_F is a measure with support F.

The faces F, F' are said to be *in special position* if the linear subspaces $L(F)$, $L(F')$ are in special position.

As an intermediate result, we state the following 'principal translative formula' for polytopes.

Theorem 4.4.3 *Let $P, P' \in \mathcal{P}^n$ be polytopes; let $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ and $j \in \{0, \dots, n\}$. Then*

$$\int_{\mathbb{R}^n} \Phi_j(P \cap P'_x, \beta \cap \beta'_x)\,\mathrm{d}x = \Phi_j(P,\beta)\Phi_n(P',\beta') + \Phi_n(P,\beta)\Phi_j(P',\beta')$$
$$+ \sum_{k=j+1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n+j-k}(P')} \gamma(F, F', P, P')[F, F']\lambda_F(\beta)\lambda_{F'}(\beta').$$

Proof By (4.22) we have

$$I := \int_{\mathbb{R}^n} \Phi_j(P \cap P'_x, \beta \cap \beta'_x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \sum_{F \in \mathcal{F}_j(P \cap P'_x)} \gamma(F, P \cap P'_x)\lambda_F(\beta \cap \beta'_x)\,\mathrm{d}x.$$

In computing this integral, we may neglect those translation vectors x for which a k-face F of P meets an m-face F'_x of P'_x with $k + m < n$ or with F, F' in special position, since

$$\{x \in \mathbb{R}^n : F \cap F'_x \neq \emptyset\} = F - F'$$

is of measure zero in each case. As the vectors x with $F \cap F'_x \neq \emptyset$ but $\operatorname{relint} F \cap \operatorname{relint} F'_x = \emptyset$ also form a set of measure zero, we may assume that each j-face of $P \cap P'_x$ is the intersection of a k-face of P and an $(n + j - k)$-face of P'_x, for some $k \in \{j, \dots, n\}$. It follows that

$$I = \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n+j-k}(P')} \int_{\mathbb{R}^n} \gamma(F \cap F'_x, P \cap P'_x)\lambda_{F \cap F'_x}(\beta \cap \beta'_x)\,\mathrm{d}x$$
$$= \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n+j-k}(P')} \gamma(F, F', P, P')J(F, F', \beta, \beta')$$

with

$$J(F, F', \beta, \beta') := \int_{\mathbb{R}^n} \lambda_{F \cap F'_x}(\beta \cap \beta'_x)\,\mathrm{d}x.$$

We fix $k \in \{j, \dots, n\}$, $F \in \mathcal{F}_k(P)$, $F' \in \mathcal{F}_{n+j-k}(P')$ and write $J(F, F', \beta, \beta') =: J$. For the computation of J we may assume that F and F' are not in special position (otherwise $J = 0$) and that $o \in \operatorname{aff} F \cap \operatorname{aff} F'$. We put $L(F) \cap L(F') =: L_1$, $L_1^\perp \cap L(F) =: L_2$, $L_1^\perp \cap L(F') =: L_3$ and write $x \in \mathbb{R}^n$ uniquely in the form $x = x_1 + x_2 + x_3$ with $x_i \in L_i$ $(i = 1, 2, 3)$. Writing $F \cap \beta = \alpha$, $F' \cap \beta' = \alpha'$ we obtain, using the definition of $[F, F']$,

$$J = [F, F'] \int_{L_3} \int_{L_2} \int_{L_1} \mathcal{H}^j(\alpha \cap \alpha'_x)\,\mathrm{d}\mathcal{H}^j(x_1)\,\mathrm{d}\mathcal{H}^{k-j}(x_2)\,\mathrm{d}\mathcal{H}^{n-k}(x_3).$$

Since $(\alpha \cap \alpha'_x) - x_2 \subset L_1$, we get

$$\begin{aligned}
&\int_{L_1} \mathcal{H}^j(\alpha \cap \alpha'_x)\, \mathrm{d}\mathcal{H}^j(x_1) \\
&= \int_{L_1} \mathcal{H}^j((\alpha - x_2) \cap (\alpha' + x_3 + x_1) \cap L_1)\, \mathrm{d}\mathcal{H}^j(x_1) \\
&= \mathcal{H}^j((\alpha - x_2) \cap L_1)\mathcal{H}^j((\alpha' + x_3) \cap L_1),
\end{aligned}$$

by Fubini's theorem. Also by Fubini's theorem,

$$\int_{L_2} \mathcal{H}^j(\alpha \cap (L_1 + x_2))\, \mathrm{d}\mathcal{H}^{k-j}(x_2) = \mathcal{H}^k(\alpha),$$
$$\int_{L_3} \mathcal{H}^j(\alpha' \cap (L_1 - x_3))\, \mathrm{d}\mathcal{H}^{n-k}(x_3) = \mathcal{H}^{n+j-k}(\alpha'),$$

thus

$$J = [F, F']\lambda_F(\beta)\lambda_{F'}(\beta').$$

If $F' \in \mathcal{F}_n(P')$, we have $\gamma(F, F', P, P') = \gamma(F, P)$, $[F, F'] = 1$ and $\lambda_{F'} = \Phi_n(P', \cdot)$, thus

$$\sum_{F \in \mathcal{F}_j(P)} \sum_{F' \in \mathcal{F}_n(P')} \gamma(F, F', P, P')J(F, F', \beta, \beta') = \Phi_j(P, \beta)\Phi_n(P', \beta').$$

This holds trivially if $\dim P' < n$, since then both sides are zero. A similar remark concerns the case $F \in \mathcal{F}_n(P)$. This completes the proof of Theorem 4.4.3. □

To perform the outer integration in (4.56) for the case of polytopes, we need the following mean value formula.

Lemma 4.4.4 *Let $P, P' \in \mathcal{P}^n$ be polytopes, let numbers $j \in \{0, \dots, n-2\}$, $k \in \{j+1, \dots, n-1\}$ and faces $F \in \mathcal{F}_k(P)$, $F' \in \mathcal{F}_{n+j-k}(P')$ be given. Then*

$$\int_{SO(n)} \gamma(F, \rho F', P, \rho P')[F, \rho F']\, \mathrm{d}\nu(\rho) = \alpha_{njk}\gamma(F, P)\gamma(F', P')$$

with a real constant α_{njk} depending only on n, j, k (its explicit value is given by (4.65)).

Proof Since a direct computation of the integral appears difficult, we proceed in an indirect way.

Denoting by σ^m the normalized spherical Lebesgue measure on m-dimensional great subspheres of $\mathbb{S}^{n-1}$, we have $\gamma(F, P) = \sigma^{n-k-1}(N(P, F) \cap \mathbb{S}^{n-1})$ by the definition of outer angles. By Lemma 4.4.1 we may neglect those rotations ρ for which F and $\rho F'$ are in special position. Hence, we may assume that

$$\gamma(F, \rho F', P, \rho P') = \sigma^{n-j-1}([N(P, F) + \rho N(P', F')] \cap \mathbb{S}^{n-1}),$$

where Theorem 2.2.1(b) was used. Now we generalize the integral to be determined. For $\omega \subset \mathbb{S}^{n-1}$, let $C(\omega) := \{\lambda x : x \in \omega, \lambda \geq 0\}$ be the cone spanned by ω. For arbitrary Borel sets $\omega \subset L(F)^\perp \cap \mathbb{S}^{n-1}$ and $\omega' \subset L(F')^\perp \cap \mathbb{S}^{n-1}$ we write

$$J(\omega, \omega') := \int_{\mathrm{SO}(n)} \sigma^{n-j-1} \left\{ [C(\omega) + \rho C(\omega')] \cap \mathbb{S}^{n-1} \right\} [F, \rho F'] \, \mathrm{d}\nu(\rho)$$

(the measurability of the integrand is proved in a standard way). First we fix ω' and consider the function $J(\cdot, \omega')$. For ν-almost all ρ, $C(\omega)$ and $\rho C(\omega')$ lie in complementary subspaces. For such ρ,

$$\left[C\left(\bigcup_m \omega_m \right) + \rho C(\omega') \right] \cap \mathbb{S}^{n-1} = \bigcup_m [C(\omega_m) + \rho C(\omega')] \cap \mathbb{S}^{n-1}$$

for any sequence $(\omega_m)_{m \in \mathbb{N}}$ of Borel sets in $L(F)^\perp \cap \mathbb{S}^{n-1}$. If the sets of this sequence are pairwise disjoint, the union on the right is disjoint up to a set of σ^{n-j-1}-measure zero. From the σ-additivity of σ^{n-j-1} and the monotone convergence theorem it follows that $J(\cdot, \omega')$ is a finite Borel measure on the sphere $L(F)^\perp \cap \mathbb{S}^{n-1}$. A rotation τ of $\mathbb{S}^{n-1}$ that carries $L(F)^\perp$ into itself and keeps $L(F)$ pointwise fixed satisfies $C(\tau\omega) + \rho C(\omega') = \tau[C(\omega) + \tau^{-1}\rho C(\omega')]$ and $[F, \rho F'] = [F, \tau^{-1}\rho F']$; hence the invariance of σ^{n-j-1} and ν implies that $J(\tau\omega, \omega') = J(\omega, \omega')$. Thus, $J(\cdot, \omega')$ must be a constant multiple of the invariant measure on the sphere $L(F)^\perp \cap \mathbb{S}^{n-1}$. Interchanging the roles of ω and ω', we arrive at

$$J(\omega, \omega') = \alpha_{njk} \sigma^{n-k-1}(\omega) \sigma^{k-j-1}(\omega')$$

with a constant α_{njk} that evidently depends only on n, j, k. The particular choice $\omega = N(P, F) \cap \mathbb{S}^{n-1}$, $\omega' = N(P', F') \cap \mathbb{S}^{n-1}$ now yields the assertion of the lemma. □

If we now put equality (4.56), Theorem 4.4.3 and Lemma 4.4.4 together, we arrive at formula (4.51) for the case where K, K' are polytopes. The coefficients are those of Lemma 4.4.4.

For general convex bodies, Theorem 4.4.2 will now be proved by approximation. For given $g \in \mathrm{G}_n$, we define a map $T_g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ by $T_g(x) := (x, g^{-1}x)$. For convex bodies $K, K' \in \mathcal{K}^n$, we denote by $\varphi_j(g, K, K', \cdot)$ the image measure of $\Phi_j(K \cap gK', \cdot)$ under the (continuous) map T_g; then

$$\varphi_j(g, K, K', \beta \times \beta') = \Phi_j(K \cap gK', \beta \cap g\beta')$$

for $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$. The set $\mathcal{D}$ of all $\alpha \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$ for which $\varphi_j(\cdot, K, K', \alpha)$ is measurable is a Dynkin system. Since $\mathcal{D}$ contains the product sets $\beta \times \beta'$ ($\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$), it contains the Dynkin system generated by these products, which is equal to the generated σ-algebra $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$. Thus, $\varphi_j(\cdot, K, K', \alpha)$ is measurable for each $\alpha \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$, and we can define a measure on $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\varphi_j(K, K', \cdot) := \int_{\mathrm{G}_n} \varphi_j(g, K, K', \cdot) \, \mathrm{d}\mu(g).$$

This measure is finite since

$$\varphi_j(g, K, K', \cdot) \le V_j(K \cap gK'). \tag{4.57}$$

In $\mathcal{K}^n$ we consider convergent sequences $K_i \to K$, $K_i' \to K'$. For μ-almost all g, either $K \cap gK' = \emptyset$ or K and gK' cannot be separated by a hyperplane, which by Theorem 1.8.10 implies that $K_i \cap gK_i' \to K \cap gK$ for $i \to \infty$. For these g, it follows from the weak continuity of the curvature measures that

$$\liminf_{i\to\infty} \Phi_j(K_i \cap gK_i', T_g^{-1}(\alpha)) \ge \Phi_j(K \cap gK', T_g^{-1}(\alpha))$$

if $\alpha \subset \mathbb{R}^n \times \mathbb{R}^n$ is open and hence also $T_g^{-1}(\alpha)$ is open. Integration with respect to g and application of Fatou's lemma yields

$$\liminf_{i\to\infty} \varphi_j(K_i, K_i', \alpha) \ge \varphi_j(K, K', \alpha).$$

Similarly, we obtain

$$\lim_{i\to\infty} \varphi_j(K_i, K_i', \mathbb{R}^n \times \mathbb{R}^n) = \varphi_j(K, K', \mathbb{R}^n \times \mathbb{R}^n)$$

if we use the bounded convergence theorem, which can be applied because of (4.57), and the fact that $V_j(K_i \cap gK_i') \to V_j(K \cap gK')$ for μ-almost all g. Thus we have proved that

$$\varphi_j(K_i, K_i', \cdot) \xrightarrow{w} \varphi_j(K, K', \cdot) \quad \text{for } i \to \infty.$$

This can be used to show that

$$\varphi_j(K, K', \cdot) = \sum_{k=j}^{n} \alpha_{njk} \Phi_k(K, \cdot) \otimes \Phi_{n+j-k}(K', \cdot) \tag{4.58}$$

if $K, K' \in \mathcal{K}^n$. In fact, both sides are finite measures on $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$. If K, K' are polytopes, both sides are equal on product sets $\beta \times \beta'$ ($\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$), since (4.51) holds for polytopes, and thus are equal on an intersection stable generating system of $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$; hence they are equal on $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$. Since both sides are weakly continuous functions of (K, K'), approximation by polytopes shows that (4.58) holds for arbitrary convex bodies K, K'. Formula (4.51) is the special case obtained by applying both sides to $\beta \times \beta'$. This completes the proof of Theorem 4.4.2, except that we still have to determine the explicit values of the constants α_{njk}.

Before we compute the numbers α_{njk}, it is convenient to deduce from Theorem 4.4.2 another integral-geometric formula for the case where the moving convex body gK' is replaced by a moving flat.

Theorem 4.4.5 *Let $K \in U(\mathcal{K}^n)$ be a set of the convex ring, let $\beta \in \mathcal{B}(\mathbb{R}^n)$, $k \in \{0, \dots, n\}$ and $j \in \{0, \dots, k\}$. Then*

$$\int_{A(n,k)} \Phi_j(K \cap E, \beta \cap E)\, \mathrm{d}\mu_k(E) = \alpha_{njk} \Phi_{n+j-k}(K, \beta)$$

(where α_{njk} is the same number as in Theorem 4.4.2).

The special case $\beta = \mathbb{R}^n$ gives

$$\int_{A(n,k)} V_j(K \cap E)\, \mathrm{d}\mu_k(E) = \alpha_{njk} V_{n+j-k}(K). \tag{4.59}$$

These equations are called *Crofton's intersection formulae*. In particular, one has

$$\int_{A(n,k)} \chi(K \cap E)\, \mathrm{d}\mu_k(E) = \alpha_{n0k} V_{n-k}(K), \tag{4.60}$$

and if K is convex, this gives

$$\alpha_{n0k} V_{n-k}(K) = \mu_k\left(\{E \in A(n,k) : K \cap E \neq \emptyset\}\right). \tag{4.61}$$

We note also the special case

$$\alpha_{n0(n-j)} \Phi_j(K,\beta) = \int_{A(n,n-j)} \Phi_0(K \cap E, \beta \cap E)\, \mathrm{d}\mu_{n-j}(E) \tag{4.62}$$

giving a mean value interpretation of the curvature measure Φ_j in terms of Φ_0, which has a simple intuitive meaning.

Proof of Theorem 4.4.5 We choose a k-dimensional linear subspace E_k of $\mathbb{R}^n$ and recall that the invariant measure μ_k can be defined as the image measure of $\lambda_{n-k} \otimes \nu$ under the map

$$\begin{array}{rccc} \gamma_k : & E_k^\perp \times \mathrm{SO}(n) & \to & A(n,k) \\ & (t,\rho) & \mapsto & \rho(E_k + t), \end{array}$$

where λ_{n-k} is the restriction of $\mathcal{H}^{n-k}$ to $\mathcal{B}(E_k^\perp)$. Similarly, we denote by λ'_k the restriction of $\mathcal{H}^k$ to $\mathcal{B}(E_k)$.

We choose a bounded Borel set $\alpha \subset E_k$ with $\lambda'_k(\alpha) = 1$ and consider the integral

$$J := \int_{\mathrm{G}_n} \Phi_j(E_k \cap gK, \alpha \cap g\beta)\, \mathrm{d}\mu(g).$$

We can choose a convex body $K' \subset E_k$ such that $\alpha \subset \operatorname{relint} K'$ and that every $g \in \mathrm{G}_n$ with $gK \cap \alpha \neq \emptyset$ satisfies $E_k \cap gK = K' \cap gK$. Therefore, the integral J can be computed by means of Theorem 4.4.2, and we obtain

$$J = \sum_{i=j}^{n} \alpha_{nji} \Phi_i(K', \alpha) \Phi_{n+j-i}(K, \beta).$$

Because $\alpha \subset \operatorname{relint} K'$, we have $\Phi_i(K', \alpha) = \lambda'_k(\alpha) = 1$ if $i = k$, and 0 otherwise, hence

$$J = \alpha_{njk} \Phi_{n+j-k}(K, \beta). \tag{4.63}$$

In the following, we write $t \in \mathbb{R}^n$ in the form $t = t_1 + t_2$ with $t_1 \in E_k^\perp$ and $t_2 \in E_k$. We find that

$$\begin{aligned}
J &= \int_{SO(n)} \int_{\mathbb{R}^n} \Phi_j(E_k \cap (\rho K + t), \alpha \cap (\rho\beta + t))\, \mathrm{d}t\, \mathrm{d}\nu(\rho) \\
&= \int_{SO(n)} \int_{E_k^\perp} \int_{E_k} \Phi_j(E_k \cap (\rho K + t_1 + t_2), \alpha \cap (\rho\beta + t_1 + t_2)) \\
&\quad \times \mathrm{d}\lambda'_k(t_2)\, \mathrm{d}\lambda_{n-k}(t_1)\, \mathrm{d}\nu(\rho) \\
&= \int_{E_k^\perp \times SO(n)} \int_{E_k} \Phi_j(E_k \cap (\rho K + t_1 + t_2), \alpha \cap (\rho\beta + t_1 + t_2)) \\
&\quad \times \mathrm{d}\lambda'_k(t_2)\, \mathrm{d}(\lambda_{n-k} \otimes \nu)(t_1, \rho).
\end{aligned}$$

For the computation of the inner integral, we put

$$\Phi_j(E_k \cap (\rho K + t_1), \cdot) =: \varphi, \qquad \rho\beta + t_1 =: \beta';$$

then

$$\begin{aligned}
&\Phi_j(E_k \cap (\rho K + t_1 + t_2), \alpha \cap (\rho\beta + t_1 + t_2)) \\
&= \Phi_j(E_k \cap (\rho K + t_1), (\alpha - t_2) \cap \beta') = \varphi((\alpha - t_2) \cap \beta').
\end{aligned}$$

Denoting the characteristic function of $A \subset \mathbb{R}^n$ by ζ_A, we obtain

$$\begin{aligned}
&\int_{E_k} \Phi_j(E_k \cap (\rho K + t_1 + t_2), \alpha \cap (\rho\beta + t_1 + t_2))\, \mathrm{d}\lambda'_k(t_2) \\
&= \int_{E_k} \varphi((\alpha - t_2) \cap \beta')\, \mathrm{d}\lambda'_k(t_2) = \int_{E_k} \int_{\mathbb{R}^n} \zeta_\alpha(x + t_2) \zeta_{\beta'}(x)\, \mathrm{d}\varphi(x)\, \mathrm{d}\lambda'_k(t_2) \\
&= \int_{\mathbb{R}^n} \int_{E_k} \zeta_{\alpha - x}(t_2) \zeta_{\beta'(x)}\, \mathrm{d}\lambda'_k(t_2)\, \mathrm{d}\varphi(x) = \int_{\mathbb{R}^n} \zeta_{\beta'}(x)\, \mathrm{d}\varphi(x) = \varphi(\beta') \\
&= \Phi_j(E_k \cap (\rho K + t_1), \rho\beta + t_1) = \Phi_j(\rho^{-1}(E_k - t_1) \cap K, \beta) \\
&= \Phi_j(K \cap \gamma_k(-t_1, \rho^{-1}), \beta).
\end{aligned}$$

This yields

$$\begin{aligned}
J &= \int_{E_k^\perp \times SO(n)} \Phi_j(K \cap \gamma_k(-t_1, \rho^{-1}), \beta)\, \mathrm{d}(\lambda_{n-k} \otimes \nu)(t_1, \rho) \\
&= \int_{E_k^\perp \times SO(n)} \Phi_j(K \cap \gamma_k(t, \rho), \beta)\, \mathrm{d}(\lambda_{n-k} \otimes \nu)(t, \rho) \\
&= \int_{A(n,k)} \Phi_j(K \cap E, \beta)\, \mathrm{d}\mu_k(E).
\end{aligned}$$

Together with (4.63) this proves the assertion of Theorem 4.4.5. □

We are now in a position to compute the constants α_{njk} appearing in Theorems 4.4.2 and 4.4.5. For the ball rB^n of radius $r > 0$ we have by (4.8), for $\varepsilon > 0$,

$$\sum_{j=0}^{n} \varepsilon^{n-j}\kappa_{n-j}V_j(rB^n) = V_n((r+\varepsilon)B^n) = \sum_{j=0}^{n}\binom{n}{j} r^j \varepsilon^{n-j}\kappa_n,$$

hence

$$V_j(rB^n) = \binom{n}{j}\frac{\kappa_n}{\kappa_{n-j}} r^j. \tag{4.64}$$

By Theorem 4.4.5, with $K = B^n$ and $\beta = \mathbb{R}^n$, we have

$$\begin{aligned}\alpha_{njk}\binom{n}{k-j}\frac{\kappa_n}{\kappa_{k-j}} &= \alpha_{njk}V_{n+j-k}(K) = \int_{A(n,k)} V_j(K\cap E)\,\mathrm{d}\mu_k(E)\\ &= \int_{\mathrm{SO}(n)}\int_{E_k^\perp} V_j(K\cap\rho(E_k+t))\,\mathrm{d}\lambda_{n-k}(t)\,\mathrm{d}\nu(\rho).\end{aligned}$$

Since $K\cap\rho(E_k+t)$ is a k-dimensional ball of radius $(1-|t|^2)^{1/2}$, we obtain

$$\begin{aligned}\int_{E_k^\perp} V_j(K\cap\rho(E_k+t))\,\mathrm{d}\lambda_{n-k}(t) &= \binom{k}{j}\frac{\kappa_k}{\kappa_{k-j}}\int_{E_k^\perp}(1-|t|^2)^{j/2}\,\mathrm{d}\lambda_{n-k}(t)\\ &= \binom{k}{j}\frac{\kappa_k}{\kappa_{k-j}}\frac{\kappa_{n+j-k}}{\kappa_j}.\end{aligned}$$

We deduce that

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k\kappa_{n+j-k}}{\binom{n}{k-j}\kappa_j\kappa_n}. \tag{4.65}$$

The Legendre relation (or duplication formula)

$$\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{2^n\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}, \quad \text{or} \quad n!\kappa_n = 2^n\pi^{(n-1)/2}\Gamma\left(\frac{n+1}{2}\right),$$

can be used to bring (4.65) into the form given in Theorem 4.4.2. The proof of Theorem 4.4.2 is now complete.

We proceed to integral-geometric formulae of a different type. An essential feature of the kinematic formulae of Theorem 4.4.2 and the Crofton formulae of Theorem 4.4.5 is that they refer to mean values of functions defined on the intersection of a fixed set and a moving set. We will now replace intersection by other geometric operations. These are the ‘linear’ operations of Minkowski addition and projection, which in the context of the present book are of particular interest. For the rest of this section, the investigations are restricted to convex bodies. First we consider rotational mean values for area measures of Minkowski sums.

Theorem 4.4.6 *Let $K, K' \in \mathcal{K}^n$ be convex bodies, $\omega, \omega' \in \mathcal{B}(\mathbb{S}^{n-1})$ and $j \in \{0, \dots, n-1\}$. Then*

$$\int_{\mathrm{SO}(n)} S_j(K + \rho K', \omega \cap \rho\omega')\,\mathrm{d}\nu(\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} S_k(K, \omega) S_{j-k}(K', \omega'). \tag{4.66}$$

Here we prefer to work with the area measures S_i and not with their renormalized versions Ψ_i, since in this way we can avoid complicated coefficients.

Proof of Theorem 4.4.6 The proof of the measurability of the integrand is similar to that for the integrand in (4.51). For given $\varepsilon > 0$, we define

$$V_\varepsilon(M, \alpha) := \mu_\varepsilon(M, \mathbb{R}^n \times \alpha) = \mathcal{H}^n(B_\varepsilon(M, \alpha))$$

for $M \in \mathcal{K}^n$ and $\alpha \in \mathcal{B}(\mathbb{S}^{n-1})$,

$$h(\rho) := V_\varepsilon(K + \rho K', \omega \cap \rho\omega')$$

for $\rho \in \mathrm{SO}(n)$ and

$$f(\rho, u) := \zeta_\omega(u)\zeta_{\omega'}(\rho^{-1}u)V_\varepsilon(K + \rho K', \mathbb{S}^{n-1})$$

for $\rho \in \mathrm{SO}(n)$ and $u \in \mathbb{S}^{n-1}$, where ζ_ω denotes the characteristic function of ω on $\mathbb{S}^{n-1}$. Then f is measurable. If we define

$$P(\rho, \alpha) := \frac{V_\varepsilon(K + \rho K', \alpha)}{V_\varepsilon(K + \rho K', \mathbb{S}^{n-1})}$$

for $\rho \in \mathrm{SO}(n)$ and $\alpha \in \mathcal{B}(\mathbb{S}^{n-1})$, then $P(\cdot, \alpha)$ is measurable and $P(\rho, \cdot)$ is a probability measure, and from

$$h(\rho) = \int_{\mathbb{S}^{n-1}} \zeta_{\omega\cap\rho\omega'}(u)\, V_\varepsilon(K + \rho K', \mathrm{d}u) = \int_{\mathbb{S}^{n-1}} f(\rho, u)\, P(\rho, \mathrm{d}u)$$

we deduce that h is measurable. From (4.6) we now infer that the function $\rho \mapsto S_j(K + \rho K', \omega \cap \rho\omega')$ is measurable.

For the proof of (4.66), we first consider the special case where $P, P' \in \mathcal{P}^n_n$ are n-dimensional polytopes and $j = n - 1$. Polytopes P, P' are said to be in *general relative position* if no pair of faces F of P and F' of P' is in special position.

Let $\rho \in \mathrm{SO}(n)$ be a rotation for which P and $\rho P'$ are in general relative position. By Lemma 4.4.1, we thus exclude only a set of rotations ρ of ν-measure zero. Let G be a facet of the polytope $P + \rho P'$; let u be the outward unit normal vector of G. Then $G = F + \rho F'$, where $F = F(P, u)$ and $\rho F' = F(\rho P', u)$. Let $\dim F = i$ and $\dim F' = k$; then $i + k = n - 1$ since P and $\rho P'$ are in general relative position. By Theorem 2.2.1(a) we have

$$u \in N(P, F) \cap N(\rho P', \rho F').$$

Conversely, if F is an i-face of P and F' is an $(n - 1 - i)$-face of P' with $N(P, F) \cap N(\rho P', \rho F') \neq \emptyset$, then $F + \rho F'$ is a j-face of $P + \rho P'$ with $j \leq n - 1$. Again excluding a set of rotations of measure zero, we may assume that $j = n-1$. Hence, for almost all ρ,

$$S_{n-1}(P + \rho P', \omega \cap \rho\omega') = \sum_{\substack{G \in \mathcal{F}_{n-1}(P+\rho P') \\ N(P+\rho P',G)\cap\omega\cap\rho\omega' \neq \emptyset}} \mathcal{H}^{n-1}(G)$$
$$= \sum_{i=0}^{n-1} \underbrace{\sum_{F \in \mathcal{F}_i(P)} \sum_{F' \in \mathcal{F}_{n-1-i}(P')}}_{A(\rho,F,F') \neq \emptyset} \mathcal{H}^{n-1}(F + \rho F')$$

with

$$A(\rho, F, F') := [N(P,F) \cap \omega] \cap \rho[N(P', F') \cap \omega'].$$

Observing that, for the rotations and faces considered, the set $A(\rho, F, F')$ is either one-pointed or empty, we arrive at

$$\int_{\mathrm{SO}(n)} S_{n-1}(P + \rho P', \omega \cap \rho\omega')\, \mathrm{d}\nu(\rho)$$
$$= \sum_{i=0}^{n-1} \sum_{F \in \mathcal{F}_i(P)} \sum_{F' \in \mathcal{F}_{n-1-i}(P')} \int_{\mathrm{SO}(n)} \mathcal{H}^{n-1}(F + \rho F')\, \mathrm{card}\, A(\rho, F, F')\, \mathrm{d}\nu(\rho).$$

We fix $i \in \{0, \dots, n-1\}$, $F \in \mathcal{F}_i(P)$ and $F' \in \mathcal{F}_{n-1-i}(P')$. For almost all ρ we have $\dim(F + \rho F') = n-1$ and hence

$$\mathcal{H}^{n-1}(F + \rho F') = [F, \rho F']\mathcal{H}^i(F)\mathcal{H}^{n-1-i}(F').$$

It remains, therefore, to compute

$$\int_{\mathrm{SO}(n)} \mathrm{card}\, A(\rho, F, F')[F, \rho F']\, \mathrm{d}\nu(\rho).$$

We argue in the same way as in the proof of Lemma 4.4.4, defining a more general integral by

$$J(\beta, \beta') := \int_{\mathrm{SO}(n)} \mathrm{card}\,(\beta \cap \rho\beta')[F, \rho F']\, \mathrm{d}\nu(\rho)$$

for Borel sets $\beta \subset L(F)^\perp \cap \mathbb{S}^{n-1}$, $\beta' \subset L(F')^\perp \cap \mathbb{S}^{n-1}$. If $\tau \in \mathrm{SO}(n)$ is a rotation that maps $L(F)^\perp$ into itself and keeps $L(F)$ pointwise fixed, then $\mathrm{card}\,(\tau\beta \cap \rho\beta') = \mathrm{card}\,(\beta \cap \tau^{-1}\rho\beta')$ and $[F, \rho F'] = [F, \tau^{-1}\rho F']$ and hence $J(\tau\beta, \beta') = J(\beta, \beta')$ by the invariance of ν. Since $J(\cdot, \beta')$ is obviously nonnegative, finite and σ-additive, it follows from the uniqueness of spherical Lebesgue measure that $J(\beta, \beta') = c\sigma^{n-1-i}(\beta)$, where the constant c depends on β'. In combination with the same argument, with the roles of β and β' interchanged, this shows that

$$J(\beta, \beta') = \alpha_{ni}\sigma^{n-1-i}(\beta)\sigma^i(\beta'),$$

where the constant α_{ni} evidently depends only on n and i. The special case $\beta = N(P, F) \cap \omega$ and $\beta' = N(P', F') \cap \omega'$ now yields

$$\int_{\mathrm{SO}(n)} S_{n-1}(P + \rho P', \omega \cap \rho\omega')\, \mathrm{d}\nu(\rho) = \sum_{k=0}^{n-1} \gamma_{nk} S_k(P, \omega) S_{n-1-k}(P', \omega')$$

with constants γ_{nk}, where (4.24) has been used. By an approximation argument strictly analogous to that used in the proof of Theorem 4.4.2, we extend this formula to general convex bodies K, K' instead of polytopes P, P'. The explicit values of the constants γ_{nk} are found by choosing balls of radii 1 and r for K and K' respectively, and $\omega = \omega' = \mathbb{S}^{n-1}$, and then comparing equal powers of r. In the resulting formula we now replace K by $K + \varepsilon B^n$ with $\varepsilon > 0$, develop both sides according to formula (4.36) and compare equal powers of ε. This yields formula (4.66) and thus completes the proof of Theorem 4.4.6. □

Formula (4.66) can serve as the source for several other integral-geometric results. These include extensions to support measures, projection formulae and kinematic formulae for non-intersecting convex bodies. We consider results of the latter type first.

For convex bodies $K, K' \in \mathcal{K}^n$ we denote by

$$d(K, K') := \min\{|x - x'| : x \in K,\ x' \in K'\}$$

their Euclidean distance. The principal kinematic formula provides an explicit expression, in terms of intrinsic volumes of K and K', for the invariant measure of the set $\{g \in \mathrm{G}_n : d(K, gK') = 0\}$. From this one can deduce a similar expression for the measure of the set $\{g \in \mathrm{G}_n : 0 < d(K, gK') \le \varepsilon\}$ for given $\varepsilon > 0$, by applying the principal kinematic formula to $K + \varepsilon B_n$ and to K, and subtracting. More generally, we are interested in the measure of the set of all rigid motions g for which $0 < d(K, gK') \le \varepsilon$ and for which the points $x \in K$, $x' \in gK'$ realizing the distance belong to certain preassigned sets. We can impose a further condition on the set of rigid motions to be measured. If $x \in K$, $x' \in K'$ are such that

$$|x - x'| = d(K, K') > 0,$$

we define a unit vector by

$$u(K, K') := \frac{x' - x}{|x' - x|}.$$

The pair (x, x') is not necessarily unique, but one readily verifies that this vector, pointing from K to K' 'in the most direct way', is unique. Clearly, we have $(x, u(K, K')) \in \operatorname{Nor} K$, the normal bundle of K, and $(x', -u(K, K')) \in \operatorname{Nor} K'$. In addition, the vector $u(K, gK')$ may be restricted to lie in preassigned sets. More generally, the following theorem treats the case where shortest distances between a fixed convex body and a disjoint moving convex body are realized within given sets of support elements.

Theorem 4.4.7 *Let $K, K' \in \mathcal{K}^n$ be convex bodies and let $\eta, \eta' \in \mathcal{B}(\Sigma)$. For $\varepsilon > 0$, let $N_\varepsilon(K, K', \eta, \eta')$ be the set of all rigid motions $g \in \mathrm{G}_n$ for which there exist points $x \in K$ and $x' \in gK'$ such that*

$$0 < |x - x'| = d(K, gK') \le \varepsilon, \quad (x, u(K, gK')) \in \eta, \quad (x', -u(K, gK')) \in g\eta'.$$

Then

$$\mu(N_\varepsilon(K, K', \eta, \eta')) = \frac{1}{n\omega_n} \sum_{i=1}^{n} \varepsilon^i \binom{n}{i} \sum_{k=0}^{n-i} \binom{n-i}{k} \Theta_k(K, \eta) \Theta_{n-i-k}(K', \eta'). \tag{4.67}$$

Before the proof, we need some preparations. In the following, measurability for functions on $\mathrm{SO}(n)$ and G_n and for subsets of these spaces refers to the completions of the measure spaces $(\mathrm{SO}(n), \mathcal{B}(\mathrm{SO}(n)), \nu)$ and $(\mathrm{G}_n, \mathcal{B}(\mathrm{G}_n), \mu)$, respectively. The completed measures are denoted by the same symbols.

If $(x, u) \in \Sigma = \mathbb{R}^n \times \mathbb{S}^{n-1}$ and $g = g_{t,\rho}$ is the rigid motion defined by $gx = \rho x + t$ with $\rho \in \mathrm{SO}(n)$ and $t \in \mathbb{R}^n$, we define $g(x, u) := (gx, \rho u)$; also $-(x, u) := (-x, -u)$. Let $K, K' \in \mathcal{K}^n$ be convex bodies. If $(x, u) \in \mathrm{Nor}\, K$ and $(x', u) \in \mathrm{Nor}\, K'$, then $(x + x', u) \in \mathrm{Nor}\,(K + K')$. Conversely, let $(y, u) \in \mathrm{Nor}\,(K + K')$. The point $y \in K + K'$ is of the form $y = x + x'$ with $x \in K$, $x' \in K'$, and necessarily $(x, u) \in \mathrm{Nor}\, K$ and $(x', u) \in \mathrm{Nor}\, K'$. In general, x and x' are not uniquely determined. If $y = z + z'$ with $(z, u) \in \mathrm{Nor}\, K$ and $(z', u) \in \mathrm{Nor}\, K'$ is a different representation, then $z - x = x' - z'$, hence K and K' contain parallel segments lying in supporting hyperplanes with the same outer normal vector u. In this case, we say that K and K' are in *singular relative position*.

We define an operation for sets of support elements which is adapted to Minkowski addition. For $\eta, \eta' \subset \Sigma$ let

$$\eta \star \eta' := \{(x + x', u) \in \Sigma : (x, u) \in \eta,\ (x', u) \in \eta'\}.$$

In particular, for $\beta, \beta' \subset \mathbb{R}^n$ and $\omega, \omega' \subset \mathbb{S}^{n-1}$ we have

$$(\beta \times \omega) \star (\beta' \times \omega') = (\beta + \beta') \times (\omega \cap \omega').$$

If $\eta \subset \mathrm{Nor}\, K$ and $\eta' \subset \mathrm{Nor}\, K'$, then $\eta \star \eta' \subset \mathrm{Nor}\,(K + K')$.

Since the sum of two Borel sets is in general not a Borel set, it is clear that $\eta \star \eta'$ need not be a Borel set if $\eta, \eta' \in \mathcal{B}(\Sigma)$. The following lemma ensures that this will cause no problems.

Lemma 4.4.8 *For convex bodies $K, K' \in \mathcal{K}^n$ and Borel sets $\eta \subset \mathrm{Nor}\, K$, $\eta' \subset \mathrm{Nor}\, K'$, the set $\eta \star \rho\eta'$ is a Borel set for ν-almost all $\rho \in \mathrm{SO}(n)$.*

Proof Let R denote the set of all rotations $\rho \in \mathrm{SO}(n)$ for which K and $\rho K'$ are not in singular relative position. Let $\rho \in R$. For $(y, u) \in \mathrm{Nor}\,(K + \rho K')$ we have $y = x + \rho x'$ with suitable $(x, u) \in \mathrm{Nor}\, K$ and $(\rho x', u) \in \mathrm{Nor}\, \rho K'$. Since $\rho \in R$, the points x and x'

are uniquely determined. Hence, we can define maps

$$\begin{aligned} f &: \mathrm{Nor}\,(K+\rho K') \to \mathrm{Nor}\,K && \text{by } f(y,u) := (x,u),\\ \bar{f} &: \mathrm{Nor}\,(K+\rho K') \to \mathrm{Nor}\,K' && \text{by } \bar{f}(y,u) := (x',\rho^{-1}u). \end{aligned}$$

It is easy to see that these maps are continuous, hence

$$\eta \star \rho\eta' = f^{-1}(\eta) \cap \bar{f}^{-1}(\eta')$$

is a Borel set. By Corollary 2.3.11 we have $\nu(\mathrm{SO}(n) \setminus R) = 0$. This proves the assertion. □

Proof of Theorem 4.4.7 Let $K, K' \in \mathcal{K}^n$ and $\varepsilon > 0$ be given. We are going to show that the set $N_\varepsilon(K, K', \eta, \eta')$ defined in Theorem 4.4.7 is measurable for arbitrary Borel sets $\eta, \eta' \in \mathcal{B}(\Sigma)$. The following proposition will be useful.

Proposition 1 Let φ be a map from $\mathcal{B}(\Sigma)$ into the set of all subsets of G_n that has the following properties.

(a) If $\eta_1, \eta_2 \in \mathcal{B}(\Sigma)$ and $\eta_1 \cap \eta_2 = \emptyset$, then $\varphi(\eta_1) \cap \varphi(\eta_2)$ is of measure zero;
(b) If $\eta_i \in \mathcal{B}(\Sigma)$ for $i \in \mathbb{N}$, then $\varphi(\bigcup_{i\in\mathbb{N}} \eta_i) = \bigcup_{i\in\mathbb{N}} \varphi(\eta_i)$;
(c) If $\eta \in \mathcal{B}(\Sigma)$ is closed, then $\varphi(\eta)$ is measurable.

Under these assumptions, the set $\varphi(\eta)$ is measurable for each $\eta \in \mathcal{B}(\Sigma)$.

In fact, it follows from (a), (b), (c) that the system of all sets $\eta \in \mathcal{B}(\Sigma)$ for which $\varphi(\eta)$ is measurable is a σ-algebra in Σ that contains the closed sets; hence this system coincides with $\mathcal{B}(\Sigma)$. This proves Proposition 1.

Now let $\eta, \eta' \subset \Sigma$ be closed and write

$$\overline{N} := N_\varepsilon(K, K', \eta, \eta') \cup N_0(K, K'),$$

where $N_0(K, K')$ is the set of all $g \in \mathrm{G}_n$ for which gK' touches K (which means that K and gK' intersect, but can be separated by a hyperplane). We want to show that $\overline{N}$ is closed. Let $(g_i)_{i\in\mathbb{N}}$ be a sequence in $\overline{N}$ that converges to some $g \in \mathrm{G}_n$. Then the sequence $(g_i K')_{i\in\mathbb{N}}$ converges to gK'. Since each body $g_i K'$ can be separated from K by a hyperplane, the same is true of gK'. Further, for each $i \in \mathbb{N}$ there exist points $x_i \in \mathrm{bd}\,K$, $x_i' \in \mathrm{bd}\,K'$ such that

$$|x_i - g_i x_i'| = d(K, g_i K') \le \varepsilon,$$

$$\left.\begin{aligned} &(x_i, u(K, g_i K')) \in \eta,\\ &(g_i x_i', -u(K, g_i K')) \in g_i \eta' \end{aligned}\right\} \quad \text{if } d(K, g_i K') > 0.$$

Since $\mathrm{bd}\,K$, $\mathrm{bd}\,K'$ are compact, there exists a sequence $(i_k)_{k\in\mathbb{N}}$ such that $(x_{i_k})_{k\in\mathbb{N}}$ converges to a point $x \in \mathrm{bd}\,K$ and $(x'_{i_k})_{k\in\mathbb{N}}$ converges to a point $x' \in \mathrm{bd}\ K'$. It follows that

$$|x - gx'| = d(K, gK') \le \varepsilon,$$

$$\left.\begin{array}{l}(x, u(K, gK')) \in \eta, \\ (gx', -u(K, gK')) \in g\eta'\end{array}\right\} \quad \text{if } d(K, gK') > 0.$$

Thus $g \in \overline{N}$, which shows that $\overline{N}$ is closed.

Now we define $\varphi(\eta) := N_\varepsilon(K, K', \eta, \eta')$ for $\eta \in \mathcal{B}(\Sigma)$, where $\eta' \subset \Sigma$ is a fixed closed set. Let $\eta_1, \eta_2 \in \mathcal{B}(\Sigma)$ be disjoint and suppose that $g \in \varphi(\eta_1) \cap \varphi(\eta_2)$. For $i = 1, 2$ there is a point $x_i \in K$ at distance $d(K, gK')$ from gK' such that $(x_i, u(K, gK')) \in \eta_i$. Since $\eta_1 \cap \eta_2 = \emptyset$, necessarily $x_1 \neq x_2$; hence K and gK' are in singular relative position. From Corollary 2.3.11 it follows that the set $\varphi(\eta_1) \cap \varphi(\eta_2)$ is of measure zero. If η is closed, then $N_\varepsilon(K, K', \eta, \eta') \cup N_0(K, K')$ is closed, as shown above, and since $N_0(K, K')$ is of measure zero (by the argument used after (4.55)), it follows that $N_\varepsilon(K, K', \eta, \eta')$ is measurable. Thus the map φ satisfies the assumptions of Proposition 1 (where (b) is satisfied trivially). Hence, for each $\eta \in \mathcal{B}(\Sigma)$ the set $N_\varepsilon(K, K', \eta, \eta')$ is measurable. Now we fix $\eta \in \mathcal{B}(\Sigma)$ and define $\psi(\eta') := N_\varepsilon(K, K', \eta, \eta')$. For η' closed, we have shown that $\psi(\eta')$ is measurable. Proposition 1 yields that $N_\varepsilon(K, K', \eta, \eta')$ is a measurable set, where now η, $\eta' \in \mathcal{B}(\Sigma)$ may be arbitrary Borel sets.

We can now apply Fubini's theorem and obtain

$$\mu(N_\varepsilon(K, K', \eta, \eta')) = \int_{\mathrm{SO}(n)} \int_{T(\rho)} \mathrm{d}\mathcal{H}^n \, \mathrm{d}\nu(\rho), \tag{4.68}$$

where

$$T(\rho) := \{t \in \mathbb{R}^n : g_{t,\rho} \in N_\varepsilon(K, K', \eta, \eta')\}.$$

This set can be described in a different way. Suppose that $t \in T(\rho)$. Then there exist points $x \in K$, $x' \in \rho' K' + t$ such that

$$0 < |x - x'| = d(K, \rho K' + t) \leq \varepsilon,$$

$$(x, u(K, \rho K' + t)) \in \eta, \qquad (x', -u(K, \rho K' + t)) \in \rho\eta' + t.$$

The vector $u = u(K, \rho K' + t)$ satisfies $(x, u) \in \operatorname{Nor} K$ and $(x', -u) \in \operatorname{Nor}(\rho K' + t)$. Writing $x' = \rho y' + t$, we have

$$(x, u) \in \eta \cap \operatorname{Nor} K, \quad (\rho y', u) \in \rho[\eta' \cap \operatorname{Nor} K'].$$

With $d = d(K, \rho K' + t)$ we get $x' = x + du$ and hence $t = x - \rho y' + du$. It is obvious that $p(K - \rho K', t) = x - \rho y'$ and $u(K - \rho K', t) = u$, thus

$$\begin{aligned}&(p(K - \rho K', t), u(K - \rho K', t)) \\ &= (x - \rho y', u) \in [\eta \cap \operatorname{Nor}\ K] \star -\rho[\eta' \cap \operatorname{Nor}\ K'],\end{aligned}$$

which shows that

$$t \in M_\varepsilon(K - \rho K', [\eta \cap \operatorname{Nor} K] \star -\rho[\eta \cap \operatorname{Nor} K']) \tag{4.69}$$

(with M_ε as in Section 4.1). If we now assume that K and $-\rho K'$ are not in singular relative position, then we can reverse the argument and conclude that (4.69) implies $t \in T(\rho)$. Thus we have

$$T(\rho) = M_\varepsilon(K - \rho K', [\eta \cap \operatorname{Nor} K] \star -\rho[\eta' \cap \operatorname{Nor} K'])$$

for a set of rotations $\rho \in \mathrm{SO}(n)$, which by Corollary 2.3.11 has full measure. Lemma 4.4.8 tells us that for almost all ρ the set $[\eta \cap \operatorname{Nor} K] \star -\rho[\eta' \cap \operatorname{Nor} K']$ is a Borel set, so that (4.5) can be applied to compute the measure $\mathcal{H}^n(T(\rho))$. Now (4.68) yields

$$\begin{aligned} &\mu(N_\varepsilon(K, K', \eta, \eta')) \qquad (4.70)\\ &= \frac{1}{n} \sum_{i=0}^{n-1} \varepsilon^{n-i} \binom{n}{i} \int_{\mathrm{SO}(n)} \Theta_i(K - \rho K', [\eta \cap \operatorname{Nor} K] \star -\rho[\eta' \cap \operatorname{Nor} K'])\, \mathrm{d}\nu(\rho). \end{aligned}$$

To compute the integrals, we first consider strictly convex bodies. Let $\pi_2 : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ denote the projection onto the second factor. If K is strictly convex and $\eta \subset \Sigma$, then

$$\eta \cap \operatorname{Nor} K = [\mathbb{R}^n \times \pi_2(\eta \cap \operatorname{Nor} K)] \cap \operatorname{Nor} K,$$

since for $u \in \mathbb{S}^{n-1}$ there is exactly one $x \in \mathbb{R}^n$ for which $(x, u) \in \operatorname{Nor} K$. Using (4.7) and the definition of S_m, we get, for $\eta \in \mathcal{B}(\Sigma)$,

$$\begin{aligned} \Theta_m(K, \eta) &= \Theta_m(K, \eta \cap \operatorname{Nor} K) = \Theta_m(K, [\mathbb{R}^n \times \pi_2(\eta \cap \operatorname{Nor} K)] \cap \operatorname{Nor} K) \\ &= \Theta_m(K, \mathbb{R}^n \times \pi_2(\eta \cap \operatorname{Nor} K)) = S_m(K, \pi_2(\eta \cap \operatorname{Nor} K)). \end{aligned}$$

Now let $K, K' \in \mathcal{K}^n$ be strictly convex; then $K - \rho K'$ is strictly convex for all $\rho \in \mathrm{SO}(n)$. Let $\eta, \eta' \in \mathcal{B}(\Sigma)$. Observing that

$$[\eta \cap \operatorname{Nor} K] \star -\rho[\eta' \cap \operatorname{Nor}\ K'] \subset \operatorname{Nor}(K - \rho K'),$$

$$\pi_2([\eta \cap \operatorname{Nor} K] \star -\rho[\eta' \cap \operatorname{Nor} K']) = \pi_2(\eta \cap \operatorname{Nor} K) \cap -\rho\pi_2(\eta' \cap \operatorname{Nor} K')$$

and $\Theta_m(-K', -\eta') = \Theta_m(K', \eta')$, we deduce from (4.70) and (4.66) that (4.67) is true.

Equality (4.67) in the general case is now proved by approximation. Again, let $K, K' \in \mathcal{K}^n$ be general convex bodies.

Proposition 2 For fixed $\eta' \in \mathcal{B}(\Sigma)$, the function $\mu(N_\varepsilon(K, K', \cdot, \eta'))$ is a finite measure on $\mathcal{B}(\Sigma)$.

For the proof, only the σ-additivity has to be shown. For $\eta_1, \eta_2 \in \mathcal{B}(\Sigma)$ with $\eta_1 \cap \eta_2 = \emptyset$ we have seen above that

$$\mu(N_\varepsilon(K, K', \eta_1, \eta') \cap N_\varepsilon(K, K', \eta_2, \eta')) = 0,$$

hence the additivity of μ implies that $\mu(N_\varepsilon(K, K', \cdot, \eta'))$ is additive. Let $(\eta_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{B}(\Sigma)$ such that $\eta_i \downarrow \emptyset$. Then

$$N_\varepsilon(K, K', \eta_i, \eta') \downarrow N := \bigcap_{j\in\mathbb{N}} N_\varepsilon(K, K', \eta_j, \eta')$$

for $i \to \infty$, hence

$$\lim_{i\to\infty} \mu(N_\varepsilon(K, K', \eta_i, \eta')) = \mu(N)$$

by the σ-additivity of μ. Let $g \in N$. For each $j \in \mathbb{N}$ there is a point $x_j \in \mathrm{bd}\, K$ at distance $d(K, gK')$ from gK' such that $(x_j, u(K, gK')) \in \eta_j$. Since $\bigcap_{j\in\mathbb{N}} \eta_j = \emptyset$, not all the points x_j coincide, thus K and gK' are in singular relative position. By Corollary 2.3.11, $\mu(N) = 0$. Thus $\mu(N_\varepsilon(K, K', \cdot, \eta'))$ is $\emptyset$-continuous and hence σ-additive. This proves Proposition 2.

Proposition 3 If the sequence $(K_i)_{i\in\mathbb{N}}$ in $\mathcal{K}^n$ converges to K, then

$$\liminf_{i\to\infty} \mu(N_\varepsilon(K_i, K', \eta, \eta')) \geq \mu(N_\varepsilon(K, K', \eta, \eta'))$$

for all open sets $\eta, \eta' \subset \Sigma$, and

$$\lim_{i\to\infty} \mu(N_\varepsilon(K_i, K', \Sigma, \Sigma)) = \mu(N_\varepsilon(K, K', \Sigma, \Sigma)).$$

For the proof, let $\eta, \eta' \subset \Sigma$ be open and denote by $N_\varepsilon^0(K, K', \eta, \eta')$ the set of all $g \in N_\varepsilon(K, K', \eta, \eta')$ for which $d(K, gK') \neq \varepsilon$ and K and gK' are not in singular relative position. Let $g \in N_\varepsilon^0(K, K', \eta, \eta')$. For each $i \in \mathbb{N}$ choose points $x_i \in \mathrm{bd}\ K_i$ and $x_i' \in \mathrm{bd}\, gK'$ such that $|x_i - x_i'| = d(K_i, gK')$. Let $x \in \mathrm{bd}\, K$, $x' \in \mathrm{bd}\, gK'$ be the unique points for which $|x - x'| = d(K, gK')$. It is easy to see that $x_i \to x$ and $x_i' \to x'$ for $i \to \infty$. If follows that $0 < d(K_i, gK) < \varepsilon$ for all i sufficiently large. Since $(x, u(K, gK')) \in \eta$, $(x', -u(K, gK')) \in g\eta'$ and η, η' are open, we also have $(x_i, u(K_i, gK')) \in \eta$ and $(x_i', -u(K_i, gK')) \in g\eta'$ for almost all i; thus $g \in N_\varepsilon(K_i, K', \eta, \eta')$ for almost all i. We have proved that

$$N_\varepsilon^0(K, K', \eta, \eta') \subset \liminf_{i\to\infty} \mu(N_\varepsilon(K_i, K', \eta, \eta'))$$

and hence that

$$\begin{aligned}\mu(N_\varepsilon(K, K', \eta, \eta')) = \mu(N_\varepsilon^0(K, K', \eta, \eta')) &\leq \mu\left(\liminf_{i\to\infty} N_\varepsilon(K_i, K', \eta, \eta')\right) \\ &\leq \liminf_{i\to\infty} \mu(N_\varepsilon(K_i, K', \eta, \eta')),\end{aligned}$$

which is the first assertion of Proposition 3. Since $N_\varepsilon(K_i, K', \Sigma, \Sigma)$ is the set of all $g \in \mathrm{G}_n$ for which gK' meets $K_i + \varepsilon B^n$ but not K_i, its measure can, by means of the principal kinematic formula, be expressed in terms of intrinsic volumes, from which its continuous dependence on K_i is clear. This proves Proposition 3.

Of course, results similar to Propositions 2 and 3 are valid with the roles of the pairs (K, η) and (K', η') interchanged.

We are now in a position to prove formula (4.67) for general convex bodies. Let us write the assertion (4.67) in the form

$$\mu(N_\varepsilon(K, K', \eta, \eta')) = \sum_{r,s=0}^{n-1} a_{rs}\Theta_r(K, \eta)\Theta_s(K', \eta'), \tag{4.71}$$

where ε is fixed and the coefficients a_{rs} are those resulting from rearranging (4.67). Then equality (4.71) is already established if K and K' are strictly convex bodies and $\eta, \eta' \in \mathcal{B}(\Sigma)$.

Let $K \in \mathcal{K}^n$ be strictly convex and $K' \in \mathcal{K}^n$ arbitrary. Let $(K'_i)_{i\in\mathbb{N}}$ be a sequence of strictly convex bodies converging to K'. By Proposition 3,

$$\liminf_{i\to\infty} \mu(N_\varepsilon(K, K'_i, \eta, \Sigma)) \geq \mu(N_\varepsilon(K, K', \eta, \Sigma))$$

for open $\eta \subset \Sigma$, and

$$\lim_{i\to\infty} \mu(N_\varepsilon(K, K'_i, \Sigma, \Sigma)) = \mu(N_\varepsilon(K, K', \Sigma, \Sigma)).$$

Thus, $\mu(N_\varepsilon(K, K'_i, \cdot, \Sigma)) \xrightarrow{w} \mu(N_\varepsilon(K, K', \cdot, \Sigma))$ for $i \to \infty$. Since (4.71) can be applied to K and K'_i, it follows that

$$\mu(N_\varepsilon(K, K', \cdot, \Sigma)) = \sum_{r,s} a_{rs}\Theta_r(K, \cdot)\Theta_s(K', \Sigma). \tag{4.72}$$

Let K and K'_i be as before and let $\eta \subset \Sigma$ be open. By Proposition 3 we have

$$\liminf_{i\to\infty} \mu(N_\varepsilon(K, K'_i, \eta, \eta')) \geq \mu(N_\varepsilon(K, K', \eta, \eta'))$$

for open $\eta' \subset \Sigma$, and from (4.72) we obtain

$$\lim_{i\to\infty} \mu(N_\varepsilon(K, K'_i, \eta, \Sigma)) = \mu(N_\varepsilon(K, K', \eta, \Sigma)).$$

Thus, $\mu(N_\varepsilon(K, K'_i, \eta, \cdot)) \xrightarrow{w} \mu(N_\varepsilon(K, K', \eta, \cdot))$ for $i \to \infty$. From (4.71) and the weak continuity of the support measures we infer that

$$\mu(N_\varepsilon(K, K', \eta, \eta')) = \sum_{r,s} a_{rs}\Theta_s(K, \eta)\Theta_s(K', \eta') \tag{4.73}$$

for all $\eta' \in \mathcal{B}(\Sigma)$. Since for fixed K, K', η' both sides, as functions of η, are measures, the equality holds for arbitrary Borel sets η.

Now let $K, K' \in \mathcal{K}^n$ both be arbitrary. Choosing a sequence $(K_i)_{i\in\mathbb{N}}$ of strictly convex bodies converging to K, we deduce from Proposition 3 that

$$\mu(N_\varepsilon(K_i, K', \cdot, \Sigma)) \xrightarrow{w} \mu(N_\varepsilon(K, K', \cdot, \Sigma))$$

and then from (4.72) that

$$\mu(N_\varepsilon(K, K', \cdot, \Sigma)) = \sum_{r,s} a_{rs}\Theta_r(K, \cdot)\Theta_s(K', \Sigma).$$

Here the roles of K and K' may be interchanged, hence

$$\mu(N_\varepsilon(K,K',\Sigma,\cdot)) = \sum_{r,s} a_{rs}\Theta_r(K,\Sigma)\Theta_s(K',\cdot). \tag{4.74}$$

Let K_i be as before, and let $\eta' \subset \Sigma$ be open. From Proposition 3 and from (4.74) we see that

$$\mu(N_\varepsilon(K_i,K',\cdot,\eta')) \xrightarrow{w} \mu(N_\varepsilon(K,K',\cdot,\eta')).$$

From (4.73) and the weak continuity of the support measures we get

$$\mu(N_\varepsilon(K,K',\eta,\eta')) = \sum_{r,s} a_{rs}\Theta_r(K,\eta)\Theta_s(K',\eta')$$

for $\eta \in \mathcal{B}(\Sigma)$. Since both sides are measures as functions of η', we deduce the equality for arbitrary $\eta' \in \mathcal{B}(\Sigma)$. This completes the proof of Theorem 4.4.7. □

Comparing the coefficients of ε^{n-j} in formulae (4.67) and (4.70), we obtain the following theorem as a corollary.

Theorem 4.4.9 *Let $K,K' \in \mathcal{K}^n$ be convex bodies, let $\eta \subset \operatorname{Nor} K$, $\eta' \subset \operatorname{Nor} K'$ be Borel sets and let $j \in \{0,\dots,n-1\}$. Then*

$$\int_{SO(n)} \Theta_j(K+\rho K',\eta \star \rho\eta')\,\mathrm{d}\nu(\rho) = \frac{1}{\omega_n}\sum_{k=0}^{j}\binom{j}{k}\Theta_k(K,\eta)\Theta_{j-k}(K',\eta'). \tag{4.75}$$

This formula generalizes (4.66). Another special case of (4.75) is the equality

$$\int_{SO(n)} C_j(K+\rho K',\beta+\rho\beta')\,\mathrm{d}\nu(\rho) = \frac{1}{\omega_n}\sum_{k=0}^{j}\binom{j}{k}C_k(K,\beta)C_{j-k}(K',\beta'), \tag{4.76}$$

which is valid for Borel sets $\beta \subset \operatorname{bd} K$, $\beta' \subset \operatorname{bd} K'$. This follows from (4.75) since $C_m(K,\beta) = \Theta_m(K,\beta\times\mathbb{S}^{n-1})$ and

$$(\beta+\rho\beta')\times\mathbb{S}^{n-1} = (\beta\times\mathbb{S}^{n-1})\star\rho(\beta'\times\mathbb{S}^{n-1}).$$

From (4.75) we shall now derive integral-geometric formulae for projections.

If $E \subset \mathbb{R}^n$ is a linear subspace, we denote by $x\,|\,E$ the image of $x \in \mathbb{R}^n$ under orthogonal projection to E, and we use a similar notation for subsets of $\mathbb{R}^n$. For sets $\eta \subset \Sigma$ we define

$$\eta\,|\,E := \{(x\,|\,E,u) : (x,u) \in \eta \text{ and } u \in E\}.$$

Theorem 4.4.10 *Let $K \in \mathcal{K}^n$ be a convex body, let $\eta \subset \operatorname{Nor} K$ be a Borel set and let $k \in \{1,\dots,n-1\}$. Then*

$$\int_{G(n,k)} \Theta'_j(K\,|\,E,\eta\,|\,E)\,\mathrm{d}\nu_k(E) = \frac{\omega_k}{\omega_n}\Theta_j(K,\eta) \tag{4.77}$$

for $j \in \{0,\dots,k-1\}$, where Θ'_j is the support measure taken with respect to the subspace E.

Special cases of (4.77) are the formulae

$$\int_{G(n,k)} S'_j(K\,|\,E, \omega \cap E)\,\mathrm{d}\nu_k(E) = \frac{\omega_k}{\omega_n} S_j(K,\omega) \tag{4.78}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and

$$\int_{G(n,k)} C'_j(K\,|\,E, \beta\,|\,E)\,\mathrm{d}\nu_k(E) = \frac{\omega_k}{\omega_n} C_j(K,\beta) \tag{4.79}$$

for Borel sets $\beta \subset \operatorname{bd} K$.

Proof of Theorem 4.4.10 We choose a fixed k-subspace $E \in G(n,k)$ and an $(n-k)$-dimensional convex body $K' \subset E^\perp$ with $\mathcal{H}^{n-k}(K') = 1$ and let η' be the set of all support elements (x,u) of K' with $x \in \operatorname{relint} K'$. From (4.5) it is then clear that

$$\Theta_j(K',\eta') := \begin{cases} n\kappa_k \binom{n}{k}^{-1} & \text{for } j = n-k, \\ 0 & \text{for } j \neq n-k. \end{cases} \tag{4.80}$$

For given $\varepsilon > 0$, let $x \in M_\varepsilon(K+K', \eta \star \eta')$. Then

$$0 < d(K+K',x) \leq \varepsilon,$$

$$(p(K+K',x), u(K+K',x)) \in \eta \star \eta'.$$

Writing $u := u(K+K',x)$, we have $p(K+K',x) = y + y'$ with $(y,u) \in \eta \subset \operatorname{Nor} K$ and $(y',u) \in \eta' \subset \operatorname{Nor} K'$. This implies $u \in E$ by the choice of η' and $(y\,|\,E, u) \in \eta\,|\,E$. Obviously we have $\left|x\,|\,E - y\,|\,E\right| = d(K+K',x)$, $p(K\,|\,E, x\,|\,E) = y\,|\,E$ and $u(K\,|\,E, x\,|\,E) = u$; thus $x\,|\,E \in M'_\varepsilon(K\,|\,E, \eta\,|\,E)$, where M'_ε is defined with respect to the subspace E. Conversely, one readily verifies that each point in $M'_\varepsilon(K\,|\,E, \eta\,|\,E)$ is the projection of a point in $M_\varepsilon(K+K', \eta \star \eta')$, thus

$$M_\varepsilon(K+K', \eta \star \eta')\,|\,E = M'_\varepsilon(K\,|\,E, \eta\,|\,E).$$

If K and K' are not in singular relative position, then the set of points in $M_\varepsilon(K+K', \eta \star \eta')$ that project to the same point of E is a translate of $\operatorname{relint} K'$; furthermore, $\eta\,|\,E$ is a Borel set. Hence, Fubini's theorem gives

$$\mathcal{H}^n(M_\varepsilon(K+K', \eta \star \eta')) = \mathcal{H}^k(M'_\varepsilon(K\,|\,E, \eta\,|\,E)),$$

and (4.5) yields

$$\frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{n-i}\binom{n}{i}\Theta_i(K+K', \eta \star \eta') = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon^{k-j}\binom{k}{j}\Theta'_j(K\,|\,E, \eta\,|\,E).$$

For ν-almost all $\rho \in \mathrm{SO}(n)$, this can be applied to ρE, $\rho K'$, $\rho\eta'$ instead of E, K', η'. If we then integrate over all rotations ρ, use (4.75) and (4.80) and compare the coefficients of equal powers of ε, we complete the proof of formula (4.77) (recall that ν_k is the image measure of ν under the map $\rho \mapsto \rho E$ from $\mathrm{SO}(n)$ to $G(n,k)$). $\square$

Theorem 4.4.7 has a counterpart where the moving convex body K' is replaced by a moving k-flat. Let $K \in \mathcal{K}^n$ be a convex body and $E \in A(n,K)$ be a k-dimensional affine subspace of $\mathbb{R}^n$, where $k = \{0,\dots,n-1\}$. If $x \in K$ and $x' \in E$ are points between which the distance is a minimum, we write $d(K,E) := |x - x'|$ and $u(K,E) := (x'-x)/|x'-x|$.

Theorem 4.4.11 *Let $K \in \mathcal{K}^n$ be a convex body and let $\eta \in \mathcal{B}(\Sigma)$ be a Borel set; let $k \in \{0,\dots,n-1\}$. For $\varepsilon > 0$, let $N_\varepsilon^k(K,\eta)$ be the set of all k-flats $E \in A(n,k)$ for which there exist points $x \in K$ and $x' \in E$ such that*

$$0 < |x - x'| = d(K,E) \le \varepsilon, \qquad (x, u(K,E)) \in \eta.$$

Then

$$\mu_k(N_\varepsilon^k(K,\eta)) = \frac{\kappa_{n-k}}{\omega_n} \sum_{i=1}^{n-k} \varepsilon^i \binom{n-k}{i} \Theta_{n-k-i}(K,\eta). \tag{4.81}$$

Proof Some of the following arguments, not surprisingly, are similar to those in the proof of Theorem 4.4.7. The convex body K and the k-flat E are said to be *in singular relative position* if some translate of E supports K (that is, meets K and lies in a supporting hyperplane of K) and contains more than one point of K. We fix a k-dimensional linear subspace L_k of $\mathbb{R}^n$ and use the fact that the invariant measure μ_k can be written as the image measure of $\lambda_{n-k} \otimes \nu$ under the map $(t,\rho) \mapsto \rho(L_k + t)$ from $L_k^\perp \times \mathrm{SO}(n)$ into $A(n,k)$, where λ_{n-k} is the restriction of $\mathcal{H}^{n-k}$ to the Borel sets in $L_k^\perp$. For $N_0^k(K)$, the set of all k-flats supporting K, we therefore obtain

$$\mu_k(N_0^k(K)) = \int_{\mathrm{SO}(n)} \int_{T(\rho)} \mathrm{d}\mathcal{H}^{n-k}\, \mathrm{d}\nu(\rho)$$

with

$$T(\rho) := \{t \in L_k^\perp : \rho(L_k + t) \in N_0^k(K)\}.$$

Since $T(\rho)$ is the relative boundary of the image of $\rho^{-1}K$ under orthogonal projection onto $L_k^\perp$, we have $\mathcal{H}^{n-k}(T(\rho)) = 0$ and thus $\mu_k(N_0^k(K)) = 0$.

We complete the measure space $(A(n,k), \mathcal{B}(A(n,k), \mu_k)$, and measurability refers to this completion. First we show that $N_\varepsilon^k(K,\eta)$ is measurable. Let $\eta \subset \Sigma$ be closed. Then $\overline{N} := N_\varepsilon^k(K,\mu) \cup N_0^k(K)$ is closed. In fact, let $(E_i)_{i\in\mathbb{N}}$ be a sequence in $\overline{N}$ converging to some k-flat E. Then $d(K,E) \le \varepsilon$, and if $d(K,E) = 0$, then E lies in a supporting hyperplane of K. For each $i \in \mathbb{N}$ there exist points $x_i \in \mathrm{bd}\, K$, $x_i' \in E_i$ such that

$$|x_i - x_i'| = d(K,E_i) \le \varepsilon, \qquad (x_i, u(K,E_i)) \in \eta \quad \text{if } d(K,E_i) > 0.$$

If $d(K,E) > 0$, then by compactness some subsequence of $(x_i, u(K,E_i))_{i\in\mathbb{N}}$ converges to an element $(x,u) \in \eta$, and clearly $x \in K$ and $u = u(K,E)$. Thus $E \in \overline{N}$, which shows that $\overline{N}$ is closed. Since $\mu_k(N_0^k(K)) = 0$, the set $N_\varepsilon(K,\eta)$ is measurable.

Let $\eta_1, \eta_2 \in \mathcal{B}(\Sigma)$ be sets with $\eta_1 \cap \eta_2 = \emptyset$. Suppose that $E \in N_\varepsilon^k(K,\eta_1) \cap N_\varepsilon^k(K,\eta_2)$. Then there are points $x_1, x_2 \in K$ at distance $d(K,E)$ from E and such that $(x_i, u(K, E)) \in \eta_i$ for $i = 1, 2$. Since $\eta_1 \cap \eta_2 = \emptyset$, we must have $x_1 \neq x_2$. Thus, the flat $E - d(K, E)u(K, E)$ supports K and contains more than one point of K; in other words, K and E are in singular relative position. From Corollary 2.3.11 it follows that $N^k_\varepsilon(K, \eta_1) \cap N^k_\varepsilon(K, \eta_2)$ has μ_k-measure zero.

By an argument analogous to that in the proof of Theorem 4.4.7, we now conclude that $N^k_\varepsilon(K, \eta)$ is measurable for each $\eta \in \mathcal{B}(\Sigma)$.

By Fubini's theorem,

$$\mu_k(N^k_\varepsilon(K, \eta)) = \int_{SO(n)} \int_{T(\rho)} \mathrm{d}\mathcal{H}^{n-k} \, \mathrm{d}\nu(\rho),$$

where now

$$T(\rho) := \{t \in L_k^\perp : \rho(L_k + t) \in N^k_\varepsilon(K, \eta)\}.$$

Let $t \in T(\rho)$. There are points $x \in K$, $x' \in \rho(L_k + t)$ such that $0 < |x - x'| = d(K, \rho(L_k + t)) \leq \varepsilon$ and $(x, u(K, \rho(L_k + t)) \in \eta$. Write $d(K, \rho(L_k + t)) = d$ and $u(K, \rho(L_k + t)) = u$. From $x' = \rho(y' + t)$ with $y' \in L_k$ and $x' = x + du$ we obtain

$$t = \rho^{-1}x + d\rho^{-1}u - y' = z + d\rho^{-1}u$$

with $z := \rho^{-1}x - y'$. Because u is orthogonal to $\rho(L_k + t)$, we have $\rho^{-1}u \in L_k^\perp$ and thus $z \in L_k^\perp$. Since $y' \in L_k$, the point z is obtained from $\rho^{-1}x$ by orthogonal projection to $L_k^\perp$, thus $z \in \rho^{-1}K \,|\, L_k^\perp$. From $(x, u) \in \eta \cap \operatorname{Nor} K$ we get $(\rho^{-1}x, \rho^{-1}u) \in \rho^{-1}\eta \cap \operatorname{Nor} \rho^{-1}K$ and hence

$$(z, \rho^{-1}u) \in (\rho^{-1}\eta \,|\, L_k^\perp) \cap \operatorname{Nor}(\rho^{-1}K \,|\, L_k^\perp).$$

Thus, $t = z + d\rho^{-1}u$ satisfies

$$t \in M'_\varepsilon(\rho^{-1}K \,|\, L_k^\perp, \rho^{-1}\eta \,|\, L_k^\perp),$$

where M'_ε is constructed in $L_k^\perp$. If we assume that K and $\rho(L_k + t)$ are not in singular relative position, then the argument can be reversed and we arrive at

$$T(\rho) = M'_\varepsilon(\rho^{-1}K \,|\, L_k^\perp, \rho^{-1}\eta \,|\, L_k^\perp).$$

We will show that the projection $\rho^{-1}\eta \,|\, L_k^\perp$ is a Borel set for almost all ρ. Let R be the set of all $\rho \in SO(n)$ for which K and ρL_k are not in singular relative position. By Corollary 2.3.11, $\nu(SO(n) \setminus R) = 0$. Let $\rho \in R$. For $(y, u) \in \operatorname{Nor}(K \,|\, \rho L_k^\perp)$ we have $y = x \,|\, \rho L_k^\perp$ with suitable $(x, u) \in \operatorname{Nor} K$. Since $\rho \in R$, the point x is uniquely determined. Hence we can define a map

$$f : \operatorname{Nor}(K \,|\, \rho L_k^\perp) \to \operatorname{Nor} K \quad \text{by } f(y, u) := (x, u).$$

This map is continuous, hence the set $\eta \,|\, \rho L_k^\perp = f^{-1}(\eta)$ is a Borel set.

For almost all ρ, we can apply (4.5) to compute $\mathcal{H}^{n-k}(T(\rho))$. Integrating over $\mathrm{SO}(n)$ and applying Theorem 4.4.10, we obtain

$$\mu(N_\varepsilon^k(K,\eta)) = \frac{1}{n-k}\sum_{j=0}^{n-k-1}\varepsilon^{n-k-j}\binom{n-k}{j}\int_{\mathrm{SO}(n)}\Theta'_j(K\,|\,\rho L_k^\perp,\eta\,|\,\rho L_k^\perp)\,\mathrm{d}\nu(\rho)$$
$$= \frac{\kappa_{n-k}}{\omega_n}\sum_{i=1}^{n-k}\varepsilon^i\binom{n-k}{j}\Theta_{n-k-i}(K,\eta).$$

This completes the proof of Theorem 4.4.11. □

As a consequence of formula (4.81) we note the limit relation

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mu_k(N_\varepsilon^k(K,\eta)) = \frac{\omega_{n-k}}{\omega_n}\Theta_{n-k-1}(K,\eta). \tag{4.82}$$

It can be viewed as an integral-geometric interpretation of the support measures.

Notes for Section 4.4

1. For more information on the topic of this section, we refer to Part II (Integral Geometry) of the book by Schneider and Weil [1740]. In particular, the Notes for §§5.1, 5.2, 5.3, 6.1, 6.2 of that book contain hints to various related topics, which are mostly outside the scope of the present book.

   An impression of more recent developments in integral geometry can be gained from the article by Bernig and Fu [211] on Hermitian integral geometry and from the survey [206] of Bernig and the lecture notes [642] of Fu, both under the title of 'Algebraic Integral Geometry'.

   In the following notes, we restrict ourselves, more or less, to topics related more closely to the present section.
2. *Integral geometry*. General older reference works for integral geometry are the books by Blaschke [252] (see also Volume 2 of his collected works, [253]), Santaló [1628], Hadwiger [911] (Chapter 6), Stoka [1819], Santaló [1630] (see also Part II of his Selected Works, [1631]). In style and attitude these books differ considerably from one another. In contrast with other treatments of integral geometry, which consider smooth submanifolds and use differential-geometric methods, this section of the present volume is restricted to convex bodies (and finite unions of them) and follows a measure-theoretic approach. It thus tries to achieve a synthesis of the viewpoints of Hadwiger [911] and Federer [556]. In a similar spirit, parts of integral geometry relevant for stochastic geometry are presented in Schneider and Weil [1740].
3. *The classical principal kinematic formula*. The specialization to total measures, that is, formula (4.52) for intrinsic volumes or its equivalent formulation in terms of quermassintegrals, includes the convex case of the classical principal kinematic formula of Blaschke, Santaló and Chern (see Santaló [1630]). For convex bodies and sets of the convex ring, a short proof of formula (4.53) (which extends to a proof of (4.52), but apparently not of (4.51)) was given by Mani–Levitska [1325]. Hadwiger [898, 901, 909, 911] developed an elegant method of proof. It goes back to ideas of Blaschke and uses Hadwiger's characterization theorem for the intrinsic volumes; see Note 5 in §6.4.
4. *Intersection formulae for curvature measures*. Federer [556] developed a theory of curvature measures for sets of positive reach and in this general setting proved the formulae of Theorems 4.4.2 and 4.4.5. A short proof was given by Rother and Zähle [1594]. For convex bodies, simpler approaches are possible. Schneider [1687] proved (4.51) as a consequence of the characterization theorem for curvature measures quoted in Section 4.2, Note 11. A slightly simpler proof was given in Schneider [1695]. The proof presented

above is taken from Schneider and Weil [1739], and Lemma 4.4.1 is taken from Goodey and Schneider [742]. A different (and similarly elementary) proof of this lemma is given in [1740], Lemma 13.2.1.

As an example of later developments, leading to very general versions of the kinematic formula (and requiring considerably deeper techniques), we mention the work of Fu [637].

For kinematic and Crofton formulae and their ramifications, we refer also to the survey article by Hug and Schneider [1015].

The method by which Theorem 4.4.5 was deduced from Theorem 4.4.2 is taken from Federer [556].

5. *Kinematic and Crofton formulae for support measures.* Theorems 4.4.2 and 4.4.5 can be extended to support measures, if the intersection of Borel sets in $\mathbb{R}^n$ is replaced by a suitable operation for sets of support elements. Such extensions are due to Glasauer [718]; see also Note 4 on p. 197 in [1740].
6. *Kinematic formulae for the convex hull operation.* Formulae of kinematic type for the convex hull of a fixed and a moving convex body were proved by Glasauer [719, 721]. They involve weighted integrals and limit procedures; cf. Note 8 on pp. 200–201 in [1740].
7. *Translative integral geometry.* The simple form of the kinematic formulae (4.51), in particular the fact that on the right-hand side the convex bodies K and K' appear separately, is a result of the integration over the group of rigid motions. If one integrates only over the translations, one still has a formula of the type

$$\int_{\mathbb{R}^n} \Phi_j(K \cap (K'+t), \beta \cap (\beta'+t))\,\mathrm{d}t = \sum_{k=j}^{n} \Phi_k^{(j)}(K, K', \beta \times \beta') \tag{4.83}$$

with finite measures $\Phi_k^{(j)}(K, K', \cdot)$ on $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$. We refer to §6.4 of the book [1740] and mention here only the following.

The global version of (4.83) reads

$$\int_{\mathbb{R}^n} V_j(K \cap (K'+t))\,\mathrm{d}t = \sum_{k=j}^{n} V_k^{(j)}(K, K') \tag{4.84}$$

with *mixed functionals* $V_k^{(j)}$ on $\mathcal{K}^n \times \mathcal{K}^n$, in particular $V_j^{(j)}(K, K') = V_j(K)V_n(K')$ and $V_n^{(j)}(K, K') = V_n(K)V_j(K')$. Explicit representations of the other mixed functionals are only known in special cases.
8. *Mixed bodies.* A translative integral for support functions leads to a notion of mixed convex bodies. The *centred support function* of the convex body $K \in \mathcal{K}^n$ is defined by $h^*(K, \cdot) := h(K - s(K), \cdot)$. (The subtraction of the Steiner point makes it translation invariant.) For convex bodies $K, M \in \mathcal{K}^n$, there are continuous functions $h_1^*(K, M; \cdot), \dots, h_n^*(K, M; \cdot)$ on $\mathbb{S}^{n-1}$ such that

$$\int_{\mathbb{R}^n} h^*(K \cap (M+t), \cdot)\,\mathrm{d}t = \sum_{k=1}^{n} h_k^*(K, M; \cdot). \tag{4.85}$$

The left-hand side of (4.85) is the support function of a convex body $T(K, M)$, which we call the *translation mixture* of K and M. It turns out that the functions $h_k^*(K, M; \cdot)$ are also support functions, hence there exists a polynomial expansion

$$T(\lambda K, \mu M) = \sum_{k=1}^{n} \lambda^k \mu^{n+1-k} T_k(K, M)$$

with convex bodies $T_k(K, M)$, which are called the *mixed bodies* of K and M. See Goodey and Weil [751] and Schneider [1725].

9. *Iterations*. The kinematic formulae (4.51) can clearly be extended to the intersections of a fixed convex body and a finite number of independently moving convex bodies. In particular, from (4.52) one deduces by induction that

$$\int_{G_n} \cdots \int_{G_n} V_j(K_0 \cap g_1 K_1 \cap \cdots \cap g_p K_p)\, \mathrm{d}\mu(g_1) \cdots \mathrm{d}\mu(g_p)$$
$$= \sum_{\substack{k_0,\dots,k_p=j \\ k_0+\cdots+k_p=pn+j}}^{n} c^{(j)}_{k_0,\dots,k_p} V_{k_0}(K_0) \cdots V_{k_p}(K_p)$$

with

$$c^{(j)}_{k_0,\dots,k_p} = \frac{\prod_{i=0}^{p} k_i!\kappa_{k_i}}{j!\kappa_j (n!\kappa_n)^p}$$

for $K_0, K_1, \dots, K_p \in \mathcal{K}^n$, $p \in \mathbb{N}$ and $j \in \{0, \dots, n\}$. Again we refer to [1740].

10. *Applications of intersection formulae in stochastic geometry*. The integral-geometric formulae for the intersection of a fixed and a moving set have remarkable applications in stochastic geometry, where they yield expectation formulae for the functional densities that can be associated with certain random sets or particle processes. We refer the reader to the book by Schneider and Weil [1740], which lays particular emphasis on the use of integral geometry in stochastic geometry.

11. *Rotational mean values*. Theorem 4.4.6 is due to Schneider [1680]. The proof given here is essentially that of Schneider [1704]. In that paper, the more general version of Theorem 4.4.9 is obtained. The special case (4.76) was proved by Weil [1942]. Also, the projection formula of Theorem 4.4.10 appears in Schneider [1704]. Its special case (4.78) had previously been proved in Schneider [1680], and case (4.79) in Weil [1942]. A certain extension of the projection formula (4.77) to sets of the convex ring, for which multiplicities have to be taken into account, is treated in Schneider [1715]. A more general integral-geometric formula derived from Theorem 4.4.6 is applied in Papaderou–Vogiatzaki and Schneider [1514] to a question on geometric collision probabilities.

12. *Distance formulae*. The proof of Theorem 4.4.7 is adapted from Schneider [1704] and Schneider [1690]. The special case $\eta = \beta \times \mathbb{S}^{n-1}$ of Theorem 4.4.11 was first proved in Schneider [1687]; the proof given above is an immediate extension. The integral-geometric interpretation (4.82) of generalized curvature measures was mentioned in Schneider [1694]; its specialization to area measures is due to Firey [602], and its specialization to Federer's curvature measures appears in Schneider [1687].

    Theorem 4.4.7 can be generalized, from indicator functions to more general measurable functions. Let $K, K' \in \mathcal{K}^n$ be convex bodies and let $g \in G_n$ be a rigid motion for which $K \cap gK' = \emptyset$. Let $x \in K$, $x' \in gK'$ be points between which the distance is minimal. Before Theorem 4.4.7, we wrote $d(K, gK') = |x' - x|$ and $u(K, gK') = (x' - x)/|x' - x|$; we also put $x(K, gK') := x$ if the pair (x, x') is unique, which is the case for μ-almost all g.

    Theorem Let $f : (0, \infty) \times \mathrm{bd}\, K \times \mathrm{bd}\, K' \to \mathbb{R}$ be a measurable function for which the integrals in (4.86) are finite. Then

$$\int_{K \cap gK' = \emptyset} f(d(K, gK'), x(K, gK'), g^{-1}x(gK', K))\, \mathrm{d}\mu(g) \tag{4.86}$$
$$= \omega_n \sum_{j=0}^{n-1} \sum_{k=0}^{j} \binom{n-1}{j} \binom{j}{k} \int_0^\infty \int_{\mathrm{bd}\, K} \int_{\mathrm{bd}\, K'} f(r, x, x')\, C_k(K, \mathrm{d}x)\, C_{j-k}(K', \mathrm{d}x') r^{n-j-1}\, \mathrm{d}r.$$

    Let $f : (0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ be a measurable function for which the integrals in (4.87) are finite; then

$$\int_{K\cap gK'=\emptyset} f(d(K,gK'),u(K,gK'),g_0^{-1}u(gK',K))\,\mathrm{d}\mu(g) \tag{4.87}$$
$$= \omega_n \sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n-1}{j}\binom{j}{k}\int_0^\infty\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} f(r,u,u')\,S_k(K,\mathrm{d}u)\,S_{j-k}(K',\mathrm{d}u')r^{n-j-1}\,\mathrm{d}r.$$

Here we avoid some complicated coefficients by using C_k, S_k instead of Φ_k, Ψ_k. A common generalization of (4.86) and (4.87) can be formulated, if support elements and support measures Θ_k are used.

The following result is obtained as a special case. Let $f : [0,\infty) \to [0,\infty)$ be a measurable function for which $f(0) = 0$ and

$$M_k(f) := k\int_0^\infty f(r)r^{k-1}\,\mathrm{d}r < \infty \quad \text{for } k = 1,\dots,n.$$

Then

$$\int_{G_n} f(d(K,gK'))\,\mathrm{d}\mu(g) = \frac{1}{\kappa_n}\sum_{k=1}^{n}\sum_{j=n+1-k}^{n}\binom{n}{k}\binom{k}{n-j}M_{k+j-n}(f)W_k(K)W_j(K').$$

This formula was first proved by Hadwiger [924]. Analogues for moving flats are due to Bokowski, Hadwiger and Wills [265], and various generalizations to Groemer [790]. Special formulae of the type (4.87) were first obtained by Hadwiger [925]; see Schneider [1683] for a short proof. A thorough study leading to the results above, extensions, translative versions and analogues for flats was made by Weil [1941, 1942, 1945].

13. *Contact measures and touching probabilities*. Some consequences of Theorems 4.4.7 and 4.4.11 can be interpreted in terms of touching or collision probabilities for convex bodies. For these, we refer to §8.5 of the book by Schneider and Weil [1740].
14. *Absolute curvature measures*. Integral-geometric formulae for absolute curvature measures of sets of positive reach are treated by Rother and Zähle [1595].
15. *Mean measure of shadow boundaries*. An interesting integral-geometric result on convex bodies, of an entirely different type, was proved by Steenaerts [1816]. In spirit, it is related to the boundary behaviour considered, from various aspects, in Chapter 2. For $K \in \mathcal{K}_n^n$ and a vector $u \in \mathbb{S}^{n-1}$, let $\Sigma(K,u)$ be the shadow boundary of K in direction u, that is,

$$\Sigma(K,u) := \{x \in \operatorname{bd} K : x + \lambda u \notin \operatorname{int} K \text{ for all } \lambda \in \mathbb{R}\}.$$

Define

$$\alpha(K) := \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-2}(\Sigma(K,u))\,\mathrm{d}\mathcal{H}^{n-1}(u)$$

and

$$\beta(K) := \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-2}(\operatorname{relbd} K \,|\, u^\perp)\,\mathrm{d}\mathcal{H}^{n-1}(u).$$

Then Steenaerts proved, confirming a conjecture of McMullen, that

$$1 \le \frac{\alpha(K)}{\beta(K)} \le \frac{\omega_n}{\pi\kappa_{n-1}}$$

for smooth K. If K is a ball, the left equality sign holds, and for K a polytope, the right equality sign is valid.
16. *Mean section bodies*. In Crofton's intersection formula (4.59) one integrates an intrinsic volume of the intersection $K \cap E$ over all k-flats $E \in A(n,k)$, where the integration is with respect to the invariant measure μ_k. One may also integrate the intersections themselves

(in the sense of Minkowski addition): for $K \in \mathcal{K}^n$ and $k \in \{1, \dots, n-1\}$, there is a unique convex body $M_k(K)$ for which

$$h(M_k(K), \cdot) = \int_{A(n,k)} h(K \cap E, \cdot)\, \mathrm{d}\mu_k(E).$$

This 'kth mean section body' $M_k(K)$ was investigated by Goodey and Weil [748]. In particular, they showed that $M_1(K)$ is always a ball, whereas $M_2(K)$ determines K uniquely. By different methods they later showed that $M_k(K)$ determines K uniquely, for $2 \leq k \leq n-1$ see [753]. See also Section 8.4, Note 4.

17. *Rotational integral geometry*. We have restricted ourselves here to translation invariant integral geometry. Motivated by requirements of local stereology (for this, see Jensen [1034]), a rotational integral geometry, involving planar sections through a fixed reference point, has been developed. The intersectional Crofton formula (4.59),

$$\int_{A(n,k)} V_j(K \cap E)\, \mathrm{d}\mu_k(E) = \alpha_{njk} V_{n+j-k}(K),$$

calls for two rotational counterparts for intrinsic volumes: to find explicit representations for the functionals $\beta_{k,j}$ defined by

$$\int_{G(n,k)} V_j(K \cap L)\, \mathrm{d}\nu_k(L) = \beta_{k,j}(K)$$

and to find functionals $\alpha_{k,j}$ such that

$$\int_{G(n,k)} \alpha_{k,j}(K \cap L)\, \mathrm{d}\nu_k(L) = V_{n+j-k}(K).$$

For sets K of positive reach, the first task was solved by Jensen and Rataj [1035]. For $j = k$, a Blaschke–Petkantschin formula yields an expression for $\beta_{k,j}(K)$, but for $j < k$ integrals of weighted curvature measures are required. The weights involve hypergeometric functions; explicit representations were given by Auneau, Rataj and Jensen [100]. For convex bodies, they also obtained a representation in terms of flag measures. The second task, determination of the functions $\alpha_{k,j}$ (which may be of greater relevance for applications), was solved by Auneau and Jensen [99], but some problems of uniqueness remain open. For the rotational integral geometry of tensor valuations, see Note 9 of Section 5.4.

4.5 Local behaviour of curvature and area measures

The notion of curvature, in its various forms existing in geometry, is designed to describe and measure, in one sense or other, local geometric shapes. In the present section we ask for information of this kind that can be gained from knowledge of the curvature measures or area measures of a convex body.

A natural first question to ask is the following. What does it mean geometrically if the mth curvature measure $C_m(K, \beta)$ vanishes on a relatively open subset β of the boundary of K? The answer is given below by the description of the support of $C_m(K, \cdot)$. The *support* of a Borel measure μ, denoted by $\operatorname{supp} \mu$, is the complement of the largest open set on which the measure vanishes.

Theorem 4.5.1 *Let $K \in \mathcal{K}^n_n$ and $m \in \{0, \dots, n-1\}$. The support of the mth curvature measure $C_m(K, \cdot)$ is the closure of the m-skeleton of K:*

$$\operatorname{supp} C_m(K,\cdot) = \operatorname{cl} \operatorname{ext}_m K.$$

Proof Let $K \in \mathcal{K}_n^n$ and $m \in \{0, \dots, n-1\}$ be given. First we show that

$$C_m(K, \operatorname{bd} K \setminus \operatorname{ext}_m K) = 0. \tag{4.88}$$

Put $\beta_m := \operatorname{bd} K \setminus \operatorname{ext}_m K$ (which is a Borel set by Section 2.1). From (4.29) we have

$$C_0(K,\beta_0) = \mathcal{H}^{n-1}(\sigma(K,\beta_0)) = 0,$$

where the latter follows from Theorem 2.2.11, since $u \in \sigma(K,\beta_0)$ implies that $F(K,u)$ contains a segment; hence u is a singular normal vector of K. Now we use a special case of Theorem 4.4.5 that can be written as

$$C_m(K,\beta) = a_{nm} \int_{A(n,n-m)} C_0(K \cap E,\beta) \, \mathrm{d}\mu_{n-m}(E), \tag{4.89}$$

with a positive constant a_{nm}. For each $E \in A(n, n-m)$ we have

$$\beta_m \cap E \subset \operatorname{relbd}(K \cap E) \setminus \operatorname{ext}_0(K \cap E),$$

since a point $x \in \beta_m \cap E$ is the centre of an $(m+1)$-dimensional ball contained in K and hence is the centre of a segment contained in $K \cap E$. It follows that

$$C_0(K \cap E,\beta_m) = C_0(K \cap E,\beta_m \cap E) = 0,$$

by the result proved above, applied to $K \cap E$. From (4.89) we deduce that

$$C_m(K, \operatorname{bd} K \setminus \operatorname{ext}_m K) = 0 \tag{4.90}$$

and hence that

$$\operatorname{supp} C_m(K,\cdot) \subset \operatorname{cl} \operatorname{ext}_m K.$$

For the proof of the opposite inclusion we need a quantitative improvement of the case $m = 0$.

Proposition Let $\beta \subset \mathbb{R}^n$ be an open ball with centre $x \in K$ and radius ρ. If $C_0(K,\beta) = 0$, then x is the centre of a segment of length $2\rho/n$ contained in K.

For the proof we first show that

$$x \in \operatorname{conv}(K \cap \operatorname{bd}\beta). \tag{4.91}$$

Suppose this were false. Clearly, $K \cap \operatorname{bd}\beta \neq \emptyset$, since otherwise $K \subset \beta$ and hence $C_0(K,\beta) > 0$. Thus, $\operatorname{conv}(K \cap \operatorname{bd}\beta)$ is a convex body not containing x. By Theorem 1.3.4, there exists a hyperplane H strongly separating x and $K \cap \operatorname{bd}\beta$. Let H^+ be the open halfspace bounded by H that contains x. Since $x \in K \cap H^+$, it is clear that the outer unit normal vectors of K at points of $K \cap H^+$ fill a neighbourhood of the inner normal vector of H^+ on $\mathbb{S}^{n-1}$. From (4.29) it follows that

$$C_0(K,\beta) \geq C_0(K, \operatorname{bd} K \cap H^+) > 0,$$

a contradiction. Thus (4.91) holds.

By Carathéodory's theorem, there are $k \le n+1$ points $y_1, \dots, y_k \in K \cap \mathrm{bd}\,\beta$ such that

$$x = \sum_{j=1}^{k} \alpha_j y_j \quad \text{with} \quad \alpha_j > 0\ (j = 1, \dots, k),\ \sum_{j=1}^{k} \alpha_j = 1.$$

From

$$\sum_{j=1}^{k} \alpha_j = \frac{1}{k-1} \sum_{j=1}^{k} (1 - \alpha_j)$$

we deduce that there exists an index i for which

$$\frac{\alpha_i}{1-\alpha_i} \ge \frac{1}{k-1}.$$

For

$$y := \frac{x - \alpha_i y_i}{1 - \alpha_i},$$

then

$$y = \frac{\alpha_1 y_1 + \cdots + \alpha_{i-1} y_{i-1} + \alpha_{i+1} y_{i+1} + \cdots + \alpha_k y_k}{1 - \alpha_i} \in \mathrm{conv}\,\{y_1, \dots, y_k\} \subset K.$$

Since $x = \alpha_i y_i + (1 - \alpha_i)$ and $|x - y_i| = \rho$, we have

$$|x - y| = \frac{\alpha_i}{1 - \alpha_i} |x - y_i| \ge \frac{\rho}{n},$$

thus the proposition is proved.

Now let $\beta \subset \mathbb{R}^n$ be an open set for which $C_m(K, \beta) = 0$. From (4.89) we have

$$\int_{A(n,n-m)} C_0(K \cap E, \beta)\, \mathrm{d}\mu_{n-m}(E) = 0,$$

hence $C_0(K \cap E, \beta) = 0$ for μ_{n-m}-almost all planes $E \in A(n, n-m)$.

Let $x \in K \cap \beta$ and let $E \in A(n, n-m)$ be an $(n-m)$-plane through x that meets $\mathrm{int}\,K$. There exists a sequence $(E_j)_{j \in \mathbb{N}}$ of $(n-m)$-planes converging to E such that $C_0(K \cap E_j, \beta) = 0$ for all j. For j sufficiently large, we may choose $x_j \in K \cap E_j \cap \beta$ such that $x_j \to x$ for $j \to \infty$, and we may further assume that, for some fixed $\rho > 0$, the open ball with centre x_j and radius ρ is contained in β. By the proposition, x_j is the centre of a segment $[a_j, b_j]$ of length $2\rho/(n-m)$ contained in $K \cap E_j$. By selecting suitable subsequences, we infer that x is the centre of a segment $[a, b] \subset K \cap E$. We have proved that every $(n-m)$-dimensional plane through x that meets $\mathrm{int}\,K$ meets K in a segment of length $2\rho/(n-m)$ with centre x. This is only possible if x is the centre of an $(m+1)$-dimensional ball contained in K. Thus $x \notin \mathrm{ext}_m\,K$, and we have proved that

$$\mathbb{R}^n \setminus \mathrm{supp}\ C_m(K, \cdot) \subset \mathbb{R}^n \setminus \mathrm{ext}_m K,$$

hence

$$\mathrm{cl\,ext}_m K \subset \mathrm{supp}\, C_m(K, \cdot).$$

This proves Theorem 4.5.1. □

In a similar way, we can obtain a counterpart to Theorem 4.5.1 for area measures. Since the proof is slightly more complicated, we formulate part of it as a lemma. For $u \in \mathbb{S}^{n-1}$ and $0 < t < 1$ we define

$$D(u,t) := \{v \in \mathbb{S}^{n-1} : \langle u, v\rangle > t\};$$

this is the open *spherical cap* with centre u and spherical radius arc cos t.

Lemma 4.5.2 *Let $K \in \mathcal{K}^n_n$, $n \geq 2$, $u \in \mathbb{S}^{n-1}$, $t \in (0,1)$ and $x \in F(K,u)$ and suppose that*

$$S_{n-1}(K, D(u,t)) = 0.$$

Then there are normal vectors $u_1, u_2 \in N(K,x) \cap \mathbb{S}^{n-1}$ such that

$$u \in \mathrm{relint\,pos}\,\{u_1, u_2\}$$

and $\langle u, u_i\rangle \leq c(n,t)$ for $i = 1, 2$, where $c(n,t) < 1$ is a constant depending only on n and t.

Proof First we prove that

$$u \in \mathrm{pos}\,(N(K,x) \cap \mathrm{bd}\, D), \tag{4.92}$$

where $D = D(u,t)$, and $\mathrm{bd}\, D$ denotes the boundary of D relative to $\mathbb{S}^{n-1}$. Suppose this is false. Then there exists a hyperplane through o that strongly separates u and $\mathrm{conv}(N(K,x) \cap \mathrm{bd}\, D)$. Hence, there are a vector $v \in \mathbb{S}^{n-1}$ with $\langle u, v\rangle > 0$ and a number $\eta > 0$ such that $\langle w, v\rangle < -\eta$ for all $w \in N(K,x) \cap \mathrm{bd}\, D$. All elements w of the compact set $\{w \in \mathrm{bd}\, D : \langle w, v\rangle \geq -\eta\}$ satisfy $w \notin N(K,x)$ and hence $\langle x, w\rangle < h(K,w)$. Therefore, we can choose a number $\varepsilon > 0$ with $\langle x, w\rangle < h(K,w) - \varepsilon$ for all $w \in \mathrm{bd}\, D$ satisfying $\langle w, v\rangle \geq -\eta$. Put

$$y := x + \alpha u + \beta v \quad \text{with } \alpha := \frac{\varepsilon\eta}{\eta + t},\ \beta := \frac{\varepsilon t}{\eta + t},$$

then $\langle y, u\rangle > \langle x, u\rangle$ and hence $y \notin K$. Let $w \in \mathrm{bd}\, D$. If $\langle w, v\rangle < -\eta$, then

$$\langle y, w\rangle = \langle x, w\rangle + \alpha\langle u, w\rangle + \beta\langle v, w\rangle < h(K,w) + \alpha t - \beta\eta = h(K,w).$$

If $\langle w, v\rangle \geq -\eta$, then

$$\langle y, w\rangle = \langle x, w\rangle + \alpha\langle u, w\rangle + \beta\langle v, w\rangle < h(K,w) - \varepsilon + \alpha + \beta = h(K,w).$$

Let $w \in \mathbb{S}^{n-1} \setminus \mathrm{cl}\, D$ and $w \neq -u$. We can choose $w_0 \in \mathrm{bd}\, D$ and $\lambda, \mu > 0$ such that $w_0 = \lambda u + \mu w$. If $\langle y, w\rangle \geq h(K,w)$, we obtain

$$\begin{aligned} h(K,w_0) > \langle y, w_0\rangle &= \lambda\langle y, u\rangle + \mu\langle y, w\rangle > \lambda h(K,u) + \mu h(K,w) \\ &\geq h(K, \lambda u + \mu w) = h(K, w_0), \end{aligned}$$

a contradiction; hence $\langle y, w\rangle < h(K, w)$. For $w = -u$ this holds trivially.

Let $B \subset \operatorname{int} K$ be a ball and put

$$A := \operatorname{conv}(B \cup \{y\}) \cap \operatorname{bd} K.$$

For each $w \in \mathbb{S}^{n-1} \setminus D$ we have proved that $\langle y, w\rangle < h(K, w)$, hence $F(K, w) \cap A = \emptyset$. This shows that $A \subset \tau(K, D)$ and hence, by (4.32), that

$$S_{n-1}(K, D) = \mathcal{H}^{n-1}(\tau(K, D)) \geq \mathcal{H}^{n-1}(A) > 0,$$

contradicting the assumption of the Lemma. Thus (4.92) is true.

From (4.92) and Carathéodory's theorem we deduce the existence of $k \leq n$ vectors $v_1, \dots, v_k \in N(K, x) \cap \operatorname{bd} D$ such that

$$u = \sum_{j=1}^{k} \alpha_j v_j \quad \text{with } \alpha_j > 0 \ (j = 1, \dots, k).$$

From $\langle v_j, u\rangle = t$ for $j = 1, \dots, k$ we get $(\alpha_1 + \cdots + \alpha_k)t = 1$ and thus

$$kt^2(\alpha_1^2 + \cdots + \alpha_k^2) \geq 1,$$

hence there exists a positive constant $c_1 = c_1(k, t) < 1$ for which

$$\sum_{j=1}^{k}(1 - \alpha_j t)^2 \leq c_1^2 \sum_{j=1}^{k}(1 - 2\alpha_j t + \alpha_j^2).$$

We deduce the existence of a number $i \in \{1, \dots, k\}$ for which

$$\frac{\langle u, u - \alpha_i v_i\rangle}{|u - \alpha_i v_i|} = \frac{1 - \alpha_i t}{\sqrt{1 - 2\alpha_i t + \alpha_i^2}} \leq c_1.$$

Put

$$u_1 := v_i, \quad u_2 := \frac{u - \alpha_i v_i}{|u - \alpha_i v_i|}$$

and

$$c(n, t) := \max\{c_1(2, t), \dots, c_1(n, t), t\}.$$

Then $u_1 \in N(K, x) \cap \mathbb{S}^{n-1}$,

$$|u - \alpha_i v_i| u_2 = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_k v_k \in N(K, x)$$

and thus $u_2 \in N(K, x) \cap \mathbb{S}^{n-1}$; finally

$$u = \alpha_i v_1 + |u - \alpha_i v_i| u_2 \in \operatorname{relint} \operatorname{pos}\{u_1, u_2\}$$

and $\langle u, u_1\rangle = t \leq c(n, t)$, $\langle u, u_2\rangle \leq c_1(k, t) \leq c(n, t)$. This completes the proof of Lemma 4.5.2. □

We can now describe the support of the mth area measure.

Theorem 4.5.3 *Let $K \in \mathcal{K}_n^n$ and $m \in \{0, \dots, n-1\}$. The support of the mth area measure $S_m(K, \cdot)$ is the closure of the set of all $(n-1-m)$-extreme unit normal vectors of K.*

Proof Let $E \in G(n, m+1)$ be an $(m+1)$-dimensional linear subspace of $\mathbb{R}^n$. A special case of (4.78) can be written in the form

$$S_m(K, \omega) = b_{nm} \int_{\mathrm{SO}(n)} S'_m(K \,|\, \rho E, \omega \cap \rho E) \, \mathrm{d}\nu(\rho) \tag{4.93}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ with a positive constant b_{nm}; here S'_m is computed in ρE. Let $\omega \subset \mathbb{S}^{n-1}$ be an open set for which $S_m(K, \omega) = 0$. Then (4.93) shows that

$$S'_m(K \,|\, \rho E, \omega \cap \rho E) = 0$$

for ν-almost all rotations $\rho \in \mathrm{SO}(n)$. Let $u \in \omega$ and let $\rho \in \mathrm{SO}(n)$ be such that $u \in \rho E$. We can choose a sequence $(\rho_j)_{j \in \mathbb{N}}$ of rotations converging to ρ such that

$$S'_m(K \,|\, \rho_j E, \omega \cap \rho_j E) = 0 \quad \text{for } j \in \mathbb{N}.$$

Writing $u_j = \rho_j \rho^{-1} u$ for $j \in \mathbb{N}$, we have $u_j \to u$ for $j \to \infty$ and $u_j \in \rho_j E$. Since ω is open, we can assume that $D(u_j, t) \subset \omega$ for all $j \in \mathbb{N}$ with some fixed $t < 1$. Choosing $x_j \in F(K, u_j)$ for $j \in \mathbb{N}$, we have

$$x_j \,|\, \rho_j E \in F(K \,|\, \rho_j E, u_j).$$

By Lemma 4.5.2, applied to $K \,|\, \rho_j E$, there are vectors

$$v_j, w_j \in N(K \,|\, \rho_j E, x_j \,|\, \rho_j E) \cap (\mathbb{S}^{n-1} \cap \rho_j E) = N(K, x_j) \cap \mathbb{S}^{n-1} \cap \rho_j E$$

such that

$$u_j \in \operatorname{relint} \operatorname{pos} \{v_j, w_j\} \tag{4.94}$$

and

$$\langle u_j, v_j \rangle \le c(m+1, t), \quad \langle u_j, w_j \rangle \le c(m+1, t). \tag{4.95}$$

After a suitable selection of subsequences and a change of notation, we may assume that $x_j \to x$, $v_j \to v$, $w_j \to w$, for $j \to \infty$, with a certain point $x \in \operatorname{bd} K$ and vectors $v, w \in \mathbb{S}^{n-1}$. Since $u_j \in N(K, x_j)$, we have $h(K, u_j) = \langle x_j, u_j \rangle$ and hence $h(K, u) = \langle x, u \rangle$; thus $u \in N(K, x)$. Similarly, we see that $v, w \in N(K, x)$. From $v_j, w_j \in \rho_j E$ we get $v, w \in \rho E$. Relation (4.94) implies $u \in \operatorname{pos}\{v, w\}$, and from (4.95) we obtain $\langle u, v \rangle \le c(m+1, t)$, $\langle u, w \rangle \le c(m+1, t) < 1$; hence $u \in \operatorname{relint} \operatorname{pos}\{u, w\}$. Thus we have proved: Any $(m+1)$-dimensional linear subspace ρE through the given vector $u \in \omega$ contains a two-dimensional subcone of $N(K, x)$, where $x \in F(K, u)$, which contains u in its relative interior. In other words, each $(m+1)$-dimensional linear subspace through u intersects the normal cone $N(K, F(K, u))$ in a cone of dimension at least two containing u in its relative interior. This is only possible if the face $T(K, u)$ of $N(K, F(K, u))$ (see Section 2.2) containing u in its relative interior has dimension at least $n + 1 - m$. By definition, this means that u is not an $(n-1-m)$-extreme normal vector of K. Denoting the set of k-extreme normal vectors of K by $\operatorname{extn}_k K$, we have proved that any open set $\omega \subset \mathbb{S}^{n-1} \setminus \operatorname{supp} S_m(K,\cdot)$ satisfies $\omega \subset \mathbb{S}^{n-1} \setminus \operatorname{extn}_{n-1-m} K$, hence

$$\operatorname{cl} \operatorname{extn}_{n-1-m} K \subset \operatorname{supp} S_m(K,\cdot).$$

Conversely, let ω be an open subset of $\mathbb{S}^{n-1} \setminus \operatorname{extn}_{n-1-m} K$. Then each vector $u \in \omega$ satisfies $\dim T(K,u) \geq n+1-m$. If $m = n-1$, we deduce from (4.32) and Theorem 2.2.5 that

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}(\tau(K,\omega)) = 0,$$

since $\tau(K,\omega)$ contains only singular points of K. Using this result and formula (4.93), we obtain $S_m(K,\omega) = 0$ for $m = 1,\dots,n-2$. This completes the proof of Theorem 4.5.3 □

In special cases, the support of the area measure $S_m(K,\cdot)$ tells us even more about the structure of K. If $P \in \mathcal{P}^n$ is a polytope, then formula (4.24) shows that the area measure $S_m(P,\cdot)$ is concentrated on the union of the spherical images of the relative interiors of the m-faces of P. The following theorem implies that only polytopes can have area measures with supports of this kind.

Theorem 4.5.4 *Let $K \in \mathcal{K}^n_n$ and $m \in \{1,\dots,n-1\}$, and suppose that the support of the area measure $S_m(K,\cdot)$ can be covered by finitely many $(n-1-m)$-dimensional great spheres. Then K is a polytope.*

Proof First let $m = n-1$. Then $S_{n-1}(K,\cdot)$ is concentrated on finitely many points $u_1,\dots,u_k \in \mathbb{S}^{n-1}$. Let

$$P := \bigcap_{i=1}^{k} H^-(K,u_i).$$

Then P is a polytope containing K. Suppose P has a vertex x that is not a point of K. The set ω of unit normal vectors (pointing towards x) of the hyperplanes strongly separating x and K is disjoint from $\{u_1,\dots,u_k\}$ and satisfies

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}(\tau(K,\omega)) > 0,$$

a contradiction. Hence each vertex of P is a point of K and we conclude that $K = P$.

We may, therefore, assume that $m \leq n-2$. Let $A_1,\dots,A_k$ be $(n-1-m)$-dimensional great spheres covering $\operatorname{supp} S_m(K,\cdot)$. We put

$$A := \bigcup_{i=1}^{k} A_i \quad \text{and} \quad \omega := \mathbb{S}^{n-1} \setminus A;$$

then $S_m(K,\omega) = 0$. From formula (4.93) we deduce

$$S'_m(K \,|\, \rho E, \omega \cap \rho E) = 0 \tag{4.96}$$

for ν-almost all rotations $\rho \in \mathrm{SO}(n)$, where $E \in G(n, m+1)$ is arbitrary but fixed.

Let $i \in \{1, \dots, k\}$ and suppose that $\rho \in \mathrm{SO}(n)$ is a rotation for which ρE meets A_i in more than two points. Then ρE meets $\operatorname{lin} A_i$ in a subspace of dimension at least two. Since $\dim \operatorname{lin} A_i = n - m$, this means that ρE and $\operatorname{lin} A_i$ are in special position. From Lemma 4.4.1 we know that this happens only for the rotations ρ in a set of measure zero. Hence, we deduce that for ν-almost all $\rho \in \mathrm{SO}(n)$, the subspace ρE meets A in at most $2k$ points. Together with (4.96) this shows that, for ν-almost all ρ, the support of $S'_m(K \,|\, \rho E, \cdot)$ contains only $2k$ points and hence (by the result proved above, applied to $K \,|\, \rho E$) the projection $K \,|\, \rho E$ is an $(m+1)$-polytope with at most $2k$ facets.

If $\rho \in \mathrm{SO}(n)$ is arbitrary, we can choose a sequence $(\rho_j)_{j \in \mathbb{N}}$ in $\mathrm{SO}(n)$ converging to ρ such that each projection $K \,|\, \rho_j E$ is an $(m+1)$-polytope with at most $2k$ facets. Since $K \,|\, \rho_j E \to K \,|\, \rho E$ (in the Hausdorff metric) for $j \to \infty$, it follows that $K \,|\, \rho E$ is an $(m+1)$-polytope with at most $2k$ facets. By a well-known theorem (see Klee [1109], Corollary 4.4), P is itself a polytope. □

Another aspect of the local behaviour of curvature measures is expressed by the fact that they cannot be positive on sets of too small a Hausdorff dimension. For instance, for $m > 0$, the measure $C_m(K, \cdot)$ cannot have point masses. This is a consequence of the more general estimates expressed in the following theorem. Its proof requires some facts from geometric measure theory; for these (as well as for the notion of rectifiability) we refer to Federer [557].

Theorem 4.5.5 *Let $K \in \mathcal{K}^n$ and $m \in \{0, \dots, n-1\}$. Then*

$$C_m(K, \beta) \leq a_1 \mathcal{H}^m(\beta) \tag{4.97}$$

for every Borel set $\beta \subset \mathbb{R}^n$, with some constant a_1 depending only on n and m, and

$$S_m(K, \omega) \leq a_2 \mathcal{H}^{n-m-1}(\omega) \tag{4.98}$$

for every $(\mathcal{H}^{n-m-1}, n-m-1)$ rectifiable set $\omega \subset \mathbb{S}^{n-1}$, with some constant a_2 depending only on n, m and K.

Proof From (4.89) we have

$$\begin{aligned} C_m(K, \beta) &= a_{nm} \int_{A(n,n-m)} C_0(K \cap E, \beta) \, \mathrm{d}\mu_{n-m}(E) \\ &\leq a_{nm} C_0(K \cap E, \mathbb{R}^n) \int_{A(n,n-m)} \operatorname{card}(E \cap \beta) \, \mathrm{d}\mu_{n-m}(E) \\ &\leq a_1 \mathcal{H}^m(\beta) \end{aligned}$$

by Federer [557] (2.10.16 and 3.3.13). Similarly, from (4.93) we obtain, with $E \in G(n, m+1)$,

$$\begin{aligned} S_m(K,\omega) &= b_{nm} \int_{\mathrm{SO}(n)} S'_m(K \,|\, \rho E, \omega \cap \rho E) \, \mathrm{d}\nu(\rho) \\ &\leq b_{nm} \max_E S'_m(K \,|\, \rho E, \mathbb{S}^{n-1} \cap \rho E) \int_{\mathrm{SO}(n)} \mathrm{card}\,(\omega \cap \rho E) \, \mathrm{d}\nu(\rho) \\ &\leq a_2 \mathcal{H}^{n-m-1}(\omega) \end{aligned}$$

by Federer [557] (Theorem 3.2.48). □

By the previous theorem, the curvature measure $C_m(K,\cdot)$ has a Radon–Nikodym derivative with respect to $\mathcal{H}^m$ on sets of σ-finite $\mathcal{H}^m$-measure, and similarly for $S_m(K,\cdot)$. The following result of Colesanti and Hug [443], which we quote without proof, provides these densities.

Theorem 4.5.6 *Let $K \in \mathcal{K}^n$ and $m \in \{0,\dots,n-1\}$. If $\beta \subset \mathrm{bd}\, K$ is a Borel set of σ-finite $\mathcal{H}^m$-measure, then*

$$\binom{n-1}{m} C_m(K,\beta) = \int_\beta \mathcal{H}^{n-1-m}(N(K,x) \cap \mathbb{S}^{n-1}) \, \mathrm{d}\mathcal{H}^m(x).$$

If $\omega \subset \mathbb{S}^{n-1}$ is a Borel set of σ-finite $\mathcal{H}^{n-m-1}$-measure, then

$$\binom{n-1}{m} S_m(K,\omega) = \int_\omega \mathcal{H}^m(F(K,u)) \, \mathrm{d}\mathcal{H}^{n-m-1}(u).$$

Notes for Section 4.5

1. Theorem 4.5.1 was proved by Schneider [1687]. Theorem 4.5.3 was conjectured by Weil [1932], p. 356, who showed it for $m = 1$, and proved in general by Schneider [1680]. The latter theorem implies, in particular, that

$$\mathrm{supp}\, S_q(K,\cdot) \subset \mathrm{supp}\, S_p(K,\cdot) \quad \text{for } q > p,$$

which had been conjectured by Weil [1932]; see also Firey [606]. Theorem 4.5.4 is due to Goodey and Schneider [742]; for $m = 1$, Weil [1932], Satz 4.4, proved a stronger result. A special case of the estimate (4.97) appears in Federer [556], p. 489. The general case and the estimate (4.98) were noted in Schneider [1693], §8. Firey [598] proved the inequality

$$S_m(K, D(u,\cos\alpha)) \leq A D(K)^m \sin^{n-m-1}\alpha \sec\alpha$$

for $K \in \mathcal{K}^n$, $m \in \{1,\dots,n-1\}$ and $\alpha \in (0,\pi/2)$, where $D(K)$ denotes the diameter of K and A is a constant depending only on n and m. From this one can also deduce an inequality of the type (4.98), namely

$$S_m(K,\omega) \leq A_{nm} D(K)^m \mathcal{H}^{n-m-1}(\omega)$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$.
2. *Differentiation of curvature measures*. Let $K \in \mathcal{K}^n$ be a convex body of class C^2_+. If $x \in \mathrm{bd}\, K$ and $(\beta_i)_{i\in\mathbb{N}}$ is a sequence of Borel subsets of $\mathrm{bd}\, K$ with positive measure and shrinking to x, then it follows from (4.25) that

$$\lim_{i\to\infty} \frac{C_m(K,\beta_i)}{\mathcal{H}^{n-1}(\beta_i)} = H_{n-1-m}(x).$$

One may ask whether this relation holds true for a general convex body K if x is a normal point (see Section 2.5), so that $H_{n-1-m}(x)$ is defined. Under certain (not too restrictive) assumptions on the sequence $(\beta_i)_{i\in\mathbb{N}}$ this can, in fact, be proved; see Aleksandrov [19] for $m = 0$ and Schneider [1691] for $m = 0, \dots, n-1$.

3. In $\mathbb{R}^3$, Aleksandrov [22] considered the *specific curvature* $C_0(K,\cdot)/C_2(K,\cdot)$, defined on the Borel sets β with $C_2(K,\beta) > 0$, and its influence on the local shape of bd K. He proved: if the specific curvature of a convex body $K \in \mathcal{K}^3_3$ is bounded on a neighbourhood of $x \in \operatorname{bd} K$, then either bd K is differentiable at x, or x is an internal point of a perfect 1-face of K. Consequences and further results can be found in Busemann [370], §5; see also Pogorelov [1538], pp. 57–60.

   Higher-dimensional results in this spirit were obtained by Burago and Kalinin [355]. One of their results says that the assumption $C_r(K,\cdot) \le \lambda C_{n-1}(K,\cdot)$, with some real λ, for a convex body $K \in \mathcal{K}^n_n$ and a number $r \in \{0, \dots, n-2\}$, implies that all normal cones of K have dimension at most $n-1-r$. In particular, K must be smooth if $r = n-2$. In this case, a stronger result is known: a ball rolls freely in K (Bangert [137], Hug [1007]).

4. *Curvature measures and singularities.* Integral representations of the curvature measures and area measures in terms of generalized principal curvatures, as they can be derived from (4.28), were systematically used by Hug [1004, 1005, 1006, 1007] to investigate the properties of curvature and area measures in relation to the sets of singular boundary points and normal vectors. For $K \in \mathcal{K}^n_n$ and $r \in \{0, \dots, n-1\}$, let $\Sigma^r(K)$ denote the set of r-singular boundary points of K. For a Borel set $\eta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and a point $x \in \mathbb{R}^n$, let $\eta_x := \{u \in \mathbb{S}^{n-1} : (x,u) \in \eta\}$ denote the x-section of η. Hug [1004] proved the formula

$$\int_{\Sigma^r(K)} \mathcal{H}^{n-1-r}(N(K,x) \cap \eta_x)\, d\mathcal{H}^r(x) = \binom{n-1}{r} \Theta_r(K, (\Sigma^r(K) \times \mathbb{S}^{n-1}) \cap \eta),$$

   from which he deduced the estimate (2.4) in Note 3 in Section 2.2. He also established the corresponding result for sets of singular normal vectors. In [1005], Hug investigated the Lebesgue decomposition (with respect to $\mathcal{H}^{n-1}$) of the curvature measure $C_r(K,\cdot)$ into an absolutely continuous and a singular part, and obtained integral representations for either part. To formulate a consequence, let $\beta \in \mathcal{B}(\mathbb{R}^n)$. If $r \in \{0, \dots, n-2\}$ and $C_r(K\cdot) \llcorner \beta \ll \mathcal{H}^{n-1} \llcorner \beta$ (where $\ll$ denotes absolute continuity), then $\mathcal{H}^r(\Sigma^r(K) \cap \beta) = 0$. An analogous result is obtained for area measures and r-singular normal vectors.

   Another result of [1005] is a counterpart to Theorem 4.5.4 for curvature measures.

   The line of research of [1005] is continued in Hug [1006]. Here the (local) absolute continuity of the Gaussian curvature measure $C_0(K,\cdot)$ is characterized by a suitably weakened form of the condition that a ball rolls freely in K. By means of integral-geometric methods, the result is extended to curvature measures of any order. A version of this result involves lower-dimensional balls touching from inside. Analogous results are obtained for area measures. In Hug [1007], transfer principles are established that allow one to translate relations between absolute continuity properties of $C_r(K,\cdot)$ and geometric properties of $K \in \mathcal{K}^n_{(o)}$ into relations between absolute continuity properties of $S_{n-1-r}(K^\circ,\cdot)$ and dual geometric properties of K°. Moreover, the implications of absolute continuity with bounded density are studied.

5

Mixed volumes and related concepts

The concept of mixed volumes, which forms a central part of the Brunn–Minkowski theory of convex bodies, arises naturally if one combines the two fundamental concepts of Minkowski addition and volume. In this chapter, we introduce mixed volumes and the closely related mixed area measures, establish their fundamental properties and study some related notions.

5.1 Mixed volumes and mixed area measures

Of several possible approaches to mixed volumes, we choose the one using strongly isomorphic polytopes, and then approximation on the basis of Theorem 2.4.15. This has several advantages: it is particularly perspicuous and elementary, it allows us to introduce the mixed area measures in a parallel fashion and it prepares the proof of the Aleksandrov–Fenchel inequality in Section 7.3.

As before, we use the symbol V_n for the volume of n-dimensional convex bodies, and we recall that $V_i(K)$, for an i-dimensional convex body K, is its i-dimensional volume.

Lemma 5.1.1 *Let $P \in \mathcal{P}^n_n$ be an n-dimensional polytope, let $F_1, \dots, F_N$ be its facets and let u_i be the outer unit normal vector of P at F_i $(i = 1, \dots, N)$. Then*

$$\sum_{i=1}^{N} V_{n-1}(F_i) u_i = o \tag{5.1}$$

and

$$V_n(P) = \frac{1}{n} \sum_{i=1}^{N} h(P, u_i) V_{n-1}(F_i). \tag{5.2}$$

Proof Let $z \in \mathbb{R}^n \setminus \{o\}$. For the orthogonal projection $P \,|\, z^\perp$ we obviously have

$$V_{n-1}(P \,|\, z^\perp) = \sum_{\langle z, u_i \rangle \geq 0} \langle z, u_i \rangle V_{n-1}(F_i) = - \sum_{\langle z, u_i \rangle < 0} \langle z, u_i \rangle V_{n-1}(F_i),$$

hence $\sum_{i=1}^{N} \langle z, u_i \rangle V_{n-1}(F_i) = 0$. Since z was arbitrary, (5.1) follows.

From (5.1) we see that the right-hand side of (5.2) does not change under translations of P. Since this is also true for the left-hand side, we may assume that $o \in \operatorname{int} P$. Then P is the union of the pyramids $\operatorname{conv}(F_i \cup \{o\})$, $i = 1, \dots, N$, which have pairwise disjoint interiors. Formula (5.2) is an immediate consequence. □

Remark 5.1.2 Formula (5.2) will be exploited in the following, but we remark already here that it extends to general convex bodies K in the form

$$V_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K,u)\, S_{n-1}(K, \mathrm{d}u). \tag{5.3}$$

In fact, for a polytope, both formulae are identical, and the general case follows by approximation, since the volume functional is continuous, the support function is continuous in both variables and the surface area measure is weakly continuous. Formula (5.19) below is a more general version of (5.3).

In the following, we assume that $\mathcal{A}$ is a given a-type of n-dimensional simple polytopes, as defined in Section 2.4. By $u_1, \dots, u_N$ we denote the unit normal vectors (of the facets) of the a-type $\mathcal{A}$. For $P \in \mathcal{A}$, we use the abbreviations

$$\begin{aligned} F_i &:= F(P, u_i), \\ F_{ij} &:= F_i \cap F_j, \\ h_i &:= h(P, u_i). \end{aligned}$$

The numbers $h_1, \dots, h_N$, which determine P uniquely, are called the *support numbers* of P. Further, we define

$$J := \{(i,j) : i, j \in \{1, \dots, N\},\ \dim F_{ij} = n-2\}.$$

Observe that J depends only on the a-type $\mathcal{A}$. For $(i,j) \in J$, let θ_{ij} be the angle between u_i and u_j, and let $v_{ij} \perp u_i$ be the unit normal vector of the $(n-1)$-polytope F_i at its $(n-2)$-face F_{ij}. We write

$$h_{ij} := h(F_i, v_{ij}).$$

Using the relation

$$u_j = u_i \cos\theta_{ij} + v_{ij} \sin\theta_{ij},$$

we take the inner product of u_j with some vector in F_{ij} and obtain

$$h_{ij} = h_j \csc\theta_{ij} - h_i \cot\theta_{ij}. \tag{5.4}$$

Lemma 5.1.3 *The volume of $P \in \mathcal{A}$ can be represented in the form*

$$V_n(P) = \sum a_{j_1 \dots j_n} h_{j_1} \cdots h_{j_n}, \tag{5.5}$$

where the sum extends over $j_1, \dots, j_n \in \{1, \dots, N\}$ and where the coefficients $a_{j_1 \dots j_n}$ are symmetric and depend only on the a-type $\mathcal{A}$.

Proof We use induction on n. For $n = 1$, the assertion is trivial. We assume that $n > 1$ and that the assertion has been proved in dimension $n-1$. Let $i \in \{1, \dots, N\}$. By the induction hypothesis, there is a representation

$$V_{n-1}(F_i) = \sum_{(i,k_r)\in J} a^{(i)}_{k_1\dots k_{n-1}} h_{ik_1} \cdots h_{ik_{n-1}}. \tag{5.6}$$

Here the coefficients $a^{(i)}_{k_1\dots k_{n-1}}$ depend only on the a-type of F_i and thus, by Lemma 2.4.10, only on the a-type $\mathcal{A}$. By (5.4),

$$h_{ik_r} = h_{k_r} \csc\theta_{ik_r} - h_i \cot\theta_{ik_r}. \tag{5.7}$$

Inserting (5.7) into (5.6) and the latter into (5.2), we obtain a representation of the form

$$V_n(P) = \sum a_{j_1\dots j_n} h_{j_1} \cdots h_{j_n}.$$

Introducing additional zero coefficients, we may assume that the summation extends formally over all $j_1, \dots, j_n \in \{1, \dots, N\}$. Clearly, the coefficients can be assumed to be symmetric in their indices. Since the numbers $a^{(i)}_{k_1\dots k_{n-1}}$ as well as the angles θ_{ij} depend only on $\mathcal{A}$, the same is true for the coefficients $a_{j_1\dots j_n}$. □

Now we assume that strongly isomorphic polytopes $P_1, \dots, P_n \in \mathcal{A}$ are given. We define $F_i^{(r)}, F_{ij}^{(r)}, h_i^{(r)}, h_{ij}^{(r)}$ for P_r in the same way as F_i, F_{ij}, h_i, h_{ij} were defined for P. Then we introduce the *mixed volume* of $P_1, \dots, P_n$ by

$$V(P_1, \dots, P_n) := \sum a_{j_1\dots j_n} h^{(1)}_{j_1} \cdots h^{(n)}_{j_n}, \tag{5.8}$$

where the coefficients $a_{j_1\dots j_n}$ are those of (5.5). Thus, $V(P_1, \dots, P_n)$ is symmetric in its arguments, and

$$V(P, \dots, P) = V_n(P). \tag{5.9}$$

Let polytopes $P_1, \dots, P_m \in \mathcal{A}$ and numbers $\lambda_1, \dots, \lambda_m \geq 0$ with $\sum \lambda_i > 0$ be given. Then $\lambda_1 P_1 + \cdots + \lambda_m P_m \in \mathcal{A}$ by Lemma 2.4.10 and the definition of strongly isomorphic polytopes, and

$$h(\lambda_1 P_1 + \cdots + \lambda_m P_m, u_i) = \lambda_1 h_i^{(1)} + \cdots + \lambda_m h_i^{(m)}.$$

From (5.5) and (5.8) we immediately obtain

$$V_n(\lambda_1 P_1 + \cdots + \lambda_m P_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(P_{i_1}, \dots, P_{i_n}). \tag{5.10}$$

The following *polarization formula* expresses the mixed volume explicitly in terms of volumes of Minkowski sums.

Lemma 5.1.4 *For $P_1, \dots, P_n \in \mathcal{A}$,*

$$V(P_1, \dots, P_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{i_1<\cdots<i_k} V_n(P_{i_1} + \cdots + P_{i_k}). \tag{5.11}$$

Proof We denote the right-hand side of (5.11) by $f(P_1, \dots, P_n)$, also in the case where one of the P_i is equal to $\{o\}$. For $\lambda_1, \dots, \lambda_n > 0$ it follows from (5.10) that the function $(\lambda_1, \dots, \lambda_n) \mapsto f(\lambda_1 P_1, \dots, \lambda_n P_n)$ is a homogeneous polynomial of degree n. This extends to $\lambda_1, \dots, \lambda_n \geq 0$. Now

$$\begin{aligned}
&(-1)^{n+1} n! f(\{o\}, P_2, \dots, P_n) \\
&= \sum_{2 \leq i \leq n} V_n(P_i) - \left[\sum_{2 \leq j \leq n} V_n(\{o\} + P_j) + \sum_{2 \leq i < j \leq n} V_n(P_i + P_j) \right] \\
&\quad + \left[\sum_{2 \leq j < k \leq n} V_n(\{o\} + P_j + P_k) + \sum_{2 \leq i < j < k \leq n} V_n(P_i + P_j + P_k) \right] \\
&\quad - \dots \\
&= 0;
\end{aligned}$$

thus $f(0P_1, \lambda_2 P_2, \dots, \lambda_n P_n)$ is identically zero for all $\lambda_2, \dots, \lambda_n$. Hence, in the polynomial $f(\lambda_1 P_1, \dots, \lambda_n P_n)$, all monomials $\lambda_{i_1} \cdots \lambda_{i_n}$ with $1 \notin \{i_1, \dots, i_n\}$ have zero coefficients. Since 1 can be replaced by each of the numbers $2, \dots, n$, we conclude that only the monomial $\lambda_1 \cdots \lambda_n$ has a non-zero coefficient, and from (5.10) we see that this is equal to $V(P_1, \dots, P_n)$. □

We note that (5.11) implies, in particular, that $V(P_1, \dots, P_n)$ does not change under arbitrary translations of any of the P_i.

Formula (5.2) extends to mixed volumes. To see this, let $P'_1, \dots, P'_{n-1}$ be strongly isomorphic $(n-1)$-polytopes. They lie in $(n-1)$-dimensional affine subspaces that are all parallel, say to H. We can define $v(P'_1, \dots, P'_{n-1})$ as the mixed volume, relative to H, of arbitrary translates of $P'_1, \dots, P'_{n-1}$ contained in H. By the translation invariance property just noted, this mixed volume is well defined. Similarly, we can define the $(n-2)$-dimensional mixed volume $v^{(n-2)}$ for strongly isomorphic $(n-2)$-polytopes.

Lemma 5.1.5 *For $P_1, \dots, P_n \in \mathcal{A}$,*

$$V(P_1, \dots, P_n) = \frac{1}{n} \sum_{i=1}^{N} h_i^{(1)} v(F_i^{(2)}, \dots, F_i^{(n)}). \tag{5.12}$$

Proof We use induction over n. The case $n = 1$ being trivial, assume that $n > 1$ and the assertion is true in dimension $n-1$. Denote the right-hand side of (5.12) by $W(P_1, \dots, P_n)$. Applying (5.2) to $\lambda_1 P_1 + \dots + \lambda_n P_n$ and using (5.10) in dimension $n-1$, we obtain

$$V_n(\lambda_1 P_1 + \dots + \lambda_n P_n) = \sum_{i_1, \dots, i_n = 1}^{n} \lambda_{i_1} \cdots \lambda_{i_n} W(P_{i_1}, \dots, P_{i_n})$$

for arbitrary $\lambda_1, \dots, \lambda_n > 0$. Hence, it suffices to show that $W(P_1, \dots, P_n)$ is symmetric in its arguments. By the inductive hypothesis,

$$\begin{aligned} W(P_1, \dots, P_n) &= \frac{1}{n} \sum_{i=1}^{n} h_i^{(1)} \frac{1}{n-1} \sum_{(i,j)\in J} h_{ij}^{(2)} v^{(n-2)}(F_{ij}^{(3)}, \dots, F_{ij}^{(n)}) \\ &= \frac{1}{n(n-1)} \sum_{\substack{(r,s)\in J \\ r<s}} \left[h_r^{(1)} h_{rs}^{(2)} + h_s^{(1)} h_{sr}^{(2)} \right] v^{(n-2)}(F_{rs}^{(3)}, \dots, F_{rs}^{(n)}) \end{aligned}$$

(for $n = 2$, the expressions $v^{(n-2)}$ have to be replaced by 1). By (5.4),

$$\begin{aligned} & h_r^{(1)} h_{rs}^{(2)} + h_s^{(1)} h_{sr}^{(2)} \\ &= h_r^{(1)} \left[h_s^{(2)} \csc\theta_{rs} - h_r^{(2)} \cot\theta_{rs} \right] + h_s^{(1)} \left[h_r^{(2)} \csc\theta_{rs} - h_s^{(2)} \cot\theta_{rs} \right], \end{aligned}$$

and this expression is symmetric in the upper indices 1 and 2. Thus,

$$W(P_1, P_2, P_3, \dots, P_n) = W(P_2, P_1, P_3, \dots, P_n),$$

and since $W(P_1, P_2, P_3, \dots, P_n)$ is symmetric in its last $n-1$ arguments, as follows from its definition and the symmetry of mixed volumes in dimension $n-1$, the proof of the lemma is complete. □

Mixed volumes are closely related to mixed area measures, which we now define for strongly isomorphic polytopes. Recall that the surface area measure $S_{n-1}(K, \cdot)$ of a convex body $K \in \mathcal{K}_n^n$ is, by Theorem 4.6, the Borel measure on the unit sphere $\mathbb{S}^{n-1}$ for which

$$S_{n-1}(K, \omega) = \mathcal{H}^{n-1}(\tau(K, \omega)) \quad \text{for } \omega \in \mathcal{B}(\mathbb{S}^{n-1}).$$

In particular, for a polytope P (not necessarily with interior points) with normal vectors $u_1, \dots, u_N$ and for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ we have

$$S_{n-1}(P, \omega) = \sum_{u_i \in \omega} V_{n-1}(F(P, u_i)). \tag{5.13}$$

For strongly isomorphic polytopes $P_1, \dots, P_{n-1} \in \mathcal{A}$ we define their *mixed area measure* by

$$S(P_1, \dots, P_{n-1}, \omega) := \sum_{u_i \in \omega} v(F_i^{(1)}, \dots, F_i^{(n-1)}), \qquad \omega \in \mathcal{B}(\mathbb{S}^{n-1}). \tag{5.14}$$

Thus, $S(P_1, \dots, P_{n-1}, \cdot)$ is a finite measure on $\mathcal{B}(\mathbb{S}^{n-1})$. For $P_1, \dots, P_m \in \mathcal{A}$ and for $\lambda_1, \dots, \lambda_m \geq 0$ we obviously have

$$S_{n-1}(\lambda_1 P_1 + \cdots + \lambda_m P_m, \cdot) = \sum_{i_1, \dots, i_{n-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n-1}} S(P_{i_1}, \dots, P_{i_{n-1}}, \cdot), \tag{5.15}$$

in analogy to equation (5.10) and as a consequence of it, applied in dimension $n-1$.

In the same way as Lemma 5.1.4 was proved (or from that lemma and from (5.14)) one obtains the following polarization formula, which expresses the mixed area measure as a linear combination of surface area measures of Minkowski sums.

Lemma 5.1.6 *For $P_1, \dots, P_{n-1} \in \mathcal{A}$,*

$$S(P_1, \dots, P_{n-1}, \cdot) = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{n-1+k} \sum_{i_1 < \dots < i_k} S_{n-1}(P_{i_1} + \dots + P_{i_k}, \cdot).$$

We note that formula (5.12), valid for $P_1, \dots, P_n \in \mathcal{A}$, can be written in the form

$$V(P_1, \dots, P_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(P_1, u)\, S(P_2, \dots, P_n, \mathrm{d}u). \tag{5.16}$$

It is now easy to extend all this to general convex bodies.

Theorem 5.1.7 (and Definition) *There is a nonnegative symmetric function $V : (\mathcal{K}^n)^n \to \mathbb{R}$, the* mixed volume*, such that, for $m \in \mathbb{N}$,*

$$V_n(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_1, \dots, i_n = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n}) \tag{5.17}$$

for arbitrary convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ and numbers $\lambda_1, \dots, \lambda_m \geq 0$.

Further, there is a symmetric map S from $(\mathcal{K}^n)^{n-1}$ into the space of finite Borel measures on $\mathbb{S}^{n-1}$, the mixed area measure*, such that, for $m \in \mathbb{N}$,*

$$S_{n-1}(\lambda_1 K_1 + \dots + \lambda_m K_m, \cdot) = \sum_{i_1, \dots, i_{n-1} = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n-1}} S(K_{i_1}, \dots, K_{i_{n-1}}, \cdot) \tag{5.18}$$

for $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$ (where we write $S(K_1, \dots, K_{n-1}, \cdot) := S(K_1, \dots, K_{n-1})(\cdot)$).

The equality

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K_1, u)\, S(K_2, \dots, K_n, \mathrm{d}u) \tag{5.19}$$

holds for $K_1, \dots, K_n \in \mathcal{K}^n$.

Proof We define

$$V(K_1, \dots, K_n) := \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{i_1 < \dots < i_k} V_n(K_{i_1} + \dots + K_{i_k}) \tag{5.20}$$

for arbitrary $K_1, \dots, K_n \in \mathcal{K}^n$, which by Lemma 5.1.4 is consistent with the definition already given for the mixed volume of strongly isomorphic polytopes.

From the continuity of Minkowski addition and the volume functional we deduce that V is a continuous function on $(\mathcal{K}^n)^n$. Let $K_1, \dots, K_n \in \mathcal{K}^n$ be given. By Theorem 2.4.15 we can find sequences $(P_{1i})_{i \in \mathbb{N}}, \dots, (P_{ni})_{i \in \mathbb{N}}$ of polytopes such that $P_{ri} \to K_r$ for $i \to \infty$ $(r = 1, \dots, n)$ and such that, for each $i \in \mathbb{N}$, the polytopes $P_{1i}, \dots, P_{ni}$ are strongly isomorphic. From (5.10) and the continuity of volume and mixed volume (which is clear from (5.20) and the continuity of Minkowski addition) we deduce that (5.17) holds.

Similarly, we define

$$S(K_1,\dots,K_{n-1},\cdot) := \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(-1)^{n+k-1}\sum_{i_1<\dots<i_k} S_{n-1}(K_{i_1}+\dots+K_{i_k},\cdot) \tag{5.21}$$

for $K_1,\dots,K_{n-1}\in\mathcal{K}^n$, which again is consistent with the former definition, because of Lemma 5.1.6. Thus, $S(K_1,\dots,K_{n-1},\cdot)$ is a signed measure on $\mathcal{B}(\mathbb{S}^{n-1})$. From the weak continuity of the surface area measure, a special case of Theorem 4.2.1, we see that the map S thus defined is weakly continuous; that is, $K_{1i}\to K_1,\dots,K_{n-1,i}\to K_{n-1}$ for $i\to\infty$ implies

$$S(K_{1i},\dots,K_{n-1,i},\cdot)\xrightarrow{w} S(K_1,\dots,K_{n-1},\cdot).$$

An approximation argument analogous to that above now shows that (5.15) leads to (5.18).

Formula (5.19) is obtained from (5.16) by approximating $K_1,\dots,K_n$ by strongly isomorphic polytopes and using the continuity of the mixed volume, the weak continuity of the mixed area measure and the fact that support functions are continuous.

It remains to show that V and S are nonnegative. That V is nonnegative on strongly isomorphic polytopes follows by an obvious induction argument, using (5.12) (where $o\in P_1$ and hence $h_i^{(1)}\ge 0$ can be assumed). The non-negativity of the mixed area measure on strongly isomorphic polytopes is then a consequence of their definition (5.14). Finally, approximation by strongly isomorphic polytopes yields the non-negativity of V and S in general. □

We remark that equality (5.14) and Lemma 5.1.5, valid so far for strongly isomorphic polytopes, can now be shown to hold for arbitrary polytopes. Let $P_1,\dots,P_n\in\mathcal{P}^n$. Applying (5.13) to $P=\lambda_1P_1+\dots+\lambda_{n-1}P_{n-1}$ with nonnegative $\lambda_1,\dots,\lambda_{n-1}$ and using (5.15) on one hand and (5.13), (5.10) (in dimension $n-1$) on the other, we find, comparing the coefficients of $\lambda_1\cdots\lambda_{n-1}$, that

$$S(P_1,\dots,P_{n-1},\omega)=\sum_{u\in\omega} v(F(P_1,u),\dots,F(P_{n-1},u)) \tag{5.22}$$

for $\omega\in\mathcal{B}(\mathbb{S}^{n-1})$. Here the sum extends formally over all $u\in\omega$, but in fact only over the finitely many normal vectors of the facets of $P_1+\dots+P_{n-1}$ contained in ω. In view of (5.22), formula (5.19) reads

$$V(P_1,\dots,P_n)=\frac{1}{n}\sum_{u\in\mathbb{S}^{n-1}} h(P_1,u)v(F(P_2,u),\dots,F(P_n,u)), \tag{5.23}$$

where the sum extends over the normal vectors of the facets of $P_2+\dots+P_n$.

We have already shown and used in the proof of Theorem 5.1.7 that the mixed volume is continuous and the mixed area measure is weakly continuous. We shall now establish some further properties of these maps.

From (5.20) and (5.21) it follows that the mixed volume and the mixed area measure do not change under an arbitrary translation of any of their arguments. Further, it follows that

$$V(K, \dots, K) = V_n(K),$$
$$V(\alpha K_1, \dots, \alpha K_n) = V(K_1, \dots, K_n)$$

for any volume-preserving affine map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, and that

$$S(K, \dots, K, \cdot) = S_{n-1}(K, \cdot),$$
$$S(\rho K_1, \dots, \rho K_{n-1}, \rho\omega) = S(K_1, \dots, K_{n-1}, \omega)$$

for any rotation $\rho \in \mathrm{SO}(n)$. But also the behaviour of the mixed area measure under general linear transformations can be described. For $K_1, \dots, K_{n-1}, L \in \mathcal{K}^n$ and $\phi \in \mathrm{GL}(n)$ we have

$$V(\phi L, \phi K_1, \dots, \phi K_{n-1}) = |\det \phi| V(L, K_1 \dots, K_{n-1})$$

and thus, by (5.19),

$$\int_{\mathbb{S}^{n-1}} h(\phi L, u)\, S(\phi K_1, \dots, \phi K_{n-1}, \mathrm{d}u) = |\det \phi| \int_{\mathbb{S}^{n-1}} h(L, v)\, S(K_1, \dots, K_{n-1}, \mathrm{d}v).$$

It follows from Lemma 1.7.8 that any continuous, positively homogeneous function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated, uniformly on $\mathbb{S}^{n-1}$, by differences of support functions. Since $h(\phi L, u) = h(L, \phi^t u)$, we conclude that

$$\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}S(\phi K_1, \dots, \phi K_{n-1}, \cdot) = |\det \phi| \int_{\mathbb{S}^{n-1}} f \circ \phi^{-t} \,\mathrm{d}S(K_1, \dots, K_{n-1}, \cdot) \tag{5.24}$$

holds for any such function f. This describes the behaviour of the mixed area measure under a linear transformation.

We turn to the monotonicity property of the mixed volume. Let $K, L \in \mathcal{K}^n$ be convex bodies such that $K \subset L$, and let $K_2, \dots, K_n \in \mathcal{K}^n$ be arbitrary. From (5.19) we obtain

$$\begin{aligned} &V(K, K_2, \dots, K_n) - V(L, K_2, \dots, K_n) \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} [h(K, u) - h(L, u)]\, S(K_2, \dots, K_n, \mathrm{d}u) \le 0, \end{aligned}$$

with equality if and only if

$$h(K, u) = h(L, u) \quad \text{for each } u \in \operatorname{supp} S(K_2, \dots, K_n, \cdot).$$

Thus,

$$K \subset L \Rightarrow V(K, K_2, \dots, K_n) \le V(L, K_2, \dots, K_n). \tag{5.25}$$

By symmetry, the mixed volume is monotonic in each of its arguments. Unfortunately, no geometric description of the support of the mixed area measure $S(K_2, \dots, K_n, \cdot)$ is known (see, however, Conjecture 7.6.14). Therefore, the complete characterization of the equality cases in (5.25) is an open problem. A special case is treated in Theorem 7.6.17.

Easier to decide is the question of when the mixed volume is strictly positive.

Theorem 5.1.8 *For $K_1, \dots, K_n \in \mathcal{K}^n$, the following assertions are equivalent:*

(a) $V(K_1, \dots, K_n) > 0$;
(b) *There are segments $S_i \subset K_i$ $(i = 1, \dots, n)$ with linearly independent directions;*
(c) $\dim(K_{i_1} + \cdots + K_{i_k}) \geq k$ *for each choice of indices* $1 \leq i_1 < \cdots < i_k \leq n$ *and for all* $k \in \{1, \dots, n\}$.

Proof Suppose (a) holds. We prove (b) by induction. The case $n = 1$ being trivial, suppose that $n > 1$ and that the assertion is true in smaller dimensions. Since a convex body can be approximated by polytopes contained in it and since the mixed volume is continuous, there are polytopes $P_i \subset K_i$ $(i = 1, \dots, n)$ such that $V(P_1, \dots, P_n) > 0$. By (5.23),

$$0 < V(P_1, \dots, P_n) = \frac{1}{n} \sum h(P_1, u_i) v(F(P_2, u_i), \dots, F(P_n, u_i)),$$

where the sum extends over finitely many unit vectors u_i. Assuming that $o \in \operatorname{relint} P_1$, all the summands are nonnegative; hence there is some j for which

$$h(P_1, u_j) > 0, \qquad v(F(P_2, u_j), \dots, F(P_n, u_j)) > 0.$$

By the induction hypothesis, the latter implies the existence of segments $S_i \subset F(P_i, u_j)$, $i = 2, \dots, n$, with independent directions. Since $h(P_1, u_j) > 0$, there is a segment $S_1 \subset P_1$ that is not orthogonal to u_j. Thus $S_i \subset K_i$ for $i = 1, \dots, n$, and the directions of $S_1, \dots, S_n$ are linearly independent. Thus (b) holds.

Suppose that (b) holds and $S_i \subset K_i$ $(i = 1, \dots, n)$ are segments with independent directions. The monotonicity of the mixed volume in each argument implies $V(K_1, \dots, K_n) \geq V(S_1, \dots, S_n)$. By (5.20),

$$n! V(S_1, \dots, S_n) = V_n(S_1 + \cdots + S_n) > 0,$$

which shows that (a) is true.

It is trivial that (b) implies (c). Conversely, if (c) holds, we may assume that $o \in \operatorname{relint} K_i$ for $i = 1, \dots, n$, and then the truth of (b) is an immediate consequence of the following lemma. □

Lemma 5.1.9 *Let $L_1, \dots, L_n$ be linear subspaces of $\mathbb{R}^n$. If*

$$\dim(L_{i_1} + \cdots + L_{i_k}) \geq k$$

for each choice of indices $1 \leq i_1 < \cdots < i_k \leq n$ and for all $k \in \{1, \dots, n\}$, then there are lines $G_i \subset L_i$ $(i = 1, \dots, n)$ such that $\dim(G_1 + \cdots + G_n) = n$.

Proof For each m-tuple $(i_1, \dots, i_m)$ from $\{2, \dots, n\}$ we have either (a) $L_1 \subset L_{i_1} + \cdots + L_{i_m}$ or (b) $\dim(L_1 \cap (L_{i_1} + \cdots + L_{i_m})) < \dim L_1$. Since L_1 is not covered by finitely many lower-dimensional subspaces, we can choose a line $G_1 \subset L_1$ such that in case (b) we always have

$$G_1 \not\subset L_{i_1} + \cdots + L_{i_m}.$$

Let $(i_2, \ldots, i_k)$ be a $(k-1)$-tuple from $\{2, \ldots, n\}$. Then either $L_1 \subset L_{i_2} + \cdots + L_{i_k}$ and hence

$$\dim(G_1 + L_{i_2} + \cdots + L_{i_k}) = \dim(L_1 + L_{i_2} + \cdots + L_{i_k}) \geq k,$$

or $L_1 \not\subset L_{i_2} + \cdots + L_{i_k}$ and thus

$$\dim(G_1 + L_{i_2} + \cdots + L_{i_k}) = 1 + \dim(L_{i_2} + \cdots + L_{i_k}) \geq 1 + (k-1) = k.$$

Repeating the procedure with L_2, and so on, we complete the proof of the lemma. □

An important property of the mixed volume is its Minkowski linearity in each argument: for $K, L, K_2, \ldots, K_n \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$ we have

$$V(\lambda K + \mu L, K_2, \ldots, K_m) = \lambda V(K, K_2, \ldots, K_m) + \mu V(L, K_2, \ldots, K_m). \tag{5.26}$$

This follows from (5.19). By symmetry, V is Minkowski linear in each of its arguments. The same holds for the mixed area measure:

$$\begin{aligned} &S(\lambda K + \mu L, K_2, \ldots, K_{n-1}, \cdot) \\ &= \lambda S(K, K_2, \ldots, K_{n-1}, \cdot) + \mu S(L, K_2, \ldots, K_{n-1}, \cdot). \end{aligned} \tag{5.27}$$

For polytopes, this is clear from (5.22) and the linearity of the mixed volume (in dimension $n-1$); the general case follows by approximation.

The polynomial expansion (5.17) can be written in a more concise form. Introducing the abbreviation

$$V(\underbrace{K_1, \ldots, K_1}_{r_1}, \ldots, \underbrace{K_k, \ldots, K_k}_{r_k}) =: V(K_1[r_1], \ldots, K_k[r_k])$$

and the multinomial coefficient

$$\binom{n}{r_1 \ldots r_m} := \begin{cases} \dfrac{n!}{r_1! \cdots r_m!} & \text{if } \sum_{j=1}^m r_j = n \text{ and } \ r_j \in \{0, 1, \ldots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

we find, using a standard combinatorial argument, for $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, that

$$\begin{aligned} &V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) \\ &= \sum_{r_1, \ldots, r_m = 0}^{n} \binom{n}{r_1 \ldots r_m} \lambda_1^{r_1} \cdots \lambda_m^{r_m} V(K_1[r_1], \ldots, K_m[r_m]). \end{aligned} \tag{5.28}$$

Similar polynomial expansions are obtained for the mixed volume if some of its arguments are held fixed. Let an integer $p \in \{1, \ldots, n\}$ and convex bodies $C_{p+1}, \ldots, C_n \in \mathcal{K}^n$ be given. Then

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m[p], C_{p+1}, \ldots, C_n) \tag{5.29}$$
$$= \sum_{r_1,\ldots,r_m=0}^{p} \binom{p}{r_1 \ldots r_m} \lambda_1^{r_1} \cdots \lambda_m^{r_m} V(K_1[r_1], \ldots, K_m[r_m], C_{p+1}, \ldots, C_n).$$

For the proof, we develop both sides of the identity

$$\begin{aligned} &V_n(\mu(\lambda_1 K_1 + \cdots + \lambda_m K_m) + \mu_{p+1} C_{p+1} + \cdots + \mu_n C_n) \\ &= V_n(\mu\lambda_1 K_1 + \cdots + \mu\lambda_m K_m + \mu_{p+1} C_{p+1} + \cdots + \mu_n C_n) \end{aligned}$$

for $\mu, \mu_{p+1}, \ldots, \mu_n \geq 0$, where we use (5.28), and then compare the coefficients of $\mu^p \mu_{p+1} \cdots \mu_n$ in the two expressions thus obtained.

It is clear that the polynomial expansion (5.18) leads to exactly analogous results for the mixed area measure; for these, we use the analogous notation.

A further important additivity property is to be noted.

Theorem 5.1.10 *For given $p \in \{1, \ldots, n\}$ and convex bodies $C_{p+1}, \ldots, C_n \in \mathcal{K}^n$, the function f defined by*

$$f(K) := V(K[p], C_{p+1}, \ldots, C_n), \quad K \in \mathcal{K}^n,$$

is additive on $\mathcal{K}^n$, thus

$$f(K \cup L) + f(K \cap L) = f(K) + f(L)$$

for $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

Proof For $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$, any $C \in \mathcal{K}^n$ and any additive function φ on $\mathcal{K}^n$ we have, by (3.1) and (3.3),

$$\varphi((K \cup L) + C) + \varphi((K \cap L) + C) = \varphi(K + C) + \varphi(L + C).$$

Since the volume is additive, for any $\mu, \mu_{p+1}, \ldots, \mu_n \geq 0$ this gives

$$\begin{aligned} &V_n(\mu(K \cup L) + \mu_{p+1} C_{p+1} + \cdots + \mu_n C_n) \\ &\quad + V_n(\mu(K \cap L) + \mu_{p+1} C_{p+1} + \cdots + \mu_n C_n) \\ &= V_n(\mu K + \mu_{p+1} C_{p+1} + \cdots + \mu_n C_n) \\ &\quad + V_n(\mu L + \mu_{p+1} C_{p+1} + \cdots + \mu_n C_n). \end{aligned}$$

Developing both sides according to (5.28) and comparing the coefficients of $\mu^p \mu_{p+1} \cdots \mu_n$, we arrive at the assertion. $\square$

Similarly, from the additivity of the area measure (a special case of Theorem 4.2.1) we infer that the map

$$K \mapsto S(K[p], C_{p+1}, \ldots, C_{n-1}, \cdot)$$

is additive on $\mathcal{K}^n$.

Finally, we note that the mixed area measure, when considered as defining a mass distribution on the sphere, always has its centroid at the origin:

$$\int_{\mathbb{S}^{n-1}} u\, S(K_1, \dots, K_{n-1}, \mathrm{d}u) = o. \tag{5.30}$$

In fact, from (5.19) and the translation invariance of the mixed volume we have

$$\begin{aligned} 0 &= V(K_1, \dots, K_{n-1}, K_n + t) - V(K_1, \dots, K_{n-1}, K_n) \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \langle u, t \rangle\, S(K_1, \dots, K_{n-1}, \mathrm{d}u) \end{aligned}$$

for arbitrary $t \in \mathbb{R}^n$, from which (5.30) follows.

Many interesting functionals of convex geometry are obtained by specializing the mixed volume, that is, restricting its arguments in some way.

In particular, we again obtain the intrinsic volumes and quermassintegrals, which were introduced in Section 4.2. In fact, comparison of (4.8) and (5.17) shows that

$$\kappa_{n-j} V_j(K) = \binom{n}{j} W_{n-j}(K) = \binom{n}{j} V(K[j], B^n[n-j]). \tag{5.31}$$

For another important special case, we introduce the notation

$$V_1(K, L) := V(K, \dots, K, L). \tag{5.32}$$

This function of two convex bodies has the representations

$$V_1(K, L) = \lim_{\varepsilon \downarrow 0} \frac{V_n(K + \varepsilon L) - V_n(K)}{\varepsilon} \tag{5.33}$$

by (5.17) and

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u)\, S_{n-1}(K, \mathrm{d}u) \tag{5.34}$$

by (5.19). The nonsymmetric function $(K, L) \mapsto V_1(K, L)$ will undergo an important generalization in Section 9.1.

Notes for Section 5.1

1. The theory of mixed volumes was created by Minkowski [1438, 1441]. A presentation of its early development and history can be found in Bonnesen and Fenchel [284]. The theory attained a more general and elegant form when the mixed area measure was introduced, independently by Aleksandrov [12] and by Fenchel and Jessen [572].

   If the mixed volume has previously been defined, then formula (5.19) can serve as a starting point for the definition of the mixed area measure. Let $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ be given. For $K \in \mathcal{K}^n$, let $h := \bar{h}_K$ and $f(h) := V(K, K_1, \dots, K_{n-1})$. This defines an additive functional f on a subset of $C(\mathbb{S}^{n-1})$, the real vector space of continuous real functions on $\mathbb{S}^{n-1}$ with the maximum norm. It can be shown that f has a unique extension to a continuous linear functional on $C(\mathbb{S}^{n-1})$. By the Riesz representation theorem, there exists a unique measure $S(K_1, \dots, K_{n-1}, \cdot)$ on $\mathcal{B}(\mathbb{S}^{n-1})$ for which, in particular,

$$V(K, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)\, S(K_1, \dots, K_{n-1}, \mathrm{d}u)$$

for all $K \in \mathcal{K}^n$. The properties of S can be deduced from those of V. By this method, the mixed area measure was introduced by Aleksandrov [12], and in a similar way by Fenchel and Jessen [572]. For a geometric interpretation, Aleksandrov first defined $S_{n-1}(K, \cdot)$, essentially by formula (4.32), and then used his definition of the mixed area measure to prove the expansion (5.18). Later, Aleksandrov [18] simplified part of his reasoning by working with weak convergence. Fenchel and Jessen [572] used the formula

$$V(L, K, \dots, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u)\, S_{n-1}(K, \mathrm{d}u)$$

to define $S_{n-1}(K, \cdot)$ and later obtained an intuitive interpretation by proving formula (4.13).

2. Mixed volumes can be introduced in different ways, but at some point a symmetry property such as that of $W(P_1, \dots, P_n)$ in the proof of Lemma 5.1.5 has to be shown. For the case of polytopes, one may compare Minkowski [1441], §21, and Aleksandrov [25], p. 411; for analytic proofs, see Bonnesen and Fenchel [284], p. 57 and the references on p. 61.
3. Lemma 5.1.9 is taken from Leichtweiß [1184], pp. 176–177. A refinement is proved and used in Schneider [1707].
4. *The mixed volume as a distribution*. The representation (5.19) exhibits the linearity of the mixed volume in its first argument, but not the symmetry of the mixed volume. To obtain a more symmetric representation, Weil [1946] expressed the mixed volume as a distribution. More generally, such a representation is possible for multilinear and continuous functions on $\mathcal{K}^n$. The following theorem is due to Goodey and Weil [744].

   Theorem Let $\varphi : (\mathcal{K}^n)^n \to \mathbb{R}$ be multilinear and continuous. Then there is a unique distribution T on $(\mathbb{S}^{n-1})^n$ that can be extended to a Banach space of functions on $(\mathbb{S}^{n-1})^n$ containing the tensor products $h_{K_1} \otimes \cdots \otimes h_{K_n}$, for all $K_1, \dots, K_n \in \mathcal{K}^n$, in such a way that

$$\varphi(K_1, \dots, K_n) = T(h_{K_1} \otimes \cdots \otimes h_{K_n}).$$

   In fact, there are real functions f_j of class C^∞ on $(\mathbb{S}^{n-1})^n$ for $j = 1, 2, \dots$ such that

$$\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} h(K_1, u_1) \cdots h(K_n, u_n) f_j(u_1, \dots, u_n)\, \mathrm{d}\sigma(u_1) \cdots \mathrm{d}\sigma(u_n) \\ \to \varphi(K_1, \dots, K_n)$$

   as $j \to \infty$, uniformly for all K_i in any fixed ball (σ is the spherical Lebesgue measure).

   Following a different line, Przesławski [1551] obtained for mixed volumes that the functions f_j in the previous theorem can even be chosen real-analytic. Przesławski developed a general approach to a class of integral representations for mixed volumes (and, more generally, for mixed moment tensors) which are symmetric in the support functions, as above. His starting point is the integral representation of the volume given by (1.56), which he combined with convex functions associated with a convex body (in the sense explained in Section 1.7), and with smoothing procedures for support functions.
5. *Characterization theorems.* Hadwiger's characterization theorem for linear combinations of intrinsic volumes (Theorem 6.4.14) was the starting point for several similar investigations. Characterizations of functionals related to mixed volumes by means of their additivity properties require strong additional assumptions. Fáry [546] characterized, for a given convex body $U \in \mathcal{K}^n$, the functionals

$$\varphi : K \mapsto \sum_{i=0}^{n} c_i V(K[i], U[n-i]),$$

   where $c_0, \dots, c_n \in \mathbb{R}$, as the translation invariant, continuous valuations φ satisfying $\varphi(K) = \varphi(L)$ whenever $V(K[i], U[n-i]) = V(L[i], U[n-i])$ for $i = 0, \dots, n$.

Firey [607] showed that an increasing Minkowski linear function $\varphi : \mathcal{K}^n \to \mathbb{R}$ that is zero on one-pointed sets must be of the form

$$\varphi(K) = V(K, L[p-1], S_{p+1}, \dots, S_n)$$

with an essentially unique convex body L and pairwise orthogonal unit segments $S_{p+1}, \dots, S_n$ that span the orthogonal complement of the affine hull of L.

6. *Characterizing the mixed volume as a multilinear functional.* As a function $F : (\mathcal{K}^n)^n \to \mathbb{R}$, the mixed volume has the following properties: in each variable, it is Minkowski linear and increasing (under set inclusion), and it is symmetric in its arguments. An attempt to characterize the mixed volume (up to trivial modifications) by these properties is doomed to failure, since there are many, and rather different, functions with these properties. Somewhat surprisingly, a mild additional vanishing condition leads to a characterization, at least for centrally symmetric convex bodies. Let $\mathcal{K}^n_c \subset \mathcal{K}^n$ denote the subset of centrally symmetric bodies. The following theorem was proved by Milman and Schneider [1433].

   Theorem Suppose that the function $F : (\mathcal{K}^n_c)^n \to \mathbb{R}$ is Minkowski additive and increasing in each variable and satisfies $F(K_1, \dots, K_n) = 0$ whenever two of the arguments $K_1, \dots, K_n$ are parallel segments. Then there is a constant $a \geq 0$ with

   $$F(K_1, \dots, K_n) = aV(K_1, \dots, K_n) \quad \text{for } K_1, \dots, K_n \in \mathcal{K}^n_c.$$

   For general convex bodies, only the two-dimensional case could be settled.

   Theorem Suppose that the function $F : (\mathcal{K}^2)^2 \to \mathbb{R}$ is Minkowski additive and increasing in each variable and satisfies $F(K_1, K_2) = 0$ if K_1, K_2 are parallel segments. Then there are constants $a \geq 0$ and $\alpha \in [0, 1]$ such that

   $$F(K_1, K_2) = aV(K_1, \alpha K_2 + (1-\alpha)(-K_2))$$

   for $K_1, K_2 \in \mathcal{K}^2$.

7. *An identity for mixed volumes*. If $K, L, K_3 \dots, K_n \in \mathcal{K}^n$ are convex bodies such that $K \cup L$ is convex, then

   $$V(K, L, K_3, \dots, K_n) = V(K \cup L, K \cap L, K_3, \dots, K_n).$$

   This follows from the additivity property of the mixed volume. It was noted by Groemer [788], p. 160, and proved in a shorter way in McMullen and Schneider [1396], p. 179. An abstract version was given by McMullen [1390], Theorem 15.

8. *Characterizations of convex bodies in terms of mixed volumes*. A convex body is determined, up to a translation, by the values of its mixed volumes with sufficiently many other convex bodies. As an example, the following result was proved in Schneider [1673]. Let $i \in \{1, \dots, n-1\}$ and let $K, K' \in \mathcal{K}^n$ be convex bodies of dimension $\geq i+1$. If

   $$V(K[i], B^n[n-i-1], L) = V(K'[i], B^n[n-i-1], L)$$

   for all two-dimensional convex bodies L, then K is a translate of K'. In fact, it is sufficient to assume the equality only for those convex bodies L that are congruent to a fixed triangle having at least one angle that is an irrational multiple of π.

   In view of such uniqueness results, it is not surprising that it is possible to characterize certain classes of convex bodies in terms of relations satisfied by mixed volumes. For example, Weil [1934] showed that $L \in \mathcal{K}^n$ is contained in some translate of $K \in \mathcal{K}^n$ if and only if

   $$V(L, C, \dots, C) \leq V(K, C, \dots, C)$$

   for all $C \in \mathcal{K}^n$. Scholtes [1743] gave a different proof, which is reproduced in the book by Pallaschke and Urbański [1498]. A stronger result, with an elementary proof, is due to Lutwak [1288]. He showed that

$$V(L, \Delta, \dots, \Delta) \le V(K, \Delta, \dots, \Delta)$$

for all simplices $\Delta \in \mathcal{K}^n$ implies that L is contained in some translate of K. Lutwak obtained this as a corollary of the following result, also proved in [1288].

Theorem (Lutwak's containment theorem) If K, L are convex bodies such that every simplex that contains K also contains a translate of L, then the body K contains a translate of L.

Continuing with criteria involving mixed volumes, each of the following conditions (a), (b), (c) is equivalent to the fact that L is a summand of K:

(a) (Weil [1934]) For each $j \in \{1, \dots, n-1\}$, the real functional

$$T_{K,L,j} = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} V(K[k], L[j-k], B^n[n-1-j], \cdot)$$

is monotonic on $\mathcal{K}^n$.

(b) (Matheron [1361]) For each $j \in \{0, \dots, n\}$ and $\lambda \in (0, 1)$,

$$V(K \div \lambda L[n-j], L[j]) = \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k \lambda^k V(K[n-k-j], L[k+j]).$$

(c) (McMullen [1391]) For all polytopes $P, Q \in \mathcal{K}^n$ with $P \subset Q$,

$$V(Q[n-1], K) - V(Q[n-1], L) - V(P[n-1], K) + V(P[n-1], L) \ge 0.$$

Other results in a similar spirit concern zonoids, generalized zonoids and centrally symmetric convex bodies. For $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$, let $V_{n-1}(K^u)$ denote the $(n-1)$-dimensional volume of the orthogonal projection $K^u = K \,|\, H_{u,0}$ of K to the hyperplane $H_{u,0}$. Weil [1938] proved that a centrally symmetric convex body K is a zonoid if and only if

$$V(K, L[n-1]) \le V(K, M[n-1])$$

for all centrally symmetric $L, M \in \mathcal{K}^n$ satisfying $V_{n-1}(L^u) \le V_{n-1}(M^u)$ for all $u \in \mathbb{S}^{n-1}$. A stronger version of this result was proved by Goodey and Zhang [756]. Goodey [731] showed that a centrally symmetric convex body $K \in \mathcal{K}^n$ is a generalized zonoid if and only if there exists a constant C such that

$$|V(K, L[n-1]) - V(K, M[n-1])| \le C \sup_{u \in \mathbb{S}^{n-1}} |V_{n-1}(L^u) - V_{n-1}(M^u)|$$

for all centrally symmetric $L, M \in \mathcal{K}^n$. He also showed that $K \in \mathcal{K}^n$ is centrally symmetric if and only if

$$V(K, L[n-1]) = V(K, M[n-1])$$

for all $L, M \in \mathcal{K}^n$ satisfying $V_{n-1}(L^u) = V_{n-1}(M^u)$ for all $u \in \mathbb{S}^{n-1}$. Similar and more general results were proved by Weil [1940] and by Goodey [734, 735]; for some necessary modifications of the latter results, see §6 of Goodey and Weil [746].

9. *Mixed area under separate rotations*. The mixed volume $V(K_1, \dots, K_n)$ is invariant under simultaneous rigid motions of the arguments, but not, of course, under separate motions applied to the arguments. Görtler [765] investigated the pairs K, L in $\mathcal{K}^2$ for which $V(K, L)$ is invariant under arbitrary rotations of one of the arguments.
10. *Complexity of computing mixed volumes.* The complexity of computing and approximating mixed volumes of polytopes and of well-presented convex bodies is discussed in Gritzmann and Klee [778] and thoroughly studied by Dyer, Gritzmann and Hufnagel [524].

11. *Polynomial volume as a characteristic property of Minkowski addition.* In their investigation of operations on convex (and other) sets (see also Section 3.1, Note 4), Gardner, Hug and Weil [679] have also studied how far the polynomial expansion of volume is typical for Minkowski addition. An operation $* : (\mathcal{K}^n_{os})^2 \to \mathcal{K}^n_{os}$ (where $\mathcal{K}^n_{os}$ is the set of o-symmetric convex bodies in $\mathbb{R}^n$) is said to have *polynomial volume* if

$$V_n(rK * sL) = \sum_{i,j=0}^{m(K,L)} a_{ij}(K,L) r^i s^j$$

holds with real coefficients $a_{ij}(K,L)$, some $m(K,L) \in \mathbb{N} \cup \{0\}$ and all $K, L \in \mathcal{K}^n_{os}$ and $r, s \geq 0$. The following result is proved in [679] (Corollary 10.5).

Theorem Let $n \geq 2$. The operation $* : (\mathcal{K}^n_{os})^2 \to \mathcal{K}^n_{os}$ is continuous, GL(n) covariant, associative and has polynomial volume and the identity property (i.e., $K*\{o\} = \{o\}*K = K$ for all $K \in \mathcal{K}^n_{os}$) if and only if it is Minkowski addition.

5.2 Extensions of mixed volumes

The mixed volume is a function on n-tuples of convex bodies that is Minkowski linear in each variable. In the present section we shall collect some information on the possibilities for extending the mixed volume to other kinds of argument, while preserving or generalizing its linearity properties.

Our first observation is a negative one: under any extension to non-convex compact sets, the mixed volume would lose its essential properties. More precisely, the following result holds (Weil [1935]).

Theorem 5.2.1 *Let $\mathcal{K}' \subset \mathcal{C}^n$ be a class of compact sets containing $\mathcal{K}^n$ and closed under Minkowski addition. If there exists a function $V : (\mathcal{K}')^n \to \mathbb{R}$ that is Minkowski additive in each variable and for which $V(K, \dots, K) = V_n(K)$ for $K \in \mathcal{K}'$, then $\mathcal{K}' = \mathcal{K}^n$.*

Proof Suppose that $\mathcal{K}'$ and V satisfy the assumptions. Since $V(\cdot, K_2, \dots, K_n)$ (where $K_2, \dots, K_n \in \mathcal{K}'$ are fixed) is Minkowski additive, the observation before Remark 3.3.1 shows that

$$V(K_1, K_2, \dots, K_n) = V(\operatorname{conv} K_1, K_2, \dots, K_n)$$

for $K_1 \in \mathcal{K}'$. Similarly,

$$V(\operatorname{conv} K_1, K_2, K_3, \dots, K_n) = V(\operatorname{conv} K_1, \operatorname{conv} K_2, K_3, \dots, K_n),$$

and so on. It follows that for $M \in \mathcal{K}'$ we have $V_n(M) = V_n(\operatorname{conv} M)$. If $\dim \operatorname{conv} M = n$, this implies $M = \operatorname{conv} M$. If $\dim \operatorname{conv} M < n$, we choose a convex body K in an affine subspace complementary to $\operatorname{aff} M$ such that $\dim (K + \operatorname{conv} M) = n$. For $M' := K + M$ we have $V_n(M') = V_n(\operatorname{conv} M')$ and thus $K + M = M' = \operatorname{conv} M' = K + \operatorname{conv} M$. Since K and $\operatorname{conv} M$ lie in complementary subspaces, this implies $M = \operatorname{conv} M$. Thus each element of $\mathcal{K}'$ is convex. □

If, however, we consider the mixed volume as a Minkowski multi-additive function not on convex bodies, but on support functions, then a natural and sometimes useful extension is possible. In the following, we identify convex bodies with their support functions, restricted to $\mathbb{S}^{n-1}$, and hence also write

$$V(K_1, \dots, K_n) = V(\overline{h}_{K_1}, \dots, \overline{h}_{K_n}) \tag{5.35}$$

for $K_1, \dots, K_n \in \mathcal{K}^n$.

The real vector space $C(\mathbb{S}^{n-1})$ of real continuous functions on the sphere $\mathbb{S}^{n-1}$ contains as a subspace the vector space $D(\mathbb{S}^{n-1})$ spanned by restrictions of support functions. If $g \in D(\mathbb{S}^{n-1})$ has the representations $g = \overline{h}_K - \overline{h}_L = \overline{h}_{K'} - \overline{h}_{L'}$ with $K, L, K', L' \in \mathcal{K}^n$, then $K + L' = K' + L$; hence it follows from the additivity of the mixed volume that

$$V(g, K_2, \dots, K_n) := V(K, K_2, \dots, K_n) - V(L, K_2, \dots, K_n)$$

does not depend on the special representation. Clearly, $V(\cdot, K_2, \dots, K_n)$ thus defined is a linear functional on $D(\mathbb{S}^{n-1})$. Similar extensions are possible in the other arguments. Explicitly, for $g_i \in D(\mathbb{S}^{n-1})$ with the representations $g_i = h_i^0 - h_i^1$, where h_i^0, h_i^1 are restrictions of support functions ($i = 1, \dots, n$), we can define the mixed volume of $g_1, \dots, g_n$ by

$$V(g_1, \dots, g_n) = \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_n} V(h_1^{\epsilon_1}, \dots, h_n^{\epsilon_n}).$$

Then V is symmetric and n-linear on $D(\mathbb{S}^{n-1})$, and satisfies (5.35). Similarly, the mixed area measure has an $(n-1)$-linear extension to $D(\mathbb{S}^{n-1})$ given by

$$S(g_1, \dots, g_{n-1}, \cdot) = \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} S(h_1^{\epsilon_1}, \dots, h_{n-1}^{\epsilon_{n-1}}, \cdot).$$

Formula (5.19) extends trivially to give

$$V(g_1, \dots, g_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} g_1(u)\, S(g_2, \dots, g_n, \mathrm{d}u)$$

for $g_1, \dots, g_n \in D(\mathbb{S}^{n-1})$.

For given convex bodies $K_2, \dots, K_n \in \mathcal{K}^n$, we can immediately obtain an extension of the functional $V(\cdot, K_2, \dots, K_n)$ to the space $C(\mathbb{S}^{n-1})$ (with the maximum norm) by defining

$$V(f, K_2, \dots, K_n) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(u)\, S(K_2, \dots, K_n, \mathrm{d}u) \tag{5.36}$$

for $f \in C(\mathbb{S}^{n-1})$. Then $V(\cdot, K_2, \dots, K_n)$ is a continuous linear functional on $C(\mathbb{S}^{n-1})$, of norm $V(B^n, K_2, \dots, K_n)$. The mixed area measure $S(K_2, \dots, K_n, \cdot)$ turns out to be the unique measure representing this linear functional according to the Riesz representation theorem. (This remark has played a role in the literature; cf. Section 5.1, Note 1).

It must, however, be pointed out that a further extension of the mixed volume to a multilinear continuous functional on $C(\mathbb{S}^{n-1})$ is not possible. It suffices to show this for $n = 2$.

Theorem 5.2.2 *There is no bilinear function $V : C(\mathbb{S}^1)^2 \to \mathbb{R}$ that is continuous in each variable and satisfies $V(\overline{h}_K, \overline{h}_L) = V(K, L)$ for $K, L \in \mathcal{K}^2$.*

Proof Suppose V exists. We identify $\mathbb{S}^1$ with $[0, 2\pi]$ via $(\cos\alpha, \sin\alpha) \mapsto \alpha$. For a continuous function f on $[0, 2\pi]$ and a convex body $K \in \mathcal{K}^2$, formula (5.19) extends to give

$$V(f, \overline{h}_K) = \frac{1}{2}\int_0^{2\pi} f(\alpha)\, S(K, \mathrm{d}\alpha).$$

For the proof, we approximate f uniformly by differences of support functions (which is possible by Lemma 1.7.8) and use the linearity and continuity of V in its first argument.

Now choose $f(\alpha) = 1 - |\sin\alpha|^{1/2}$ and let $0 < \varepsilon < 1$. Let K be the segment with endpoints $\pm(0, a)$ where $a = 1/\varepsilon$. The measure $S(K, \cdot)$ is concentrated at the points $\alpha = 0$ and $\alpha = \pi$ and associates a weight 2α with each of these points. Let L be obtained from K by a rotation around its centre by an angle φ, where $\sin\varphi = \varepsilon^2$. Then K and L are Hausdorff distance ε apart. On the other hand,

$$V(f, \overline{h}_K) - V(f, \overline{h}_L) = 2a - 2a(1 - \sqrt{\sin\varphi}) = 2.$$

Thus, the function $g_\varepsilon := \overline{h}_K - \overline{h}_L$ satisfies $\|g_\varepsilon\| = \varepsilon$ and $V(f, g_\varepsilon) = 2$, which for $\varepsilon \to 0$ contradicts the continuity of V in its second variable. □

If convex bodies are identified not with their support functions but with their characteristic functions, then a different extension problem arises for the mixed volume. We shall describe briefly the theory developed by Groemer [788] for this case. For $K \in \mathcal{K}^n$, let $K^\bullet$ be the characteristic function of K, and let $\mathscr{V}(\mathcal{K}^n)$ be the real vector space spanned by these functions for $K \in \mathcal{K}^n$. The first step is to show that there exists a unique bilinear map $\psi : \mathscr{V}(\mathcal{K}^n)^2 \to \mathscr{V}(\mathcal{K}^n)$ such that

$$\psi(K^\bullet, L^\bullet) = (K + L)^\bullet \quad \text{for } K, L \in \mathcal{K}^n.$$

The construction of this map uses the Euler characteristic on $\mathscr{V}(\mathcal{K}^n)$ as established by Groemer [782, 786]. The vector space $\mathscr{V}(\mathcal{K}^n)$, together with the multiplication $\times$ defined by $f \times g = \psi(f, g)$, is a commutative algebra over $\mathbb{R}$ with unit element. The Euler characteristic is an algebra homomorphism. To every affine map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ there exists a unique linear map $\overline{\alpha} : \mathscr{V}(\mathcal{K}^n) \to \mathscr{V}(\mathcal{K}^n)$ such that $\overline{\alpha}(K^\bullet) = (\alpha K)^\bullet$ for $K \in \mathcal{K}^n$. In the special case $\alpha x = \lambda x$ with $\lambda \geq 0$ one writes $\overline{\alpha}(f) := \lambda \circ f$ for $f \in \mathscr{V}(\mathcal{K}^n)$. Obviously, there is a unique linear functional $\overline{V}_n$ on $\mathscr{V}(\mathcal{K}^n)$ such that $\overline{V}_n(K^\bullet) = V_n(K)$ for $K \in \mathcal{K}^n$. Groemer showed that (5.28) can be generalized as follows. For nonnegative integers $r_1, \dots, r_n$ with $r_1 + \cdots + r_n = n$, there exists exactly one n-linear map

$$v_{(r_1,\dots,r_n)} : \mathcal{V}(\mathcal{K}^n)^n \to \mathbb{R}$$

such that

$$v_{(r_1,\dots,r_n)}(K_1^\bullet, \dots, K_n^\bullet) = V(K_1[r_1], \dots, K_n[r_n])$$

for $K_1, \dots, K_n \in \mathcal{K}^n$. For $f_1, \dots, f_n \in \mathcal{V}(\mathcal{K}^n)$ and $\lambda_1, \dots, \lambda_n \geq 0$ we have

$$\overline{V}_n((\lambda_1 \circ f_1) \times \cdots \times (\lambda_n \circ f_n)) = \sum \binom{n}{r_1 \dots r_n} \lambda_1^{r_1} \cdots \lambda_n^{r_n} v_{(r_1,\dots,r_n)}(f_1, \dots, f_n).$$

For the further investigation of these generalized mixed volumes we refer to the original article of Groemer [788] and also to Panina [1503].

Next, we consider a very special situation where the mixed volume can be extended in a natural way. This extension will be particularly useful for the proof of the Aleksandrov–Fenchel inequality in Section 7.3.

Let $\mathcal{A}$ be a given a-type of strongly isomorphic simple n-dimensional polytopes. As in Section 5.1, we let $u_1, \dots, u_N$ be the normal vectors of the facets of any $P \in \mathcal{A}$, and we use the notation of the first part of Section 5.1. The N-tuple

$$\overline{P} := (h_1, \dots, h_N) \in \mathbb{R}^N$$

is called the *support vector* of P. We recall that the support numbers of the facet F_i are given by

$$h_{ij} = h_j \csc \theta_{ij} - h_i \cot \theta_{ij} \quad \text{for } (i, j) \in J,$$

and furthermore that

$$V(P_1, \dots, P_n) = \sum a_{j_1 \dots j_n} h_{j_1}^{(1)} \cdots h_{j_n}^{(n)}, \tag{5.37}$$

$$v(F_i^{(1)}, \dots, F_i^{(n-1)}) = \sum a_{k_1 \dots k_{n-1}}^{(i)} h_{ik_1}^{(1)} \cdots h_{ik_{n-1}}^{(n-1)} \tag{5.38}$$

for $P_1, \dots, P_n \in \mathcal{A}$, where the coefficients are symmetric and depend only on the a-type $\mathcal{A}$. The summations in (5.37) and (5.38) can be considered to extend respectively over all $j_1, \dots, j_n$ and all $k_1, \dots, k_{n-1}$, from 1 to N, if zero coefficients are introduced and, say, $h_{ij}^{(r)} := 0$ if $(i, j) \notin J$.

We can consider V, as given by (5.37), to be defined on n-tuples of support vectors, and we now extend this to arbitrary N-tuples

$$X_r = (x_1^{(r)}, \dots, x_N^{(r)}) \in \mathbb{R}^N, \quad r = 1, \dots, n,$$

by putting

$$V(X_1, \dots, X_n) := \sum a_{j_1 \dots j_n} x_{j_1}^{(1)} \cdots x_{j_n}^{(n)}. \tag{5.39}$$

Then V is an N-linear function on $\mathbb{R}^N$. Furthermore, for $X = (x_1, \dots, x_N) \in \mathbb{R}^N$ we define

$$x_{ij} := \begin{cases} x_j \csc\theta_{ij} - x_i \cot\theta_{ij} & \text{if } (i,j) \in J, \\ 0 & \text{if } (i,j) \notin J, \end{cases}$$

and

$$\Lambda_i X := (x_{i1}, \dots, x_{iN}), \tag{5.40}$$

so that $\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$ is a linear map. Then we put

$$v(\Lambda_i X_1, \dots, \Lambda_i X_{n-1}) := \sum a^{(i)}_{k_1 \dots k_{n-1}} x^{(1)}_{ik_1} \cdots x^{(n-1)}_{ik_{n-1}}. \tag{5.41}$$

Lemma 5.1.5 now extends to give

$$V(X_1, \dots, X_n) = \frac{1}{n} \sum_{i=1}^{N} x_i^{(1)} v(\Lambda_i X_2, \dots, \Lambda_i X_n). \tag{5.42}$$

This follows from the facts that the equality is true for support vectors, that $\overline{P} + \varepsilon X_r$ is a support vector if $P \in \mathcal{A}$ and ε is in a suitable neighbourhood of 0 (as follows from Lemma 2.4.13) and that both sides are n-linear functions.

Up to now in this section, the interest was in extensions of the mixed volume as an n-linear function of n variables. The special mixed volume

$$(M, K) \mapsto V_1(M, K) = V(M, \dots, M, K)$$

has extensions to certain nonconvex sets M, which are important for several applications. Recall, first, that for convex bodies $M, K \in \mathcal{K}^n$ we have

$$V_1(M, K) = \lim_{\varepsilon \downarrow 0} \frac{V_n(M + \varepsilon K) - V_n(M)}{\varepsilon} \tag{5.43}$$

by (5.33), and

$$V_1(M, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) \, S_{n-1}(M, \mathrm{d}u) \tag{5.44}$$

by (5.34). Using (4.32), the latter formula can be transformed into

$$V_1(M, K) = \frac{1}{n} \int_{\mathrm{bd}\, M} h_K(u_M(x)) \, \mathrm{d}\mathcal{H}^{n-1}(x), \tag{5.45}$$

where $u_M(x)$ is the outer unit normal vector of M at $x \in \mathrm{bd}\, M$.

One can use (5.45) to define $V_1(M, K)$ if M is a compact set with a boundary which is a piecewise C^1 hypersurface (but K remains a convex body). This was done (and applied) by Zhang [2060], who proved for this case that also the limit relation (5.43) remains valid.

Notes for Section 5.2

1. The multilinear extension of V and S to differences of support functions appears in Aleksandrov [12]. It was further investigated (and used for characterizations of convex functions within the vector space of differences of support functions) by Weil [1933]; see also Weil [1934].

   The assertion of Theorem 5.2.2 was first proved, in a different way, by Meier in 1982 (unpublished). The construction used above is taken from Schneider [1672], in a modified form.
2. To the extension of the mixed volume to the vector space $\mathscr{V}(\mathcal{K}^n)$ spanned by the indicator functions of convex bodies, Chen [412] has contributed a new aspect. On $\mathscr{V}(\mathcal{K}^n)$ he introduced the so-called Geissinger multiplication and showed that the mixed volume of convex bodies can be represented as the volume of the Geissinger product of their indicator functions. This led to some generalizations, for example, the definition of mixed volumes for relatively open bounded convex sets.
3. *Integrals of functions over Minkowski linear combinations.* Let $\mathscr{C} = (K_1, \dots, K_m)$ be a given m-tuple of convex bodies, and let $F : \mathbb{R}^n \to \mathbb{C}$ be a continuous function. For $\lambda_1, \dots, \lambda_m \geq 0$, let
$$(M_{\mathscr{C}}F)(\lambda_1, \dots, \lambda_m) := \int_{\sum_{i=1}^m \lambda_i K_i} F(x)\, \mathrm{d}x.$$
   If $F \equiv 1$, then $M_{\mathscr{C}}F$ is a homogeneous polynomial in $\lambda_1, \dots, \lambda_m$ and the coefficients yield the mixed volumes. Alesker [31] investigated $M_{\mathscr{C}}F$ as a function of $\lambda_1, \dots, \lambda_m$ if F is smooth or analytic.

5.3 Special formulae for mixed volumes

The purpose of this section is to collect a number of special formulae for mixed volumes and mixed area measures and, in particular, for their specializations to intrinsic volumes or quermassintegrals (in this section, we prefer the latter normalization, for simplicity). These formulae relate mixed volumes to the notions of Section 2.5, that is, to local curvature functions in the case of convex bodies of class C^2_+, and also to the integral-geometric considerations of Section 4.4. Moreover, some special representations are treated that are valid for generalized zonoids.

5.3.1 Formulae involving curvature functions

In this subsection, it is supposed that the convex bodies $K, L, K_i \in \mathcal{K}^n_n$ are all of class C^2_+. First we recall that in this case, by (4.25) and (4.26), the curvature measures and area measures have the explicit representations
$$C_m(K,\beta) = \int_{\beta \cap \mathrm{bd}\, K} H_{n-1-m}\, \mathrm{d}\mathcal{H}^{n-1}, \tag{5.46}$$
$$S_m(K,\omega) = \int_{\omega} s_m\, \mathrm{d}\mathcal{H}^{n-1} \tag{5.47}$$
for $\beta \in \mathcal{B}(\mathbb{R}^n)$, $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and $m = 0, 1, \dots, n-1$. Applying (5.47) for $m = n-1$ to the body $K = \lambda_1 K_1 + \cdots + \lambda_{n-1} K_{n-1}$ with $\lambda_1, \dots, \lambda_{n-1} \geq 0$ and using (5.18) and (2.65), we find that

$$S(K_1, \dots, K_{n-1}, \omega) = \int_\omega s(K_1, \dots, K_{n-1}, u)\, \mathrm{d}u. \tag{5.48}$$

Formula (5.19) for the mixed volume now reads

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K_1, u) s(K_2, \dots, K_n, u)\, \mathrm{d}u. \tag{5.49}$$

Since the mixed volume is symmetric in its arguments, this yields integral formulae such as

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} h(K_1, u) s(K_2, K_3, \dots, K_n, u)\, \mathrm{d}u \\ &= \int_{\mathbb{S}^{n-1}} h(K_2, u) s(K_1, K_3, \dots, K_n, u)\, \mathrm{d}u. \end{aligned} \tag{5.50}$$

The area measures of lower order are obtained from the mixed area measure by taking a special case: in analogy to formula (2.67), namely

$$s_j(K, u) = s(\underbrace{K, \dots K}_{j}, \underbrace{B^n, \dots, B^n}_{n-1-j}, u), \tag{5.51}$$

we have

$$S_j(K, \omega) = S(\underbrace{K, \dots K}_{j}, \underbrace{B^n, \dots, B^n}_{n-1-j}, \omega). \tag{5.52}$$

This follows immediately from (5.48), or (for general convex bodies) from a comparison of the local Steiner formula (4.36) for $m = n - 1$, that is,

$$S_{n-1}(K + \rho B^n, \omega) = \sum_{j=0}^{n-1} \rho^{n-1-j} \binom{n-1}{j} S_j(K, \omega),$$

with the polynomial expansion (5.18). Consequently, the quermassintegrals $W_1, \dots, W_n$, which in Section 4.2 were introduced by

$$W_i(K) := \frac{1}{n} S_{n-i}(K, \mathbb{S}^{n-1}), \quad i = 1, \dots, n,$$

turn out (again) to be special mixed volumes: by (5.52) and (5.19),

$$W_i(K) = V(\underbrace{K, \dots, K}_{n-i}, \underbrace{B^n, \dots, B^n}_{i}) \tag{5.53}$$

for $i = 1, \dots, n$, but also for $i = 0$, since $W_0(K) = V_n(K)$, by definition. Of course, (5.53) is valid for arbitrary convex bodies K, and it also follows directly by comparing the expansions (4.8) and (5.17).

From (5.47) and (5.46) we see that

$$W_i(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} s_{n-i}\, \mathrm{d}\mathcal{H}^{n-1}, \tag{5.54}$$

$$W_i(K) = \frac{1}{n} \int_{\mathrm{bd}\, K} H_{i-1}\, \mathrm{d}\mathcal{H}^{n-1} \tag{5.55}$$

for $i = 1, \dots, n$. Using (5.50), we obtain another representation, namely

$$W_i(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K s_{n-i-1} \, d\mathcal{H}^{n-1} \tag{5.56}$$

for $i = 0, \dots, n-1$. From (5.55) we see again that $nW_1(K)$ is the surface area of K, and the case $i = n-1$ of (5.56), together with $h_K(u) + h_K(-u) = w(K, u)$, shows that

$$W_{n-1}(K) = \frac{\kappa_n}{2} w(K), \tag{5.57}$$

where $w(K)$ is the mean width of K (by approximation, this extends to general convex bodies).

Using (2.62) and (2.51), we can transform formula (5.56) into

$$W_i(K) = \frac{1}{n} \int_{\mathrm{bd}\, K} q_K H_i \, d\mathcal{H}^{n-1}, \tag{5.58}$$

where $q_K := h_K \circ u_K$, thus $q_K(x) = \langle x, u_K(x) \rangle$ for $x \in \mathrm{bd}\, K$.

In differential geometry, the equalities resulting from (5.54), (5.56) and from (5.55), (5.58), respectively,

$$\int_{\mathbb{S}^{n-1}} s_j \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} h_K s_{j-1} \, d\mathcal{H}^{n-1} \tag{5.59}$$

and

$$\int_{\mathrm{bd}\, K} H_{j-1} \, d\mathcal{H}^{n-1} = \int_{\mathrm{bd}\, K} q_K H_j \, d\mathcal{H}^{n-1}, \tag{5.60}$$

for $j = 1, \dots, n-1$, are known as *Minkowskian integral formulae*.

The case $i = n-2$ of (5.56) can be given a form that is quite useful. By (2.56) we know that

$$s_1 = h_K + \frac{1}{n-1} \Delta_S h_K$$

on $\mathbb{S}^{n-1}$, where Δ_S is the spherical Laplace operator on $\mathbb{S}^{n-1}$. This yields

$$W_{n-2}(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \left(h_K + \frac{1}{n-1} \Delta_S h_K \right) d\mathcal{H}^{n-1}. \tag{5.61}$$

More generally, it follows from (5.49) and (5.51) that

$$V(K, L, B^n, \dots, B^n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \left(h_L + \frac{1}{n-1} \Delta_S h_L \right) d\mathcal{H}^{n-1}. \tag{5.62}$$

Using Green's formula on $\mathbb{S}^{n-1}$, we can rewrite (5.61) as

$$W_{n-2}(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(h_K^2 - \frac{1}{n-1} \nabla_S h_K \right) d\mathcal{H}^{n-1}, \tag{5.63}$$

where ∇_S denotes the second Beltrami operator (square of the gradient) on $\mathbb{S}^{n-1}$. Taking a local parametrization $N : M \to \mathbb{R}^n$ of the sphere $\mathbb{S}^{n-1}$ and using the notation of Section 2.5, in particular $X = x_K \circ N$ and $e_{ij} = \langle N_i, N_j \rangle$, we have, for $h = h_K \circ N$,

$$(\nabla_S h_K) \circ N = \sum_{i,j=1}^{n-1} e_{ij} h^i h^j$$

with

$$h^i = \sum_{j=1}^{n-1} e^{ij} h_j.$$

One easily finds that

$$X = hN + \sum_{j=1}^{n-1} h^j N_j$$

and thus

$$(\nabla_S h_K) \circ N = \Big| \sum h^j N_j \Big|^2 = |X - hN|^2 = |X|^2 - h^2.$$

Thus, $(\nabla_S h_K)(u)$ is equal to the squared distance between the point where the support plane $H(K, u)$ touches K and the foot of the perpendicular to $H(K, u)$ through o.

Heil [952] has pointed out that formula (5.63) holds for arbitrary convex bodies K. Observe that in $\mathbb{R}^3$ this is a formula for the surface area.

5.3.2 Formulae involving integrations

In the second subsection we now treat the relationship of mixed volumes and of mixed area measures to the integral-geometric considerations of Section 4.4. In particular, we collect some observations on translative integral geometry and on rotational mean values. The convex bodies K, K', K_i occurring in the following are arbitrary.

First we recall the principal kinematic formula (the case $j = 0$ and $\beta = \beta' = \mathbb{R}^n$ of Theorem 4.4.2), written in terms of quermassintegrals:

$$\int_{\mathrm{G}_n} \chi(K \cap gK')\, \mathrm{d}\mu(g) = \frac{1}{\kappa_n} \sum_{k=0}^{n} \binom{n}{k} W_k(K) W_{n-k}(K'). \tag{5.64}$$

If here the motion group G_n is replaced by the group of translations, which may be identified with $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} \chi(K \cap (K' + t))\, \mathrm{d}t = \mathcal{H}^n\{t \in \mathbb{R}^n : K \cap (K' + t) \neq \emptyset\} = V_n(K - K')$$

$$= \sum_{k=0}^{n} \binom{n}{k} V(K[k], -K'[n-k]). \tag{5.65}$$

Thus, mixed volumes appear inevitably in translative integral geometry. (See also Section 4.4, Note 7.)

Since the proof given in Section 4.4 for the principal kinematic formula first treated the translative case, it yields, incidentally, a new representation for the mixed volume in the special case where the arguments are two polytopes. Suppose that $P, P' \in \mathcal{P}^n$ are polytopes. Theorem 4.4.3, for $j = 0, \beta = \beta' = \mathbb{R}^n$, gives

$$\int_{\mathbb{R}^n} \chi(P \cap (P' + t))\,\mathrm{d}t = \chi(P)V_n(P') + V_n(P)\chi(P')$$
$$+ \sum_{k=1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k}(P')} \gamma(F, F', P, P')[F, F']V_k(F)V_{n-k}(F').$$

Using (5.65) for $K = P$, $K' = P'$ and then replacing P by λP in both equations and comparing the coefficients of equal powers of λ, we obtain

$$\begin{aligned}&\binom{n}{k} V(P[k], -P'[n-k])\\ &= \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k}(P')} \gamma(F, F', P, P')[F, F']V_k(F)V_{n-k}(F').\end{aligned} \tag{5.66}$$

It is not clear whether this is a special case of some formula which would hold for general convex bodies.

Another translative integral formula, but now referring to subspaces, is obtained as follows. With $k \in \{1, \dots, n-1\}$, let $E_k \subset \mathbb{R}^n$ be a k-dimensional linear subspace, and let U_k be a convex body contained in E_k. By $v^{(k)}, W_j^{(k)}$ we denote, respectively, the mixed volume and the jth quermassintegral, each with respect to a k-dimensional affine subspace. For $K \in \mathcal{K}^n$ and $\lambda \geq 0$ we have

$$\begin{aligned}&\sum_{j=0}^{n} \binom{n}{j} \lambda^j V(K[n-j], U_k[j]) = V_n(K + \lambda U_k)\\
&= \int_{E_k^\perp} V_k((K + \lambda U_k) \cap (E_k + t))\,\mathrm{d}\mathcal{H}^{n-k}(t)\\
&= \int_{E_k^\perp} V_k([K \cap (E_k + t)] + \lambda U_k)\,\mathrm{d}\mathcal{H}^{n-k}(t)\\
&= \int_{K|E_k^\perp} \sum_{j=0}^{k} \binom{k}{j} \lambda^j v^{(k)}(K \cap (E_k + t)[k-j], U_k[j])\,\mathrm{d}\mathcal{H}^{n-k}(t).\end{aligned}$$

By comparison, the following two consequences are obtained. Choosing for U_k the k-dimensional unit ball $B_k = B^n \cap E_k$, we find that

$$\int_{E_k^\perp} W_j^{(k)}(K \cap (E_k + t))\,\mathrm{d}\mathcal{H}^{n-k}(t) = \binom{n}{j}\binom{k}{j}^{-1} V(K[n-j], B_k[j]) \tag{5.67}$$

for $j = 0, \dots, k-1$. This is another instance where a formula in translative integral geometry requires mixed volumes.

The case $j = k$ yields

$$\binom{n}{k} V(K[n-k], U_k[k]) = V_k(U_k) V_{n-k}(K \,|\, E_k^\perp).$$

Here we replace K by $\sum \lambda_i K_i$ with $K_1, \dots, K_{n-k} \in \mathcal{K}^n$ and U_k by $\sum \mu_i L_i$ with $L_1, \dots, L_k \subset E_k$ and, after developing, compare the coefficients. We obtain the following reduction theorem for mixed volumes.

Theorem 5.3.1 *Let $k \in \{1, \dots, n-1\}$, let $E \subset \mathbb{R}^n$ be a k-dimensional linear subspace and let $L_1, \dots, L_k, K_1, \dots, K_{n-k} \in \mathcal{K}^n$ be convex bodies with $L_i \subset E$ for $i = 1, \dots, k$. Then*

$$\binom{n}{k} V(L_1, \dots, L_k, K_1, \dots, K_{n-k}) = v_E(L_1, \dots, L_k) v_{E^\perp}(K_1 \,|\, E^\perp, \dots, K_{n-k} \,|\, E^\perp),$$

where v_E denotes the mixed volume in E and $v_{E^\perp}$ denotes the mixed volume in $E^\perp$.

In particular, changing the notation and choosing, for $E \in G(n,k)$, a convex body $U \subset E^\perp$ with $V_{n-k}(U) = 1$, we have

$$v_E(K_1 \,|\, E, \dots, K_k \,|\, E) = \binom{n}{k} V(K_1, \dots, K_k, U[n-k]), \tag{5.68}$$

which shows that k-dimensional mixed volumes of orthogonal projections to E can be expressed by mixed volumes in $\mathbb{R}^n$.

We turn to rotational mean values. If in (5.65) we replace K' by $-\rho\lambda K'$ with $\rho \in \mathrm{SO}(n)$ and $\lambda \geq 0$, integrate over $\mathrm{SO}(n)$ with respect to the invariant measure ν, use (5.64) and finally compare the coefficients of equal powers of λ, we end up with the equality

$$\begin{aligned} &\int_{\mathrm{SO}(n)} V(K[m], \rho K'[n-m]) \, \mathrm{d}\nu(\rho) \\ &= \frac{1}{\kappa_n} V(K[m], B^n[n-m]) V(B^n[m], K'[n-m]) \end{aligned} \tag{5.69}$$

for $m = 0, \dots, n$. Here we replace K, K' by $\sum \lambda_i K_i$ and $\sum \mu_j K'_j$, respectively, expand both sides and compare the coefficients, to obtain (after changing the notation)

$$\begin{aligned} &\int_{\mathrm{SO}(n)} V(K_1, \dots, K_m, \rho K_{m+1}, \dots, \rho K_n) \, \mathrm{d}\nu(\rho) \\ &= \frac{1}{\kappa_n} V(K_1, \dots, K_m, B^n[n-m]) V(B^n[m], K_{m+1}, \dots, K_n). \end{aligned} \tag{5.70}$$

A mean value formula for projections is obtained as follows. Let E_k be a k-dimensional linear subspace of $\mathbb{R}^n$ ($k \in \{1, \dots, n-1\}$) and choose a convex body $U_{n-k} \subset E_k^\perp$ with $V_{n-k}(U_{n-k}) = 1$. Then

$$\begin{aligned}&\int_{\mathrm{SO}(n)} v^{(k)}(K_1 \,|\, \rho E_k, \dots, K_k \,|\, \rho E_k)\, \mathrm{d}\nu(\rho)\\&= \int_{\mathrm{SO}(n)} \binom{n}{k} V(K_1, \dots, K_k, \rho U_{n-k}[n-k])\, \mathrm{d}\nu(\rho)\\&= \frac{\binom{n}{k}}{\kappa_n} V(K_1, \dots, K_k, B^n[n-k]) V(B^n[k], U_{n-k}[n-k]).\end{aligned}$$

Here the formulae (5.68) and (5.70) were used. Since

$$V(B^n[k], U_{n-k}[n-k]) = W_k(U_{n-k}) = \frac{\kappa_k}{\binom{n}{k}} V_{n-k}(U_{n-k}) = \frac{\kappa_k}{\binom{n}{k}},$$

we arrive at the formula

$$\int_{G(n,k)} v^{(k)}(K_1 \,|\, E, \dots, K_k \,|\, E)\, \mathrm{d}\nu_k(E) = \frac{\kappa_k}{\kappa_n} V(K_1, \dots, K_k, B^n[n-k]). \tag{5.71}$$

As a special case, we note that

$$\int_{G(n,k)} W^{(k)}_{k-j}(K \,|\, E)\, \mathrm{d}\nu_k(E) = \frac{\kappa_k}{\kappa_n} W_{n-j}(K) \tag{5.72}$$

for $j = 0, \dots, k$, which is known as *Kubota's integral recursion*. The further specialization $k = j = n-1$ is *Cauchy's surface area formula*. It says that the surface area $S(K)$ of K is given by

$$S(K) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} V_{n-1}(K \,|\, u^\perp)\, \mathrm{d}u. \tag{5.73}$$

Similar formulae for mixed area measures are obtained if we use the equality (a special case of Theorem 4.4.6)

$$\int_{\mathrm{SO}(n)} S_{n-1}(K + \rho K', \omega \cap \rho\omega')\, \mathrm{d}\nu(\rho) = \frac{1}{\omega_n} \sum_{k=0}^{n-1} \binom{n-1}{k} S_k(K, \omega) S_{n-1-k}(K', \omega'),$$

valid for Borel sets $\omega, \omega' \in \mathcal{B}(\mathbb{S}^{n-1})$. In the usual way, by 'mixing', we deduce

$$\begin{aligned}&\int_{\mathrm{SO}(n)} S(K_1, \dots, K_m, \rho K_{m+1}, \dots, \rho K_{n-1}, \omega \cap \rho\omega')\, \mathrm{d}\nu(\rho) \qquad (5.74)\\&= \frac{1}{\omega_n} S(K_1, \dots, K_m, B^n[n-1-m], \omega) S(B^n[m], K_{m+1}, \dots, K_{n-1}, \omega')\end{aligned}$$

for $m = 0, \dots, n-1$.

We denote by $s^{(k)}, S^{(k)}_j$, respectively, the mixed area measure and the area measure of order j, each with respect to a k-dimensional affine subspace. For projections to a k-dimensional linear subspace E_k, formula (4.78) states that

$$\int_{G(n,k)} S^{(k)}_j(K \,|\, E, \omega \cap E)\, \mathrm{d}\nu_k(E) = \frac{\omega_k}{\omega_n} S_j(K, \omega) \tag{5.75}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and $j = 0, \dots, k-1$. From the case $j = k-1$ we deduce, again in the usual way, the rotational mean value formula

$$\int_{G(n,k)} s^{(k)}(K_1 \,|\, E, \dots, K_{k-1} \,|\, E, \omega \cap E) \, \mathrm{d}\nu_k(E)$$
$$= \frac{\omega_k}{\omega_n} S(K_1, \dots, K_{k-1}, B^n[n-k], \omega). \tag{5.76}$$

Of particular importance are projections onto hyperplanes. For these, we introduce some simplified notation. For $K \in \mathcal{K}^n$ and a unit vector $u \in \mathbb{S}^{n-1}$, we denote by K^u the image of K under orthogonal projection to the linear subspace orthogonal to u; thus $K^u = K \,|\, u^\perp$. For the $(n-1)$-dimensional mixed volume in a hyperplane we write v instead of $v^{(n-1)}$. If U is a segment of unit length parallel to u, then

$$v(K_1^u, \dots, K_{n-1}^u) = nV(K_1, \dots, K_{n-1}, U) \tag{5.77}$$

by (5.68). We may assume that U has centre o; then $h(U, v) = \frac{1}{2}|\langle u, v\rangle|$ and hence, by (5.19),

$$v(K_1^u, \dots, K_{n-1}^u) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, S(K_1, \dots, K_{n-1}, \mathrm{d}v). \tag{5.78}$$

By integration we immediately obtain

$$\int_{\mathbb{S}^{n-1}} v(K_1^u, \dots, K_{n-1}^u) \, \mathrm{d}u = n\kappa_{n-1} V(K_1, \dots, K_{n-1}, B^n), \tag{5.79}$$

which is a special case of (5.71) in a slightly different formulation.

Formula (5.78) shows that the function $u \mapsto v(K_1^u, \dots, K_{n-1}^u)$ is the restriction of a support function to $\mathbb{S}^{n-1}$. In particular, the function $u \mapsto V_{n-1}(K \,|\, u^\perp)$ defines a support function. The convex body with this support function is called the *projection body* of K and denoted by ΠK, thus

$$h(\Pi K, u) = V_{n-1}(K \,|\, u^\perp) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, S_{n-1}(K, \mathrm{d}v) \quad \text{for } u \in \mathbb{S}^{n-1}. \tag{5.80}$$

According to Theorem 3.5.3, the projection body ΠK is a zonoid with centre at the origin. We shall come back to projection bodies in Section 10.9.

5.3.3 Formulae for generalized zonoids

In the last part of this section we derive formulae for mixed volumes and mixed area measures of zonoids and generalized zonoids. For several applications, it is useful to have explicit expressions for these functions in terms of the generating measures. Recall that the convex body Z is a generalized zonoid with centre o if its support function has a representation

$$h(Z, u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}\rho(v) \quad \text{for } u \in \mathbb{R}^n,$$

with an even finite signed Borel measure ρ on $\mathbb{S}^{n-1}$. This even signed measure is uniquely determined by Z and is called the generating measure of Z. The body Z is a zonoid if its generating measure is a positive measure. Any translate of a body represented as above is also called a generalized zonoid.

For $x_1, \dots, x_k \in \mathbb{R}^n$ we denote by $D_k(x_1, \dots, x_k)$ the k-dimensional volume of the parallelepiped spanned by $x_1, \dots, x_k$. Denoting by $[x, y]$ the closed segment with endpoints $x, y \in \mathbb{R}^n$, we thus have

$$D_k(x_1, \dots, x_k) = V_k([o, x_1] + \cdots + [o, x_k]).$$

Recall that $v^{(k)}$ denotes the mixed volume in a k-dimensional space.

Theorem 5.3.2 *For $i = 1, \dots, j \leq n$ let $Z_i \in \mathcal{K}^n$ be a generalized zonoid with generating measure ρ_i and let $K_1, \dots, K_{n-j} \in \mathcal{K}^n$. Then*

$$\begin{aligned} &V(Z_1, \dots, Z_j, K_1, \dots, K_{n-j}) \qquad (5.81)\\ &= \frac{2^j (n-j)!}{n!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_j(u_1, \dots, u_j)\\ &\quad \times v^{(n-j)}(K_1 \,|\, \mathrm{lin}\{u_1, \dots, u_j\}^\perp, \dots, K_{n-j} \,|\, \mathrm{lin}\{u_1, \dots, u_j\}^\perp)\\ &\quad \times \mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_j(u_j). \end{aligned}$$

Proof Let $u_1, \dots, u_j \in \mathbb{S}^{n-1}$ be linearly independent, let $E := \mathrm{lin}\{u_1, \dots, u_j\}$ and $L_i := [-u_i, u_i]$ for $i = 1, \dots, j$. By Theorem 5.3.1, and because of the relation $j! v^{(j)}(L_1, \dots, L_j) = V_j(L_1 + \cdots + L_j)$, we have

$$\binom{n}{j} V(L_1, \dots, L_j, K_1, \dots, K_{n-j}) = \frac{2^j}{j!} D_j(u_1, \dots, u_j) v^{(n-j)}(K_1 \,|\, E, \dots, K_{n-j} \,|\, E).$$

This formula also holds true if $u_1, \dots, u_j$ are linearly dependent, because then both sides are zero.

We first assume that $K_1 = \cdots = K_{n-j} =: K$, $Z_1 = \cdots = Z_j =: Z$ and Z is a zonotope, say $Z = S_1 + \cdots + S_m$, where

$$S_k = \alpha_k [-v_k, v_k] \quad \text{with } v_k \in \mathbb{S}^{n-1},\ \alpha_k > 0$$

$(k = 1, \dots, m)$. Then

$$h(Z, \cdot) = \sum_{k=1}^{m} \alpha_k |\langle \cdot, v_k \rangle| = \int_{\mathbb{S}^{n-1}} |\langle \cdot, v \rangle| \, \mathrm{d}\rho(v),$$

where ρ is concentrated at $\pm v_i$ and assigns mass $\alpha_i/2$ to each of these points. Then we have

$$\begin{aligned}
&V(Z[j], K[n-j]) \\
&= V\left(\sum_{k_1} \alpha_{k_1}[-v_{k_1}, v_{k_1}], \dots, \sum_{k_j} \alpha_{k_j}[-v_{k_j}, v_{k_j}], K[n-j]\right) \\
&= \frac{2^j(n-j)!}{n!} \sum_{k_1,\dots,k_j} \alpha_{k_1} \cdots \alpha_{k_j} D_j(v_{k_1}, \dots, v_{k_j}) V_{n-j}(K \,|\, \mathrm{lin}\{v_{k_1}, \dots, v_{k_j}\}^\perp) \\
&= \frac{2^j(n-j)!}{n!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_j(u_1, \dots, u_j) V_{n-j}(K \,|\, \mathrm{lin}\{u_1, \dots, u_j\}^\perp) \\
&\quad \times \, \mathrm{d}\rho(u_1) \cdots \mathrm{d}\rho(u_j).
\end{aligned}$$

Now let Z be a zonoid. We can approximate Z by a sequence $(\bar{Z}_i)_{i\in\mathbb{N}}$ of zonotopes whose generating measures $\bar{\rho}_i$ converge weakly to the generating measure ρ of Z (compare the proof of Theorem 3.5.3). Then the product measures $\bar{\rho}_i^n$ converge weakly to ρ^n. Hence, the previous integral representation of $V(Z[j], K[n-j])$ extends to arbitrary zonoids. By 'mixing' (that is, replacing Z by a Minkowski combination of zonoids, and also replacing K by a Minkowski combination of convex bodies, expanding, and comparing coefficients), we obtain the general formula (5.81) for zonoids $Z_1, \dots, Z_j$ and convex bodies $K_1, \dots, K_{n-j}$.

Let Z_1 be a generalized zonoid, while $Z_2, \dots, Z_n$ are still zonoids. From the Hahn–Jordan decomposition of the signed measure ρ_1 we obtain zonoids Z_1^+, Z_1^- with $Z_1^+ = Z_1 + Z_1^-$. Writing

$$V(Z_1, Z_2, \dots, Z_n) = V(Z_1^+, Z_2, \dots, Z_n) - V(Z_1^-, Z_2, \dots, Z_n)$$

and applying (5.81) to the right-hand side, we obtain the representation (5.81) for $V(Z_1, \dots, Z_2)$. Using this, we can replace Z_2 by a generalized zonoid, and so on. □

Note that special cases of Theorem 5.3.2 are the formula

$$V(Z_1, \dots, Z_n) = \frac{2^n}{n!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_n(u_1, \dots, u_n) \, \mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_n(u_n) \tag{5.82}$$

for the mixed volume of generalized zonoids $Z_1, \dots, Z_n$, and the representation

$$V_j(Z) = \frac{2^j}{j!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_j(u_1, \dots, u_j) \, \mathrm{d}\rho(u_1) \cdots \mathrm{d}\rho(u_j) \tag{5.83}$$

of the jth intrinsic volume of a generalized zonoid Z.

Now we turn to mixed area measures. For a linearly independent $(n-1)$-tuple $(u_1, \dots, u_{n-1}) \in (\mathbb{S}^{n-1})^{n-1}$, the vector $T(u_1, \dots, u_{n-1}) \in \mathbb{S}^{n-1}$ is defined as the unit vector orthogonal to $u_1, \dots, u_{n-1}$ and such that the n-tuple

$$(u_1, \dots, u_{n-1}, T(u_1, \dots, u_{n-1}))$$

has the same orientation as some fixed basis of $\mathbb{R}^n$. Then T is continuous and hence can be extended (trivially) to a Borel measurable map from $(\mathbb{S}^{n-1})^{n-1}$ into $\mathbb{S}^{n-1}$. In particular, for a Borel measure φ on $(\mathbb{S}^{n-1})^{n-1}$ the image measure $T\left(\int D_{n-1}\,\mathrm{d}\varphi\right)$ of the indefinite integral $\int D_{n-1}\,\mathrm{d}\varphi$ is well defined by

$$T\left(\int D_{n-1}\,\mathrm{d}\varphi\right)(\omega) = \int_{T^{-1}(\omega)} D_{n-1}\,\mathrm{d}\varphi$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$.

Theorem 5.3.3 *For $i = 1, \dots, n-1$, let $Z_i \in \mathcal{K}^n$ be a generalized zonoid with generating measure ρ_i. Then*

$$S(Z_1, \dots, Z_{n-1}, \cdot) = \frac{2^n}{(n-1)!} T\left(\int D_{n-1}\,\mathrm{d}(\rho_1 \otimes \cdots \otimes \rho_{n-1})\right). \tag{5.84}$$

If ρ_i has the continuous density g_i (w.r.t. spherical Lebesgue measure σ), for $i = 1, \dots, n-1$, then $S(Z_1, \dots, Z_{n-1}, \cdot)$ has the density g given by

$$g(u) = \frac{2^{n-1}}{(n-1)!} \int_{\mathbb{S}^{n-1}\cap u^\perp} \cdots \int_{\mathbb{S}^{n-1}\cap u^\perp} D_{n-1}(v_1, \dots, v_{n-1})^2 g_1(v_1) \cdots g_{n-1}(v_{n-1})$$
$$\times\, \mathrm{d}\sigma_u(v_1) \cdots \mathrm{d}\sigma_u(v_{n-1})$$

for $u \in \mathbb{S}^{n-1}$, where σ_u denotes the $(n-2)$-dimensional spherical Lebesgue measure on $\mathbb{S}^{n-1} \cap u^\perp$.

Proof First we assume that Z is a zonotope and its generating measure ρ is concentrated at $\pm v_k$ and assigns mass $\alpha_k/2$ to each of these points ($k = 1, \dots, m$).

The area measure of Z is, according to (5.22), given by

$$S_{n-1}(Z, \omega) = \sum_{u\in\omega} V_{n-1}(F(Z, u))$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. The face $F(Z, u)$ is a translate of the sum of the faces of the segments $S_1, \dots, S_m$ that are orthogonal to u. Using the discrete case of (5.82) in dimension $n-1$, we obtain

$$\begin{aligned} S_{n-1}(Z, \omega) &= \sum_{u\in\omega} \sum_{\substack{1\le k_1<\cdots<k_{n-1}\le m\\ v_{k_1},\dots,v_{k_{n-1}}\perp u}} V_{n-1}(S_{k_1} + \cdots + S_{k_{n-1}}) \\ &= 2^{n-1} \sum_{u\in\omega} \sum_{\substack{1\le k_1<\cdots<k_{n-1}\le m\\ v_{k_1},\dots,v_{k_{n-1}}\perp u}} D_{n-1}(v_{k_1}, \dots, v_{k_{n-1}})\, \alpha_{k_1} \cdots \alpha_{k_{n-1}} \\ &= \frac{2^n}{(n-1)!} \sum_{\substack{k_1,\dots,k_{n-1}\\ T(v_{k_1},\dots,v_{k_{n-1}})\in\omega}}^{m} D_{n-1}(v_{k_1}, \dots, v_{k_{n-1}})\, \alpha_{k_1} \cdots \alpha_{k_{n-1}}, \end{aligned}$$

hence

$$S_{n-1}(Z, \omega) = \frac{2^n}{(n-1)!} T\left(\int D_{n-1} \mathrm{d}\rho^{n-1}\right)(\omega). \tag{5.85}$$

Let Z be a zonoid. Again, we approximate Z by a sequence $(\bar{Z}_i)_{i\in\mathbb{N}}$ of zonotopes whose generating measures $\bar{\rho}_i$ converge weakly to the generating measure ρ of Z. For any continuous real function f on $\mathbb{S}^{n-1}$ we have

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} f(x)\,\mathrm{d}T\left(\int D_{n-1}\,\mathrm{d}\bar{\rho}_i^{n-1}\right)(x)\\ &= \int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}} f(T(u_1,\dots,u_{n-1}))D_{n-1}(u_1,\dots,u_{n-1})\,\mathrm{d}\bar{\rho}_i(u_1)\cdots\,\mathrm{d}\bar{\rho}_i(u_{n-1}),\end{aligned}$$

and the function $f(T(\cdot,\dots,\cdot))D_{n-1}(\cdot,\dots,\cdot)$ is continuous on $(\mathbb{S}^{n-1})^{n-1}$, since D_{n-1} is continuous and vanishes where T is discontinuous. Hence, we have $T\left(\int D_{n-1}\,\mathrm{d}\bar{\rho}_i^{n-1}\right) \xrightarrow{w} T\left(\int D_{n-1}\,\mathrm{d}\rho^{n-1}\right)$. Since $(S_{n-1}(\bar{Z}_i,\cdot))_{i\in\mathbb{N}}$ converges weakly to $S_{n-1}(Z,\cdot)$, equality (5.85) for the zonoid Z follows.

The extension to $n-1$ generalized zonoids is achieved as in the proof of the previous theorem. This completes the proof of (5.84).

Now suppose that ρ_i has the continuous density g_i, $i=1,\dots,n-1$. For any even continuous function f on $\mathbb{S}^{n-1}$ we get from (5.84)

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} f(u)\,\mathrm{d}S(Z_1,\dots,Z_{n-1},u)\\ &= \frac{2^n}{(n-1)!}\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}} f(T(u_1,\dots,u_{n-1}))D_{n-1}(u_1,\dots,u_{n-1})\\ &\quad\times g_1(u_1)\cdots g_{n-1}(u_{n-1})\,\mathrm{d}u_1\cdots\mathrm{d}u_{n-1}.\end{aligned}$$

We put

$$F(u_1,\dots u_{n-1}) := f(T(u_1,\dots,u_{n-1}))D_{n-1}(u_1,\dots,u_{n-1})g_1(u_1)\cdots g_{n-1}(u_{n-1})$$

and use the formula

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}} F(u_1,\dots,u_{n-1})\,\mathrm{d}u_1\cdots\mathrm{d}u_{n-1}\\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}\cap u^\perp}\cdots\int_{\mathbb{S}^{n-1}\cap u^\perp} F(v_1,\dots,v_{n-1})D_{n-1}(v_1,\dots,v_{n-1})\\ &\quad\times \mathrm{d}\sigma_u(v_1)\cdots\mathrm{d}\sigma_u(v_{n-1})\,\mathrm{d}u.\end{aligned}$$

This formula follows immediately from the linear Blaschke–Petkantschin formula of integral geometry; see [1740], Theorem 7.2.1. It results that

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} f(u)\,\mathrm{d}S(Z_1,\dots,Z_{n-1},u)\\ &= \frac{2^{n-1}}{(n-1)!}\int_{\mathbb{S}^{n-1}} f(u)\int_{\mathbb{S}^{n-1}\cap u^\perp}\cdots\int_{\mathbb{S}^{n-1}\cap u^\perp} D_{n-1}(v_1,\dots,v_{n-1})^2\\ &\quad\times g_1(v_1)\cdots g_{n-1}(v_{n-1})\,\mathrm{d}\sigma_u(v_1)\cdots\mathrm{d}\sigma_u(v_{n-1})\,\mathrm{d}u.\end{aligned}$$

Since this holds for all even continuous functions f, the assertion follows. □

If some of the generalized zonoids in Theorem 5.3.3 are balls, the number of integrations can be reduced. The following theorem contains, in particular, a description of the area measures $S_j(Z, \cdot)$ of a generalized zonoid Z.

Theorem 5.3.4 *For $i = 1, \dots, j \le n-1$, let $Z_i \in \mathcal{K}^n$ be a generalized zonoid with generating measure ρ_i. Then*

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}S(Z_1, \dots, Z_j, B^n[n-1-j], \cdot)\\ &= \frac{2^j (n-1-j)!}{(n-1)!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{S}^{n-1} \cap \operatorname{lin}\{u_1,\dots,u_j\}^\perp} f \,\mathrm{d}\mathcal{H}^{n-1-j}\right]\\ &\quad \times D_j(u_1, \dots, u_j) \,\mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_j(u_j)\end{aligned}$$

for every function $f \in C(\mathbb{S}^{n-1})$.

Proof For $f = h(K, \cdot)$ with $K \in \mathcal{K}^n$, Theorem 5.3.2 gives

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}S(Z_1, \dots, Z_j, B^n[n-1-j], \cdot) = nV(Z_1, \dots, Z_j, B^n[n-1-j], K)\\ &= \frac{2^j (n-j)!}{(n-1)!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_j(u_1, \dots, u_j)\\ &\quad \times v^{(n-j)}(B^n \,|\, \operatorname{lin}\{u_1, \dots, u_j\}^\perp[n-1-j], K \,|\, \operatorname{lin}\{u_1, \dots, u_j\}^\perp)\\ &\quad \times \mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_j(u_j).\end{aligned}$$

Here,

$$\begin{aligned}&v^{(n-j)}(B^n \,|\, \operatorname{lin}\{u_1, \dots, u_j\}^\perp[n-1-j], K \,|\, \operatorname{lin}\{u_1, \dots, u_j\}^\perp)\\ &= \frac{1}{n-j} \int_{\mathbb{S}^{n-1} \cap \operatorname{lin}\{u_1,\dots,u_j\}^\perp} f \,\mathrm{d}\mathcal{H}^{n-1-j}.\end{aligned}$$

This yields the assertion for the case when f is a support function. The extension to differences of support functions and then to arbitrary continuous functions, using Lemma 1.7.8 and approximation, is routine. $\square$

The next theorem expresses the mixed volumes of projections of zonoids in terms of their generating measures. Note that, for complementary linear subspaces F, E, the bracket $[F, E]$, introduced before Theorem 4.4.3, denotes the factor by which the $(\dim F)$-dimensional volume is multiplied under orthogonal projection from F to $E^\perp$; we have $[F, E] = [E, F]$.

Theorem 5.3.5 *Let $j \in \{1, \dots, n-1\}$, let $E \in G(n, j)$ be a j-dimensional linear subspace, and denote by v_E the mixed volume in E. If $Z_i \in \mathcal{K}^n$ is a generalized zonoid with generating measure ρ_i $(i = 1, \dots, j)$, then*

$$v_E(Z_1 \,|\, E, \dots, Z_j \,|\, E) \tag{5.86}$$
$$= \frac{2^j}{j!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_j(u_1, \dots, u_j)\, [\mathrm{lin}\,\{u_1, \dots, u_j\}^\perp, E]\, \mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_j(u_j).$$

Proof Let $B_{E^\perp} = E^\perp \cap B^n$ be the unit ball in $E^\perp$. It is a zonoid; let ρ be its generating measure. By Theorems 5.3.1 and 5.3.2, we obtain

$$\kappa_{n-j} v_E(Z_1 \,|\, E, \dots, Z_j \,|\, E)$$
$$= \binom{n}{j} V(Z_1, \dots, Z_j, B_{E^\perp}, \dots, B_{E^\perp})$$
$$= \binom{n}{j} \frac{2^n}{n!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_n(u_1, \dots, u_n)\, \mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_j(u_j)\, \mathrm{d}\rho(u_{j+1}) \cdots \mathrm{d}\rho(u_n).$$

Here we insert

$$D_n(u_1, \dots, u_n) = D_j(u_1, \dots, u_j) D_{n-j}(u_{j+1}, \dots, u_n)[\mathrm{lin}\,\{u_1, \dots, u_j\}^\perp, E],$$

which follows immediately from the definitions, and use (5.81) for $n - j$ copies of the ball $B_{E^\perp}$ in $E^\perp$, to carry out the integrations with respect to ρ. This yields the result. □

One can give formula (5.86) a more concise form by using the map

$$\Lambda_j : (u_1, \dots, u_j) \mapsto \mathrm{lin}\,\{u_1, \dots, u_j\}$$

from the set of linearly independent j-tuples of vectors in $\mathbb{S}^{n-1}$ to the Grassmannian $G(n, j)$, to define a measure $\rho_1 \star \cdots \star \rho_j$ on $G(n, j)$ by

$$(\rho_1 \star \cdots \star \rho_j)(A) := \frac{2^j}{j! \kappa_j} \int_{\Lambda_j^{-1}(A)} D_j(u_1, \dots, u_j)\, \mathrm{d}(\rho_1 \otimes \cdots \otimes \rho_j)(u_1, \dots, u_j)$$

for $A \in \mathcal{B}(G(n, j))$. Then

$$v_E(Z_1 \,|\, E, \dots, Z_j \,|\, E) = \kappa_j \int_{G(n,j)} [E, L^\perp]\, d(\rho_1 \star \cdots \star \rho_j)(L) \quad \text{for } E \in G(n, j).$$

In particular, for a zonoid Z with generating measure ρ, the measure $\rho_{(j)} := \rho \star \cdots \star \rho$ (j factors) is known as the *jth projection generating measure* of Z.

Notes for Section 5.3

1. For dimensions two and three, many of the special formulae of this section can be found in Blaschke [241]; see also Bonnesen and Fenchel [284]. For the integral-geometric formulae, references are found in the notes for Section 4.4. Special cases of Theorem 5.3.2 appear in Weil [1937] (with partly different proofs), and some cases already in Blaschke [241]; see also Weil [1931], Matheron [1358], Goodey and Weil [749], Vitale [1889]. Theorem 5.3.3 also appears in Weil [1937].
2. *Differentiation of mixed area measures*. Let $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ be convex bodies of class C^2_+. If $u \in \mathbb{S}^{n-1}$ and $(\omega_i)_{i \in \mathbb{N}}$ is a sequence of Borel subsets of $\mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(\omega_i) > 0$ shrinking to u, then (5.48) shows that

$$\lim_{i \to \infty} \frac{S(K_1, \dots, K_{n-1}, \omega_i)}{\mathcal{H}^{n-1}(\omega_i)} = s(K_1, \dots, K_{n-1}, u).$$

Aleksandrov [19] extended this to general convex bodies $K_1, \dots, K_{n-1}$. Almost everywhere on $\mathbb{S}^{n-1}$ their support functions are simultaneously twice differentiable. At each point u where this happens, $s(K_1, \dots, K_{n-1}, u)$ can be defined by (2.68), and then the limit relation above is valid for suitably chosen sequences $(\omega_i)_{i\in\mathbb{N}}$.

3. *Minkowskian integral formulae*. The formulae

$$\int_{\mathbb{S}^{n-1}} s_j \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} h_K s_{j-1} \, \mathrm{d}\mathcal{H}^{n-1} \tag{5.87}$$

and

$$\int_{\mathrm{bd}\,K} H_{j-1} \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\mathrm{bd}\,K} q_K H_j \, \mathrm{d}\mathcal{H}^{n-1} \tag{5.88}$$

are the prototypes for a number of integral formulae that have found applications in global differential geometry. We give a simple example of a typical application. Suppose that K is a convex body of class C_+^2 whose kth elementary symmetric function of the principal radii of curvature is constant, say

$$s_k = 1,$$

for some $k \in \{1, \dots, n-1\}$. Newton's inequalities for elementary symmetric functions yield

$$s_{k-1}^{1/(k-1)} \geq s_k^{1/k} = 1$$

if $k \geq 2$, hence $s_{k-1} \geq s_0$, and

$$s_1 \geq s_k^{1/k} = 1 = s_k.$$

We may assume that $o \in \mathrm{int}\,K$, hence $h_K > 0$. Applying (5.59) twice, we obtain (integrating over $\mathbb{S}^{n-1}$ with respect to $\mathcal{H}^{n-1}$)

$$\int s_k = \int h_K s_{k-1} \geq \int h_K s_0 = \int s_1 \geq \int s_k,$$

hence $s_1 \equiv s_k^{1/k}$. If $k = 1$, we have

$$s_1 = s_1^2 \geq s_2$$

and

$$\int s_1 \geq \int s_2 = \int h_K s_1 = \int h_K = \int s_1,$$

hence $s_1 \equiv s_2^{1/2}$. In either case, we deduce that every boundary point of K is an umbilic, and this is known to imply that K is a ball. Similarly, (5.60) can be used to show that K (of class C_+^2) must be a ball if H_k is constant for some $k \in \{1, \dots, n-1\}$.

Apparently, this simple argument was first used (in the more general setting of relative differential geometry) by Süss [1827].

Formulae (5.59) and (5.60) express the symmetry of the mixed volume in special cases. As mentioned in Section 5.1, Note 2, this symmetry can be proved analytically. In differential geometry, the usual way to prove such integral formulae is to apply Stokes' theorem to a suitably chosen differential form. Various such choices yield useful generalizations and analogues of the Minkowskian integral formulae. A systematic exposition of the derivation and application of integral formulae in global differential geometry can be found in Chapter 3 of Huck *et al.* [1000]. To illustrate the wide applicability of generalized Minkowskian integral formulae, we mention a few papers: Chern [419] (pairs of convex hypersurfaces), Simon [1792] (a systematic study), Schneider [1657] (affine differential geometry), Katsurada [1069] (Riemannian spaces).

The extension of the Minkowskian integral formula (5.60) to convex bodies without differentiability assumptions, also in spaces of constant curvature, was studied and applied by Kohlmann [1128, 1130].

Other extensions of the Minkowskian integral formulae yield representations of the quermassintegrals of the types (5.54) and (5.56), but involving higher powers of the support function (and other expressions). General versions of such formulae were derived by Bokowski and Heil [266], who extended and unified earlier results of Simon [1792], Shahin [1772], Yano and Tani [1999], Firey [605]. The formulae are complicated, but Bokowski and Heil [266] made an elegant application to inequalities for quermassintegrals.

4. The reduction formula of Theorem 5.3.1 appears in Fedotov [560].
5. *Quermassintegrals of bodies of constant width.* The quermassintegrals $W_0, \dots, W_n$ of a convex body of constant width b satisfy $[(n+1)/2]$ independent linear relations, namely

$$2W_{n-k} = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} b^{k-i}\, W_{n-i} \quad \text{for } 0 \le k \le n \text{ with } k \text{ odd.}$$

For $n = 3$, this includes Blaschke's relation,

$$V_3 = \frac{1}{2} bS - \frac{1}{3}\pi b^3,$$

where S is the surface area. Proofs for the general case were given by Dinghas [480], Santaló [1625] and Debrunner [472].

6. *The Wills functional*. The special linear combination of quermassintegrals given by

$$W(K) = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{\kappa_i} W_i(K) = \sum_{j=0}^{n} V_j(K)$$

has some remarkable properties; for instance,

$$W(K) = \int_{\mathbb{R}^n} e^{-\pi d(K,x)^2}\, \mathrm{d}x$$

and, hence, $W(K \oplus L) = W(K)W(L)$ for an orthogonal direct sum $K \oplus L$. For these and further properties, see Hadwiger [926]. The functional W attracted considerable attention after Wills [1981] had conjectured that $G(K) \le W(K)$ for $K \in \mathcal{K}^n$, where $G(K)$ denotes the number of points with integer coordinates (with respect to some orthonormal basis of $\mathbb{R}^n$) in K. The conjecture is easily established for $n = 2$ and was proved for $n = 3$ by Overhagen [1496], but Hadwiger [927] found counterexamples for $n \ge 441$. Related results and many connections between discrete and convex geometry can be found in the survey article by Wills [1982]. Much of the work described there is concerned with relations between lattice point inequalities and quermassintegrals, both topics created by Minkowski. Wills [1983] established inequalities between the zeros of the polynomial $\sum_{j=0}^{n} V_j(K)(-x)^j$ and the successive minima of K, as defined by Minkowski in the geometry of numbers.

An application of the Wills functional to Gaussian processes appears in Vitale [1890].

7. *The Gaussian representation of the intrinsic volumes.* The following representation of the intrinsic volumes is sometimes useful. If $\Gamma_{n,k}$ is a $k \times n$ random matrix with independent $\mathrm{N}(0,1)$ Gaussian entries, then the kth intrinsic volume of $K \in \mathcal{K}^n$ is given by

$$V_k(K) = \frac{(2\pi)^{k/2}}{\kappa_k k!} \mathbb{E}\, \mathcal{H}^k(\Gamma_{n,k} K),$$

where $\mathbb{E}$ denotes mathematical expectation (see Tsirelson [1856], Vitale [1892]).

Using this, Paouris and Pivovarov [1511] showed the following. If $\phi \in \mathrm{GL}(n)$ is contracting, that is, $|\phi x| \le |x|$ for $x \in \mathbb{R}^n$, then $V_k(\phi K) \le V_k(K)$ for $K \in \mathcal{K}^n$ and $k = 1, \dots, n$.

8. *Further integral-geometric formulae*. Goodey and Weil [745] and Weil [1950] obtained some Crofton-type formulae for mixed volumes, of which we mention the following. Let $j \in \{1, \dots, n-1\}$ and $K, L \in \mathcal{K}^n$. Then
$$\int_{A(n,j)} V(K \cap E[j], L[n-j]) \, d\mu_j(E) = \frac{\kappa_j \kappa_{n-j}}{\binom{n}{j}^2 \kappa_n} V_n(K) V_{n-j}(L).$$
If K and L are centrally symmetric, then
$$\int_{A(n,n-j+1)} \int_{A(n,j+1)} V(K \cap E[j], L \cap F[n-j]) \, d\mu_{j+1}(E) \, d\mu_{n-j+1}(F)$$
$$= \frac{n(n-1)\alpha_{n0(j+1)}}{4\binom{n}{j}\kappa_{n-2}} V(\Pi K, \Pi L, B^n, \dots, B^n),$$
where ΠK denotes the projection body of K.
9. *Representations of mixed volumes of polytopes*. For polytopes $P, Q \in \mathcal{P}^n$ and for $k \in \{1, \dots, n-1\}$, the mixed volume $V(P[k], Q[n-k])$ can be represented in a way similar to (5.66), but depending on a vectorial parameter. Suppose that $x \in \mathbb{R}^n \setminus \{o\}$ is a vector in general position with respect to P and Q, which means that there do not exist faces F of P and G of Q such that $\dim F + \dim G > n$ and
$$x \in \operatorname{lin} [N(P, F) + N(Q, G)].$$
Then, as proved by Betke [216],
$$\binom{n}{k} V(P[k], Q[n-k]) = \underbrace{\sum_{F \in \mathcal{F}_k(P)} \sum_{G \in \mathcal{F}_{n-k}(Q)}}_{x \in N(P,F) - N(Q,G)} [F, G] V_k(F) V_{n-k}(G).$$
Here, as indicated, the summation extends only over those pairs F, G for which x is in the difference of the corresponding normal cones. By integration over all unit vectors x (almost all of which are in general position with respect to P and Q), formula (5.66) is obtained again.

Betke's formula for the mixed volume of two polytopes can be extended as follows.

Theorem Let $P_1, \dots, P_k \in \mathcal{P}^n$ be polytopes ($k \geq 2$). Choose vectors $x_1, \dots, x_k \in \mathbb{R}^n$, not all zero, such that
$$x_1 + \cdots + x_k = o$$
and
$$\bigcap_{i=1}^{k} [\operatorname{relint} N(P_i, F_i) - x_i] = \emptyset$$
whenever $F_i \in \mathcal{F}(P_i)$ and $\dim F_1 + \cdots + \dim F_k > n$. If $m_1, \dots, m_k \in \{0, \dots, n-1\}$ and $m_1 + \cdots + m_k = n$, then
$$\binom{n}{m_1 \dots m_k} V(P_1[m_1], \dots, P_k[m_k]) = \sum V_n(F_1 + \cdots + F_k),$$
where the summation extends over the k-tuples $(F_1, \dots, F_k)$ of faces $F_i \in \mathcal{F}_{m_i}(P_i)$, $i = 1, \dots, k$, for which $\dim (F_1 + \cdots + F_k) = n$ and
$$\bigcap_{i=1}^{k} [N(P_i, F_i) - x_i] \neq \emptyset.$$

This was proved in Schneider [1719] and appears as part of a theory of certain general mixed functionals in Schneider [1722]. The latter yields also a series of representations for the mixed functionals in (4.84) in the case of polytopes.

For the terms $F_1 + \cdots + F_k$ appearing in the sum above, the faces $F_1, \dots, F_k$ are in complementary subspaces (due to the summation rule and the choice of the vectors $x_1, \dots, x_k$). Therefore, the result can also be written in the form

$$\binom{n}{m_1 \dots m_k} V(P_1[m_1], \dots, P_k[m_k]) = \sum [F_1, \dots, F_k] V_{m_1}(F_1) \cdots V_{m_k}(F_k),$$

with the same summation rule. Here, $[F_1, \dots, F_k]$ denotes the volume of the parallelepiped that is the sum of unit cubes in the affine hulls of $F_1, \dots, F_k$. Use of these representations is made in Dyer, Gritzmann and Hufnagel [524].

10. Hug, Rataj and Weil [1013] used flag measures to obtain an integral representation of the mixed volumes of two convex bodies that are supposed to be in a certain general relative position.
11. *Volumes and quermassintegrals of projections*. Extending definition (5.80), one may define a convex body $\Pi_i K$, for $K \in \mathcal{K}^n$ and $i = 1, \dots, n-1$, by

$$h(\Pi_i K, u) = w_{n-i-1}(K^u) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \, S_i(K, \mathrm{d}v)$$

for $u \in \mathbb{S}^{n-1}$, where $w_{n-i-1}(K^u)$ denotes the $(n-i-1)$th quermassintegral of $K^u = K \,|\, H_{u,0}$. Thus, the zonoid $\Pi_i K$ comprises all the information on the $(n-i-1)$th quermassintegrals of the orthogonal projections of K onto hyperplanes. If conclusions are to be drawn from partial information about volumes or other quermassintegrals of projections, this is equivalent to obtaining information on K from partial knowledge of $\Pi_i K$. Investigations of this kind, in several different directions, are found in Betke and McMullen [218], Bourgain and Lindenstrauss [315, 316], Campi [381, 382], Chakerian [399], Firey [589, 601], Goodey [736], Goodey and Groemer [737], Groemer [797], Schneider [1662, 1684], Schneider and Weil [1737]; see also the surveys on zonoids by Schneider and Weil [1738], Goodey and Weil [749].

5.4 Moment vectors, curvature centroids, Minkowski tensors

In combining the notion of volume with the Minkowski addition of convex bodies, one is led naturally to mixed volumes and, as a special case, to quermassintegrals. In a similar way, a theory of mixed moment vectors can be developed, which by specialization leads to curvature centroids. This can be continued: from volumes and moment vectors to integrals of tensors. In this section, we give a brief introduction to these natural tensorial generalizations of the quermassintegrals. Beginning with tensors of rank two, this theory becomes distinctly different in character. For this reason, we consider moment vectors and the counterparts of higher rank separately.

5.4.1 Moment vectors and curvature centroids

We give no proofs in this subsection. They can essentially be modelled after those for mixed volumes, and can also be found in Schneider [1667].

For $K \in \mathcal{K}^n$ and for $r \in \{\dim K, \dots, n\}$ we define

$$z_{r+1}(K) := \int_K x \, \mathrm{d}\mathcal{H}^r(x)$$

and call $z_{r+1}(K)$, for $r = \dim K$, the *moment vector* of K. This vector depends on the position of K relative to the origin; in fact

$$z_{r+1}(K+t) = z_{r+1}(K) + V_r(K)t \qquad (r = \dim K)$$

for $t \in \mathbb{R}^n$. If $P \in \mathcal{P}^n_n$ is an n-dimensional polytope with facets $F_1, \dots, F_N$ and corresponding outer unit normal vectors $u_1, \dots, u_N$, then

$$z_{n+1}(P) = \frac{1}{n+1} \sum_{i=1}^{N} h(P, u_i) z_n(F_i). \tag{5.89}$$

There is a symmetric function $z : (\mathcal{K}^n)^{n+1} \to \mathbb{R}^n$ such that

$$z_{n+1}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_{n+1}}^{m} \lambda_{i_1} \cdots \lambda_{i_{n+1}} z(K_{i_1}, \dots, K_{i_{n+1}}) \tag{5.90}$$

for $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$. The map z is called the *mixed moment vector*. We have

$$z(K_1, \dots, K_{n+1}) = \frac{1}{(n+1)!} \sum_{k=1}^{n+1} (-1)^{n+k+1} \sum_{i_1 < \cdots < i_k} z_{n+1}(K_{i_1} + \cdots + K_{i_k}) \tag{5.91}$$

and, with notation similar to that in Section 5.1,

$$\begin{aligned} &z(\lambda_1 K_1 + \cdots + \lambda_m K_m[p], \overline{K}_{p+1}, \dots, \overline{K}_{n+1}) \\ &= \sum_{r_1,\dots,r_m=0}^{p} \binom{p}{r_1 \dots r_m} \lambda_1^{r_1} \cdots \lambda_m^{r_m} z\left(K_1[r_1], \dots, K_m[r_m], \overline{K}_{p+1}, \dots, \overline{K}_{n+1}\right) \end{aligned} \tag{5.92}$$

for $1 \leq p \leq n+1$.

We list some properties of the map z. For $K, K_1, \dots, K_{n+1} \in \mathcal{K}^n$ we have

$$z(K, \dots, K) = z_{n+1}(K)$$

and

$$z(\phi K_1, \dots, \phi K_{n+1}) = |\det \phi| \phi z(K_1, \dots, K_{n+1})$$

for any linear map $\phi : \mathbb{R}^n \to \mathbb{R}^n$. Under translations, z behaves according to

$$\begin{aligned} &z(K_1 + t_1, \dots, K_{n+1} + t_{n+1}) \\ &= z(K_1, \dots, K_{n+1}) + \frac{1}{n+1} \sum_{i=1}^{n+1} V(K_1, \dots, \check{K}_i, \dots, K_{n+1}) t_i \end{aligned} \tag{5.93}$$

($\check{K}_i$ means that K_i is deleted), for $t_1, \dots, t_{n+1} \in \mathbb{R}^n$. Furthermore, z is continuous, and for fixed $p \in \{1, \dots, n+1\}$ and $K_{p+1}, \dots, K_{n+1}$, the map

$$K \mapsto z(K[p], K_{p+1}, \dots, K_{n+1})$$

is additive on $\mathcal{K}^n$. If $V(K_1, \dots, \check{K}_i, \dots, K_{n+1}) = 0$ for $i = 1, \dots, n+1$, then $z(K_1, \dots, K_{n+1}) = o$.

In analogy with the quermassintegral $W_r(K) = V(K[n-r], B^n[r])$, we define the rth *quermassvector* of K by

$$q_r(K) := \frac{n+1}{n+1-r} z(K[n+1-r], B^n[r]) \tag{5.94}$$

for $r = 0, \dots, n$; in particular, $q_0(K) = z_{n+1}(K)$. From (5.92), we have the Steiner-type formula

$$z_{n+1}(K + \lambda B^n) = \sum_{r=0}^{n} \binom{n}{r} \lambda^r q_r(K), \tag{5.95}$$

since $z_{n+1}(B^n) = o$. It is consistent with the earlier definition of $z_{r+1}(K)$ to extend z_{r+1} to convex bodies K with $\dim K > r$ as follows:

$$z_{r+1}(K) := \frac{\binom{n}{r}}{\kappa_{n-r}} q_{n-r}(K) \tag{5.96}$$

for $r = 0, \dots, n$. The vector $z_{r+1}(K)$ is called the *intrinsic* $(r+1)$-*moment* of K. If $P \in \mathcal{P}^n$ is a polytope, then

$$z_{r+1}(P) = \sum_{F \in \mathcal{F}_r(P)} \gamma(F, P) z_{r+1}(F), \tag{5.97}$$

in analogy with (4.23). From (5.96) and (5.97) we obtain a representation for $q_r(P)$, which in view of (4.22) can be written in the form

$$q_r(P) = \frac{1}{n} \int_{\mathbb{R}^n} x \, C_{n-r}(P, \mathrm{d}x).$$

Using approximation and the weak continuity of the curvature measures, we deduce that

$$q_r(K) = \frac{1}{n} \int_{\mathbb{R}^n} x \, C_{n-r}(K, \mathrm{d}x) \tag{5.98}$$

for arbitrary $K \in \mathcal{K}^n$.

Let $\mathcal{K}^n_{n-r} := \{K \in \mathcal{K}^n : \dim\ K \geq n-r\}$. For $r = 0, \dots, n$ and $K \in \mathcal{K}^n_{n-r}$, we define

$$p_r(K) := \frac{q_r(K)}{W_r(K)} = \frac{(n+1) z(K[n+1-r], B^n[r])}{(n+1-r) V(K[n-r], B^n[r])}.$$

Thus, $p_0(K) = c(K)$ is the ordinary centroid (centre of gravity) of K,

$$c(K) = \frac{\int_K x \, \mathrm{d}x}{\int_K \mathrm{d}x},$$

and for $r = 1, \dots, n$,

$$p_r(K) = \frac{\int_K x \, C_{n-r}(K, \mathrm{d}x)}{\int_K C_{n-r}(K, \mathrm{d}x)} \tag{5.99}$$

is the centroid of the mass distribution that is given by the curvature measure $C_{n-r}(K, \cdot)$. The points $p_1(K), \dots, p_n(K)$ are, therefore, called the *curvature centroids* of K. In particular, $p_1(K)$ is the *area centroid* of K. The centroid of the Gaussian curvature measure $C_0(K,\cdot)$ is the Steiner point, that is,

$$p_n(K) = s(K) \tag{5.100}$$

for $K \in \mathcal{K}^n$. This can be seen as follows. First let K be of class C^2_+. We use the notation of Section 2.5. By (2.39), $\xi(x) = \nabla h_K(x)$. We integrate this over the unit ball B^n. (More precisely, we apply the divergence theorem to each of the vector fields $h_K e_i$, where $(e_1,\dots,e_n)$ is an orthonormal basis of $\mathbb{R}^n$, and to the domain $\{x \in \mathbb{R}^n : r \le |x| \le 1\}$, and then let $r > 0$ tend to 0.) The result is

$$\int_{B^n} \xi(x)\,\mathrm{d}x = \int_{\mathbb{S}^{n-1}} h_K(u)u\,\mathrm{d}u = \kappa_n s(K),$$

according to the definition (1.31). On the other hand, using $\xi(\lambda u) = \xi(u)$ for $\lambda > 0$,

$$\int_{B^n} \xi(x)\,\mathrm{d}x = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \xi(u)\,\mathrm{d}u = \frac{1}{n}\int_{\mathrm{bd}\,K} xH_{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

by (2.62). This yields

$$\kappa_n s(K) = \frac{1}{n}\int_K x\,C_0(K,\mathrm{d}x) = q_n(K),$$

which by approximation extends to arbitrary $K \in \mathcal{K}^n$ and thus proves (5.100).

We mention some properties of the curvature centroids. From (5.99) it is clear that

$$p_r(K) \in \operatorname{relint} K. \tag{5.101}$$

For each similarity $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, we have $p_r(\alpha K) = \alpha p_r(K)$. The map p_r is continuous on $\mathcal{K}^n_r$, and $W_r p_r = q_r$ is additive. For $\lambda \ge 0$,

$$p_r(K+\lambda B^n) = \frac{\sum_{i=r}^{n}\binom{n-r}{i-r}\lambda^{i-r}W_i(K)p_i(K)}{\sum_{i=r}^{n}\binom{n-r}{i-r}\lambda^{i-r}W_i(K)}, \tag{5.102}$$

so that $p_r(K+\lambda B^n)$ is a convex combination of $p_r(K),\dots,p_n(K)$.

5.4.2 Minkowski tensors

For $r \in \mathbb{N}_0$, we denote by $\mathbb{T}^r$ the vector space (with its standard topology) of symmetric tensors of rank r on $\mathbb{R}^n$. Since we use the standard scalar product of $\mathbb{R}^n$ to identify this space with its dual space, we need not distinguish between covariant and contravariant tensors, and we can view each element of $\mathbb{T}^r$ as a symmetric r-linear functional on $(\mathbb{R}^n)^r$. If a, b are symmetric tensors on $\mathbb{R}^n$ and $\odot$ denotes the symmetric tensor product, we use the abbreviations

$$a\odot b =: ab, \qquad \underbrace{a\odot\cdots\odot a}_{r} =: a^r, \qquad a^0 := 1.$$

The normalization of the symmetric tensor product is such that $a^r = a \otimes \cdots \otimes a$ (r factors). The metric tensor on $\mathbb{R}^n$ is denoted by Q, thus

$$Q(a,b) := \langle a,b\rangle \qquad \text{for } a,b \in \mathbb{R}^n.$$

For $K \in \mathcal{K}^n$ and $r \in \mathbb{N}_0$, we define

$$\Psi_r(K) := \frac{1}{r!}\int_K x^r \,\mathrm{d}x, \tag{5.103}$$

so that $\Psi_r(K)$ is a tensor of rank r; in particular, $\Psi_0(K) = V_n(K)$ and $\Psi_1(K) = z_{n+1}(K)$.

The normalizing factor $1/r!$ has the effect that the formula for the polynomial behaviour of Ψ_r under translations takes a simple form, namely

$$\Psi_r(K+t) = \sum_{j=0}^{r} \frac{1}{j!}\Psi_{r-j}(K)t^j \qquad \text{for } t \in \mathbb{R}^n. \tag{5.104}$$

We prove a representation of the tensor function Ψ_r and two relations for it.

Theorem 5.4.1 *For $K \in \mathcal{K}^n$,*

$$\Psi_r(K) = \frac{1}{r!(n+r)}\int_\Sigma x^r\langle x,u\rangle\, \Theta_{n-1}(K,\mathrm{d}(x,u)) \tag{5.105}$$

for $r \in \mathbb{N}_0$. For $r \in \mathbb{N}$,

$$t\,\Psi_{r-1}(K) = \frac{1}{r!}\int_\Sigma x^r\langle u,t\rangle\, \Theta_{n-1}(K,\mathrm{d}(x,u)) \tag{5.106}$$

for $t \in \mathbb{R}^n$ and

$$Q\Psi_{r-1}(K) = \frac{1}{r!}\int_\Sigma x^r u\, \Theta_{n-1}(K,\mathrm{d}(x,u)). \tag{5.107}$$

Proof For the proof, we assume that K is smooth; from this, the general case follows by approximation, since Ψ_r is continuous. We write $x = x_1e_1 + \cdots + x_ne_n$, where $(e_1,\dots,e_n)$ is the standard orthonormal basis of $\mathbb{R}^n$. For fixed $i_1,\dots,i_r \in \{1,\dots,n\}$, we apply the divergence theorem to the vector field $x \mapsto x_{i_1}\cdots x_{i_r}x$. Denoting by $u(K,x)$ the unique outer unit normal vector of K at $x \in \mathrm{bd}\,K$, we obtain

$$(n+r)\int_K x_{i_1}\cdots x_{i_r}\,\mathrm{d}x = \int_{\mathrm{bd}\,K} x_{i_1}\cdots x_{i_r}\langle x,u(K,x)\rangle\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

By Lemma 4.2.2 and (4.31), this can be written in the form (5.105).

The relation (5.106) is proved by induction with respect to r. For $r = 1$, it follows by an application of the divergence theorem to the vector field $x \mapsto x_it$, for each fixed $i \in \{1,\dots,n\}$. For the induction step, we use (5.104) and (5.105) to write

$$\sum_{j=0}^{r}\frac{1}{j!}\Psi_{r-j}(K)t^j = \frac{1}{r!(n+r)}\int_\Sigma (x+t)^r\langle x+t,u\rangle\,\Theta_{n-1}(K,\mathrm{d}(x,u)).$$

We expand the right-hand side and use (5.105) again, together with the induction hypothesis. After rearranging, this completes the induction.

For the proof of (5.107), we apply the divergence theorem to the vector field $x \mapsto x_{i_1} \cdots x_{i_r} e_j$. We get a relation which can be written as

$$\sum_{k=1}^{r} Q(e_{i_k}, e_j)\Psi_{r-1}(e_{i_1}, \dots, \check{e}_{i_k}, \dots, e_{i_r})$$
$$= \frac{1}{(r-1)!} \int_{\mathrm{bd}\, K} x_{i_1} \cdots x_{i_r} \langle u(K, x), e_j \rangle \, \mathrm{d}\mathcal{H}^{n-1}(x)$$

($\check{e}_{i_k}$ denotes that e_{i_k} is deleted). By the definition of the symmetric tensor product, this relation, which holds for all $i_1, \dots, i_r, j \in \{1, \dots, n\}$, is equivalent to (5.107). □

We turn to a Steiner formula for the tensor functional Ψ_r. In order to express its coefficients in a convenient way, we introduce further tensor functionals. It has a simplifying effect to work with the support measures Λ_m defined by (4.18) and to choose a particular normalization. We define

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_\Sigma x^r u^s \, \Lambda_k(K, \mathrm{d}(x, u)) \tag{5.108}$$

for $k = 1, \dots, n-1$ and $r, s \in \mathbb{N}_0$. It is further convenient to define

$$\Phi_n^{r,0}(K) := \Psi_r(K) \tag{5.109}$$

$$\Phi_k^{r,s} := 0 \quad \text{if } k \notin \{0, \dots, n\} \text{ or } r \notin \mathbb{N}_0 \text{ or } s \notin \mathbb{N}_0 \text{ or } k = n,\ s \neq 0.$$

This allows us in the following to extend summations over s formally over all nonnegative integers.

We notice that

$$\Phi_k^{0,0}(K) = V_k(K)$$

is the kth intrinsic volume of K, and

$$\Phi_k^{1,0}(K) = z_{k+1}(K)$$

is the intrinsic $(k+1)$-moment of K. Since the tensor functionals $\Phi_k^{r,s}$ extend the intrinsic volumes or Minkowski functionals, they have been called *Minkowski tensors*.

Further we remark that (5.107) can now be written in the form

$$Q\Phi_n^{r-1,0} = 2\pi\Phi_{n-1}^{r,1}. \tag{5.110}$$

Theorem 5.4.2 *For $r \in \mathbb{N}_0$, a convex body $K \in \mathcal{K}^n$, and $\rho \geq 0$, the extended Steiner formula*

$$\Psi_r(K + \rho B^n) = \sum_{k=0}^{n+r} \rho^{n+r-k} \kappa_{n+r-k} V_k^{(r)}(K) \tag{5.111}$$

holds, where the tensorial coefficients are given by

$$V_k^{(r)} = \sum_{s\in\mathbb{N}_0} \Phi_{k-r+s}^{r-s,s}. \tag{5.112}$$

They satisfy

$$QV_k^{(r-2)} = 2\pi \sum_{s\in\mathbb{N}_0} s\Phi_{k-r+s}^{r-s,s}. \tag{5.113}$$

Proof Writing $K_\rho := K + \rho B^n$, we have

$$\Psi_r(K+\rho B^n) = \Psi_r(K) + \frac{1}{r!}\int_{K_\rho\setminus K} x^r \,\mathrm{d}x.$$

Applying the local Steiner formula (4.38) coordinate-wise, we obtain

$$\begin{aligned}
\int_{K_\rho\setminus K} x^r \,\mathrm{d}x &= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\rho t^{n-j-1} \int_\Sigma (x+tu)^r \,\phi_j(K, \mathrm{d}(x,u))\,\mathrm{d}t \\
&= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\rho t^{n-j-1} \int_\Sigma \sum_{s=0}^{r} \binom{r}{s} x^{r-s}u^s t^s \,\phi_j(K, \mathrm{d}(x,u))\,\mathrm{d}t \\
&= \sum_{j=0}^{n-1}\sum_{s=0}^{r} \omega_{n-j}\binom{r}{s}\frac{\rho^{n-j+s}}{n-j+s}\int_\Sigma x^{r-s}u^s\,\phi_j(K,\mathrm{d}(x,u)).
\end{aligned}$$

Introducing the index $k = j + r - s$ and observing (5.108), we obtain (5.111).

To prove (5.113), we write (5.107), for a strictly convex body K, in the form

$$Q\Psi_{r-1}(K) = \frac{1}{r!}\int_{\mathbb{S}^{n-1}} x(K,u)^r u\, S_{n-1}(K,\mathrm{d}u).$$

Together with (5.111) this gives, for $\rho \geq 0$,

$$\begin{aligned}
&\sum_{k=0}^{n+r-1} \rho^{n+r-1-k}\kappa_{n+r-1-k} QV_k^{(r-1)}(K) = Q\Psi_{r-1}(K+\rho B^n) \\
&= \frac{1}{r!}\int_{\mathbb{S}^{n-1}} x(K+\rho B^n, u)^r u\, S_{n-1}(K+\rho B^n, \mathrm{d}u) \\
&= \frac{1}{r!}\sum_{i=0}^{n-1}\binom{n-1}{i}\int_{\mathbb{S}^{n-1}}\sum_{j=0}^{r}\binom{r}{j} x(K,u)^j u^{r-j+1}\, S_i(K,\mathrm{d}u)\rho^{r-j+n-1-i} \\
&= \frac{1}{r!}\sum_{k=0}^{n+r-1}\rho^{n+r-1-k}\sum_{s=1}^{r+1}\binom{r}{s-1}\binom{n-1}{k-r-1+s}\int_\Sigma x^{r+1-s}u^s\,\Theta_{k-r-1+s}(K,\mathrm{d}(x,u)).
\end{aligned}$$

Comparison gives

$$\begin{aligned}
&\kappa_{n+r-1-k} QV_k^{(r-1)}(K) \\
&= \frac{1}{r!}\sum_{s=1}^{r+1}\binom{r}{s-1}\binom{n-1}{k-r-1+s}\int_\Sigma x^{r+1-s}u^s\,\Theta_{k-r-1+s}(K,\mathrm{d}(x,u)).
\end{aligned}$$

Using the identity $2\pi\kappa_m = \omega_{m+2}$, we deduce the relation (5.113). For general convex bodies, it is obtained by approximation. □

Remark 5.4.3 Defining a $\mathbb{T}^r$-valued Borel measure $S^r_{n-1}(K, \cdot)$ on $\mathbb{S}^{n-1}$ by

$$S^r_{n-1}(K, \omega) := \frac{1}{r!} \int_{\mathbb{R}^n \times \omega} x^r \,\Theta_{n-1}(K, \mathrm{d}(x, u)) \qquad \text{for } \omega \in \mathbb{S}^{n-1},$$

we can write (5.105) in the form

$$\Psi_r(K) = \frac{1}{n + r} \int_{\mathbb{S}^{n-1}} h(K, u)\, S^r_{n-1}(K, \mathrm{d}u),$$

which extends (5.3).

From (5.112) and (5.113) we immediately get the following.

Corollary 5.4.4 (McMullen) *For $r \in \mathbb{N}$ with $r \geq 2$ and $k \in \{0, \dots, n + r - 2\}$,*

$$Q \sum_{s \in \mathbb{N}_0} \Phi^{r-s,s-2}_{k-r+s} = 2\pi \sum_{s \in \mathbb{N}_0} s\Phi^{r-s,s}_{k-r+s}. \tag{5.114}$$

Relations (5.114) were proved by McMullen [1393], in a different way (first for polytopes). For $r = 1$, relation (5.114) also holds, but only expresses the already known fact that

$$\int_{\mathbb{S}^{n-1}} u\, S_j(K, \mathrm{d}u) = o$$

for $j = 0, \dots, n - 1$.

Remark 5.4.5 If in the McMullen relations (5.114) we choose $k = n + r - 2$ (and afterwards replace r by $r + 1$), we obtain

$$Q\Phi^{r-1,0}_n = 2\pi\Phi^{r,1}_{n-1},$$

that is, (5.110) again. As the above proof shows, this single relation generates the full series of relations (5.114), by applying it to a parallel body, expanding, and comparing coefficients.

We shall return to Minkowski tensors and the McMullen relations (5.114) in Note 7 of Section 6.4.

Notes for Section 5.4

1. *Steiner point and curvature centroids*. The point later called the Steiner point was first introduced, as a curvature centroid, by Steiner [1817], for planar convex bodies that are either a polygon or sufficiently smooth. The characterization given by equation (A.7) in the Appendix appears, for $n = 2$, in Kubota [1150] and implicitly in Meissner [1400], pp. 3, 11. Relation (A.7) is essentially the same as (1.31). Equality (5.100), that is, the equivalence of definition (5.99) for $r = n$ with (1.31), was proved for some special cases by Bose and Roy [308], Gericke [696], Shephard [1779]. Bose and Roy [308] and Roy [1599] have similar integral representations for the other curvature centroids in $\mathbb{R}^3$. An extension of the Steiner point to not necessarily convex hypersurfaces was treated by Flanders [615]. Some related integral identities are derived in Hadwiger and Meier [930].

2. *Extremal properties of the Steiner point*. Originally, Steiner was led to the Steiner point by considering certain extremal problems. Other proofs of the extremal properties established by him as well as further such properties are found in Ferrers [574], Meissner [1400], Nakajima [1466], Hayashi [943], Su [1824, 1825], Arnold [74].
3. *Curvature centroids of parallel bodies*. Equation (5.102) appears in special cases (mostly for $n = 2$) in Kubota [1150], Blaschke [250, 252], Bose and Roy [306, 307, 308]. Related observations on the relative positions of the various curvature centroids of a convex body and its parallel bodies were made by Duporcq [521, 522], Kubota [1150], Nikliborc [1473], Blaschke [250], Bose and Roy [306, 307, 308].
4. *Curvature centroids of bodies of constant width*. In analogy with Note 5 of Section 5.3, the curvature centroids $p_0, \dots, p_n$ of a convex body of constant width b satisfy

$$2W_{n-k}p_{n-k} - \sum_{i=0}^{k-1}(-1)^i\binom{k}{i}b^{k-i}W_{n-i}p_{n-i} = o$$

for $1 \leq k \leq n$, k odd. For $n = 2$, this states that the Steiner point and perimeter centroid coincide; this was noted by Meissner [1400] and proved in a different way by Bose and Roy [306]. For $n = 3$, one finds that the Steiner point coincides with the mean curvature centroid (see Bose and Roy [308]) and that centroid, surface-area centroid and Steiner point are collinear. A proof of the general relations appears in Schneider [1667].
5. *Integral-geometric formulae for quermassvectors*. The special case $\beta' = \mathbb{R}^n$ of the integral-geometric formula (4.51) states that

$$\int_{G_n} \Phi_j(K \cap gK', \beta)\, d\mu(g) = \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K')\Phi_k(K, \beta)$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$. As functions of β, both sides of this equality define finite measures on $\mathcal{B}(\mathbb{R}^n)$. Integrating the identity function on $\mathbb{R}^n$ with respect to these measures and observing (4.19) and (5.98), one obtains integral-geometric formulae for the quermassvectors, which are analogous to (4.52). In a similar way, Theorem 4.4.5 and formula (4.76) (with $\beta' = \mathbb{R}^n$) lead to formulae for quermassvectors or, equivalently, for curvature centroids. In a different way, such formulae were derived by Hadwiger and Schneider [932] and by Schneider [1668]; special cases for low dimensions appeared earlier in Blaschke [252] and Müller [1455].
6. *Position of the curvature centroids*. Let $K \in \mathcal{K}^n_n$ be a convex body with centroid o. Then a result going back to Minkowski [1435] (cf. Bonnesen and Fenchel [284], p. 53) states that

$$\frac{1}{n+1}w(K,u) \leq h(K,u) \leq \frac{n}{n+1}w(K,u)$$

for $u \in \mathbb{S}^{n-1}$. For a convex body with centroid c this implies, in particular, that

$$B\left(c, \frac{1}{n+1}\Delta(K)\right) \subset K \subset B\left(c, \frac{n}{n+1}D(K)\right).$$

Similar estimates for the position of the perimeter centroid in $\mathbb{R}^2$ and the area centroid in $\mathbb{R}^3$ were obtained by v. Sz.-Nagy [1836].

Rather precise information on the position of the curvature centroids is available for convex bodies with curvature restrictions. Let $K \in \mathcal{K}^n$ be of class C^2_+ and assume that there are constants R_1, R_2 such that $0 < R_1 \leq \rho \leq R_2$ whenever ρ is a radius of curvature of bd K. Then

$$B(c, R_1) \subset K \subset B(c, R_2)$$

whenever c is one of the curvature centroids $p_1(K), \dots, p_n(K)$. This was proved by Schneider [1708], generalizing a result of Goodman [759] for the perimeter centroid in the plane.

7. *Centroids in stereology*. Possible stereological applications of curvature centroids in dimensions two and three were discussed by Davy [469, 470, 471].
8. The Steiner formula (5.111) for tensor valuations was proved in Schneider [1723], in a more complicated way. The present proof is taken from lecture notes of E. B. V. Jensen.
9. *Integral geometry of tensor valuations.* Integral-geometric formulae of Crofton and kinematic type for tensor valuations of rank at least two were first investigated by Müller [1455]. Following a suggestion by Blaschke, he considered Minkowski tensors of rank two in the plane. He obtained most of the valid kinematic formulae in this case, but not all. Higher dimensions and higher ranks were considered by Schneider [1723], which was continued by Schneider and Schuster [1735]. The complete set of Crofton and kinematic formulae for Minkowski tensors of all ranks and in all dimensions was established by Hug, Schneider and Schuster [1021]. The results are rather complicated and don't seem to exhibit a perspicuous structure.

   The Minkowski tensor $\Phi_{n-1}^{0,2}$ was used in Schneider and Schuster [1736] to associate with a stationary random process of convex particles in $\mathbb{R}^n$ a tensorial density, as a means of describing and measuring the anisotropy and mean orientation of the particles. A Crofton formula was used to obtain a stereological formula allowing an estimation of this tensorial density from measurements in planar sections.

   In local stereology, integral-geometric formulae for rotations about a fixed point are of importance. The rotational integral geometry of modified Minkowski tensors was investigated by Auneau-Cognacq, Ziegel and Jensen [101].
10. For polytopes, McMullen [1393] has given a representation for the tensor functionals $\Phi_k^{r,s}$ which can be considered as a generalization of the formula (4.23) for the intrinsic volumes of polytopes. Let $P \in \mathcal{P}^n$. For $k \in \{0, \dots, n-1\}$ and a face $F \in \mathcal{F}_k$, define

$$\Upsilon_r(F) := \frac{1}{r!} \int_F x^r \, d\mathcal{H}^k(x)$$

and

$$\Theta_s(P, F) := \frac{1}{s!} \int_{N(P,F)} x^s e^{-\pi|x|^2} \, d\mathcal{H}^{n-k}(x).$$

Then McMullen showed that

$$\Phi_k^{r,s}(P) = \sum_{F \in \mathcal{F}^k(P)} \Upsilon_r(F) \Theta_s(P, F).$$

11. *Mixed moment tensors.* Starting from the moment tensor Ψ_r instead of the volume functional, a theory of mixed moment tensors and of tensorial mixed area measures can be developed, parallel to the approach in Section 5.1. This has been carried out in the introductory part of Schuster [1752]. As a result, for each $r \in \mathbb{N}_0$ there exists a unique symmetric mapping $\Psi^r : (\mathcal{K}^n)^{n+r} \to \mathbb{T}^r$, the *rth mixed moment tensor*, such that, for $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$,

$$\Psi_r(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_1,\dots,i_{n+r}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n+r}} \Psi^r(K_{i_1}, \dots, K_{i_{n+r}}). \tag{5.115}$$

Further, there exists a unique symmetric mapping S^r from $(\mathcal{K}^n)^{n+r-1}$ to the space of r-tensor valued Borel measures on $\mathcal{B}(S^{n-1})$ such that, for $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$,

$$S_{n-1}^r(\lambda_1 K_1 + \dots + \lambda_m K_m, \cdot) = \sum_{i_1,\dots,i_{n+r-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n+r-1}} S^r(K_{i_1}, \dots, K_{i_{n+r-1}}, \cdot) \tag{5.116}$$

(compare Remark 5.4.3 for the measure S^r_{n-1}). Moreover,

$$\Psi^r(K_1,\dots,K_{n+r}) = \frac{1}{n+r}\int_{\mathbb{S}^{n-1}} h(K_1,u)\,S^r(K_2,\dots,K_{n+r},\mathrm{d}u).$$

The coefficients of the Steiner formula (5.111) can now be written as

$$V_k^{(r)}(K) = \frac{1}{\kappa_{n+r-k}}\binom{n+r}{k}\Psi^r(K[k],B^n[n+r-k]).$$

A different approach to mixed moment tensors was followed by Przesławski [1551], who obtained symmetric representations in terms of the support functions (compare Note 4 of Section 5.1).

5.5 Mixed discriminants

The *mixed discriminant* in dimension n (already mentioned in Section 2.5) is the symmetric function D of n real symmetric $n\times n$ matrices $A_r = (a^r_{ij})^n_{i,j=1}$, $r = 1,\dots,n$, which is defined by

$$D(A_1,\dots,A_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\begin{vmatrix} a_{11}^{\sigma(1)} & \cdots & a_{1n}^{\sigma(n)} \\ \vdots & & \vdots \\ a_{n1}^{\sigma(1)} & \cdots & a_{nn}^{\sigma(n)}\end{vmatrix}, \tag{5.117}$$

where S_n is the group of all permutations of the set $\{1,\dots,n\}$. For $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ we have

$$\det(\lambda_1A_1+\cdots+\lambda_mA_m) = \sum_{i_1,\dots,i_n=1}^{m}\lambda_{i_1}\cdots\lambda_{i_n}D(A_{i_1},\dots,A_{i_n}), \tag{5.118}$$

which also defines the symmetric function D uniquely, namely,

$$D(A_1,\dots,A_n) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda_1\cdots\partial\lambda_n}\det(\lambda_1A_1+\cdots+\lambda_nA_n).$$

If q is a quadratic form on $\mathbb{R}^n$, it can be written as $q(x) = \sum_{i,j=1}^n a_{ij}\xi_i\xi_j$ for $x = \xi_1e_1+\cdots+\xi_ne_n$ (with respect to some orthonormal basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$), with a symmetric real matrix $A = (a_{ij})^n_{i,j=1}$. The determinant of this matrix does not depend on the choice of the basis and is known as the discriminant of q. This explains the terminology of 'mixed discriminant'. If $q_1,\dots,q_n$ are quadratic forms on $\mathbb{R}^n$ with corresponding symmetric matrices $A_1,\dots,A_n$, then $D(A_1,\dots,A_n)$ is also called the mixed discriminant of $q_1,\dots,q_n$.

It is for several reasons that mixed discriminants have their place in the Brunn–Minkowski theory of convex bodies. First, there is a strong analogy between mixed volumes and mixed discriminants, under formal aspects as well as regarding similar sets of inequalities satisfied by both functions. Second, in establishing these inequalities for mixed volumes, Aleksandrov in his second proof made essential use of mixed discriminants. And, third, in various special investigations in the Brunn–Minkowski theory, mixed discriminants have been used.

We shall derive Aleksandrov's inequalities for mixed discriminants from more general inequalities for hyperbolic polynomials.

Let p be a homogeneous polynomial of degree $k \geq 1$ on $\mathbb{R}^n$. For given $e \in \mathbb{R}^n$, the polynomial p is called *hyperbolic in direction e* if $p(e) > 0$ and for each $x \in \mathbb{R}^n$ the univariate polynomial

$$t \mapsto p(x+te), \qquad t \in \mathbb{R}, \tag{5.119}$$

has only real roots. Together with p, we consider the derivative polynomials, which for $m \in \{1, \dots, k\}$ and $e_1, \dots, e_m \in \mathbb{R}^n$ are defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} p(x+te_1)\Big|_{t=0} =: p^{(1)}_{e_1}(x)$$

and inductively by

$$\frac{\mathrm{d}}{\mathrm{d}t} p^{(m-1)}_{e_1,\dots,e_{m-1}}(x+te_m)\Big|_{t=0} =: p^{(m)}_{e_1,\dots,e_m}(x)$$

for $m \geq 2$.

The following theorem is due to Gårding [670]. Our proof partly follows Hörmander [988] (pp. 64–65), but mostly follows Renegar [1571].

Theorem 5.5.1 (and Definition) *Let p be a homogeneous polynomial of degree $k \geq 1$ on $\mathbb{R}^n$ which is hyperbolic in some direction $e \in \mathbb{R}^n$, and define*

$$\mathcal{H}(p,e) := \{x \in \mathbb{R}^n : p(x+te) = 0 \ \Rightarrow \ t < 0\}.$$

Then the following hold.

(a) *$\mathcal{H}(p,e)$ is the connected component of the set $\{x \in \mathbb{R}^n : p(x) \neq 0\}$ that contains e.*

(b) *The polynomial p is hyperbolic in any direction $b \in \mathcal{H}(p,e)$, and $\mathcal{H}(p,b) = \mathcal{H}(p,e) =: \mathcal{H}(p)$. This is an open convex cone, called the* hyperbolicity cone *of p.*

(c) *For $m \in \{1, \dots, k-1\}$ and $e_1, \dots, e_m \in \mathcal{H}(p)$, the derivative polynomial $p^{(m)}_{e_1,\dots,e_m}$ is hyperbolic in every direction $b \in \mathcal{H}(p)$.*

Proof (a) First we note that $e \in \mathcal{H}(P,e)$, since $p(e+te) = (1+t)^k p(e)$. Then we use that the largest zero of (5.119) is a continuous function of x. From this it follows that $\mathcal{H}(p,e)$ is open. It also follows for $x \in \operatorname{cl} \mathcal{H}(p,e)$ that $p(x+te) \neq 0$ for $t > 0$. Thus, $\mathcal{H}(p,e)$ is open and closed in $\{x \in \mathbb{R}^n : p(x) \neq 0\}$. Let $x \in \mathcal{H}(p,e)$. Then $x + te \in \mathcal{H}(p,e)$ for $t \geq 0$, hence $p(\lambda x + (1-\lambda)e + te) = \lambda^k p(x + ((1-\lambda+t)/\lambda)e) \neq 0$ for $0 < \lambda \leq 1$ and $t \geq 0$, thus $\lambda x + (1-\lambda)e \in \mathcal{H}(p,e)$. This shows that $\mathcal{H}(p,e)$ is starshaped with respect to e. In particular, $\mathcal{H}(p,e)$ is connected and hence coincides with the connected component in $\{x \in \mathbb{R}^n : p(x) \neq 0\}$ containing e.

(b) Let $b \in \mathcal{H}(p,e)$. Then $p(b) > 0$, as follows from (a) and $p(e) > 0$. Let $y \in \mathbb{R}^n$. For the proof that p is hyperbolic in direction b, we have to show that $r \mapsto p(y+rb)$ has only real roots.

Let i denote the imaginary unit. We assert that for all real $\alpha > 0$ and $s \geq 0$, all roots of

$$r \mapsto p(rb + sy + \alpha \mathrm{i} e) \tag{5.120}$$

have negative imaginary parts. For $s = 0$ this is true, since $b \in \mathcal{H}(p, e)$ and p is homogeneous. Suppose that, for some $s > 0$, one of the roots of (5.120) has a nonnegative imaginary part. Since the roots of (5.120) depend continuously on s, there exists a value of s yielding a real root; that is, there exist $s' > 0$ and $r' \in \mathbb{R}$ with $p(r'b + s'y + \alpha \mathrm{i} e) = 0$. This means that $\alpha \mathrm{i}$ is a root of $t \mapsto p(z + te)$, where $z := r'b + s'y$, a contradiction, since p is hyperbolic in direction e. This proves that indeed all roots of (5.120) have negative imaginary parts. This holds, in particular, for $s = 1$ and all $\alpha > 0$. Letting α tend to zero, we conclude by continuity that all roots of $r \mapsto p(y + rb)$ have nonpositive imaginary parts. Since these roots occur in conjugate pairs, all of them are real. This proves that p is hyperbolic in direction b.

It follows from (a) that $\mathcal{H}(p, b) = \mathcal{H}(p, e)$ and hence this set can be written as $\mathcal{H}(p)$. It is an open cone, and as shown in the proof of (a), it is starshaped with respect to each of its elements and hence convex.

(c) Let $e \in \mathcal{H}(p)$ and $x \in \mathbb{R}^n$. Let $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_k(x)$ be the roots of $t \mapsto p(x + te)$. From Rolle's theorem it follows that all roots of $t \mapsto p_e^{(1)}(x + te)$ are real and, if they are denoted by $\lambda'(x) \leq \cdots \leq \lambda'_{k-1}(x)$, that

$$\lambda_1(x) \leq \lambda'_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_{k-1}(x) \leq \lambda'_{k-1}(x) \leq \lambda_k(x) \tag{5.121}$$

and

$$\{\lambda_j(x) = \lambda'_j(x) \;\text{ or }\; \lambda'_j(x) = \lambda_{j+1}(x)\} \Leftrightarrow \lambda_j(x) = \lambda'_j(x) = \lambda_{j+1}(x). \tag{5.122}$$

Thus, $p_e^{(1)}$ is hyperbolic in direction e. If $x \in \mathcal{H}(p)$, then $\lambda_k(x) < 0$ and therefore $\lambda'_{k-1}(x) < 0$, hence $x \in \mathcal{H}(p_e^{(1)})$, thus $\mathcal{H}(p) \subset \mathcal{H}(p_e^{(1)})$, and $p_e^{(1)}$ is hyperbolic in each direction $b \in \mathcal{H}(p)$. The full assertion (c) now follows by induction. □

Under the assumptions of the previous theorem, we denote by $\operatorname{mult}(x, e)$ the multiplicity of 0 as a root of $t \mapsto p(x + te)$, where $x \in \mathbb{R}^n$. For $x \in \operatorname{bd} \mathcal{H}(p)$ (implying $p(x) = 0$) we have

$$\operatorname{mult}(x, e) = 1 \Leftrightarrow \nabla p(x) \neq o. \tag{5.123}$$

Indeed, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} p(x + te)\Big|_{t=0} = \langle \nabla p(x), e \rangle.$$

Hence, $\nabla p(x) = o$ implies $\operatorname{mult}(x, e) > 1$. Conversely, if $\nabla p(x) \neq o$, then $\{y \in \mathbb{R}^n : \langle \nabla p(x), y \rangle = 0\}$ is a tangent plane of the cone $\operatorname{cl} \mathcal{H}(p)$ and hence a support plane, and thus does not contain the interior point e. This yields $\operatorname{mult}(x, e) = 1$.

Lemma 5.5.2 (and Definition) *Let p, e be as above and $x \in \mathbb{R}^n$.*

(a) *The multiplicity* $\mathrm{mult}\,(x,e)$ *does not depend on* $e \in \mathcal{H}(p)$. *It is denoted by* $\mathrm{mult}\,(x)$.

For $m = 1, \dots, k$, *let*

$$\partial^m \mathcal{H}(p) := \{x \in \mathrm{bd}\,\mathcal{H}(p) : \mathrm{mult}\,(x) = m\}.$$

(b) *For* $m \geq 2$,

$$\partial^m \mathcal{H}(p_e^{(1)}) = \partial^{m+1}\mathcal{H}(p). \tag{5.124}$$

For $k \geq 2$ *and* $e_1, \dots, e_{k-1} \in \mathcal{H}(p)$,

$$\partial^1 \mathcal{H}(p^{(k-1)}_{e_1,\dots,e_{k-1}}) = \partial^2 \mathcal{H}(p^{(k-2)}_{e_1,\dots,e_{k-2}}) = \cdots = \partial^k \mathcal{H}(p). \tag{5.125}$$

(c) *The set* $\partial^k \mathcal{H}(p)$ *is the lineality space of the cone* $\mathrm{cl}\,\mathcal{H}(p)$.

Proof (a) Let $e_1, e_2 \in \mathcal{H}(p)$, and for given x let m_i be the multiplicity of 0 as a root of $t \mapsto p(x + te_i)$ $(i = 1, 2)$. We assume $m_1 \leq m_2$, without loss of generality, and use induction over m_1. If $m_1 = 0$, then $p(x) \neq 0$ and hence $m_2 = 0$. Let $m_1 \geq 1$. Then 0 is a simple root of $t \mapsto p^{(m_1-1)}_{e_1,\dots,e_1}(x + te_1)$. Since $e_1, e_2 \in \mathcal{H}(p) \subset \mathcal{H}(p^{(m_1-1)}_{e_1,\dots,e_1})$, it follows from (5.123) (applied to $p^{(m_1-1)}_{e_1,\dots,e_1}$) that 0 is a simple root of $t \mapsto p^{(m_1-1)}_{e_1,\dots,e_1}(x + te_2)$. Now

$$(p^{(m_1-1)}_{e_1,\dots,e_1})^{(1)}_{e_2} = (p^{(1)}_{e_2})^{(m_1-1)}_{e_1,\dots,e_1},$$

hence $(p^{(1)}_{e_2})^{(m_1-1)}_{e_1,\dots,e_1}(x) \neq 0$. This means that for $t \mapsto p^{(1)}_{e_2}(x + te_1)$, the number 0 is a root of multiplicity $\leq m_1 - 1$. Since $e_1, e_2 \in \mathcal{H}(p) \subset \mathcal{H}(p^{(1)}_{e_2})$, the inductive hypothesis (applied to $p^{(1)}_{e_2}$) yields that 0 is a root of multiplicity $\leq m_1 - 1$ for $t \mapsto p^{(1)}_{e_2}(x + te_2)$. We conclude that $m_2 \leq m_1$ and thus $m_1 = m_2$. This completes the induction.

(b) Relation (5.124) follows immediately from (5.121) and (5.122), and (5.125) is a consequence of this and Theorem 5.5.1(c).

(c) Let $x \in \partial^k \mathcal{H}(p)$. Then for $e \in \mathcal{H}(p)$ we have $p(x + te) = ct^k$ with $c \neq 0$. By homogeneity, for $\alpha \in \mathbb{R}$ this gives $p(\alpha x + e + te) = c(1 + t)^k$ and therefore $\alpha x + e \in \mathcal{H}(p)$. This shows that x is contained in the lineality space of $\mathrm{cl}\,\mathcal{H}(p)$.

Conversely, let x be in this lineality space. For all $\alpha \in \mathbb{R}$ we then have $\alpha x + e \in \mathcal{H}(p)$, hence $p(\alpha x + e) > 0$. Since the polynomial $s \mapsto p(sx + e)$ has only real roots, it must be constant, thus $p(sx + e) = p(e)$ for all $s \in \mathbb{R}$. By homogeneity, $p(x + te) = t^k p(e)$, hence $x \in \partial^k \mathcal{H}(p)$. □

We recall the notion of polarization. Let p be a real homogeneous polynomial of degree k on $\mathbb{R}^n$. Given a basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$, it can be written as

$$p(x) = \sum_{j_1,\dots,j_k=1}^{n} a_{j_1\dots j_k} \xi_{j_1} \cdots \xi_{j_k}, \qquad x = \xi_1 e_1 + \cdots + \xi_n e_n, \tag{5.126}$$

with symmetric real coefficients $a_{j_1\dots j_k}$. For $x_i = \xi_1^{(i)} e_1 + \cdots + \xi_n^{(i)} e_n$, $i = 1, \dots, k$, the *polarization* $\tilde{p}$ of p is defined by

$$\tilde{p}(x_1, \dots, x_k) := \sum_{j_1,\dots,j_k=1}^{n} a_{j_1\dots j_k} \xi^{(1)}_{j_1} \cdots \xi^{(k)}_{j_k}. \tag{5.127}$$

Then

$$\tilde{p} \text{ is symmetric, } k\text{-linear, } \tilde{p}(x, \dots, x) = p(x). \tag{5.128}$$

From (5.126) and (5.127) we obtain

$$p(\lambda_1 x_1 + \cdots + \lambda_m x_m) = \sum_{i_1,\dots,i_k=1}^{m} \lambda_{i_1} \cdots \lambda_{i_k} \tilde{p}(x_{i_1}, \dots, x_{i_k})$$

for $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. The coefficient of $\lambda_1 \cdots \lambda_k$ is $k!\tilde{p}(x_1, \dots, x_k)$, hence $\tilde{p}$ is uniquely determined by the properties (5.128) and does not depend on the choice of the basis.

Now we formulate the fundamental inequalities for hyperbolic polynomials.

Theorem 5.5.3 *Let p be a hyperbolic homogeneous polynomial of degree $k \geq 1$ on $\mathbb{R}^n$, and let $\tilde{p}$ be its polarization. Let $y, x_2, \dots, x_k \in \mathcal{H}(p)$, $x \in \operatorname{cl}\mathcal{H}(p)$, and $z \in \mathbb{R}^n$. Then*

$$\tilde{p}(x, x_2, \dots, x_k) \geq 0. \tag{5.129}$$

Equality holds if and only if x is in the lineality space of $\operatorname{cl}\mathcal{H}(p)$.

Further,

$$\tilde{p}(y, z, x_3, \dots, x_k)^2 \geq \tilde{p}(y, y, x_3, \dots, x_k)\tilde{p}(z, z, x_3, \dots, x_k). \tag{5.130}$$

Equality holds if and only if $z + \alpha y$, for some $\alpha \in \mathbb{R}$, is in the lineality space of $\operatorname{cl}\mathcal{H}(p)$.

Proof For $m \in \{1, \dots, k\}$ and $e_1, \dots, e_{k-1} \in \mathcal{H}(p)$, one easily proves by induction that

$$p^{(m)}_{e_1,\dots,e_m}(x) = k(k-1)\cdots(k-m+1)\tilde{p}(e_1, \dots, e_m, \underbrace{x, \dots, x}_{k-m}).$$

Hence, by Theorem 5.5.1, the linear function $x \mapsto \tilde{p}(x, x_2, \dots, x_k)$ is hyperbolic in any direction from $\mathcal{H}(p)$ and hence positive on $\mathcal{H}(p)$. This yields (5.129). Equality holds if and only if $p^{(k-1)}_{x_2,\dots,x_k}(x) = 0$, which is equivalent to $x \in \partial^1\mathcal{H}(p^{(k-1)}_{x_2,\dots,x_k})$. By Lemma 5.5.2, this is equivalent to x being an element of the lineality space of $\operatorname{cl}\mathcal{H}(p)$.

Further, Theorem 5.5.1 yields that the polynomial $x \mapsto \tilde{p}(x, x, x_3, \dots, x_k)$ is hyperbolic in direction y. Thus, the polynomial

$$\begin{aligned} t \mapsto& \tilde{p}(z + ty, z + ty, x_3, \dots, x_k) \\ &= \tilde{p}(z, z, x_3, \dots, x_k) + 2t\tilde{p}(y, z, x_3, \dots, x_k) + t^2\tilde{p}(y, y, x_3, \dots, x_k) \end{aligned}$$

has only real roots and hence has a nonnegative discriminant. This is the inequality (5.130). Equality holds if and only if this polynomial has a root of multiplicity 2, which is equivalent to the fact that the line $\{z + ty : t \in \mathbb{R}\}$ meets the set $\partial^2\mathcal{H}(p^{(k-2)}_{x_3,\dots,x_k})$. By Lemma 5.5.2, this is equivalent to the fact that this line meets the lineality space of $\operatorname{cl}\mathcal{H}(p)$. $\square$

We apply the preceding results to mixed discriminants. Let $A = (a_{ij})_{i,j=1}^n$ be a real symmetric $n \times n$ matrix. We identify A with the vector

$$(a_{11}, \dots, a_{1n}, a_{22}, \dots, a_{2n}, \dots, a_{nn}) \in \mathbb{R}^N, \qquad N = n(n+1)/2.$$

Then $A \mapsto \det A$ is a homogeneous polynomial of degree n on $\mathbb{R}^N$. Its polarization is the mixed discriminant. The polynomial det is hyperbolic in direction of the $n \times n$ unit matrix, since a real symmetric matrix has only real eigenvalues. The hyperbolicity cone of det is the cone of positive definite matrices; its closure is the cone of positive semi-definite matrices. The lineality space of the latter cone contains only the zero matrix O. Therefore, we obtain the following result.

Theorem 5.5.4 (Aleksandrov) *Let $A, B, C, A_2, \dots, A_n$ be real symmetric $n \times n$ matrices, where $A, A_2, \dots, A_n$ are positive definite, C is positive semi-definite and B is arbitrary. Then*

$$D(C, A_2, \dots, A_n) \geq 0. \tag{5.131}$$

Equality holds if and only if $C = O$.

Further,

$$D(A, B, A_3, \dots, A_n)^2 \geq D(A, A, A_3, \dots, A_n) D(B, B, A_3, \dots, A_n). \tag{5.132}$$

Equality holds if and only if $B = \lambda A$ with a real number λ.

Notes for Section 5.5

1. *Aleksandrov's inequalities for mixed discriminants.* Aleksandrov's [16] proof of the inequalities (5.132) (which is reproduced in Busemann [370] and Leichtweiß [1192]) exhibits some formal analogies to his proof of the inequalities for mixed volumes of strongly isomorphic polytopes. No really simple proof seems to be known (the one attempted by Schneider [1653] is erroneous, and the more general result stated there does not hold).

   New interest in the inequalities for mixed discriminants arose when Egorychev [535] used a special case in his proof of van der Waerden's conjecture on the permanents of doubly stochastic matrices; this led to simple proofs of that special case. We refer the reader to the references given in the survey article of Lagarias [1165] and to Gurvits [871] for a more general inequality.
2. *Hyperbolic polynomials.* The approach to quadratic inequalities via hyperbolic polynomials is based on the fundamental paper by Gårding [670]. He proved essentially all results on homogeneous hyperbolic polynomials which are presented in this section (though we rather followed the proofs given by Renegar [1571]), with the exception of the quadratic inequalities. Instead, he proved the inequality

   $$\tilde{p}(x_1, \dots, x_k)^k \geq p(x_1) \cdots p(x_k) \tag{5.133}$$

   for a homogeneous hyperbolic polynomial of degree $k > 1$ and for $x_1, \dots, x_k \in \mathcal{H}(p)$. Equality holds if and only if $x_1, \dots, x_k$ are pairwise proportional modulo the lineality space of $\operatorname{cl} \mathcal{H}(p)$.

   The quadratic inequalities (5.130) for homogeneous hyperbolic polynomials were proved by Khovanskiĭ [1072]. He also indicated a shorter proof of the basic underlying results of Gårding, but did not carry it out. The basic facts on hyperbolic polynomials and the inequality (5.133) are also proved in Hörmander [988] (pp. 63–65). Referring to this source, Klartag [1098] (Appendix) also indicated briefly how Aleksandrov's inequalities for mixed discriminants can be obtained.

3. For real symmetric $n \times n$ matrices $A_r = (a^r_{ij})^n_{i,j=1}$, $r = 1, \dots, n$, it follows from the definition of the mixed discriminant and the linearity properties of the determinant that there is a representation

$$D(A_1, \dots, A_n) = \sum_{i,j=1}^{n} D_{ij}(A_1, \dots, A_{n-1}) a^n_{ij}.$$

Here

$$D_{ii}(A_1, \dots, A_{n-1}) = \frac{1}{n} D'(A_1^{<i>}, \dots, A_{n-1}^{<i>}),$$

where D' denotes the mixed discriminant in $n-1$ dimensions and the matrix $A_r^{<i>}$ arises from A_r by deleting its ith row and column. This representation (together with a unimodular transformation bringing A_n to diagonal form) can be used (and was used by Aleksandrov [16]) to give an inductive proof of the fact that $D(A_1, \dots, A_n) > 0$ if $A_1, \dots, A_n$ are positive definite.

4. *Positive semi-definite matrices.* By continuity, the inequalities (5.131) and (5.132) still hold if the matrices $A, A_2, \dots, A_n$ are only assumed to be positive semi-definite. The assertions on the equality cases, however, are then no longer true. The complete equality conditions in the case of positive semi-definite matrices (or quadratic forms) were found by Panov [1509] (announced in [1508]). For further results on mixed discriminants of positive semi-definite matrices, we refer to Bapat [139].

6

Valuations on convex bodies

Besides Minkowski additivity of mappings on the space of convex bodies, we have already met several examples of a different kind of additivity. A mapping from a family $\mathscr{S}$ of sets into some abelian group is called additive, or a valuation, if the sum of its values at $K \cup M$ and $K \cap M$ equals the sum of the values at K and M, whenever $K, M, K \cup M, K \cap M \in \mathscr{S}$. On suitable domains, a valuation is also known as a finite measure. Valuations on $\mathcal{K}^n$ play an important role in the geometry of convex bodies. In this chapter, we establish some basic results on valuations. They have proved fundamental and stimulating for the exciting recent developments in the theory of valuations and their applications. These, however, are outside the scope of this book and we must restrict ourselves to a brief sketch in Section 6.5, with hints to the original literature.

6.1 Basic facts and examples

First, we recall some definitions from Section 4.3. A function φ defined on a family $\mathscr{S}$ of sets and with values in an (additively written) abelian group is called *additive* or a *valuation* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{6.1}$$

whenever $K, L, K \cup L, K \cap L \in \mathscr{S}$. If $\emptyset \in \mathscr{S}$, we moreover assume that $\varphi(\emptyset) = 0$. For the range of a valuation, we will tacitly allow also an abelian semigroup with cancellation law (as $\mathcal{K}^n$, for example), since it can be embedded in an abelian group.

A trivial example of a valuation is the characteristic function. Let $\mathscr{S}$ be a family of subsets of a set S. It is now convenient to denote the characteristic function of $A \subset S$ by $A^\bullet$, thus

$$A^\bullet(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in S \setminus A. \end{cases}$$

Then the mapping $A \mapsto A^\bullet$ is obviously a valuation (with values, say, in the additive group of real functions on S).

The family $\mathscr{S}$ is called *intersectional* if $\emptyset \in \mathscr{S}$ and if $A, B \in \mathscr{S}$ implies $A \cap B \in \mathscr{S}$. A valuation φ on an intersectional family $\mathscr{S}$ is called *fully additive on* $\mathscr{S}$ if it satisfies the following extension of (6.1) (which is the special case $m = 2$):

$$\begin{aligned}\varphi(K_1 \cup \cdots \cup K_m) &= \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le m} \varphi(K_{i_1} \cap \cdots \cap K_{i_r}) \\ &= \sum_{v \in S(m)} (-1)^{|v|-1} \varphi(K_v) \qquad (6.2)\end{aligned}$$

for all $K_1, \dots, K_m \in \mathscr{S}$ with $K_1 \cup \cdots \cup K_m \in \mathscr{S}$ and all $m \in \mathbb{N}$. We recall the notation used here. The set $S(m)$ is the family of nonempty subsets of $\{1, \dots, m\}$, and for $v = \{i_1, \dots, i_r\} \in S(m)$ we denote by $|v|$ the number of elements in v and write $K_v := K_{i_1} \cap \cdots \cap K_{i_r}$ (for given $K_1, \dots, K_m$). If φ is fully additive on $\mathscr{S}$, then it is also said to satisfy the *inclusion–exclusion principle on* $\mathscr{S}$.

For an intersectional family $\mathscr{S}$, we denote by $U(\mathscr{S})$ the lattice of all finite unions of elements from $\mathscr{S}$. If φ is a valuation on $U(\mathscr{S})$, then it is fully additive on $U(\mathscr{S})$, as follows easily by induction, and hence it is fully additive on $\mathscr{S}$. What can be said about the converse statement will be considered in the next section.

Of particular interest for us are, of course, valuations on the space $\mathcal{K}^n$ of convex bodies and on the convex ring $U(\mathcal{K}^n)$. We recall the examples of valuations that we have already met in previous chapters.

By (3.4), the identity map $K \mapsto K$ is a valuation on $\mathcal{K}^n$, and hence the support function mapping $K \mapsto h_K$ is a valuation. Therefore, it is clear from (1.30) and (1.31) that the mean width and the Steiner point are also valuations. It also follows (using Theorem 1.7.2, for example) that the mapping $K \mapsto F(K, u)$ is a valuation for each fixed $u \in \mathbb{S}^{n-1}$. On the space $\mathcal{K}^n_{(o)}$ of convex bodies with o as interior point, the polarity $K \mapsto K^\circ$ is a valuation, by Theorem 1.6.3 and (3.4). In Theorem 4.1.3 it was shown that the map $K \mapsto \mu_\rho(K, \cdot)$ is a valuation, where the measure $\mu_\rho(K, \cdot)$ is the local parallel volume of K at distance ρ. The proof was based on the fact that $K \mapsto p(K, x)$ is a valuation for each $x \in \mathbb{R}^n$, where $p(K, \cdot)$ is the nearest-point map of K. Theorem 4.1.3 was used to prove, in Theorem 4.2.1, that the mth support measure map, $K \mapsto \Theta_m(K, \cdot)$, is a valuation, and this implies the same property for the curvature measures and the area measures. Lemma 4.2.6 showed that certain measure-valued mappings μ with $\mathrm{d}\mu(K, \cdot) = G(h_K)\, \mathrm{d}S_{n-1}(K, \cdot)$ are valuations. Theorem 4.3.1 established the Euler characteristic χ, as the unique valuation on the convex ring $U(\mathcal{K}^n)$ that satisfies $\chi(K) = 1$ for all convex bodies $K \in \mathcal{K}^n$ (and $\chi(\emptyset) = 0$). It was used in Section 4.3 for constructing additive extensions of the support measures to the convex ring. According to Section 5.1, the mixed volume and the mixed area measure define valuations, via

$$K \mapsto V(K[p], C_{p+1}, \dots, C_n) \quad \text{and} \quad K \mapsto S(K[p], C_{p+1}, \dots, C_{n-1}, \cdot),$$

for any fixed $p \in \{1, \dots, n-1\}$ and $C_{p+1}, \dots, C_n \in \mathcal{K}^n$. Further examples are given by the mixed moment vectors and the Minkowski tensors of Section 5.4.

We remark that the method of Theorem 4.3.1, of constructing an additive extension of the Euler characteristic from $\mathcal{K}^n$ to $U(\mathcal{K}^n)$, can also be used for an explicit additive extension of the support function to the convex ring. For $K \in U(\mathcal{K}^n)$ and for $x \in \mathbb{R}^n \setminus \{o\}$ we define, following Mani [1323],

$$h(K, x) := \sum_{\lambda \in \mathbb{R}} \lambda \left[\chi(K \cap H_{x,\lambda}) - \lim_{\mu \downarrow \lambda} \chi(K \cap H_{x,\mu}) \right]. \tag{6.3}$$

For $K \in \mathcal{K}^n$, the right-hand side evidently gives the value of the support function of K at x, thus the definition is consistent. Since the additivity of χ carries over to $h(\cdot, x)$, we see that (6.3) extends the map $K \mapsto h(K, \cdot)$ as a valuation from $\mathcal{K}^n$ to $U(\mathcal{K}^n)$. Also for $K \in U(\mathcal{K}^n)$, we call $h(K, \cdot)$, as defined by (6.3), the support function of K. It should, however, be observed that it no longer determines K uniquely; for instance, $h(\{o\}, \cdot) = 0 = h(\emptyset, \cdot)$. Observe also that

$$h(K + t, x) = h(K, x) + \chi(K)\langle t, x \rangle$$

for $t \in \mathbb{R}^n$.

We conclude this section with two observations connecting Minkowski additivity with additivity in the valuation sense.

Lemma 6.1.1 *Let φ be a valuation on $\mathcal{K}^n$. If $C \in \mathcal{K}^n$ is a fixed convex body and if*

$$\varphi_C(K) := \varphi(K + C) \quad \textit{for } K \in \mathcal{K}^n,$$

then φ_C is a valuation on $\mathcal{K}^n$.

Proof Applying φ to (3.1) and to (3.3) for convex bodies K, L, C and adding, we obtain the assertion. □

For functions on $\mathcal{K}^n$, additivity is a weaker property than Minkowski additivity.

Theorem 6.1.2 *Every Minkowski additive function on $\mathcal{K}^n$ with values in an abelian group is fully additive.*

Proof By formula (6.3), the support function was extended as a valuation to $U(\mathcal{K}^n)$, hence it satisfies the inclusion–exclusion principle. In particular, for $K \in \mathcal{K}^n$ of the form $K = K_1 \cup \cdots \cup K_m$ with $K_i \in \mathcal{K}^n$ $(i = 1, \ldots, m)$ this implies that

$$h(K, \cdot) = \sum_{v \in S(m)} (-1)^{|v|-1} h(K_v, \cdot),$$

hence

$$h(K, \cdot) + \sum_{\substack{v \in S(m) \\ |v| \text{ even}}} h(K_v, \cdot) = \sum_{\substack{v \in S(m) \\ |v| \text{ odd}}} h(K_v, \cdot).$$

Since $h(\emptyset, \cdot) = 0$ by (6.3), it follows that

$$K + \sum_{\substack{v \in S(m) \\ |v| \text{ even}}}' K_v = \sum_{\substack{v \in S(m) \\ |v| \text{ odd}}}' K_v,$$

where the sums $\sum'$ extend only over those $v \in S(m)$ for which $K_v \neq \emptyset$. If now φ is a Minkowski additive function on $\mathcal{K}^n$, we apply it to both sides of this equation. Defining $\varphi(\emptyset) := 0$, we obtain

$$\varphi(K) = \sum_{v \in S(m)} (-1)^{|v|-1} \varphi(K_v),$$

which shows that φ is fully additive. □

Notes for Section 6.1

1. For the early history of the theory of valuations on convex bodies, its relations to the dissection theory of polytopes and its development until 1993, we refer the reader to the survey articles by McMullen and Schneider [1396] and by McMullen [1392]. In the notes for the sections in this chapter, we mention only a few of the older results, which were relevant for later investigations, and we mainly concentrate on developments after 1993.
2. *Minkowski additivity and general additivity*. Theorem 6.1.2 was first formulated in McMullen and Schneider [1396], Theorem (5.20). It generalizes earlier results of Sallee [1608], Mani [1323] and Spiegel [1802], referring to special cases.

6.2 Extensions

The previous section has already mentioned examples of valuations on special intersectional classes $\mathcal{S}$ which admit additive extensions to the lattice $U(\mathcal{S})$ of finite unions of elements from $\mathcal{S}$. The present section deals with general theorems on the possibility of such extensions. We consider first an abstract result, for arbitrary $\mathcal{S}$, and then special results for the families $\mathcal{P}^n$ of polytopes and $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$.

Let $\mathcal{S}$ be an intersectional family of sets. We recall that $K^\bullet$ denotes the characteristic function of the set $K \in \mathcal{S}$; we consider it as defined on the union of the sets in $\mathcal{S}$. As noted, the mapping $K \mapsto K^\bullet$ is additive. It is even fully additive; thus

$$(K_1 \cup \cdots \cup K_m)^\bullet = \sum_{v \in S(m)} (-1)^{|v|-1} K_v^\bullet \tag{6.4}$$

holds for $K_1, \dots, K_m \in \mathcal{S}$. This is an immediate consequence of the binomial theorem. By $U^\bullet(\mathcal{S})$ we denote the $\mathbb{Z}$-module spanned by the characteristic functions of the elements of $\mathcal{S}$, and by $\overline{U}(\mathcal{S})$ the system of all sets with characteristic function in $U^\bullet(\mathcal{S})$. We have $U(\mathcal{S}) \subset \overline{U}(\mathcal{S})$ by (6.4).

Let φ be a valuation on $\mathcal{S}$ with values in an abelian group (which can canonically be considered as a $\mathbb{Z}$-module). We define the functional $\varphi^\bullet$ by

$$\varphi^\bullet(K^\bullet) := \varphi(K) \quad \text{for } K \in \mathcal{S}. \tag{6.5}$$

The following extension theorem is due to Groemer [789].

Theorem 6.2.1 (Groemer) *Let $\mathcal{S}$ be an intersectional family of sets, and let φ be a mapping from $\mathcal{S}$ into an abelian group (considered as a $\mathbb{Z}$-module), such that* $\varphi(\emptyset) = 0$. *Then the following statements are equivalent.*

(a) *φ is fully additive on $\mathcal{S}$;*
(b) *If $a_1K_1^\bullet + \cdots + a_mK_m^\bullet = 0$ with $K_i \in \mathcal{S}$ and $a_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$, then $a_1\varphi(K_1) + \cdots + a_m\varphi(K_m) = 0$;*
(c) *$\varphi^\bullet$ has a $\mathbb{Z}$-linear extension to $U^\bullet(\mathcal{S})$;*
(d) *φ has an additive extension to $\overline{U}(\mathcal{S})$, and hence in particular to $U(\mathcal{S})$.*

Proof First we assume that (a) holds and that (b) is false. Then there is a pair of relations

$$a_1K_1^\bullet + \cdots + a_mK_m^\bullet = 0, \qquad a_1\varphi(K_1) + \cdots + a_m\varphi(K_m) \neq 0 \tag{6.6}$$

with $m \in \mathbb{N}$, $K_1, \ldots, K_m \in \mathcal{S}$ and $a_1, \ldots, a_m \in \mathbb{Z}$. We extend the sequence $(K_1, \ldots, K_m)$ to a sequence of (not necessarily distinct) elements of $\mathcal{S}$, in the following way. We choose a bijection ν_2 from the set of pairs (i, j) of integers with $1 \leq i < j \leq m$ to the interval $[m + 1, m_1]$, $m_1 = m + \binom{m}{2}$, of $\mathbb{N}$, and we put $K_{\nu_2(i,j)} := K_i \cap K_j$. Then we choose a bijection ν_3 from the set of triples (i, j, k) of integers with $1 \leq i < j < k \leq m$ to the interval $[m_1 + 1, m_2]$, $m_2 = m_1 + \binom{m}{3}$, of $\mathbb{N}$, and we put $K_{\nu_3(i,j,k)} := K_i \cap K_j \cap K_k$, and so on. The last term of the constructed sequence is $K_p = K_1 \cap \cdots \cap K_m$ with $p = 2^m - 1$. This construction implies: for any integers $r, t_1, \ldots, t_j$ with $j \geq 1$ and $1 \leq r < t_1, \ldots, t_j \leq p$, there is an integer $s > r$ with $K_r \cap K_{t_1} \cap \cdots \cap K_{t_j} = K_s$.

There exist pairs of relations of the form

$$\sum_{i=r}^{p} c_iK_i^\bullet = 0 \quad \text{with } c_i \in \mathbb{Z},\ c_r \neq 0, \qquad \sum_{i=r}^{p} c_i\varphi(K_i) \neq 0 \tag{6.7}$$

with $r \in \{1, \ldots, p\}$, since (6.6) can be written in this form. Of all such pairs of relations, we take for (6.7) one with maximal r. Then necessarily $r < p$, since $c_pK_p^\bullet = 0$ with $c_p \neq 0$ would imply $K_p = \emptyset$ and hence $c_p\varphi(K_p) = 0$, a contradiction. The first relation of (6.7) is only possible if each point of K_r is contained in some K_{r+k} with $k \geq 1$; hence, writing $K_r \cap K_{r+k} =: M_k$ we have

$$K_r = \bigcup_{k=1}^{p-r} M_k.$$

Since φ is fully additive on $\mathcal{S}$, this gives

$$\varphi(K_r) = \sum_{v \in S(p-r)} (-1)^{|v|-1}\varphi(M_v).$$

Because also the map $K \mapsto K^\bullet$ is fully additive, we similarly have

$$K_r^\bullet = \sum_{v \in S(p-r)} (-1)^{|v|-1}M_v^\bullet.$$

Each set M_v appearing here is of the form $M_v = K_r \cap K_{t_1} \cap \cdots \cap K_{t_j}$ with $j \geq 1$ and $t_1, \ldots, t_j > r$ and hence is equal to some K_s with $s > r$, as mentioned. Hence, there are representations

$$K_r^\bullet = \sum_{s=r+1}^{p} d_s K_s^\bullet, \qquad \varphi(K_r) = \sum_{s=r+1}^{p} d_s \varphi(K_s)$$

with integer coefficients d_s. Inserting these representations into (6.7), we obtain a pair of relations of the same type, but with r replaced by a larger integer. This contradicts the maximality of r, and this contradiction shows that (a) implies (b).

Assume that (b) holds. A function $f \in U^\bullet(\mathscr{S})$ can be written as

$$f = \sum_{i=1}^{m} a_i K_i^\bullet$$

with $m \in \mathbb{N}$, $K_i \in \mathscr{S}$ and $a_i \in \mathbb{Z}$ for $i = 1, \dots, m$. We can define

$$\varphi^\bullet(f) := \sum_{i=1}^{m} a_i \varphi(K_i),$$

since by (b) this does not depend on the chosen representation of f. The function $\varphi^\bullet$ thus defined on $U^\bullet(\mathscr{S})$ satisfies $\varphi^\bullet(K^\bullet) = \varphi(K)$ for $K \in \mathscr{S}$ and is $\mathbb{Z}$-linear, by definition. Thus (c) holds.

Assume that (c) holds. Denoting the $\mathbb{Z}$-linear extension of $\varphi^\bullet$ by the same symbol, we define

$$\varphi(K) := \varphi^\bullet(K^\bullet) \quad \text{for } K \in \overline{U}(\mathscr{S}),$$

which is consistent with (6.5). For $K, L \in \overline{U}(\mathscr{S})$ we obtain

$$\begin{aligned} &\varphi(K \cup L) + \varphi(K \cap L) = \varphi^\bullet((K \cup L)^\bullet) + \varphi^\bullet((K \cap L)^\bullet) \\ &= \varphi^\bullet((K \cup L)^\bullet + (K \cap L)^\bullet) = \varphi^\bullet(K^\bullet + L^\bullet) = \varphi^\bullet(K^\bullet) + \varphi^\bullet(L^\bullet) = \varphi(K) + \varphi(L). \end{aligned}$$

Thus φ is additive on $\overline{U}(\mathscr{S})$.

Finally, it has already been remarked that the implication (d)⇒(a) easily follows by induction. □

Remark 6.2.2 If the function φ in Theorem 6.2.1 takes its values in a real vector space, then the theorem holds in a modified form, after replacing $\mathbb{Z}$ by $\mathbb{R}$ in (b) and (c) and $U^\bullet(\mathscr{S})$ by the real vector space $V(\mathscr{S})$ spanned by the characteristic functions of the elements of $\mathscr{S}$. This is the original version of Groemer's theorem. In [792], Groemer has characterized the system of sets with characteristic function in $V(\mathscr{S})$ as the smallest ring containing $\mathscr{S}$. (A system $\mathscr{R}$ of subsets of a set is called a *ring* if $A, B \in \mathscr{R}$ implies $A \cup B \in \mathscr{R}$ and $A \setminus B \in \mathscr{R}$, hence also $A \cap B \in \mathscr{R}$.)

A particularly useful extension theorem can be proved for valuations on the system $\mathcal{P}^n$ of convex polytopes in $\mathbb{R}^n$. Even a seemingly weaker form of additivity is sufficient for that. A function φ on $\mathcal{P}^n$ with values in an abelian group is called *weakly additive* if, after supplementing the definition by $\varphi(\emptyset) = 0$, for each $P \in \mathcal{P}^n$ and each hyperplane H with corresponding closed halfspaces H^+, H^- the relation

$$\varphi(P) = \varphi(P \cap H^+) + \varphi(P \cap H^-) - \varphi(P \cap H) \tag{6.8}$$

is satisfied. Every valuation on $\mathcal{P}^n$ is weakly additive, and the converse holds in a stronger form.

Theorem 6.2.3 *Every weakly additive function on $\mathcal{P}^n$ with values in an abelian group is fully additive on $\mathcal{P}^n$.*

Proof Let φ be a weakly additive function on $\mathcal{P}^n$. We have to show that for all polytopes $P, P_1, \dots, P_m \in \mathcal{P}^n$ with $P = P_1 \cup \cdots \cup P_m$, and using the notations $(P_1, \dots, P_m) =: \tau$ and

$$\varphi(P, \tau) := \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le m} \varphi(P_{i_1} \cap \cdots \cap P_{i_r}), \tag{6.9}$$

we have

$$\varphi(P, \tau) = \varphi(P). \tag{6.10}$$

For the proof, we use induction over n. For $n = 0$, the assertion is trivial, so we assume that $n \ge 1$ and that the assertion is true in spaces of smaller dimension. Without loss of generality, we may assume that $\dim P = n$. As a first special case, we assume that one of the polytopes $P_1, \dots, P_m$, say P_1, is equal to P. Then any summand

$$(-1)^{r-1} \varphi(P_{i_1} \cap \cdots \cap P_{i_r}) \quad \text{with } i_1 > 1$$

in the right-hand side of (6.9) is cancelled by the summand

$$(-1)^r \varphi(P_1 \cap P_{i_1} \cap \cdots \cap P_{i_r}) = (-1)^r \varphi(P_{i_1} \cap \cdots \cap P_{i_r}).$$

Only the summand $\varphi(P_1) = \varphi(P)$ remains; hence (6.10) holds. Thus, the first case can now be excluded.

For given n, we prove (6.10) by induction over m. The case $m = 1$ being trivial, we assume that $m > 1$ and the assertion is true for all representations of P as unions of less than m polytopes. As a second special case, we assume that one of the polytopes $P_1, \dots, P_m$, say P_m, is of dimension less than n. Since P is the closure of its interior, we must have $P = P_1 \cup \cdots \cup P_{m-1}$ and $P_m \subset P$, hence $P_m = (P_1 \cap P_m) \cup \cdots \cup (P_{m-1} \cap P_m)$. Since these are unions of less than m polytopes, the induction hypothesis gives

$$\varphi(P) = \sum_{r=1}^{m-1} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le m-1} \varphi(P_{i_1} \cap \cdots \cap P_{i_r})$$

and

$$0 = \varphi(P_m) - \sum_{r=1}^{m-1} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le m-1} \varphi(P_{i_1} \cap \cdots \cap P_{i_r} \cap P_m).$$

Adding these two equations, we obtain (6.10). Hence, the second case can also be excluded.

Since now P_1 is an n-polytope properly contained in P, it has a facet whose affine hull, H, meets $\operatorname{int} P$. The two closed halfspaces H^+, H^- bounded by H are denoted such that $P_1 \subset H^+$. Since φ is weakly additive,

$$\varphi(P) = \varphi(P \cap H^+) + \varphi\left(\bigcup_{i=1}^{m}(P_i \cap H^-)\right) - \varphi\left(\bigcup_{i=1}^{m}(P_i \cap H)\right).$$

Relation (6.10) can be applied to the middle term on the right-hand side, since $\dim(P_1 \cap H^-) < n$ (second special case). It can also be applied to the last term, since it holds in smaller dimensions. Together with

$$\varphi(Q \cap H^-) - \varphi(Q \cap H) = \varphi(Q) - \varphi(Q \cap H^+)$$

for any $Q = P_{i_1} \cap \cdots \cap P_{i_r}$, this yields

$$\varphi(P) - \varphi(P, \tau) = \varphi(P \cap H^+) - \varphi(P \cap H^+, \tau^+),$$

where τ^+ denotes the tuple $(P_1 \cap H^+, \ldots, P_m \cap H^+)$. We have $P_1 \subset P \cap H^+$. If P_1 is properly contained in $P \cap H^+$, we can repeat the procedure with $P, P_1, \ldots, P_m$ replaced by their intersections with H^+. After finitely many such repetitions, the remaining polytope coincides with P_1. In this situation, relation (6.10) holds, by the first special case considered above. This completes the double induction and hence the proof. □

We can now extend valuations on $\mathcal{P}^n$ additively to finite unions of relatively open polytopes. This is of advantage, since it allows us to work with disjoint decompositions and thus to avoid the clumsy inclusion–exclusion principle.

By an *ro-polytope* we understand the relative interior of a convex polytope. The set of all ro-polytopes in $\mathbb{R}^n$ is denoted by $\mathcal{P}^n_{ro}$. The elements of $U(\mathcal{P}^n_{ro})$, the finite unions of relatively open polytopes, are called *ro-polyhedra*. For a polytope $P \in \mathcal{P}^n$, we denote by P_0 its relative interior, and for an ro-polytope Q, we denote by $\overline{Q}$ its closure; this is a polytope. We recall that $\mathcal{F}_k(P)$ denotes the set of k-dimensional faces of a polytope $P \in \mathcal{P}^n$, $k = 0, \ldots, \dim P$. We have the disjoint decomposition

$$P = \biguplus_{k=0}^{\dim P} \ \biguplus_{F \in \mathcal{F}_k(P)} F_0, \tag{6.11}$$

where the symbol $\biguplus$ denotes a disjoint union. Equation (6.11) shows that $\mathcal{P}^n \subset U(\mathcal{P}^n_{ro})$. By induction on the dimension, one obtains that $Q^\bullet \in U^\bullet(\mathcal{P}^n)$ for any ro-polytope Q. Using the fact that every ro-polytope is the intersection of finitely many hyperplanes and open halfspaces, it is easy to see that any ro-polyhedron can be written as a disjoint union of finitely many ro-polytopes. Therefore, $U(\mathcal{P}^n_{ro}) \subset \overline{U}(\mathcal{P}^n)$. Now Theorems 6.2.3 and 6.2.1 immediately yield the following.

Corollary 6.2.4 *Any weakly additive function on the set $\mathcal{P}^n$ of polytopes with values in an abelian group has an additive extension to the set $U(\mathcal{P}^n_{ro})$ of ro-polyhedra.*

The value of such an extension at an ro-polytope can be expressed in a simple way by values of the valuation at polytopes, as we now explain. First we observe that the extension of the Euler characteristic to $U(\mathcal{P}^n_{ro})$ satisfies

$$\chi(Q) = (-1)^{\dim Q} \quad \text{for } Q \in \mathcal{P}^n_{ro}. \tag{6.12}$$

For the proof, we note that the inductive proof of Theorem 4.3.1 (which established the extension of the Euler characteristic to the convex ring) can verbally be copied, replacing $U(\mathcal{K}^n)$ by $U(\mathcal{P}^n_{ro})$, to show the existence and uniqueness of a valuation χ on $U(\mathcal{P}^n_{ro})$ with $\chi(\emptyset) = 0$, $\chi(P) = 1$ for $P \in \mathcal{P}^n$ and $\chi(Q) = (-1)^{\dim Q}$ for $Q \in \mathcal{P}^n_{ro}$.

Next, we note that by applying the extended Euler characteristic to the disjoint decomposition (6.11), we obtain

$$\sum_{k=0}^{n} (-1)^k \sum_{F \in \mathcal{F}_k(P)} 1 = 1 \quad \text{for } P \in \mathcal{P}^n,$$

which is known as the *Euler relation*. We write it in the more concise form

$$\sum_{F \leq P} (-1)^{\dim F} = 0,$$

denoting by $\sum_{F \leq P}$ a summation over all faces F of P, including P and the empty face, which by definition has dimension -1. Similarly, we write $\sum_{G \leq F \leq P}$ for a summation over all faces F of P that contain the given face G of P. With this notation, the Euler relation extends to the more general formula

$$\sum_{G \leq F \leq P} (-1)^{\dim F} = 0 \quad \text{for any face } G < P. \tag{6.13}$$

Of course, $G < P$ means that G is a face of P, but different from P. For the proof of (6.13) we assume, without loss of generality, that $\dim P = n$ and $o \in \operatorname{int} P$, and use the polar polytope P°. The bijective correspondence $F \leftrightarrow \widehat{F}$ between the faces of P and their conjugate faces, which are faces of P°, satisfies $F \leq G \Leftrightarrow \widehat{G} \leq \widehat{F}$ and $\dim \widehat{F} = n - 1 - \dim F$ (compare Theorem 2.1.4 and (2.28)). Hence, for $G < P$ we obtain

$$\begin{aligned} \sum_{G \leq F \leq P} (-1)^{\dim F} &= (-1)^n + \sum_{G \leq F < P} (-1)^{\dim F} = (-1)^n + \sum_{\emptyset < \widehat{F} \leq \widehat{G}} (-1)^{n-1-\dim \widehat{F}} \\ &= (-1)^{n-1} \left[-1 + \sum_{\emptyset < \widehat{F} \leq \widehat{G}} (-1)^{\dim \widehat{F}} \right] = 0, \end{aligned}$$

the latter by the Euler relation for $\widehat{G}$. This proves (6.13).

Now let φ be a valuation on $U(\mathcal{P}^n_{ro})$. We state that

$$\varphi(Q) = \sum_{F \leq \overline{Q}} (-1)^{\dim Q - \dim F} \varphi(F) \qquad \text{for } Q \in \mathcal{P}^n_{ro}. \tag{6.14}$$

For the proof, we note that

$$\varphi(F) = \sum_{G \le F} \varphi(G_0)$$

for $F \le \overline{Q}$, by (6.11), and we use the general Euler relation (6.13) to obtain

$$\begin{aligned}\sum_{F \le \overline{Q}} (-1)^{\dim Q - \dim F} \varphi(F) &= \sum_{F \le \overline{Q}} (-1)^{\dim Q - \dim F} \sum_{G \le F} \varphi(G_0) \\ &= \sum_{G \le \overline{Q}} \varphi(G_0) \sum_{G \le F \le \overline{Q}} (-1)^{\dim Q - \dim F} = \varphi((\overline{Q})_0) = \varphi(Q)\end{aligned}$$

and thus (6.14).

We turn to valuations on the space $\mathcal{K}^n$ and formulate the following result.

Theorem 6.2.5 (Groemer's extension theorem) *Every continuous valuation on the space $\mathcal{K}^n$ of convex bodies with values in a topological vector space has an additive extension to the convex ring $U(\mathcal{K}^n)$.*

Here continuity on $\mathcal{K}^n$ refers, of course, to the Hausdorff metric. In fact, a weaker continuity assumption is sufficient, as the proof shows. We say that the function φ from $\mathcal{K}^n$ into a topological (Hausdorff real) vector space is σ-continuous if for every decreasing sequence $(K_i)_{i \in \mathbb{N}}$ in $\mathcal{K}^n$ one has

$$\lim_{i \to \infty} \varphi(K_i) = \varphi\left(\bigcap_{i \in \mathbb{N}} K_i \right).$$

Groemer [789] proved that every σ-continuous valuation on $\mathcal{K}^n$ with values in some topological vector space has an additive extension to $U(\mathcal{K}^n)$. His proof is reproduced in [1740], Theorem 14.4.2. We do not repeat it here, since no use of this extension theorem will be made in this book.

It seems to be unknown whether every valuation on $\mathcal{K}^n$ has an additive extension to $U(\mathcal{K}^n)$.

Notes for Section 6.2

1. *Extensions of valuations*. The first extension theorem of the types considered here was obtained by Volland [1900]. He proved that every valuation on the class $\mathcal{P}^n$ of polytopes in $\mathbb{R}^n$ admits a unique additive extension to the class $U(\mathcal{P}^n)$ of finite unions of polytopes. (Related but simpler extension results can be found in Hadwiger [911], p. 81, and in Böhm and Hertel [263], p. 47.) Volland in his proof first showed that every valuation on $\mathcal{P}^n$ is fully additive; then he proved part of Theorem 6.2.1, namely that a valuation on an intersectional class $\mathscr{S}$ has an additive extension to $U(\mathscr{S})$ if and only if it is fully additive. Volland's extension theorem was rediscovered by Perles and Sallee [1523]. Before that, it had been observed by Sallee [1609] that weak additivity of a function on $\mathcal{P}^n$ implies additivity.

   Corollary 6.2.4, saying that any weakly additive function on $\mathcal{P}^n$ has an additive extension to the class $U(\mathcal{P}^n_{ro})$ of ro-polyhedra, was deduced here from the extension theorems of Volland and Groemer. A different version of the proof (but also based on Volland's ideas) appears in Schneider [1706]. The proof given above for Theorem 6.2.3 is a modification of Volland's proof.

In contrast to the case of polytopes, a weak valuation on the space of convex bodies need not be a valuation. An example showing this (which is due to Groemer) is described in [1396], p. 173.

2. *The Euler characteristic*. The elementary proof, due to Hadwiger [906], for the existence of the Euler characteristic on $U(\mathcal{K}^n)$ (Theorem 4.3.1 and the version for $U(\mathcal{P}^n_{ro})$ mentioned in the present section) appears in the literature in various forms; see Hadwiger [912, 914, 919] and Hadwiger and Mani [928]. See also Hadwiger [923] for an elementary treatment of the Euler characteristic for polygons in the plane. Applications of the Euler characteristic on the convex ring to some questions of combinatorial geometry were treated by Hadwiger [893, 906, 914] and Klee [1112]. Klee's paper put the Euler characteristic in a lattice-theoretic setting, and the general treatment of the Euler characteristic in combinatorial and algebraic terms was continued by Rota [1590]; see also Rota [1589]. Eckhoff [527] solved a problem posed by Hadwiger [906] and Mani [929], namely to determine sharp lower and upper bounds for the Euler characteristic of a union of k convex bodies in $\mathcal{K}^n$.

   The extended Euler characteristic on $U(\mathcal{P}^n_{ro})$ was considered by Lenz [1201] and Groemer [782] and in special cases (but with a different sign for odd-dimensional ro-polytopes) also by Hadwiger [919, 922].

   Some ideas from the proofs of this section can be used to extend the notion of an Euler characteristic. First, Groemer [786] proved an abstract theorem from which results on Euler characteristics in special concrete situations can be deduced. The class $\mathscr{S}$ of subsets of S is called *separable* if to any two disjoint sets $A, B \in \mathscr{S}$ there exists a pair $X, Y \subset S$ such that $X \cap C \in \mathscr{S}$ and $Y \cap C \in \mathscr{S}$ for every $C \in \mathscr{S}$, $A \subset X$, $A \cap Y = \emptyset$, $B \subset Y$, $B \cap X = \emptyset$, $X \cup Y = S$, and $Z \cap X \neq \emptyset$, $Z \cap Y \neq \emptyset$ for $Z \in \mathscr{S}$ only if $Z \cap X \cap Y \neq \emptyset$. Groemer proved the following.

   Theorem Let $\mathscr{S}$ be a separable intersectional class of subsets of S. There exists exactly one linear functional χ on the vector space $V(\mathscr{S})$ such that $\chi(K^\bullet) = 1$ for every nonempty set K of $\mathscr{S}$.

   Second, Hadwiger's recursive procedure for the introduction of the Euler characteristic can also be modified to extend the Euler characteristic to a linear functional on the vector space $V(\mathscr{S})$, for suitable $\mathscr{S}$. We refer to Hadwiger [913], Groemer [782], Lenz [1201]. Further investigations of an extended notion of the Euler characteristic in certain situations are due to Groemer [783, 787]. An extension of Hadwiger's approach appears in Chen [411].

3. A thorough study of the extension of the Euler characteristic, and of many operations over convex sets to multilinear mappings on linear spaces, by replacing convex sets by their characteristic functions, was made by Przesławski [1550]. One feature of this paper is that it studies compact and relatively open bounded convex sets simultaneously. In a similar vein, Chen [413] investigated the real vector space generated by the characteristic functions of closed and of relatively open convex sets, and introduced an algebra structure with a multiplication corresponding to Minkowski addition (where a problem with the definition of the product of the characteristic functions of a closed and a relatively open convex set has to be overcome).

4. *Euler-type relations.* By Corollary 6.2.4, any valuation on the class $\mathcal{P}^n$ of convex polytopes has an additive extension to the class $U(\mathcal{P}^n_{ro})$ of ro-polyhedra, and by (6.14), the extension satisfies

$$\varphi(Q) = \sum_{F \leq \overline{Q}} (-1)^{\dim Q - \dim F} \varphi(F) \qquad \text{for } Q \in \mathcal{P}^n_{ro}.$$

   For a given valuation φ on $\mathcal{P}^n$, Sallee [1609] defined

$$\varphi^*(P) := \sum_{F \leq \overline{Q}} (-1)^{\dim F} \varphi(F) \qquad \text{for } P \in \mathcal{P}^n.$$

Hence, $\varphi^*(P) = (-1)^{\dim P}\varphi(\operatorname{relint} P)$. Using the additivity of φ on $U(\mathcal{P}^n_{ro})$, it is easy to see that φ^* is a valuation on $\mathcal{P}^n$. Sallee proved this in a different way.

The function φ on $\mathcal{P}^n$ is said to satisfy an *Euler-type relation* if $\varphi^*(P) = \varepsilon\varphi(\eta P)$ for all $P \in \mathcal{P}^n$, where $\varepsilon, \eta \in \{1, -1\}$ are fixed. Shephard [1779, 1783] showed that the Steiner point and the mean width satisfy Euler-type relations; see also Shephard [1785]. The following general result of this kind is due to McMullen [1380, 1383].

Theorem If φ is a translation invariant valuation on $\mathcal{P}^n$ which is homogeneous of degree r, then $\varphi^*(P) = (-1)^r\varphi(-P)$ for all $P \in \mathcal{P}^n$.

See also the survey by Sallee [1613]. A general study of Euler-type relations for valuations on polyhedra was made by Klain [1085].

5. *An identity for polytopes.* An Euler-type relation (see the previous note) is satisfied, in particular, by the support function. If $P \in \mathcal{P}^n$ and $u \in \mathbb{R}^n$, then

$$\sum_{j=0}^{n}(-1)^j \sum_{F\in\mathcal{F}_j(P)} h(F,u) = -h(P,-u),$$

as proved by Shephard [1785]. This can be written as an identity for Minkowski sums, namely

$$-P + \sum_{2|\dim F} F = \sum_{2\nmid\dim F} F,$$

where the sums extend over all nonempty faces of P of respectively even and odd dimensions (including 0 and n).

6. *Integrals based on valuations.* Let $\mathscr{S}$ be an intersectional class of subsets of some set, and let φ be a function from $\mathscr{S}$ into a topological vector space. If the conditions of Theorem 6.2.1 are satisfied, one can define a φ-integral on the vector space $V(\mathscr{S})$ in the following way. Let $f \in V(\mathscr{S})$, then $f = a_1K_1^\bullet + \cdots + a_mK_m^\bullet$ with $K_1, \dots, K_m \in \mathscr{S}$ and $a_1, \dots, a_m \in \mathbb{R}$. The definition

$$\int f\, \mathrm{d}\varphi := a_1\varphi(K_1) + \cdots + a_m\varphi(K_m)$$

makes sense, since it does not depend on the choice of the representation of f. This definition is due to Groemer [789]. Later, such an integration with respect to a valuation, mainly the Euler characteristic, was rediscovered and applied in various ways; see, for example, Viro [1882], Khovanskiĭ and Pukhlikov [1552, 1073]. An application to mixed volumes of polyhedral functions (in the sense of Groemer [788]) was made by Panina [1503].

7. *Groemer's extension of the quermassintegrals*. The following extension of the quermassintegrals (which apparently does not generalize to curvature measures) was proposed by Groemer [782]. He introduced a vector space A^n of real functions on $\mathbb{R}^n$ with a pseudonorm such that A^n contains $\overline{U}(\mathcal{P}^n)$ (see Section 6.1) as a proper dense subspace. The elements of A^n are called 'approximable' functions. The system $\mathcal{S}_A$ of subsets of $\mathbb{R}^n$ whose characteristic functions are approximable contains the convex ring $U(\mathcal{K}^n)$ and, for example, the relative interiors of convex bodies. Groemer showed that the quermassintegrals can be extended from $\mathcal{K}^n$ to continuous linear functionals on A^n. In particular, this yields an additive extension of the quermassintegral W_i to the class $\mathcal{S}_A$.

6.3 Polynomiality

For valuations defined on subsets of $\mathbb{R}^n$, the interplay between properties of the valuations and the vector space structure of $\mathbb{R}^n$ is particularly interesting and fruitful. The domains of the valuations considered in the following will always be closed under homotheties. A valuation φ on such a domain is called *translation invariant* if $\varphi(K + t) = \varphi(K)$ for all K in the domain of φ and all $t \in \mathbb{R}^n$. If the range of φ is a rational vector space (a real vector space), then φ is called *rational homogeneous* (*homogeneous*) of degree r if $\varphi(\lambda K) = \lambda^r \varphi(K)$ for all K in the domain of φ and all rational (respectively, all real) $\lambda \geq 0$; here r can be any real number. It turns out that, for translation invariant valuations (satisfying some natural assumptions), only a few degrees of homogeneity are possible.

In the first part, we consider valuations taking their values in a rational vector space, which we denote by X. In the following, direct sums of convex sets play an important role. A (not necessarily closed) convex set $Z \subset \mathbb{R}^n$ is called an *s-cylinder* if $Z = Q_1 + \cdots + Q_s$ with (possibly one-pointed) convex sets $Q_i \subset L_i$, where $L_1, \ldots, L_s$ are independent proper linear subspaces of $\mathbb{R}^n$. We denote by $\mathscr{Z}_s^n$ the set of all s-cylinders in $\mathbb{R}^n$. Note that $\mathscr{Z}_s^n \subset \mathscr{Z}_r^n$ for $r \leq s$.

Theorem 6.3.1 (and Definition) *Let $\varphi : \mathcal{P}^n \to X$ be a translation invariant valuation. Then there are translation invariant valuations $\varphi_0, \ldots, \varphi_n : \mathcal{P}^n \to X$ such that φ_r is rational homogeneous of degree r $(r = 0, \ldots, n)$ and*

$$\varphi(\lambda P) = \sum_{r=0}^{n} \lambda^r \varphi_r(P) \quad \text{for } P \in \mathcal{P}^n \text{ and rational } \lambda \geq 0.$$

In particular, $\varphi = \varphi_0 + \cdots + \varphi_n$ (the McMullen decomposition *of φ).*

Proof We use the so-called canonical simplex dissection (as vigorously employed by Hadwiger [911]), but in a modified version using disjoint decompositions.

Let $S \in \mathcal{K}^n$ be an n-dimensional simplex, without loss of generality given by

$$S = \left\{ \sum_{i=1}^{n} x_i e_i : 0 \leq x_1 \leq \cdots \leq x_n \leq 1 \right\},$$

where $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$. By S_0 we denote the interior of S. The set

$$S' := \left\{ \sum_{i=1}^{n} x_i e_i : 0 \leq x_1 < \cdots < x_n < 1 \right\}$$

is the union of S_0 and the relative interior of a facet of S and hence belongs to $U(\mathcal{P}_{ro}^n)$. We call it a *half-open simplex*.

For an interval $I = \{r, r + 1, \ldots, s\} \subset \{1, \ldots, n\}$ we define

$$\Delta(I) := \left\{ \sum_{i=r}^{s} x_i e_i : 0 \leq x_r < \cdots < x_s < 1 \right\}.$$

This set is a half-open simplex of dimension $s - r + 1$. If δ is a decomposition of $\{1, \ldots, n\}$ into nonempty intervals $I_1, \ldots, I_p$, we put

$$Z(\delta) := \Delta(I_1) + \cdots + \Delta(I_p).$$

This set belongs to $\mathscr{Z}_p^n$.

Let $k \in \mathbb{N}$ be given. The half-open interval $[0, k) \subset \mathbb{R}$ is the disjoint union of the half-open intervals

$$J_m := [m-1, m), \quad m = 1, \dots, k.$$

Let $x = \sum_{i=1}^{n} x_i e_i \in kS'$. Then $x_i \in [0, k)$ for $i = 1, \dots, n$, thus x determines indices $1 \le j_1 < \dots < j_p \le k$, $p \in \{1, \dots, n\}$, such that $J_{j_1}, \dots, J_{j_p}$ are precisely the intervals J_m that have a nonempty intersection with $\{x_1, \dots, x_n\}$. It also determines a decomposition δ of $\{1, \dots, n\} \subset \mathbb{N}$ into intervals $I_1, \dots, I_p$ such that $i \in I_r$ implies $x_i \in J_{j_r}$ for $r = 1, \dots, p$. The set of all $x \in kS'$ that induce the same p-tuple $(J_{j_1}, \dots, J_{j_p})$ and the same decompositon δ is given by $Z(\delta) + t(j_1, \dots, j_p, \delta)$, with a suitable translation vector $t(j_1, \dots, j_p, \delta)$. Therefore, we have a disjoint decomposition

$$kS' = \biguplus_{p=1}^{n} \biguplus_{1 \le j_1 < \dots < j_p \le k} \biguplus_{\delta \in D_p} [Z(\delta) + t(j_1, \dots, j_p, \delta)],$$

where D_p denotes the set of all decompositions of $\{1, \dots, n\}$ into p nonempty intervals.

Now let $\varphi : \mathcal{P}^n \to X$ be a translation invariant valuation. By Corollary 6.2.4, it has an additive extension (denoted by the same symbol) to $U(\mathcal{P}^n_{ro})$. Clearly, the extension is also translation invariant. From the disjoint decomposition above we obtain

$$\varphi(kS') = \sum_{p=1}^{n} \binom{k}{p} \sum_{\delta \in D_p} \varphi(Z(\delta)). \tag{6.15}$$

Proposition For every relatively open simplex T, there exist $a_0, \dots, a_n \in X$, depending only on φ and T, such that

$$\varphi(kT) = \sum_{i=0}^{n} k^i a_i \quad \text{for each } k \in \mathbb{N}.$$

We prove this by induction over $\dim T$. The case $\dim T = 0$ is trivial. Let $\dim T = 1$. Then kT is the disjoint union of k translates of T and of $k-1$ singletons, hence

$$\varphi(kT) = k\varphi(T) + (k-1)\varphi(\{o\}) = k[\varphi(T) + \varphi(\{o\})] - \varphi(\{o\}).$$

Assume that $\dim T \ge 2$ and the proposition has been proved for ro-simplices of smaller dimensions. Without loss of generality, we may assume that $T = S_0$. By (6.15), $\varphi(kS')$ is a polynomial in k of degree at most n, whose coefficients depend only on φ and S_0. The half-open simplex S' is the disjoint union of the open simplex S_0 and of the relative interior F_0 of a facet F of the simplex S. From $\varphi(kS_0) = \varphi(kS') - \varphi(kF_0)$ and the inductive hypothesis we now obtain that the proposition is true.

Let $P \in \mathcal{P}^n$. The polytope P has a disjoint decomposition into relatively open simplices (of dimensions $0, \dots, \dim P$). It follows from the proposition that there are elements $b_0, \dots, b_n \in X$, depending only on φ and P and possibly on the decomposition, such that

$$\varphi(kP) = \sum_{j=0}^{n} k^j b_j \quad \text{for each } k \in \mathbb{N}.$$

Since the matrix $(k^j)_{k=1,\dots,n+1}^{j=0,\dots,n}$ has an inverse, we obtain representations

$$b_j = \sum_{k=1}^{n+1} c_{jk}\varphi(kP), \quad j = 0, \dots, n, \tag{6.16}$$

with certain constants c_{jk}. This shows that each b_j depends only on φ and P, hence we can define $\varphi_j(P) := b_j$. It follows immediately that this defines a translation invariant valuation φ_j on $\mathcal{P}^n$ which satisfies $\varphi_j(mP) = m^j\varphi_j(P)$ for $m \in \mathbb{N}$. Replacing P by $(1/k)P$ with $k \in \mathbb{N}$, we see that $\varphi_j(\lambda P) = \lambda^j\varphi_j(P)$ for rational $\lambda > 0$. We have $\varphi(P) = \sum_{j=0}^{n} \varphi_j(P)$ and hence

$$\varphi(\lambda P) = \sum_{j=0}^{n} \lambda^j\varphi_j(P) \quad \text{for rational } \lambda > 0.$$

Since $\varphi(\lambda\{o\}) = \varphi(\{o\})$, we see that $\varphi_j(\{o\}) = 0$ for $j = 1, \dots, n$ and $\varphi_0(\{o\}) = \varphi(\{o\})$. Therefore, $\varphi(\lambda P) = \sum_{j=0}^{n} \lambda^j\varphi_j(P)$ holds also for $\lambda = 0$. This completes the proof. □

Corollary 6.3.2 *Let $\varphi : \mathcal{P}^n \to X$ be a translation invariant valuation which is homogeneous of degree r, for some $r \in \{1, \dots, n\}$.*

(a) *If $P \in \mathcal{P}^n$ and* $\dim P < r$*, then $\varphi(P) = 0$.*

(b) *If $\varphi(P) = 0$ for all $P \in \mathcal{P}^n \cap \mathcal{Z}_r^n$, then $\varphi = 0$.*

Proof (a) The restriction of φ to the polytopes in a given $(r-1)$-dimensional subspace L is a valuation which is invariant under the translations of L into itself. By Theorem 6.3.1, it is a sum of valuations of homogeneity degrees $0, \dots, r-1$. For a polytope $P \subset L$, this together with $\varphi(\lambda P) = \lambda^r\varphi(P)$ shows that $\varphi(P) = 0$. Since L was arbitrary, the translation invariance of φ shows that $\varphi(P) = 0$ whenever $\dim P \le r - 1$.

(b) Suppose that $\varphi(P) = 0$ for all $P \in \mathcal{P}^n \cap \mathcal{Z}_r^n$. If $P \in \mathcal{P}^n \cap \mathcal{Z}_s^n$ with $s \ge r$, then all faces of P belong to $\mathcal{Z}_r^n$ (by Theorem 1.7.5(c) and since $\mathcal{Z}_s^n \subset \mathcal{Z}_r^n$), hence also the relative interiors of these faces belong to $\mathcal{Z}_r^n$. The proof of Theorem 6.3.1, in particular (6.15), where now $\varphi(Z(\delta)) = 0$ whenever $\delta \in D_p$ with $p \ge r$, shows that φ cannot be homogeneous of degree r, except if it vanishes identically. □

From Theorem 6.3.1 we obtain polynomial expansions for valuations of Minkowski linear combinations. Let $\varphi : \mathcal{P}^n \to X$ be a translation invariant valuation on $\mathcal{P}^n$ which is rational homogeneous of degree $m \in \{1, \dots, n\}$. For fixed $P_2 \in \mathcal{P}^n$, the mapping $P \mapsto \varphi(P + P_2)$ is a translation invariant valuation (Lemma 6.1.1), hence there is a polynomial expansion $\varphi(\lambda P_1 + P_2) = \sum_{i=0}^{n} \lambda^i\psi_i(P_1, P_2)$ for rational $\lambda \ge 0$. The mapping $P \mapsto \psi_i(P_1, P)$ is again a translation invariant valuation. Repeating the argument, one sees that for $k \in \mathbb{N}$ and polytopes $P_1, \dots, P_k \in \mathcal{P}^n$, the value $\varphi(\lambda_1 P_1 + \dots + \lambda_k P_k)$ is a polynomial in the rational numbers $\lambda_1, \dots, \lambda_k \geq 0$. Since φ is rational homogeneous of degree m, this polynomial is homogeneous of degree m. Hence, using multinomial coefficients, it can be written in the form

$$\varphi(\lambda_1 P_1 + \dots + \lambda_k P_k) = \sum_{r_1,\dots,r_k=0}^{m} \binom{m}{r_1 \dots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \varphi_{r_1,\dots,r_k}(P_1, \dots, P_k). \tag{6.17}$$

Remark 6.3.3 The case $m = 1$ immediately shows the following: *Every translation invariant valuation on $\mathcal{P}^n$ which is rational homogeneous of degree one is Minkowski additive.*

Continuing with the expansion (6.17), we choose $k = m$ and put

$$\overline{\varphi}(P_1, \dots, P_m) := \varphi_{1,\dots,1}(P_1, \dots, P_m).$$

Clearly, this function is symmetric in its arguments. The coefficients in (6.17) (again for arbitrary k) are given by

$$\varphi_{r_1,\dots,r_k}(P_1, \dots, P_k) = \overline{\varphi}(\underbrace{P_1, \dots, P_1}_{r_1}, \dots, \underbrace{P_k, \dots, P_k}_{r_k}) =: \overline{\varphi}(P_1[r_1], \dots, P_k[r_k]).$$

This follows by expanding both sides of

$$\begin{aligned} &\varphi((\lambda_1 + \dots + \lambda_{r_1})P_1 + (\lambda_{r_1+1} + \dots + \lambda_{r_1+r_2})P_2 + \dots \\ &\quad + (\lambda_{m-r_k+1} + \dots + \lambda_m)P_k) \\ &= \varphi(\lambda_1 P_1 + \dots + \lambda_{r_1} P_1 + \lambda_{r_1+1} P_2 + \dots + \lambda_{r_1+r_2} P_2 + \dots \\ &\quad + \lambda_{m-r_k+1} P_k + \dots + \lambda_m P_k) \end{aligned}$$

according to (6.17) and comparing the coefficients of the monomial $\lambda_1 \cdots \lambda_m$. For $r \in \{1, \dots, m\}$ and fixed polytopes $Q_{r+1}, \dots, Q_m$, the mapping

$$P \mapsto \overline{\varphi}(P[r], Q_{r+1}, \dots, Q_m)$$

is translation invariant and rational homogeneous of degree r; as for the mixed volume in Section 5.1, one shows that it is a valuation. In particular, $\overline{\varphi}$ is Minkowski additive in each of its arguments. We collect the obtained results in the following theorem.

Theorem 6.3.4 *Let $\varphi : \mathcal{P}^n \to X$ (with X a rational vector space) be a translation invariant valuation which is rational homogeneous of degree $m \in \{1, \dots, n\}$. Then there exists a symmetric mapping $\overline{\varphi} : (\mathcal{P}^n)^m \to X$ which is translation invariant and Minkowski additive in each variable, such that*

$$\varphi(\lambda_1 P_1 + \dots + \lambda_k P_k) = \sum_{r_1,\dots,r_k=0}^{m} \binom{m}{r_1 \dots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \overline{\varphi}(P_1[r_1], \dots, P_k[r_k])$$

holds for all $P_1, \dots, P_k \in \mathcal{P}^n$ and all rational $\lambda_1, \dots, \lambda_k \geq 0$. The mapping $P \mapsto \varphi(P[r], Q_{r+1}, \dots, Q_m)$ is a translation invariant valuation, rational homogeneous of degree r, for each $r \in \{1, \dots, m\}$ and fixed polytopes $Q_{r+1}, \dots, Q_m$.

We call $\overline{\varphi}$ the *mixed valuation induced by* φ.

The preceding theorems imply similar results for continuous valuations on the space $\mathcal{K}^n$ of convex bodies with values in a real topological vector space.

Theorem 6.3.5 *Let φ be a translation invariant, continuous valuation on $\mathcal{K}^n$ with values in a real topological vector space. Then there are continuous, translation invariant valuations $\varphi_0, \dots, \varphi_n$ on $\mathcal{K}^n$ such that φ_i is homogeneous of degree i $(i = 0, \dots, n)$ and*

$$\varphi(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K) \quad \textit{for } K \in \mathcal{K}^n \textit{ and } \lambda \geq 0.$$

In particular, $\varphi = \varphi_0 + \cdots + \varphi_n$.

Proof For given φ, let $\varphi_0, \dots, \varphi_n$ be the valuations on $\mathcal{P}^n$ according to Theorem 6.3.1, thus

$$\varphi(\lambda P) = \sum_{i=0}^{n} \lambda^i \varphi_i(P) \quad \text{for } P \in \mathcal{P}^n \text{ and rational } \lambda \geq 0.$$

By continuity, this holds for all real $\lambda \geq 0$. With the coefficients c_{jk} from (6.16) we have

$$\varphi_j(P) = \sum_{k=1}^{n+1} c_{jk} \varphi(kP), \qquad j = 0, \dots, n, \qquad \text{for } P \in \mathcal{P}^n.$$

We define

$$\varphi_j(K) := \sum_{k=1}^{n+1} c_{jk} \varphi(kK), \qquad j = 0, \dots, n, \qquad \text{for } K \in \mathcal{K}^n.$$

Using approximation of convex bodies by polytopes and the continuity of φ, we easily obtain the remaining assertions. □

Theorem 6.3.6 *Let $\varphi : \mathcal{K}^n \to X$ (with X a topological vector space) be a continuous, translation invariant valuation which is homogeneous of degree $m \in \{1, \dots, n\}$. Then there exists a continuous symmetric mapping $\overline{\varphi} : (\mathcal{K}^n)^m \to X$ which is translation invariant and Minkowski additive in each variable, such that*

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_k K_k) = \sum_{r_1, \dots, r_k = 0}^{m} \binom{m}{r_1 \dots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \overline{\varphi}(K_1[r_1], \dots, K_k[r_k])$$

holds for all $K_1, \dots, K_k \in \mathcal{K}^n$ and all real $\lambda_1, \dots, \lambda_k \geq 0$. The mapping $K \mapsto \varphi(K[r], M_{r+1}, \dots, M_m)$ is a continuous, translation invariant valuation, homogeneous of degree r, for each $r \in \{1, \dots, m\}$ and fixed convex bodies $M_{r+1}, \dots, M_m$.

To deduce this from Theorem 6.3.4 by polytopal approximation, one can argue similarly to the proof of Theorem 5.1.7: that is, make use of a polarization formula analogous to Lemma 5.1.4.

Again we call $\overline{\varphi}$ the *mixed valuation induced by* φ.

Note for Section 6.3

1. *Polynomial expansion of translation invariant valuations.* The polynomiality result of Theorems 6.3.1, 6.3.4 has an interesting history. As early as 1945, Hadwiger [888] stated essentially the result of Theorem 6.3.4, in a very short note and without proof. In his later work, a proof only appears for simple valuations (see [911], p. 54). At an Oberwolfach conference in 1974, McMullen explicitly asked the following question on translation invariant, continuous valuations φ on $\mathcal{K}^n$ (see Problem 48 in [841]): 'Do they behave like mixed volumes, in the sense that they satisfy polynomial relations $\varphi(\lambda_1 P_1 + \cdots + \lambda_r \varphi_r P_r) = \sum_{\alpha_1,\ldots,\alpha_r \geq 0} \lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r} \varphi_{\alpha_1,\ldots,\alpha_r}(P_1, \ldots, P_r)$ where the functions $\varphi_{\alpha_1,\ldots,\alpha_r}(P_1, \ldots, P_r)$ are independent of $\lambda_1, \ldots, \lambda_r$?' He gave an affirmative answer in the same year; see McMullen [1380, 1383]. His proof uses the bilinear angle sum relations established before by him and requires, therefore, that the valuations take their values in a real vector space. The proof of Theorem 6.3.1 given above is essentially due to Spiegel [1803], who, however, used closed polytopes and hence worked with the inclusion–exclusion principle. We have chosen this proof here since it is particularly natural and also more in the spirit of Hadwiger's book [911], where it could well have had its place. A further proof was given by Meier [1399], who, more generally, considered decompositions of Minkowski sums of polytopes and introduced mixed polyhedra. (The interested reader should compare McMullen [1390], §15.)

6.4 Translation invariant, continuous valuations

Many of the geometrically significant valuations on the space of convex bodies are translation invariant and continuous. The present section is devoted to a study of such valuations. Under various additional assumptions, they can be classified, and corresponding characterization theorems can be proved. In this section (with the exception of the Notes), all valuations have their values in $\mathbb{R}$.

A valuation φ on $\mathcal{K}^n$ or $\mathcal{P}^n$ is called *simple* if $\varphi(K) = 0$ whenever $\dim K < n$. A useful tool for the investigation of translation invariant, simple valuations is the dissection theory of polytopes. A *dissection* of the polytope P is a set $\{P_1, \ldots, P_m\}$ of polytopes such that $P = \bigcup_{i=1}^m P_i$ and $\dim (P_i \cap P_j) < n$ for $i \neq j$. Two polytopes $P, Q \in \mathcal{P}^n$ are called T-*equidissectable* if there are dissections $\{P_1, \ldots, P_m\}$ of P and $\{Q_1, \ldots, Q_m\}$ of Q such that Q_i is a translate of P_i for $i = 1, \ldots, m$. Obviously, T-equidissectability is an equivalence relation.

Lemma 6.4.1 *If φ is a translation invariant, simple valuation on $\mathcal{P}^n$ and the polytopes $P, Q \in \mathcal{P}^n$ are* T-*equidissectable, then $\varphi(P) = \varphi(Q)$.*

Proof Let $\{P_1, \ldots, P_m\}$ be a dissection of P such that $\{P_1 + t_1, \ldots, P_m + t_m\}$, with suitable translation vectors $t_1, \ldots, t_m$, is a dissection of Q. By Corollary 6.2.4, the valuation φ has an additive extension to $U(\mathcal{P}^n)$. The inclusion–exclusion principle (6.2) immediately gives

$$\begin{aligned}
\varphi(P) &= \varphi(P_1 \cup \cdots \cup P_m) = \varphi(P_1) + \cdots + \varphi(P_m) \\
&= \varphi(P_1 + t_1) + \cdots + \varphi(P_m + t_m) = \varphi((P_1 + t_1) \cup \cdots \cup (P_m + t_m)) \\
&= \varphi(Q),
\end{aligned}$$

as asserted. $\square$

The following basic result of dissection theory was discovered by Hadwiger [900]. His proof, which we follow, is also found in his book [911], Section 1.3.3.

Lemma 6.4.2 *Any two parallelotopes of equal volume in $\mathcal{P}^n_n$ are* T*-equidissectable.*

Proof We need only consider parallelotopes $P \in \mathcal{P}^n_n$ with one vertex at the origin, and such a parallelotope can be represented in the form

$$P = \langle a_1, \dots, a_n \rangle := \left\{ \sum_{i=1}^{n} \alpha_i a_i : 0 \le \alpha_i \le 1,\ i = 1, \dots, n \right\}$$

with linearly independent vectors $a_1, \dots, a_n$. Transforming the parallelotope $P = \langle a_1, \dots, a_n \rangle$ into the parallelotope $Q = \langle a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, a_n \rangle$ with $i \ne j$ and $\lambda \in \mathbb{R}$ is called a *shearing*, and P and Q are said to be *equivalent by shearing*, written as $P \approx_S Q$. They have the same volume. Clearly, $\approx_S$ is an equivalence relation. Similarly, we write $P \approx_T Q$ if P and Q are T-equidissectable.

Proposition If $P \approx_S Q$, then $P \approx_T Q$.

Suppose that $P \approx_S Q$. In one dimension, there is nothing to prove. For the two-dimensional case, let $P = \langle a_1, a_2 \rangle$ and, say, $Q = \langle a_1, a_2 + \lambda a_1 \rangle$. If $|\lambda| \le 1$, the relation $P \approx_T Q$ is obvious. If $|\lambda| > 1$, we choose $k \in \mathbb{N}$ with $\lambda = k\mu$ and $|\mu| \le 1$, put $Q_j := \langle a_1, a_2 + j\mu a_1 \rangle$ and have $P \approx_T Q_1 \approx_T Q_2 \approx_T \cdots \approx_T Q_k = Q$. If $n > 2$, we observe that, for example, $\langle a_1, a_2, a_3, \dots, a_n \rangle = \langle a_1, a_2 \rangle \oplus \langle a_3, \dots, a_n \rangle$ and $\langle a_1, a_2 + \lambda a_1, a_3, \dots, a_n \rangle = \langle a_1, a_2 + \lambda a_1 \rangle \oplus \langle a_3, \dots, a_n \rangle$. The assertion of the proposition now follows from the two-dimensional case.

For the proof of the lemma, we assume that two given parallelotopes $P = \langle a_1, \dots, a_n \rangle$ and $Q = \langle b_1, \dots, b_n \rangle$ have the same positive volume. Without loss of generality (because this can be achieved by changing the numeration of the vectors $a_1, \dots, a_n$) we assume that $\operatorname{span}\{a_1, \dots, a_{n-1}\} \ne \operatorname{span}\{b_1, \dots, b_{n-1}\}$. Under this assumption, a simple continuity argument shows the existence of a unit vector w such that for the projections $\overline{a}_i = a_i \,|\, w^\perp$ and $\overline{b}_i = b_i \,|\, w^\perp$ we have $V_{n-1}(\langle \overline{a}_1, \dots, \overline{a}_{n-1} \rangle) = V_{n-1}(\langle \overline{b}_1, \dots, \overline{b}_{n-1} \rangle) > 0$. We can write $P = \langle \overline{a}_1 + \alpha_1 w, \dots, \overline{a}_n + \alpha_n w \rangle$ with suitable $\alpha_1, \dots, \alpha_n$. Since $\overline{a}_1, \dots, \overline{a}_{n-1}$ are linearly independent, there is a representation $-\overline{a}_n = \sum_{i=1}^{n-1} \lambda_i \overline{a}_i$. By $n-1$ shearings we can transform the parallelotope P into the parallelotope $P_1 = \langle \overline{a}_1 + \alpha_1 w, \dots, \overline{a}_{n-1} + \alpha_{n-1} w, \xi w \rangle$ with $\xi = \alpha_n + \sum_{i=1}^{n-1} \lambda_i \alpha_i$. Here $\xi \ne 0$, since P_1 has positive volume. By another $n-1$ shearings we can transform P_1 into the parallelotope $P_2 = \langle \overline{a}_1, \dots, \overline{a}_{n-1}, \xi w \rangle$. We have $P_2 \approx_S P$ and hence $V_n(P_2) = V_n(P)$ and $P_2 \approx_T P$, the latter by the proposition. Similarly, there is a parallelotope $Q_2 = \langle \overline{b}_1, \dots, \overline{b}_{n-1}, \eta w \rangle$ with $Q_2 \approx_S Q$ and hence $V_n(Q_2) = V_n(Q)$ and $Q_2 \approx_T Q$. From $V_n(P_2) = V_n(Q_2)$ and $V_{n-1}(\langle \overline{a}_1, \dots, \overline{a}_{n-1} \rangle) = V_{n-1}(\langle \overline{b}_1, \dots, \overline{b}_{n-1} \rangle)$ it follows that $|\xi| = |\eta|$, say $\xi = \eta$ (since this can be achieved by replacing P by $-P$). By the induction hypothesis, the parallelotopes $\langle \overline{a}_1, \dots, \overline{a}_{n-1} \rangle$ and $\langle \overline{b}_1, \dots, \overline{b}_{n-1} \rangle$, which have the same $(n-1)$-volume, are T-equidissectable in $w^\perp$. Since $P_2 = \langle \overline{a}_1, \dots, \overline{a}_{n-1} \rangle \oplus \xi[o, w]$ and $Q_2 = \langle \overline{b}_1, \dots, \overline{b}_{n-1} \rangle \oplus \xi[o, w]$, the parallelotopes P_2 and Q_2 are T-equidissectable in

$\mathbb{R}^n$. Therefore, $P \approx_T Q$, which completes the induction and thus the proof of the lemma. □

Using this, we can derive the following result of Hadwiger [911], p. 79.

Theorem 6.4.3 (Hadwiger) *Let φ be a translation invariant valuation on $\mathcal{P}^n$. If φ is homogeneous of degree n, then $\varphi = cV_n$ with a real constant c.*

Proof By Corollary 6.3.2(a), the valuation φ vanishes on polytopes of dimension less than n, thus it is simple. Let c be its value at a fixed unit cube C, then $\varphi(C) = cV_n(C)$ and hence $\varphi(\lambda C) = cV_n(\lambda C)$ for $\lambda \geq 0$, by homogeneity of degree n. By $\psi := \varphi - cV_n$ we define a valuation on $\mathcal{P}^n$ which is also translation invariant and homogeneous of degree n. It satisfies $\psi(\lambda C) = 0$ for all $\lambda \geq 0$, hence by Lemmas 6.4.1 and 6.4.2 it vanishes at all parallelotopes. By Corollary 6.3.2(b), ψ vanishes identically. This completes the proof. □

We prove another classical result of Hadwiger, concerning valuations on polytopes with a weak continuity property. For $P \in \mathcal{P}^n$, we denote by $N(P)$ the set of outer unit normal vectors of the facets of the polytope P, and if P' is a polytope in an $(n-1)$-dimensional subspace H, then $N'(P')$ has the same meaning for P' with respect to the subspace H. For an affine subspace $A \subset \mathbb{R}^n$, we denote by $\mathcal{P}(A)$ the set of all polytopes whose affine hull is contained in a translate of A.

For functions on $\mathcal{P}^n$, the following weak continuity property is often appropriate. Let $U := \{u_1, \dots, u_m\}$ be a finite set of unit vectors positively spanning $\mathbb{R}^n$. We say that a function φ on $\mathcal{P}^n$ is *weakly continuous* if, for any such U, the function

$$(\eta_1, \dots, \eta_m) \mapsto \varphi(\{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq \eta_i,\ i = 1, \dots, m\})$$

is continuous on the set of $(\eta_1, \dots, \eta_m)$ for which the argument of φ is not empty. Intuitively speaking, φ is weakly continuous if it is continuous under parallel displacements of the facets of a polytope. If φ is continuous, then it is obviously weakly continuous.

Let **SV** denote the set of all simple valuations $\varphi : \mathcal{P}^n \to \mathbb{R}$ that are translation invariant and weakly continuous. For an affine subspace A, let $\mathbf{SV}(A)$ be the set of all translation invariant valuations on $\mathcal{P}(A)$ that are simple and weakly continuous within each translate of A. The following theorem is due to Hadwiger [903].

Theorem 6.4.4 (Hadwiger) *Each $\varphi \in \mathbf{SV}$ has a representation*

$$\varphi(P) = cV_n(P) + \sum_{u \in N(P)} g(u)\, \varphi'_u(F(P,u)), \qquad P \in \mathcal{P}^n, \tag{6.18}$$

with a constant c, an odd function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ and $\varphi'_u = \varphi'_{-u} \in \mathbf{SV}(u^\perp)$.

Let $u_0 \in \mathbb{S}^{n-1}$. For a polytope $P' \subset u_0^\perp$ and for $h \geq 0$, we define the cylinder $[P', h] := P' + \{\lambda u_0 : 0 \leq \lambda \leq h\}$.

Lemma 6.4.5 *To each $\varphi' \in \mathbf{SV}(u_0^\perp)$ there exists $\varphi \in \mathbf{SV}$ such that*

$$\varphi([P', h]) = h\varphi'(P') \tag{6.19}$$

for all polytopes $P' \subset u_0^\perp$ and all $h \geq 0$.

We prove the theorem and the lemma simultaneously by induction with respect to the dimension.

Proof of Theorem 6.4.4 and Lemma 6.4.5 Let $n = 1$. Then $P \in \mathcal{P}^1$ is a closed segment and $\varphi(P)$ depends only on the length α of P, by the translation invariance, thus $\varphi(P) = f(\alpha)$. By simple additivity, the function f satisfies Cauchy's functional equation, $f(\alpha+\beta) = f(\alpha)+f(\beta)$, and the continuity gives $f(\alpha) = c\alpha$ (see, e.g., Aczél [4]), thus $\varphi(P) = cV_1(P)$, with a constant c. This is the assertion of Theorem 6.4.4 for $n = 1$. The assertion of Lemma 6.4.5 is clear for $n = 1$, since we have $\varphi' = c$ and can set $\varphi([P', h]) = ch$.

Assume now that $n \geq 2$ and that the theorem and the lemma have been proved in smaller dimensions. Let u_0 and $\mathbf{SV}(u_0^\perp)$ be as above. Let $\varphi \in \mathbf{SV}$. We define $\varphi'(P') := \varphi([P', 1])$ for polytopes $P' \subset u_0^\perp$. Then $\varphi' \in \mathbf{SV}(u_0^\perp)$. For a fixed polytope P' in $u_0^\perp$ and for arbitrary $\alpha \geq 0$, let $\varphi([P', \alpha]) =: f(\alpha)$. Similarly as above, one shows that $f(\alpha) = f(1)\alpha$ and hence $\varphi([P', h]) = h\varphi([P', 1]) = h\varphi'(P')$.

By the inductive hypothesis we have, for polytopes $P' \subset u_0^\perp$,

$$\varphi'(P') = cV_{n-1}(P') + \sum_{u' \in N'(P')} g'(u')\, \varphi''_{u'}(F(P', u'))$$

with a constant c, an odd function g' on $\mathbb{S}^{n-1} \cap u_0^\perp$ and with $\varphi''_{u'} \in \mathbf{SV}(u_0^\perp \cap u'^\perp)$, hence, by the previous relation,

$$\varphi([P', h]) = cV_{n-1}(P')h + \sum_{u' \in N'(P')} g'(u')\, \varphi''_{u'}(F(P', u'))h.$$

Since, by the inductive hypothesis, Lemma 6.4.5 holds in dimension $n-1$, there is a valuation $\varphi'_{u'} \in \mathbf{SV}(u'^\perp)$ such that $\varphi'_{u'}([P'', h]) = h\varphi''_{u'}(P'')$ for all cylinders $[P'', h]$ with polytopes P'' in $u_0^\perp \cap u'^\perp$. This gives

$$\varphi([P', h]) = cV_n([P', h]) + \sum_{u' \in N'(P')} g'(u')\, \varphi'_{u'}([F(P', u'), h]).$$

Now we define

$$\Phi(P) := cV_n(P) + \sum_{u \in N(P)} g(u)\, \Phi'_u(F(P, u)), \qquad P \in \mathcal{P}^n, \tag{6.20}$$

with

$$(g(u), \Phi'_u(F(P, u))) := \begin{cases} (g'(u), \varphi'_u(F(P, u))) & \text{if } \langle u, u_0 \rangle = 0, \\ (0, 0) & \text{if } \langle u, u_0 \rangle \neq 0. \end{cases}$$

Then $\Phi \in \mathbf{SV}$. Since $F([P', h], u') = [F(P', u'), h]$ for $\langle u', u_0 \rangle = 0$, we find that $\varphi([P', h]) = \Phi([P', h])$. The valuation ψ defined by

$$\psi(P) := \varphi(P) - \Phi(P)$$

belongs to $\mathbf{SV}$ and vanishes on all cylinders $[P', h]$ with $P' \subset u_0^\perp$.

Let $u \in \mathbb{S}^{n-1}$ be a vector with $\langle u, u_0 \rangle > 0$. Let $Q \in \mathcal{P}(u^\perp)$, and without loss of generality $Q \subset \{x \in \mathbb{R}^n : \langle x, u_0 \rangle > 0\}$. Let P' be the orthogonal projection of Q to $u_0^\perp$, and let $S := \mathrm{conv}\,(Q \cup P')$. We call each such S a *skew cylinder*. The value $\psi(S)$ depends only on Q, since ψ is translation invariant and vanishes on cylinders of the form $[P', h]$. Therefore, we can define $\psi'_u(Q) := \psi(S)$, and it is easy to see that this defines a valuation $\psi'_u \in \mathbf{SV}(u^\perp)$. Now we define

$$\Psi(P) := \sum_{u \in N(P)} f(u)\, \psi'_u(F(P, u)),$$

where

$$(f(u), \psi'_u(F(P, u))) := \begin{cases} (1, \psi'_u(F(P, u))) & \text{if } \langle u, u_0 \rangle > 0, \\ (-1, \psi'_{-u}(F(P, u))) & \text{if } \langle u, u_0 \rangle < 0, \\ (0, 0) & \text{if } \langle u, u_0 \rangle = 0. \end{cases}$$

It is easy to check that $\Psi \in \mathbf{SV}$ and that for each skew cylinder S we have $\psi(S) = \Psi(S)$. Let $P \in \mathcal{P}^n$, and without loss of generality $P \subset \{x \in \mathbb{R}^n : \langle x, u_0 \rangle > 0\}$. Then there are skew cylinders $S_1, \dots, S_m, T_1, \dots, T_k$ such that $P \cup \bigcup_{i=1}^m S_i = \bigcup_{j=1}^k T_j$ and $\dim\,(P \cap S_i) < n$, $\dim\,(S_i \cap S_j) < n$, $\dim\,(T_i \cap T_j) < n$ for $i \neq j$. From simple additivity (and the extension theorem 6.2.3) it follows that $\psi(P) = \Psi(P)$. This gives $\varphi(P) = \Phi(P) + \Psi(P)$, which yields the required representation.

Now we complete the induction step for the lemma. Let $\varphi' \in \mathbf{SV}(u_0^\perp)$ be given. By the inductive hypothesis, Theorem 6.4.4 holds in dimension $n - 1$, hence there is a corresponding representation

$$\varphi'(P') = cV_{n-1}(P') + \sum_{u' \in N'(P')} g'(u')\, \varphi''_{u'}(F(P', u'))$$

for polytopes $P' \subset u_0^\perp$, thus

$$h\varphi'(P') = cV_{n-1}(P')h + \sum_{u' \in N'(P')} g'(u')\, \varphi''_{u'}(F(P', u'))h.$$

By the induction hypothesis, Lemma 6.4.5 holds in dimension $n - 1$, hence for given $u' \in u_0^\perp$ there exists $\varphi'_{u'} \in \mathbf{SV}(u'^\perp)$ such that $\varphi'_{u'}([F(P', u'), h]) = \varphi''_{u'}(F(P', u'))h$ for all cylinders $[F(P', u'), h]$. Since $V_{n-1}(P')h = V_n([P', h])$ and $[F(P', u'), h] = F([P', h], u')$, we get

$$h\varphi'(P') = cV_n([P', h]) + \sum_{u' \in N'(P')} g'(u')\, \varphi'_{u'}(F([P', h], u')).$$

We define

$$\varphi(P) := cV_n(P) + \sum_{u \in N(P)} g(u)\, \varphi'_u(F(P,u))$$

with

$$(g(u), \varphi'_u(F(P,u))) := \begin{cases} (g'(u), \varphi'_u(F(P,u))) & \text{if } \langle u, u_0 \rangle = 0, \\ (0,0) & \text{if } \langle u, u_0 \rangle \neq 0. \end{cases}$$

Then we have $\varphi \in \mathbf{SV}$ and $\varphi([P', h]) = h\varphi'(P')$. This completes the induction. □

Theorem 6.4.4 gives a recursive representation for the valuations of $\mathbf{SV}$. From this, an explicit representation can be obtained. Let $\mathcal{U}^s$ be the Stiefel manifold of all s-frames, that is, ordered s-tuples of orthonormal vectors, in $\mathbb{R}^n$, $s = 1, \dots, n$. For $U = (u_1, \dots, u_s) \in \mathcal{U}^s$ and a polytope $P \in \mathcal{P}^n$, the face P_U is defined inductively by $P_{u_1} := F(P, u_1)$ and $P_{(u_1,\dots,u_k)} := F(P_{(u_1,\dots,u_{k-1})}, u_k)$, $k = 2, \dots, s$. We say that the s-frame $(u_1, \dots, u_s) \in \mathcal{U}^s$ is *P-tight* if $\dim P_{(u_1,\dots,u_r)} = n - r$ holds for $r = 1, \dots, s$. By $\mathcal{U}^s(P) \subset \mathcal{U}^s$ we denote the set of P-tight s-frames. Clearly, this is a finite set. A function $\eta : \mathcal{U}^s \to \mathbb{R}^n$ is called *odd* if $\eta(\epsilon_1 u_1, \dots, \epsilon_s u_s) = \epsilon_1 \cdots \epsilon_s \eta(u_1, \dots, u_s)$ for $\epsilon_i = \pm 1$.

Theorem 6.4.6 *A function $\varphi : \mathcal{P}^n \to \mathbb{R}$ is a weakly continuous, translation invariant, simple valuation if and only if there are a constant c and odd functions η_s : $\mathcal{U}^s \to \mathbb{R}$, $s = 1, \dots, n$, such that*

$$\varphi(P) = cV_n(P) + \sum_{s=1}^{n} \sum_{U \in \mathcal{U}^s(P)} \eta_s(U) V_{n-s}(P_U), \qquad P \in \mathcal{P}^n. \tag{6.21}$$

(*The term with $s = n$ vanishes if $n \geq 1$.*)

Proof First, let c and $\eta_1, \dots, \eta_n$ be as in the theorem, and define φ by (6.21). Let $\dim P < n$. Then $V_n(P) = 0$, and if there exists some $U = (u_1, \dots, u_s) \in \mathcal{U}^s(P)$, then $\dim P = n - 1$, $U' := (-u_1, u_2, \dots, u_s) \in \mathcal{U}^s(P)$ and $P_U = P_{U'}$, hence $\eta_s(U)V_{n-s}(P_U) + \eta_s(U')V_{n-s}(P_{U'}) = 0$. It follows that $\varphi(P) = 0$. With similar arguments, one sees that the function φ is weakly additive, hence by Theorem 6.2.3 it is additive. It is not difficult to check that φ is weakly continuous. Clearly, φ is translation invariant.

The term with $s = n$ in (6.21) vanishes if $n \geq 1$ since $(u_1, \dots, u_n) \in \mathcal{U}^n(P)$ implies $(u_1, \dots, u_{n-1}, -u_n) \in \mathcal{U}^n(P)$.

Conversely, let $\varphi : \mathcal{P}^n \to \mathbb{R}$ be a weakly continuous, translation invariant, simple valuation. We prove the existence of the representation (6.21) by induction with respect to the dimension, using Theorem 6.4.4. The case $n = 1$ is clear. Suppose that the assertion has been proved in dimension $n - 1$. By Theorem 6.4.4, there are a constant c, an odd function $g : \mathbb{S}^{n-1} \to \mathbb{R}^n$ and for each $u \in \mathbb{S}^{n-1}$ a function $\varphi'_u = \varphi'_{-u} \in \mathbf{SV}(u^\perp)$ such that

$$\varphi(P) = cV_n(P) + \sum_{u\in N(P)} g(u)\varphi'_u(F(P,u)) \qquad \text{for } P \in \mathcal{P}^n.$$

By the induction hypothesis, for each $u \in \mathbb{S}^{n-1}$ there are a constant c_u and odd functions $\eta^u_r : \mathcal{U}^r \cap (u^\perp)^r \to \mathbb{R}$, $r = 1, \dots, n-1$, such that, for $P \in \mathcal{P}^n$,

$$\varphi'_u(F(P,u)) = c_u V_{n-1}(F(P,u)) + \sum_{r=2}^{n} \sum_{(u,U)\in\mathcal{U}^r(F(P,u))} \eta^u_{r-1}(U) V_{n-r}(F(P,u)_U).$$

We define $\eta_1(u) := g(u)c_u$ and $\eta_r(u, u_1, \dots, u_{r-1}) := g(u)\eta^u_{r-1}(u_1, \dots, u_{r-1})$ for $r = 2, \dots, n$. Then

$$\begin{aligned}\varphi(P) &= cV_n(P) + \sum_{u\in N(P)} g(u)c_u V_{n-1}(F(P,u)) \\ &\quad + \sum_{u\in N(P)} \sum_{r=2}^{n} \sum_{(u,U)\in\mathcal{U}^r(F(P,u))} g(u)\eta^u_{r-1}(U)V_{n-r}(F(P,u)_U) \\ &= cV_n(P) + \sum_{U\in\mathcal{U}^1(P)} \eta_1(U)V_{n-1}(P_U) + \sum_{r=2}^{n} \sum_{U\in\mathcal{U}^r(P)} \eta_r(U)V_{n-r}(P_U).\end{aligned}$$

This is the required representation in dimension n. □

Theorem 6.4.6 was extended by McMullen [1386] to non-simple valuations. We quote his result and refer to the original paper for the proof.

Theorem 6.4.7 (McMullen) *A function $\varphi : \mathcal{P}^n \to \mathbb{R}$ is a weakly continuous, translation invariant valuation if and only if*

$$\varphi(P) = \sum_{r=0}^{n} \sum_{F\in\mathcal{F}_r(P)} V_r(F)\,\theta_r(N(P,F)) \qquad \textit{for } P \in \mathcal{P}^n,$$

where θ_r is a simple real valuation on the system of all closed polyhedral convex cones of dimension at most $n - r$.

Here, as well as in Theorems 6.4.4 and 6.4.6, the target space $\mathbb{R}$ of the valuation φ may be replaced by any real topological vector space.

From now on, we consider only continuous valuations on $\mathcal{K}^n$.

Theorem 6.4.8 *Let φ be a continuous, translation invariant valuation on $\mathcal{K}^n$. If φ is homogeneous of degree n, then $\varphi = cV_n$ with a real constant c.*

Proof This follows from Theorem 6.4.3 and continuity.

Without using Theorem 6.4.3 (and thus Lemma 6.4.2), we may conclude as follows. The n-homogeneous valuation φ is simple, by Corollary 6.3.2(a). On polytopes, the continuous valuation φ is weakly continuous, hence Theorem 6.4.6 gives $\varphi(P) = cV_n(P)$ for $P \in \mathcal{P}^n$, with a constant c. The continuity of φ gives the assertion. □

By **Val** (or **Val** $(\mathbb{R}^n)$, if necessary) we denote the real vector space of translation invariant, continuous real valuations on $\mathcal{K}^n$. Denoting by $\mathbf{Val}_m$ the subspace of valuations which are homogeneous of degree m, we have from Theorem 6.3.5 the McMullen decomposition

$$\mathbf{Val} = \bigoplus_{m=0}^{n} \mathbf{Val}_m. \tag{6.22}$$

A further decomposition is possible. A valuation φ on $\mathcal{K}^n$ or $\mathcal{P}^n$ is called *even* (*odd*) if $\varphi(-K) = \varphi(K)$ (respectively, $\varphi(-K) = -\varphi(K)$) for all K in the domain of φ. Obviously, every valuation φ can uniquely be written as the sum of an even and an odd valuation, with the same continuity and invariance properties as φ. Thus, $\mathbf{Val}_m = \mathbf{Val}_m^+ \oplus \mathbf{Val}_m^-$, where $\mathbf{Val}_m^+$ ($\mathbf{Val}_m^-$) is the vector space of even (respectively, odd) translation invariant, continuous real valuations on $\mathcal{K}^n$ that are homogeneous of degree m.

We note that the case $\lambda = 0$ of Theorem 6.3.5 gives $\varphi(\{o\}) = \varphi_0(K)$, hence φ_0 is constant, or in other words the space $\mathbf{Val}_0$ is spanned by the Euler characteristic. By Theorem 6.4.8, the space $\mathbf{Val}_n$ is spanned by the volume functional. The space $\mathbf{Val}_{n-1}$ is of infinite dimension, but its elements can still be described explicitly. The following result and its proof are due to McMullen [1385].

Theorem 6.4.9 (McMullen) *Let $\varphi \in \mathbf{Val}_{n-1}$. There exists a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u) \qquad \text{for } K \in \mathcal{K}^n.$$

The function f is uniquely determined up to adding the restriction of a linear function.

Proof From Theorem 6.4.8, applied in $(n-1)$-dimensional subspaces, it follows that for $u \in \mathbb{S}^{n-1}$ and polytopes $Q \subset u^\perp$ we have $\varphi(Q) = c(u)V_{n-1}(Q)$, with a number $c(u)$, where $c(u) = c(-u)$. We define

$$\psi(P) := \varphi(P) - \frac{1}{2} \sum_{u \in N(P)} \varphi(F(P,u)) = \varphi(P) - \frac{1}{2} \sum_{u \in N(P)} c(u)V_{n-1}(F(P,u))$$

for $P \in \mathcal{P}^n$. Then ψ is a valuation on $\mathcal{P}^n$, as is easy to check. It is translation invariant and vanishes at polytopes of dimension $n-1$, by definition, and also at polytopes of smaller dimension, by Corollary 6.3.2(a). Thus, ψ is simple. From the continuity of φ we obtain that ψ is weakly continuous. Since it is homogeneous of degree $n-1$, we deduce from Theorem 6.4.6 that it has a representation

$$\psi(P) = \sum_{u \in N(P)} g(u)V_{n-1}(F(P,u))$$

with an odd function $g : \mathbb{S}^{n-1} \to \mathbb{R}$. For the valuation φ this yields a representation of the form

$$\varphi(P) = \sum_{u \in N(P)} f(u) V_{n-1}(F(P, u)), \qquad P \in \mathcal{P}^n, \tag{6.23}$$

with a function $f : \mathbb{S}^{n-1} \to \mathbb{R}$. We show that any such function f is continuous. For the proof, let $(v_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{S}^{n-1}$ converging to u_0. Let T be a simplex circumscribed to the unit ball B^n and with outer unit normal vectors $u_0, u_1, \dots, u_n$. For sufficiently large j, the vectors $v_j, u_1, \dots, u_n$ are the unit normal vectors of a simplex T_j circumscribed to B^n. We have $T_j \to T$ and hence $\varphi(T_j) \to \varphi(T)$ for $j \to \infty$, thus

$$\begin{aligned} &f(v_j) V_{n-1}(F(T_j, v_j)) + \sum_{i=1}^n f(u_i) V_{n-1}(F(T_j, u_i)) \\ \to &f(u_0) V_{n-1}(F(T, u_0)) + \sum_{i=1}^n f(u_i) V_{n-1}(F(T, u_i)). \end{aligned}$$

Now $F(T_j, u_i) \to F(T, u_i)$ for $i = 1, \dots, n$ and $F(T_j, v_j) \to F(T, u_0)$, and since $V_{n-1}(F(T, u_0)) > 0$, it follows that $f(v_j) \to f(u_0)$. Thus, f is continuous.

Equation (6.23) can be written in the form

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u) \tag{6.24}$$

for polytopes $K \in \mathcal{P}^n$. By the continuity of φ and the weak continuity of the surface area measure S_{n-1}, this equation extends to arbitrary convex bodies K, by approximation. This is the asserted representation.

Now let f_1, f_2 be two continuous functions, both yielding the representation (6.24) for φ. Then the continuous function $f := f_1 - f_2$ satisfies

$$\int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u) = 0 \tag{6.25}$$

for all convex bodies K. Let $(e_1, \dots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$. Choosing for K an $(n-1)$-dimensional convex body in $e_i^\perp$, we see that $f(-e_i) = -f(e_i)$. Let $u = \sum_{i=1}^n \eta_i e_i \in \mathbb{S}^{n-1}$ be a vector with $\eta_i \neq 0$ for $i = 1, \dots, n$. If we set $e_i' := -(\operatorname{sgn} \eta_i) e_i$, then there is an n-simplex T with outer unit normal vectors $u, e_1', \dots, e_n'$. Let $A_0 := V_{n-1}(F(T, u))$ and $A_i := V_{n-1}(F(T, e_i'))$. From (5.1) we have

$$A_0 u + \sum_{i=1}^n A_i e_i' = o,$$

hence scalar multiplication with the vector $a := \sum_{j=1}^n f(e_j) e_j$ gives

$$0 = A_0 \langle a, u \rangle + \sum_{i=1}^n A_i (-(\operatorname{sgn} \eta_i)) f(e_i) = A_0 \langle a, u \rangle + \sum_{i=1}^n A_i f(e_i').$$

From (6.25)) we get

$$A_0 f(u) + \sum_{i=1}^{n} A_i f(e_i') = 0.$$

Comparison gives $f(u) = \langle a, u\rangle$, and by continuity this holds for all $u \in \mathbb{S}^{n-1}$. Thus, $f_1 = f_2 + \langle a, \cdot\rangle$. This proves the final statement of the theorem. □

Another subclass of **Val** with a complete classification, besides the valuations with homogeneity degrees n, $n-1$ or 0, are the simple valuations. Since every valuation is the sum of an even and an odd valuation, we may consider even and odd valuations separately. First we treat the even case. The following 'volume characterization theorem' and its proof are due to Klain [1081].

Theorem 6.4.10 (Klain) *Let $\varphi \in$ **Val**$^+$ be simple. Then $\varphi(K) = cV_n(K)$ for all $K \in \mathcal{K}^n$, with a real constant c.*

Proof Let φ satisfy the assumptions, let $C \subset \mathbb{R}^n$ be a fixed unit cube and put $c := \varphi(C)$ and $\psi(K) := \varphi(K) - cV_n(K)$ for $K \in \mathcal{K}^n$. Then ψ has the same properties as φ and vanishes at C. The assertion of the theorem follows, therefore, from the following proposition.

Proposition If φ satisfies the assumptions of the theorem and vanishes at a fixed unit cube, then $\varphi = 0$.

Let φ satisfy all assumptions of the proposition. Then φ vanishes at any unit cube, by Lemmas 6.4.2 and 6.4.1. We prove the proposition by induction with respect to the dimension. If $n = 1$, then φ vanishes at singletons and at unit segments, hence also at segments of rational length and by continuity at all one-dimensional convex bodies. Let $n > 1$ and assume that the proposition has been proved in smaller dimensions. Let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and $I = [o, s]$ with a unit vector s orthogonal to H. The function φ' defined by $\varphi'(K) := \varphi(K + I)$ for convex bodies $K \subset H$ has all the properties of the proposition relative to H and vanishes at unit cubes in H. By the inductive hypothesis, $\varphi' = 0$ and thus $\varphi(K + I) = 0$ for $K \subset H$. Arguing similarly as in the case $n = 1$, we see that $\varphi(K + S) = 0$ for any closed segment S orthogonal to H. Since H was arbitrary, φ vanishes at all right convex cylinders. With H and K as before, let $S = [o, s]$, where s is any vector not parallel to H. For a sufficiently large integer $k > 0$, the cylinder $Z := K + kS$ can be cut by a hyperplane H' orthogonal to s and such that the two closed halfspaces H^-, H^+ bounded by H' satisfy $K \subset H^-$ and $K + ks \subset H^+$. Then $\overline{Z} := [(Z \cap H^-) + ks] \cup (Z \cap H^+)$ is a right cylinder, and we obtain $k\varphi(K + S) = \varphi(Z) = \varphi(\overline{Z}) = 0$. Thus, φ vanishes at arbitrary convex cylinders.

Let $P \in \mathcal{P}^n$ and S be a segment. The Minkowski sum $P + S$ has a dissection $\{P, C_1, \dots, C_m\}$ where each C_i is a convex cylinder. Since φ is simple and vanishes at convex cylinders, we deduce (using the extension theorem of Corollary 6.2.4) that $\varphi(P + S) = \varphi(P)$. By induction, we conclude that $\varphi(P + Z) = \varphi(P)$ if Z is a zonotope, and approximation together with continuity yields $\varphi(K + Z) = \varphi(K)$ for arbitrary $K \in \mathcal{K}^n$ and zonoids Z. If K is a generalized zonoid, there are zonoids Z_1, Z_2 with $K + Z_1 = Z_2$, and it follows that $\varphi(K) = \varphi(K + Z_1) = \varphi(Z_2) = 0$. Since the generalized zonoids are dense in the set of all centrally symmetric convex bodies (Corollary 3.5.7), we deduce that $\varphi(K) = 0$ for all centrally symmetric convex bodies K.

Let $T = \operatorname{conv}\{o, v_1, \dots, v_n\}$ be an n-simplex. Setting $v := v_1 + \cdots + v_n$ and $T' := \operatorname{conv}\{v, v - v_1, \dots, v - v_n\}$, we have $T' = -T + v$. The parallelotope $P := [o, v_1] + \cdots + [o, v_n]$ has a dissection into T, T' and a centrally symmetric polytope Q that lies between the hyperplanes spanned by $v_1, \dots, v_n$ and $v - v_1, \dots, v - v_n$, respectively. We deduce that $0 = \varphi(P) = \varphi(T) + \varphi(Q) + \varphi(T') = \varphi(T) + \varphi(T')$. Since φ is even, also $\varphi(T) = \varphi(T')$, hence $\varphi(T) = 0$. By dissecting a polytope P into simplices, we obtain $\varphi(P) = 0$, and by continuity we conclude that $\varphi(K) = 0$ for all convex bodies K. This completes the inductive proof of the proposition and thus the proof of the theorem. □

Before continuing with simple valuations, we note a useful consequence of Theorem 6.4.10. Let $m \in \{1, \dots, n-1\}$, and let $\varphi \in \mathbf{Val}_m$. The restriction of φ to the convex bodies in a fixed subspace $L \in G(n, m)$ is a constant multiple of the m-dimensional volume, by Theorem 6.4.8. Hence, there exists a constant $c_\varphi(L)$ with $\varphi(K) = c_\varphi(L) V_m(K)$ for $K \in \mathcal{K}(L)$. Since φ is continuous, this defines a continuous function c_φ on the Grassmannian $G(n, m)$. It is called the *Klain function* of φ. Evidently, $\mathrm{K}_m : \varphi \mapsto c_\varphi$ defines a linear mapping $\mathbf{Val}_m \to C(G(n, m))$. The following theorem shows that the restriction of this mapping to $\mathbf{Val}_m^+$ is injective, as noted by Klain [1084]. This restriction is now called the *Klain embedding* of $\mathbf{Val}_m^+$, or the *Klain map*.

Theorem 6.4.11 *A valuation in $\mathbf{Val}_m^+$ is uniquely determined by its Klain function* ($m \in \{1, \dots, n-1\}$).

Proof Let φ be a valuation in $\mathbf{Val}_m^+$ with Klain function zero. The restriction of φ to the convex bodies in a fixed m-dimensional subspace vanishes. Therefore, the restriction of φ to the convex bodies in a fixed $(m+1)$-dimensional subspace is simple and hence, by Theorem 6.4.10, is a constant multiple of $(m+1)$-dimensional volume and thus homogeneous of degree $m+1$. Since it is also homogeneous of degree m, it must vanish identically. Continuing in this way (stepwise increasing the dimension by one), we finally show that φ vanishes on $\mathcal{K}^n$. This implies the assertion. □

The question arises how a valuation as in Theorem 6.4.11 can be reconstructed from its Klain function. The following theorem, also due to Klain [1084], gives a partial answer, namely a reconstruction on a dense subclass of the set of centrally symmetric convex bodies. (We need not assume evenness here; the assertion concerns only the even part of the valuation.)

Theorem 6.4.12 *Let $\varphi \in \mathbf{Val}_m$ with $m \in \{1, \dots, n-1\}$, let $c_\varphi : G(n,m) \to \mathbb{R}$ be its Klain function. If $Z \in \mathcal{K}^n$ is a generalized zonoid with generating signed measure ρ, then*

$$\varphi(Z) = \frac{2^m}{m!} \int_{\mathbb{S}^{n-1}} c_\varphi(\mathrm{lin}\{u_1, \dots, u_m\}) D_m(u_1, \dots, u_m)\, \mathrm{d}\rho(u_1) \cdots \mathrm{d}\rho(u_m).$$

Proof Let $\overline{\varphi}$ be the mixed valuation induced by φ, according to Theorem 6.3.6. Let $L \in G(n,m)$. We have $\varphi(K) = c_\varphi(L) V_m(K)$ for $K \in \mathcal{K}(L)$. Since φ and V_m have the same polynomial expansion property under Minkowski linear combinations, we deduce that

$$\overline{\varphi}(K_1, \dots, K_m) = c_\varphi(L) v^{(m)}(K_1, \dots, K_m)$$

for convex bodies $K_1, \dots, K_m \in \mathcal{K}(L)$, where $v^{(m)}$ denotes the mixed volume in L. In particular, if $v_1, \dots, v_m \in L$ are unit vectors, without loss of generality linearly independent, and $S_i = \alpha_i[-v_i, v_i]$ with $\alpha_i > 0$ ($i = 1, \dots, m$), then

$$\begin{aligned}
\overline{\varphi}(S_1, \dots, S_m) = c_\varphi(L) v^{(m)}(S_1, \dots, S_m) &= c_\varphi(L) \frac{1}{m!} V_m(S_1 + \cdots + S_m) \\
&= c_\varphi(L) \frac{1}{m!} D_m(v_1, \dots, v_m) 2^m \alpha_1 \cdots \alpha_m \\
&= \frac{2^m}{m!} c_\varphi(\mathrm{lin}\{v_1, \dots, v_m\}) D_m(v_1, \dots, v_m)\, \alpha_1 \cdots \alpha_m.
\end{aligned}$$

Now let Z be a zonotope, say $Z = S_1 + \cdots + S_k$ with segments $S_i = \alpha_i[-v_i, v_i]$, where $v_i \in \mathbb{S}^{n-1}$ and $\alpha_i > 0$ ($i = 1, \dots, k$). The generating measure ρ of Z is concentrated at $\pm v_1, \dots, \pm v_k$ and assigns mass $\alpha_i/2$ to each of v_i and $-v_i$. Together with Theorem 6.3.6, the formula derived above yields

$$\begin{aligned}
\varphi(Z) = \varphi(S_1 + \cdots + S_k) &= \sum_{i_1, \dots, i_m = 1}^{k} \overline{\varphi}(S_{i_1}, \dots, S_{i_m}) \\
&= \sum_{i_1, \dots, i_m = 1}^{k} \frac{2^m}{m!} c_\varphi(\mathrm{lin}\{v_{i_1}, \dots, v_{i_m}\}) D_m(v_{i_1}, \dots, v_{i_m})\, \alpha_{i_1} \cdots \alpha_{i_m} \\
&= \frac{2^m}{m!} \int_{\mathbb{S}^{n-1}} c_\varphi(\mathrm{lin}\{u_1, \dots, u_m\}) D_m(u_1, \dots, u_m)\, \mathrm{d}\rho(u_1) \cdots \mathrm{d}\rho(u_m).
\end{aligned}$$

By approximation, we extend this result from zonotopes to zonoids. Let $Z_1, \dots, Z_m$ be zonoids, where Z_i has generating measure ρ_i. Inserting $Z = \lambda_1 Z_1 + \cdots + \lambda_m Z_m$ with $\lambda_i > 0$ in the previous equation, developing both sides and comparing coefficients, we obtain

$$\begin{aligned}
&\overline{\varphi}(Z_1, \dots, Z_m) \\
&= \frac{2^m}{m!} \int_{\mathbb{S}^{n-1}} c_\varphi(\mathrm{lin}\{u_1, \dots, u_m\}) D_m(u_1, \dots, u_m)\, \mathrm{d}\rho_1(u_1) \cdots \mathrm{d}\rho_m(u_m).
\end{aligned}$$

Since $\overline{\varphi}$ is Minkowski additive in each argument, we can replace each Z_i, one after the other, by a generalized zonoid and thus complete the proof of the theorem. □

We return to simple valuations, consider the odd case and prove the following result (Schneider [1720]).

Theorem 6.4.13 *Let $\varphi \in \mathbf{Val}^-$ be simple. Then there exists an odd continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ such that*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, S_{n-1}(K, \mathrm{d}u)$$

for all $K \in \mathcal{K}^n$.

Proof Let φ satisfy the assumptions. By Theorem 6.3.5 there are valuations $\varphi_r \in \mathbf{Val}_r$, $r = 0, \dots, n$, with

$$\varphi(\lambda K) = \sum_{r=0}^{n} \lambda^r \varphi_r(K) \qquad \text{for } K \in \mathcal{K}^n,\ \lambda \geq 0.$$

Clearly, each φ_r is simple and odd. By Theorem 6.4.8, φ_n is a multiple of the volume, and since it is odd, it must be zero. Since φ_0 is a multiple of the Euler characteristic, it is even and hence also vanishes. By Theorem 6.4.9,

$$\varphi_{n-1}(K) = \int_{\mathbb{S}^{n-1}} g(u)\, S_{n-1}(K, \mathrm{d}u), \qquad K \in \mathcal{K}^n,$$

with a continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$. Since φ_{n-1} is simple, the function g must be odd. Thus, to prove the theorem, it remains to show that $\varphi_r = 0$ for $1 \leq r \leq n-2$. We assume $n \geq 3$ in the following and make the inductive assumption that the assertion has been proved in dimensions less than n.

Let $r \in \{1, \dots, n-2\}$ be given. For given $k \in \{1, \dots, n-1\}$, we write $\mathbb{R}^n$ as a direct sum, $\mathbb{R}^n = E_k \oplus F_{n-k}$, of subspaces of dimensions k and $n-k$, respectively. Let $M \subset F_{n-k}$ be a convex body. Then $K \mapsto \varphi_r(K + M)$ for $K \subset E_k$ defines a translation invariant, continuous, simple valuation on the convex bodies in E_k, hence the inductive assumption gives a representation

$$\varphi_r(K + M) = c_M V_k(K) + \int_{\mathbb{S}^{n-1} \cap E_k} g_M(u)\, S^{E_k}_{k-1}(K, \mathrm{d}u) \tag{6.26}$$

with a constant c_M and an odd continuous function g_M on $\mathbb{S}^{n-1} \cap E_k$, which may depend on M. Here $S^{E_k}_{k-1}$ denotes the surface area measure in the subspace E_k. Similarly, we obtain

$$\varphi_r(K + M) = b_K V_{n-k}(M) + \int_{\mathbb{S}^{n-1} \cap F_{n-k}} f_K(u)\, S^{F_{n-k}}_{n-k-1}(M, \mathrm{d}u) \tag{6.27}$$

with a constant b_K and an odd continuous function f_K on $\mathbb{S}^{n-1} \cap F_{n-k}$.

Now let the convex bodies $K \subset E_k$ and $M \subset F_{n-k}$ be centrally symmetric and satisfying $V_k(K) \neq 0$, $V_{n-k}(M) \neq 0$. Then the integrals in (6.26) and (6.27) are zero, hence $\varphi_r(K + M) = c_M V_k(K) = b_K V_{n-k}(M)$, which gives

$$\frac{c_M}{V_{n-k}(M)} = \frac{b_K}{V_k(K)}.$$

As this number depends neither on K nor on M, it is a constant c, thus

$$\varphi_r(K+M) = cV_k(K)V_{n-k}(M).$$

For $\lambda > 0$ we have

$$\lambda^r \varphi_r(K+M) = \varphi_r(\lambda(K+M)) = cV_k(\lambda K)V_{n-k}(\lambda M) = c\lambda^n V_k(K)V_{n-k}(M)$$

and hence $\varphi_r(K+M) = 0 = c$. This shows that $c_M = b_K = 0$ if K and M are centrally symmetric.

If $K \subset E_k$ is centrally symmetric and $V_k(K) \neq 0$, we choose a centrally symmetric convex body $M \subset F_{n-k}$ with $V_{n-k}(M) \neq 0$ and deduce that $b_K = 0$. Thus, $b_K = 0$ whenever $K \subset E_k$ is centrally symmetric and $V_k(K) \neq 0$. Similarly, $c_M = 0$ whenever $M \subset F_{n-k}$ is centrally symmetric and $V_{n-k}(M) \neq 0$.

Let $K \subset E_k$ be an arbitrary convex body. We choose a centrally symmetric convex body $M \subset F_{n-k}$ with $V_{n-k}(M) \neq 0$. Then we have

$$\varphi_r(K+M) = \int_{\mathbb{S}^{n-1}\cap E_k} g_M(u)\, S_{k-1}^{E_k}(K, \mathrm{d}u) = b_K V_{n-k}(M).$$

From this and the homogeneity of φ_r we get $b_{\lambda K} = \lambda^{r-n+k} b_K$ for $\lambda > 0$. On the other hand,

$$b_{\lambda K} V_{n-k}(M) = \int_{\mathbb{S}^{n-1}\cap E_k} g_M(u)\, S_{k-1}^{E_k}(\lambda K, \mathrm{d}u) = \lambda^{k-1} b_K V_{n-k}(M).$$

Since $r \neq n-1$, we conclude that $b_K = 0$. Similarly we obtain that $c_M = 0$ for arbitrary convex bodies $M \subset F_{n-k}$. Thus, we arrive at the representations

$$\varphi_r(K+M) = \int g_M(u)\, S_{k-1}^{E_k}(K, \mathrm{d}u) = \int f_K(u)\, S_{n-k-1}^{F_{n-k}}(M, \mathrm{d}u),$$

valid for arbitrary convex bodies $K \subset E_k$ and $M \subset F_{n-k}$, where the integrations, as in the following, extend over the corresponding unit spheres. Using these representations and the facts that g_{-M} and f_K are odd functions on $\mathbb{S}^{n-1}$, we obtain

$$\begin{aligned}
\varphi_r(-(K+M)) &= \int g_{-M}(u)\, S_{k-1}^{E_k}(-K, \mathrm{d}u) = -\int g_{-M}(-u)\, S_{k-1}^{E_k}(K, \mathrm{d}(-u)) \\
&= -\int f_K(-u)\, S_{n-k-1}^{F_{n-k}}(-M, \mathrm{d}(-u)) = \int f_K(u)\, S_{n-k-1}^{F_{n-k}}(M, \mathrm{d}u) \\
&= \varphi_r(K+M).
\end{aligned}$$

We have shown that the valuation ψ_r defined by $\psi_r(L) := \varphi_r(L) - \varphi_r(-L)$ for $L \in \mathcal{K}^n$ vanishes on direct sums of convex bodies. Now let T be an n-simplex. The simplex decomposition used in the proof of Theorem 6.3.1, in particular equation (6.15) for $k = 2$ (where for simple valuations we may work with closed simplices), shows that $\psi_r(2T) = 2\psi_r(T)$. By the homogeneity of φ_r we also have $\psi_r(2T) = 2^r \psi_r(T)$. Now we assume first that $r \neq 1$. Then we conclude that $\psi_r(T) = 0$. By dissection of polytopes into simplices and using continuity, we get $\psi_r = 0$, hence the valuation

φ_r is even. By Theorem 6.4.10, it must be a constant multiple of the volume and hence homogeneous of degree n. Since $r \neq n$, we deduce that $\varphi_r = 0$.

It remains to consider the case $r = 1$. By Remark 6.3.3, φ_1 is Minkowski additive. Since $n \geq 3$ and φ_1 is simple, it vanishes at two-dimensional convex bodies. Hence, it vanishes at sums of triangles, and by continuity it vanishes at triangle bodies, by additivity then also at generalized triangle bodies. By Corollary 3.5.12, the set of generalized triangle bodies is dense in $\mathcal{K}^n$. It follows that $\varphi_1 = 0$. This completes the proof of the theorem. □

Strengthening the assumption of translation invariance for valuations to rigid motion invariance has a dramatic effect: the space $\mathbf{Val}^{\mathrm{SO}(n)}$ of valuations in $\mathbf{Val}$ which are also invariant under proper rotations is of finite dimension. This is shown by Hadwiger's [911] famous characterization theorem for the intrinsic volumes. The shorter proof that we give here is due to Klain [1081].

Theorem 6.4.14 (Hadwiger) *Let φ be a continuous valuation on $\mathcal{K}^n$ which is invariant under proper rigid motions. Then there are real constants $c_0, \dots, c_n$ such that*

$$\varphi(K) = \sum_{i=0}^{n} c_i V_i(K)$$

for all $K \in \mathcal{K}^n$.

Proof First we prove the following special case.

Proposition If the simple, continuous valuation $\varphi : \mathcal{K}^n \to \mathbb{R}$ is invariant under proper rigid motions and vanishes at unit cubes, then $\varphi = 0$.

For the proof, let T be an n-simplex. It follows from Theorem 6.4.10 that $\varphi(T) + \varphi(-T) = 0$. If n is even, then $-T$ is obtained from T by a proper rigid motion, thus $\varphi(-T) = \varphi(T)$ and hence $\varphi(T) = 0$. Let n be odd. (The following nice argument goes back to Sah [1603], p. 17.) Let z be the centre of the inscribed ball of T, and let p_i be the point where this ball touches the facet F_i of T, $i = 1, \dots, n+1$. For $i \neq j$, let $Q_{ij} := \operatorname{conv}((F_i \cap F_j) \cup \{z, p_i, p_j\})$. The polytope Q_{ij} is invariant under reflection in the hyperplane spanned by $F_i \cap F_j$ and z. The set $\{Q_{ij} : 1 \leq i < j \leq n+1\}$ is a dissection of T. Since $-Q_{ij}$ is the image of Q_{ij} under a proper rigid motion (a reflection in a hyperplane followed by a reflection in a point), we have $\varphi(-T) = \sum \varphi(-Q_{ij}) = \sum \varphi(Q_{ij}) = \varphi(T)$. Thus $\varphi(T) = 0$ for all simplices T. As usual (dissection of polytopes and approximation) this yields $\varphi = 0$ and thus completes the proof of the proposition.

From the proposition, Hadwiger's characterization theorem follows by induction with respect to the dimension. For $n = 0$, the assertion is trivial. Suppose that $n > 0$ and the assertion has been proved in smaller dimensions. Let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. By the inductive assumption, there are constants $c_0, \dots, c_{n-1}$ such that $\varphi(K) = \sum_{i=0}^{n-1} c_i V_i(K)$ for all convex bodies $K \subset H$ (note that $V_i(K)$ is independent of the dimension of the surrounding space in which it is computed). By rigid motion invariance, the equation $\varphi(K) = \sum_{i=0}^{n-1} c_i V_i(K)$ holds for all $K \in \mathcal{K}^n$ of dimension less than n. Therefore, the valuation ψ defined by $\psi(K) := \varphi(K) - \sum_{i=0}^{n} c_i V_i(K)$ for $K \in \mathcal{K}$, where c_n is determined so that φ vanishes at a fixed unit cube, is simple. Since it also satisfies all the other assumptions of the proposition, we conclude that $\psi = 0$. This completes the induction and thus the proof of the theorem. □

Notes for Section 6.4

1. *Weakly continuous valuations.* According to Theorem 6.4.6, a weakly continuous, translation invariant simple valuation on $\mathcal{P}^n$ can be homogeneous of any degree from $\{1, \dots, n\}$ without being identically zero. A comparison with Theorems 6.4.10 and 6.4.13 shows that such a valuation in general does not have an extension to a continuous valuation on $\mathcal{K}^n$.
   The general form of weakly continuous, translation invariant valuations on $\mathcal{P}^n$ is given by McMullen's Theorem 6.4.7. The continuous extendability of a subclass of these valuations was investigated by Alesker [50]. Let $k \in \{1, \dots, n-2\}$. For $P \in \mathcal{P}^n$, let
$$\varphi(P) := \sum_{F \in \mathcal{F}_k(P)} V_k(F) \int_{\mathbb{S}^{n-1} \cap N(P,F)} f(\overline{F}, u) \, \mathrm{d}\mathcal{H}^{n-k-1}(u),$$
   where $\overline{F} = \mathrm{lin}(F - F)$ and f is a given real function with suitable smoothness properties. This defines a weakly continuous, translation invariant valuation φ on $\mathcal{P}^n$, wich is homogeneous of degree k. Alesker found that in general there is a non-trivial obstruction to an extension of φ to a continuous valuation on $\mathcal{K}^n$. For $n = 3$, he found a necessary and sufficient condition on f in order that a continuous extension to $\mathcal{K}^n$ exists.
2. With quite a different approach, the continuous extension of translation invariant valuations from polytopes to general convex bodies was investigated by Hinderer, Hug and Weil [979].
3. *The Klain embedding.* Let $m \in \{1, \dots, n-1\}$ be fixed. The Klain map $\mathrm{K}_m : \varphi \mapsto c_\varphi$ of $\mathbf{Val}_m^+$ into $C(G(n,m))$ is injective, according to Theorem 6.4.11, but not surjective. Its image on smooth valuations (see Section 6.5) was determined by Alesker and Bernstein [53].
   The continuity of the inverse Klain map was investigated by Parapatits and Wannerer [1519]. They were able to characterize the centrally symmetric convex bodies K with the property that there exists a constant $C \geq 0$ (depending on K) such that
$$|\varphi(K)| \leq C \|\mathrm{K}_m(\varphi)\| \qquad \text{for all } \varphi \in \mathbf{Val}_m^+,$$
   where $\|\cdot\|$ denotes the maximum norm. These bodies are a strict subclass of all centrally symmetric convex bodies. As a corollary, the inverse of the Klain map is not continuous for $m \neq n-1$. The authors also characterized the centrally symmetric convex bodies K (again a strict subclass) for which $\mathrm{K}_m(\varphi) \geq 0$ implies $\varphi(K) \geq 0$ for all $\varphi \in \mathbf{Val}_m^+$. Moreover, they obtained the following result. If $n \geq 3$, then there exists a positive, even, translation invariant, continuous valuation on $\mathcal{K}^n$ such that not all of its homogeneous components in the McMullen decomposition are positive.
4. *Hadwiger's characterization theorem.* The first attempt to characterize the linear combinations of the intrinsic volumes was made by Blaschke [252], §43. He considered motion invariant valuations on polyhedral complexes in $\mathbb{R}^3$ ('addierbare Komplexfunktionen') with an additional property (local boundedness) and aimed at a classification. He needed, however, the additional assumption of equiaffine invariance for the volume component of his valuation. The final characterization theorem 6.4.14 was proved by Hadwiger [901] for $n = 3$ and then in [904] for general dimensions; the proof appears also in [911]. A variant of Hadwiger's proof, along similar lines but with several simplifications, was published by Chen [414].

Hadwiger himself (see [911]) has proved a counterpart to his characterization theorem, where the assumption of continuity is replaced by monotonicity. McMullen [1383] has shown directly that a monotonic translation invariant real valuation on $\mathcal{K}^n$ is continuous.

The short and elegant proof of Hadwiger's theorem given by Klain [1081], which we presented above, together with his volume characterization theorem (Theorem 6.4.10), mark the beginning of a revival of the theory of valuations, followed by an exciting development.

5. *Integral-geometric applications of Hadwiger's theorem.* Hadwiger, beginning with [898], has used his characterization theorem to give short proofs of some integral-geometric results, for example of the Crofton formula (4.59) and the kinematic formula (4.52). To explain this principle in a simple example, note that the left side of (4.59), as a function of K, defines a rigid motion invariant simple valuation, which moreover is homogeneous of degree $n + j - k$. By Hadwigers characterization theorem, it must be a constant multiple of the intrinsic volume V_{n+j-k}, and the constant factor can then be determined by choosing balls as arguments. A similar argument, applied twice, yields the kinematic formula.

   As a more sophisticated example of this approach, where no other proof is known, we mention Hadwiger's [909, 911] 'general integral-geometric theorem'.

   Theorem If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous valuation, then

$$\int_{G_n} \varphi(K \cap gM)\, d\mu(g) = \sum_{k=0}^{n} \varphi_{n-k}(K) V_k(M)$$

   for $K, M \in \mathcal{K}^n$, where the coefficients are given by

$$\varphi_{n-k}(K) = \int_{A(n,k)} \varphi(K \cap E)\, d\mu_k(E).$$

   For the details we refer, e.g., to Schneider and Weil [1740], §5.1.

6. *A general translative integral-geometric theorem.* A translative counterpart to Hadwiger's general integral-geometric theorem is possible for simple valuations. The following was proved by Schneider [1726].

   Theorem Let $\varphi : \mathcal{K}^n \to \mathbb{R}$ be a continuous simple valuation. Then

$$\int_{\mathbb{R}^n} \varphi(K \cap (M + x))\, dx = \varphi(K)V_n(M) + \int_{\mathbb{S}^{n-1}} f_{K,\varphi}(u)\, S_{n-1}(M, du)$$

   for $K, M \in \mathcal{K}^n$, where the odd function $f_{K,\varphi} : \mathbb{S}^{n-1} \to \mathbb{R}$ is given by

$$f_{K,\varphi}(u) = \int_{-h(K,-u)}^{h(K,u)} \varphi(K \cap H^-(u,t))\, dt - \varphi(K)h(K,u).$$

   The proof is based on the characterization theorems 6.4.10 and 6.4.13.

7. *Analogues of Hadwiger's characterization theorem.* Hadwiger's characterization of the intrinsic volumes has been a model for the characterization of similar objects by the valuation property and rigid motion invariance or equivariance. Characterizations of the local versions of the intrinsic volumes, namely the curvature, area or support measures, were already mentioned in Notes 11 and 12 of Section 4.2.

   One can also characterize the quermassvectors (see Subsection 5.4.1) in an analogous way.

   Theorem Let $n \geq 2$. If the valuation $\varphi : \mathcal{K}^n \to \mathbb{R}^n$ is continuous, equivariant under rotations and such that $\varphi(K + t) - \varphi(K)$ is parallel to t for all $t \in \mathbb{R}^n$, then there are real constants $c_0, \dots, c_n$ such that

$$\varphi(K) = \sum_{r=0}^{n} c_r q_r(K) \quad \text{for } K \in \mathcal{K}^n.$$

This theorem appears in Hadwiger and Schneider [932]. Its proof is based on the following result of Schneider [1668], which is a counterpart to Theorem 3.3.3, with Minkowski additivity replaced by the valuation property.

Theorem Let $n \geq 2$. If the valuation $\varphi : \mathcal{K}^n \to \mathbb{R}^n$ is continuous and equivariant under rigid motions, then φ is the Steiner point map.

8. *Polynomial valuations.* Generalizing the translation invariance or covariance, a real valuation φ on $\mathcal{P}^n$ or $\mathcal{K}^n$ is called *polynomial* of degree k if, for every fixed K, the function $x \mapsto \varphi(K + x)$ is a polynomial of degree at most k on $\mathbb{R}^n$. Polynomial valuations were introduced and investigated, in a more general form, by Pukhlikov and Khovanskiĭ [1552]. For fully addditive valuations φ on certain subclasses of $\mathcal{P}^n$ they showed that, if φ is polynomial of degree k, then $\varphi(\sum_{i=1}^m \lambda_i K_i)$, for fixed $K_1, \dots, K_m$ and integers $\lambda_1, \dots, \lambda_m \geq 0$, is a polynomial in $\lambda_1, \dots, \lambda_m$ of degree at most $n + k$. For another proof, see Alesker [30].

   Alesker [32] has investigated polynomial valuations with additional invariance properties. He proved the following.

   Theorem Let φ be a continuous polynomial valuation on $\mathcal{K}^n$ which is $\mathrm{SO}(n)$ invariant if $n \geq 3$ and $\mathrm{O}(n)$ invariant if $n = 2$. Then there exist polynomials $p_0, \dots, p_{n-1}$ in two variables such that

$$\varphi(K) = \sum_{j=0}^{n-1} \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} p_j\left(|x|^2, \langle x, u\rangle\right) \Lambda_j(K, \mathrm{d}(x, u))$$

   for $K \in \mathcal{K}^n$.

   A modified version classifies the continuous polynomial $\mathrm{SO}(2)$ invariant valuations on $\mathcal{K}^2$.

9. *Characterization of Minkowski tensors.* The next natural step, after characterizing intrinsic volumes and quermassvectors, is to ask for an analogous characterization of the Minkowski tensors $\Phi_k^{r,s}$, given by (5.108) and (5.109). The situation becomes more complicated. Let φ be a tensor valuation on $\mathcal{K}^n$, that is, a valuation with values in $\mathbb{T} := \bigoplus_{r \in \mathbb{N}_0} \mathbb{T}^r$ (for the notation used here, see Subsection 5.4.2). It is called *isometry covariant* if $\varphi(gK) = g\varphi(K)$ for all $K \in \mathcal{K}^n$ and all $g \in \mathrm{O}(n)$ and if it has polynomial behaviour under translations. The latter means that there is a number $s \in \mathbb{N}$ such that φ maps into $\bigoplus_{r=0}^s \mathbb{T}^r$ and there are functions $\varphi_j : \mathcal{K}^n \to \bigoplus_{r=0}^s \mathbb{T}^r$ such that

$$\varphi(K + t) = \sum_{r=0}^{s} \varphi_{s-r}(K) t^r$$

   for $K \in \mathcal{K}^n$ and $t \in \mathbb{R}^n$ (recall that t^r denotes the r-fold symmetric tensor product of the vector t). The following characterization theorem was proved by Alesker [33] (based on [32], and announced in [31]).

   Theorem Let $p \in \mathbb{N}_0$, and let $\varphi : \mathcal{K}^n \to \mathbb{T}^p$ be a continuous, isometry covariant valuation. Then φ is a linear combination, with constant real coefficients, of the tensor valuations $Q^l \Phi_k^{r,s}$, where l, k, r, s are such that $2l + r + s = p$.

   This extends Hadwiger's characterization theorem (case $p = 0$), but was proved by different means. It is not surprising that, for $p \geq 2$, powers of the metric tensor Q appear, since this constant tensor is isometry covariant.

   In contrast to the cases $p = 0$ and $p = 1$, for $p > 1$ the valuations listed in the characterization theorem are not linearly independent, in view of the McMullen relations (5.4.4). It was proved by Hug, Schneider and Schuster [1020] that the McMullen relations are essentially (up to multiplication by powers of Q and linear combinations) the only linear dependences between the tensors $Q^l \Phi_k^{r,s}$. Another contrast to the cases $p = 0, 1$ is the fact that the characterization theorem does not seem to be of great use in proving integral-geometric identities, such as Crofton formulae. For the integral geometry of tensor valuations, see Note 9 of Section 5.4.

One may conjecture that Alesker's characterization theorem quoted above has a local counterpart. Let $\Gamma : \mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{T}^p$ be a mapping such that $\Gamma(K, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$, and that Γ is isometry covariant, locally defined and weakly continuous. It was asked in Schneider [1731] whether Γ must be a linear combination of the measure-valued tensor valuations $Q^l \phi_k^{r,s}$, $2l + r + s = p$, where

$$\phi_k^{r,s}(K, \eta) := \int_\eta x^r u^s \Lambda_k(K, \mathrm{d}(x, u)).$$

A characterization theorem of this kind was proved in [1731], but with $\mathcal{K}^n$ replaced by $\mathcal{P}^n$, without a valuation or weak continuity assumption, but instead with the assumption that $\Gamma(P, \cdot)$ is concentrated on the normal bundle $\operatorname{Nor} P$. The characterized tensor valuations are of a more general type.

10. *An analogue of Hadwiger's characterization theorem for non-Euclidean spaces?* It has repeatedly been asked whether there is an analogue of Hadwiger's characterization theorem in spherical or hyperbolic space (for the possible counterparts of the intrinsic volumes, see Note 10 in Section 7.4). For two-dimensional spherical space, such a result is Theorem 11.3.1 in Klain and Rota [1093]. A corresponding theorem in two-dimensional hyperbolic space was proved by Klain [1090]. The higher-dimensional cases are wide open.
11. *Valuations on convex sets of oriented hyperplanes.* Aiming at a 'dual' version of Hadwiger's characterization theorem, Gates, Hug and Schneider [689] considered convex sets of oriented hyperplanes in $\mathbb{R}^n$, introduced a class of valuations on them, and conjectured a characterization theorem. They proved it for $n = 2$, where the result says that a continuous, rigid motion invariant valuation on compact convex sets of oriented lines in the plane is a linear combination of three functionals: the restriction of the Haar measure the total angular measure and the Euler characteristic.
12. *Minkowski valuations.* A valuation on $\mathcal{K}^n$ or $\mathcal{P}^n$ with values in $(\mathcal{K}^n, +)$, where $+$ is Minkowski addition, is called a *Minkowski valuation*. Examples are given by the Minkowski endomorphisms introduced in Section 3.3, as follows from equation (3.4). Let **MVal** denote the set of all continuous, translation invariant Minkowski valuations on $\mathcal{K}^n$ and $\mathbf{MVal}_m$ the subset of Minkowski valuations that are homogeneous of degree m. To get examples, let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(\langle \cdot, v \rangle)$ is a support function for one (and hence for all) $v \in \mathbb{S}^{n-1}$. Then

$$h(\Phi K, u) = \int_{\mathbb{S}^{n-1}} f(\langle u, v \rangle)\, S_j(K, \mathrm{d}v)$$

defines a rotation equivariant Minkowski valuation $\Phi \in \mathbf{MVal}_j$.

A simple additional geometric assumption suffices to single out, essentially, the projection body operator. The following was proved by Schneider and Schuster [1733]. Let $n \geq 3$. If $\Phi \in \mathbf{MVal}$ is rotation equivariant and maps polytopes to polytopes, then

$$\Phi = c_1 \Pi + c_2 \mathcal{I} + c_3(-\mathcal{I})$$

with constants $c_1, c_2, c_3 \geq 0$, where Π is the projection body operator and $\mathcal{I}(K) = K - s(K)$ for $K \in \mathcal{K}^n$.

Let $\Phi \in \mathbf{MVal}$. From the McMullen decomposition, one can deduce (see [1733]) that there exist convex bodies $L_0, L_n \in \mathcal{K}^n$ and functions $g_i : \mathcal{K}^n \times \mathbb{R}^n$ such that

$$h(\Phi(K), \cdot) = h(L_0, \cdot) + \sum_{i=1}^{n-1} g_i(K, \cdot) + V_n(K) h(L_n, \cdot)$$

for all $K \in \mathcal{K}^n$, and the following holds. Each function $g_i(K, \cdot)$ is a difference of support functions; the map $K \mapsto g_i(K, \cdot)$ is a continuous, translation invariant valuation that is homogeneous of degree i. It was asked in [1733] whether each $g_i(K, \cdot)$ is a support function. This was disproved by Parapatits and Wannerer [1519]. Under the additional assumption of rotation equivariance, the question remains open.

More on polynomiality with convex coefficients can be said in the case of zonoids. The following results were proved by Parapatits and Schuster [1518]. Let $\Phi \in \mathbf{MVal}$. If $Z \in \mathcal{K}$ is a zonoid, then there exist $\Phi_Z^{(j)} \in \mathbf{MVal}_j$, $j = 0, \dots, n$, such that

$$\Phi(K + rZ) = \sum_{j=0}^{n} r^{n-j} \Phi_Z^{(j)}(K) \qquad \text{for } K \in \mathcal{K}^n.$$

If $Z_1, \dots, Z_m$ are zonoids and $\lambda_1, \dots, \lambda_m \geq 0$, then $\Phi(\lambda_1 Z_1 + \cdots + \lambda_m Z_m)$ is a polynomial in $\lambda_1, \dots, \lambda_m$ of degree at most n whose coefficients are convex bodies.

13. *Valuations compatible with affine transformations.* While many Minkowski valuations are rotation equivariant, fewer of them show a simple behaviour under affine transformations. Therefore, an assumption of $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$ covariance or contravariance opens the way to interesting characterizations of several geometrically important convex body-valued valuations. These will be considered in Section 10.16. Also real-valued valuations with special behaviour under affine transformations will be treated there.

6.5 The modern theory of valuations

Since the second half of the 1990s, the theory of valuations has undergone a rapid development, in the beginning mainly due to the work of Alesker, with remarkable expansions in depth, scope and applications. While starting from the classical theory of valuations on convex bodies, most of the employed tools and the fields of application of this new theory are far outside the range of the present book. We confine ourselves, therefore, to giving hints to the essential references, partially guided by Alesker's survey [45] and lecture notes [48].

In this section, the values of the valuations in **Val** may be in $\mathbb{C}$. For $\varphi \in \mathbf{Val}$, let

$$\|\varphi\| := \sup \{|\varphi(K)| : K \in \mathcal{K}(B^n)\}.$$

This defines a norm on **Val** (that $\|\varphi\| = 0$ implies $\varphi = 0$ follows from the McMullen decomposition (6.22)). Convergence with respect to this norm is equivalent to uniform convergence on compact sets of convex bodies.

A natural representation ρ of the locally compact group $\mathrm{GL}(n)$ on **Val** is defined by

$$(\rho(g)\varphi)(K) := \varphi(g^{-1}K), \quad K \in \mathcal{K}^n,\ \varphi \in \mathbf{Val},\ g \in \mathrm{GL}(n).$$

The representation ρ is continuous. Each space $\mathbf{Val}_m^{\pm}$ is a closed, invariant subspace of **Val**. This representation has been introduced by Alesker [35]. He proved the following result, which turned out to be the key to much of the later development.

Theorem 6.5.1 (Alesker's irreducibility theorem) *The natural representation of* $\mathrm{GL}(n)$ *on each vector space* $\mathbf{Val}_m^+$ *and* $\mathbf{Val}_m^-$ *is irreducible.*

As an immediate consequence, Alesker [35] obtained that the linear combinations of the valuations $K \mapsto V(K[m], C_{m+1}, \dots, C_n)$ (with $C_{m+1}, \dots, C_n \in \mathcal{K}_n^n$) are dense in $\mathbf{Val}_m$. This proved a conjecture by McMullen [1385]. It also follows that the valuations φ_A, defined by $\varphi_A(K) := V_n(K + A)$, $A \in \mathcal{K}^n$, span a dense subspace of **Val**.

The decomposition of the space **Val** into a sum of SO(n) irreducible subspaces was determined by Alesker, Bernig and Schuster [52]. They have applications to symmetry properties of rigid motion invariant and homogeneous bivaluations, which in turn are used to obtain Brunn–Minkowski type inequalities for Minkowski valuations.

The following notion is fundamental to further development. A valuation $\varphi \in$ **Val** is *smooth* if the mapping $g \mapsto \rho(g)\varphi$ from GL(n) into the Banach space **Val** is infinitely differentiable. The space $\mathbf{Val}^\infty$ of smooth valuations is an invariant, dense subspace of **Val**. It carries a natural linear topology, stronger than that induced from **Val**, making it a Fréchet space. Denoting by $\mathbf{Val}_m^\infty$, $\mathbf{Val}_m^{+,\infty}$, $\mathbf{Val}_m^{-,\infty}$ the subspaces of smooth valuations in $\mathbf{Val}_m$, $\mathbf{Val}_m^+$, $\mathbf{Val}_m^-$, respectively, one has the decomposition

$$\mathbf{Val}^\infty = \bigoplus_{m=0}^{n} \mathbf{Val}_m^\infty, \qquad \mathbf{Val}_m^\infty = \mathbf{Val}_m^{+,\infty} \oplus \mathbf{Val}_m^{-,\infty}.$$

The irreducibility theorem holds also for $\mathbf{Val}^\infty$.

In [38], Alesker introduced on $\mathbf{Val}^\infty$ a canonical continuous product $\mathbf{Val}^\infty \times \mathbf{Val}^\infty \to \mathbf{Val}^\infty$, with which $\mathbf{Val}^\infty$ becomes a graded algebra, with the Euler characteristic as unit, and satisfying $\mathbf{Val}_i^\infty \cdot \mathbf{Val}_j^\infty \subset \mathbf{Val}_{i+j}^\infty$. There is a Poincaré duality: for $i \in \{0, \dots, n\}$, the product $\mathbf{Val}_i^\infty \times \mathbf{Val}_{n-i}^\infty \to \mathbf{Val}_n^\infty$ is a perfect pairing, that is, for each non-zero $\varphi \in \mathbf{Val}_i^\infty$ there exists $\psi \in \mathbf{Val}_{n-i}^\infty$ such that $\varphi \cdot \psi \neq 0$.

The product introduced by Alesker is uniquely determined by its values on smooth valuations of the form φ_A, φ_B, and here one has

$$(\varphi_A \cdot \varphi_B)(K) = V_{2n}(\{(x + a, x + b) \in \mathbb{R}^n \times \mathbb{R}^n : x \in K,\ a \in A,\ b \in B\}).$$

For the valuations in **Val**, and mainly for $\mathbf{Val}^\infty$, some useful operations have been introduced, such as pull-back, push-forward, a Fourier-type transformation by Alesker [49] and a convolution by Bernig and Fu [210].

The following results are two versions of Alesker's hard Lefschetz theorem for valuations. Define $L : \mathbf{Val}^\infty \to \mathbf{Val}^\infty$ by $L\varphi := \varphi \cdot V_1$ (where V_1 is the first intrinsic volume), and define $\Lambda : \mathbf{Val}^\infty \to \mathbf{Val}^\infty$ by

$$(\Lambda\varphi)(K) := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \varphi(K + \varepsilon B^n).$$

(The existence of the limit follows from Theorems 6.3.1 and 6.3.5.)

Theorem 6.5.2 (i) *Let* $0 \leq i < n/2$. *The mapping*

$$L^{n-2i} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{n-i}^\infty$$

is an isomorphism.

(ii) *Let $n/2 < i \leq n$. The mapping*

$$\Lambda^{2i-n} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{n-i}^\infty$$

is an isomorphism.

For even valuations, Alesker proved (i) in [39]. One of the tools was the range characterization of the cosine transform on higher rank Grassmannians, achieved by Alesker and Bernstein [53]. Version (ii) was proved by Alesker [36] for even valuations and by Bernig and Bröcker [209] in general. Bernig and Fu [210] showed that the two versions are equivalent under the Fourier transform for valuations, which they already had for even valuations and which was later established in the general case by Alesker [49]; this then also yielded the odd case of (i).

A general theory of valuations on manifolds, with the system of convex bodies replaced by the system of compact submanifolds with corners, was developed in a series of papers by Alesker [42, 43, 44] and by Alesker and Fu [55]. In Alesker's words [45]: 'the author's feeling is that the notion of valuation equips smooth manifolds with a new, general and rich structure'. The investigations were continued by Bernig [204], Alesker [47], Alesker and Bernig [51].

Returning to vector spaces, let G be a compact subgroup of the orthogonal group $\mathrm{O}(n)$ and denote by $\mathbf{Val}^G \subset \mathbf{Val}$ the subspace of G-invariant valuations. Alesker [34] proved the following theorem (see [45] for the 'only if' part).

Theorem 6.5.3 (Alesker) *The space $\mathbf{Val}^G$ has finite dimension if and only if G acts transitively on the sphere $\mathbb{S}^{n-1}$.*

If G acts transitively on $\mathbb{S}^{n-1}$, then $\mathbf{Val}^G \subset \mathbf{V}^\infty$, as shown by Alesker [36, 38]. According to Bernig [203], all valuations in $\mathbf{Val}^G$ are even. The algebra structure of $\mathbf{Val}^G$ was investigated by Alesker [36, 38, 39] and by Fu [640].

The compact connected Lie groups acting transitively on spheres have been completely classified; there are six infinite series, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, $\mathrm{Sp}(n)\cdot\mathrm{U}(1)$, and three exceptional groups, G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$. Theorem 6.5.3 is an invitation to a far-reaching extension of Hadwiger's characterization theorem and its applications: to determine the valuations in $\mathbf{Val}^G$ if G is one of these groups, and to develop the corresponding integral geometry, in particular with a view to kinematic and Crofton formulae. The following parts of this programme have been carried out.

The structure theory of $\mathbf{Val}^{\mathrm{U}(m)}$ and its implications on Hermitian integral geometry have been investigated in great depth. We mention the work of Alesker [36], Fu [640], Bernig and Fu [211]. Wannerer [1924] studied the Hermitian analogues of the Euclidean area measures of convex bodies. Bernig, Fu and Solanes [212] utilized Alesker's theory of valuations on manifolds for an investigation of the integral geometry of complex space forms. In particular, they computed the kinematic formulae for invariant valuations and invariant curvature measures in these spaces. Crofton-type formulae in complex space forms and their relations to valuations were studied by Abardia, Gallego and Solanes [3].

A basis of $\mathbf{Val}_k^{\mathrm{SU}(2)}(\mathbb{C}^2)$ was found by Alesker [40] and its algebra structure was computed by Bernig [204]. Bernig [202] found the dimension of $\mathbf{Val}_k^{\mathrm{SU}(n)}(\mathbb{C}^n)$, constructed an explicit basis and obtained kinematic formulae. Valuations invariant under the quaternionic groups $\mathrm{Sp}(n)$ or $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ were constructed by Alesker [41]. Further, Bernig [205] studied the spaces of $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot \mathrm{U}(1)$ or $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ invariant, translation invariant, continuous valuations on the quaternionic vector space $\mathbb{H}^n$ and obtained combinatorial dimension formulae. In [46], Alesker constructed valuations in $\mathbf{Val}^G$ where G is the exceptional group Spin(9), which acts transitively on $\mathbb{S}^{15} \subset \mathbb{R}^{16}$. Bernig [203] described the G_2 and Spin(7) invariant valuations, including the algebra structure and integral-geometric applications.

Fu [641] gave a brief description of the importance of Alesker's new theory of valuations for integral geometry. Comprehensive surveys of this new direction of integral geometry are presented by Bernig [206] and Fu [642], both under the title of 'Algebraic integral geometry'.

Notes for Section 6.5

1. *Rotation equivariant Minkowski valuations.* Some of the new results on valuations were used by Schuster to obtain a representation for certain Minkowski valuations (for these, see Note 12 of Section 6.4). For $i \in \{1, \dots, n-1\}$, let $\mathrm{O}(i) \times \mathrm{O}(n-i)$ be the subgroup of $\mathrm{O}(n)$ mapping some fixed i-subspace of $\mathbb{R}^n$ into itself. Identifying the sphere $\mathbb{S}^{n-1}$ with $\mathrm{O}(n)/\mathrm{O}(n-1)$ and the Grassmannian $G(n,i)$ with $\mathrm{O}(n)/\mathrm{O}(i)\times\mathrm{O}(n-i)$, one can define a convolution $*$ of continuous functions on $G(n,i)$ with $\mathrm{O}(i)\times\mathrm{O}(n-i)$ invariant signed measures or functions on $\mathbb{S}^{n-1}$. The following was proved by Schuster [1749]. Let $\Phi \in \mathbf{MVal}_i$ be $\mathrm{O}(n)$ equivariant, smooth (in a sense derived from smoothness for valuations in $\mathbf{Val}$) and even. Then there exists an $\mathrm{O}(i)\times\mathrm{O}(n-i)$ invariant, even, smooth function f on $\mathbb{S}^{n-1}$ such that
$$h(\Phi(K),\cdot) = V_i(K\,|\,\cdot) * f \qquad \text{for } K \in \mathcal{K}^n.$$
According to Wannerer [1922], the function f is uniquely determined.
2. Wannerer [1922] succeeded in proving the following Hadwiger-type theorem for smooth Minkowski valuations.

   Theorem Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a smooth, translation invariant, $\mathrm{SO}(n)$ equivariant Minkowski valuation. There are real constants c_0, c_n and functions $f_1, \dots, f_{n-1} \in C[-1,1] \cap C^\infty(-1,1)$ such that
$$h(\Phi K,\cdot) = c_0 + \sum_{i=1}^{n-1} \int_{\mathbb{S}^{n-1}} f_i(\langle \cdot, v\rangle)\, S_i(K, \mathrm{d}v) + c_n V_n(K).$$
   The constants c_0, c_n are uniquely determined; the functions f_i are unique up to restrictions of linear functions to $[-1,1]$.

7

Inequalities for mixed volumes

7.1 The Brunn–Minkowski theorem

Studying the behaviour of volume under Minkowski addition, one is led to the notion of mixed volumes on one hand and to the Brunn–Minkowski theorem on the other. In its simplest form, this theorem says that for two convex bodies $K, L \in \mathcal{K}^n$, each of volume 1, the Minkowski sum $\frac{1}{2}(K + L)$ has volume at least 1, and its volume is equal to 1 only if K and L are translates. We shall give three proofs, each of which has its merits and illuminates the scenery from a different point of view. The Brunn–Minkowski theorem is the starting point for a rich theory of geometric (and also analytic) inequalities, with many applications to extremal, uniqueness and other problems.

We mention already here that, according to the introductory character of this book, we present only basic facts about the Brunn–Minkowski inequality. For a fuller picture of its extensions and ramifications we refer to the excellent survey article by Gardner [674].

Theorem 7.1.1 (Brunn–Minkowski) *For convex bodies $K_0, K_1 \in \mathcal{K}^n$ and for $0 \le \lambda \le 1$,*

$$V_n((1-\lambda)K_0 + \lambda K_1)^{1/n} \ge (1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}. \tag{7.1}$$

Equality for some $\lambda \in (0, 1)$ holds if and only if K_0 and K_1 either lie in parallel hyperplanes or are homothetic.

If $K_0, K_1 \in \mathcal{K}^n$ are given, we often write

$$K_\lambda := (1-\lambda)K_0 + \lambda K_1 \quad \text{for } 0 \le \lambda \le 1. \tag{7.2}$$

For $\sigma, \tau \in [0, 1]$ and $0 \le \lambda \le 1$, we have

$$(1-\lambda)K_\sigma + \lambda K_\tau = (1-\alpha)K_0 + \alpha K_1$$

with $\alpha = (1-\lambda)\sigma + \lambda\tau$. Applying Theorem 7.1.1 to K_σ and K_τ, where $\sigma, \tau \in [0, 1]$ are arbitrary, we deduce that the function

$$\lambda \mapsto V_n((1-\lambda)K_0 + \lambda K_1)^{1/n}$$

is concave on $[0, 1]$, and linear only in the cases described in Theorem 7.1.1. (As is usual in this context, we say 'linear' instead of 'affine'.)

The first proof we shall give for the Brunn–Minkowski theorem is a classical version due to Kneser and Süss [1122], which is also reproduced in Bonnesen and Fenchel [284]. It still has some advantages.

Proof of Theorem 7.1.1 If K_0 and K_1 lie in parallel hyperplanes, then K_λ lies in a hyperplane and hence $V_n(K_\lambda) = 0$ for $0 \le \lambda \le 1$. Thus equality holds in (7.1). This is also true, trivially, if K_0 and K_1 are homothetic.

Suppose that $\dim K_0 < n$ and $\dim K_1 < n$. Then (7.1) holds trivially, and if equality holds, then K_λ lies in a hyperplane, hence K_0 and K_1 lie in parallel hyperplanes.

If, say, $\dim K_0 < n$ and $\dim K_1 = n$, then the inclusion $K_\lambda \supset (1-\lambda)x + \lambda K_1$, for arbitrary $x \in K_0$, implies that

$$V_n(K_\lambda) \ge V_n((1-\lambda)x + \lambda K_1) = \lambda^n V_n(K_1),$$

with equality if and only if $K_0 = \{x\}$. In that case, K_0 and K_1 are homothetic, by definition.

Hence, from now on we can assume that $\dim K_0 = \dim K_1 = n$. We can also assume that $V_n(K_0) = V_n(K_1) = 1$. Indeed, if (7.1) is proved under this assumption and if $K_0, K_1 \in \mathcal{K}^n_n$ are arbitrary, then we put

$$\overline{K}_i := V_n(K_i)^{-1/n} K_i \quad \text{for } i = 0, 1, \quad \overline{\lambda} := \frac{\lambda V_n(K_1)^{1/n}}{(1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}}.$$

From $V_n((1-\overline{\lambda})\overline{K}_0 + \overline{\lambda}\,\overline{K}_1)^{1/n} \ge 1$ we then obtain (7.1); the results for the case of equality can also be generalized.

Theorem 7.1.1 is proved by induction with respect to the dimension. The case $n = 1$ being trivial, we assume that $n \ge 2$ and that the theorem is true in dimension $n - 1$. We choose $u \in \mathbb{S}^{n-1}$ and write $H_{u,\alpha} =: H(\alpha)$, $H^-_{u,\alpha} =: H^-(\alpha)$ and, furthermore, $\alpha_\lambda := -h(K_\lambda, -u)$ and $\beta_\lambda := h(K_\lambda, u)$. For $\zeta \in \mathbb{R}$ and $i = 0, 1$, we define

$$v_i(\zeta) := V_{n-1}(K_i \cap H(\zeta)), \quad w_i(\zeta) := V_n(K_i \cap H^-(\zeta)),$$

so that

$$w_i(\zeta) := \int_{\alpha_i}^{\zeta} v_i(t)\,\mathrm{d}t.$$

On (α_i, β_i) the function v_i is continuous, hence w_i is differentiable and

$$w_i'(\zeta) = v_i(\zeta) > 0 \quad \text{for } \alpha_i < \zeta < \beta_i.$$

Let z_i be the inverse function of w_i; then

$$z_i'(\tau) = \frac{1}{v_i(z_i(\tau))} \quad \text{for } 0 < \tau < 1.$$

Writing

$$k_i(\tau) := K_i \cap H(z_i(\tau)), \qquad z_\lambda(\tau) := (1-\lambda)z_0(\tau) + \lambda z_1(\tau),$$

we have

$$K_\lambda \cap H(z_\lambda(\tau)) \supset (1-\lambda)k_0(\tau) + \lambda k_1(\tau).$$

We deduce that

$$\begin{aligned} V_n(K_\lambda) &= \int_{\alpha_\lambda}^{\beta_\lambda} V_{n-1}(K_\lambda \cap H(\zeta))\,\mathrm{d}\zeta = \int_0^1 V_{n-1}(K_\lambda \cap H(z_\lambda(\tau)))z'_\lambda(\tau)\,\mathrm{d}\tau \\ &\geq \int_0^1 V_{n-1}\left((1-\lambda)k_0(\tau) + \lambda k_1(\tau)\right)\left[\frac{1-\lambda}{v_0(z_0(\tau))} + \frac{\lambda}{v_1(z_1(\tau))}\right]\mathrm{d}\tau \\ &\geq \int_0^1 \left[(1-\lambda)v_0^{1/(n-1)} + \lambda v_1^{1/(n-1)}\right]^{n-1}\left[\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right]\mathrm{d}\tau, \end{aligned}$$

where in the final line we have used the induction hypothesis and abbreviated $v_i(z_i(\tau))$ by v_i. The inequality

$$\left[(1-\lambda)v_0^p + \lambda v_1^p\right]^{1/p}\left[\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right] \geq 1$$

holds for $v_0, v_1, p > 0$ and $0 < \lambda < 1$. For the proof, one takes logarithms and notes that log is concave and increasing. Equality holds only for $v_0 = v_1$.

It follows that $V_n(K_\lambda) \geq 1$, and this completes the inductive proof of the Brunn–Minkowski inequality.

Suppose that the equality $V_n(K_\lambda) = 1$ holds for some $\lambda \in (0,1)$. Then $v_0(z_0(\tau)) = v_1(z_1(\tau))$ and hence $z'_0(\tau) = z'_1(\tau)$ for $0 \leq \tau \leq 1$; thus $z_1(\tau) - z_0(\tau)$ is constant. We may assume that K_0 and K_1 have their centroids at the origin. Then

$$0 = \int_{K_i} \langle x, u\rangle\,\mathrm{d}x = \int_{\alpha_i}^{\beta_i} V_{n-1}(K_i \cap H(\zeta))\zeta\,\mathrm{d}\zeta = \int_0^1 z_i(\tau)\,\mathrm{d}\tau$$

for $i = 0, 1$ and hence $z_0(\tau) = z_1(\tau)$ for $0 \leq \tau \leq 1$. This yields $\beta_0 = \beta_1$, thus $h(K_0, u) = h(K_1, u)$. Since $u \in \mathbb{S}^{n-1}$ was arbitrary, we conclude that $K_0 = K_1$. □

We note that the Brunn–Minkowski theorem can be expressed in several equivalent forms:

(a) For all $K_0, K_1 \in \mathcal{K}^n$,

$$V_n(K_0 + K_1)^{1/n} \geq V_n(K_0)^{1/n} + V_n(K_1)^{1/n}.$$

(b) For all $K_0, K_1 \in \mathcal{K}^n$ and all $s, t \geq 0$,

$$V_n(sK_0 + tK_1)^{1/n} \geq sV_n(K_0)^{1/n} + tV_n(K_1)^{1/n}.$$

(c) For all $K_0, K_1 \in \mathcal{K}^n$ and all $\lambda \in [0,1]$,

$$V_n(K_0) = V_n(K_1) = 1 \Rightarrow V_n((1-\lambda)K_0 + \lambda K_1) \geq 1.$$

(d) For all $K_0, K_1 \in \mathcal{K}^n$ and all $\lambda \in [0,1]$,

$$V_n((1-\lambda)K_0 + \lambda K_1) \geq \min\{V_n(K_0), V_n(K_1)\}.$$

(e) For all $K_0, K_1 \in \mathcal{K}^n$ and all $\lambda \in [0,1]$,

$$V_n((1-\lambda)K_0 + \lambda K_1) \geq V_n(K_0)^{1-\lambda} V_n(K_1)^{\lambda}.$$

In fact, (a) implies (b) because of the homogeneity of the volume, and (b) trivially implies (c) and (d). That (c) implies (a) was shown in the proof of Theorem 7.1.1 under the assumption that $V_n(K_i) > 0$; this yields the general case. Clearly, (d) immediately gives (c). By the arithmetic-geometric mean inequality, the Brunn–Minkowski inequality implies (e), and (e) gives (d).

The inequality under (e), where, in contrast to (a), the dimension does not appear (except in the notation for the volume), is known as the multiplicative form of the Brunn–Minkowski theorem.

We shall now sketch a second proof of the Brunn–Minkowski theorem, essentially following Knothe [1125]. As above, it is sufficient to consider two convex bodies $K, L \in \mathcal{K}^n$ satisfying $V_n(K) = V_n(L) = 1$. We write $x \in \mathbb{R}^n$ in the form $x = (x_1, \dots, x_n)$, where the x_i are the coordinates of x with respect to a given orthonormal basis, and define

$$K_{x_1,\dots,x_j} := \{y \in K : y_i = x_i \text{ for } 1 \leq i \leq j\}$$

for $j = 1, \dots, n$, and similarly for L. On $\operatorname{int} K$ we can define a function f_1 by

$$\int_{-\infty}^{x_1} V_{n-1}(K_t)\,\mathrm{d}t = \int_{-\infty}^{f_1(x)} V_{n-1}(L_t)\,\mathrm{d}t.$$

If $j \in \{2, \dots, n\}$ and $f_1, \dots, f_{j-1}$ have already been defined, we define f_j by

$$\int_{-\infty}^{x_j} V_{n-j}(K_{x_1,\dots,x_{j-1},t})\,\mathrm{d}t = \frac{V_{n-j+1}(K_{x_1,\dots,x_{j-1}})}{V_{n-j+1}(L_{f_1(x),\dots,f_{j-1}(x)})} \int_{-\infty}^{f_j(x)} V_{n-j}(L_{f_1(x),\dots,f_{j-1}(x),t})\,\mathrm{d}t.$$

Then $f_j(x)$ depends only on $x_1, \dots, x_j$, and for $x \in \operatorname{int} K$ we obviously have

$$\frac{\partial f_j}{\partial x_j}(x) = \frac{V_{n-j}(K_{x_1,\dots,x_j})}{V_{n-j}(L_{f_1(x),\dots,f_j(x)})} \frac{V_{n-j+1}(L_{f_1(x),\dots,f_{j-1}(x)})}{V_{n-j+1}(K_{x_1,\dots,x_{j-1}})} > 0$$

and thus

$$\prod_{j=1}^{n} \frac{\partial f_j}{\partial x_j} = 1. \tag{7.3}$$

The map $F : \operatorname{int} K \to \operatorname{int} L$ defined by $F(x) = (f_1(x), \dots, f_n(x))$ is bijective.

Now let $\lambda \in [0, 1]$. Then $[(1-\lambda)\mathrm{id} + \lambda F]K \subset K_\lambda := (1-\lambda)K + \lambda L$ and hence

$$\begin{aligned} V_n(K_\lambda) &\geq V_n([(1-\lambda)\mathrm{id} + \lambda F]K) \\ &= \underbrace{\int \cdots \int}_{\mathrm{int}\, K} \left|\mathrm{Jac}[(1-\lambda)\mathrm{id} + \lambda F]\right| \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int \cdots \int \prod_{j=1}^{n} \left[(1-\lambda) + \lambda \frac{\partial f_j}{\partial x_j}(x)\right] \mathrm{d}x_1 \cdots \mathrm{d}x_n. \end{aligned}$$

(In using the Jacobian, we do not need to know whether $\partial f_j / \partial x_i$ exists for $i < j$. In fact, since $f_j(x)$ depends only on $x_1, \ldots, x_j$, the transition from the first to the third line does not require the general transformation rule for multiple integrals, but is obtained by using successively the substitution rule for one-variable integrals.)

Now

$$\log \left[(1-\lambda) + \lambda \frac{\partial f_j}{\partial x_i}\right] \geq \lambda \log \frac{\partial f_j}{\partial x_j}$$

since log is a concave function, and together with (7.3) this gives

$$V_n(K_\lambda) \geq 1.$$

If equality holds, we must have

$$\frac{\partial f_j}{\partial x_j}(x) = 1 \quad \text{for } x \in \mathrm{int}\, K \text{ and } j = 1, \ldots, n$$

and thus $f_1(x) = x_1 + a_1$ with a constant a_1. Since any coordinate axis may play the role of the first one, we deduce that L is a translate of K.

The preceding proof of the Brunn–Minkowski theorem can be used to obtain extensions of that theorem, with the volume replaced by the integral of a function with suitable properties (see Note 2 below).

The third proof of the Brunn–Minkowski inequality that we present is based on the following analytic inequality.

Theorem 7.1.2 (Prékopa–Leindler inequality) *Let $0 < \lambda < 1$, and let f, g, h be nonnegative, Lebesgue integrable real functions on $\mathbb{R}^n$ with*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda \quad \textit{for } x, y \in \mathbb{R}^n. \tag{7.4}$$

Then

$$\int_{\mathbb{R}^n} h(x)\, \mathrm{d}x \geq \left(\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\, \mathrm{d}x\right)^{\lambda}. \tag{7.5}$$

This inequality reminds one of Hölder's inequality. The latter states that for the function

$$h(x) := f(x)^{1-\lambda} g(x)^\lambda$$

one has

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \le \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{\lambda}.$$

Thus, the Prékopa–Leindler inequality is an estimate in the opposite direction, which is possible under the strong assumption (7.4).

Before we prove the Prékopa–Leindler inequality, we show how it implies the Brunn–Minkowski inequality. Let $0 < \lambda < 1$, and let $A, B \subset \mathbb{R}^n$ be bounded, measurable sets with the property that also $(1-\lambda)A + \lambda B$ is measurable. Put $f := \mathbf{1}_A$, $g := \mathbf{1}_B$ and $h := \mathbf{1}_{(1-\lambda)A+\lambda B}$ (where $\mathbf{1}$ denotes the characteristic function). We assert that

$$h((1-\lambda)x + \lambda y) \ge f(x)^{1-\lambda} g(y)^{\lambda} \quad \text{for } x, y \in \mathbb{R}^n.$$

In fact, if the right-hand side is 0, this is true, and otherwise

$$f(x)^{1-\lambda} g(y)^{\lambda} = 1 \Rightarrow x \in A \text{ and } y \in B \Rightarrow (1-\lambda)x + \lambda y \in (1-\lambda)A + \lambda B$$

and thus $h((1-\lambda)x + \lambda y) = 1$. Now the Prékopa–Leindler inequality gives

$$V_n((1-\lambda)A + \lambda B) \ge V_n(A)^{1-\lambda} V_n(B)^{\lambda}. \tag{7.6}$$

As noted above, this implies the Brunn–Minkowski inequality (in this step, convexity is not needed). In this way, the Prékopa–Leindler inequality implies the Brunn–Minkowski inequality for not necessarily convex sets.

In the following proof, we follow Gardner [673] (see also the proof in Pisier [1535], which follows Ball [116]).

Proof of Theorem 7.1.2 It is sufficient to prove the assertion for continuous, positive, integrable functions; the general case then follows by standard measure-theoretic arguments. We prove (7.5) by induction with respect to the dimension. First let $n = 1$. Putting

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x =: F, \qquad \int_{\mathbb{R}} g(x)\,\mathrm{d}x =: G,$$

we have $F, G > 0$. For $t \in (0,1)$ let $u(t), v(t)$ be defined by

$$\frac{1}{F}\int_{-\infty}^{u(t)} f(x)\,\mathrm{d}x = \frac{1}{G}\int_{-\infty}^{v(t)} g(x)\,\mathrm{d}x = t.$$

The functions $u, v : (0,1) \to \mathbb{R}$ are increasing and differentiable; differentiation gives

$$\frac{f(u(t))u'(t)}{F} = \frac{g(v(t))v'(t)}{G} = 1.$$

Let

$$w(t) := (1-\lambda)u(t) + \lambda v(t).$$

By the arithmetic-geometric mean inequality we have

$$w'(t) = (1-\lambda)u'(t) + \lambda v'(t) \geq u'(t)^{1-\lambda}v'(t)^{\lambda} = \left(\frac{F}{f(u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda}$$

for $t \in (0,1)$ and hence

$$\begin{aligned}\int_{\mathbb{R}} h(x)\,\mathrm{d}x &\geq \int_0^1 h(w(t))w'(t)\,\mathrm{d}t \\ &\geq \int_0^1 f(u(t))^{1-\lambda}g(v(t))^{\lambda}\left(\frac{F}{f(u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda}\mathrm{d}t = F^{1-\lambda}G^{\lambda}.\end{aligned}$$

This proves (7.5) for $n = 1$.

Let $n > 1$ and assume that the assertion of Theorem 7.1.2 holds in spaces of smaller dimension. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1}\times\mathbb{R}$. For $s \in \mathbb{R}$ we write $h_s(z) := h(z,s)$ for $z \in \mathbb{R}^{n-1}$, and f_s, g_s are defined similarly. Let $x, y \in \mathbb{R}^{n-1}$, $a, b \in \mathbb{R}$ and $c := (1-\lambda)a + \lambda b$. Then

$$\begin{aligned}&h_c((1-\lambda)x + \lambda y) = h((1-\lambda)x + \lambda y, (1-\lambda)a + \lambda b) \\ &= h((1-\lambda)(x,a) + \lambda(y,b)) \geq f(x,a)^{1-\lambda}g(y,b)^{\lambda} = f_a(x)^{1-\lambda}f_b(y)^{\lambda}.\end{aligned}$$

By the induction hypothesis,

$$\int_{\mathbb{R}^{n-1}} h_c(x)\,\mathrm{d}x \geq \left(\int_{\mathbb{R}^{n-1}} f_a(x)\,\mathrm{d}x\right)^{1-\lambda}\left(\int_{\mathbb{R}^{n-1}} g_b(x)\,\mathrm{d}x\right)^{\lambda}.$$

We put

$$H(c) := \int_{\mathbb{R}^{n-1}} h_c(x)\,\mathrm{d}x, \quad F(a) := \int_{\mathbb{R}^{n-1}} f_a(x)\,\mathrm{d}x, \quad G(b) := \int_{\mathbb{R}^{n-1}} g_b(x)\,\mathrm{d}x.$$

Then the previous inequality can be written as

$$H((1-\lambda)a + \lambda b) \geq F(a)^{1-\lambda}G(b)^{\lambda}.$$

Using Fubini's theorem and the already settled case $n = 1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} h_c(z)\,\mathrm{d}z\,\mathrm{d}c = \int_{\mathbb{R}} H(c)\,\mathrm{d}c \\ &\geq \left(\int_{\mathbb{R}} F(a)\,\mathrm{d}a\right)^{1-\lambda}\left(\int_{\mathbb{R}} G(b)\,\mathrm{d}b\right)^{\lambda} = \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{\lambda}.\end{aligned}$$

This finishes the proof. □

Before investigating conclusions from the Brunn–Minkowski theorem more systematically in the next section, we want to present here a particularly striking one, the isoperimetric property of the ball.

The surface area $S(K)$ of a convex body $K \in \mathcal{K}^n_n$ is, by definition, the $(n-1)$-dimensional Hausdorff measure of its boundary, $S(K) = \mathcal{H}^{n-1}(\mathrm{bd}\,K)$. From (4.32) and (4.8) we have

$$S(K) = \lim_{\varepsilon \downarrow 0} \frac{V_n(K + \varepsilon B^n) - V_n(K)}{\varepsilon}, \tag{7.7}$$

and this is all we need to know at present about the surface area. Starting from (7.7), the Brunn–Minkowski inequality immediately yields

$$S(K) \geq \lim_{\varepsilon \downarrow 0} \frac{\left(V_n(K)^{1/n} + \varepsilon V_n(B^n)^{1/n}\right)^n - V_n(K)}{\varepsilon} = nV_n(K)^{(n-1)/n} V_n(B^n)^{1/n}.$$

Since $S(B^n) = nV_n(B^n)$, this can be written as

$$\left(\frac{S(K)}{S(B^n)}\right)^{1/(n-1)} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{1/n}. \tag{7.8}$$

This is the *isoperimetric inequality*, saying that among all convex bodies of given positive volume the balls have the smallest surface area. To show that balls are the only extremal bodies, the proof must be refined slightly; we leave this to the next section.

Along with a sharp inequality comes the question of its stability. We discuss this briefly for the Brunn–Minkowski inequality. The uniqueness assertion of Theorem 7.1.1 states that two n-dimensional convex bodies K_0, K_1 that satisfy (7.1) with equality for some $\lambda \in (0, 1)$, must be homothetic. If equality holds only approximately, can one assert that K_0 and K_1 are nearly homothetic? Explicit estimates giving positive answers are usually called stability results. To formulate a more perspicuous special case, assume that $V_n(K_0) = V_n(K_1) = 1$ and $0 < \lambda < 1$. Then Theorem 7.1.1 reduces to the assertion that

$$V_n((1 - \lambda)K_0 + \lambda K_1) \geq 1, \tag{7.9}$$

with equality only if K_0 and K_1 are translates, hence only if $\delta(\tilde{K}_0, \tilde{K}_1) = 0$, where $\tilde{K}_i$ denotes the translate of K_i having its centroid at the origin. A stability result for the inequality (7.9) would ensure that

$$V_n((1 - \lambda)K_0 + \lambda K_1) \leq 1 + \varepsilon \quad \Rightarrow \quad \delta(\tilde{K}_0, \tilde{K}_1) \leq f(\varepsilon),$$

with some explicit function f (possibly involving limitations on the size or degeneracy of K_0, K_1) that satisfies $f(\varepsilon) \to 0$ for $\varepsilon \to 0$. The following result of this type is due to Groemer [793] (it is a special case of his Theorem 3).

Theorem 7.1.3 *Let $K_0, K_1 \in \mathcal{K}^n$ be convex bodies with $V_n(K_0) = V_n(K_1) = 1$, and define $D := \max\{D(K_0), D(K_1)\}$. If*

$$V_n((1 - \lambda)K_0 + \lambda K_1)^{1/n} \leq 1 + \varepsilon \tag{7.10}$$

for some $\varepsilon > 0$ and some $\lambda \in (0, 1)$, then

$$\delta\left(\tilde{K}_0, \tilde{K}_1\right) \leq \eta_n \left(\frac{1}{\sqrt{\lambda(1 - \lambda)}} + 2\right) D\varepsilon^{1/(n+1)} \tag{7.11}$$

with $\eta_n = 6.00025n$.

We refer to Groemer [793] for the proof. It follows the approach of Kneser and Süss (our first proof given above for the Brunn–Minkowski theorem), but introduces explicit estimates. In [793] one also finds some consequences of the theorem, for instance, a formulation for convex bodies of arbitrary volumes. We remark also that Theorem 7.1.3 can be formulated as a strengthened version of (7.9), namely as the inequality

$$V_n((1-\lambda)K_0 + \lambda K_1)^{1/n} \geq 1 + \left\{ \frac{\sqrt{\lambda(1-\lambda)}}{\eta_n[1+2\sqrt{\lambda(1-\lambda)}]D} \right\}^{n+1} \delta\left(\tilde{K}_0, \tilde{K}_1\right)^{n+1}, \qquad (7.12)$$

which is valid for $V_n(K_0) = V_n(K_1) = 1$ and $0 \leq \lambda \leq 1$. This inequality is obtained from (7.11) by setting $\varepsilon = V_n((1-\lambda)K_0 + \lambda K_1)^{1/n} - 1$.

It may seem more natural, in a stability estimate for a volume inequality, to measure the deviation of convex bodies, not in terms of the Hausdorff metric, but in terms of volumes of symmetric differences. Figalli, Maggi and Pratelli [576, 577] have used a new mass transportation proof of the Brunn–Minkowski inequality to obtain the following stability result. For $K, L \in \mathcal{K}_n^n$ they define the *relative asymmetry* of $K, L \in \mathcal{K}_n^n$ by

$$A(K, L) := \inf_{x \in \mathbb{R}^n} \frac{V_n(K \Delta (\lambda L + x))}{V_n(K)}, \quad \text{where } \lambda^n = \frac{V_n(K)}{V_n(L)} \qquad (7.13)$$

(here $A \Delta B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference). Let

$$\beta(K, L) := \frac{V_n(K+L)^{1/n}}{V_n(K)^{1/n} + V_n(L)^{1/n}} - 1$$

denote the *Brunn–Minkowski deficit* of K and L. Then for convex bodies K, L with $V_n(K) = V_n(L) > 0$, the inequality

$$A(K, L) \leq C n^{3.5} \sqrt{\beta(K, L)} \qquad (7.14)$$

holds, with some constant C independent of the dimension. For simplicity, we have formulated this inequality only for the case of equal volumes, but it holds in a more general version. It was proved by Figalli, Maggi and Pratelli [576, 577], with a slightly less favourable dependence on n, and was improved to the form (7.14) by Segal [1765].

Notes for Section 7.1

1. For the reader who wants to get a full impression of the impact of the Brunn–Minkowski inequality in geometry and analysis, there is no better recommendation than to read the already mentioned survey article by Gardner [674]. This article describes many applications, which therefore will not be mentioned in this book. Also the survey by Barthe [159], where the Brunn–Minkowski theorem is the starting point for a fascinating tour through analytic inequalities, is highly recommended. This holds also for the seminar exposition by Maurey [1366], which emphasizes the mass transportation aspect.
2. *The Brunn–Minkowski theorem for convex bodies*. The theorem now named after Brunn and Minkowski was discovered (for dimensions ≤ 3) by Brunn [349, 350]. Its importance was emphasized by Minkowski, who gave an analytic proof for the n-dimensional case (Minkowski [1440]) and characterized the equality case; for the latter, see also Brunn

[351]. Alternative proofs were given by Blaschke [241], §22, who used symmetrization, and, for $n = 3$, by Hilbert [973], who applied his theory of integral equations on the sphere. An improved version is due to Bonnesen [282], Chapter VI. The first proof presented above was given by Kneser and Süss [1122]. All this can be found in Bonnesen and Fenchel [284]. The proof by Kneser and Süss was extended by Hadwiger [911], §6.4.5, to obtain general classes of concave functionals on convex bodies.

Knothe [1125] extended his proof, sketched above, to yield the following more general result. For each $K \in \mathcal{K}_n^n$, let $\rho(K, \cdot)$ be a real function on K and suppose that ρ is nonnegative and continuous in both variables, that

$$\rho(\lambda K + a, \lambda x + a) = \lambda^m \rho(K, x)$$

for $\lambda > 0$ and $a \in \mathbb{R}^n$, where $m > 0$ is a given number, and furthermore that

$$\log \rho(K_\lambda, (1-\lambda)x_0 + \lambda x_1) \geq (1-\lambda) \log \rho(K_0, x_0) + \lambda \log \rho(K_1, x_1)$$

for $x_0 \in K_0$, $x_1 \in K_1$, $\lambda \in [0, 1]$. (An example, with $m = 1$, is given by $\rho(K, x) := d(x, \operatorname{bd} K)$ for $x \in K$, where d denotes the distance.) Under these assumptions, Knothe showed that

$$\left(\int_{K_\lambda} \rho(K_\lambda, x)\, \mathrm{d}x\right)^{\frac{1}{n+m}} \geq (1-\lambda)\left(\int_{K_0} \rho(K_0, x)\, \mathrm{d}x\right)^{\frac{1}{n+m}} + \lambda\left(\int_{K_1} \rho(K_1, x)\, \mathrm{d}x\right)^{\frac{1}{n+m}}$$

for $K_0, K_1 \in \mathcal{K}_n^n$ and $\lambda \in [0, 1]$. Related results, also for non-convex sets, are treated by Dinghas [493].

Knothe's method is also used in Bourgain and Lindenstrauss [316] (see also Bourgain [313]). These authors call the map F appearing in the proof the *Knothe map*. Dinghas [493] called a closely related map the *Brunn–Minkowski–Schmidt map*. He had previously employed it in [485], where he referred to it as an oral communication by E. Schmidt. For more history, see Dinghas [493], footnote 18.

3. Alternative approaches to the Brunn–Minkowski theorem and to its equality conditions were given by McMullen [1395], working with polytopes, and by Klain [1091], working with projections.
4. *Approaches via mass transportation.* Knothe's proof of the Brunn–Minkowski inequality is an early example of a method which nowadays is known as mass transportation. Gromov (Appendix I in [1432]) showed how Knothe's method yields more general inequalities, of Sobolev and isoperimetric type. Whereas the Jacobian in Knothe's approach is upper triangular (with respect to a suitable basis), in the more recent applications of mass transportation, employing the Brenier map, it is symmetric and positive definite. A proof of the Brunn–Minkowski inequality using the Brenier map is nicely presented in an expositary article by Ball [125]. Before that, McCann [1370] used a related aproach to obtain more general inequalities.
5. A stability result for the Brunn–Minkowski inequality, different from Groemer's one quoted as Theorem 7.1.3, was proved earlier by Diskant [501]. Suppose that $K, L \in \mathcal{K}^n$ satisfy $V_n(K) = V_n(L) = 1$. There are constants $\varepsilon_0 > 0$ and $c > 0$, depending on n and the inradius and circumradius of K, such that the inequality

$$V_n((1-\lambda)K + \lambda L) \leq 1 + \varepsilon \quad \text{for all } 0 \leq \lambda \leq 1$$

with some $\varepsilon \in [0, \varepsilon_0]$ implies $\delta(K', L) \leq c\varepsilon^{1/n}$ for a suitable translate K' of K. Note that, in contrast to Groemer's result, the assumption is made for all $\lambda \in [0, 1]$.

An introductory survey of stability properties of geometric inequalities was given by Groemer [795].

6. Dar [466] conjectured an interesting stronger version of the Brunn–Minkowski inequality, namely

$$V_n(K+L)^{1/n} \geq M(K,L)^{1/n} + \left(\frac{V_n(K)V_n(L)}{M(K,L)}\right)^{1/n}$$

for $K, L \in \mathcal{K}_n^n$, where

$$M(K,L) := \max_{x \in \mathbb{R}^n} V_n(K \cap (L+x)).$$

He verified this in some special cases.

Segal [1765] pointed out that the truth of Dar's conjecture would imply an improved version of the stability estimate (7.14), with $Cn^{3.5}$ replaced by Cn.

7. For convex bodies $K, L \in \mathcal{K}_n^n$ and for $\rho \geq 0$, define

$$\alpha_K(L,\rho) := V_n(K \cap \rho L),$$
$$\eta_K(L,\rho) := \mathcal{H}^{n-1}(K \cap \operatorname{bd}(\rho L)).$$

Campi, Gardner and Gronchi [386] noted that if $o \in L$ then it follows from the Brunn–Minkowski theorem that the function $\alpha_K(L,\cdot)^{1/n}$ is concave, and further that here the exponent $1/n$ is best possible, but can be improved in special situations. They made a thorough investigation of the concavity properties of the functions $\alpha_K(L,\cdot)$ and $\eta_K(L,\cdot)$.

8. *From the Brunn–Minkowski inequality to Poincaré-type inequalities.* By the Brunn–Minkowski inequality, the function $V_n^{1/n}$ is concave on $\mathcal{K}_n^n$. As observed by Colesanti [437], on the class of convex bodies of class C_+^2 one can compute the second variation of $V_n^{1/n}$, and since this has to be negative semi-definite, an analytic inequality results. It has the form of a Poincaré-type inequality for C^1 functions on the boundary of a convex body of class C_+^2. A generalization of this inequality, starting from Brunn–Minkowski inequalities for intrinsic volumes, was obtained by Colesanti and Saorín Gómez [448].

9. Colesanti, Hug and Saorín Gómez [445] found that a version of the Brunn–Minkowski inequality can serve to characterize certain mixed volumes. They considered functionals of the form

$$\mathcal{F}(K) = \int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u), \qquad K \in \mathcal{K}^n,$$

with $f \in C(\mathbb{S}^{n-1})$. They proved that if $n \geq 3$ and if $\mathcal{F}$ satisfies

$$\mathcal{F}((1-\lambda)K_0 + \lambda K_1) \geq \min\{\mathcal{F}(K_0), \mathcal{F}(K_1)\}, \qquad K_0, K_1 \in \mathcal{K}^n,\ \lambda \in [0,1], \tag{7.15}$$

then f is the restriction of a support function. Together with Theorem 6.4.9, this implies the following.

Theorem Let $\mathcal{F}$ be a continuous, translation invariant valuation on $\mathcal{K}^n$, $n \geq 3$, which is homogeneous of degree $n-1$ and satisfies (7.15). Then there exists a convex body $L \in \mathcal{K}^n$ such that $\mathcal{F}(K) = V(K[n-1], L)$ for $K \in \mathcal{K}^n$.

10. *Non-convex sets.* As we have seen, the Brunn–Minkowski inequality is not restricted to convex sets. There are, in fact, very general versions of it, even for non-measurable sets. They are connected with the names of Lusternik, Henstock, Macbeath, Dinghas, Hadwiger and Ohmann. Hadwiger and Ohmann [931] gave a particularly beautiful proof, which is reproduced in the survey by Gardner [674] (§4) and at several other places (see the references in [674]). A general treatment of the Brunn–Minkowski theorem for non-convex sets would lead us too far from our topic, the geometry of convex bodies; hence we refer only to Hadwiger [911], Dinghas [494], Burago and Zalgaller [357] for details and references, and to Uhrin [1860, 1861, 1862] for some more recent achievements. And again, we refer to the references in Gardner [674].

11. *The Prékopa–Leindler inequality.* The following formulation of the Prékopa–Leindler theorem appears in Brascamp and Lieb [332]. If f, g are nonnegative measurable functions on $\mathbb{R}^n$, if $\lambda \in (0,1)$ and if k is defined by

$$k(x) := \operatorname{ess\,sup}_{y \in \mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{1-\lambda} g\left(\frac{y}{\lambda}\right)^{\lambda},$$

then k is measurable and

$$\|k\|_1 \geq \|f\|_1^{1-\lambda} \|g\|_1^{\lambda}. \tag{7.16}$$

This implies a more general version of the Brunn–Minkowski inequality (see Gardner [674], Theorem 9.2). Proofs, extensions, analogues and hints about the applications of such results are found in Brascamp and Lieb [332], Das Gupta [467], Uhrin [1861].

Equality conditions for the Prékopa–Leindler inequality are found in Dubuc [516]. Stability estimates for this inequality and their applications appear in the papers by Ball and Böröczky [126, 127].

12. *Milman's reverse Brunn–Minkowski inequality*. If $K, L \in \mathcal{K}^n$ are convex bodies of volume 1, the Brunn–Minkowski inequality says that their sum $K + L$ has volume at least 2^n. The question of an upper bound does not make sense: the volume of $K + L$ can be arbitrarily large. However, the following is true. If $K, L \in \mathcal{K}_n^n$ are centrally symmetric bodies with interior points, there exists a volume-preserving linear transformation Λ of $\mathbb{R}^n$ such that

$$V_n(K + \Lambda L)^{1/n} \leq C[V_n(K)^{1/n} + V_n(L)^{1/n}],$$

where C is a constant independent of n. This theorem is due to Milman [1427]. Different approaches to its proof and relations to other theorems with applications in the local theory of Banach spaces are described in the book by Pisier [1535]. An extension to certain nonconvex sets was proved by Bastero, Bernués and Peña [170].

13. *The Brunn–Minkowski inequality for volume differences.* If $K, L, D, D' \in \mathcal{K}^n$ are such that D and D' are homothetic and $V_n(D) \leq V_n(K)$, $V_n(D') \leq V_n(L)$, it follows from the Brunn–Minkowski theorem and an algebraic inequality of Bellman that

$$[V_n(K+L) - V_n(D+D')]^{1/n} \geq [V_n(K) - V_n(D)]^{1/n} + [V_n(L) - V_n(D')]^{1/n}.$$

This was observed by Leng [1199]. The same proof scheme has later been used in similar situations, for other functionals and other types of addition (as treated in Chapter 9), to extend known inequalities to differences.

14. *Moments of inertia.* For a convex body $K \in \mathcal{K}_n^n$, let $c(K)$ be its centroid (centre of gravity), and define

$$I(K) := \int_K |x - c(K)|^2 \, \mathrm{d}x;$$

thus $I(K)$ is the polar moment of inertia of K with respect to its centroid. Hadwiger [910] has proved that $I^{1/(n+2)}$ is a concave function, and has obtained several related inequalities.

15. *Brunn–Minkowski type inequalities for variational functionals.* Inequalities of the Brunn–Minkowski type have been proved for some functionals from physics and calculus of variations, such as capacities, torsional rigidity and first eigenvalue of the Laplacian. We refer to Borell [287], Caffarelli, Jerison and Lieb [378], Colesanti and Salani [447], Colesanti [435], Colesanti and Cuoghi [438], Colesanti, Cuoghi and Salani [439]. Further references are found in these papers.

16. *Restricted Minkowski addition and convolution bodies.* For $K, L \in \mathcal{K}^n$ (or for more general sets) and for $\theta \in [0,1]$, define

$$K +_\theta L := \left\{ x \in K + L : V_n(K \cap (x - L)) \geq \theta \max_{x \in K+L} V_n(K \cap (x - L)) \right\}.$$

This modification of the Minkowski addition is related to the notion of *convolution bodies*, which in various forms were studied by Schmuckenschläger [1649, 1650], Meyer, Reisner and Schmuckenschläger [1415], Tsolomitis [1857].

Alonso-Gutiérrez, Jiménez and Villa [65] have studied inequalities of the form

$$V_n(K +_\theta L)^{1/2} \geq \varphi(\theta)^{1/n}\left(V_n(K)^{1/n} + V_n(L)^{1/n}\right),$$

together with various other inequalities involving the volume of $K +_\theta L$.

For certain restricted Minkowski sums (of a different kind), a version of the Brunn–Minkowski inequality was proved by Szarek and Voiculescu [1833], and a version of the Prékopa–Leindler inequality by Barthe [160].

17. *Random sets.* A Brunn–Minkowski inequality for set-valued expectations of compact random sets was proved by Vitale in [1887] and a stronger version in [1888].

7.2 The Minkowski and isoperimetric inequalities

We now have two fundamental pieces of information about the volume of the Minkowski convex combination $(1-\lambda)K_0 + \lambda K_1$ of two convex bodies K_0, K_1: it is a polynomial in λ, and its nth root is a concave function. These facts taken together have interesting consequences for mixed volumes.

Let $K_0, K_1 \in \mathcal{K}_n^n$ be n-dimensional convex bodies and let $K_\lambda := (1-\lambda)K_0 + \lambda K_1$ for $0 \leq \lambda \leq 1$. By (5.28),

$$V_n(K_\lambda) = \sum_{i=0}^{n} \binom{n}{i} (1-\lambda)^{n-i}\lambda^i V_{(i)}(K_0, K_1)$$

with

$$V_{(i)}(K_0, K_1) := V(K_0[n-i], K_1[i]) =: V_{(i)}. \tag{7.17}$$

When we use the (classical) notation $V_{(i)}$, it will be clear from the context that K_0, K_1 are given and held fixed. We recall that in (5.32) we have already introduced the special notation

$$V_1(K_0, K_1) = V_{(1)}(K_0, K_1) = V(K_0[n-1], K_1).$$

(These two different notations have been chosen in order to avoid ambiguities in later extensions.)

The function f defined by

$$f(\lambda) := V_n(K_\lambda)^{1/n} - (1-\lambda)V_n(K_0)^{1/n} - \lambda V_n(K_1)^{1/n}$$

for $0 \leq \lambda \leq 1$ is concave, by Theorem 7.1, and it satisfies $f(0) = f(1) = 0$. Hence, its derivative at 0,

$$f'(0) = V_{(0)}^{(n-1)/n}\left[V_{(1)} - V_{(0)}^{(n-1)/n} V_{(n)}^{1/n}\right],$$

fulfils $f'(0) \geq 0$, and $f'(0) = 0$ only if f is identically 0. The latter implies equality in (7.1). Furthermore,

$$f''(0) = -(n-1)V_{(0)}^{(2n-1)/n}\left[V_{(1)}^2 - V_{(0)}V_{(2)}\right] \leq 0.$$

This yields the following.

Theorem 7.2.1 (Minkowski's inequalities) *For n-dimensional convex bodies $K, L \in \mathcal{K}_n^n$,*

$$V(K,\dots,K,L)^n \geq V_n(K)^{n-1}V_n(L). \tag{7.18}$$

Equality holds if and only if K and L are homothetic. Further,

$$V(K,\dots,K,L)^2 \geq V_n(K)V(K,\dots,K,L,L). \tag{7.19}$$

One often calls (7.18) 'Minkowski's first inequality', and (7.19) is known as 'Minkowski's second inequality'.

The equality cases of (7.19) are harder to identify; this will be done in Theorem 7.6.19. More general quadratic inequalities for mixed volumes are proved in the next section. These again imply (7.18). However, (7.19) is required for their proof.

We remark that Minkowski's first inequality directly implies the Brunn–Minkowski inequality. In fact, for $K, L, M \in \mathcal{K}^n$, inequality (7.18) yields

$$V_1(M, K+L) = V_1(M,K) + V_1(M,L) \geq V_n(M)^{(n-1)/n}\left[V_n(K)^{1/n} + V_n(L)^{1/n}\right].$$

With $M = K + L$, this gives the Brunn–Minkowski inequality.

By considering the special case where K or L is equal to a ball, Minkowski's inequality (7.18) reduces to well-known and fundamental geometric inequalities for convex bodies. Recall that

$$S(K) := nW_1(K) = nV(K[n-1], B^n)$$

is the surface area of K and

$$w(K) = \frac{2}{\kappa_n}W_{n-1}(K) = \frac{2}{\kappa_n}V(K, B^n[n-1])$$

is the mean width of K. Hence, (7.18) implies, as already mentioned, the isoperimetric inequality, which states that the volume V_n and surface area S of an n-dimensional convex body satisfy

$$\left(\frac{S}{\omega_n}\right)^n \geq \left(\frac{V_n}{\kappa_n}\right)^{n-1}, \tag{7.20}$$

with equality (by Theorem 7.2.1) if and only if K is a ball. It also implies *Urysohn's inequality* for volume and mean width,

$$\left(\frac{w}{2}\right)^n \geq \frac{V_n}{\kappa_n}, \tag{7.21}$$

with the same equality cases. A weaker form is the isodiametric (or Bieberbach's) inequality

$$\left(\frac{D}{2}\right)^n \geq \frac{V_n}{\kappa_n} \tag{7.22}$$

for the diameter D of K, which follows trivially since $D(K) \geq w(K)$.

The improved version of the Brunn–Minkowski theorem given by Theorem 7.1.3 also leads to a stability result for Minkowski's inequality (7.18). For a convex body $K \in \mathcal{K}_n^n$, we use the normalization defined by

$$\widetilde{K} := V_n(K)^{-1/n}[K - c(K)], \tag{7.23}$$

where $c(K)$ denotes the centroid of K. Thus $\widetilde{K}$ has centroid o and volume 1.

Theorem 7.2.2 *Let $K, L \in \mathcal{K}_n^n$ and set $D := \max\{D(\widetilde{K}), D(\widetilde{L})\}$. Then*

$$\frac{V(K, \dots, K, L)^n}{V_n(K)^{n-1}V_n(L)} - 1 \geq \frac{\gamma_n}{D^{n+1}}\delta(\widetilde{K}, \widetilde{L})^{n+1}, \tag{7.24}$$

where $\gamma_n = 2n/(4\eta_n)^{n+1}$ and η_n is given in Theorem (7.1.3).

Although we did not reproduce Groemer's proof of Theorem 7.1.3, we give the simple argument of Groemer [796] to deduce (7.24).

Proof First suppose that $V_n(K) = V_n(L) = 1$ and $c(K) = c(L) = o$, and put $f(\lambda) := V_n((1-\lambda)K + \lambda L)^{1/n}$ and

$$u(\lambda) := \left\{\frac{\sqrt{\lambda(1-\lambda)}}{\eta_n[1 + 2\sqrt{\lambda(1-\lambda)}]D}\right\}^{n+1} \delta(K, L)^{n+1}.$$

Then $f(\lambda) \geq 1 + u(\lambda)$ by (7.12), and $f(0) = f(1) = 1$. By the Brunn–Minkowski theorem, f is concave, hence $f'(0) \geq [f(h) - f(0)]/h$ for $0 < h \leq 1$ and in particular $f'(0) \geq 2[f(1/2) - 1] \geq 2u(1/2)$. Since $f'(0) = V(K, \dots, K, L) - 1$, we obtain

$$V(K, \dots, K, L) \geq 1 + \frac{\gamma_n}{nD^{n+1}}\delta(K, L)^{n+1}$$

and thus

$$V(K, \dots, K, L)^n \geq 1 + \frac{\gamma_n}{D^{n+1}}\delta(K, L)^{n+1}.$$

Inequality (7.24) for convex bodies of arbitrary volume follows easily. □

As in the case of the Brunn–Minkowski theorem, one might prefer measuring the deviation in terms of volume, namely by the relative asymmetry (7.13). In fact, the estimate (7.14) was deduced by Figalli, Maggi and Pratelli [577] from a stability result for the first Minkowski inequality. For convex bodies $K, L \in \mathcal{K}_n^n$, define their *relative isoperimetric deficit* by

$$\vartheta(K, L) := \frac{V_1(K, L)}{V_n(K)^{(n-1)/n}V_n(L)^{1/n}} - 1.$$

In the version of Segal [1765], the mentioned stability result reads

$$A(K,L) \le C_1 n^{3.5} \sqrt{\vartheta(K,L)},$$

with some universal constant C_1.

Next, we formulate a theorem that strengthens Minkowski's first inequality (7.18) in several respects. It may look rather complicated at first sight, but it has interesting consequences and admits specializations of intuitive appeal.

Theorem 7.2.3 *Let $K, L \in \mathcal{K}_n^n$ be convex bodies, and let $\Omega \subset \mathbb{S}^{n-1}$ be a closed set such that*

$$K = \bigcap_{u \in \Omega} H^-(K,u). \tag{7.25}$$

If $\overline{L}$ is defined by

$$\overline{L} := \bigcap_{u \in \Omega} H^-(L,u), \tag{7.26}$$

then

$$V_1(K,L)^{\frac{n}{n-1}} - V_n(K)V_n(\overline{L})^{\frac{1}{n-1}} \ge \left[V_1(K,L)^{\frac{1}{n-1}} - r(K,L)V_n(\overline{L})^{\frac{1}{n-1}}\right]^n, \tag{7.27}$$

where r(K, L) denotes the inradius of K relative to L.

This theorem is due to Diskant [502]; the special case $\Omega = \mathbb{S}^{n-1}$ (in which $\overline{L} = L$) had been treated before by Diskant [500]. We shall present Diskant's proof in Section 7.5.

Solving inequality (7.27) for $r(K,L)$ and choosing $\Omega = \mathbb{S}^{n-1}$, we obtain

$$r(K,L) \ge \left(\frac{V_1(K,L)}{V_n(L)}\right)^{\frac{1}{n-1}} - \frac{[V_1(K,L)^{\frac{n}{n-1}} - V_n(K)V_n(L)^{\frac{1}{n-1}}]^{\frac{1}{n}}}{V_n(L)^{\frac{1}{n-1}}}, \tag{7.28}$$

which will be used in the proof of Theorem 8.5.1.

To interpret inequality (7.27), we suppose that the assumptions of Theorem 7.2.3 are satisfied. We may assume that $r(K,L)L \subset K$. If $x \in \mathbb{R}^n \setminus K$, there exists $u \in \Omega$ with $x \notin H^-(K,u)$ and hence

$$x \notin H^-(r(K,L)L,u) = H^-(r(K,L)\overline{L},u).$$

It follows that

$$r(K,L)\overline{L} \subset K.$$

Further, we remark that

$$V_1(K,L) = V_1(K,\overline{L}).$$

This is easy to prove, but we refer to Section 7.5, where the result is obtained in the course of the proof of Theorem 7.2.3. Using Theorem 7.2.1, we obtain

$$V_1(K,L)^n = V_1(K,\overline{L})^n \ge V_n(K)^{n-1}V_n(\overline{L}) \ge r(K,L)^{n(n-1)}V_n(\overline{L})^n.$$

Hence, for the right-hand side of (7.27) we have

$$V_1(K,L)^{1/(n-1)} - r(K,L)V_n(\overline{L})^{1/(n-1)} \geq 0,$$

with equality if and only if K is homothetic to $\overline{L}$. Since $V_n(\overline{L}) \geq V_n(L)$, it is now clear that inequality (7.27) is an improvement of Minkowski's inequality (7.18).

Inequality (7.27) can be used to obtain some stability results; see Note 1 and Theorem 8.5.1. Here we discuss only the special case obtained by taking for L the unit ball $B = B^n$.

Let $K \in \mathcal{K}_n^n$ be given. Denoting its volume by V, its surface area by S and its inradius by r, we obtain from (7.27) the inequality

$$\left(\frac{S}{\omega_n}\right)^{\frac{n}{n-1}} - \left(\frac{V_n(\overline{B})}{\kappa_n}\right)^{\frac{1}{n-1}} \frac{V}{\kappa_n} \geq \left[\left(\frac{S}{\omega_n}\right)^{\frac{1}{n-1}} - r\left(\frac{V_n(\overline{B})}{\kappa_n}\right)^{\frac{1}{n-1}}\right]^n. \tag{7.29}$$

The special case $\Omega = \mathbb{S}^{n-1}$ gives

$$\left(\frac{S}{\omega_n}\right)^{\frac{n}{n-1}} - \frac{V}{\kappa_n} \geq \left[\left(\frac{S}{\omega_n}\right)^{\frac{1}{n-1}} - r\right]^n. \tag{7.30}$$

Since $\omega_n r^{n-1}$ is the surface area of the inball of K, it is clear that the right-hand side of (7.30) is nonnegative and is zero only if K is a ball. Thus, (7.30) estimates the 'isoperimetric deficit' $(S/\omega_n)^{n/(n-1)} - (V/\kappa_n)$ in terms of a quantity that has a simple geometric meaning and is obviously positive if K is not a ball.

Similarly, choosing $K = B^n$ and $\Omega = \mathbb{S}^{n-1}$ in (7.27), we obtain an improvement of the Urysohn inequality between the mean width w and the volume V of a convex body of circumradius R, namely

$$\left(\frac{w}{2}\right)^{\frac{n}{n-1}} - \left(\frac{V}{\kappa_n}\right)^{\frac{1}{n-1}} \geq \left[\left(\frac{w}{2}\right)^{\frac{1}{n-1}} - \frac{1}{R}\left(\frac{V}{\kappa_n}\right)^{\frac{1}{n-1}}\right]^n. \tag{7.31}$$

Next, if Ω is a closed set satisfying (7.25), we note that (7.29) implies

$$S^n/V^{n-1} \geq n^n V(\overline{B}), \tag{7.32}$$

with equality if and only if K is homothetic to $\overline{B}$. One says (see Section 7.5) that the convex body K is *determined by* Ω if (7.25) holds. Thus, among all convex bodies determined by a given closed set $\Omega \subset \mathbb{S}^{n-1}$, precisely the bodies circumscribed about a ball, that is, homothetic to

$$\bigcap_{u\in\Omega} H^-(B^n, u),$$

have the smallest 'isoperimetric ratio' S^n/V^{n-1}. For finite sets Ω, this is a theorem first proved (for $n = 3$) by Lindelöf [1218] and in a different way by Minkowski [1435]; for general sets Ω it was proved by Aleksandrov [15].

The smallest closed set Ω that one can take for a given convex body K so that (7.25) holds is the closure of the set of outer unit normal vectors at regular boundary points of K. If Ω is this set, then

$$K_* := \bigcap_{u \in \Omega} H^-(B^n, u)$$

is called the *form body* of K. Inequality (7.32) now reads

$$S^n / V^{n-1} \geq n^n V_n(K_*), \tag{7.33}$$

with equality if and only if K is homothetic to its form body. Using Theorem 2.2.10, one sees that this is the case if and only if K is a tangential body of a ball. Inequality (7.33) strengthens the isoperimetric inequality for convex bodies with singularities.

Notes for Section 7.2

1. *Minkowski's inequalities.* In the plane $\mathbb{R}^2$, there is only one Minkowskian inequality, namely

$$V(K_1, K_2)^2 \geq V_2(K_1)V_2(K_2)$$

for the *mixed area* $V(K_1, K_2)$, with equality if and only if K_1 and K_2 are homothetic. There are several proofs and improvements in the literature; see Bonnesen and Fenchel [284], §51, and Note 3 below.

Upper bounds for the mixed area in terms of perimeters were obtained by Betke and Weil [219]. For convex bodies $K_1, K_2 \in \mathcal{K}^2$ with perimeters $L(K_1)$, $L(K_2)$, they showed that

$$V(K_1, K_2) \leq \frac{1}{8} L(K_1)L(K_2),$$

with equality if and only if K_1 and K_2 are orthogonal segments (possibly degenerate), and

$$V(K, -K) \leq \frac{\sqrt{3}}{18} L(K)^2.$$

Here the equality sign holds if K is an equilateral triangle, and probably only in this case, but this has only been proved if K is a polygon.

For $n = 3$, the cubic inequality (7.18) is a consequence of the quadratic inequalities of type (7.19). These inequalities were first proved by Minkowski [1438] (announced in [1436, 1437]; see Bonnesen and Fenchel [284], §52, for more information). For general dimension n, (7.18) was proved by Süss [1829], using the Brunn–Minkowski theorem, and Bonnesen [283] noticed that it can be obtained from the Brunn–Minkowski theorem in the simple way described in the proof of Theorem 7.2.1.

The *stability problem* for the Minkowski inequality (7.18) requires us to find explicit estimates for the deviation of suitable homothets of convex bodies $K, L \in \mathcal{K}^n_n$ in terms of the 'deficit'

$$\frac{V_1(K, L)^n}{V_n(K)^{n-1}V_n(L)} - 1.$$

Theorem 7.2.2 gives a result of this type, which is due to Groemer [796]. The first result of this kind had already been obtained by Minkowski [1438], §6, who proved his inequality (7.18), for $n = 3$, in the following sharper form. Let $K, L \in \mathcal{K}^3_3$ have coinciding centroids, and let D be the maximum of

$$\frac{\sqrt[3]{V_3(K)}h(L, \cdot)}{\sqrt[3]{V_3(L)}h(K, \cdot)}$$

over $\mathbb{S}^2$; then (inequality (76) of Minkowski [1438])

$$\frac{V(K,K,L)}{\sqrt[3]{V_3(K)^2V_3(L)}} - 1 \geq \frac{1}{2^{10}3^4 7^{4/3}} \frac{(D-1)^6}{D^5}.$$

Stability estimates for the Minkowski inequality (7.18) in dimension n were obtained by Volkov [1898] and Diskant [500]; see also Diskant [501] and Bourgain and Lindenstrauss [316].

Minkowski's inequality (7.18) is a useful tool for the solution of many different geometric extremum problems. Some examples are found in Ohmann [1483].

2. *The Blaschke diagram*. For a three-dimensional convex body $K \in \mathcal{K}^3$, one often uses the classical notation $V(K,K,K) = V$ (volume), $3V(K,K,B^3) = S$ (surface area), $3V(K,B^3, B^3) = M$ ('integral of mean curvature'). The quadratic Minkowskian inequalities, special cases of (7.19), now state that

$$M^2 \geq 4\pi S,$$

where equality holds if and only if K is a ball, and

$$S^2 \geq 3VM,$$

where equality holds if and only if K is a cap body of a ball (equality conditions are discussed in §7.6). These quadratic inequalities imply the two cubic ones,

$$S^3 \geq 36\pi V^2, \quad M^3 \geq 48\pi^2 V.$$

On the other hand, the two quadratic inequalities are not a complete system of inequalities satisfied by the three functionals V, S, M of convex bodies. Following Blaschke, one considers in an (x,y)-plane all points with coordinates

$$x = \frac{4\pi S}{M^2}, \quad y = \frac{48\pi^2 V}{M^3},$$

for all possible three-dimensional convex bodies, and finds that part of the boundary of the resulting 'Blaschke diagram' must correspond to a third sharp inequality, of the form $V \geq f(S,M)$, which is still unknown. This problem is neatly discussed in Hadwiger [908], §§28–29. Later investigations resulted in the non-sharp inequality

$$V \geq \frac{S}{12\pi^2 M}\left(\frac{\pi^3}{2}S - M^2\right),$$

proved by Groemer [781] (extended to higher dimensions by Firey [587]), and in the interesting conjecture, due to Sangwine-Yager [1620], that

$$\frac{S}{M} \leq \frac{8}{\pi^2}\frac{M}{4\pi} + \left(1 - \frac{8}{\pi^2}\right)\frac{3V}{S}.$$

3. *Diskant's inequality*. Diskant's [502] improvement (7.27) of Minkowski's inequality extends and unifies many earlier (and even some later) results.

The two-dimensional case, for $\Omega = \mathbb{S}^1$, can be written in the form

$$A_{01}^2 - A_0 A_1 \geq (A_{01} - rA_1)^2 \tag{7.34}$$

with $A_{01} := V(K,L)$, $A_0 := V_2(K)$, $A_1 := V_2(L)$, $r := r(K,L)$; an equivalent formulation is

$$A_0 - 2rA_{01} + r^2 A_1 \leq 0. \tag{7.35}$$

If $K \in \mathcal{K}_2^2$ has area A, perimeter L and inradius r, a special case of (7.34) gives

$$L^2 - 4\pi A \geq (L - 2\pi r)^2. \tag{7.36}$$

For various proofs of this inequality, see the references in Note 4, but also Bonnesen and Fenchel [284], p. 113, and Hadwiger [886, 894].

Inequality (7.30) was first proved by Hadwiger [897] for $n = 3$ and by Dinghas [486] for general n; a short proof is given in Hadwiger [911], §6.5.2. Dinghas [486] had already obtained a slightly weaker inequality of the form (7.29).

Inequality (7.33) was proved in Hadwiger [911], §6.5.5. Several special cases had been treated before by Bol [268], Bol and Knothe [273], Dinghas [481, 482, 483, 484, 488, 489, 490], but the easier deductibility from Minkowski's inequalities was not always noticed.

The improvement of the isoperimetric inequality by (7.33) is strong if K has sharp singularities. Improvements that are strong if K has large flat pieces in the boundary were treated by Hadwiger [896]; see also Ohmann [1483].

4. *Bonnesen-type inequalities*. Let $n = 2$, $K_0, K_1 \in \mathcal{K}_2^2$, $V(K_0, K_1) =: A_{01}$, $V_2(K_0) =: A_0$, $V_2(K_1) := A_1$, $r(K_0, K_1) := r$ and $r(K_1, K_0) =: 1/R$, so that R is the circumradius of K_0 relative to K_1; in other words, $K_0 \subset RK_1'$ for a suitable translate K_1' of K_1, and R is the smallest number with this property. With these notations, inequality (7.35) can be generalized to

$$A_0 - 2\rho A_{01} + \rho^2 A_1 \leq 0, \tag{7.37}$$

or equivalently

$$A_{01}^2 - A_0 A_1 \geq (A_{01} - \rho A_1)^2, \tag{7.38}$$

for all numbers ρ in the interval $[r, R]$. The two cases $\rho = r$ and $\rho = R$ together with $(a^2 + b^2)/2 \geq [(a + b)/2]^2$ yield

$$A_{01}^2 - A_0 A_1 \geq \frac{A_1^2}{4}(R - r)^2. \tag{7.39}$$

If $K_1 = B^2$, these inequalities read

$$A - \rho L + \pi\rho^2 \leq 0, \tag{7.40}$$

$$L^2 - 4\pi A \geq (L - 2\pi\rho)^2 \tag{7.41}$$

for $r \leq \rho \leq R$, and

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{7.42}$$

where r, R are respectively the usual inradius and circumradius. The last inequality has a sharper version, namely

$$L^2 - 4\pi A \geq 4\pi (R_0 - r_0)^2, \tag{7.43}$$

where r_0 and $R_0, r_0 \leq R_0$, are the radii of two concentric circles enclosing bd K_0 such that $R_0 - r_0$ is minimal.

All these results are known as Bonnesen's inequalities. Inequalities (7.41) and thus (7.42) were proved by Bonnesen [280] and are reproduced in Bonnesen [282], pp. 60–63 (see also Eggleston [532], pp. 108–110); for (7.43), see Bonnesen [281] and [282], pp. 70–74. An elegant integral-geometric proof of (7.42), due to Santaló, can be found in Blaschke [252], §11. Osserman [1494] gives a comprehensive survey of inequalities equivalent to (7.41) and their extensions, including non-convex curves and curves on surfaces. New treatments of Bonnesen-type inequalities were given by Gage [657] and by Fuglede [647], who considers non-convex curves, and more recently by Klain [1087, 1089].

An integral-geometric proof of (7.39) appears in Blaschke [252], §15, where the inequality itself is ascribed to Bonnesen. Also related to integral geometry are the proofs given by Hadwiger [885], Fejes Tóth [565], Flanders [616]. Fourier series are used by Bol [267] and Wallen [1903].

The equality conditions for the fundamental inequality (7.37) are as follows. Equality holds only if either $\rho = r$ or $\rho = R$. For $\rho = r$, equality holds if and only if K_0 is the sum of a homothet of K_1 and a line segment (possibly degenerate). For $\rho = R$, equality holds if and only if K_1 is the sum of a homothet of K_0 and a line segment. This was proved by Bol [267], and a different proof was given by Böröczky, Lutwak, Yang and Zhang [295], Theorem 4.1. It follows that equality in (7.39) holds if and only if K_0 and K_1 are either homothetic or are parallelograms with parallel sides.

In higher dimensions, the inequality (7.30), that is,

$$\left(\frac{S}{\omega_n}\right)^{\frac{n}{n-1}} - \frac{V}{\kappa_n} \geq \left[\left(\frac{S}{\omega_n}\right)^{\frac{1}{n-1}} - r\right]^n$$

for the volume V and surface area S of a convex body can be considered as a generalization of (7.41) for the value $\rho = r$, the inradius. An extension of (7.40) for $\rho = r$ is the inequality

$$V - rS + (n-1)\kappa_n r^n \leq 0, \tag{7.44}$$

where equality holds only for the ball. Inequality (7.44) was conjectured by Wills [1980] and proved independently by Bokowski [264] and Diskant [503] and in a slightly sharper form by Osserman [1494]. In each case, the inequality is deduced from (7.29). Further strengthenings of the inequality were obtained by Brannen [330].

For $n = 2$, inequality (7.44) remains true if the inradius r is replaced by the circumradius R, giving

$$A - RL + \pi R^2 \leq 0; \tag{7.45}$$

however, this does not hold in dimensions $n \geq 3$. Instead, one has

$$(n-1)V - 2RS + (n+1)\kappa_n R^n \geq 0 \tag{7.46}$$

for $n \geq 2$ (note the reversed inequality sign). This was proved by Bokowski and Heil [266], who obtained the following more general result.

Theorem For an arbitrary convex body $K \in \mathcal{K}^n, n \geq 2$, with circumradius R and quermassintegrals $W_0, \dots, W_n$, the inequalities

$$c_{ijk}R^i W_i + c_{jki}R^j W_j + c_{kij}R^k W_k \geq 0 \tag{7.47}$$

hold for $0 \leq i < j < k \leq n$ and $c_{pqr} := (r-q)(p+1)$.

Inequality (7.46) is a special case of (7.47), as is the inequality

$$iW_{i-1} - 2(i+1)RW_i + (i+2)R^2 W_{i+1} \geq 0$$

for $i = 1, \dots, n-1$.

For $n = 2$, (7.47) reduces to

$$A - 2RL + 3\pi R^2 \geq 0, \tag{7.48}$$

with equality only for circles; this is due to Favard [551].

An extension of (7.45) to higher dimensions that does hold is the inequality

$$W_{n-2} - \beta RW_{n-1} + (\beta - 1)R^2 W_n \leq 0 \tag{7.49}$$

with $\beta = n/(n-1)$. More generally, Sangwine-Yager [1619] established the inequalities

$$V_n(K) - nrV_1(K, L) + (n-1)r^2 V_{(2)}(K, L) \leq 0 \tag{7.50}$$

and

$$(n-1)V_{(n-2)}(K,L) - nRV_{(n-1)}(K,L) + R^2V_n(L) \le 0 \tag{7.51}$$

for $K, L \in \mathcal{K}^n$, with $r = r(K,L)$ and $R = 1/r(K,L)$. Equality in (7.50) holds if K is an $(n-2)$-tangential body of L, and in (7.51) if L is an $(n-2)$-tangential body of K. The special case $n = 3, L = B^3$ of (7.50) has already been treated by Sangwine-Yager [1618], where the author was able to show that equality holds only for cap bodies of balls.

Heil [952] investigated whether (7.49) can be improved for $n \ge 3$ by increasing β and, by an interesting application of the calculus of variations, he found that this is the case, at least if the body considered is centrally symmetric or if R is replaced by half the diameter.

For further Bonnesen inequalities, see Note 4 of §7.7.

5. In the plane, we again denote area and perimeter by A and L, respectively. For a convex body $K \in \mathcal{K}^2$, Alvino, Ferone and Nitsch [66] defined the *Hausdorff asymmetry index* by

$$\delta_a(K) := \min\{\delta(K, B(z,\rho)) : z \in \mathbb{R}^2, A(B(z,\rho)) = A(K)\}.$$

They proved that among the convex domains $K \in \mathcal{K}^2$ with given Hausdorff asymmetry index, there is, up to a similarity, a unique one that minimizes the isoperimetric deficit, and they showed how to construct it. A corollary of their results is the sharp inequality

$$L(K)^2 - 4\pi A(K) \ge 16\,\delta_a(K)^2.$$

6. *The isoperimetric inequality*. The isoperimetric inequality has many facets, and with all its versions and ramifications it can easily fill a book by itself. In fact, isoperimetric inequalities are one of the central themes of the book by Burago and Zalgaller [357], which a reader interested in a broad overview of isoperimetric problems should consult first. An impressive survey of the various ramifications of the isoperimetric inequality, mainly from the viewpoint of analysis and differential geometry, and its applications to these fields, is given by Osserman [1493]. The impact of isoperimetric inequalities on analysis is described by Chavel [408].

Minkowski's discovery that the isoperimetric inequality for convex bodies can be deduced from the Brunn–Minkowski theorem has strongly influenced the later development. Minkowski's inequality (7.18) can be interpreted as an isoperimetric inequality for a suitably defined notion of relative surface area, and versions of the Brunn–Minkowski theorem for non-convex sets lead to corresponding isoperimetric inequalities (for a brief version of the proof, including the Brunn–Minkowski inequality, see Federer [557], §§3.2.41–3.2.43). Here the uniqueness question often poses major problems. We refer the reader to the treatments in Hadwiger [911] and Dinghas [494]. Further references are Busemann [366], Ohmann [1477], Baebler [105], Barthel and Bettinger [166, 167].

7. *Minkowski's first inequality and the isoperimetric inequality for non-convex sets.* Although non-convex sets are generally outside the scope of this book, we give references here to Minkowski's first inequality and the isoperimetric inequality for such sets.

A very general treatment was given by Busemann [366], in the course of his investigations of the isoperimetric inequality in Minkowski spaces. Given a convex body $K \in \mathcal{K}^n_n$ and an arbitrary bounded set $M \subset \mathbb{R}^n$, Busemann defined the Minkowski area $B(M)$ of M relative to K as

$$B(M) := \liminf_{\varepsilon \downarrow 0} \varepsilon^{-1}(|M + \varepsilon K| - |\mathrm{cl}\, M|),$$

where $|\cdot|$ denotes the outer Lebesgue measure. From Lusternik's [1260] version of the Brunn–Minkowski theorem, Busemann immediately deduced that

$$B(M) \ge n|\mathrm{cl}\, M|^{(n-1)/n}|K|^{1/n}.$$

He was able to give a complete characterization of the equality cases.

The 'classical isoperimetric theorem', in the terminology of today, is the one for sets of finite perimeter, first proved by de Giorgi in 1958. For information and further references, we refer to the book by Chavel [408], the Handbook article by Talenti [1837] and the lecture notes by Fusco [654].

8. *The isoperimetric inequality in spherical and hyperbolic spaces.* The extremal property of the ball in these cases was established by Schmidt [1645]; see also Burago and Zalgaller [356], §10.2. In spherical space $\mathbb{S}^n$, the following fact is known also as the 'isoperimetric inequality'. Let $A \subset \mathbb{S}^n$ be measurable, and let $C \subset \mathbb{S}^n$ be a ball with $\sigma_n(A) = \sigma_n(C)$, where σ_n denotes the spherical volume in $\mathbb{S}^n$. For $\varepsilon > 0$, let A_ε be the set of all points of $\mathbb{S}^n$ with distance from A at most ε. Then $\sigma(A_\varepsilon) \geq \sigma_n(C_\varepsilon)$. Proofs by different types of symmetrization are found in Schmidt [1645], Figiel, Lindenstrauss and Milman [578] (Appendix), Benyamini [195], Schechtman [1641].
9. Isoperimetric problems in Minkowski spaces are the subject of papers by Busemann [364, 366], Barthel [165], Holmes and Thompson [986]; they are thoroughly treated in the book by Thompson [1845].
10. For isoperimetric inequalities in mathematical physics and in analysis that are analogous to the geometric one, but concern quantitities with various physical meanings, we refer to the classical book of Pólya and Szegö [1542] and to the more recent one of Bandle [131], and further to the Handbook article by Talenti [1838].
11. *Stability estimates for the isoperimetric inequality*. If the isoperimetric inequality

$$S^n - n^n \kappa_n V_n^{n-1} \geq 0$$

for the surface area S and the volume V_n of an n-dimensional convex body $K \in \mathcal{K}_n^n$ holds with equality, then K is a ball. A stability estimate for this inequality would be any result of the kind

$$S^n - n^n \kappa_n V_n^{n-1} \leq \varepsilon \Rightarrow \Delta(K, B_K) \leq f(\varepsilon),$$

where Δ is some metric on the space of convex bodies (or some other measure for the deviation of two convex bodies), B_K is a suitable ball and f is an explicitly known function, non-decreasing and satisfying $\lim_{\varepsilon\downarrow 0} f(\varepsilon) = f(0) = 0$. This function may involve constants that depend on some given bounds for ε and for the inradius and circumradius of K as well as on the dimension.

In the plane, the Bonnesen inequalities (7.42) and (7.43) immediately yield such stability estimates. The difference $R-r$ of circumradius and inradius is, of course, a satisfactory measure for the deviation of a convex body from a suitable ball. Inequality (7.42) has no immediate extension to higher dimensions, since the example of a convex body close to a segment shows that for $n \geq 3$ the isoperimetric deficit $S^n - n^n \kappa_n V_n^{n-1}$ can be arbitrarily small while $R - r \geq 1$, say. However, one can show that

$$S^n - n^n \kappa_n V_n^{n-1} \geq c(n) r^{n(n-1)} \left(\frac{R-r}{r} \right)^{(n+3)/2} \tag{7.52}$$

for $K \in \mathcal{K}_n^n$, where $c(n)$ is an explicitly known constant depending only on n. For the class of convex bodies with $r \geq c_0 > 0$, c_0 given, inequality (7.52) provides a stability estimate for the isoperimetric inequality. It was obtained by Groemer and Schneider [804], as a consequence of more general results (see also §7.6, in particular (7.125)).

Some results in the literature can be expressed in the form

$$S^n - n^n \kappa_n V_n^{n-1} \geq c\delta(K, B_K)^\alpha, \tag{7.53}$$

where B_K denotes a suitable ball, c may depend on n and bounds for inradius and circumradius of K, and α depends only on n. An inequality of type (7.53) becomes sharper as its exponent α becomes smaller. The specializations of the stability results for the Minkowski inequality due to Volkov [1898] and Diskant [500, 501] mentioned in Note 1, as well as an argument of Osserman [1495] using (7.30), yield estimates of this kind, all with $\alpha \geq n$. In Groemer and Schneider [804], an estimate with $\alpha = (n+3)/2$ was achieved, and it was shown that one cannot have $\alpha < (n+1)/2$. By analytic methods, Fuglede [645, 646] obtained stability estimates for the isoperimetric inequality and under additional assumptions also for non-convex sets, which in a certain sense are of optimal orders.

For the sharp inequality bounding the area of a plane domain of given constant width from below, stability estimates were obtained by Groemer [794].

Very strong stability versions (involving the symmetric difference metric) of the isoperimetric inequality for sets of finite perimeter were obtained by Fusco, Maggi and Pratelli [655] (see also Maggi [1315]), where symmetrization techniques are applied, and by Figalli, Maggi and Pratelli [577] with the aid of mass transportation. The latter paper deals with the anisotropic isoperimetric inequality, where the definition of the perimeter involves a weight function depending on the normal direction.

Stability estimates of Bonnesen type for the isoperimetric inequality for n-dimensional domains with special properties were proved by Rajala and Zhong [1556].

From stability estimates for the isoperimetric inequality, Maggi, Ponsiglione and Pratelli [1316] deduced stability estimates for the isodiametric inequality.

12. *Improved versions of the isoperimetric inequality for restricted classes of convex bodies.* It is clear that on any class of convex bodies that does not contain bodies arbitrarily close to balls, a strengthening of the isoperimetric inequality must be possible (in principle). Simple examples can be found in Hadwiger [911], §§6.5.3, 6.5.4, 6.5.6. A beautiful inequality of this type was proved by Fejes Tóth [564].

   Theorem The surface area S and volume V of a three-dimensional convex polytope with k facets satisfy

   $$\frac{S^3}{V^2} \geq 54(k-2)\tan\alpha_k(4\sin^2\alpha_k - 1), \qquad \alpha_k = \frac{\pi}{6}\frac{k}{k-2}.$$

   Equality holds precisely for the simple regular polytopes (tetrahedron, cube, dodecahedron).

   Further related results (and some interesting problems) are found in the survey article by Florian [621].

   Another example is Hadwiger's improvement of the isoperimetric inequality for 'half-bodies'. A convex body $K \in \mathcal{K}_n^n$ is called a half-body if it has a supporting hyperplane H such that the union of K and its image under reflection in H is convex. Hadwiger [915] proved that the volume and surface area of such a body satisfy

   $$2S^n \geq n^n(\kappa_n + 2\kappa_{n-1})V_n^{n-1}.$$

   Equality holds if K is the union of a half-ball of radius ρ and a right circular cylinder of height ρ attached to the equator $(n-1)$-ball of the half-ball. A different proof will be given in Section 7.7, Note 5.

13. *The ratio of volume and surface area*. While we are on the topic of volume and surface area, we mention that the ratio V_n/S has the property of quasimonotonicity: if $K, L \in \mathcal{K}_n^n$ and $K \subset L$, then

   $$\frac{V_n(K)}{S(K)} \leq n\frac{V_n(L)}{S(L)},$$

   and here the factor n cannot be replaced by a smaller number. This was proved by Wills [1980].

14. *Quermassintegrals and other functionals*. Bonnesen-type inequalities are, as described in Note 4, a class of inequalities involving volume, surface area and possibly other quermassintegrals, together with some other functional of convex bodies, such as inradius or circumradius. There are several inequalities of a similar character, that is, connecting some of the quermassintegrals with additional geometric quantities, for example minimal width, diameter or the sizes of suitably chosen projections or plane sections. Some of these results are strengthenings of isoperimetric inequalities; others prefer various different viewpoints.

In the plane, Santaló [1629] started an investigation of complete sets of inequalities between any three of the functionals area, perimeter, diameter, minimal width, circumradius, inradius. This was continued in a series of papers by Hernández Cifre [959, 960], Hernández Cifre and Segura Gomis [970], Hernández Cifre, Salinas and Segura Gomis [962, 963, 964], Hernández Cifre, Pastor, Salinas Martínez and Segura Gomis [961], Böröczky, Hernández Cifre and Salinas [292]. A few similar inequalities for centrally symmetric convex bodies in $\mathbb{R}^n$ are proved by Hernández Cifre, Salinas and Segura Gomis [964].

For various further inequalities that relate one or more quermassintegrals or mixed volumes to other quantities of convex bodies, we refer to Benson [194], Boček [262], Bonnesen [282] (Chapter 6), Chakerian [400], Favard [553], Firey [590, 591, 597], Hadwiger [911] (p. 292), Knothe [1125] (inequalities (41), (43)), Nádenîk [1458, 1459], Petermann [1524].

15. *Quermassintegrals and mixed volumes of polar bodies*. For $0 \le i, j < n$ and $i + j \ge n - 1$, one has

$$W_i(K)^{n-j} W_j(K^\circ)^{n-i} \ge \kappa_n^{2n-i-j}$$

for $K \in \mathcal{K}^n_{(o)}$. Equality holds precisely for centred balls. For $i + j = n - 1$, this is due to Firey [603] and for $i = j = n - 1$ to Lutwak [1261]. The general result was deduced from Firey's case, independently by Heil [951] and Lutwak [1265]. Firey [586] proved the sharp inequalities $V(K, K^\circ) \ge \kappa_2$ for $n = 2$ and $V(K, K^\circ, B^3) \ge \kappa_3$ for $n = 3$.

For the problem of the infimum of $W_0(K)W_0(K^\circ)$, see §10.7.

Heil [951] remarks that for $n > 3$ the minimum of $W_1(K)W_1(K^\circ)$ is not attained by balls.

Further inequalities of this type were proved by Ghandehari [699], for example, for n convex bodies $K, K_1, \dots, K_{n-1} \in \mathcal{K}^n_{(o)}$ the inequality

$$V(K, K_1, \dots, K_{n-1}) V(K^\circ, K_1, \dots, K_{n-1}) \ge V(B^n, K_1, \dots, K_{n-1})^2,$$

and also inequalities involving dual mixed volumes. Further extensions to the L_p theory (see Section 9.1) are found in Zhao, Chen and Cheung [2066].

7.3 The Aleksandrov–Fenchel inequality

We have seen in the previous section that the Brunn–Minkowski theorem implies the quadratic inequality

$$V(K_0, K_1, \dots, K_1)^2 \ge V(K_0, K_0, K_1, \dots, K_1) V_n(K_1).$$

This is merely a special case of a system of quadratic inequalities satisfied by general mixed volumes.

Theorem 7.3.1 (Aleksandrov–Fenchel inequality) *For $K_1, K_2, K_3, \dots, K_n \in \mathcal{K}^n$,*

$$V(K_1, K_2, K_3, \dots, K_n)^2 \ge V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n). \tag{7.54}$$

Clearly, equality holds in (7.54) if K_1 and K_2 are homothetic. However, examples show that this is by no means the only possibility for equality. The complete classification of the equality cases is an unsolved problem. We shall return to this question and some partial results in Section 7.6.

The proof we shall give for Theorem 7.3.1 is due to Aleksandrov [13]. It uses strongly isomorphic polytopes and approximation.

As in the last part of Section 5.2, we assume that an a-type, $\mathcal{A}$, of strongly isomorphic simple n-dimensional polytopes is given. We use the notation of Sections 5.1 and 5.2; in particular, $u_1, \dots, u_N$ are the unit normal vectors corresponding to the facets of $\mathcal{A}$, the N-tuple $\overline{P} = (h_1, \dots, h_N)$ with $h_i := h(P, u_i)$ is the support vector of $P \in \mathcal{A}$ and the mixed volume of N-tuples $X_1, \dots, X_n \in \mathbb{R}^N$ is defined by (5.39). We shall often identify $P \in \mathcal{A}$ with its support vector $\overline{P}$; that is, if in $V(X_1, \dots, X_n)$ or $v(\Lambda_i X_1, \dots, \Lambda_i X_{n-1})$ (defined by (5.40) and (5.41)) one of the arguments X_r is a support vector $\overline{P}_r$, we replace X_r by P_r and $\Lambda_i X_r$ by $F_i^{(r)} = F(P_r, u_i)$. Further, we say that $Z = (\zeta_1, \dots, \zeta_N) \in \mathbb{R}^N$ is the support vector of a point $z \in \mathbb{R}^n$ if $\zeta_i = h(\{z\}, u_i)$ for $i = 1, \dots, N$, thus if

$$Z = (\langle z, u_1 \rangle, \dots, \langle z, u_N \rangle).$$

The following theorem is a sharper version of a special case of Theorem 7.3.1, and the latter can be deduced from it.

Theorem 7.3.2 *If $P, P_3, \dots, P_n$ are strongly isomorphic polytopes of the simple a-type $\mathcal{A}$ and if $Z \in \mathbb{R}^N$, then*

$$V(Z, P, P_3, \dots, P_n)^2 \geq V(Z, Z, P_3, \dots, P_n) V(P, P, P_3, \dots, P_n).$$

The equality sign holds if and only if $Z = \lambda \overline{P} + A$, where $\lambda \in \mathbb{R}$ and A is the support vector of a point.

Taking for Z the support vector of another polytope in $\mathcal{A}$, we obtain inequality (7.54) for the special case of simple strongly isomorphic polytopes. The general case then follows from the approximation theorem 2.4.15 and the continuity of the mixed volume. The limit procedure is responsible for the fact that the cases of equality cause problems.

Proof of Theorem 7.3.2 We introduce a symmetric bilinear form Φ on $\mathbb{R}^N$ by

$$\Phi(X, Y) := V(X, Y, P_3, \dots, P_n) \quad \text{for } X, Y \in \mathbb{R}^N$$

if $n \geq 2$ (with $P_3, \dots, P_n$ omitted if $n = 2$). It suffices to prove the following proposition.

Proposition 1 If $\Phi(Z, P) = 0$, then $\Phi(Z, Z) \leq 0$, and equality holds if and only if Z is the support vector of a point.

In fact, suppose that Proposition 1 is true. If $Z \in \mathbb{R}^N$ is given, define

$$\lambda := \frac{\Phi(Z, P)}{\Phi(P, P)} \quad \text{and} \quad Z' := Z - \lambda \overline{P}$$

(observe that $\Phi(P, P) = V(P, P, P_3, \dots, P_n) > 0$). Then $\Phi(Z', P) = 0$ and hence $\Phi(Z', Z') \leq 0$, with equality if and only if Z' is the support vector of a point. From

$$\Phi(Z', Z') = \Phi(Z, Z) - \frac{\Phi(Z, P)^2}{\Phi(P, P)}$$

the assertion of the theorem follows.

To prove Proposition 1, we first consider the special case $P_3 = \cdots = P_n = P$ of the theorem.

Proposition 2 The inequality

$$V(Z, P, \dots, P)^2 \geq V(Z, Z, P, \dots, P)V_n(P) \tag{7.55}$$

holds. If $n = 2$, equality holds if and only if $Z = \lambda\overline{P} + A$, where $\lambda \in \mathbb{R}$ and A is the support vector of a point.

For the proof, we note that, by Lemma 2.4.13, $\overline{P} + \varepsilon Z$ is the support vector of a polytope $Q \in \mathcal{A}$ if $|\varepsilon| > 0$ is sufficiently small. From Minkowski's inequality (7.19) we infer that

$$\begin{aligned} 0 &\leq V(Q, P, \dots, P)^2 - V(Q, Q, P, \dots, P)V_n(P) \\ &= \varepsilon^2[V(Z, P, \dots, P)^2 - V(Z, Z, P, \dots, P)V_n(P)]. \end{aligned}$$

This proves (7.55). If $n = 2$, then by Theorem 7.2.1 (first part) equality in (7.55) holds if and only if Q and P are homothetic. This completes the proof of Proposition 2.

We prove Proposition 1 by induction with respect to the dimension. For $n = 2$, the assertion is true by Proposition 2. We assume that $n \geq 3$ and that the assertion of Proposition 1 is valid in smaller dimensions.

For each $i \in \{1, \dots, N\}$ we define a symmetric bilinear form φ_i on $\mathbb{R}^N$ by

$$\varphi_i(X, Y) := v(\Lambda_i X, \Lambda_i Y, F_i^{(4)}, \dots, F_i^{(n)}) \quad \text{for } X, Y \in \mathbb{R}^N$$

(with $F_i^{(4)}, \dots, F_i^{(n)}$ omitted if $n = 3$).

Proposition 3 The vector $Z \in \mathbb{R}^N$ is an eigenvector of the bilinear form Φ with eigenvalue 0 if and only if Z is the support vector of a point.

For the proof, we first note that

$$\Phi(X, Y) = \frac{1}{n} \sum_{i=1}^{N} x_i \varphi_i(Y, P_3)$$

by (5.42). Since $\varphi_i(\cdot, P_3)$ is linear, it is of the form

$$\varphi_i(Y, P_3) = \sum_{j=1}^{N} b_{ij} y_j,$$

thus

$$\Phi(X, Y) = \frac{1}{n} \sum_{i,j=1}^{N} b_{ij} x_i y_j. \tag{7.56}$$

Here $b_{ij} = b_{ji}$, because Φ is symmetric. Stating that $Z = (\zeta_1, \dots, \zeta_N) \neq 0$ is an eigenvector of Φ corresponding to the eigenvalue 0 means that

$$\sum_{j=1}^{N} b_{ij}\zeta_j = 0 \quad \text{for } i = 1, \dots, N,$$

or equivalently that

$$\varphi_i(Z, P_3) = 0 \quad \text{for } i = 1, \dots, N. \tag{7.57}$$

If Z is the support vector of the point z, then

$$\varphi_i(Z, P_3) = v(\{z\}, F_i^{(3)}, \dots, F_i^{(n)}) = 0.$$

Suppose, conversely, that (7.57) holds. By the induction hypothesis, this implies $\varphi_i(Z, Z) \leq 0$. Without loss of generality, we may assume that $h(P_3, u_i) > 0$; then

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{N} \zeta_i\varphi_i(Z, P_3) = \Phi(Z, Z) \\ &= V(Z, Z, P_3, \dots, P_n) = V(P_3, Z, Z, P_4, \dots, P_n) \\ &= \frac{1}{n}\sum_{i=1}^{N} h(P_3, u_i)\varphi_i(Z, Z) \leq 0 \end{aligned}$$

and hence $\varphi_i(Z, Z) = 0$. By the induction hypothesis, this implies that $\Lambda_i Z$ is the support vector, relative to the a-type of F_i, of a point z_i. Explicitly, this means that

$$\Lambda_i Z = (\langle z_i, v_{i1}\rangle, \dots, \langle z_i, v_{iN}\rangle),$$

with $v_{ij} := o$ for $(i, j) \notin J$ (where we use the notation of Section 5.1). We choose $\varepsilon \neq 0$ so that $\overline{P}_3 + \varepsilon Z$ is a support vector of a polytope $Q \in \mathcal{A}$. The equality $\Lambda_i(\overline{P}_3 + \varepsilon Z) = \Lambda_i\overline{P}_3 + \varepsilon\Lambda_i Z$ yields

$$h(F(Q, u_i), v_{ij}) = h(F_i^{(3)}, v_{ij}) + \varepsilon\langle z_i, v_{ij}\rangle = h(F_i^{(3)} + \varepsilon z_i, v_{ij})$$

and thus $F(Q, u_i) = F_i^{(3)} + t_i$ with a vector t_i ($= \varepsilon z_i + \alpha_i u_i$ for some $\alpha_i \in \mathbb{R}$). For $(i, j) \in J$ we conclude that the $(n-2)$-face $G := F(Q, u_i) \cap F(Q, u_j)$ satisfies $G = F_{ij}^{(3)} + t_i$ and analogously $G = F_{ij}^{(3)} + t_j$, hence $t_i = t_j$. Since any two facets of P_3 can be joined by a chain of facets such that any two consecutive facets in the chain have an $(n-2)$-dimensional intersection, we conclude that $t_i = t_j$ for all i, j. Thus Q is a translate of P_3 and, hence, Z is the support vector of a point. This completes the proof of Proposition 3.

Besides Φ, we now introduce a second symmetric bilinear form Ψ on $\mathbb{R}^N$, by means of

$$\Psi(X, Y) := \frac{1}{n}\sum_{i=1}^{N} \frac{\varphi_i(P, P_3)}{h(P, u_i)} x_i y_i \quad \text{for } X, Y \in \mathbb{R}^N.$$

Here we assume, without loss of generality, that $h(P, u_i) > 0$ for $i = 1, \dots, N$. Since $\varphi_i(P, P_3) > 0$, the form Ψ is positive definite.

We consider the eigenvalues $\lambda_1 > \lambda_2 > \dots$ of Φ relative to Ψ and make use of the fact that

$$\begin{aligned}\lambda_1 &= \max\{\Phi(X, X) : \Psi(X, X) = 1\},\\ \lambda_2 &= \max\{\Phi(X, X) : \Psi(X, X) = 1,\\ &\qquad \text{and } \Psi(X, Y) = 0 \text{ for all } Y \text{ in the } \lambda_1\text{-eigenspace}\}.\end{aligned} \tag{7.58}$$

In analogy with (7.56) we write

$$\Psi(X, Y) = \frac{1}{n}\sum_{i,j=1}^{N} c_{ij} x_i y_j,$$

where

$$c_{ij} := \begin{cases} \dfrac{\varphi_i(P, P_3)}{h(P, u_i)} & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases}$$

Then $Z = (\zeta_1, \dots, \zeta_N) \in \mathbb{R}^N$ is an eigenvector of Φ relative to Ψ with eigenvalue λ if and only if

$$\sum_{j=1}^{N} (b_{ij} - \lambda c_{ij})\zeta_j = 0 \quad \text{for } i = 1, \dots, N,$$

or equivalently if

$$\varphi_i(Z, P_3) = \lambda \frac{\varphi_i(P, P_3)}{h(P, u_i)} \zeta_i \quad \text{for } i = 1, \dots, N.$$

In particular, $\lambda = 1$ is an eigenvalue with corresponding eigenvector $Z = \overline{P}$.

Proposition 4 The only positive eigenvalue of Φ relative to Ψ is 1, and it is simple.

For the proof, we first assume that $P = P_3 = \cdots = P_n$. Suppose Proposition 4 were false in this case. If there is a positive eigenvalue $\mu \neq 1$, then there exists $Z \in \mathbb{R}^N$ with $\Psi(Z, P) = 0$ and $\Phi(Z, Z) = \mu\Psi(Z, Z) > 0$. If 1 is a multiple eigenvalue, the corresponding eigenspace is at least two-dimensional and hence contains a vector Z with $\Psi(Z, P) = 0$ and $\Phi(Z, Z) = \Psi(Z, Z) > 0$. Thus, in either case we conclude from

$$\Psi(Z, P) = \frac{1}{n}\sum_{i=1}^{N} \frac{\varphi_i(P, P)}{h(P, u_i)} \zeta_i h(P, u_i) = V(Z, P, \dots, P)$$

that $V(Z, P, \dots, P) = 0$. Since $V(Z, Z, P, \dots, P) = \Phi(Z, Z) > 0$, this contradicts Proposition 2.

Now let $P_3, \dots, P_n \in \mathcal{A}$ be arbitrary. For $\vartheta \in [0, 1]$ let $P_r(\vartheta) := (1-\vartheta)P + \vartheta P_r$, $r = 3, \dots, n$. The coefficients of the corresponding forms Φ, Ψ depend continuously on ϑ, hence the same is true for the relative eigenvalues. By Proposition 3, the number 0 is always an eigenvalue with multiplicity n. It follows that the sum of the multiplicities of the positive eigenvalues is independent of ϑ. Since it is equal to 1 for $\vartheta = 0$, it must be equal to 1 for $\vartheta = 1$. This proves Proposition 4.

Proposition 4 implies that the eigenspace corresponding to the eigenvalue 1 coincides with lin $\{\overline{P}\}$ and that the second eigenvalue is not positive, and hence that $\Phi(Z,Z) \leq 0$ for all Z satisfying $\Psi(Z,P) = 0$; the latter is equivalent to $\Phi(Z,P) = 0$. Thus $\Phi(Z,P) = 0$ implies $\Phi(Z,Z) \leq 0$. Suppose that we have equality for some $Z \neq 0$. Since at Z the maximum in (7.58) is attained, Z is an eigenvector with eigenvalue 0. By Proposition 3, Z is the support vector of a point. This completes the proof of Theorem 7.3.2. □

Notes for Section 7.3

1. *Proofs of the Aleksandrov–Fenchel inequality*. The proof of Theorem 7.3.1 presented above is due to Aleksandrov [13]. Aleksandrov [16] gave a second proof, which uses bodies of class C^2_+ and Aleksandrov's inequalities for mixed discriminants (see Theorem 5.5.4). Although the underlying classes of convex bodies are quite different, there are certain analogies between the two proofs. The principal ideas can be traced back to the proof given by Hilbert [973] for Minkowski's quadratic inequalities in three-space. The second of Aleksandrov's proofs is reproduced in Busemann [370] and in Hörmander [988] (with modifications), and the first in Leichtweiß [1184]. Of the several existing proofs for the Aleksandrov–Fenchel inequality, the one presented above is historically the first and without doubt the most elementary one.

   It has become customary to talk of the 'Aleksandrov–Fenchel' inequality, because Fenchel [567] also stated the inequality and sketched a proof. We quote from Busemann [370], p. 51: 'Fenchel's proof is very sketchy, a detailed version has never appeared and it is not quite clear what it would involve'. Hadwiger [911], p. 290, characterizes Fenchel's note as 'schwer verständlich' (hard to understand). In honour of Fenchel, it seems justified to talk of the Aleksandrov–Fenchel inequality, although it is doubtful whether he had a complete proof.
2. *Connections with other fields.* Surprising connections exist between the Aleksandrov–Fenchel inequality and algebraic geometry. Independently, Khovanskiĭ and Teissier [1840, 1841] found algebraic approaches to the Aleksandrov–Fenchel inequality, yielding a new proof via the Hodge index theorem. A detailed version, written by Khovanskiĭ, can be found in the book by Burago and Zalgaller [357], §27. Here mixed volumes of Newton polyhedra associated with Laurent polynomials (Bernštein [213]) play an essential role. This line of research was carried further by Gromov [805]. By what he called a modern re-edition of Aleksandrov's (second) proof ('exterior products of differential forms instead of mixed discriminants of quadratic forms'), he obtained simultaneously the Hodge–Teissier–Khovanskiĭ inequality and the Aleksandrov–Fenchel inequality.
3. For coaxial bodies of revolution, Hadwiger [911], §§6.5.8, 6.5.9, obtained improved versions of the Aleksandrov–Fenchel inequality in an elementary way.
4. *A mass transportation approach.* For the inequality

   $$V(K_1, \dots, K_n)^n \geq V_n(K_1) \cdots V_n(K_n)$$

   for convex bodies $K_1, \dots, K_n \in \mathcal{K}^n$, which is a consequence of the Aleksandrov–Fenchel inequality, Alesker, Dar and Milman [54] gave a proof that uses ideas from mass transportation. For this, they prove the following theorem. If K, L are open, bounded convex subsets of volume 1 in $\mathbb{R}^n$, then there exists a C^1-diffeomorphism $F : K \to L$ preserving the Lebesgue measure such that

   $$K + \lambda L = \{x + \lambda F(x) : x \in K\} \qquad \text{for } \lambda > 0.$$

The proof is based on applications of the Brenier map and a result taken from Gromov [805] (obtainable by combining Corollary 1.7.11 and Lemma 1.7.12).

In their proof, the authors use a symmetric representation of the mixed volume of convex bodies $K_1, \dots, K_n$ of the same positive volume, namely

$$V(K_1, \dots, K_n) = \int_{\mathbb{R}^n} D(\operatorname{Hess} f_1, \dots, \operatorname{Hess} f_n)\, \mathrm{d}\mathcal{H}^n,$$

where D denotes the mixed discriminant and $f_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions of class C^2 with the property that ∇f_i maps $\mathbb{R}^n$ bijectively onto $\operatorname{int} K_i$ ($i = 1, \dots, n$).

5. *Non-convex domains.* By PDE or differential-geometric methods, the Aleksandrov–Fenchel inequality or its special cases referring to the intrinsic volumes have been extended to certain classes of non-convex domains with sufficiently smooth boundaries (with corresponding extensions of the mixed volumes). The convexity assumption is replaced here by the assumption that suitable elementary symmetric functions of curvatures, or similar functions derived from Hessians, are nonnegative. Different approaches have been followed by Trudinger [1852] (according to [857], [863], the argument is incomplete), Guan and Li [857], Guan, Ma, Trudinger and Zhu [863], Chang and Wang [407].
6. *Full sets of inequalities.* If $p \geq 2$ convex bodies $K_1, \dots, K_p$ in $\mathbb{R}^n$ are given, there are $N = \binom{n+p-1}{n}$ mixed volumes $V(K_{i_1}, \dots, K_{i_n})$ ($1 \leq i_1 \leq \dots \leq i_n \leq p$) that can be formed. The Aleksandrov–Fenchel inequality, applied to these values in all possible ways permitted by the symmetry properties, yields a certain system of quadratic inequalities. This set of inequalities is said to be a *full set* if any given set of N quantities satisfying these inequalities can arise as the set of mixed volumes of some system of p convex bodies. Heine [954] for $n = 2$ and Shephard [1775] for $n \geq 2$ investigated whether the known inequalities are a full set. Shephard showed by an interesting construction that the mentioned inequalities are a full set for $p = 2$. He also showed that they are not a full set for $p = n + 2$. For arbitrary p, the answer is still unknown.
7. *Applications.* Several applications of the Aleksandrov–Fenchel inequality to the geometry of convex bodies appear in later sections. Here we mention briefly a few examples of applications in other fields. It has been applied to differential-geometric uniqueness theorems (Schneider [1652]), to extremal problems for geometric probabilities (Thomas [1844], Schneider [1702]) and to combinatorial questions, in particular to showing that certain sequences of combinatorial interest are log-concave (Stanley [1812, 1813], Kahn and Saks [1056]).
8. Analogues of mixed volumes and the Aleksandrov–Fenchel inequality for $(n+1)$-tuples of functions on $\mathbb{R}^n$ with certain concavity properties were suggested and discussed by Klartag [1098].

7.4 Consequences and improvements

Since the mixed volume $V(K_1, K_2, \dots, K_n)$ is symmetric in its arguments, it is clear that from the Aleksandrov–Fenchel inequality

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geq V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n)$$

many other inequalities can be deduced, by repeated application. We first mention some of the more frequently occurring of such derived inequalities.

Some preliminary remarks are in order. A finite sequence $(a_0, a_1, \dots, a_m)$ of real numbers is called *concave* if

$$a_{i-1} - 2a_i + a_{i+1} \leq 0 \quad \text{for } i = 1, \dots, m-1,$$

or, equivalently, if

$$a_0 - a_1 \le a_1 - a_2 \le \cdots \le a_{m-1} - a_m.$$

For $0 \le i < j < k \le m$ we deduce, taking arithmetic means, that a concave sequence $(a_0, a_1, \ldots, a_m)$ satisfies

$$\frac{(a_i - a_{i+1}) + \cdots + (a_{j-1} - a_j)}{j-i} \le \frac{(a_j - a_{j+1}) + \cdots + (a_{k-1} - a_k)}{k-j},$$

hence

$$\frac{a_i - a_j}{j-i} \le \frac{a_j - a_k}{k-j}$$

and thus

$$(k-j)a_i + (i-k)a_j + (j-i)a_k \le 0. \tag{7.59}$$

Equality holds if and only if

$$a_{r-1} - 2a_r + a_{r+1} = 0 \quad \text{for } r = i+1, \ldots, k-1.$$

The sequence $(a_0, a_1, \ldots, a_m)$ of positive numbers is called *log-concave* if the sequence $(\log a_0, \ldots, \log a_m)$ is concave, or equivalently if

$$a_i^2 \ge a_{i-1}a_{i+1} \quad \text{for } i = 1, \ldots, m-1.$$

In this case, for $0 \le i < j < k \le m$,

$$a_i^{k-j} a_j^{i-k} a_k^{j-i} \le 1 \tag{7.60}$$

or, written with positive exponents,

$$a_j^{k-i} \ge a_i^{k-j} a_k^{j-i}. \tag{7.61}$$

Equality holds if and only if

$$a_r^2 = a_{r-1}a_{r+1} \quad \text{for } r = i+1, \ldots, k-1.$$

If a number $m \in \{1, \ldots, n\}$ and convex bodies $K_0, K_1, K_{m+1}, \ldots K_n \in \mathcal{K}^n$ are given, we often use the abbreviations

$$\mathcal{C} := (K_{m+1}, \ldots, K_n)$$

and

$$V_{(i)} := V_{(i)}(K_0, K_1, \mathcal{C}) := V(K_0[m-i], K_1[i], K_{m+1}, \ldots, K_n) \tag{7.62}$$

for $i = 0, \ldots, m$. This notation extends that of (7.17).

If the numbers

$$a_i := V_{(i)}(K_0, K_1, \mathcal{C}), \quad i = 0, \ldots, m,$$

are positive, then the sequence $(a_0, \dots, a_m)$ is log-concave by the Aleksandrov–Fenchel inequality, hence

$$V_{(j)}^{k-i} \geq V_{(i)}^{k-j} V_{(k)}^{j-i} \tag{7.63}$$

if $0 \leq i < j < k \leq m$. By approximation, this inequality holds also if one of the numbers involved is zero. The case $m = n$, $i = 0$, $j = n - 1$, $k = n$ is Minkowski's inequality (7.18) again (but without information on the equality case, if derived in this way).

At this point, we can formulate a more general inequality, namely

$$V(K_1, \dots, K_n)^m \geq \prod_{r=1}^{m} V(K_r[m], K_{m+1}, \dots, K_n) \tag{7.64}$$

for $K_1, \dots, K_n \in \mathcal{K}^n$ and $m = 2, \dots, n$. For $m = 2$, this is Minkowski's inequality (7.18). Suppose that (7.64) holds for some $m \geq 2$. From (7.63) we obtain, setting $(K_{m+2}, \dots, K_n) =: \mathscr{C}$,

$$V(K_r[m], K_{m+1}, \mathscr{C})^{m+1} \geq V(K_r[m+1], \mathscr{C})^m V(K_{m+1}[m+1], \mathscr{C}). \tag{7.65}$$

Raising (7.64) to power $m + 1$, applying (7.65) and taking the mth root, we obtain (7.64) with m replaced by $m + 1$, which finishes the induction.

We return to (7.63). Again choosing $m = n$ in (7.63), but now taking $K_1 = B^n$, we find that the quermassintegrals $W_0, \dots, W_n$ of the convex body $K_0 \in \mathcal{K}^n$ satisfy

$$W_j^{k-i} \geq W_i^{k-j} W_k^{j-i} \tag{7.66}$$

for $0 \leq i < j < k \leq n$. The case $k = n$ deserves special mention; since $W_n = \kappa_n$, it reduces to an inequality between only two quermassintegrals:

$$\kappa_n^i W_j^{n-i} \geq \kappa_n^j W_i^{n-j} \tag{7.67}$$

for $0 \leq i < j < n$. Assuming $\dim K_0 \geq n - i$, we have $W_i > 0$. In this case, equality in (7.67) holds if and only if $W_r^2 = W_{r-1} W_{r+1}$ for $r = i + 1, \dots, n - 1$. We shall see later (Section 7.6) that equality in the inequality $W_{n-1}^2 \geq W_{n-2} W_n$ holds only if K_0 is a ball. Hence, equality in (7.67) (for $\dim K_0 \geq n - i$) characterizes balls. In particular, among all convex bodies of given (positive) volume, precisely the balls have smallest jth quermassintegral for $j = 1, \dots, n - 1$ (case $i = 0$), and among all convex bodies of given mean width (a fortiori, of given diameter) precisely the balls have greatest ith quermassintegral for $i = 0, \dots, n - 2$ (case $j = n - 1$).

Next, we derive from the Aleksandrov–Fenchel inequality an improved version, from which some useful consequences can be drawn.

Lemma 7.4.1 *Let $K_0, K_1, \dots, K_n \in \mathcal{K}^n$ and write $(K_3, \dots, K_n) =: \mathscr{C}$ and*

$$U_{ij} := V(K_i, K_j, \mathscr{C}) = V(K_i, K_j, K_3, \dots, K_n) \tag{7.68}$$

for $i, j = 0, 1, 2$. Then

$$(U_{00} U_{12} - U_{01} U_{02})^2 \leq \left(U_{01}^2 - U_{00} U_{11}\right)\left(U_{02}^2 - U_{00} U_{22}\right). \tag{7.69}$$

Proof For $\lambda_1, \lambda_2 \geq 0$, the Aleksandrov–Fenchel inequality gives

$$\begin{aligned}
0 &\leq V(K_1 + \lambda_2 K_0, K_2 + \lambda_1 K_0, \mathcal{C})^2 \\
&\quad - V(K_1 + \lambda_2 K_0, K_1 + \lambda_2 K_0, \mathcal{C}) V(K_2 + \lambda_1 K_0, K_2 + \lambda_1 K_0, \mathcal{C}) \\
&= (U_{12} + \lambda_2 U_{02} + \lambda_1 U_{01} + \lambda_1\lambda_2 U_{00})^2 \\
&\quad - (U_{11} + 2\lambda_2 U_{01} + \lambda_2^2 U_{00})(U_{22} + 2\lambda_1 U_{02} + \lambda_1^2 U_{00}) \\
&= \text{absolute} + \text{linear terms} + A\lambda_1^2 + 2B\lambda_1\lambda_2 + C\lambda_2^2
\end{aligned}$$

(the higher-degree terms cancel), where

$$A := U_{01}^2 - U_{00}U_{11}, \quad B := U_{12}U_{00} - U_{01}U_{02}, \quad C := U_{02}^2 - U_{00}U_{22}.$$

Replacing λ_i by $t\lambda_i$ and letting $t \to \infty$, we deduce that $A\lambda_1^2 + 2B\lambda_1\lambda_2 + C\lambda_2^2 \geq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

For $\lambda_1, \lambda_2 \geq 0$ also

$$\begin{aligned}
0 &\leq V(\lambda_1 K_1 + \lambda_2 K_2, K_0, \mathcal{C})^2 - V(\lambda_1 K_1 + \lambda_2 K_2, \lambda_1 K_1 + \lambda_2 K_2, \mathcal{C}) V(K_0, K_0, \mathcal{C}) \\
&= A\lambda_1^2 - 2B\lambda_1\lambda_2 + C\lambda_2^2.
\end{aligned}$$

Now we can conclude that $Ax^2 + 2Bx + C \geq 0$ for all real x. This implies $B^2 - AC \leq 0$, which is (7.69). □

For a first important consequence, we assume that the inequality

$$U_{01}^2 \geq U_{00}U_{11} \tag{7.70}$$

holds with equality. Then (7.69) gives

$$U_{00}U_{12} - U_{01}U_{02} = 0$$

for all convex bodies K_2. By (5.19), this implies that the signed measure

$$\mu := U_{00} S(K_1, \mathcal{C}, \cdot) - U_{01} S(K_0, \mathcal{C}, \cdot)$$

satisfies

$$\int_{\mathbb{S}^{n-1}} h_K \, \mathrm{d}\mu = 0$$

for all $K \in \mathcal{K}^n$. Using Lemma 1.7.8 we conclude that $\int f \, \mathrm{d}\mu = 0$ for all $f \in C(\mathbb{S}^{n-1})$ and hence that $\mu = 0$. If we now assume that

$$U_{01} > 0, \tag{7.71}$$

this implies that the measures $S(K_0, \mathcal{C}, \cdot)$ and $S(K_1, \mathcal{C}, \cdot)$ are proportional (by (7.71), none of them is zero). Conversely, let $S(K_0, \mathcal{C}, \cdot) = \alpha S(K_1, \mathcal{C}, \cdot)$ with $\alpha > 0$. Then integration of $h(K_0, \cdot)$ and $h(K_1, \cdot)$, respectively, gives $U_{00} = \alpha U_{01}$ and $U_{01} = \alpha U_{11}$ and hence equality in (7.70). Thus we have proved the following.

Theorem 7.4.2 *Let $K, L, K_3, \dots, K_n \in \mathcal{K}^n$ and $\mathcal{C} := (K_3, \dots, K_n)$, and assume that*

$$V(K, L, \mathcal{C}) > 0. \tag{7.72}$$

Then equality holds in the inequality

$$V(K, L, \mathcal{C})^2 \geq V(K, K, \mathcal{C})V(L, L, \mathcal{C}) \tag{7.73}$$

if and only if

$$S(K, \mathcal{C}, \cdot) = \alpha S(L, \mathcal{C}, \cdot) \tag{7.74}$$

with some $\alpha > 0$.

Equality (7.74) may be considered as a kind of generalized Euler–Lagrange equation for the extremum problem connected with the equality case in (7.73).

To derive further consequences of (7.69), we now assume that

$$U_{00} > 0, \quad U_{01} > 0, \quad U_{02} > 0 \tag{7.75}$$

and write (7.69) in the form

$$(U_{00}U_{12} - U_{01}U_{02})^2 \leq U_{01}^2 U_{02}^2 \left(1 - \frac{U_{00}U_{11}}{U_{01}^2}\right)\left(1 - \frac{U_{00}U_{22}}{U_{02}^2}\right).$$

Taking square roots (the negative one on the left-hand side) and applying the inequality $4ab \leq (a+b)^2$, where a, b are the brackets on the right-hand side, we obtain

$$\frac{U_{11}}{U_{01}^2} - \frac{2U_{12}}{U_{01}U_{02}} + \frac{U_{22}}{U_{02}^2} \leq 0. \tag{7.76}$$

This improves the Aleksandrov–Fenchel inequality

$$U_{12}^2 \geq U_{11}U_{22}. \tag{7.77}$$

If equality holds in (7.76), then equality must hold in the arithmetic-geometric mean inequality applied in the proof, hence

$$\frac{U_{11}}{U_{01}^2} = \frac{U_{22}}{U_{02}^2}. \tag{7.78}$$

Together with (7.76) this yields equality in (7.77). Conversely, if (7.77) holds with equality, then the quadratic equation

$$U_{11}x^2 - 2U_{12}x + U_{22} = 0$$

has only one real root, hence (7.76) holds with equality.

We collect the results obtained so far, but change notation ($K_1 = K$, $K_2 = L$, $K_0 = M$).

Theorem 7.4.3 *Let $K, L, M, K_3, \dots, K_n \in \mathcal{K}^n$ be convex bodies, write $\mathcal{C} := (K_3, \dots, K_n)$ and suppose that*

$$V(K, M, \mathcal{C}) > 0, \quad V(L, M, \mathcal{C}) > 0, \tag{7.79}$$

$$V(M, M, \mathscr{C}) > 0. \tag{7.80}$$

Then

$$V(K, L, \mathscr{C})^2 \geq V(K, K, \mathscr{C})V(L, L, \mathscr{C}) \tag{7.81}$$

and

$$\frac{V(K, K, \mathscr{C})}{V(K, M, \mathscr{C})^2} - \frac{2V(K, L, \mathscr{C})}{V(K, M, \mathscr{C})V(L, M, \mathscr{C})} + \frac{V(L, L, \mathscr{C})}{V(L, M, \mathscr{C})^2} \leq 0. \tag{7.82}$$

The following assertions are equivalent:

(a) *equality in (7.81);*

(b) *equality in (7.82).*

The particular advantage of inequality (7.82) lies in the fact that the convex body M is at one's disposal. Observe that (b) for one convex body M implies (a), which in turn implies (b) for all convex bodies M.

Corollary 7.4.4 *Under the assumptions of Theorem 7.4.3,*

$$\frac{V(K + L, K + L, \mathscr{C})}{V(K + L, M, \mathscr{C})} \geq \frac{V(K, K, \mathscr{C})}{V(K, M, \mathscr{C})} + \frac{V(L, L, \mathscr{C})}{V(L, M, \mathscr{C})}. \tag{7.83}$$

Proof Inequality (7.82) can be written as

$$2V(K, L, \mathscr{C}) \geq V(K, K, \mathscr{C})\frac{V(L, M, \mathscr{C})}{V(K, M, \mathscr{C})} + V(L, L, \mathscr{C})\frac{V(K, M, \mathscr{C})}{V(L, M, \mathscr{C})}.$$

Adding $V(K, K, \mathscr{C}) + V(L, L, \mathscr{C})$ to both sides and then dividing by $V(K, M, \mathscr{C}) + V(L, M, \mathscr{C}) = V(K + L, M, \mathscr{C})$, we obtain the assertion. □

For the derivation of inequality (7.82) and the equivalence of (a) and (b) we had to assume non-degeneracy in the form (7.79), (7.80). If (7.79) holds, but not necessarily (7.80), the inequality (7.82) is still true, by approximation. The implication (b) ⇒ (a), however, is no longer valid. We shall now investigate equality in (7.82) in the case $V(M, M, \mathscr{C}) = 0$. This will play a role in Section 7.7, where M is a segment.

Let $K, L, M, \mathscr{C}$ be as in Theorem 7.4.3, but without assuming (7.80). We write $K = C_1$ and $L = C_2$, choose $C_3 \in \mathcal{K}_n^n$ arbitrarily and define

$$W_{ij} := V(C_i, C_j, \mathscr{C}), \qquad q_i := V(C_i, M, \mathscr{C})$$

for $i, j = 1, 2, 3$,

$$K_\alpha := \sum_{i=1}^{3} x_{\alpha i} C_i \quad \text{with } x_{\alpha i} > 0$$

for $\alpha = 1, 2$, and

$$\begin{aligned} U_{\alpha\beta} &:= V(K_\alpha, K_\beta, \mathscr{C}) = \sum_{i,j=1}^{3} x_{\alpha i} x_{\beta j} W_{ij}, \\ p_\alpha &:= V(K_\alpha, M, \mathscr{C}) = \sum_{i=1}^{3} x_{\alpha i} q_i. \end{aligned} \tag{7.84}$$

By (7.82), applied to K_1, K_2 instead of K, L, we have

$$\frac{U_{11}}{p_1^2} - \frac{2U_{12}}{p_1 p_2} + \frac{U_{22}}{p_2^2} \leq 0. \tag{7.85}$$

We choose real numbers μ_i and positive numbers ν_i and put $x_{1i} := \nu_i$, $x_{2i} := t\nu_i + \mu_i$, where t is so large that $x_{2i} > 0$ ($i = 1, 2, 3$). Inserting (7.84) into (7.41), we obtain

$$\begin{aligned} &\left(\sum W_{ij}\nu_i\nu_j\right)\left(\sum q_i\mu_i\right)^2 + \left(\sum W_{ij}\mu_i\mu_j\right)\left(\sum q_i\nu_i\right)^2 \\ &- 2\left(\sum W_{ij}\nu_i\mu_j\right)\left(\sum q_i\nu_i\right)\left(\sum q_i\mu_i\right) \leq 0 \end{aligned}$$

(the terms containing t cancel). Choosing $\nu_1 = 1$, $\mu_1 = 0$ and letting $\nu_2, \nu_3 \to 0$, an inequality results that is quadratic in μ_2 and μ_3. If now equality holds in (7.82), the coefficient of μ_2^2 is zero. Since the inequality holds for arbitrary real numbers μ_2, μ_3, the coefficient of $\mu_2\mu_3$ must vanish, too, which gives

$$\frac{W_{11}q_2 - W_{12}q_1}{q_1} = \frac{W_{13}q_2 - W_{23}q_1}{q_3}.$$

Denoting this number by k, we see that k is independent of C_3 and that

$$W_{13}q_2 - W_{23}q_1 - kq_3 = 0.$$

By (5.19), this can be written in the form

$$\int_{\mathbb{S}^{n-1}} h(C_3, \cdot)\, \mathrm{d}[q_2 S(K, \mathscr{C}, \cdot) - q_1 S(L, \mathscr{C}, \cdot) - kS(M, \mathscr{C}, \cdot)] = 0.$$

Since this holds for all $C_3 \in \mathcal{K}_n^n$, we deduce that

$$q_2 S(K, \mathscr{C}, \cdot) - q_1 S(L, \mathscr{C}, \cdot) = kS(M, \mathscr{C}, \cdot). \tag{7.86}$$

Conversely, suppose that (7.86) is satisfied. Integrating the support functions of K and L with this measure, we obtain

$$W_{11}q_2 - W_{12}q_1 = kq_1, \quad W_{12}q_2 - W_{22}q_1 = kq_2,$$

hence

$$W_{11}q_2^2 - 2W_{12}q_1q_2 + W_{22}q_1^2 = 0,$$

which is (7.82) with equality.

Condition (7.86), which has thus been shown to be equivalent to equality in (7.82), can be written in the form

$$
\begin{aligned}
S(q_2K,\mathscr{C},\cdot) &= S(q_1L+kM,\mathscr{C},\cdot) \qquad \text{if } k \ge 0\\
S(q_1L,\mathscr{C},\cdot) &= S(q_2K+|k|M,\mathscr{C},\cdot) \quad \text{if } k \le 0.
\end{aligned} \tag{7.87}
$$

We turn to a different class of consequences of the Aleksandrov–Fenchel inequality, which concern a generalization of the Brunn–Minkowski theorem.

Theorem 7.4.5 (General Brunn–Minkowski theorem) *Let a number $m \in \{1,\dots,n\}$ and convex bodies $K_0, K_1, K_{m+1}, \dots, K_n \in \mathcal{K}^n$ be given; define $K_\lambda := (1-\lambda)K_0 + \lambda K_1$ and*

$$f(\lambda) := V(K_\lambda[m], K_{m+1}, \dots, K_n)^{1/m} \tag{7.88}$$

for $0 \le \lambda \le 1$. Then f is a concave function on $[0,1]$.

Proof We have to show that $f''(\lambda) \le 0$. It suffices to prove this for $\lambda = 0$: if $0 < \lambda < 1$, we put $\overline{K}_\tau := (1-\tau)K_\lambda + \tau K_1$ and

$$h(\tau) := V(\overline{K}_\tau[m], K_{m+1}, \dots, K_n)^{1/m}.$$

Then $f(\lambda+\mu) = h(\mu/(1-\lambda))$, hence $f''(\lambda) \le 0$ follows from $h''(0) \le 0$. Now

$$f''(0) = (m-1)V_{(0)}^{(1/m)-2}[V_{(0)}V_{(2)} - V_{(1)}^2] \le 0$$

with $V_{(i)}$ given by (7.62). We have assumed that $V_{(0)} > 0$; the general case is then obtained by approximation. □

For $m = n$, Theorem 7.4.5 reduces to the Brunn–Minkowski theorem, and in that case we know that for n-dimensional convex bodies K_0, K_1 the function f is linear only if K_0 and K_1 are homothetic. An analogous assertion is not true for the general Brunn–Minkowski theorem, but some conditions equivalent to linearity of f can be formulated and are of interest.

In analogy to (7.62), that is,

$$V_{(i)} := V(K_0[m-i], K_1[i], \mathscr{C}),$$

we write

$$S_{(i)} := S(K_0[m-1-i], K_1[i], \mathscr{C}, \cdot) \tag{7.89}$$

for $i = 0, \dots, m-1$; hence $\mathscr{C} = (K_{m+1}, \dots, K_n)$.

Theorem 7.4.6 *Under the assumptions of Theorem 7.4.5 and*

$$V_{(0)} > 0, \quad V_{(m)} > 0,$$

the following conditions are equivalent:

(a) *The function f is linear;*
(b) $V_{(i)}^2 = V_{(i-1)} V_{(i+1)}$ *for* $i = 1, \dots, m-1$*;*
(c) $V_{(1)}^m = V_{(0)}^{m-1} V_{(m)}$*;*
(d) *The measures* $S_{(0)}$ *and* $S_{(m-1)}$ *are proportional.*

Proof Suppose that f is linear; hence

$$V(K_\lambda[m], \mathscr{C}) = \left[(1-\lambda)V_{(0)}^{1/m} + \lambda V_{(m)}^{1/m}\right]^m$$

for $0 \le \lambda \le 1$. By (5.29), this gives

$$\sum_{i=0}^{m} \binom{m}{i} (1-\lambda)^{m-i} \lambda^i V_{(i)} = \sum_{i=0}^{m} \binom{m}{i} (1-\lambda)^{m-i} \lambda^i V_{(0)}^{(m-i)/m} V_{(m)}^{i/m},$$

hence

$$V_{(i)} = V_{(0)}^{(m-i)/m} V_{(m)}^{i/m} \quad \text{for } i = 0, \dots, m.$$

From

$$V_{(i)}^{2m} = V_{(0)}^{2(m-i)} V_{(m)}^{2i}, \qquad V_{(i-1)}^m = V_{(0)}^{m-i+1} V_{(m)}^{i-1}, \qquad V_{(i+1)}^m = V_{(0)}^{m-i-1} V_{(m)}^{i+1}$$

we deduce (b).

Since the function f is concave, it satisfies $f'(0) \ge f(1) - f(0)$, with equality if and only if f is linear. As in the proof of (7.18), this yields

$$V_{(1)}^m \ge V_{(0)}^{m-1} V_{(m)}, \tag{7.90}$$

with equality if and only if f is linear. Thus (a) and (c) are equivalent.

If (b) holds, then Theorem 7.4.2 yields that $S_{(i-1)}$ and $S_{(i)}$ are proportional for $i = 1, \dots, m-1$. This implies (d).

Suppose that (d) holds. After a dilatation applied to K_0 we may assume that $S_{(0)} = S_{(m-1)}$. Integrating $h(K_1, \cdot)$ with respect to this measure, we deduce that $V_{(1)} = V_{(m)}$ and hence, by (7.90), that

$$V_{(m)}^m = V_{(1)}^m \ge V_{(0)}^{m-1} V_{(m)},$$

thus $V_{(m)} \ge V_{(0)}$. Interchanging the roles of K_0 and K_1, we obtain $V_{(0)} \ge V_{(m)}$ and thus $V_{(m)} = V_{(0)}$. This implies equality in (7.90), thus (c) holds. □

Theorem 7.6.9 will describe an important situation where the conditions (a) – (d) of Theorem 7.4.6 are satisfied.

By Theorem 7.4.2, equality in the Aleksandrov–Fenchel inequality (7.73) implies the equality (7.74) between two mixed area measures. In the special case where one of the bodies involved in the $(n-2)$-tuple $\mathscr{C}$ is a ball, the latter equality in turn implies equality in a set of lower-dimensional Aleksandrov–Fenchel inequalities for projections. This opens the way to proofs employing induction with respect to the dimension. We state the crucial step as a lemma.

As in Section 5.3, we use the abbreviation K^u for the projection $K\,|\,u^\perp$, and we denote the $(n-1)$-dimensional mixed volume by v. If $\mathscr{C} = (K_1, \dots, K_m)$, we write $\mathscr{C}^u := (K_1^u, \dots, K_m^u)$, and the unit ball B^n in $\mathbb{R}^n$ is denoted in brief by B.

Lemma 7.4.7 *Let $n \geq 3$, let $K, L, K_4, \dots, K_n \in \mathcal{K}^n$ be convex bodies and $\mathscr{C} = (K_4, \dots, K_n)$, and suppose that $v(K^u, B^u, \mathscr{C}^u) > 0$, $v(L^u, B^u, \mathscr{C}^u) > 0$. If*

$$S(K, B, \mathscr{C}, \cdot) = S(L, B, \mathscr{C}, \cdot), \tag{7.91}$$

then

$$v(K^u, L^u, \mathscr{C}^u)^2 = v(K^u, K^u, \mathscr{C}^u)v(L^u, L^u, \mathscr{C}^u) \tag{7.92}$$

for all $u \in \mathbb{S}^{n-1}$.

Proof By inequality (7.82), applied in $H_{u,0}$ for any $u \in \mathbb{S}^{n-1}$, we have

$$\frac{v(K^u, K^u, \mathscr{C}^u)}{v(K^u, B^u, \mathscr{C}^u)^2} - \frac{2v(K^u, L^u, \mathscr{C}^u)}{v(K^u, B^u, \mathscr{C}^u)v(L^u, B^u, \mathscr{C}^u)} + \frac{v(L^u, L^u, \mathscr{C}^u)}{v(L^u, B^u, \mathscr{C}^u)^2} \leq 0.$$

By formula (5.78), assumption (7.91) implies

$$v(K^u, B^u, \mathscr{C}^u) = v(L^u, B^u, \mathscr{C}^u),$$

hence

$$v(K^u, K^u, \mathscr{C}^u) - 2v(K^u, L^u, \mathscr{C}^u) + v(L^u, L^u, \mathscr{C}^u) \leq 0. \tag{7.93}$$

By (5.79), this yields

$$V(K, K, B, \mathscr{C}) - 2V(K, L, B, \mathscr{C}) + V(L, L, B, \mathscr{C}) \leq 0. \tag{7.94}$$

Again using (7.91), from (5.19) we obtain

$$V(K, K, B, \mathscr{C}) = V(K, L, B, \mathscr{C}) = V(L, L, B, \mathscr{C}).$$

Thus (7.94) holds with equality, which implies equality in (7.93). By Theorem 7.4.3, this implies equality (7.92). □

Notes for Section 7.4

1. The general inequality (7.64) was already noted by Aleksandrov [18].
2. *Log-linear inequalities for quermassintegrals*. Inequality (7.66) says that the quermassintegrals $W_0, \dots, W_n$ of a convex body in $\mathcal{K}^n$ satisfy

$$W_i^{j-k} W_j^{k-i} W_k^{i-j} \geq 1$$

for $0 \leq i < j < k \leq n$. Gritzmann [776] investigated systematically all inequalities of this type. Let $0 \leq i < j < k \leq n-1$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $c > 0$. Then Gritzmann showed that there is an inequality of the type

$$W_i^\alpha(K) W_j^\beta(K) W_k^\gamma(K) \geq c, \tag{7.95}$$

valid for all convex bodies $K \in \mathcal{K}_n^n$, if and only if

$$\alpha \leq 0, \quad \alpha + \beta + \gamma \geq 0, \quad \alpha(n-i) + \beta(n-j) + \gamma(n-k) = 0.$$

Furthermore, in each case the constant c can be chosen such that equality holds for the ball, and all the valid inequalities of type (7.95) are generated by the $n-1$ special Aleksandrov–Fenchel inequalities

$$W_i(K)^2 \geq W_{i-1}(K)W_{i+1}(K) \quad \text{for } i = 1, \dots, n-1.$$

From this result, Gritzmann also derived a characterization of all log-linear inequalities for the diameter D and two quermassintegrals. Examples are Bieberbach's inequality (7.22), Kubota's [1153] inequality

$$\left(\frac{D}{2}\right)^{n-1} \geq \frac{S}{\omega_n},$$

and the inequality

$$S^{n-1} > \kappa_{n-1} D (nV_n)^{n-2},$$

which was proved in an entirely different way by Gritzmann, Wills and Wrase [780].

3. Theorem 7.4.2 appears in Aleksandrov [13] and Fenchel and Jessen [572]. Aleksandrov's proof is different. As mentioned before, if the question of the equality case in (7.73) is considered as a minimum problem with side condition, then equality (7.74) can be considered as a generalized Euler–Lagrange equation for this problem. Correspondingly, Aleksandrov employed a variational argument. The method used above goes back to Favard [553] and Fenchel [568].

   It appears that the algebraic manipulations leading to inequalities (7.69), (7.76) and finally to Theorem 7.4.3 were, in special cases, first applied by Favard [552] and slightly extended by Matsumura [1364]; see also Bonnesen and Fenchel [284], §51. Full use of this method was made by Favard [553].

   The proof of the equivalence of the equality in (7.82) (without the assumption (7.80)) and the equation (7.86), as given above, extends an argument due to Favard [553].
4. Motivated by analogy to a determinant inequality, Milman has asked whether

$$\frac{V_k(K+L)}{V_{k-1}(K+L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)} \tag{7.96}$$

   holds for all convex bodies $K, L \in \mathcal{K}^n$ and for $k = 2, \dots, n$ (the case $k = 1$ is trivial). For $k = 2$, an affirmative answer is given by inequality (7.83) (choosing $\mathcal{C} = (B^n, \dots, B^n)$ and $M = B^n$, and multiplying by a suitable constant). This is due to Fradelizi, Giannopoulos and Meyer [627]. If L is a ball, then (7.96) holds for all $k \in \{1, \dots, n\}$, as shown by Giannopoulos, Hartzoulaki and Paouris [703]. Both results were noted independently by Schwella [1763]. Fradelizi, Giannopoulos and Meyer [627] proved that (7.96) holds for all $K, L \in \mathcal{K}^n$ only if $k = 1$ or $k = 2$. They also obtained the following result. If $0 \leq k \leq p \leq n$, $K \in \mathcal{K}^n$ and $K\,|\,E$ denotes the orthogonal projection of K to some p-dimensional subspace E of $\mathbb{R}^n$, then

$$\frac{V_{n-k}(K)}{V_n(K)} > \frac{1}{\binom{n-p+k}{n-p}} \frac{V_{p-k}(K\,|\,E)}{V_p(K\,|\,E)}.$$

   The numerical factor is best possible.
5. *The general Brunn–Minkowski theorem.* Theorem 7.4.5 is a corollary of the Aleksandrov–Fenchel inequalities and consequently appeared in print right after their proof; see Fenchel [568] and Aleksandrov [13]. Theorem 7.4.6 is essentially taken from the latter paper.

   Conversely, the method of proof by which Minkowski's second inequality (7.19) was derived from the Brunn–Minkowski theorem can also be used to show that the general Brunn–Minkowski theorem implies the Aleksandrov–Fenchel inequality.

   Stability estimates for some special cases of the general Brunn–Minkowski theorem were obtained by Diskant [504, 507]. More general results of this kind were obtained by Schneider [1716].

6. The proof of Lemma 7.4.7, which in the form given above appears in Schneider [1701], extends an argument due to Favard [553].
7. *Integral-geometric proofs*. For the special case $i = n-2$ (and hence $j = n-1$) of inequality (7.67), that is,

$$W_{n-1}^2 \geq \kappa_n W_{n-2},$$

two integral-geometric proofs, due to Santaló and to Blaschke, respectively, are known for $n = 3$. The n-dimensional extensions can be found in Gericke [695].
8. *A localized inequality.* Inequality (7.67) can (in view of (4.9)) be written as

$$V_n(K)^{m/n} \leq \beta(m,n) V_m(K)$$

for $K \in \mathcal{K}^n$ and $m \in \{1, \dots, n-1\}$, where the constant $\beta(m,n)$ is such that equality holds for balls. Of this inequality, Ferrari, Franchi and Lu [573] have proved a localized version. Let $K \in \mathcal{K}_n^n$, $x \in \mathrm{bd}\, K$, $r > 0$, $B := B(x,r)$ and $B_0 := \mathrm{int}\, B$. Let $\Phi_m^s(K \cap B, \cdot)$ denote the singular part of the curvature measure $\Phi_m(K \cap B, \cdot)$ with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ on $\mathrm{bd}(K \cap B)$. Then they proved that

$$V_n(K \cap B)^{m/n} \leq \alpha(m,n) \left[\Phi_m(K, B_0) + \Phi_m^s(K \cap B, \mathrm{bd}\, K \cap \mathrm{bd}\, B) \right],$$

with an explicitly given constant $\alpha(m,n)$ (which is not sharp, though).
9. *Non-convex sets*. Ohmann [1478, 1479, 1482] used an extension of Kubota's integral recursion (formula (5.72) for $j = k$) to define quermassintegrals for non-convex sets, and he was able to extend some of the inequalities (7.67) to this general situation.
10. *Intrinsic volumes in non-Euclidean spaces.* In Euclidean space, there are different approaches to the intrinsic volumes of convex bodies: the Steiner formula (4.1), curvature integrals (4.25) (for $\beta = \mathbb{R}^n$), integral-geometric intersection formulae (4.60). In spherical or hyperbolic space, all these approaches make sense, but lead to different functionals; there are, however, relations between them (for spherical space see, for example, the introductory part of [668]). An investigation of these functionals within integral geometry is found in Santaló [1630]. From other points of view, such as inequalities or characterization theorems, there remains much to explore. An analogue of the Urysohn inequality in spherical and hyperbolic space is proved in Gao, Hug and Schneider [668]. Inequalities between quermassintegrals and between total mean curvatures in hyperbolic space were obtained by Gallego and Solanes [663].

7.5 Wulff shapes

Some of the deeper investigations of inequalities for mixed volumes and of the equality cases make essential use of the method of inner parallel bodies. In this section we study a more general concept, which associates a convex body with any positive continuous function on the unit sphere, and we collect some of the necessary tools and auxiliary results for its application. In particular, we provide a useful variational formula (Lemma 7.5.3), the postponed proof of Theorem 7.2.3 and a result (Lemma 7.5.4) that will be needed in the next section.

Let us assume that a closed subset $\Omega \subset \mathbb{S}^{n-1}$ of the unit sphere, not lying in a closed hemisphere, and a positive continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ are given. (Only the values of f on Ω will be needed, but without loss of generality we may assume that f is defined on all of $\mathbb{S}^{n-1}$.) The closed convex set

$$K := \bigcap_{u \in \Omega} H_{u,f(u)}^- \tag{7.97}$$

is bounded, since Ω positively spans $\mathbb{R}^n$, and it has o as an interior point, since the restriction of f to Ω has a positive lower bound. The body K is called the *Wulff shape* associated with (Ω, f) (or just with f, if $\Omega = \mathbb{S}^{n-1}$), since a special case of this construction appears in Wulff's [1992] work on crystallography. (See the short but illuminating discussion in Gardner [674], Section 6. See also Gruber [834], Theorem 8.13, for one version of Wulff's theorem.) We also say that K is *determined by* Ω if K is the Wulff shape associated with (Ω, f) for some positive continuous function f, or equivalently, if

$$K = \bigcap_{u \in \Omega} H^-_{u,h_K(u)}.$$

The Wulff shape has also been called the *Aleksandrov body*, since it plays a role in the variational lemma 7.5.3, due to Aleksandrov.

Let $K, L \in \mathcal{K}^n_n$. In Section 3.1, the system $\{K_\rho\}$ of parallel bodies of K relative to L was defined by

$$K_\rho := \begin{cases} K + \rho L & \text{for } 0 \leq \rho < \infty, \\ K \div -\rho L & \text{for } -r(K, L) \leq \rho < 0, \end{cases}$$

where $r(K, L)$ is the inradius of K relative to L. By (3.19) and Theorem 1.33(b),

$$K_\rho = \bigcap_{u \in \mathbb{S}^{n-1}} H^-_{u,h(K,u)+\rho h(L,u)}$$

for $-r(K, L) \leq \rho < \infty$. Hence, if we arrange by suitable translations that $o \in \operatorname{int} r(K, L)L \subset K$, then for $\rho > -r(K, L)$ the body K_ρ is the Wulff shape associated with $(\mathbb{S}^{n-1}, h_K + \rho h_L)$. Thus, the operation (7.97) is a very general version of the formation of parallel bodies.

In the following, the set Ω is fixed. The functions appearing in the construction of Wulff shapes are always continuous and positive.

Lemma 7.5.1 *If K is the Wulff shape associated with (Ω, f), then*

$$S_{n-1}(K, \mathbb{S}^{n-1} \setminus \Omega) = 0 \tag{7.98}$$

and

$$V_n(K) = \frac{1}{n} \int_\Omega f(u)\, S_{n-1}(K, \mathrm{d}u). \tag{7.99}$$

Proof Let $x \in \operatorname{bd} K$. Then there exists a vector $u \in \Omega$ such that $x \in H_{u,f(u)}$, since otherwise $\langle x, u\rangle < f(u)$ for all $u \in \Omega$ and hence $\langle x, u\rangle + \varepsilon < f(u)$ for all $u \in \Omega$ with some $\varepsilon > 0$, which would imply $x \in \operatorname{int} K$.

Let $v \in \mathbb{S}^{n-1} \setminus \Omega$ and $x \in K \cap H(K, v)$. Then x lies in the support plane $H(K, v)$ and, by the preceding remark, also in a different support plane. Thus x is a singular point of K. From (4.32) and Theorem 2.2.5 the assertion (7.98) follows.

If $w \in \Omega$ is such that $h(K, w) \neq f(w)$, then any point $x \in K \cap H(K, w)$ lies in $H(K, w)$ and in some distinct support plane $H_{u,f(u)}$, hence x is singular. Thus

$$S_{n-1}(K, \{w \in \Omega : h(K, w) \neq f(w)\}) = 0. \tag{7.100}$$

Now we conclude from

$$V_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)\, S_{n-1}(K, \mathrm{d}u)$$

together with (7.98) and (7.100) that (7.99) holds. □

Lemma 7.5.2 *If K_j is the Wulff shape associated with (Ω, f_j) for $j \in \mathbb{N}_0$ and if $(f_j)_{j\in\mathbb{N}}$ converges uniformly to f_0, then $(K_j)_{j\in\mathbb{N}}$ converges to K_0.*

Proof It is easy to see that each interior point of K_0 is an interior point of K_j for almost all j. The assertion then follows with the aid of Theorem 1.8.8. □

The next lemma is a useful variational formula. We assume that a continuous function $G : [-\varepsilon, \varepsilon]\times\Omega \to \mathbb{R}$ is given, with some $\varepsilon > 0$, and that there is a continuous function $g : \Omega \to \mathbb{R}$ such that

$$\lim_{t\downarrow 0} \frac{G(t,\cdot)}{t} = g \qquad \text{uniformly on } \Omega. \tag{7.101}$$

We write $V(f)$ for the volume of the Wulff shape associated with (Ω, f). For all sufficiently small $|t|$, the function $f + G(t,\cdot)$ is positive, and hence $V(f + G(t,\cdot))$ is defined.

Lemma 7.5.3 *Let K be the Wulff shape associated with (Ω, f). If G is as above, then*

$$\lim_{t\downarrow 0} \frac{V(f + G(t,\cdot)) - V(f)}{t} = \int_\Omega g(u)\, S_{n-1}(K, \mathrm{d}u). \tag{7.102}$$

The same assertion holds if in (7.101) and (7.102) the one-sided limit $\lim_{t\downarrow 0}$ *is replaced by* $\lim_{t\uparrow 0}$ *or by* $\lim_{t\to 0}$.

Proof For sufficiently small $t > 0$, let K_t be the Wulff shape associated with $(\Omega, f + G(t,\cdot))$, and define

$$V_j(t) := V(K_t[j], K[n-j]) \qquad \text{for } j = 0, \dots, n.$$

In particular, $V(f) = V_n(K) = V_0(t) =: V_0$.

From $h(K_t, u) \le f(u) + G(t, u)$ for $u \in \Omega$ and from (7.98) we have

$$\int_{\mathbb{S}^{n-1}} h(K_t, u)\, S_{n-1}(K, \mathrm{d}u) \le \int_\Omega [f(u) + G(t, u)]\, S_{n-1}(K, \mathrm{d}u).$$

Together with (7.99) this gives

$$V_1(t) - V_0 \le \frac{1}{n} \int_\Omega G(t, u)\, S_{n-1}(K, \mathrm{d}u).$$

From (7.101) we deduce that

$$\limsup_{t\downarrow 0} \frac{V_1(t) - V_0}{t} \le \frac{1}{n} \int_\Omega g(u)\, S_{n-1}(K, \mathrm{d}u). \tag{7.103}$$

Again from Lemma 7.5.1,

$$\frac{1}{n}\int_{\Omega}[f(u)+G(t,u)]\,S_{n-1}(K_t,\mathrm{d}u)=V_n(t).$$

From $h(K,u)\le f(u)$ for $u\in\Omega$ and (7.98) we have

$$\int_{\Omega}f(u)\,S_{n-1}(K_t,\mathrm{d}u)\ge\int_{\mathbb{S}^{n-1}}h(K,u)\,S_{n-1}(K_t,\mathrm{d}u).$$

Subtraction gives

$$\frac{1}{n}\int_{\Omega}G(t,u)\,S_{n-1}(K_t,\mathrm{d}u)\le V_n(t)-V_{n-1}(t).$$

From $K_t\to K$ for $t\to 0$, the weak continuity of S_{n-1}, (7.98) (for K and K_t) and (7.101) we obtain

$$\frac{1}{n}\int_{\Omega}g(u)\,S_{n-1}(K,\mathrm{d}u)\le\liminf_{t\downarrow 0}\frac{V_n(t)-V_{n-1}(t)}{t}. \tag{7.104}$$

With the aid of Minkowski's inequality (7.18) we get

$$\begin{aligned}[V_1(t)-V_0]\sum_{k=0}^{n-1}[V_1(t)/V_0]^k &=\left[V_1(t)^n-V_0^n\right]V_0^{1-n}\\ &\ge\left[V_0^{n-1}V_n(t)-V_0^n\right]V_0^{1-n}=V_n(t)-V_0.\end{aligned}$$

Since $K_t\to K$ for $t\to 0$, we deduce that

$$n\limsup_{t\downarrow 0}\frac{V_1(t)-V_0}{t}\ge\limsup_{t\downarrow 0}\frac{V_n(t)-V_0}{t}. \tag{7.105}$$

In a similar way (namely, replacing the pair $(V_1(t),V_0)$ by $(V_{n-1}(t),V_n(t))$ in the preceding argument), we find that

$$n\liminf_{t\downarrow 0}\frac{V_n(t)-V_{n-1}(t)}{t}\le\liminf_{t\downarrow 0}\frac{V_n(t)-V_0}{t}. \tag{7.106}$$

Applying successively (7.104), (7.106) and $\liminf\le\limsup$, then (7.105) and (7.103), we conclude that in all these inequalities the equality sign is valid, hence

$$\lim_{t\downarrow 0}\frac{V_n(t)-V_0}{t}=\int_{\Omega}g(u)\,S_{n-1}(K,\mathrm{d}u).$$

This yields the main assertion of the lemma.

The corresponding assertion for left-sided derivatives follows upon replacing $G(t,u)$ by $G(-t,u)$, and both results give the corresponding result for limits. □

We are now in a position to present the proof of Theorem 7.2.3, which was postponed.

Proof of Theorem 7.2.3 By assumption, $K, L \in \mathcal{K}_n^n$ are n-dimensional convex bodies, and $\Omega \subset \mathbb{S}^{n-1}$ is a closed subset such that

$$K = \bigcap_{u \in \Omega} H^-(K, u).$$

Then Ω is not contained in a closed hemisphere, since K is bounded, and with the terminology introduced above we see that K is the Wulff shape associated with (Ω, h_K). The body $\overline{L}$ is defined by

$$\overline{L} := \bigcap_{u \in \Omega} H^-(L, u).$$

The relative inradius $r(K, L)$ of K relative to L will be abbreviated by r. Without loss of generality, we may assume that $o \in \operatorname{int} rL \subset K$.

If $x \in r\overline{L}$, then $\langle x, u \rangle \le h(rL, u) \le h(K, u)$ for $u \in \Omega$ and hence $x \in K$. Thus $r\overline{L} \subset K$ and $r(K, \overline{L}) = r$.

For $-r < \lambda \le 0$, let K_λ be the Wulff shape associated with the pair $(\Omega, h_K + \lambda h_L)$. By Lemma 7.5.3 (applied with $f = h_K + \lambda h_L$ and $G(t, \cdot) = t h_L$),

$$\frac{\mathrm{d} V_n(K_\lambda)}{\mathrm{d}\lambda} = \int_\Omega h(\overline{L}, u)\, S_{n-1}(K_\lambda, \mathrm{d}u), \tag{7.107}$$

and, by (7.98),

$$\int_\Omega h(\overline{L}, u)\, S_{n-1}(K_\lambda, \mathrm{d}u) = \int_{\mathbb{S}^{n-1}} h(\overline{L}, u)\, S_{n-1}(K_\lambda, \mathrm{d}u) = n V_1(K_\lambda, \overline{L}). \tag{7.108}$$

We assert that

$$\lim_{\lambda \to -r} V_n(K_\lambda) = 0. \tag{7.109}$$

For the proof, let $A := \{u \in \Omega : h(K, u) = h(rL, u)\}$. Then $o \in \operatorname{conv} A$, since otherwise one could construct a homothet of L contained in K and larger than rL. There is a finite subset $A' \subset A$ such that $o \in \operatorname{conv} A'$ and hence a finite subset $\Omega' \subset \Omega$ such that

$$P := \bigcap_{u \in \Omega'} H^-(K, u)$$

is a polytope containing no homothet of L larger than rL. If P_λ is the Wulff shape associated with $(\Omega', h_K + \lambda h_L)$, then $K_\lambda \subset P_\lambda$ for $-r < \lambda \le 0$, and it is easy to see that $V_n(P_\lambda) \to 0$ for $\lambda \to -r$. This proves (7.109).

From (7.107), (7.108) and (7.109) we now obtain

$$V_n(K) = n \int_{-r}^{0} V_1(K_\lambda, \overline{L})\, \mathrm{d}\lambda. \tag{7.110}$$

By the general Brunn–Minkowski theorem 7.4.5,

$$V_1(K_\lambda + |\lambda| \overline{L}, \overline{L})^{1/(n-1)} \ge V_1(K_\lambda, \overline{L})^{1/(n-1)} + |\lambda| V_n(\overline{L})^{1/(n-1)}.$$

From $K_\lambda + |\lambda|\overline{L} \subset K$, we have $V_1(K_\lambda + |\lambda|\overline{L}, \overline{L}) \le V_1(K, \overline{L})$ and hence

$$V_1(K_\lambda, \overline{L})^{1/(n-1)} \le V_1(K, \overline{L})^{1/(n-1)} + \lambda V_n(\overline{L})^{1/(n-1)}.$$

Inserting this into (7.110), we obtain

$$V_n(K) \le n \int_{-r}^{0} \left[V_1(K,\overline{L})^{1/(n-1)} + \lambda V_n(\overline{L})^{1/(n-1)}\right]^{n-1} \mathrm{d}\lambda.$$

Since $V_1(K,\overline{L}) = V_1(K,L)$ by (7.98), we arrive at

$$V_n(K)V_n(\overline{L})^{1/(n-1)} \le V_1(K,L)^{n/(n-1)} - \left[V_1(K,L)^{1/(n-1)} - rV_n(\overline{L})^{1/(n-1)}\right]^n,$$

which is the assertion of Theorem 7.2.3. □

As we have seen, and shall further see in the proof of Theorem 7.6.19, differentiability assertions with respect to the parameter of a system of inner parallel bodies play an essential role. Another result of that type is needed in the proof of Theorem 7.6.7 and will now be proved.

Let $A, C \in \mathcal{K}_n^n$ be convex bodies with interior points. We say that A is *adapted to* C if to each point $x \in \operatorname{bd} C$ there is a point $y \in \operatorname{bd} A$ such that the normal cones at these points satisfy $N(C,x) \subset N(A,y)$. For example, it follows from Theorem 2.2.1 that A is adapted to C if A is a summand of C, but this is only a very special case.

In Section 3.1, the Minkowski difference of the two convex bodies C and A was defined by

$$C \div A = \bigcap_{a\in A} (C-a) = \{x \in \mathbb{R}^n : A + x \subset C\}.$$

For $0 \le \tau \le r(C,A)$ (the inradius of C relative to A) we define

$$C_\tau := (C \div \tau A) + \tau A$$

and

$$h_u(\tau) := h(C_\tau, u) \quad \text{for } u \in \mathbb{S}^{n-1}.$$

Lemma 7.5.4 *If A is adapted to C, then $h'_u(0) = 0$ for each $u \in \mathbb{S}^{n-1}$.*

Proof Let $u \in \mathbb{S}^{n-1}$ and $0 < \tau < r(C,A)$ be given. We can choose $z \in \mathbb{R}^n$ such that $\tau A + z \subset C$ and $\langle z, u\rangle$ is maximal. Let

$$U_\tau := \{v \in \mathbb{S}^{n-1} : h(C,v) = h(\tau A + z, v)\}.$$

We assert that

$$u \in \operatorname{pos} U_\tau. \tag{7.111}$$

Assume this is false. Then u and the closed convex cone $\operatorname{pos} U_\tau$, which is pointed since $\tau < r(C,A)$, can be separated strongly by a hyperplane. Hence, there is a vector $w \in \mathbb{R}^n$ such that $\langle w,u\rangle > 0$ and $\langle w,v\rangle < 0$ for each $v \in U_\tau \setminus \{o\}$. On the closed hemisphere $\mathbb{S}^{n-1} \cap H^+_{w,0}$, the continuous function $h(C,\cdot) - h(\tau A + z, \cdot)$ is positive and hence attains a positive minimum ε. Then

$$h(C,v) - h(\tau A + z, v) \ge \varepsilon\langle w,v\rangle$$

for all $v \in \mathbb{S}^{n-1}$ and hence $\tau A + z + \varepsilon w \subset C$. Since $\langle z + \varepsilon w, u\rangle > \langle z, u\rangle$, this contradicts the choice of z. Hence, the assumption was false, and (7.111) holds.

By (7.111) and Carathéodory's theorem, we can choose (not necessarily distinct) unit vectors $v_1, \dots, v_n \in U_\tau$ and numbers $\lambda_1, \dots, \lambda_n \geq 0$ such that

$$u = \sum_{i=1}^{n} \lambda_i v_i.$$

For $x \in C$ we have

$$\langle x, v_i\rangle \leq h(C, v_i) = h(\tau A + z, v_i),$$

hence

$$\langle x, u\rangle \leq \sum_{i=1}^{n} \lambda_i h(\tau A + z, v_i)$$

and thus

$$h(C, u) \leq \sum_{i=1}^{n} \lambda_i h(\tau A + z, v_i).$$

By (3.15) we have $h_u(\tau) \leq h_u(0)$. Further, $\tau A + z \subset C$, hence $z \in C \div \tau A$ and thus $\tau A + z \subset C_\tau$, from which we infer that

$$\begin{aligned} h_u(0) - h_u(\tau) \leq h(C, u) - h(\tau A + z, u) &\leq \sum_{i=1}^{n} \lambda_i h(\tau A + z, v_i) - h(\tau A + z, u) \\ &= \tau \left[\sum_{i=1}^{n} \lambda_i h(A, v_i) - h(A, u) \right]. \end{aligned}$$

We deduce that

$$0 \geq \frac{h_u(\tau) - h_u(0)}{\tau} \geq h(A, u) - \sum_{i=1}^{n} \lambda_i h(A, v_i). \tag{7.112}$$

Now we let τ tend to zero. The vectors v_i and numbers λ_i chosen above depend on τ. For each $\tau \in (0, r(C, A))$ we make a definite choice of unit vectors $v_1(\tau), \dots, v_n(\tau)$ and corresponding numbers $\lambda_1(\tau), \dots, \lambda_n(\tau)$. We assert that

$$\lim_{\tau \to 0} \sum_{i=1}^{n} \lambda_i(\tau) h(A, v_i(\tau)) = h(A, u). \tag{7.113}$$

Suppose this were false. Then there exists a number $\alpha > 0$ and a sequence $(\tau_j)_{j \in \mathbb{N}}$ such that $\tau_j \to 0$ for $j \to \infty$ and

$$h(A, u) - \sum_{i=1}^{n} \lambda_i(\tau_j) h(A, v_i(\tau_j)) \leq -\alpha \tag{7.114}$$

for $j \in \mathbb{N}$. For each τ_j we choose a vector z_j such that $\tau_j A + z_j \subset C$ and $\langle z_j, u\rangle$ is maximal. After selecting subsequences and changing the notation we may assume that

$$\lim_{j\to\infty} z_j = z, \qquad \lim_{j\to\infty} v_i(\tau_j) = v_i \quad \text{for } i = 1,\dots,n,$$

and also that the compact sets U_{τ_j} converge to a set U. Since $\operatorname{pos} U$ is pointed, there are a unit vector b and a number $\beta > 0$ such that $\langle b, v\rangle \geq \beta$ for $v \in U_{\tau_j}$ and all sufficiently large j. From

$$\langle b, u\rangle = \sum_{i=1}^{n} \lambda_i(\tau_j)\langle b, v_i(\tau_j)\rangle$$

it follows that $\lambda_i(\tau_j) \in [0, \langle b, u\rangle/\beta]$ for large j. Hence, we may also assume that

$$\lim_{j\to\infty} \lambda_i(\tau_j) = \lambda_i \quad \text{for } i = 1,\dots,n.$$

Assuming further that $o \in A$, we have $z_j \in \tau_j A + z_j$, and we deduce that

$$z \in F(C,u), \qquad u \in N(C,z),$$

$$v_i \in N(C,z) \quad \text{for } i = 1,\dots,n,$$

$$u = \sum_{i=1}^{n} \lambda_i v_i.$$

Since A is adapted to C, there is a point $y \in \operatorname{bd} A$ for which $N(C,z) \subset N(A,y)$. Thus $u \in N(A,y)$, $v_i \in N(A,y)$ for $i = 1,\dots,n$ and

$$h(A,u) - \sum_{i=1}^{n} \lambda_i h(A,v_i) = \langle y, u\rangle - \sum_{i=1}^{n} \lambda_i \langle y, v_i\rangle = 0.$$

But (7.114) yields

$$h(A,u) - \sum_{i=1}^{n} \lambda_i h(A,v_i) < 0.$$

This contradiction proves (7.113), which together with (7.112) completes the proof. □

Notes for Section 7.5

1. Lemmas 7.5.1 to 7.5.3 (the latter with a slight generalization) and their proofs are taken from Aleksandrov [15]. The proof of Theorem 7.2.3 given here is due to Diskant [502]. Lemma 7.5.4, with essentially the same proof, appears in Schneider [1713].
2. *Inner parallel bodies*. In the geometry of convex bodies, there are many more applications of inner parallel bodies. Here we give a list of references containing such applications: Bol [269, 270, 271, 272], Chakerian [401, 402], Chakerian and Sangwine-Yager [406], Czipszer [457], Dinghas [483], Diskant [502, 503], Fáry [547], Hadwiger [908, 911], Matheron [1360, 1361], Oshio [1484, 1485, 1486], Rényi [1572], Sangwine-Yager [1616, 1617, 1618, 1619], Schneider [1710, 1713], v. Sz.-Nagy [1834]. The application by Fáry [548]

contains a serious error (there are counterexamples to the higher-order differentiability properties stated in Theorem 4).

3. *Inclusion measures.* Inner parallel bodies, in particular formula (7.110), can be used to obtain information on the *inclusion measure*

$$m(L \subset K) := \mu(\{g \in \mathrm{G}_n : gL \subset K\})$$

for convex bodies $K, L \in \mathcal{K}^n$, where μ is the Haar measure on the motion group G_n, normalized as in §4.4. In this way, Zhang [2055] obtained the estimates (assuming $V_n(K) \geq V_n(L)$, with S the surface area and w the mean width)

$$V_n(K) + (n-1)V_n(L) - \tfrac{1}{2}S(K)w(L) \leq m(L \subset K) \leq \left[V_n(K)^{1/n} - V_n(L)^{1/n}\right]^n.$$

He characterized the equality cases. Further, he obtained different sufficient conditions for K to contain a congruent copy of L. The Bonnesen-type inequalities of Corollary 1 in [2055], obtained as a by-product, are slightly weaker than (7.30) and (7.31).

For further information on inclusion measures, we refer to the survey by Zhang and Zhou [2061] and to the estimates obtained by Xiong, Cheung and Li [1996].

4. *H-convex sets*. The notion of a convex body determined by Ω can be generalized. Let $H \subset \mathbb{S}^{n-1}$ be an arbitrary subset. Any halfspace $H^-_{u,\alpha}$ with $u \in H$ and $\alpha \in \mathbb{R}$ is called H-convex, and the intersection of any family of H-convex halfspaces is called an H-convex set. A thorough study, mainly from the combinatorial viewpoint, of the family of H-convex sets, for given H, was made in the books by Boltyanski and Soltan [276] and by Boltyanski, Martini and Soltan [277].

5. *Wulff shape*. For $K \in \mathcal{K}^n_{(o)}$ and a continuous nonnegative function g on $\mathbb{S}^{n-1}$, let $F_t K$ be the Wulff shape associated with $(\mathbb{S}^{n-1}, h_K + tg)$ for $t \geq 0$, thus

$$F_t K = \bigcap_{u \in \mathbb{S}^{n-1}} H^-_{u,h(K,u)+tg(u)}.$$

The map $F : \mathcal{K}^n_{(o)} \times [0,\infty) \to \mathcal{K}^n_{(o)}$ defined in this way was studied by Willson [1985]. In particular, he proved the semigroup property $F_s F_t K = F_{s+t} K$ and described applications to the Wulff shape in the theory of the growth of physical crystals.

For an application of the Brunn–Minkowski theory to the Wulff shape, see also Dinghas [485] and Gruber [834], Theorem 8.13. More references are found in Gardner [674], §6.

7.6 Equality cases and stability

The Aleksandrov–Fenchel inequality

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geq V(K_1, K_1, K_3, \dots, K_n)V(K_2, K_2, K_3, \dots, K_n)$$

holds with equality if K_1 and K_2 are homothetic. For special choices of the bodies $K_3, \dots, K_n$ this is the only case of equality (certainly for $n = 2$, by Theorem 7.2.1). In the following, we shall first study these special cases, and the corresponding stability results and their applications. The second part of the present section is devoted to the (still incomplete) investigation of the cases of equality in general.

The classical result in this context states that equality implies homothety of K_1 and K_2 if $K_3, \dots, K_n$ are balls. We give two proofs for this result and start with a lemma.

Lemma 7.6.1 *Let $n \geq 3$; let $K, L \in \mathcal{K}^n$ be convex bodies such that their projections K^u and L^u are translates, for each $u \in \mathbb{S}^{n-1}$. Then K and L are translates.*

Proof For $u \in \mathbb{S}^{n-1}$ we define

$$G(u) := \{z \in \mathbb{R}^n : K^u = (L+z)^u\};$$

then $G(u)$ is a line of direction u. Let $u, v \in \mathbb{S}^{n-1}$ be linearly independent. Let $z \in G(u)$, thus $K^u = (L+z)^u$. Since $(L+z)^v$ is a translate of L^v, there is a vector $y \in \mathbb{R}^n$ with $K^v = (L+z+y)^v$. For all $w \in \mathbb{S}^{n-1}$ with $w \perp u$ and $w \perp v$ we have $h(K,w) = h(L+z,w)$, because $K^u = (L+z)^u$ and $w \perp u$, and $h(K,w) = h(L+z+y,w)$, because $K^v = (L+z+y)^v$ and $w \perp v$; hence $\langle y, w\rangle = 0$. Since this holds for all w with $w \perp u, v$, we infer that $y = \lambda u + \mu v$ with $\lambda, \mu \in \mathbb{R}$. The vector $x := z + y - \mu v$ satisfies $x \in G(u)$; further, $K^v = (L+x)^v$ and hence $x \in G(v)$. Thus we have proved that $G(u) \cap G(v) \neq \emptyset$ for linearly independent u, v.

Now let $u_1, u_2 \in \mathbb{S}^{n-1}$ be linearly independent; then $G(u_1) \cap G(u_2) = \{t\}$ for some t. If $v \in \mathbb{S}^{n-1}$ is linearly independent of u_1 and u_2, the line $G(v)$, which has to meet $G(u_1)$ and $G(u_2)$, necessarily passes through t. Hence, $K^v = (L+t)^v$ and thus $h(K,w) = h(L+t,w)$ for all $w \perp v$. The set of all w for which this holds is dense in $\mathbb{S}^{n-1}$, hence $K = L + t$. □

In the following, the unit ball B^n is briefly denoted by B.

Theorem 7.6.2 *If $K, L \in \mathcal{K}^n$ are convex bodies for which equality holds in*

$$V(K,L,B,\dots,B)^2 \geq V(K,K,B,\dots,B)V(L,L,B,\dots,B), \tag{7.115}$$

then K and L are homothetic.

Proof We use induction with respect to the dimension. For $n = 2$, the result follows from Theorem 7.2.1 if $\dim K = \dim L = 2$, and from Theorem 5.1.8 if this is not the case (recall that, by definition, a singleton and an arbitrary convex body are always homothetic). Suppose that $n \geq 3$ and the assertion is true in smaller dimensions. If (7.115) holds with equality, then by Theorem 7.4.2 the measures $S(K,\mathcal{B},\cdot)$ and $S(L,\mathcal{B},\cdot)$ are proportional, where $\mathcal{B}$ stands for the $(n-2)$-tuple $(B,\dots,B)$. Without loss of generality, we may assume that

$$S(K,\mathcal{B},\cdot) = S(L,\mathcal{B},\cdot). \tag{7.116}$$

Then Lemma 7.4.7 yields

$$v(K^u, L^u, B^u[n-3])^2 = v(K^u, K^u, B^u[n-3])v(L^u, L^u, B^u[n-3])$$

for $u \in \mathbb{S}^{n-1}$. We can choose u from a dense subset of $\mathbb{S}^{n-1}$ such that $\dim K^u \geq 1$, $\dim L^u \geq 1$. Then it follows from the induction hypothesis that K^u and L^u are homothetic. From (7.116) and (5.78) we get $v(K^u, \mathcal{B}^u) = v(L^u, \mathcal{B}^u)$, hence K^u and L^u are, in fact, translates. By continuity, this holds for all $u \in \mathbb{S}^{n-1}$. By Lemma 7.6.1, K and L are translates. □

The second proof we shall give for Theorem 7.6.2 yields more, namely a stability version. It depends upon the following analytic inequality, which is closely related to a result known as Wirtinger's lemma (or the Poincaré inequality).

Lemma 7.6.3 *Let f be a real function of class C^2 on $\mathbb{S}^{n-1}$ satisfying*

$$\int f \, d\sigma = 0 \tag{7.117}$$

and

$$\int f(u)u \, d\sigma(u) = o, \tag{7.118}$$

where σ denotes spherical Lebesgue measure and the integrations extend over $\mathbb{S}^{n-1}$. Then

$$\int f\left(f + \frac{1}{n-1}\Delta_S f\right) d\sigma + \frac{n+1}{n-1} \int f^2 \, d\sigma \leq 0. \tag{7.119}$$

Proof We use spherical harmonics (see the Appendix for details). Let $X_m := \pi_m f$ (defined by (A.3)) for $m = 0, 1, 2, \ldots$, then $X_0 = 0$ and $X_1 = 0$ by (7.117) and (7.118); hence the Fourier series of f with respect to the system of spherical harmonics is

$$f \sim \sum_{m=2}^{\infty} X_m.$$

By Green's formula on the sphere $\mathbb{S}^{n-1}$ and by (A.2),

$$(\Delta_S f, X_m) = (f, \Delta_S X_m) = -m(m+n-2)(f, X_m),$$

hence

$$\Delta_S f \sim -\sum_{m=2}^{\infty} m(m+n-2) X_m.$$

The Parseval relation (A.5) gives

$$\int f^2 \, d\sigma = \sum_{m=2}^{\infty} \int X_m^2 \, d\sigma,$$

$$\int f \Delta_S f d\sigma = -\sum_{m=2}^{\infty} m(m+n-2) \int X_m^2 \, d\sigma.$$

This yields

$$\begin{aligned} &\int f\left(f + \frac{1}{n-1}\Delta_S f\right) d\sigma + \frac{n+1}{n-1} \int f^2 \, d\sigma \\ &= \frac{1}{n-1} \sum_{m=2}^{\infty} [2n - m(m+n-2)] \int X_m^2 \, d\sigma \leq 0. \end{aligned}$$

□

If the preceding result is applied to differences of support functions, one obtains estimates for the deviation of convex bodies; however, these are not immediately in terms of the Hausdorff metric δ but in terms of the L_2 metric δ_2. This is defined by

$$\delta_2(K, L)^2 := \int_{\mathbb{S}^{n-1}} |h_K - h_L|^2 \, d\sigma. \tag{7.120}$$

Clearly, $\delta_2 \leq \sqrt{\omega_n}\,\delta$. Estimates in the opposite direction are given in the following two lemmas, which we state without proof.

Lemma 7.6.4 *For $K, L \in \mathcal{K}^n$,*

$$\delta_2(K, L)^2 \geq \alpha_n D(K \cup L)^{1-n} \delta(K, L)^{n+1}$$

with

$$\alpha_n = \frac{\omega_n \mathrm{B}(3, n-1)}{\mathrm{B}\left(\frac{1}{2}, \frac{n-1}{2}\right)},$$

where $\mathrm{B}(\cdot, \cdot)$ *is the beta integral.*

For a convex body K, we introduce its *Steiner ball* $B(K)$ as the ball that has the same Steiner point and mean width as K, thus

$$B(K) := B(s(K), w(K)/2). \tag{7.121}$$

Lemma 7.6.5 *If $K \in \mathcal{K}^n$ and $B(K)$ is the Steiner ball of K, then*

$$\delta_2(K, B(K))^2 \geq \beta_n \left(\frac{\kappa_n}{W_{n-1}(K)} \right)^{(n-1)/2} \delta(K, B(K))^{(n+3)/2}$$

with $\beta_n = \gamma_n(\omega_n/2\kappa_{n-1})$, where

$$\gamma_n(c) = \frac{\omega_{n-1}}{(n^2-1)(n+3)} \min\left[\frac{3}{\pi^2 n(n+2)2^n}, \frac{16(c+2)^{(n-3)/2}}{(c+1)^{n-2}} \right].$$

Lemma 7.6.4 was proved (in a more general form) by Vitale [1885] and Lemma 7.6.5 by Groemer and Schneider [804]. For the special case of the deviation from a suitable ball, Lemma 7.6.5 gives a better estimate, since the exponent $(n+3)/2$ is smaller than the exponent $n+1$ of $\delta(K, L)$ in Lemma 7.6.4.

When stability estimates are obtained in terms of the L_2 metric, we shall refrain from translating them into Hausdorff metric estimates, but the reader should keep in mind that this is always possible, as a result of Lemmas 7.6.4 and 7.6.5.

To deduce such a stability estimate from Lemma 7.6.3, we associate, with each convex body $K \in \mathcal{K}^n$ of dimension at least one, its normalized homothetic copy $\overline{K}$, defined by

$$\overline{K} := \frac{K - s(K)}{w(K)}.$$

Let $K, L \in \mathcal{K}^n$ be convex bodies of dimension at least one, and put

$$V_{ij} := V(K[i], L[j], B[n-i-j]).$$

Then equality (7.115) reads

$$V_{11}^2 - V_{20}V_{02} \geq 0.$$

The following theorem contains a strengthened version of this inequality.

Theorem 7.6.6 *For convex bodies $K, L \in \mathcal{K}^n$ of dimension ≥ 1,*

$$V_{11}^2 - V_{20}V_{02} \geq \frac{n+1}{n(n-1)} w(K)^2 V_{02} \delta_2(\overline{K}, \overline{L})^2. \tag{7.122}$$

Proof First we assume that K and L are of class C_+^2. The function $f := h(\overline{K}, \cdot) - h(\overline{L}, \cdot)$ satisfies (7.117) and (7.118), by (A.6) and (A.7). Hence, Lemma 7.6.3 yields

$$\int \left(f^2 + \frac{1}{n-1} f \Delta_S f \right) \mathrm{d}\sigma + \frac{n+1}{n-1} \delta_2(\overline{K}, \overline{L})^2 \leq 0.$$

By formula (5.62),

$$\begin{aligned} &\frac{1}{n} \int \left(f^2 + \frac{1}{n-1} f \Delta_S f \right) \mathrm{d}\sigma \\ &= V(\overline{K}, \overline{K}, B[n-2]) - 2V(\overline{K}, \overline{L}, B[n-2]) + V(\overline{L}, \overline{L}, B[n-2]) \\ &= \frac{V_{20}}{w(K)^2} - \frac{2V_{11}}{w(K)w(L)} + \frac{V_{02}}{w(L)^2}. \end{aligned}$$

Applying the identity

$$b^2 - ac = -c(a - 2b + c) + (b - c)^2$$

with

$$a = \frac{w(L)}{w(K)} V_{20}, \quad b = V_{11}, \quad c = \frac{w(K)}{w(L)} V_{02},$$

we arrive at the inequality

$$V_{11}^2 - V_{20}V_{02} \geq \frac{n+1}{n(n-1)} w(K)^2 V_{02} \delta_2(\overline{K}, \overline{L})^2 + \left(V_{11} - \frac{w(K)}{w(L)} V_{02} \right)^2,$$

of which (7.122) is a weaker version. By approximation, this inequality is now extended to general convex bodies K and L. $\square$

The special case of (7.122) where L is a ball can be written in a more convenient form, using the Steiner ball of K. Since

$$\delta_2(\overline{K}, \overline{B})^2 = w(K)^{-2} \delta_2(K, B(K))^2,$$

as a corollary of Theorem 7.6.6 we obtain, for the quermassintegrals W_{n-1}, W_{n-2} of the convex body $K \in \mathcal{K}^n$, the inequality

$$\frac{1}{\kappa_n} W_{n-1}^2 - W_{n-2} \geq \frac{n+1}{n(n-1)} \delta_2(K, B(K))^2. \tag{7.123}$$

From this, we can deduce a stability result for the more general inequality (7.67) between two quermassintegrals of convex bodies. Let $K \in \mathcal{K}_n^n$ and write $W_k = W_k(K)$; let $0 \leq i < j < n$. Using the special Aleksandrov–Fenchel inequality $W_k^2 \geq W_{k-1}W_{k+1}$, we obtain

$$\frac{W_{j+1}^{n-j}}{W_j^{n-j-1}W_n} = \left(\frac{W_{j+1}^2}{W_jW_{j+2}}\right)^{n-j-1}\left(\frac{W_{j+2}^2}{W_{j+1}W_{j+3}}\right)^{n-j-2}\cdots\left(\frac{W_{n-1}^2}{W_{n-2}W_n}\right) \geq \frac{W_{n-1}^2}{W_{n-2}W_n}$$

and

$$\frac{W_j^{j-i+1}}{W_iW_{j+1}^{j-i}} \geq 1,$$

the latter from (7.66). This gives

$$\left(\frac{W_{j+1}^{n-j}}{W_j^{n-j-1}W_n}\right)^{j-i}\left(\frac{W_j^{j-i+1}}{W_iW_{j+1}^{j-i}}\right)^{n-j} \geq \left(\frac{W_{n-1}^2}{W_{n-2}W_n}\right)^{j-i},$$

hence

$$\frac{W_j^{n-i}}{W_i^{n-j}} \geq \left(\frac{W_{n-1}^2}{W_{n-2}}\right)^{j-i}$$

and thus

$$\kappa_n^{i-j}W_j^{n-i} - W_i^{n-j} \geq \frac{W_i^{n-j}}{W_{n-2}^{j-i}}\left[\left(\frac{W_{n-1}^2}{\kappa_n}\right)^{j-i} - W_{n-2}^{j-i}\right].$$

Using the identity

$$\frac{a^k - b^k}{a - b} = \sum_{r=0}^{k-1} a^{k-1-r}b^r =: s_k(a, b)$$

with $k = j - i$, $a = W_{n-1}^2/\kappa_n$, $b = W_{n-2}$ and then (7.123), we arrive at

$$\kappa_n^{i-j}W_j^{n-i} - W_i^{n-j} \geq \gamma_{n,i,j}(K)\delta_2(K, B(K))^2 \tag{7.124}$$

with

$$\gamma_{n,i,j}(K) = \frac{n+1}{n(n-1)}\frac{W_i^{n-j}}{W_{n-2}^{j-i}}s_{j-i}\left(\frac{W_{n-1}^2}{\kappa_n}, W_{n-2}\right).$$

This is the announced stability result for two quermassintegrals.

The case $i = 0$, $j = 1$ of inequality (7.124) is a strengthened version of the isoperimetric inequality, namely

$$\left(\frac{S}{\omega_n}\right)^n - \left(\frac{V_n}{\kappa_n}\right)^{n-1} \geq \frac{n+1}{n(n-1)\kappa_n^{n-1}}\frac{V_n^{n-1}}{W_{n-2}}\delta_2(K, B(K))^2. \tag{7.125}$$

Up to now, the considerations of this section centred around the fact that equality in the Aleksandrov–Fenchel inequality

$$V(K, L, C_1, \dots, C_{n-2})^2 \geq V(K, K, C_1, \dots, C_{n-2})V(L, L, C_1, \dots, C_{n-2}), \tag{7.126}$$

under the special assumption $C_1 = \cdots = C_{n-2} = B^n$ (unit ball), holds only if K and L are homothetic. Using this result, we shall now show that the same conclusion can be drawn if we assume only that $C_1, \dots, C_{n-2}$ are smooth. We first prove a more general result, which states that equality in (7.126) is preserved under replacement of the $(n-2)$-tuple $(C_1, \dots, C_{n-2})$ by any other $(n-2)$-tuple $(A_1, \dots, A_{n-2})$ of convex bodies such that A_i is adapted to C_i, in the sense defined in Section 7.5.

Theorem 7.6.7 *Let $C_1, \dots, C_{n-2} \in \mathcal{K}^n_n$, set $\mathscr{C} = (C_1, \dots, C_{n-2})$ and let $K, L \in \mathcal{K}^n$ be convex bodies satisfying*

$$V(K, L, \mathscr{C})^2 = V(K, K, \mathscr{C})V(L, L, \mathscr{C}). \tag{7.127}$$

If $\mathscr{A} = (A_1, \dots, A_{n-2})$, where $A_i \in \mathcal{K}^n_n$ is adapted to C_i for $i = 1, \dots, n-2$, then also

$$V(K, L, \mathscr{A})^2 = V(K, K, \mathscr{A})V(L, L, \mathscr{A}). \tag{7.128}$$

Proof Let $n \geq 3$, without loss of generality. If $\dim K = 0$ or $\dim L = 0$, then K and L are homothetic. If, say, $\dim K = 1$, then $V(K, K, \mathscr{C}) = 0$, hence $V(K, L, \mathscr{C}) = 0$. Since $C_1, \dots, C_{n-2}$ are n-dimensional, this is only possible if either $\dim L = 0$ or L is a segment parallel to K. Thus K and L are homothetic, and (7.128) holds. We may, therefore, assume in the following that $\dim K \geq 2$ and $\dim L \geq 2$.

We write

$$C_1 = C, \quad A_1 = A \quad \text{and} \quad (C_2, \dots, C_{n-2}) = \mathscr{C}',$$

thus

$$V(K, L, C, \mathscr{C}')^2 = V(K, K, C, \mathscr{C}')V(L, L, C, \mathscr{C}'). \tag{7.129}$$

By Theorem 7.4.2 we may assume, after a dilatation of K, that

$$S(K, C, \mathscr{C}', \cdot) = S(L, C, \mathscr{C}', \cdot). \tag{7.130}$$

Let $Q \in \mathcal{K}^n$ be a convex body with $\dim Q \geq 1$. By (7.82), with $\mathscr{C}$ replaced by $(Q, \mathscr{C}')$ and M replaced by C, we have

$$\frac{V(K, K, Q, \mathscr{C}')}{V(K, Q, C, \mathscr{C}')^2} - \frac{2V(K, L, Q, \mathscr{C}')}{V(K, Q, C, \mathscr{C}')V(L, Q, C, \mathscr{C}')} + \frac{V(L, L, Q, \mathscr{C}')}{V(L, Q, C, \mathscr{C}')^2} \leq 0. \tag{7.131}$$

By (7.130) and (5.19),

$$V(K, Q, C, \mathscr{C}') = V(L, Q, C, \mathscr{C}'), \tag{7.132}$$

hence

$$V(K, K, Q, \mathscr{C}') - 2V(K, L, Q, \mathscr{C}') + V(L, L, Q, \mathscr{C}') \leq 0. \tag{7.133}$$

Again by (7.130) and (5.19),

$$V(K,K,C,\mathscr{C}') - 2V(K,L,C,\mathscr{C}') + V(L,L,C,\mathscr{C}') = 0. \tag{7.134}$$

We define

$$\Phi(Q) := V(K,K,Q,\mathscr{C}') - 2V(K,L,Q,\mathscr{C}') + V(L,L,Q,\mathscr{C}')$$

for $Q \in \mathcal{K}^n$ and

$$g(\tau) := \Phi(C_\tau)$$

for $0 < \tau < r(C,A)$, with $C_\tau := (C \div \tau A) + \tau A$ as in Section 7.5; further,

$$f(\tau) := \Phi(C \div \tau A).$$

Then

$$f(\tau) \leq 0 \tag{7.135}$$

and

$$\Phi(A) \leq 0 \tag{7.136}$$

by (7.133). By the linearity of the mixed volume,

$$g(\tau) = f(\tau) + \tau\Phi(A). \tag{7.137}$$

We assert that

$$g'(0) = 0. \tag{7.138}$$

For the proof, let

$$\varphi(\tau) := V(C_\tau, \overline{\mathscr{C}}),$$

where $\overline{\mathscr{C}}$ is an arbitrary $(n-1)$-tuple of convex bodies. Then, by (5.19) and with $h_u(\tau) := h(C_\tau, u)$,

$$\frac{\varphi(\tau) - \varphi(0)}{\tau} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \frac{h_u(\tau) - h_u(0)}{\tau}\, S(\overline{\mathscr{C}}, \mathrm{d}u).$$

We show that the integrand remains bounded for $\tau \to 0$. Choose a positive homothet $\lambda A + z$ contained in C. For given $u \in \mathbb{S}^{n-1}$, choose $x \in F(C,u)$. For $0 < \tau < \lambda$, let $\tau A + z_\tau$ be obtained from $\lambda A + z$ by dilatation with centre x. Then $\tau A + z_\tau \subset C$ and hence $h(C_\tau, u) \geq h(\tau A + z_\tau, u)$. Writing

$$h(C,u) - h(\tau A + z_\tau, u) =: \alpha, \qquad h(C,u) - h(\lambda A + z, u) =: \beta,$$

we have $\alpha/\beta = \tau/\lambda$ and hence

$$0 \leq h_u(0) - h_u(\tau) = h(C,u) - h(C_\tau, u) \leq \alpha = \beta\tau/\lambda \leq \gamma\tau,$$

where the constant γ is independent of u. This proves that the integrand remains bounded. Hence, we can apply the bounded convergence theorem together with Lemma 7.5.4 to deduce that $\varphi'(0) = 0$. Since $\mathscr{C}'$ was arbitrary, this proves (7.138).

From (7.137) and (7.138) we get $f'(0) + \Phi(A) = 0$, hence $f'(0) \geq 0$ by (7.136). On the other hand, $f(0) = 0$ by (7.134) and $f(\tau) \leq 0$ by (7.135). This yields $f'(0) = 0$ and thus $\Phi(A) = 0$, hence

$$V(K, K, A, \mathscr{C}') - 2V(K, L, A, \mathscr{C}') + V(L, L, A, \mathscr{C}') = 0.$$

This, together with (7.132) for $Q = A$, shows that there is equality in (7.131) for $Q = A$. By Theorem 7.4.3, this implies

$$V(K, L, A, \mathscr{C}')^2 = V(K, K, A, \mathscr{C}')V(L, L, A, \mathscr{C}')$$

or, returning to the former notation,

$$\begin{aligned} &V(K, L, A_1, C_2, \ldots, C_{n-2})^2 \\ &\quad = V(K, K, A_1, C_2, \ldots, C_{n-2})V(L, L, A_1, C_2, \ldots, C_{n-2}). \end{aligned}$$

This equality has been deduced under the assumption that (7.127) holds and that A_1 is adapted to C_1. In a similar way, each of the bodies $C_2, \ldots, C_{n-2}$ can be replaced by another one adapted to it. This proves the theorem. □

Now we can prove the announced result.

Theorem 7.6.8 *If equality holds in*

$$V(K, L, C_1, \ldots, C_{n-2})^2 \geq V(K, K, C_1, \ldots, C_{n-2})V(L, L, C_1, \ldots, C_{n-2}), \tag{7.139}$$

where $C_1, \ldots, C_{n-2}$ are smooth convex bodies, then K and L are homothetic.

Proof If C_i is smooth, then the unit ball B^n is adapted to C_i; hence, by Theorem 7.6.7, equality in (7.139) implies equality in (7.115). By Theorem 7.6.2, K and L are homothetic. □

Introducing smoothness conditions, we are now in a position to complete the results on the general Brunn–Minkowski theorem that are collected in Theorem 7.4.6.

Theorem 7.6.9 *Let a number $m \in \{2, \ldots, n\}$ and an $(n-m)$-tuple $\mathscr{C} = (K_{m+1}, \ldots, K_n)$ of smooth convex bodies be given, and let $K_0, K_1 \in \mathcal{K}^n$ be convex bodies of dimension $\geq m$. Then the conditions* (a) – (d) *of* Theorem 7.4.6 *hold only if K_0 and K_1 are homothetic.*

Proof We use the notation introduced before Theorem 7.4.6 and note that $V_{(i)} \neq 0$ for $i = 0, \ldots, m$, since $\dim K_j \geq m$ for $j = 0, 1$ and $\dim K_j = n$ for $j = m + 1, \ldots, n$. Assume that the equivalent conditions (a) – (d) of that theorem are satisfied. Since (b) holds, that is,

$$V_{(k)}^2 = V_{(k-1)}V_{(k+1)} \quad \text{for } k = 1, \ldots, m - 1, \tag{7.140}$$

then by Theorem 7.4.2 the measures $S_{(k)}$ and $S_{(k-1)}$ are proportional. The proportionality factor is obtained by integrating $h(K_0, \cdot)$, and we obtain

$$V_{(k-1)}S_{(k)} = V_{(k)}S_{(k-1)}.$$

Integrating the support function of the unit ball B^n with this measure, we get

$$V_{(k-1)}V(K_0[m-1-k], K_1[k], B^n, \mathscr{C}) = V_{(k)}V(K_0[m-k], K_1[k-1], B^n, \mathscr{C}).$$

If we define

$$\overline{V}_{(k)} := V(K_0[m-1-k], K_1[k], B^n, \mathscr{C}),$$

this reads

$$V_{(k-1)}\overline{V}_{(k)} = V_{(k)}\overline{V}_{(k-1)} \quad \text{for } k = 1, \dots, m-1,$$

hence

$$V_{(k)}\overline{V}_{(k+1)} = V_{(k+1)}\overline{V}_{(k)} \quad \text{for } k = 0, \dots, m-2.$$

Together with (7.140), this yields

$$\overline{V}^2_{(k)} = \overline{V}_{(k-1)}\overline{V}_{(k+1)} \quad \text{for } k = 1, \dots, m-2.$$

Thus, we are again in the situation described by (7.140), but with m replaced by $m-1$ and $\mathscr{C}$ replaced by $(B^n, \mathscr{C})$. Repeating the procedure, we end up with the equality

$$V(K_0, K_1, B^n[m-2], \mathscr{C})^2 = V(K_0, K_0, B^n[m-2], \mathscr{C})V(K_1, K_1, B^n[m-2], \mathscr{C}).$$

By Theorem 7.6.8, this implies that K_0 and K_1 are homothetic. □

Corollary 7.6.10 *Let $m \in \{2, \dots, n\}$, and let $\mathscr{C} = (K_{m+1}, \dots, K_n)$ be an $(n-m)$-tuple of smooth convex bodies. Let $K_0, K_1 \in \mathcal{K}^n$ be convex bodies of dimension $\geq m$. Then the inequality*

$$V(K_0[n-m-1], K_1, \mathscr{C})^{n-m} \geq V(K_0[n-m], \mathscr{C})^{n-m-1}V(K_1[n-m], \mathscr{C}) \tag{7.141}$$

holds with equality if and only if K_0 and K_1 are homothetic.

Proof In the same way as for the equality conditions in Minkowski's inequality (7.18), one sees that equality in (7.141) holds if and only if the function f defined by (7.88) is linear. The assertion follows now from Theorem 7.6.9. □

The functional $W_i(\cdot, \cdot)$ defined by

$$W_i(K, L) := V(K[n-i-1], L, B^n[i]) \tag{7.142}$$

for $i \in \{0, \dots, n-1\}$ and $K, L \in \mathcal{K}^n$, is known as a *mixed quermassintegral.*

Corollary 7.6.11 *Let $i \in \{0, \dots, n-2\}$, and let $K, L \in \mathcal{K}^n$ be convex bodies of dimension at least $n-i$. Then the inequality*

$$W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1}W_i(L) \tag{7.143}$$

holds, and equality holds if and only if K and L are homothetic.

Theorem 7.6.8 generalizes Theorem 7.6.2, in which the bodies $C_1, \dots, C_{n-2}$ were assumed to be balls. For the latter result, Theorem 7.6.6 gives a strengthened version in the form of a stability result. We shall now briefly discuss the possibilities of improving Theorem 7.6.8 in a similar way. For convex bodies $K, L, C_1, \dots, C_{n-2} \in \mathcal{K}^n$ we introduce the deficit Δ by

$$\begin{aligned}&\Delta(K, L, C_1, \dots, C_{n-2})\\ &:= V(K, L, C_1, \dots, C_{n-2})^2 - V(K, K, C_1, \dots, C_{n-2})V(L, L, C_1, \dots, C_{n-2}),\end{aligned}$$

so that the Aleksandrov–Fenchel inequality can be written as

$$\Delta(K, L, C_1, \dots, C_{n-2}) \geq 0. \tag{7.144}$$

Inequality (7.122) tells us that

$$\Delta(K, L, B^n, \dots, B^n) \geq \frac{n+1}{n(n-1)} a(K, L)\delta_2(\overline{K}, \overline{L})^2 \tag{7.145}$$

(if $\dim K, L \geq 1$), where

$$a(K, L) := \max\{w(K)^2 V(L, L, B^n[n-2]), w(L)^2 V(K, K, B^n[n-2])\}.$$

For general convex bodies $C_1, \dots, C_{n-2}$ it is possible (as we shall see later in this section) that equality holds in (7.144) without K and L being homothetic. This implies that Theorem 7.6.8 cannot be improved to a stability version of the type (7.145) that would hold for all bodies $C_1, \dots, C_{n-2}$ in a dense subclass of $\mathcal{K}^n$. However, one can show stability for (7.144) under the assumption that the convex bodies $C_1, \dots, C_{n-2}$ are ρ-smooth, for some fixed $\rho > 0$. The convex body $K \in \mathcal{K}^n_n$ is called *ρ-smooth* if ρB^n is a summand of K, or equivalently (by Theorem 3.2.2) if to each boundary point x of K there is a ball $\rho B^n + t$ of radius ρ such that $x \in \rho B^n + t \subset K$.

For given numbers $r, R > 0$, we denote by $\mathcal{K}^n(r, R)$ the set of all convex bodies in $\mathcal{K}^n$ that contain some ball of radius r and are contained in some ball of radius R.

Theorem 7.6.12 *Let $\rho, r, R > 0$ and an integer $p \in \{1, \dots, n-2\}$ be given. If $K, L, C_1, \dots, C_p \in \mathcal{K}^n(r, R)$ and if $C_1, \dots, C_p$ are ρ-smooth, then*

$$\Delta(K, L, C_1, \dots, C_p, B^n, \dots, B^n) \geq \rho^{4(2^p-1)} \gamma \Delta(K, L, B^n, \dots, B^n)^{2^p}, \tag{7.146}$$

where the constant γ depends only on n, p, r, R.

Together with (7.145), the inequality (7.146), with $p = n-2$, yields a stability estimate for the Aleksandrov–Fenchel inequality (7.144), provided that $C_1, \dots, C_{n-2}$ are ρ-smooth. It is inevitable that the factor of $\gamma\Delta$ on the right-hand side of an inequality of the type (7.146) tends to zero if ρ tends to zero; therefore, Theorem 7.6.8 is not a corollary of Theorem 7.6.12.

We shall not reproduce here the proof of Theorem 7.6.12. It can be found in Schneider [1714]. The result is a quantitative improvement of one obtained earlier in Schneider [1701]. A result of a similar type was proved by Arnold [75].

In the second part of this section, we shall now investigate the problem of equality in the Aleksandrov–Fenchel inequality

$$V(K, L, \mathscr{C})^2 \geq V(K, K, \mathscr{C})V(L, L, \mathscr{C}) \tag{7.147}$$

in the case where $\mathscr{C} = (C_1, \dots, C_{n-2})$ and the bodies $C_1, \dots, C_{n-2}$ are not necessarily smooth. In that case, equality in (7.147) may hold even if K and L are not homothetic. An example is given by Corollary 7.6.18 below. Before treating such examples and special cases, we formulate a general conjecture.

Conjecture 7.6.13 If $K, L \in \mathcal{K}^n$ and $C_1, \dots, C_{n-2} \in \mathcal{K}^n_n$, then equality in (7.147) holds if and only if, after applying a suitable homothety to K or L, the support functions satisfy $h(K, u) = h(L, u)$ for all $(C_1, \dots, C_{n-2}, B)$-extreme unit vectors u.

The notion of $(C_1, \dots, C_{n-1})$-extreme vectors was defined in Section 2.2. In particular, u is $(C_1, \dots, C_{n-2}, B)$-extreme if and only if there exist $(n-1)$-dimensional linear subspaces $E_1, \dots, E_{n-2}$ of $\mathbb{R}^n$ such that

$$T(C_i, u) \subset E_i \quad \text{for } i = 1, \dots, n-2$$

and $\dim E_1 \cap \dots \cap E_{n-2} = 2$. Here $T(C_i, u)$ is the touching cone of C_i at u.

Without the assumption in Conjecture 7.6.13 that $C_1, \dots, C_{n-2}$ are n-dimensional, more cases of equality in (7.147) are possible. This was pointed out by Ewald [541]. For example, let $n = 3$, let $S_1, S_2 \subset \mathbb{R}^3$ be segments of independent directions and let $M \subset \mathcal{K}^3$ be an arbitrary convex body. Then put $K := M + S_1$, $L := M + S_2$ and $C := S_1 + S_2$. Since $V(M, S_i, S_i) = 0$, one computes $V(K, K, C) = V(L, L, C) = V(K, L, C)$ and thus $V(K, L, C)^2 = V(K, K, C)V(L, L, C)$. The condition of Conjecture 7.6.13, however, is not satisfied.

If $C_1, \dots, C_{n-2}$ are smooth, then each vector $u \in \mathbb{S}^{n-1}$ is $(C_1, \dots, C_{n-2}, B)$-extreme, hence Theorem 7.6.8 would follow from Conjecture 7.6.13, if true. In fact, Theorem 7.6.7 would also follow.

The set of $(\mathscr{C}, B)$-extreme unit vectors appears in Conjecture 7.6.13 since it is probably closely related to the mixed area measure $S(\mathscr{C}, B, \cdot)$, according to another conjecture (of which a special case is verified by Theorem 4.5.3).

Conjecture 7.6.14 Let $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ be convex bodies. The support of their mixed area measure, $\operatorname{supp} S(K_1, \dots, K_{n-1}, \cdot)$, is the closure of the set of $(K_1, \dots, K_{n-1})$-extreme unit vectors.

Before discussing further the relation between the two conjectures, we state a result on supports of mixed area measures from which further consequences can be drawn.

Lemma 7.6.15 *Let $K, L, C_1, \dots, C_{n-2} \in \mathcal{K}^n$ and $\mathscr{C} = (C_1, \dots, C_{n-2})$. If L is smooth and strictly convex, then*

$$\operatorname{supp} S(K, \mathscr{C}, \cdot) \subset \operatorname{supp} S(L, \mathscr{C}, \cdot).$$

Proof Suppose the assertion is false. Then we can choose a vector $u_0 \in \operatorname{supp} S(K, \mathscr{C}, \cdot) \setminus \operatorname{supp} S(L, \mathscr{C}, \cdot)$ and an open neighbourhood α of u_0 in $\mathbb{S}^{n-1}$ such that $S(L, \mathscr{C}, \alpha) = 0$. Let

$$L' := \{x \in L : \langle x, u_0 \rangle \le h(L, u_0) - \varepsilon\},$$

where $\varepsilon > 0$ is so small that each outer unit normal vector to L at a point of $\operatorname{cl}(\operatorname{bd} L \setminus L')$ belongs to α; this is possible since L is smooth. For $u \in \mathbb{S}^{n-1} \setminus \alpha$, the unique boundary point of L at which u is an outer normal vector is also a boundary point of L'. We deduce that $S(L, \mathscr{C}, \beta) = S(L', \mathscr{C}, \beta)$ for every Borel set $\beta \subset \mathbb{S}^{n-1} \setminus \alpha$. This yields

$$0 \le V(B, L, \mathscr{C}) - V(B, L', \mathscr{C}) = \frac{1}{n} S(L, \mathscr{C}, \alpha) - \frac{1}{n} S(L', \mathscr{C}, \alpha)$$

and hence $S(L', \mathscr{C}, \alpha) = 0$. Thus $S(L, \mathscr{C}, \cdot) = S(L', \mathscr{C}, \cdot)$, giving $V(K, L, \mathscr{C}) = V(K, L', \mathscr{C})$. On the other hand, we can choose an open neighbourhood β of u_0 such that $h(L, u) > h(L', u)$ for $u \in \beta$. Since $u_0 \in \operatorname{supp} S(K, \mathscr{C}, \cdot)$, we have $S(K, \mathscr{C}, \beta) > 0$, which implies

$$\begin{aligned} V(K, L, \mathscr{C}) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u)\, S(K, \mathscr{C}, \mathrm{d}u) \\ &> \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L', u)\, S(K, \mathscr{C}, \mathrm{d}u) = V(K, L', \mathscr{C}). \end{aligned}$$

This contradiction completes the proof. $\square$

Instead of the condition appearing in Conjecture 7.6.13, let us now assume that the following is satisfied:

$$h(K, u) = h(L, u) \quad \text{for each } u \in \operatorname{supp} S(\mathscr{C}, B, \cdot).$$

Then for any convex body $M \in \mathcal{K}^n$ we have

$$\begin{aligned} &\frac{1}{n} \int_{\mathbb{S}^{n-1}} h(M, u)\, [S(K, \mathscr{C}, \mathrm{d}u) - S(L, \mathscr{C}, \mathrm{d}u)] = V(M, K, \mathscr{C}) - V(M, L, \mathscr{C}) \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} [h(K, u) - h(L, u)]\, S(M, \mathscr{C}, \mathrm{d}u) = 0, \end{aligned}$$

because $\operatorname{supp} S(M, \mathscr{C}, \cdot) \subset \operatorname{supp} S(B, \mathscr{C}, \cdot)$ by Lemma 7.6.15. Since M was arbitrary, we deduce that $S(K, \mathscr{C}, \cdot) = S(L, \mathscr{C}, \cdot)$ and thus equality holds in (7.147).

One may conjecture that the converse of this observation is also true.

Conjecture 7.6.16 If $K, L \in \mathcal{K}^n$, $C_1, \dots, C_{n-2} \in \mathcal{K}^n_n$ and equality holds in (7.147), then, after applying a suitable homothety to K or L,

$$h(K, u) = h(L, u) \quad \text{for each } u \in \operatorname{supp} S(\mathscr{C}, B, \cdot).$$

The truth of Conjectures 7.6.14 and 7.6.16 would imply the truth of Conjecture 7.6.13.

The problem of equality in the Aleksandrov–Fenchel inequality is related to another unsolved problem. This is the equality case for (5.25), which expresses the monotonicity of mixed volumes. We shall first prove a special result concerning this problem. It is useful in treating a special case of Conjecture 7.6.13, and its corollary gives a first example of non-trivial equality in (7.147). The result deals with the special mixed volumes $V_{(i)}(K, L) = V(K[n-i], L[i])$; for $L = B$ these are the quermassintegrals $W_i(K)$. If K, L are convex bodies such that $L \subset K$, then the monotonicity (5.25) implies

$$V_{(0)}(K, L) \geq V_{(1)}(K, L) \geq \cdots \geq V_{(n)}(K, L). \tag{7.148}$$

The following theorem characterizes the pairs K, L for which $V_{(p-1)}(K, L) = V_{(p)}(K, L)$ for some p. For $n = 3$, $p = 2$ it goes back to Minkowski [1441]; the general case is essentially due to Favard [553].

Theorem 7.6.17 *Let $K, L \in \mathcal{K}^n_n$ be convex bodies satisfying $L \subset K$; let $p \in \{1, \dots, n\}$. Then*

$$V_{(p-1)}(K, L) = V_{(p)}(K, L) \tag{7.149}$$

holds if and only if K is an $(n-p)$-tangential body of L; in this case, $V_{(0)}(K, L) = V_{(1)}(K, L) = \cdots = V_{(p)}(K, L)$.

Proof Assume that (7.149) holds. First let $p = 1$. As remarked before (5.25), $V_{(0)}(K, L) = V_{(1)}(K, L)$ implies

$$h(K, u) = h(L, u) \quad \text{for } u \in \operatorname{supp} S_{n-1}(K, \cdot).$$

By Theorem 4.5.3, $\operatorname{supp} S_{n-1}(K, \cdot)$ is the closure of the set of extreme unit normal vectors of K, hence each extreme supporting hyperplane of K supports L. By definition, K is an $(n-1)$-tangential body of L.

Now let $p \geq 1$. For $n = 2$, only $p = 1$ is possible, hence the assertion is true. We assume that $n > 2$ and that the assertion (i.e., sufficiency of (7.149)) has been proved in dimension $n-1$. We may also assume $p \geq 2$. From $V_{(p-2)}(K, L) \geq V_{(p-1)}(K, L) = V_{(p)}(K, L)$ and the Aleksandrov–Fenchel inequality we have

$$V_{(p-1)}(K, L)^2 = V_{(p-2)}(K, L) V_{(p)}(K, L).$$

By Theorem 7.4.2 (where (7.149) implies $\alpha = 1$) and (5.78) this yields

$$v(K^u[n-p+1], L^u[p-2]) = v(K^u[n-p], L^u[p-1]) \tag{7.150}$$

for each $u \in \mathbb{S}^{n-1}$.

Let $H(K, v)$ be a $(p-1)$-extreme support plane of K. Choose a unit vector u orthogonal to v and, if $\dim T(K, v) \geq 2$, in the linear hull of the touching cone $T(K, v)$. Then $H' := H(K, v) \cap H_{u,0}$ is a $(p-2)$-extreme support plane of K^u (relative to $H_{u,0}$). Since $L^u \subset K^u$, we deduce from (7.150) and the induction hypothesis that K^u is an $[(n-1)-(p-1)]$-tangential body of L^u. Hence, the $(p-2)$-extreme support plane H' of

K^u supports L^u. This implies that $H(K, v)$ supports L. Thus K is an $(n-p)$-tangential body of L. This completes the induction.

Conversely, suppose that K is an $(n-p)$-tangential body of L for some $p \in \{1, \dots, n\}$. Then

$$h(K, u) = h(L, u) \quad \text{for } u \in \operatorname{extn}_{p-1} K,$$

where $\operatorname{extn}_j K$ denotes the set of j-extreme unit normal vectors of K. For $0 \leq j \leq p-1$, we deduce from Lemma 7.6.15 and Theorem 4.5.3 that

$$\begin{aligned} \operatorname{supp} S(K[n-j-1], L[j], \cdot) &\subset \operatorname{supp} S(K[n-j-1], B[j], \cdot) \\ &= \operatorname{cl} \operatorname{extn}_j K \subset \operatorname{cl} \operatorname{extn}_{p-1} K \end{aligned}$$

and hence that $h(K, u) = h(L, u)$ for u in the support of $S(K[n-j-1], L[j], \cdot)$. Thus

$$\int_{\mathbb{S}^{n-1}} h(K, u)\, S(K[n-j-1], L[j], \mathrm{d}u) = \int_{\mathbb{S}^{n-1}} h(L, u)\, S(K[n-j-1], L[j], \mathrm{d}u)$$

for $j = 0, \dots, p-1$, which by (5.19) gives

$$V_{(0)}(K, L) = V_{(1)}(K, L) = \cdots = V_{(p)}(K, L), \tag{7.151}$$

as asserted. □

Corollary 7.6.18 *Let $K, L \in \mathcal{K}^n_n$. If $i \in \{1, \dots, n-1\}$ and K is homothetic to an $(n-i-1)$-tangential body of L, then*

$$V_{(i)}(K, L)^2 = V_{(i-1)}(K, L) V_{(i+1)}(K, L). \tag{7.152}$$

Proof We may assume that rL is a maximal homothetic copy of L contained in K. By assumption, K is an $(n-i-1)$-tangential body of a homothet of L and necessarily of rL. From (7.151), with L replaced by rL, we obtain

$$V_{(0)}(K, L) = rV_{(1)}(K, L) = \cdots = r^{i+1} V_{(i+1)}(K, L) \tag{7.153}$$

and thus (7.152). □

The next theorem establishes a converse to Corollary 7.6.18, for $i = 1$. It thus settles a special but important case of the general equality problem for the Aleksandrov–Fenchel inequality. It was conjectured (for $n = 3$) by Minkowski [1438] and proved by Bol [272]. His proof (which we follow) is a nice application of the powerful method of inner parallel bodies.

Theorem 7.6.19 *Let $L \in \mathcal{K}^n_n$ be an n-dimensional convex body and $K \in \mathcal{K}^n$ an arbitrary convex body. In the inequality*

$$V(K, \dots, K, L)^2 \geq V_n(K) V(K, \dots, K, L, L) \tag{7.154}$$

equality holds if and only if either $\dim K < n-1$ *or K is homothetic to an $(n-2)$-tangential body of L.*

Proof If $\dim K < n-1$, both sides of (7.154) are zero. If K is homothetic to an $(n-2)$-tangential body of L, then equality in (7.154) is a special case of Corollary 7.6.18. We assume now that equality holds in (7.154) and that $\dim K \geq n-1$. Since $\dim K = n-1$ would imply $V_n(K) = 0$ and $V(K, \dots, K, L) > 0$, this case cannot occur.

We use the system of parallel bodies K_λ of K relative to L, as defined in Section 3.1. In particular, $K_\lambda = K \div -\lambda L$ for $-r \leq \lambda \leq 0$, where $r = r(K, L)$ denotes the inradius of K relative to L. Let $V_i(\lambda) := V_i(K_\lambda, L) = V(K_\lambda[n-i], L[i])$ and $f_i(\lambda) := V_i(\lambda)^{1/(n-i)}$. By the concavity of the system of parallel bodies (Lemma 3.1.13), the monotonicity of mixed volumes and the general Brunn–Minkowski theorem 7.4.5, the function f_i is concave on $[-r, \infty)$. Therefore, $-f_i$ has the properties collected in Theorem 1.5.4, some of which will be used in the following.

We have

$$\frac{\mathrm{d}V_0(\lambda)}{\mathrm{d}\lambda} = nV_1(\lambda), \tag{7.155}$$

which is a very special case of Lemma 7.5.3. For the left derivative of V_1 we write

$$\left(\frac{\mathrm{d}V_1(\lambda)}{\mathrm{d}\lambda}\right)_l =: (n-1)V_2^*(\lambda) \tag{7.156}$$

for $\lambda > -r$, and we define

$$\Delta(\lambda) := V_1(\lambda)^2 - V_0(\lambda)V_2^*(\lambda).$$

Then

$$V_2^*(\lambda) \geq V_2(\lambda) \tag{7.157}$$

since

$$V_1(K_\lambda + \alpha L, L) = \sum_{i=1}^{n-1} \binom{n-1}{i} \alpha^i V_{i+1}(\lambda)$$

for $\alpha \geq 0$; furthermore, $K_\lambda + \alpha L \subset K_{\lambda+\alpha}$ by (3.15) and hence

$$V_1(K_\lambda + \alpha L, L) \leq V_1(\lambda + \alpha) \quad \text{for } \alpha \geq 0,$$

with equality for $\alpha = 0$, and this gives

$$\begin{aligned}(n-1)V_2^*(\lambda) &= \left[\frac{\mathrm{d}}{\mathrm{d}\alpha} V_1(\lambda+\alpha)\right]_l^{\alpha=0} \geq \left[\frac{\mathrm{d}}{\mathrm{d}\alpha} V_1(\lambda+\alpha)\right]_r^{\alpha=0} \\ &\geq \left[\frac{\mathrm{d}}{\mathrm{d}\alpha} V_1(K_\lambda + \alpha L, L)\right]_r^{\alpha=0} = (n-1)V_2(\lambda).\end{aligned}$$

The fact that f_0 is concave and hence satisfies $[f_0'(\lambda)]_l' \leq 0$ yields

$$\Delta(\lambda) \geq 0, \tag{7.158}$$

by (7.155) and (7.156).

Proposition The function Δ is non-decreasing on $(-r, \infty)$.

For the proof, let $-r < \lambda_1 < \lambda_2$ and assume, on the contrary, that $\Delta(\lambda_2) < \Delta(\lambda_1)$. Then $\Delta(\lambda_1) > 0$ by (7.158). Put $b := \inf\{\lambda : \lambda > \lambda_1, \Delta(\lambda) < \Delta(\lambda_1)\}$. We write $f_1 =: f$, thus

$$V_1(\lambda) = f(\lambda)^{n-1}. \tag{7.159}$$

Since f is concave and increasing,

$$f_l'(\lambda) \geq f_r'(\lambda) \geq 0 \quad \text{for } \lambda > -r \tag{7.160}$$

and

$$f_r'(\alpha_1) \geq f_l'(\alpha_2) \quad \text{for } -r < \alpha_1 < \alpha_2. \tag{7.161}$$

By (7.159) and (7.156),

$$V_2^*(\lambda) = f(\lambda)^{n-2} f_l'(\lambda). \tag{7.162}$$

If $\lambda_1 < \lambda < b$, then (7.162), (7.160) and (7.161) yield

$$\begin{aligned}\Delta(\lambda_1) \leq \Delta(\lambda) &= V_1(\lambda)^2 - V_0(\lambda)V_2^*(\lambda) = V_1(\lambda)^2 - V_0(\lambda)f(\lambda)^{n-2}f_l'(\lambda)\\ &\leq V_1(\lambda)^2 - V_0(\lambda)f(\lambda)^{n-2}f_r'(\lambda) \leq V_1(\lambda)^2 - V_0(\lambda)f(\lambda)^{n-2}f_l'(b).\end{aligned}$$

With $\lambda \to b$ we obtain

$$0 < \Delta(\lambda_1) \leq V_1(b)^2 - V_0(b)f(b)^{n-2}f_l'(b) = \Delta(b). \tag{7.163}$$

Now we consider the function Γ defined by

$$\Gamma(\lambda) := V_1(\lambda)^2 - V_0(\lambda)f(\lambda)^{n-2}f_r'(b)$$

for $\lambda > -r$. By (7.160) we have

$$\Delta(b) \leq \Gamma(b), \tag{7.164}$$

and by (7.161),

$$\Gamma(\lambda) \leq \Delta(\lambda) \quad \text{for } \lambda > b. \tag{7.165}$$

The function Γ has a right derivative at b, for which we obtain, using (7.165) and (7.161),

$$\begin{aligned}\Gamma_r'(b) &= (2n-2)f(b)^{2n-3}f_r'(b) - nV_1(b)f(b)^{n-2}f_r'(b)\\ &\quad - V_0(b)(n-2)f(b)^{n-3}f_r'(b)^2\\ &= (n-2)V_1(b)^{-1}f(b)^{n-2}f_r'(b)\Gamma(b) > 0,\end{aligned}$$

since $f_r'(b) > 0$ (because $f_l'(\lambda) > 0$ for all $\lambda > -r$ by (7.162)) and $\Gamma(b) > 0$ by (7.164) and (7.163). Hence, there is a number $\delta > 0$ such that

$$\Gamma(\lambda) \geq \Gamma(b) \quad \text{for } b \leq \lambda \leq b + \delta. \tag{7.166}$$

For $b < \lambda \le b + \delta$ we now deduce from (7.165), (7.166), (7.164) and (7.163) that

$$\Delta(\lambda) \ge \Gamma(\lambda) \ge \Gamma(b) \ge \Delta(b) \ge \Delta(\lambda_1),$$

which contradicts the definition of b. This contradiction proves the proposition.

We have assumed equality in (7.154); thus $V_1(0)^2 - V_0(0)V_2(0) = 0$. By (7.157) and (7.158) this implies $\Delta(0) = 0$ and

$$V_2^*(0) = V_2(0). \tag{7.167}$$

The proposition yields $\Delta(\lambda) = 0$ for $\lambda \in (-r, 0)$. According to the deduction of (7.158), this is only possible if f_0 is a linear function on $(-r, 0)$. For $\lambda \to -r$, $V_0(\lambda)$ converges to 0 (see (7.109)), hence

$$V_n(K_\lambda) = \left(1 + \frac{\lambda}{r}\right)^n V_n(K). \tag{7.168}$$

The body K contains a translate of rL and after a translation we may assume that $rL \subset K$. Then $(-\lambda)L + [1 + (\lambda/r)]K \subset K$ for $-r \le \lambda \le 0$ and hence $[1 + (\lambda/r)]K \subset K_\lambda$; thus

$$\left(1 + \frac{\lambda}{r}\right)^n V_n(K) \le V_n(K_\lambda).$$

Since equality holds here, each K_λ is homothetic to K. By Lemma 3.1.14, K is homothetic to a tangential body of L.

From (7.167) we have

$$V_1(\lambda) = \left(1 + \frac{\lambda}{r}\right)^{n-1} \frac{1}{r} V_n(K), \quad V_2^*(\lambda) = \left(1 + \frac{\lambda}{r}\right)^{n-2} \frac{1}{r^2} V_n(K).$$

For $\lambda = 0$, together with $V_i(0) = V_{(i)}(K, L)$, (7.168) and (7.167), this yields

$$V_{(0)}(K, rL) = V_{(1)}(K, rL) = V_{(2)}(K, rL).$$

Since $rL \subset K$, Theorem 7.6.17 now shows that K is an $(n-2)$-tangential body of rL. This completes the proof. □

A converse to another special case of Corollary 7.6.18 is contained in the following theorem. It concerns the case where L is a ball. Unfortunately, for a technical reason we have to assume central symmetry for K, which is very probably unnecessary.

Theorem 7.6.20 *If $K \in \mathcal{K}^n$ is centrally symmetric and $i \in \{1, \dots, n-1\}$, then the equality*

$$W_i(K)^2 = W_{i-1}(K)W_{i+1}(K) \tag{7.169}$$

holds if and only if either $\dim K < n - i$ *or K is an $(n - i - 1)$-tangential body of a ball.*

Proof Since $W_i(K) > 0$ if and only if $\dim K \ge n - i$, we may assume that $\dim K \ge n - i + 1$. That (7.169) holds for $(n - i - 1)$-tangential bodies of balls is a special case of Corollary 7.6.18.

We prove by induction with respect to n that (7.169) and the assumption $\dim K \geq n-i+1$ imply that K is an $(n-i-1)$-tangential body of a ball ($i = 1, \dots, n-1$). For $n = 2$, the assertion is true by Theorem 7.2.1. Assume that $n \geq 3$, that the assertion is true in dimensions less than n and that (7.169) holds for some $i \in \{1, \dots, n-1\}$, where $\dim K \geq n-i+1$. If $i = 1$, then K is an $(n-2)$-tangential body of a ball, by Theorem 7.6.19. Let $i \geq 2$. From (7.169) and Theorem 7.4.2 it follows that, after applying a suitable dilatation to K, we may assume that

$$S_{n-i}(K, \cdot) = S_{n-i-1}(K, \cdot). \tag{7.170}$$

By Lemma 7.4.7, this implies

$$w_{i-1}(K^u)^2 = w_{i-2}(K^u)w_i(K^u)$$

for $u \in \mathbb{S}^{n-1}$, where we have written

$$w_k(K^u) := v(K^u[n-1-k], B^u[k]).$$

By the induction hypothesis, K^u is an $(n-i-1)$-tangential body of a homothet of B^u. Let r_u be the radius of this homothet. Equations (7.153) in dimension $n-1$ give

$$r_u^{i-1} w_{i-1}(K^u) = r_u^i w_i(K^u).$$

From (7.170) and (5.78) we have $w_{i-1}(K^u) = w_i(K^u)$, thus $r_u = 1$.

Let B_r be the maximal ball in K with centre at the symmetry centre of K; here r denotes the radius of B_r. The bodies K and B_r have a common supporting hyperplane H, and by symmetry a second one parallel to it, say H'. Let u be a unit vector parallel to H. The intersection $H \cap H_{u,0}$ is a common support plane (in $H_{u,0}$) of K^u and B_r^u and hence an extreme support plane of K^u. The same holds true for $H' \cap H_{u,0}$. Since both $(n-2)$-planes must support some ball in $H_{u,0}$ of radius 1, it follows that $r = 1$. Hence, we may assume that $B_r = B$. Now let $u \in \mathbb{S}^{n-1}$ be arbitrary. The projection K^u is an $(n-i-1)$-tangential body of a unit ball and necessarily of B^u.

The argument is completed in a way similar to the proof of Theorem 7.6.17. Let $H(K, v)$ be an i-extreme support plane of K. We choose a unit vector u orthogonal to v and, if $\dim T(K, v) \geq 2$, in the linear hull of $T(K, v)$. Then $H' := H(K, v) \cap H_{u,0}$ is an $(i-1)$-extreme support plane of K^u. Since K^u is an $(n-i-1)$-tangential body of B^u, we deduce that H' supports B^u and hence $H(K, v)$ supports B. Thus K is an $(n-i-1)$-tangential body of B. □

The general conjecture 7.6.13, with distinct bodies $C_1, \dots, C_{n-2}$, has so far only been verified under strong restrictions for these bodies. One example is given by the following result, which extends the equality assertion of Theorem 7.3.2.

Theorem 7.6.21 *Conjecture 7.6.13 is true if $C_1, \dots, C_{n-2}$ are strongly isomorphic simple polytopes (and K, L are arbitrary convex bodies).*

Proof Let $P_3, \dots, P_n \in \mathcal{K}_n^n$ be strongly isomorphic simple polytopes. We have to apply some results from Sections 5.1, 5.2 and 7.3, and therefore use the notation introduced there. In particular, $\mathcal{A}$ is the a-type of $P_3, \dots, P_n$ and $P \in \mathcal{A}$ is an arbitrary member. Further, $u_1, \dots, u_N, F_i, F_{ij}, v_{ij}, \theta_{ij}, J, F_i^{(r)} := F(P_r, u_i)$ etc. are defined as in Section 5.1. We abbreviate $(P_3, \dots, P_n)$ by $\mathcal{C}$.

Let $K, L \in \mathcal{K}^n$ be convex bodies. After applying a dilatation to K or L, we may assume that

$$V(K, K, \mathcal{C}) = V(L, L, \mathcal{C}). \tag{7.171}$$

We may also assume that $V(K, L, \mathcal{C}) > 0$, since otherwise K and L are parallel segments (possibly degenerate), and the assertion is trivial. By Theorem 7.4.2, equality in the inequality

$$V(K, L, \mathcal{C})^2 \geq V(K, K, \mathcal{C})V(L, L, \mathcal{C}) \tag{7.172}$$

holds if and only if

$$S(K, \mathcal{C}, \cdot) = S(L, \mathcal{C}, \cdot). \tag{7.173}$$

We must, therefore, examine the measure $S(K, P_3, \dots, P_n, \cdot)$. First,

$$S(K, P_3, \dots, P_n, \{u_j\}) = v(F(K, u_i), F_i^{(3)}, \dots, F_i^{(n)}) \tag{7.174}$$

for $i = 1, \dots, N$, by the definition of the mixed area measure. For $(i, j) \in J$ (recall that this means that $F_{ij} := F(P, u_i) \cap F(P, u_j)$ is an $(n-2)$ face of P), we denote by σ_{ij} the spherical image of F_{ij} and by π_{ij} the orthogonal projection onto the two-dimensional linear subspace L_{ij} orthogonal to F_{ij}. Observe that σ_{ij} and π_{ij} depend only on (i, j) and the a-type $\mathcal{A}$.

Let $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ be a Borel set. If $\omega \subset \operatorname{relint} \sigma_{ij}$, then

$$S(K, P_3, \dots, P_n, \omega) = v^{(n-2)}(F_{ij}^{(3)}, \dots, F_{ij}^{(n)}) s(\pi_{ij} K, \omega \cap L_{ij}), \tag{7.175}$$

where s denotes the area measure in L_{ij}. If $\omega \cap \sigma_{ij} = \emptyset$ for all $(i, j) \in J$, then

$$S(K, P_3, \dots, P_n, \omega) = 0. \tag{7.176}$$

The last two equalities are proved by computing $S_{n-1}(\lambda K + \lambda_3 P_3 + \cdots + \lambda_n P_n, \omega)$ for $\lambda, \lambda_3, \dots, \lambda_n > 0$, using (4.32) and Fubini's theorem, and then applying (5.18).

Now we assume equality in (7.172) and hence that (7.173) holds. Suppose that $(i, j) \in J$ and that $\omega \subset \operatorname{relint} \sigma_{ij}$ is a Borel set. Then (7.175) yields $s(\pi_{ij} K, \omega) = s(\pi_{ij} L, \omega)$. It follows from Theorem 8.3.3 (to be proved later) that the convex arcs in $\pi_{ij} K$ and $\pi_{ij} L$ that are the reverse spherical images of $\operatorname{relint} \sigma_{ij}$ must be translates of each other. Hence, there exists a vector $t_{ij} \in \mathbb{R}^n$, orthogonal to F_{ij}, such that

$$h(K, u) - h(L, u) = \langle t_{ij}, u \rangle \quad \text{for } u \in \sigma_{ij} \tag{7.177}$$

(first for $u \in \operatorname{relint} \sigma_{ij}$, then by continuity for $u \in \sigma_{ij}$). Define

$$h(K, u_i) - h(L, u_i) = \langle t_{ij}, u_i \rangle =: \alpha_i,$$

$$h(F(K, u_i), v_{ij}) - h(F(L, u_i), v_{ij}) =: \lambda_{ij}$$

for $i \in \{1, \dots, N\}$ and $(i, j) \in J$. From (7.173), (7.174) and (5.19) we deduce that, for each fixed $i \in \{1, \dots, N\}$,

$$\sum_{(i,j)\in J} \lambda_{ij} v^{(n-2)}(F_{ij}^{(3)}, \dots, F_{ij}^{(n)}) = 0. \tag{7.178}$$

Taking directional derivatives in (7.177), we infer from Theorem 1.7.2 that $\lambda_{ij} = \langle t_{ij}, v_{ij}\rangle$. Since t_{ij} is a linear combination of u_i and v_{ij}, we deduce that $t_{ij} = \alpha_i u_i + \lambda_{ij} v_{ij}$. As $t_{ij} = t_{ji} = \alpha_j u_i + \lambda_{ji} v_{ji}$, multiplication by u_j yields

$$\lambda_{ij} = \alpha_j \csc\theta_{ij} - \alpha_i \cot\theta_{ij} \tag{7.179}$$

for $(i, j) \in J$. Writing $Z := (\alpha_1, \dots, \alpha_N)$, we see from (5.40) that

$$\Lambda_i Z = (\lambda_{i1}, \dots, \lambda_{iN})$$

with $\lambda_{ij} := 0$ if $(i, j) \notin J$. With φ_i as defined in the proof of Theorem 7.3.2, we have

$$\varphi_i(Z, P_3) = v(\Lambda_i Z, F_i^{(3)}, \dots, F_i^{(n)}) = \frac{1}{n-1} \sum_{(i,j)\in J} \lambda_{ij} v^{(n-2)}(F_{ij}^{(3)}, \dots, F_{ij}^{(n)}),$$

hence $\varphi_i(Z, P_3) = 0$ by (7.178). This means that Z is an eigenvector with eigenvalue 0 of the bilinear form Φ used in the proof of Theorem 7.3.2; hence, from that proof (Proposition 3) it follows that Z is the support vector of a point. Thus, there exists a vector $z \in \mathbb{R}^n$ such that $\alpha_i = \langle z, u_i\rangle$ for $i = 1, \dots, n$. After applying a suitable translation to K, we may assume that $z = o$. Then $\lambda_{ij} = 0$ by (7.179) and hence $t_{ij} = o$; thus (7.177) now reads

$$h(K, u) = h(L, u) \quad \text{for } u \in \sigma_{ij},\ (i, j) \in J. \tag{7.180}$$

The $(P_3, \dots, P_n, B)$-extreme unit normal vectors are precisely the vectors in $\bigcup_{(i,j)\in J} \sigma_{ij}$.

Conversely, from (7.180) we deduce (7.173) and hence equality in (7.172). This completes the proof. □

Another case where the truth of Conjecture 7.6.13 has been established is the following result (Schneider [1707]), which we formulate here without proof.

Theorem 7.6.22 *If $K, L \in \mathcal{K}_n^n$ are centrally symmetric and $C_1, \dots, C_{n-2} \in \mathcal{K}_n^n$ are zonoids, then Conjecture 7.6.13 holds for these bodies.*

Notes for Section 7.6

1. Lemma 7.6.1 is due to Süss [1830]; proofs also appear in Aleksandrov [13] and Leichtweiß [1184], pp. 241–243. Corresponding stability results were proved by Groemer [791]; see also Groemer [798].
2. The first proof given above for Theorem 7.6.2 is essentially due to Favard [553] (compare also Aleksandrov [13] and Leichtweiß [1185]). The second proof, using Lemma 7.6.3, extends a method of Kubota [1154]. Theorem 7.6.6 was proved by Schneider [1712] and independently by Goodey and Groemer [737]. The stability estimate (7.124) was derived in Groemer and Schneider [804].

   An alternative approach to a stability estimate of the type of Theorem 7.6.6 is due to Meyer and Reisner [1411], who applied a perturbation argument to the Blaschke–Santaló inequality.

For the stability estimate of Theorem 7.6.6, Arnold and Wellerding [76] proved a variant that involves the Sobolev distance.

3. Theorems 7.6.7 and 7.6.8 and their proofs are taken from Schneider [1713]. The proof of Theorem 7.6.9 (given Theorem 7.6.8) is modelled on the procedure of Fenchel [568].
4. *Equality cases in the Aleksandrov–Fenchel inequality*. Conjectures 7.6.13, 7.6.14 and 7.6.16 and Lemma 7.6.15 were formulated in Schneider [1701], where credit for 7.6.16 and a special case of 7.6.15 is given to A. Loritz. Conjecture 7.6.13 states that the equality

$$V(K, L, \mathscr{C})^2 = V(K, K, \mathscr{C})V(L, L, \mathscr{C}),$$

where $K, L \in \mathscr{K}^n$, $\mathscr{C} = (C_1, \dots, C_{n-2})$ and $C_1, \dots, C_{n-2} \in \mathcal{K}_n^n$, holds if and only if suitable homothets of K and L have the same $(\mathscr{C}, B)$-extreme supporting hyperplanes. When this conjecture was formulated, the following classical cases in favour of it were known.
   (a) $\mathscr{C}$ consists of balls. This is the case mentioned in Note 2, first proved by Kubota [1154], with a different proof given by Favard [553].
   (b) $\mathscr{C}$ consists of strongly isomorphic simple polytopes, and K, L are polytopes having the same system of normal vectors to the facets as the polytopes of $\mathscr{C}$. This comes out in Aleksandrov's first proof for the Aleksandrov–Fenchel inequality, as presented in §7.3. Theorem 7.6.21 extends this result to arbitrary convex bodies K, L.
   (c) $\mathscr{C}$ consists of bodies of class C^2_+, and K, L have support functions of class C^2. This case is a by-product of Aleksandrov's [16] second proof and was also noted by Favard [555].
   (d) $C_1 = \cdots = C_{n-2} = K$. This is the case treated in Theorem 7.6.19, which is due to Bol [272]. Corollary 7.6.18 (for $n = 3$) had already been discovered by Minkowski [1438], §7. A special case of Theorem 7.6.19 was treated by Knothe [1124] in a different way.
5. *Tangential bodies*. Theorem 7.6.17, as mentioned, is due to Favard [553], but we found it necessary to give a different proof for the necessity of condition (7.149).

   If $K \in \mathcal{K}_n^n$ is a convex body with inradius r, then its quermassintegrals satisfy

$$W_{n-2}(K) - rW_{n-1}(K) \geq 0.$$

   If equality holds here, then Theorem 7.6.17 implies that K is a cap body of a ball. Sangwine-Yager [1621] obtained an explicit estimate showing that K must be close to some cap body of a ball if $W_{n-2}(K) - rW_{n-1}(K)$ is small.

   Theorem 7.6.20 is taken from Schneider [1701].
6. The proof of Theorem 7.6.21 was given in Schneider [1718]. The second part of the proof, in which it is shown that the vectors t_{ij} associated with the $(n-2)$-faces F_{ij} of P are all equal to a fixed vector, is similar to (but different from) a method of Aleksandrov [25], Chapter XI, §3, which shows the infinitesimal rigidity of convex polytopes in $\mathbb{R}^3$ with the aid of mixed volumes. (Aleksandrov's rigidity study of polytopes was extended by Bauer [178].) A connection between rigidity of polytopes and mixed volumes was first exploited by Weyl [1971]; his proof is sketched in Efimow [531], §100. Among more recent investigations on relations between rigidity and Brunn–Minkowski theory, we mention Filliman [581].
7. *Differentiability properties of intrinsic volumes in systems of parallel bodies.* In the proof of Theorem 7.6.19, differentiability properties of the concave functions

$$f_i(\lambda) := V_{(i)}(K_\lambda, L)^{1/(n-i)} = V(K_\lambda[n-i], L[i])^{1/(n-i)}$$

   play a crucial role; here $\{K_\lambda : -r(K, L) \leq \lambda < \infty\}$ is the system of (inner and outer) parallel bodies of $K \in \mathcal{K}^n$ relative to a gauge body $L \in \mathcal{K}_n^n$. The one-sided derivatives of the mentioned functions satisfy

$$(f_i)'_l \geq (f_i)'_r \geq (n-i)f_{i+1}$$

   for $i = 0, \dots, n-1$. For given $L \in \mathcal{K}_n^n$ and for $p \in \{0, \dots, n-1\}$, let $\mathscr{R}_p$ denote the set of all convex bodies $K \in \mathcal{K}^n$ that satisfy

$$(f_i)'_l(\lambda) = (f_i)'_r(\lambda) = (n-i)f_{i+1}(\lambda)$$

for $0 \le i \le p$ and $-r(K,L) \le \lambda < \infty$. Then $\mathscr{R}_0 = \mathcal{K}^n$, and the inclusions $\mathscr{R}_{i+1} \subset \mathscr{R}_i$, $i = 0,\dots,n-2$, are strict. For $n = 3$ and $L = B^3$, these classes were first introduced by Hadwiger [908], §23, who was able (*loc. cit*, §29) to solve within each of these classes the problem of a complete set of inequalities between the intrinsic volumes. Hernández Cifre and Saorín [966, 968, 967, 969, 1633] studied the classes $\mathscr{R}_p$ in the general case and related them to several other questions on the behaviour of intrinsic volumes in systems of inner parallel bodies. One result of [966] says that the only sets in $\mathscr{R}_{n-1}$ are the outer parallel bodies of convex bodies of dimension less than n.

7.7 Linear inequalities

Under certain circumstances, the quadratic Aleksandrov–Fenchel inequality

$$V(K,L,\mathscr{C})^2 \ge V(K,K,\mathscr{C})V(L,L,\mathscr{C}), \tag{7.181}$$

where $K, L, K_3, \dots, K_n \in \mathcal{K}^n$ and $\mathscr{C} = (K_3,\dots,K_n)$, admits the linear improvement

$$2V(K,L,\mathscr{C}) \ge V(K,K,\mathscr{C}) + V(L,L,\mathscr{C}). \tag{7.182}$$

A sufficient condition for the validity of (7.182) is that there exists a convex body M fulfilling the conditions

$$V(K,M,\mathscr{C}) = V(L,M,\mathscr{C}) > 0. \tag{7.183}$$

This follows immediately from (7.82). For example, M could be a segment. If u is a unit vector parallel to it, then by (5.77), condition (7.183) is equivalent to

$$v(K^u,\mathscr{C}^u) = v(L^u,\mathscr{C}^u) > 0.$$

A special case of the latter inequality occurs when $K^u = L^u$, that is, K and L have the same circumscribed cylinder of direction u. In this case, the Brunn–Minkowski theorem for K and L also holds in an improved version: the volume of $(1-\vartheta)K + \vartheta L$, and not only its nth root, is a concave function of ϑ for $0 \le \vartheta \le 1$. This is clear from Fubini's theorem and the fact that each line G of direction u satisfies

$$G \cap [(1-\vartheta)K + \vartheta L] \supset (1-\vartheta)(G \cap K) + \vartheta(G \cap L).$$

Less immediate is the fact that the general Brunn–Minkowski theorem (Theorem 7.4.5) also admits an improved version of this type. In the following, we describe a theory of linear inequalities for mixed volumes of bodies inscribed in a cylinder; this unifies some special results in the literature obtained by other methods.

We assume that a unit vector $u \in \mathbb{S}^{n-1}$ and an $(n-1)$-dimensional convex body C in the hyperplane $H_{u,0}$ orthogonal to u are given. Then we define

$$\mathcal{K}_C := \{K \in \mathcal{K}^n : K^u = C\}$$

and call such a set of convex bodies a *canal class*. The $(n-1)$-dimensional mixed volume of convex bodies in $H_{u,0}$ is denoted by v. If U is a segment of length 1 parallel to u, then

$$v(K_1^u,\dots,K_{n-1}^u) = nV(K_1,\dots,K_{n-1},U) \tag{7.184}$$

by (5.77). The mixed area measure in $H_{u,0}$ is denoted by s and, without loss of generality, is considered as a measure on $\mathbb{S}^{n-1}$ that is concentrated on the great subsphere

$$s_u^{n-2} := \mathbb{S}^{n-1} \cap H_{u,0}.$$

Then we have

$$s(K_1^u, \dots, K_{n-2}^u, \cdot) = (n-1)S(K_1, \dots, K_{n-2}, U, \cdot) \tag{7.185}$$

on $\mathbb{S}^{n-1}$. This follows from (5.18) and the equality

$$S_{n-1}(K + \lambda U, \cdot) = S_{n-1}(K, \cdot) + \lambda s(K^u, \dots, K^u, \cdot),$$

which is clear from (4.32), if one then replaces K by $\lambda_1 K_1 + \cdots + \lambda_{n-2} K_{n-2}$ and compares the coefficients.

Theorem 7.7.1 *Let $\mathscr{C}$ be an $(n-2)$-tuple of convex bodies in $\mathcal{K}^n$ satisfying*

$$v(C, \mathscr{C}^u) > 0. \tag{7.186}$$

For K, L in the canal class $\mathcal{K}_C$, the inequality

$$V(K, K, \mathscr{C}) - 2V(K, L, \mathscr{C}) + V(L, L, \mathscr{C}) \leq 0 \tag{7.187}$$

is true. Here equality holds if and only if

$$S(K, \mathscr{C}, \cdot) = S(L + aU, \mathscr{C}, \cdot) \tag{7.188}$$

or

$$S(L, \mathscr{C}, \cdot) = S(K + aU, \mathscr{C}, \cdot) \tag{7.189}$$

with some number $a \geq 0$.

Proof By (7.184) and (7.186), each convex body $A \in \mathcal{K}_C$ satisfies

$$nV(A, U, \mathscr{C}) = v(A^u, \mathscr{C}^u) = v(C, \mathscr{C}^u) > 0.$$

In particular, $V(K, U, \mathscr{C}) = V(L, U, \mathscr{C}) > 0$, hence (7.82) with $M = U$ yields inequality (7.187) (observe that (7.80) is not necessary for the validity of (7.82), as remarked after the formulation of Theorem 7.4.3). As shown in Section 7.4, equality in (7.82) is equivalent to (7.87), with a suitable number k. The numbers q_1, q_2 occurring there are, in our present case, given by $q_1 = q_2 = v(C, \mathscr{C}^u)/n$. Hence, the first equation of (7.87) is equivalent to (7.188) with suitable $a \geq 0$, and the second equation of (7.87) is equivalent to (7.189). □

Next we show that for canal classes the general Brunn–Minkowski theorem 7.4.5 holds in a stronger form, and there is also an analogue of Theorem 7.4.6. Let a number $m \in \{2, \dots, n\}$, an $(n-m)$-tuple $\mathscr{C} = (K_{m+1}, \dots, K_n)$ of convex bodies

$K_{m+1}, \dots, K_n \in \mathcal{K}^n$, and two bodies $K_0, K_1 \in \mathcal{K}_C$ be given. We use the notation of Section 7.4, in particular

$$V_{(i)} := V(K_0[m-i], K_1[i], \mathscr{C}) \qquad \text{for } i = 0, \dots, m,$$
$$S_{(i)} := S(K_0[m-1-i], K_1[i], \mathscr{C}, \cdot) \qquad \text{for } i = 0, \dots, m-1,$$

and in addition we define a measure γ on $\mathbb{S}^{n-1}$ by

$$\gamma := s(C[m-2], \mathscr{C}^u, \cdot).$$

By (7.187), the sequence $(V_{(0)}, \dots, V_{(m)})$ is concave, hence (7.59) tells us that, for $0 \le i < j < k \le m$,

$$(k-j)V_{(i)} + (i-k)V_{(j)} + (j-i)V_{(k)} \le 0. \tag{7.190}$$

This is a stronger version of (7.63). The following result is the general Brunn–Minkowski theorem for canal classes.

Theorem 7.7.2 *Let a number $m \in \{2, \dots, n\}$ and convex bodies $K_0, K_1 \in \mathcal{K}_C$ and $K_{m+1}, \dots, K_n \in \mathcal{K}^n$ be given and define*

$$\mathscr{C} := (K_{m+1}, \dots, K_n), \qquad K_\lambda := (1-\lambda)K_0 + \lambda K_1$$

and

$$f(\lambda) := V(K_\lambda[m], \mathscr{C})$$

for $0 \le \lambda \le 1$. *Then f is a concave function on* $[0, 1]$.

Under the assumption

$$v(C[m-1], \mathscr{C}^u) > 0,$$

the following assertions are equivalent:

(a) *The function f is linear;*

(b) *The sequence $(V_{(0)}, \dots, V_{(m)})$ is linear, that is,*

$$V_{(0)} - V_{(1)} = V_{(1)} - V_{(2)} = \cdots = V_{(m-1)} - V_{(m)};$$

(c) $(m-1)V_{(0)} - mV_{(1)} + V_{(m)} = 0$;

(d) $S_{(k-1)} - S_{(k)} = a\gamma$ *for* $k = 1, \dots, m-1$ *with a constant a;*

(e) $S_{(0)} = S_{(m-1)}$ *on* $\mathbb{S}^{n-1} \setminus s_u^{n-2}$.

Proof We have

$$f''(0) = m(m-1)[V_{(0)} - 2V_{(1)} + V_{(2)}],$$

hence $f''(0) \le 0$ by (7.187). As in the proof of Theorem 7.4.5, we deduce that f is concave.

The concavity of f yields $f'(0) \ge f(1) - f(0)$, thus

$$(m-1)V_{(0)} - mV_{(1)} + V_{(m)} \le 0, \tag{7.191}$$

with equality if and only if f is linear. Hence (a) and (c) are equivalent.

Inequality (7.191) is also a special case of (7.190) and hence equality holds here (see the remark after (7.59)) if and only if equality holds in $V_{(k-1)} - 2V_{(k)} + V_{(k+1)} \leq 0$ for $k = 1, \ldots, m-1$. Thus (b) and (c) are equivalent.

Suppose that (b) holds. By Theorem 7.7.1, this implies

$$S_{(k-1)} - S_{(k)} = a_k \gamma \tag{7.192}$$

for $k = 1, \ldots, m-1$, with a constant a_k possibly depending on k. Integrating the support function of K_1 with the measure given by (7.192) we obtain

$$V_{(k)} - V_{(k+1)} = \frac{n-1}{n} a_k v(C[m-1], \mathscr{C}^u).$$

For $k \leq m-2$ we integrate the support function of K_0 with the measure given by (7.192), but with k replaced by $k+1$, and obtain

$$V_{(k)} - V_{(k+1)} = \frac{n-1}{n} a_{k+1} v(C[m-1], \mathscr{C}^u).$$

Thus $a_1 = \cdots = a_{m-1}$, and (d) holds. Trivially, (d) implies (e), since γ is concentrated on s_u^{n-2}.

Suppose that (e) holds. Then

$$\int_{\mathbb{S}^{n-1} \setminus s_u^{n-2}} [h(K_0, v) - h(K_1, v)] \, \mathrm{d}(S_{(0)} - S_{(m-1)})(v) = 0.$$

Since $h(K_0, v) = h(K_1, v)$ for $v \in s_u^{n-2}$, we also have

$$\int_{s_u^{n-2}} [h(K_0, v) - h(K_1, v)] \, \mathrm{d}(S_{(0)} - S_{(m-1)})(v) = 0.$$

Thus,

$$V_{(0)} - V_{(1)} - V_{(m-1)} + V_{(m)} = 0.$$

Adding (7.191) and the inequality

$$(m-1)V_{(m)} - mV_{(m-1)} + V_{(0)} \leq 0,$$

which also follows from (7.190), we see that equality holds in (7.191), which is condition (c). This completes the proof of Theorem 7.7.2. □

There is also a counterpart to Theorem 7.6.9. We say that the convex bodies $K_0, K_1 \in \mathcal{K}_C$ are equivalent by *telescoping* if $K_0 = K_1 + \lambda U$ or $K_1 = K_0 + \lambda U$ with $\lambda \geq 0$, where U is a unit segment parallel to u.

Theorem 7.7.3 *If the assumptions of Theorem 7.7.2 are satisfied and if the bodies $K_{m+1}, \ldots, K_n$ are smooth, then the conditions* (a) *to* (e) *of Theorem 7.7.2 hold if and only if K_0 and K_1 are equivalent by telescoping.*

Proof It is clear that condition (a) is satisfied if K_0 and K_1 are equivalent by telescoping. To prove the other direction, we use induction over m. Assume that conditions (a) – (e) of Theorem 7.7.2 are fulfilled. First let $m = 2$. Then condition (d) can be written in the form

$$S(K_0, \mathscr{C}, \cdot) - S(K_1, \mathscr{C}, \cdot) = a(n-1)S(U, \mathscr{C}, \cdot).$$

If $a \geq 0$, this is equivalent to

$$S(K_0, \mathscr{C}, \cdot) = S(K_1 + a(n-1)U, \mathscr{C}, \cdot).$$

By Theorems 7.4.2 and 7.6.8, the bodies K_0 and $K_1 + a(n-1)U$ are homothetic, and since $K_0, K_1 \in \mathcal{K}_C$, this implies that K_0 and K_1 are equivalent by telescoping. If $a < 0$, the argument is similar.

Now assume that $m \geq 3$ and that the assertion of the theorem is true for $m-1$. By condition (d),

$$S_{(k-1)} - S_{(k)} = a\gamma \quad \text{for } k \in \{1, \dots, m-1\}.$$

Integrating the support function of the unit ball B, we obtain

$$\begin{aligned} &V(K_0[m-k], K_1[k-1], B, \mathscr{C}) - V(K_0[m-k-1], K_1[k], B, \mathscr{C}) \\ &= \frac{n-1}{n} av(C[m-2], B^u, \mathscr{C}^u), \end{aligned}$$

which is independent of k. Thus, writing

$$\overline{V}_k := V(K_0[m-1-k], K_1[k], B, \mathscr{C}),$$

we have

$$\overline{V}_{(0)} - \overline{V}_{(1)} = \overline{V}_{(1)} - \overline{V}_{(2)} = \cdots = \overline{V}_{(m-2)} - \overline{V}_{(m-1)}.$$

By the induction hypothesis, this implies that K_0 and K_1 are equivalent by telescoping. □

Notes for Section 7.7

1. The results and proofs of this section are taken from a paper (Schneider [1711]) which partially extends ideas of Favard [553]. The above investigation, together with the remark that (7.183) implies (7.182), unifies and generalizes several special results on linear improvements of inequalities for mixed volumes scattered in the literature. These are obtained by different methods. Some particular results are not covered by the above. We refer the reader to Bonnesen and Fenchel [284], p. 99, Geppert [694], Bol [267], Hadwiger [911], (6.6.4) and p. 279, and Dinghas [487].
2. Ohmann [1480] proved a linear improvement (under a suitable assumption) of the Brunn–Minkowski inequality for closed sets.
3. As remarked above, for $K_0, K_1 \in \mathcal{K}_C$ the improved version of the Brunn–Minkowski theorem is easily obtained; it says that $V_n((1-\vartheta)K_0 + \vartheta K_1)$ is a concave function of ϑ on $[0, 1]$. As in the proof of Theorem 7.2.1, this implies the inequalities

$$(n-1)V_{(0)} - nV_{(1)} + V_{(n)} \leq 0, \tag{7.193}$$

$$V_{(0)} - 2V_{(1)} + V_{(2)} \le 0 \tag{7.194}$$

for $V_{(i)} := V(K_0[n-i], K_1[i])$; see Bonnesen and Fenchel [284], p. 94. If instead of $K_0^u = K_1^u$ one assumes only that

$$V_{n-1}(K_0^u) = V_{n-1}(K_1^u), \tag{7.195}$$

then, as described in Bonnesen and Fenchel [284], p. 95, a volume-preserving symmetrization procedure transforms $K_\vartheta = (1-\vartheta)K_0 + \vartheta K_1$ into K'_ϑ such that $(K'_0)^u = (K'_1)^u$ and

$$V_n(K'_\vartheta) \ge (1-\vartheta)V_n(K'_0) + \vartheta V_n(K'_1).$$

It follows that (7.193) holds also under the assumption (7.195). However, $V_n(K'_\vartheta)$ need not be a concave function of ϑ, so that (7.194) cannot be deduced if merely (7.195) is assumed, contrary to an assertion made in Bonnesen and Fenchel [284], p. 95. This error was pointed out by Diskant [508], who constructed a counterexample.

4. Let $K_0, K_1 \in \mathcal{K}_n^n$. Without the assumption (7.195), a linear improvement of the Brunn–Minkowski inequality (due to Bonnesen) is still possible, but involving the volumes of the projections, namely

$$\frac{V_n(K_0+K_1)}{\left[V_{n-1}(K_0^u)^{1/(n-1)} + V_{n-1}(K_1^u)^{1/(n-1)}\right]^{n-1}} \ge \frac{V_n(K_0)}{V_{n-1}(K_0^u)} + \frac{V_n(K_1)}{V_{n-1}(K_1^u)} \tag{7.196}$$

(see Bonnesen and Fenchel [284], §50 and the references given there). The equality case of (7.196) has only recently been settled. According to Freiman, Grynkiewicz, Serra and Stanchescu [634], equality holds if and only if there are homothetic convex bodies K'_0, K'_1 and segments U_0, U_1 parallel to u such that $K_0 = K'_0 + U_0$ and $K_1 = K'_1 + U_1$. The equality case for the analogous inequality, with projections replaced by maximal sections, was determined by Böröczky and Serra [299].

5. *An isoperimetric problem of Hadwiger*. As mentioned in Section 7.2, Note 12, Hadwiger [915] proved that the volume and surface area of a half-body satisfy

$$2S^n \ge n^n(\kappa_n + 2\kappa_{n-1})V_n^{n-1}, \tag{7.197}$$

and that equality holds if K is the union of a half-ball of radius ρ, say, and a right cylinder of height ρ attached to it. Hadwiger did not prove that equality holds only in this case. The following proof of (7.197) proceeds in a different way and settles the equality case. Moreover, it could be generalized to quermassintegrals other than the surface area.

As Hadwiger did, we consider, more generally, the class $\mathcal{K}_S$ of convex bodies K with the property that K has a supporting hyperplane H such that

$$V_{n-1}(K \cap H) \ge V_{n-1}(K \cap H')$$

for each hyperplane H' parallel to H. Let $K \in \mathcal{K}_S$ be given and let H be as above. By Schwarz symmetrization (we assume here that the reader is familiar with this) with respect to a line G orthogonal to H we obtain from K a convex body K_0 with $V_n(K_0) = V_n(K)$ and $S(K_0) \le S(K)$; here strict inequality holds unless K is a body of revolution with axis parallel to G. Moreover, $K_0 \in \mathcal{K}_S$; hence K_0 belongs to the canal class $\mathcal{K}_C$, where $C = H \cap K_0$ is an $(n-1)$-dimensional ball of radius, say ρ. By K_1 we denote the half-ball of radius ρ with basis C and such that $K_1 \subset H^-$, where H^- is the halfspace bounded by H that contains K_0.

By (7.190) and Theorems 7.7.2 and 7.7.3, with $m = n$ and empty $\mathscr{C}$, we have

$$(n-1)V_{(0)} - nV_{(1)} + V_{(n)} \le 0,$$

with equality if and only if K_0 and K_1 are equivalent by telescoping. Now

$$V_{(0)} = V_n(K_0), \qquad V_{(n)} = V_n(K_1) = \frac{1}{2}\kappa_n\rho^n,$$

$$V_{(1)} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K_1,u)\,S_{n-1}(K_0,\mathrm{d}u) = \frac{1}{n}\rho[S(K_0) - \kappa_{n-1}\rho^{n-1}].$$

The latter equality is easily seen if one chooses the origin at the centre of C. With $V_n(K_0) = V_n(K) =: V_n$ and $S(K_0) \leq S(K) =: S$ we obtain

$$(n-1)V_n - \rho S + \rho^n\left(\frac{1}{2}\kappa_n + \kappa_{n-1}\right) \leq 0.$$

Writing the inequality

$$(n-1)a - n\rho b + \rho^n c \leq 0 \tag{7.198}$$

in the form

$$\rho b \geq \frac{n-1}{n}a + \frac{1}{n}\rho^n c,$$

taking logarithms and using the concavity of log, we obtain $b^n \geq a^{n-1}c$, with equality if and only if it holds in (7.198) and $a = \rho^n C$. Hence, inequality (7.197) follows. If equality holds in (7.197), then $S = S(K_0)$, K_0 and K_1 are equivalent by telescoping, and $V_n = \rho^n(\frac{1}{2}\kappa_n + \kappa_{n-1})$. The second condition implies that K_0 is, up to a translation, the union of K_1 and a circular cylinder attached to C, and the third condition implies that the cylinder has height ρ. Finally, $S = S(K_0)$ yields that K must be a translate of K_0.

8

Determination by area measures and curvatures

The theory of mixed volumes is a powerful tool for treating some questions on closed convex hypersurfaces from the point of view of differential geometry, but in a general form without differentiability assumptions. Under smoothness assumptions, the results we have in mind concern the determination of closed convex hypersurfaces from curvature functions, such as Gauss curvature, mean curvature and their generalizations. Here 'determination' comprises questions of existence, uniqueness and stability. Without differentiability assumptions, the usual curvature functions, namely the elementary symmetric functions of the principal curvatures on the boundary of a convex body or of the principal radii of curvature on the spherical image, have to be replaced by curvature measures and area measures, respectively. The area measures are particularly accessible to the Brunn–Minkowski theory. In Section 8.1 we treat uniqueness theorems for these. Section 8.2 is devoted to Minkowski's existence theorem for convex bodies with given surface area measure (area measure of order $n-1$) and Section 8.3 deals with area measures of order one, where the existence problem is known as the Christoffel problem. The intermediate cases, area measures of orders strictly between 1 and $n-1$, are briefly considered in Section 8.4. The final section is devoted to corresponding stability estimates and to a few uniqueness results for curvature measures.

8.1 Uniqueness results

We start with the uniqueness assertion for the determination of a convex body by its surface area measure. Although this result will be improved and generalized by later theorems, we give its formulation and proof separately, to show in a basic example the close connection with results on mixed volumes.

Theorem 8.1.1 *If $K, L \in \mathcal{K}^n_n$ are convex bodies with*

$$S_{n-1}(K, \cdot) = S_{n-1}(L, \cdot), \tag{8.1}$$

then K and L are translates of each other.

Proof We integrate the support function $h(K, \cdot)$ over $\mathbb{S}^{n-1}$, with the measure $S_{n-1}(K, \cdot) = S_{n-1}(L, \cdot)$, use (5.19) and apply Minkowski's inequality (7.18), to obtain

$$V_n(K)^n = V(K, L, \dots, L)^n \geq V_n(K)V_n(L)^{n-1}. \tag{8.2}$$

Thus $V_n(K) \geq V_n(L)$, and analogously we obtain $V_n(L) \geq V_n(K)$. Hence, equality holds in (8.2), which implies that K and L are homothetic. Because of (8.1), they must be translates. □

Suppose that K and L are convex bodies of class C^2_+. Then the assumption (8.1) is, by (4.26), equivalent to the assumption that the function s_{n-1}, the product of the principal radii of curvature as a function of the outer unit normal vector, is the same for K and L. Equivalently, the Gauss–Kronecker curvatures of the convex hypersurfaces $\mathrm{bd}\, K$ and $\mathrm{bd}\, L$ coincide at points with the same outer normal vector. This is the classical assumption for the uniqueness part of Minkowski's problem in differential geometry.

The following consequence of Theorem 8.1.1 is often useful.

Theorem 8.1.2 *If $K, L \in \mathcal{K}^n$ are convex bodies with the property that*

$$V_1(K, M) = V_1(L, M) \qquad \text{for all } M \in \mathcal{K}^n,$$

then K and L are translates of each other.

Proof The assumption implies that

$$\int_{\mathbb{S}^{n-1}} f \, \mathrm{d}[S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot)] = 0,$$

first if f is a support function, then by linearity if f is a difference of support functions, and finally by Lemma 1.7.8 if $f \in C(\mathbb{S}^{n-1})$ is arbitrary. A standard measure-theoretic argument gives $S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot) = 0$, from which the assertion follows. □

The uniqueness assertion of Theorem 8.1.1 can be generalized considerably, namely to certain mixed area measures.

Theorem 8.1.3 *Let $m \in \{2, \dots, n\}$ and an $(n-m)$-tuple $\mathcal{C} = (K_{m+1}, \dots, K_n)$ of smooth convex bodies be given. If $K, L \in \mathcal{K}^n$ are convex bodies of dimension at least m satisfying*

$$S(K[m-1], \mathcal{C}, \cdot) = S(L[m-1], \mathcal{C}, \cdot), \tag{8.3}$$

then K and L are translates of each other.

Proof The assumption (8.3) says that condition (d) in Theorem 7.4.6 is satisfied. By Theorem 7.6.9, this implies that K and L are homothetic. By (8.3), they must be translates. □

The special case $K_{m+1} = \cdots = K_n = B^n$ gives the Aleksandrov–Fenchel–Jessen theorem.

Corollary 8.1.4 (Aleksandrov–Fenchel–Jessen) *Let $m \in \{2, \dots, n\}$. If $K, L \in \mathcal{K}^n$ are convex bodies of dimension at least m satisfying*

$$S_{m-1}(K, \cdot) = S_{m-1}(L, \cdot), \tag{8.4}$$

then K and L are translates of each other.

A special case is the differential-geometric version: if K and L are convex bodies of class C^2_+ such that, at points with the same outer normal, the elementary symmetric functions of order $m-1$ of the principal radii of curvature are the same, then K and L differ only by a translation.

The following theorem is an important consequence of the Aleksandrov–Fenchel–Jessen theorem. Recall that $K \,|\, u^\perp$ denotes the image of K under orthogonal projection to $u^\perp$, the hyperplane through o orthogonal to the vector u.

Corollary 8.1.5 (Aleksandrov's projection theorem) *Let $i \in \{1, \dots, n-1\}$. Let $K, L \in \mathcal{K}^n$ be centrally symmetric convex bodies of dimension at least $i+1$. If*

$$V_i(K \,|\, u^\perp) = V_i(L \,|\, u^\perp) \qquad \text{for all } u \in \mathbb{S}^{n-1},$$

then K and L are translates of each other.

Proof By formula (5.78) and the uniqueness part of Theorem 3.5.4 (for which the central symmetry is needed), the assumption leads to $S_i(K, \cdot) = S_i(L, \cdot)$. The Aleksandrov–Fenchel–Jessen theorem gives the assertion. □

The theory of mixed volumes is flexible enough to permit also applications to non-closed convex hypersurfaces. As an example, we prove a version of the Aleksandrov–Fenchel–Jessen theorem for convex hypersurfaces with boundary and with spherical image in a hemisphere.

Let a unit vector $u \in \mathbb{S}^{n-1}$ and an $(n-1)$-dimensional convex body $C \subset H_{u,0}$ be given. By a convex hypersurface of the canal class $\mathcal{K}_C^+$ we mean a hypersurface F with boundary in $\mathbb{R}^n$ that by orthogonal projection to $H_{u,0}$ is mapped bijectively onto C and is such that the set

$$K(F) := \{x - \lambda u : x \in F,\ \lambda \geq 0\}$$

is convex. Let $\Omega_u^+ := \{x \in \mathbb{S}^{n-1} : \langle x, u \rangle > 0\}$. We define the jth-order area measure of F by

$$S_j(F, \omega) := S_j(K(F), \omega) \quad \text{for } \omega \in \mathcal{B}(\Omega_u^+).$$

Thus $S_j(F, \cdot)$ is defined on the open half-sphere Ω_u^+. Observe that we do not demand that the hypersurface F be tangential to the right cylinder with cross-section C. Thus, the relative interiors of two convex hypersurfaces of the canal class $\mathcal{K}_C^+$ may a priori well have different spherical images.

Theorem 8.1.6 *Let F_0, F_1 be convex hypersurfaces of the canal class $\mathcal{K}_C^+$ satisfying*

$$S_{m-1}(F_0, \cdot) = S_{m-1}(F_1, \cdot)$$

for some $m \in \{2, \dots, n\}$. Then there is a translation parallel to u carrying F_0 into F_1.

Proof We choose $\lambda \in \mathbb{R}$ with $F_0, F_1 \subset H^+_{u,\lambda}$ and put $K_i := K(F_i) \cap H^+_{u,\lambda}$ for $i = 0, 1$. Then K_0 and K_1 are convex bodies belonging to the canal class $\mathcal{K}_C$ defined in Section 7.7. We have $S_{m-1}(K_0, \omega) = S_{m-1}(K_1, \omega)$ for Borel sets $\omega \subset \mathbb{S}^{n-1} \setminus H_{u,0}$. This holds by assumption if $\omega \subset \Omega^+_u$. For $\omega \subset \Omega^-_u := \{x \in \mathbb{S}^{n-1} : \langle x, u \rangle < 0\}$ it is true since $F(K_0, u) = F(K_1, u)$ for $u \in \Omega^-_u$. Thus condition (e) in Theorem 7.7.2 (with $K_{m+1} = \dots = K_n = B^n$) is satisfied. By Theorem 7.7.3, K_0 and K_1 are equivalent by telescoping. This proves the assertion. □

The Aleksandrov–Fenchel–Jessen theorem shows that differential-geometric results on the determination of closed convex hypersurfaces by curvature functions can be extended to general convex bodies $K \in \mathcal{K}^n_n$ without any smoothness assumptions, if the curvature functions are generalized to area measures. This may not be too surprising. A more surprising aspect of the extension to general convex bodies is the phenomenon that most convex bodies show an unexpected rigidity in their determination by lower-order area measures: already inequalities suffice for a determination up to homothety. The following two theorems and their proofs are taken from Schneider [1719].

Theorem 8.1.7 *Let $i \in \{1, \dots, n-2\}$. There is a dense G_δ set $\mathcal{A} \subset \mathcal{K}^n$ such that each $K \in \mathcal{A}$ has the following property. If $L \in \mathcal{K}^n$ is a convex body of dimension at least $i + 1$ with*

$$S_i(L, \cdot) \le S_i(K, \cdot),$$

then L is homothetic to K.

Since, by Theorem 2.7.1, the set of smooth and strictly convex bodies is a dense G_δ set, we can conclude that also the smooth and strictly convex bodies with the property of Theorem 8.1.7 make up a dense G_δ set.

The theorem is a consequence of the following result on polytopes. Let $P \in \mathcal{P}^n_n$ be a polytope. For a proper face F of P we use the abbreviation

$$\nu(P, F) := N(P, F) \cap \mathbb{S}^{n-1},$$

so that $\nu(P, F) = \sigma(P, \operatorname{relint} F)$ is the spherical image of the relative interior of F. Recall that, for a polytope $P \in \mathcal{P}^n$ and a Borel set $\omega \subset \mathbb{S}^{n-1}$, we have

$$S_i(P, \omega) = \binom{n-1}{i}^{-1} \sum_{F \in \mathcal{F}_i(P)} V_i(F) \mathcal{H}^{n-1-i}(\nu(P, F) \cap \omega) \tag{8.5}$$

by (4.20) and (4.24). Hence, the support of the ith area measure of P is given by

$$\operatorname{supp} S_i(P, \cdot) = \bigcup_{F \in \mathcal{F}_i(P)} \nu(P, F) =: \sigma_{n-1-i}(P),$$

where $\sigma_{n-1-i}(P)$ denotes the spherical image of the relative interiors of all i-faces of P.

Theorem 8.1.8 *Let $i \in \{1, \dots, n-1\}$ and let $P \in \mathcal{P}^n_n$ be a polytope with the property that all its $(i+1)$-faces are simplices. If $K \in \mathcal{K}^n$ is a convex body of dimension at least $i+1$ satisfying*

$$S_i(K, \cdot) \leq S_i(P, \cdot), \tag{8.6}$$

then K is homothetic to P.

Proof If (8.6) holds, then

$$\operatorname{supp} S_i(K, \cdot) \subset \operatorname{supp} S_i(P \cdot) = \sigma_{n-1-i}(P).$$

By Theorem 4.5.4, K is a polytope, and it satisfies

$$\sigma_{n-1-i}(K) \subset \sigma_{n-1-i}(P). \tag{8.7}$$

If $i = n-1$, then P is by assumption an n-simplex. Its surface area measure is concentrated in $n+1$ points, hence by (8.7) the same holds true for K. Since the normal vectors of the facets of a simplex determine the homothety class of the simplex uniquely, K is homothetic to P. We can, therefore, assume in the following that $i < n-1$.

With each face $F \in \mathcal{F}_i(P)$, we associate a number $\tau(F)$ in the following way. If

$$\sigma_{n-1-i}(K) \cap \operatorname{relint} \nu(P, F) = \emptyset, \tag{8.8}$$

we put $\tau(F) := 0$. If (8.8) does not hold, there exists a face $G \in \mathcal{F}_i(K)$ with

$$\nu(K, G) \cap \operatorname{relint} \nu(P, F) \neq \emptyset.$$

Let

$$A := \mathbb{S}^{n-1} \setminus \bigcup_{F \neq F' \in \mathcal{F}_i(P)} \nu(P, F');$$

then A is an open subset of $\mathbb{S}^{n-1}$ and

$$A \cap \sigma_{n-1-i}(P) = \operatorname{relint} \nu(P, F).$$

Since $\nu(K, G) \subset \sigma_{n-1-i}(P)$ by (8.7), it follows that

$$\nu(K, G) \cap A \subset \operatorname{relint} \nu(P, F).$$

Suppose this inclusion is strict. Then $\operatorname{relint} \nu(P, F)$ contains a relative boundary point u of $\nu(K, G)$. The support set $F(K, u)$ is a face of K of dimension $> i$. This face contains an i-face $G'(K)$ of K such that $\nu(K, G') \cap A$ is not contained in $\operatorname{relint} \nu(P, F)$, a contradiction. Thus,

$$\nu(K, G) \cap A = \operatorname{relint} \nu(P, F). \tag{8.9}$$

In particular, the face G is uniquely determined, and we can define

$$\tau(F) := \frac{V_i(G)}{V_i(F)}.$$

This completes the definition of τ.

We assert that a stronger version of (8.9) is true, namely

$$\nu(K, G) = \nu(P, F). \tag{8.10}$$

Suppose that this is false. Since relint $\nu(P, F) \subset \nu(K, G)$ by (8.9), there is a point v in the relative boundary of $\nu(P, F)$ that is not a relative boundary point of $\nu(K, G)$. Such a point can, in fact, be chosen in the relative interior of a (spherical) $(n-i-2)$-face ϕ of $\nu(P, F)$. The face $S := F(P, v)$ satisfies $\nu(P, S) = \phi$ and hence is an $(i+1)$-face, by assumption a simplex. It has $i+2$ faces of dimension i, say $F_1 = F, F_2, \dots, F_{i+2}$. Since $v \in \operatorname{relint} \nu(P, S)$, we can choose a neighbourhood U of v in $\mathbb{S}^{n-1}$ that does not meet $\nu(P, F')$ for any i-face F' of P different from $F_2, \dots, F_{i+2}$. Since $\nu(K, G) \subset \sigma_{n-1-i}(P)$, we deduce that

$$U \cap \nu(K, G) \subset \bigcup_{r=1}^{i+2} \nu(P, F_r).$$

Since S is a simplex, the linear subspaces $\operatorname{lin} N(P, F_r)$, $r = 1, \dots, i+2$, are pairwise different, hence $U \cap \nu(K, G) \subset \nu(P, F)$. Thus the point v is a relative boundary point of $\nu(K, G)$. This contradiction shows that (8.10) holds.

Now let S be an arbitrary $(i+1)$-face of P. The simplex S has $i+2$ faces $F_1, \dots, F_{i+2}$ of dimension i. Choose

$$w \in \operatorname{relint} \nu(P, S). \tag{8.11}$$

Suppose, say, that $\tau(F_1) > 0$. Then there is an i-face G_1 of K such that

$$\nu(K, G_1) = \nu(P, F_1). \tag{8.12}$$

Since w is a relative boundary point of $\nu(K, G_1)$, the face $F(K, w)$ of K is of dimension greater than i. From (8.11) and (8.12) we deduce that $F(K, w)$ is an $(i+1)$-face of K. It has at least $i+2$ faces $G_1, \dots, G_{i+2}$ of dimension i. Since $w \in \nu(K, G_r) \subset \sigma_{n-1-i}(P)$ by (8.7), we can assume, after renumbering, that

$$\nu(K, G_r) = \nu(P, F_r) \tag{8.13}$$

for $r = 1, \dots, i+2$. The face $F(K, w)$ cannot have further i-faces and hence is a simplex. By (8.13), the affine hull of this simplex is parallel to the affine hull of S, and the affine hull of G_r is parallel to the affine hull of F_r ($r = 1, \dots, i+2$). Hence, $F(K, w)$ is homothetic to S. This implies that $V_i(G_r)/V_i(F_r)$ is independent of r and thus shows that $\tau(F_1) = \cdots = \tau(F_{i+2})$. This has been proved under the assumption that at least one $\tau(F_r)$ is positive. If this does not hold, then $\tau(F_r) = 0$ for $r = 1, \dots, i+2$, hence $\tau(F_1) = \cdots = \tau(F_{i+2})$ holds again.

Since any two $(i+1)$-faces of P can be connected by a chain of $(i+1)$-faces such that any two consecutive ones have a common i-face, we conclude that $\tau(F)$ does not depend on F, say $\tau(F) = \tau$ for all i-faces F of P. Since K does have i-faces, the case $\tau = 0$ is impossible.

To each face $F \in \mathcal{F}_i(P)$, there exists a unique face $G \in \mathcal{F}_i(K)$ such that

$$\nu(P, F) = \nu(K, G),$$

and we have $V_i(G) = \tau V_i(F)$. Every i-face of K corresponds in this way to some i-face of P. It follows from (8.5) that $S_i(K, \cdot) = \tau S_i(P, \cdot) = S_i(\tau^{1/i} P, \cdot)$. By the Aleksandrov–Fenchel–Jessen theorem, K and $\tau^{1/i} P$ are translates. This completes the proof. □

Now we can finish the proof of Theorem 8.1.7.

Proof of Theorem 8.1.7 Let $\mathcal{A}$ denote the set of all convex bodies $K \in \mathcal{K}^n$ with the property that $S_i(L, \cdot) \leq S_i(K, \cdot)$ for a body $L \in \mathcal{K}^n$ of dimension at least $i + 1$ implies that L is homothetic to K. By Theorem 8.1.8, $\mathcal{A}$ is dense in $\mathcal{K}^n$ if $i < n - 1$.

For $k \in \mathbb{N}$, let $\mathcal{A}_k$ be the set of all convex bodies $K \in \mathcal{K}^n$ for which there exists a convex body $L \in \mathcal{K}^n$ that contains an $(i + 1)$-ball of radius $1/k$ and satisfies

$$S_i(L, \cdot) \leq S_i(K, \cdot), \qquad \delta(L, K) \leq k, \qquad \delta(L, K') \geq 1/k$$

for all homothets K' of K. We assert that

$$\mathcal{A} = \mathcal{K}^n \setminus \bigcup_{k \in \mathbb{N}} \mathcal{A}_k. \tag{8.14}$$

For the proof, suppose that $K \in \mathcal{A}_k$ for some $k \in \mathbb{N}$. Then there exists a convex body $L \in \mathcal{K}^n$ such that $\dim L \geq i + 1$, $S_i(L, \cdot) \leq S_i(K, \cdot)$ and $\delta(L, K') \geq 1/k$ for all homothets K' of K. In particular, L is not homothetic to K, hence $K \notin \mathcal{A}$.

Conversely, let $K \in \mathcal{K} \setminus \bigcup_{k \in \mathbb{N}} \mathcal{A}_k$. Let $L \in \mathcal{K}^n$ be a body of dimension at least $i + 1$ satisfying $S_i(L, \cdot) \leq S_i(K, \cdot)$. There exists a number k_0 such that L contains an $(i + 1)$-ball of radius $1/k$ and $\delta(L, K) \leq k$ for all $k \geq k_0$. Let $K \in \mathbb{N}$ and $k \geq k_0$. Since $K \notin \mathcal{A}_k$, we can choose a homothet K_k of K with $\delta(L, K_k) < 1/k$. Since $\delta(L, K_k) \to 0$ for $k \to \infty$, the sequence $(K_k)_{k \geq k_0}$ converges to L. On the other hand, each K_k and hence L is homothetic to K. This shows that $K \in \mathcal{A}$ and thus establishes (8.14).

To prove that $\mathcal{A}_k$ is closed, let $(K_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{A}_k$ converging to a convex body K. For each $j \in \mathbb{N}$, there exists a convex body $L_j \in \mathcal{K}^n$ that contains an $(i + 1)$-ball B_j of radius $1/k$ and is such that $S_i(L_j, \cdot) \leq S_i(K_j, \cdot)$, $\delta(L_j, K_j) \leq k$ and $\delta(L_j, K_j') \geq 1/k$ for all homothets K_j' of K_j. From $\delta(L_j, K_j) \leq k$ and the convergence $K_j \to K$ it follows that the sequence $(L_j)_{j \in \mathbb{N}}$ is bounded. Choosing a suitable subsequence and changing the notation, we may assume that $L_j \to L$ and $B_j \to B$ $(j \to \infty)$ for convex bodies $L, B \in \mathcal{K}^n$. Then B is an $(i + 1)$-ball of radius $1/k$ contained in L. From the weak continuity of S_i and the continuity of δ we conclude that $S_i(L, \cdot) \leq S_i(K, \cdot)$ and $\delta(L, K) \leq k$. Let K' be any homothet of K, say $K' = \lambda K + t$ with $\lambda \geq 0$ and $t \in \mathbb{R}^n$. Since $\lambda K_j + t$ is a homothet of K_j, we have $\delta(L_j, \lambda K_j + t) \geq 1/k$, from which we deduce that $\delta(L, K') \geq 1/k$. This shows that $K \in \mathcal{A}_k$. We have shown that $\mathcal{A}_k$ is closed, which completes the proof. □

Notes for Section 8.1

1. *Minkowski's uniqueness theorem.* The method of proof for the uniqueness theorem 8.1.1 is the original one found by Minkowski [1435, 1438].

From a differential-geometric viewpoint, a paper worthy of recommendation that has historical interest is Stoker [1821]. It treats the uniqueness theorems of Minkowski and Christoffel and the infinitesimal rigidity for closed convex surfaces in $\mathbb{R}^3$, showing clearly the interrelations between these problems and how the methods for their proofs are related to the classical geometry of convex bodies.

2. *The Aleksandrov–Fenchel–Jessen theorem.* Corollary 8.1.4, the classical Aleksandrov–Fenchel–Jessen theorem, was proved independently by Aleksandrov [13] and by Fenchel and Jessen [572] (see also Busemann [370], p. 70, Leichtweiß [1184], p. 319, and Leichtweiß [1185]). Theorem 8.1.3 for the special case where the bodies of $\mathscr{C}$ are of class C^2_+ was obtained by Aleksandrov [16]; the general version for bodies that are only smooth, as given here, is new.

   For the special case of the Aleksandrov–Fenchel–Jessen theorem where the bodies K and L are of class C^2_+, a differential-geometric proof was given by Chern [419].

   A version of the Aleksandrov–Fenchel–Jessen theorem for convex hypersurfaces with boundaries was proved by Busemann [371]. His assumption of identical boundaries is rather strong; it seems more natural to assume that the support functions coincide at the boundary of the spherical image. For differential-geometric uniqueness results of this type, see Aleksandrov [24] and Oliker [1489, 1490]. The uniqueness theorem 8.1.6 for general convex hypersurfaces with boundaries is due to Schneider [1711].

   One may ask whether the Aleksandrov–Fenchel–Jessen theorem can be generalized by imposing more general relations between area measures. For example, if $K, L \in \mathcal{K}^n_n$ are convex bodies satisfying

$$\sum_{j=1}^{n-1} \alpha_j S_j(K,\cdot) = \sum_{j=1}^{n-1} \alpha_j S_j(L,\cdot),$$

   where the constants $\alpha_1, \dots, \alpha_{n-1}$ are nonnegative and not all zero, must K be a translate of L? An affirmative answer for $n = 3$ is contained in Schneider [1681].

3. *Aleksandrov's projection theorem.* This was proved by Aleksandrov [13]; see also Chakerian [399].

   The fascinating topic of the determination of convex bodies from the volumes of projections belongs to the wide field that has now been established as 'geometric tomography'. Since there is the excellent treatise by Gardner [675], we touch such topics only briefly in this book. For the determination from projection volumes, we refer the reader in particular to Chapter 3 of [675] and the extensive notes therein.

   In particular, we want to draw the reader's attention to Note 3.6 of [675], which describes contributions to Nakajima's problem. This problem asked whether a three-dimensional convex body of constant width and constant brightness must be a ball. An affirmative solution was achieved by Howard [990], and several results on the determination of higher-dimensional convex bodies by two projection functions were obtained by Howard and Hug [991, 992] and by Hug [1009].

4. The following result includes Aleksandrov's projection theorem and can thus be considered as a generalization. If the orthogonal projections to r-dimensional subspaces of the body $K \in \mathcal{K}^n_n$ have the same r-dimensional volume as the projections of a centrally symmetric convex body M, then the quermassintegrals satisfy $W_j(M) \geq W_j(K)$, for any $0 \leq j \leq n-r$, with equality if and only if K and M are translates. This was proved by Chakerian and Lutwak [405], together with more general results. 'Dual' analogues of these results (in the sense of §9.3) were obtained by Lv and Leng [1308].

5. Theorem 8.1.7 has been extended (for $n \geq 4$) by Bauer [176] in the following way. Let $n \geq 4$. For most pairs $(K, M) \in \mathcal{K}^n \times \mathcal{K}^n$, the inequality $S_i(L,\cdot) \leq S_i(K,\cdot) + S_i(M,\cdot)$ for a convex body L of dimension at least $i+1$ implies that L is homothetic either to K or to M.

6. Hug [1005] has proved the following counterpart to Theorem 8.1.8. Let $P \in \mathcal{P}^n$ be a polytope and let $r \in \{0, \dots, n-1\}$. If $K \in \mathcal{K}^n$ is a convex body of dimension at least $r+1$ satisfying $C_r(K,\cdot) \leq C_r(P,\cdot)$, then $K = P$.

8.2 Convex bodies with given surface area measures

8.2.1 Minkowski's existence theorem

If φ is the area measure $S_{n-1}(K, \cdot)$ of the n-dimensional convex body $K \in \mathcal{K}_n^n$, then

$$\int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\varphi(u) = o, \tag{8.15}$$

which is a special case of (5.30). From the geometric meaning of the area measure it is clear that it cannot be concentrated on any great subsphere of $\mathbb{S}^{n-1}$. Minkowski's existence theorem says that these two simple necessary conditions are also sufficient in order that φ be the area measure of a convex body $K \in \mathcal{K}_n^n$ (by Theorem 8.1.1, this convex body is unique up to a translation). This beautiful and useful result was proved by Minkowski for special cases, in which, however, the essential ideas were already present; it was later extended independently by Aleksandrov and by Fenchel and Jessen, when the full notion of area measure was available.

Although Minkowski's existence theorem is not an application of the inequalities for mixed volumes, its proof is motivated by Minkowski's inequality. In fact, if $K \in \mathcal{K}_n^n$ with $S_{n-1}(K, \cdot) = \varphi$ exists, then any convex body $L \in \mathcal{K}_n^n$ satisfies

$$\frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u) \, \mathrm{d}\varphi(u) = V(L, K, \dots, K) \geq V_n(L)^{1/n} V_n(K)^{1-1/n},$$

with equality if and only if L is homothetic to K. Thus, up to homothety the convex body K is characterized by the fact that it minimizes the functional

$$L \mapsto \int_{\mathbb{S}^{n-1}} h(L, u) \, \mathrm{d}\varphi(u)$$

under the side condition $V_n(L) = 1$.

Following Minkowski [1435, 1438], we first treat the case of discrete measures and polytopes. Approximation then extends the result to general measures and convex bodies. A different approach, due to Aleksandrov, working with general measures from the beginning, will be used in the treatment of the L_p Minkowski problem (Theorem 9.2.1).

Theorem 8.2.1 (Minkowski) *Let $u_1, \dots, u_N \in \mathbb{S}^{n-1}$ be pairwise distinct vectors linearly spanning $\mathbb{R}^n$, and let $f_1, \dots, f_N$ be positive real numbers such that*

$$\sum_{i=1}^{N} f_i u_i = o. \tag{8.16}$$

Then there is a polytope $P \in \mathcal{P}_n^n$ having (precisely) $u_1, \dots, u_N$ as its normal vectors and satisfying

$$V_{n-1}(F(P, u_i)) = f_i \tag{8.17}$$

for $i = 1, \dots, N$.

Proof For an N-tuple $A = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ with nonnegative components $\alpha_1, \dots, \alpha_N$ we define

$$P(A) := \bigcap_{i=1}^{N} H^-_{u_i,\alpha_i}.$$

From $\operatorname{lin}\{u_1, \dots, u_N\} = \mathbb{R}^N$ and (8.16) it follows that $\operatorname{pos}\{u_1, \dots, u_N\} = \mathbb{R}^n$; hence $P(A)$ is a polytope, with $o \in P(A)$. Its normal vectors are among the vectors $u_1, \dots, u_N$, and

$$h(P(A), u_i) \le \alpha_i \quad \text{for } i = 1, \dots, N,$$

with equality if $V_{n-1}(F(P(A)), u_i) > 0$. Let

$$M := \{A \in \mathbb{R}^N : \alpha_i \ge 0, V_n(P(A)) \ge 1\};$$

then M is closed, since $V_n(P(A))$ depends continuously on A. The linear function Φ defined by

$$\Phi(A) := \frac{1}{n} \sum_{i=1}^{N} \alpha_i f_i, \quad A = (\alpha_1, \dots, \alpha_N),$$

attains a minimum on M, because $f_i > 0$ for all i. Let μ^{n-1} be this minimum and suppose that it is attained at $A^* = (\alpha_1^*, \dots, \alpha_N^*)$; put $P(A^*) =: P^*$. We assert that μP^* solves the problem. By (8.16), $\Phi(A)$ is not changed under a translation of $P(A)$, hence we may assume that P^* has o as an interior point and, hence, that $\alpha_1^*, \dots, \alpha_N^* > 0$. Write

$$V_{n-1}(F(P^*, u_i)) := f_i^* \quad \text{for } i = 1, \dots, N.$$

We have $V_n(P^*) = 1$, since otherwise λP^* with suitable $\lambda < 1$ would satisfy $V_n(\lambda P^*) \ge 1$ and yield a value less than μ^{n-1} for the function Φ. From

$$V_n(P^*) = \frac{1}{n} \sum_{i=1}^{N} h(P^*, u_i) f_i^*$$

and $h(P^*, u_i) = \alpha_i^*$ if $f_i^* \neq 0$ it follows that

$$\frac{1}{n} \sum_{i=1}^{N} \alpha_i^* f_i^* = 1. \tag{8.18}$$

Since $\Phi(A^*) = \mu^{n-1}$, we have

$$\frac{1}{n} \sum_{i=1}^{N} \alpha_i^* f_i = \mu^{n-1}. \tag{8.19}$$

Now consider the hyperplanes in $\mathbb{R}^N$ given by

$$H_1 := \left\{ A \in \mathbb{R}^N : \frac{1}{n} \sum_{i=1}^N \alpha_i f_i = \mu^{n-1} \right\},$$

$$H_2 := \left\{ A \in \mathbb{R}^N : \frac{1}{n} \sum_{i=1}^N \alpha_i f_i^* = 1 \right\}.$$

From (8.18) and (8.19) it follows that $A^* \in H_1 \cap H_2$. Suppose $A = (\alpha_1, \dots, \alpha_N) \in H_1$, where $\alpha_1, \dots, \alpha_N \geq 0$. Then $V_n(P(A)) \leq 1$, since otherwise μ^{n-1} would not be the minimum of Φ on M. For $0 \leq \vartheta \leq 1$ we have $(1-\vartheta)A^* + \vartheta A \in H_1$. Further, a point $y \in (1-\vartheta)P^* + \vartheta P(A)$ satisfies

$$\langle y, u_i \rangle \leq (1-\vartheta)\alpha_i^* + \vartheta \alpha_i \quad \text{for } i = 1, \dots, N,$$

hence

$$(1-\vartheta)P^* + \vartheta P(A) \subset P((1-\vartheta)A^* + \vartheta A).$$

It follows that $V_n((1-\vartheta)P^* + \vartheta P(A)) \leq 1$ for $0 \leq \vartheta \leq 1$, which together with $V_n(P^*) = 1$ gives

$$V(P(A), P^*, \dots, P^*) \leq 1. \tag{8.20}$$

Since $\alpha_1^*, \dots, \alpha_N^* > 0$, there is a neighbourhood U of A^* in $\mathbb{R}^N$ such that, for $A \in H_1 \cap U$, the polytope $P(A)$ satisfies $\alpha_1, \dots, \alpha_N > 0$ and

$$V_{n-1}(F(P(A), u_i)) > 0, \quad \text{if } f_i^* > 0.$$

If this holds, then $h(P(A), u_i) = \alpha_i$, hence

$$V(P(A), P^*, \dots, P^*) = \frac{1}{n} \sum_{i=1}^N \alpha_i f_i^*.$$

Inequality (8.20) now shows that $H_1 \cap U \subset H_2^-$, where H_2^- is one of the two closed halfspaces bounded by H_2. Since $A^* \in H_1 \cap H_2$, we conclude that $H_1 = H_2$ and hence that

$$f_i = \mu^{n-1} f_i^* = V_{n-1}(F(\mu P^*, u_i))$$

for $i = 1, \dots, N$. This completes the proof that μP^* is the desired polytope. □

The general case of Minkowski's existence theorem will now be obtained by approximation.

Theorem 8.2.2 *Let φ be a measure on $\mathcal{B}(\mathbb{S}^{n-1})$ with the properties*

$$\int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\varphi(u) = o \tag{8.21}$$

and $\varphi(s) < \varphi(\mathbb{S}^{n-1})$ for each great subsphere s of $\mathbb{S}^{n-1}$. Then there is a convex body $K \in \mathcal{K}_n^n$ for which $S_{n-1}(K, \cdot) = \varphi$.

Proof For each $k \in \mathbb{N}$, we decompose the sphere $\mathbb{S}^{n-1}$ into finitely many pairwise disjoint Borel sets of diameter at most $1/k$ and with spherically convex closure. We fix k and denote by $\Delta_1, \dots, \Delta_N$ those sets of the decomposition on which φ is positive. For $i = 1, \dots, N$, let

$$c_i := \frac{1}{\varphi(\Delta_i)} \int_{\Delta_i} u \, \mathrm{d}\varphi(u);$$

then $c_i \neq o$ since Δ_i lies in an open hemisphere, and hence $c_i = f_i u_i$ with $u_i \in \mathbb{S}^{n-1}$ and $f_i > 0$. Observing that $c_i \in \operatorname{conv} \Delta_i$, one finds that $1 - (2k^2)^{-1} \leq f_i \leq 1$. For $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, let

$$\varphi_k(\omega) := \sum_{u_i \in \omega} \varphi(\Delta_i) f_i.$$

Then φ_k is a measure on $\mathcal{B}(\mathbb{S}^{n-1})$ satisfying

$$\int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\varphi_k(u) = \int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\varphi(u) = o.$$

For $g \in C(\mathbb{S}^{n-1})$ we have

$$\int_{\mathbb{S}^{n-1}} g(u) \, \mathrm{d}\varphi_k(u) - \int_{\mathbb{S}^{n-1}} g(u) \, \mathrm{d}\varphi(u) = \sum_{i=1}^{N} \int_{\Delta_i} [f_i g(u_i) - g(u)] \, \mathrm{d}\varphi(u)$$

and

$$|f_i g(u_i) - g(u)| \leq |g(u_i) - g(u)| + \|g\|/2k^2.$$

If $u \in \Delta_i$, then $|u_i - u| \leq 1/k$, since $u_i \in \operatorname{pos} \Delta_i$. From the uniform continuity of g we deduce that $\int g \, \mathrm{d}\varphi_k \to \int g \, \mathrm{d}\varphi$ for $k \to \infty$, hence

$$\varphi_k \xrightarrow{w} \varphi \quad \text{for } k \to \infty.$$

There is a number $a > 0$ such that

$$\int_{\mathbb{S}^{n-1}} \langle u, v \rangle_+ \, \mathrm{d}\varphi(u) \geq a \quad \text{for } v \in \mathbb{S}^{n-1}$$

(where the subscript $_+$ denotes the positive part), since the integral is a continuous function of v that is positive, by the properties of φ. The estimate above shows that

$$\lim_{k \to \infty} \int_{\mathbb{S}^{n-1}} \langle u, v \rangle_+ \, \mathrm{d}\varphi_k(u) = \int_{\mathbb{S}^{n-1}} \langle u, v \rangle_+ \, \mathrm{d}\varphi(u)$$

uniformly in v. Hence, there exists k_0 such that

$$\int_{\mathbb{S}^{n-1}} \langle u, v \rangle_+ \, \mathrm{d}\varphi_k(u) > \frac{a}{2}$$

for $k \geq k_0$ and all $v \in \mathbb{S}^{n-1}$. Thus, for $k \geq k_0$, the measure φ_k is not concentrated on any great subsphere.

By Theorem 8.2.1, for $k \geq k_0$ there is a polytope $P_k \in \mathcal{P}_n^n$ with $S_{n-1}(P_k, \cdot) = \varphi_k$ and, without loss of generality, $o \in P_k$. Since $\varphi_k(\mathbb{S}^{n-1}) \leq \varphi(\mathbb{S}^{n-1})$, the surface areas of the polytopes P_k remain bounded; hence, by the isoperimetric inequality, $V_n(P_k) \le b$ with some constant b. Let $x \in P_k$ and write $x = |x|v$ with $v \in \mathbb{S}^{n-1}$; then

$$h(P_k, u) \ge h(\operatorname{conv}\{o, x\}, u) = |x|\langle u, v\rangle_+$$

for $u \in \mathbb{S}^{n-1}$ and hence

$$b \ge V_n(P_k) \ge \frac{|x|}{n} \int_{\mathbb{S}^{n-1}} \langle u, v\rangle_+ \, \mathrm{d}\varphi_k(u) \ge \frac{|x|}{n} \frac{a}{2},$$

thus $P_k \subset B(o, 2nb/a)$. By the Blaschke selection theorem 1.8.7, there is a subsequence of $(P_k)_{k \ge k_0}$ converging to some convex body $K \in \mathcal{K}^n$. From the weak convergence of the sequence $(\varphi_k)_{k \ge k_0}$ to φ we conclude that it satisfies $S_{n-1}(K, \cdot) = \varphi$. □

A direct variational proof for Minkowski's existence theorem, which became important for later developments, was given by Aleksandrov [15], §3. Let φ satisfy the assumptions of Theorem 8.2.2. With a positive, continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ one associates, as in (7.97), the convex body

$$K := \bigcap_{u \in \mathbb{S}^{n-1}} H^-_{u, f(u)},$$

the Aleksandrov body or Wulff shape of f. Its volume is denoted by $V(f)$. Aleksandrov considers the minimum of the functional

$$\Phi(f) := \int_{\mathbb{S}^{n-1}} f \, \mathrm{d}\varphi$$

over all positive, continuous functions f on $\mathbb{S}^{n-1}$ with $V(f) = 1$. He shows that it is sufficient to seek this minimum among support functions, and that it is attained. Aleksandrov's variational lemma, which is our Lemma 7.5.3, then shows that a positive dilatate of an extremal body has surface area measure equal to φ.

The details of this proof may be seen from its later extension, the proof of Theorem 9.1.4.

8.2.2 Blaschke addition

Minkowski's theorem makes it possible to define a new addition for n-dimensional convex bodies. For convex bodies $K, L \in \mathcal{K}^n_n$, the sum of their surface area measures satisfies the conditions of Theorem 8.2.2, hence there exists a convex body $M \in \mathcal{K}^n_n$ for which

$$S_{n-1}(M, \cdot) = S_{n-1}(K, \cdot) + S_{n-1}(L, \cdot).$$

By Theorem 8.1.1, the body M is uniquely determined up to a translation, and we may assume that it has its centroid at the origin. This body M is called the *Blaschke sum* of K and L and is denoted by $K \# L$. (Usually when working with Blaschke sums, translations do not matter.)

Although the surface area measure is a metric notion, Blaschke addition properly belongs to affine geometry, in the sense that, for $\phi \in \mathrm{SL}(n)$, the body $\phi K \# \phi L$ is a translate of $\phi(K\#L)$. In fact, for any $M \in \mathcal{K}^n$,

$$\begin{aligned} V_1(\phi(K\#L), \phi M) &= V_1(K\#L, M) = V_1(K, M) + V_1(L, M) \\ &= V_1(\phi K, \phi M) + V_1(\phi L, \phi M) = V_1(\phi K \# \phi L, \phi M). \end{aligned}$$

Theorem 8.1.2 gives the assertion (this argument is taken from Lutwak [1281]).

In this brief subsection, we collect a few results on Blaschke addition. First, for Blaschke addition, there is a counterpart to the Brunn–Minkowski theorem.

Theorem 8.2.3 (Kneser–Süss inequality) *For $K, L \in \mathcal{K}_n^n$,*

$$V_n(K\#L)^{(n-1)/n} \geq V_n(K)^{(n-1)/n} + V_n(L)^{(n-1)/n}. \tag{8.22}$$

Equality in (8.22) holds if and only if K and L are homothetic.

Proof For $M \in \mathcal{K}_n^n$, Minkowski's inequality (7.18) yields

$$V_1(K\#L, M) = V_1(K, M) + V_1(L, M) \geq V_n(M)^{1/n} \left[V_n(K)^{(n-1)/n} + V_n(L)^{(n-1)/n}\right].$$

With $M = K\#L$, this gives the inequality (8.22), together with the equality condition. $\square$

Next, we consider the analogues of the Minkowski endomorphisms studied in Section 3.3, with Minkowski addition replaced by Blaschke addition. A *Blaschke endomorphism* of $\mathcal{K}_n^n$ is defined as a mapping $T : \mathcal{K}_n^n \to \mathcal{K}_n^n$ which satisfies $T(K\#M) = TK\#TM$ for $K, M \in \mathcal{K}_n^n$ and is continuous and $\mathrm{SO}(n)$ equivariant. To formulate a classification result, we recall that in Section 3.3 we have defined the convolution $f * \mu$ of a function $f \in C(\mathbb{S}^{n-1})$ and a zonal signed measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$, where $p \in \mathbb{S}^{n-1}$ is some fixed point. (We also use the further terminology introduced in Section 3.3.) Now let $\nu \in \mathcal{M}(\mathbb{S}^{n-1})$ and let μ be as above. By the Riesz representation theorem there exists a unique signed measure $\nu * \mu \in \mathcal{M}(\mathbb{S}^{n-1})$ with

$$\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}(\nu * \mu) = \int_{\mathbb{S}^{n-1}} f * \mu \,\mathrm{d}\nu$$

for all $f \in C(\mathbb{S}^{n-1})$. This defines the convolution $\nu * \mu$ of $\nu \in \mathcal{M}(\mathbb{S}^{n-1})$ and $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$. The following theorem is due to Kiderlen [1075].

Theorem 8.2.4 *A mapping $T : \mathcal{K}_n^n \to \mathcal{K}_n^n$ is a Blaschke endomorphism if and only if there is a weakly positive measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ such that*

$$S_{n-1}(TK, \cdot) = S_{n-1}(K, \cdot) * \mu \qquad \text{for } K \in \mathcal{K}_n^n.$$

The measure μ is uniquely determined up to the addition of a linear measure.

We may also combine the two types of addition for convex bodies. A mapping $T : \mathcal{K}_n^n \to \mathcal{K}_n^n$ is called a *Blaschke–Minkowski homomorphism* if it satisfies $T(K\#M) = TK + TM$ for $K, M \in \mathcal{K}_n^n$ and is continuous and $\mathrm{SO}(n)$ equivariant. These mappings are particularly interesting, because there are prominent examples among them, such as the projection body mapping defined by (5.80). For a description in the style of the previous theorem, we denote by $C(\mathbb{S}^{n-1}, p) \subset C(\mathbb{S}^{n-1})$ the subspace of the zonal functions (with pole p), that is, those which are invariant under the rotations fixing p. For $f \in C(\mathbb{S}^{n-1}, p)$ and $\nu \in \mathcal{M}(\mathbb{S}^{n-1})$, we define $\nu * f := \nu * \int_{(\cdot)} f \, \mathrm{d}\sigma$, where σ denotes spherical Lebesgue measure. A *linear zonal function* is a function of the form $c\langle p, \cdot\rangle$ with $c \in \mathbb{R}$, and $f \in C(\mathbb{S}^{n-1}, p)$ is *weakly positive* if it is nonnegative up to the addition of a linear zonal function. The following two theorems are due to Schuster [1747].

Theorem 8.2.5 *If $T : \mathcal{K}^n_n \to \mathcal{K}^n_n$ is a Blaschke–Minkowski homomorphism, then there is a weakly positive function $g \in C(\mathbb{S}^{n-1}, p)$ such that*

$$h_{TK} = S_{n-1}(K, \cdot) * g \qquad \text{for } K \in \mathcal{K}^n_n.$$

The function g is uniquely determined up to the addition of a linear zonal function.

There seems to be no simple and intuitive characterization of the functions g appearing in Theorem 8.2.5. This is different in the following special case.

Theorem 8.2.6 *The mapping $T : \mathcal{K}^n_n \to \mathcal{K}^n_n$ is a Blaschke–Minkowski homomorphism satisfying $TK = T(-K)$ for all $K \in \mathcal{K}^n_n$ if and only if there is a centrally symmetric convex body L with $h_L \in C(\mathbb{S}^{n-1}, p)$ such that*

$$h_{TK} = S_{n-1}(K, \cdot) * h_L \qquad \text{for } K \in \mathcal{K}^n_n.$$

The body L is uniquely determined up to a translation.

If the mapping T of the previous theorem has the property that its range contains a polytope, then it is a multiple of the projection body map. This follows from a characterization theorem of Schneider and Schuster [1733].

Notes for Section 8.2

1. *Minkowski's existence theorem.* The polytopal case of Minkowski's existence theorem, Theorem 8.2.1, is due (for $n = 3$) to Minkowski [1435]; see also Minkowski [1438], §9. We have essentially followed the presentation of the proof in Bonnesen and Fenchel [284], §60. Related but slightly different versions of the proof have been given, for instance, by Aleksandrov [25], Chapter VII, §2, and McMullen [1376], §7. Aleksandrov [17] found a different proof, using his 'so-called mapping lemma', an application of the 'invariance of domain'. This proof appears also in Aleksandrov [25], Chapter VII, §1, and Pogorelov [1539], §2. Similar existence results for unbounded polyhedra can be found in Aleksandrov [25], Chapter VII, §3. Minkowski [1438], §10, extended his result to convex bodies more general than polytopes, namely to convex bodies K for which, in present-day terminology, the area measure $S_{n-1}(K, \cdot)$ has a continuous density with respect to spherical Lebesgue measure.

   Once the notion of the area measure had been introduced, the general version of Minkowski's result, Theorem 8.2.2, could be deduced from the polytope case by means of approximation, using the weak continuity of S_{n-1}. This was done by Fenchel and Jessen [572] (above we have followed their approach; see also Fenchel [569]) and later by Aleksandrov [18]. Independently of Fenchel and Jessen, Aleksandrov [15], §3, gave his proof by a variational argument that is mentioned above. A similar attempt, but with an insufficient variation argument, was made by Süss [1828]. Related existence theorems for infinite convex surfaces are stated without proof in Aleksandrov [25], pp. 305–306. A similar existence result for convex caps was obtained by Busemann [371].

For polytopes, Klain [1086] gave a modified treatment of the Minkowski problem. His proof yields simultaneously the Minkowski existence theorem and the Minkowski inequality with equality conditions.

2. *Minkowski's problem in differential geometry.* If K is a convex body of class C^2_+, then its area measure $S_{n-1}(K, \cdot)$ has a continuous density (with respect to spherical Lebesgue measure) given by s_{n-1}, the product of the principal radii of curvature. Its reciprocal value is the Gauss–Kronecker curvature, as a function of the outer normal. Thus Minkowski's existence theorem is related to the problem of the existence of a convex hypersurface with Gauss–Kronecker curvature prescribed as a function of the normal vector. Classical contributions to this problem are the papers by Lewy [1209], Miranda [1443] and Nirenberg [1474]. If one wants to utilize the general existence result, Theorem 8.2.2, to solve this problem, one needs regularity results for convex surfaces with sufficiently smooth curvature functions. For such results and for more information on Minkowski's problem and related problems from the viewpoint of differential geometry, we refer to the books of Pogorelov [1538, 1539], to Gluck [722] and the survey article of Gluck [723], to Cheng and Yau [418] and to the regularity results for Monge–Ampère equations due to Caffarelli. It follows from Caffarelli's work (see Theorem 0.7 in Jerison [1036] and the references given there) that, for given nonnegative integer k and $0 < \alpha < 1$, the solution is of class $C^{k+2,\alpha}$ if the given Gauss curvature on the spherical image is of class $C^{k,\alpha}$.

   A solution of Minkowski's problem in the smooth category by means of a modified Gauss curvature flow was obtained by Chou and Wang [427].
3. *Analogues of Minkowski's existence theorem.* Minkowski's existence theorem and its proofs have been the model for similar investigations, in which the volume and the surface area measure are replaced by a physical quantity and a measure derived from its first variation. In this sense, Jerison [1036] treated a Minkowski problem for electrostatic capacity. In Jerison [1037], this and similar problems are approached via direct methods of the calculus of variations. This approach is systematized by Colesanti [435], who also explains the connection to Brunn–Minkowski type inequalities for variational functionals (cf. Note 15 of §7.1). In this way, the Minkowski problem for the torsional rigidity is solved by Colesanti and Fimiani [440].
4. *Algorithmic version*. The essential idea of the proof of Theorem 8.2.1 is the minimization of the linear functional Φ over the region $M \subset \mathbb{R}^N$, which is convex by the Brunn–Minkowski theorem. Known iterative methods for such optimization problems can therefore be used to derive algorithms for a constructive solution of Minkowski's problem for polytopes. Such an algorithm in $\mathbb{R}^3$, of interest for computer vision problems, was described by Little [1227]. The role of possible applications to robot vision is discussed in Horn [989], Chapter 16. The algorithmic complexity of Minkowski's reconstruction problem was investigated by Gritzmann and Hufnagel [777]. The survey article by Alexandrov, Kopteva and Kutateladze [57] describes an algorithm for the construction of three-dimensional polytopes with given outer normals and areas of the facets, and presents some pictures of Blaschke sums of polytopes.

   An algorithm for the numerical treatment of Minkowski's problem with smooth data in $\mathbb{R}^3$, using approximation of the support function by finite sums of spherical harmonics, was suggested by Lamberg and Kaasalainen [1167].
5. *Applications.* An elegant application of Minkowski's existence theorem, Blaschke addition and Minkowski's inequality to a problem in discrete geometry is made in a paper by Böröczky, Bárány, Makai and Pach [291]. Another application is in Fáry and Makai [549].
6. *Applications in stochastic geometry.* Minkowski's existence theorem has found applications in stochastic geometry. The principal idea is to associate with a measure on the unit sphere, which may appear as a directional distribution of a stationary random hyperplane process or the mean area measure of a particle process, an auxiliary convex body with

this measure as area measure. It may then be possible to apply results on convex bodies in order to obtain information on the original random structures. A typical example is Theorem 4.6.9 in Schneider and Weil [1740]. We refer to that book for some other uses of such an auxiliary convex body (see, e.g., [1740], Note 1 of §4.6 and the end of Note 9 of §10.5, on page 514), which is usually called a *Blaschke body*, because of its connection with Blaschke addition. The application of Minkowski's existence theorem to random hyperplane systems was initiated in Schneider [1697] and has more recently played a role in the investigation of random hyperplane mosaics; see Hug and Schneider [1017, 1018].

7. *Blaschke addition.* The addition of convex bodies now called Blaschke addition is mentioned briefly in Blaschke [241], p. 112. For the case of polytopes it had occurred already in the work of Minkowski [1435], p. 117. Theorem 8.2.3 was proved by Kneser and Süss [1122]; weaker forms had been proved previously by Herglotz (unpublished) and Süss [1830].

   Blaschke sums have been investigated under various aspects and have been applied in several different ways. We refer to the work of Firey and Grünbaum [610], Firey [589, 592, 594], Grünbaum [848], Chapter 15.3, Schneider [1656, 1705], Chakerian [400], Kutateladze and Rubinov [1162], Kutateladze [1160, 1161], Goikhman [726], Bronshtein [344], Goodey and Schneider [742], McMullen [1385], Lutwak [1272, 1274, 1277, 1281], Campi, Colesanti and Gronchi [383], Weil [1951], Goodey, Kiderlen and Weil [740]. An addition of Blaschke type is also used in Bronshtein [343].

8. *A characterization of Blaschke addition.* Gardner, Parapatits and Schuster [681] characterized the Blaschke addition on the set $\mathcal{K}^n_{(os)}$ of o-symmetric convex bodies in $\mathcal{K}^n_n$ in the following way. On $\mathcal{K}^n_{(os)}$, the Lévy–Prokhorov metric δ_{LP} is defined by

$$\delta_{LP}(K, L) := \delta_P(S_{n-1}(K, \cdot), S_{n-1}(L, \cdot)) \qquad \text{for } K, L \in \mathcal{K}^n_{(os)},$$

   where δ_P is defined below, before Theorem 8.5.3.

   Theorem If $n \geq 3$, then $* : \mathcal{K}^n_{(os)} \times \mathcal{K}^n_{(os)} \to \mathcal{K}^n_{(os)}$ is uniformly continuous in the Lévy–Prokhorov metric and $\mathrm{GL}(n)$ covariant if and only if for all $K, L \in \mathcal{K}^n_{(os)}$ we have either $K * L = aK$ for some $a > 0$, or $K * L = bL$ for some $b > 0$, or $K * L = aK \# bL$ for some $a, b > 0$.

   The authors gave examples to show the necessity of the assumptions, in particular, that the uniform continuity cannot be replaced by continuity.

9. *Mixed bodies*. For convex bodies $K_1, \dots, K_{n-1} \in \mathcal{K}^n_n$, the mixed area measure

$$S(K_1, \dots, K_{n-1}, \cdot)$$

   satisfies the assumptions of Minkowski's theorem; hence there is a convex body $[K_1, \dots, K_{n-1}]$, unique up to a translation, for which

$$S_{n-1}([K_1, \dots, K_{n-1}], \cdot) = S(K_1, \dots, K_{n-1}, \cdot).$$

   $[K_1, \dots, K_{n-1}]$ is called the *mixed body* of $K_1, \dots, K_{n-1}$. Mixed bodies were introduced, in special cases, by Firey [592], and were extensively studied by Lutwak [1272], who obtained a number of inequalities for these bodies.

10. *An existence result for the curvature measure* C_0. For a convex body $K \in \mathcal{K}^n_{(o)}$, let $f(u) := \rho(K, u)u$ for $u \in \mathbb{S}^{n-1}$, so that $f(u) \in \mathrm{bd}\, K$.

   Theorem Let κ be a measure on $\mathcal{B}(\mathbb{S}^{n-1})$. There exists a convex body $K \in \mathcal{K}^n_{(o)}$ for which $C_0(K, f(\omega)) = \kappa(\omega)$ for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ if and only if the following two conditions are satisfied:

(a) $\kappa(\mathbb{S}^{n-1}) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$;
(b) $\kappa(\mathbb{S}^{n-1} \setminus \omega) > \mathcal{H}^{n-1}(\omega^*)$ whenever $\omega \subset \mathbb{S}^{n-1}$ is spherically convex and ω^* denotes the set polar to ω.

The body K is unique up to a dilatation.

Existence was proved by Aleksandrov [17], first for polytopes and then by approximation in general. The polytopal case is an existence result for polytopes with vertices on given rays through o and preassigned curvatures at these vertices. Uniqueness up to a dilatation was shown in Aleksandrov [21].

Analogous results for unbounded convex surfaces and orthogonal projection onto a plane were obtained by Aleksandrov [21] and [25], Chapter IX. Oliker [1492] treated, by means of a variational method, an existence problem similar to Aleksandrov's theorem.

The theorems of Minkowski and Aleksandrov are existence results, which a priori give no additional information on the convex bodies whose existence they ensure. It is an interesting problem to find conditions on the given measures from which conclusions on the shape of the solution bodies can be drawn. For Aleksandrov's theorem, such a result was obtained by Treibergs [1851]. For $\omega \subset \mathbb{S}^{n-1}$ and given $\alpha, 0 < \alpha < \pi/2$, let ω_α be the set of all points of $\mathbb{S}^{n-1}$ at spherical distance $\leq \alpha$ from ω. Treibergs showed that if in Aleksandrov's theorem stated above one assumes in addition that

$$\kappa(\omega) \leq \mathcal{H}^{n-1}(\omega_\alpha) \quad \text{for all } \omega \in \mathcal{B}(\mathbb{S}^{n-1}),$$

then the body K satisfies

$$\frac{\sup\{\rho(K,u) : u \in \mathbb{S}^{n-1}\}}{\inf\{\rho(K,u) : u \in \mathbb{S}^{n-1}\}} \leq c(\alpha, n)$$

with a real constant $c(\alpha, n)$ depending only on α and n.

8.3 The area measure of order one

The area measure of order one of the convex body $K \in \mathcal{K}^n$ is the special mixed area measure

$$S_1(K, \cdot) = S(K, B^n, \dots, B^n).$$

By (5.27) it has the important linearity property

$$S_1(\lambda K + \mu L, \cdot) = \lambda S_1(K, \cdot) + \mu S_1(L, \cdot) \tag{8.23}$$

for $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$. For this reason, the area measure of order one exhibits some special features and suggests a separate treatment, which does not make use of the theory of mixed volumes. We remark that the Minkowski linearity (8.23) can also be deduced, first for bodies of class C^2_+ from (4.26) and the linearity of s_1, which is evident from (2.54), and then in general by approximation, using the weak continuity of the area measures.

8.3.1 The length measure in the plane

First we consider the area measure in the plane. Here we use the special notation

$$S_1(K, \cdot) =: S_K \qquad \text{for } K \in \mathcal{K}^2$$

and call S_K the *length measure* of K. The reason is clear: for a Borel set $\omega \subset \mathbb{S}^1$, the value $S_K(\omega)$ is the length of the boundary part of K which is the reverse spherical image $\tau(K, \omega)$.

In the plane, Minkowski's existence theorem takes the following form.

Theorem 8.3.1 *Let φ be a finite Borel measure on $\mathbb{S}^1$ satisfying*

$$\int_{\mathbb{S}^1} u \, \mathrm{d}\varphi(u) = o. \tag{8.24}$$

Then there exists a convex body $K \in \mathcal{K}^2$ for which $S_K = \varphi$. It is uniquely determined up to a translation.

The difference from the n-dimensional case is that (8.24) is the only restriction on the measure φ. In fact, suppose that φ does not satisfy the second condition of Theorem 8.2.2, thus φ is concentrated at two antipodal points z and $-z$ of $\mathbb{S}^1$. It must assign equal measure ξ to each of these points, hence a segment of length ξ orthogonal to z has length measure φ. This segment is uniquely determined by φ, up to translation; therefore, the uniqueness part of Minkowski's theorem in the plane involves no dimensional restriction. Concerning the existence part of the proof, we remark that the polygonal case (the two-dimensional case of Theorem 8.2.1) is almost trivial; compare the construction after (3.23).

The linearity of S_K allows the following corollary to Minkowski's theorem.

Corollary 8.3.2 *Let $K, M \in \mathcal{K}^2$. If*

$$S_K(\omega) \leq S_M(\omega)$$

for all $\omega \in \mathcal{B}(\mathbb{S}^1)$, then K is a summand of M.

Proof By assumption, $\varphi := S_M - S_K \geq 0$, and, since $\int_{\mathbb{S}^1} u \, \mathrm{d}\varphi(u) = o$, there exists $L \in \mathcal{K}^2$ with $\varphi = S_L$. Since $S_{K+L} = S_K + S_L = S_M$, the bodies $K + L$ and M are translates of each other, which gives the assertion. □

We choose an orthonormal basis (e_1, e_2) of $\mathbb{R}^2$ and represent each unit vector $u \in \mathbb{S}^1$ in the form

$$u = (\cos\theta_u)e_1 + (\sin\theta_u)e_2, \qquad 0 \leq \theta_u < 2\pi.$$

For a convex body $K \in \mathcal{K}^2$ of class C^2_+, formula (4.26) reduces to

$$S_K(\omega) = \int_\omega r_K \, \mathrm{d}\mathcal{H}^1,$$

where r_K denotes the radius of curvature function of K, and by (2.60) this is given by

$$r_K(u) = (h'' + h)(\theta_u), \qquad h(\theta_u) := h_K(u).$$

It is, therefore, not surprising that the support function of K can be obtained by a simple integration from the length measure of K. For a general convex body $K \in \mathcal{K}^2$, this representation reads as follows.

Theorem 8.3.3 *Let $K \in \mathcal{K}^2$. Determine x_0 by*

$$F(K, e_1) = \operatorname{conv}\{x_0, x_0 + S_K(\{e_1\})e_2\}$$

and let $N_u := \{v \in \mathbb{S}^1 : 0 \le \theta_v < \theta_u\}$. Then

$$h(K, u) = \langle x_0, u\rangle + \int_{N_u} \sin(\theta_u - \theta_v)\, \mathrm{d}S_K(v) \tag{8.25}$$

for $u \in \mathbb{S}^1$.

Proof If K is a polygon, the assertion follows by elementary geometry. Let $K \in \mathcal{K}^2$ be arbitrary. For $u = e_1$, formula (8.25) is correct. Let $u \in \mathbb{S}^1 \setminus \{e_1\}$ be given. We can choose a sequence $(P_i)_{i\in\mathbb{N}}$ of polygons converging to K and such that

$$F(K, e_1) = F(P_i, e_1), \qquad F(K, u) = F(P_i, u) \tag{8.26}$$

and hence

$$S_K(\{e_1\}) = S_{P_i}(\{e_1\}), \qquad S_K(\{u\}) = S_{P_i}(\{u\}) \tag{8.27}$$

for all $i \in \mathbb{N}$. We define the restrictions

$$\mu := S_K \llcorner \operatorname{int} N_u, \qquad \mu_i := S_{P_i} \llcorner \operatorname{int} N_u, \quad i \in \mathbb{N}.$$

The sequence $(S_{P_i})_{i\in\mathbb{N}}$ converges weakly to S_K, by Theorem 4.2.1. By (8.27), the sequence $(S_{P_i}(\cdot \setminus \{e_1, u\}))_{i\in\mathbb{N}}$ also converges weakly to $S_K(\cdot \setminus \{e_1, u\})$. If $\omega \subset \mathbb{S}^1$ is a Borel set whose boundary has μ measure zero, then the boundary of $\omega \cap \operatorname{int} N_u$ has $S_K(\cdot \setminus \{e_1, u\})$ measure zero, hence $\lim_{i\to\infty} \mu_i(\omega) = \mu(\omega)$. Thus, the sequence $(\mu_i)_{i\in\mathbb{N}}$ converges weakly to μ. From this and from (8.25) for P_i, (8.26) and (8.27) we get, for $i \to \infty$,

$$\begin{aligned} h(P_i, u) &= \langle x_0, u\rangle + \int_{\mathbb{S}^1} \sin(\theta_u - \theta_v)\, \mathrm{d}\mu_i(v) + \int_{\{e_1\}} \sin(\theta_u - \theta_v)\, \mathrm{d}S_K(v) \\ &\to \langle x_0, u\rangle + \int_{\mathbb{S}^1} \sin(\theta_u - \theta_v)\, \mathrm{d}\mu(v) + \int_{\{e_1\}} \sin(\theta_u - \theta_v)\, \mathrm{d}S_K(v) \\ &= \langle x_0, u\rangle + \int_{N_u} \sin(\theta_u - \theta_v)\, \mathrm{d}S_K(v). \end{aligned}$$

Together with $h(P_i, u) \to h(K, u)$, this completes the proof. □

The following corollary is similar to Corollary 8.3.2, but with a weaker assumption we have only a weaker conclusion.

Corollary 8.3.4 *Let $K, M \in \mathcal{K}^2$. If there exists $z_0 \in \mathbb{S}^1$ such that*

$$S_K(\omega) \le S_M(\omega)$$

for all $\omega \in \mathcal{B}(\mathbb{S}^1)$ with $z_0 \notin \omega$, then K is contained in a translate of M.

Proof We choose the orthonormal basis (e_1, e_2) of $\mathbb{R}^2$ such that $e_1 = -z_0$. By Theorem 8.3.3 we can assume, after applying suitable translations to K and M, that

$$h(K,u) = \int_{N_u} \sin(\theta_u - \theta_v)\, \mathrm{d}S_K(v), \quad h(M,u) = \int_{N_u} \sin(\theta_u - \theta_v)\, \mathrm{d}S_M(v) \tag{8.28}$$

for $u \in \mathbb{S}^1$. From

$$\int_{\mathbb{S}^1} v\, \mathrm{d}S_K(v) = o$$

we obtain also that

$$h(K,u) = -\int_{\mathbb{S}^1 \setminus N_u} \sin(\theta_u - \theta_v)\, \mathrm{d}S_K(v), \quad h(M,u) = -\int_{\mathbb{S}^1 \setminus N_u} \sin(\theta_u - \theta_v)\, \mathrm{d}S_M(v) \tag{8.29}$$

for $u \in \mathbb{S}^1$. Let $P := \{v \in \mathbb{S}^1 : 0 \leq \theta_v < \pi\}$. For $u \in P$ we deduce from (8.28) that $h(K,u) \leq h(M,u)$, and for $u \in \mathbb{S}^1 \setminus P$ we draw from (8.29) the same conclusion. By the continuity of the support function, the inequality extends to $u = z_0$. This shows that $K \subset M$. □

The length measure is an appropriate tool to treat some questions involving Minkowski addition of planar convex bodies. The next theorem shows that the ordered semigroup $(\mathcal{K}^2, +, \subset)$ has some formal properties in common with the multiplicative ordered semigroup of positive integers, in so far as there are analogues of the greatest common divisor and the least common multiple. We emphasize, however, that these results do not extend to higher dimensions.

Recall that the convex body A is a summand of the convex body K, and K is an antisummand of A, if there exists a convex body B with $A + B = K$.

Theorem 8.3.5 *Let $K, M \in \mathcal{K}^2$.*

There exists a common summand C of K and M such that any common summand of K and M is contained in a translate of C.

There exists a common antisummand D of K and M such that any common antisummand of K and M contains a translate of D.

The bodies C and D are uniquely determined up to translations.

Proof Let (P, N) be a Hahn decomposition for the signed measure $S_K - S_M$. Thus, P and N are disjoint Borel subsets of $\mathbb{S}^1$ with $P \cup N = \mathbb{S}^1$ and such that $(S_K - S_M)(\omega \cap P) \geq 0$ and $(S_K - S_M)(\omega \cap N) \leq 0$ for all $\omega \in \mathcal{B}(\mathbb{S}^1)$. We define a measure μ by

$$\mu(\omega) := S_K(\omega \cap N) + S_M(\omega \cap P), \qquad \omega \in \mathcal{B}(\mathbb{S}^1).$$

For $\omega \in \mathcal{B}(\mathbb{S}^1)$,

$$\mu(\omega) = S_K(\omega \cap N) + S_M(\omega \cap P) \leq S_K(\omega \cap N) + S_K(\omega \cap P) = S_K(\omega),$$
$$\mu(\omega) = S_K(\omega \cap N) + S_M(\omega \cap P) \leq S_M(\omega \cap N) + S_M(\omega \cap P) = S_M(\omega).$$

We can determine a unit vector $z \in \mathbb{S}^1$ and a number $\xi \geq 0$ with

$$\int_{\mathbb{S}^1} u \, \mathrm{d}(\mu + \xi\delta_z)(u) = o, \tag{8.30}$$

where δ_z is the Dirac measure at z (that is, $\delta_z(\omega) = \mathbf{1}_\omega(z)$ for $\omega \in \mathcal{B}(\mathbb{S}^1)$). There exists a convex body $L \in \mathcal{K}^2$ with $S_L = \mu + \xi\delta_z$. Its face $F(L, z)$ has length at least ξ. Hence, if t denotes a vector orthogonal to z of length ξ, then the convex body $C := L \cap (L + t)$ satisfies $S_C(\{z\}) = \mu(\{z\})$. For $\omega \in \mathcal{B}(\mathbb{S}^1)$, the relation $S_C(\omega) \leq S_L(\omega) = \mu(\omega)$ holds if either ω or $-\omega$ is contained in $\{u \in \mathbb{S}^1 : \langle u, t\rangle > 0\}$. Hence, we have

$$S_C(\omega) \leq \mu(\omega) \quad \text{if } \omega \in \mathcal{B}(\mathbb{S}^1),\ -z \notin \omega. \tag{8.31}$$

If $S_C(\{-z\}) > 0$, then $F(C, -z) = F(L, -z) \cap F(L + t, -z)$ and, therefore, $S_C(\{-z\}) \leq S_L(\{-z\}) = \mu(\{-z\})$. The inequality $S_C(\{-z\}) \leq \mu(\{-z\})$ holds trivially if $S_C(\{-z\}) = 0$. Altogether, we have obtained that $S_C(\omega) \leq \mu(\omega)$ for $\omega \in \mathcal{B}(\mathbb{S}^1)$. Since $\mu \leq S_K$ and $\mu \leq S_M$, the body C is a common summand of K and M, by Corollary 8.3.2.

Let A be any common summand of K and M. Then $S_A \leq S_K$, $S_A \leq S_M$ and hence, for $\omega \in \mathcal{B}(\mathbb{S}^1)$,

$$S_A(\omega) = S_A(\omega \cap N) + S_A(\omega \cap P) \leq S_K(\omega \cap N) + S_M(\omega \cap P) = \mu(\omega)$$

and thus $S_A \leq S_L$. Therefore, A is a summand of L.

Let $x_0, x_0 + \lambda t$ with $\lambda \geq 1$ be the endpoints of the face $F(L, z)$. By Theorem 3.2.2 there exist vectors v, w such that $x_0 \in A + v \subset L$ and $x_0 + \lambda t \in A + w \subset L$. From $F(A + v, z) \subset F(L, z)$ and $F(A + w, z) \subset F(L, z)$ together with $S_L(\{z\}) = \mu(\{z\}) + \xi \geq S_A(\{z\}) + \xi$ we deduce that $A + v + t \subset L$. Therefore, $A + v + t \subset L \cap (L + t) = C$. This proves the existence of a largest common summand C. Its uniqueness up to translations is clear from the definition.

In order to prove the existence of a smallest common antisummand, we define

$$\nu := S_K + S_M - \mu. \tag{8.32}$$

Thus,

$$\nu(\omega) = S_K(\omega \cap P) + S_M(\omega \cap N)$$

for $\omega \in \mathcal{B}(\mathbb{S}^1)$, hence ν is a positive measure, and similarly as above we obtain that $\nu \geq S_K$ and $\nu \geq S_M$. From (8.32) and (8.30) we get

$$\int_{\mathbb{S}^1} u \, \mathrm{d}(\nu + \xi\delta_{-z})(u) = o.$$

Therefore, there exists a convex body $D \in \mathcal{K}^2$ with $S_D = \nu + \xi\delta_{-z}$. From $S_K \leq \nu \leq S_D$ and $S_M \leq \nu \leq S_D$ it follows that D is a common antisummand of K and M.

Let B be any common antisummand of K and M. Then $S_K \leq S_B$, $S_M \leq S_B$ and, for $\omega \in \mathcal{B}(\mathbb{S}^1)$,

$$S_B(\omega) = S_B(\omega \cap P) + S_B(\omega \cap N) \geq S_K(\omega \cap P) + S_M(\omega \cap N) = \nu(\omega)$$

and thus

$$S_B(\omega) \geq S_D(\omega) \quad \text{for } \omega \in \mathcal{B}(\mathbb{S}^1) \text{ with } -z \notin \omega.$$

By Corollary 8.3.4, the body B contains a translate of D. This proves the existence of D, and the uniqueness up to translations is again clear. □

8.3.2 The Christoffel problem

Now we turn to the first-order area measure $S_1(K, \cdot)$ in n-dimensional space. First we give a proof for the unique determination of a convex body K by this measure, up to translations, which does not use the theory of mixed volumes. It appears that the shortest proof of this fact is the one using spherical harmonics. In the following, σ denotes spherical Lebesgue measure on $\mathbb{S}^{n-1}$.

Theorem 8.3.6 *If $K, L \in \mathcal{K}^n$ are convex bodies with $S_1(K, \cdot) = S_1(L, \cdot)$, then K and L are translates of each other.*

Proof Let Y_m be a spherical harmonic of degree m. First let K be of class C^2_+. Then it follows from (4.26) and (2.56) that

$$\int_{\mathbb{S}^{n-1}} Y_m(u)\, S_1(K, \mathrm{d}u) = \int_{\mathbb{S}^{n-1}} Y_m \left(h_K + \frac{1}{n-1} \Delta_S h_K \right) \mathrm{d}\sigma.$$

Using Green's formula, $(Y_m, \Delta_S h_K) = (\Delta_S Y_m, h_K)$ and $\Delta_S Y_m = -m(m + n - 2) Y_m$ (Appendix equation (A.2)), we obtain

$$\int_{\mathbb{S}^{n-1}} Y_m(u)\, S_1(K, \mathrm{d}u) = \lambda_{n,m} \int_{\mathbb{S}^{n-1}} h_K Y_m \,\mathrm{d}\sigma \tag{8.33}$$

with a real number $\lambda_{n,m}$ satisfying $\lambda_{n,m} \neq 0$ for $m \neq 1$. By approximation, (8.33) extends to arbitrary convex bodies. The assumption of the theorem now yields

$$\lambda_{n,m} \int_{\mathbb{S}^{n-1}} (h_K - h_L) Y_m \,\mathrm{d}\sigma = 0.$$

Since $m \in \{0, 1, \dots\}$ was arbitrary, it follows from the completeness of the system of spherical harmonics that $\bar{h}_K - \bar{h}_L$ is a spherical harmonic of degree 1, hence $h_K - h_L = \langle \cdot, t \rangle$ with $t \in \mathbb{R}^n$. This is equivalent to $K = L + t$. □

The n-dimensional existence problem for S_1 consists in finding necessary and sufficient conditions for a Borel measure φ on $\mathbb{S}^{n-1}$ in order that there exists a convex body $K \in \mathcal{K}^n$ for which $S_1(K, \cdot) = \varphi$. In the smooth version, one asks which conditions a real function f on $\mathbb{S}^{n-1}$ must satisfy in order that there exists a (sufficiently smooth) convex body K such that f is equal to $s_1(K, \cdot)$, the mean radius of curvature of K as a function of the outer unit normal vector. The three-dimensional case was treated by Christoffel [429]; the general case is, therefore, now called *Christoffel's problem*. In the smooth case, this problem comes down to solving the elliptic linear differential equation

$$\frac{1}{n-1}\Delta_S h + h = f \tag{8.34}$$

on $\mathbb{S}^{n-1}$ and expressing the condition that the solution h has to be the restriction of a support function in terms of conditions on the given function f (the latter was not done by Christoffel). An approach via a Green function is possible, also in the general case. In different versions, this approach was followed, independently, by Firey [596] and Berg [198]. Firey based his treatment on the differential equation $\Delta\xi(u) = \nabla f(u)$, where $\xi(u)$ is defined as in Section 2.5 and f is continuously differentiable and is extended to $\mathbb{R}^n \setminus \{o\}$ as a positively homogeneous function of degree -1. Berg embedded his treatment of the Christoffel problem in a theory of potentials and subharmonic functions on the sphere $\mathbb{S}^{n-1}$. He constructed real C^∞ functions g_n on $(-1,1)$, for $n = 2,3,\dots$, such that

$$h(K,u) = \int_{\mathbb{S}^{n-1}} g_n(\langle u,v\rangle)\, S_1(K,\mathrm{d}v) + \langle s(K),u\rangle \tag{8.35}$$

for $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$.

From this result and the fact that

$$\int_{\mathbb{S}^{n-1}} |g_n(\langle u,v\rangle)|\,\mathrm{d}v < \infty, \tag{8.36}$$

also shown in [198], a stability version of the uniqueness theorem 8.3.6 can be derived, as follows.

Theorem 8.3.7 *If $K, K' \in \mathcal{K}^n$ are convex bodies with coinciding Steiner points and satisfying*

$$|S_1(K,\omega) - S_1(K',\omega)| \le \varepsilon\mathcal{H}^{n-1}(\omega) \tag{8.37}$$

for all $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ and a given number $\varepsilon > 0$, then

$$\delta(K,K') \le c_n\varepsilon$$

with some constant c_n depending only on the dimension n.

Proof From (8.35) and the assumption $s(K) = s(K')$ we obtain

$$|h(K,u) - h(K',u)| \le \int_{\mathbb{S}^{n-1}} |g_n(\langle u,v\rangle)|\,\mathrm{d}\mu(v),$$

where μ is the variation of the signed measure $S_1(K,\cdot) - S_1(K',\cdot)$. The assumption (8.37) yields $\mu(\omega) \le \varepsilon\mathcal{H}^{n-1}(\omega)$. The existence of the constant c_n now follows from (8.36). □

A short approach to the result (8.35) appears in Grinberg and Zhang [775]. Their formulation of the solution of Christoffel's problem, due to Berg and Firey in different versions, is equivalent to the following one.

Theorem 8.3.8 *A nonnegative measure φ on $\mathbb{S}^{n-1}$ is the first-order area measure of a convex body if and only if*

$$h(u) = \int_{\mathbb{S}^{n-1}} g_n(\langle u, v\rangle)\, \mathrm{d}\varphi(v), \qquad u \in \mathbb{S}^{n-1}, \tag{8.38}$$

is the restriction of a support function.

Of course, this 'solution' of Christoffel's problem has a slightly formal character, as long as there is no efficient criterion on φ to decide whether the convolution of φ with g_n given by (8.38) yields a support function. But perhaps one cannot expect such a criterion.

The situation is different in the smooth case and for polytopes. In the first case, a sufficient but not necessary condition was found by Pogorelov [1536] (see also [1538], p. 442). It holds for measures with a sufficiently smooth density and has the form of an inequality for the second directional derivatives of this density.

For polytopes, Christoffel's problem can be given an independent treatment by direct elementary methods. This has been done by Schneider [1682]. We formulate here the result, referring to [1682] for the proof. Let $n \geq 3$. By a spherical complex we understand a finite system $\mathscr{C}$ of spherically convex polytopes in $\mathbb{S}^{n-1}$ (intersections of $\mathbb{S}^{n-1}$ with pointed convex polyhedral cones in $\mathbb{R}^n$), called the *cells* of $\mathscr{C}$, with the following properties (a), (b), (c).

(a) Each face of a cell of $\mathscr{C}$ is a cell of $\mathscr{C}$.
(b) The intersection of any two cells of $\mathscr{C}$ is either empty or a cell of $\mathscr{C}$.
(c) The union of all cells of $\mathscr{C}$ is $\mathbb{S}^{n-1}$.

Let $\mathscr{C}$ be such a spherical complex. Let Z be any $(n-3)$-dimensional cell of $\mathscr{C}$ and C any $(n-2)$-dimensional cell of Z containing it. Then there is a unique unit vector $u(Z, C)$ orthogonal to $\operatorname{lin} Z$ which points from an arbitrary point in $\operatorname{relint} Z$ into the relative interior of C. The first-order area measure $\varphi = S_1(P, \cdot)$ of a polytope $P \in \mathcal{K}^n$ has the following properties.

(A) The support of φ is the union of the $(n-2)$-dimensional cells of some spherical complex $\mathscr{C}$.
(B) To each $(n-2)$-dimensional cell C of $\mathscr{C}$ there is a positive number $\alpha(C)$ such that $\varphi \llcorner C = \alpha(C)\mathcal{H}^{n-2} \llcorner C$.
(C) For each $(n-3)$-dimensional cell Z of $\mathscr{C}$,

$$\sum \alpha(C) u(Z, C) = o,$$

where the summation extends over the $(n-2)$-dimensional cells of $\mathscr{C}$ containing Z.

Theorem 8.3.9 *The conditions (A), (B), (C) are necessary and sufficient for a finite Borel measure φ on the sphere $\mathbb{S}^{n-1}$ to be the first-order area measure of a polytope.*

Notes for Section 8.3

1. *The length measure in the plane*. Although Berg's formula (8.35) provides an integral representation of the support function in terms of the first-order area measure in all dimensions, we have treated Theorem 8.3.3 separately, giving a complete proof. Formula (8.25) is not very common in the convexity literature. One finds it in Levin [1204] (chap. 1, §19) and Letac [1202]. The latter paper is a detailed study of the length measure in the plane. Our proof of Theorem 8.3.3 and Corollary 8.3.4 follows Bauer [177]; the Corollary first appeared, with a different proof, in Scholtes [1743], as an extension of a Lemma by Aleksandrov ([25], p. 239).

   The length measure has repeatedly been used for treating questions on the Minkowski addition of planar convex bodies. For example, Kallay [1058] characterized the extreme convex sets K in the set of bodies in $\mathcal{K}^2$ with a given width function in terms of a property of the Radon–Nikodym derivative of the length measure S_K.

   The length measure is particularly useful for the treatment of asymmetry classes and of minimal and reduced pairs in the plane. Here we continue Notes 11, 12, 13 from Section 3.2, and refer to these for the terminology.

   Asymmetry classes. Let $K \in \mathcal{K}^2$, and let ν^+ and ν^- be the positive and negative parts of the signed measure $S_K - S_{-K}$, the difference of the length measures of K and $-K$. Then ν^+ is the length measure of a convex body $M \in \mathcal{K}^2$, and it turns out that M is a strongly minimal member of the asymmetry class $[K]$ (Schneider [1674]).

   Minimal pairs. The following is Bauer's [177] treatment of minimal pairs in the plane. Let $K \in \mathcal{K}^2$, and let $S_K - S_M = \nu^+ - \nu^-$ be the Jordan decomposition. One can determine a vector $z \in \mathbb{S}^1$ and a number $\xi \geq 0$ such that $\nu^+ + \xi\delta_z$ (with the Dirac measure δ_z) and $\nu^- + \xi\delta_z$ satisfy the assumption of Minkowski's theorem, hence there exist convex bodies L^+, L^- with these length measures. They can be chosen such that $(L^+, L^-) \sim (K, M)$. It turns out that (L^+, L^-) is a minimal member of the equivalence class $[K, M]$ and is unique up to translations. Bauer deduced that the pair (K, M) is minimal if and only if there is a point $z \in \mathbb{S}^1$ such that the restrictions of the length measures S_K and S_M to $\mathbb{S}^1 \setminus \{z\}$ are mutually singular.

   Reduced pairs. Similarly, Bauer proved that the pair (K, M) of convex bodies in $\mathcal{K}^2$ is reduced if and only if the length measures S_K and S_M are mutually singular.

   Largest common summands and smallest common antisummands. The first part of Theorem 8.3.5 is due to Grzybowski [852], with a different proof. The second part of this theorem follows from the result of Scholtes [1743], also proved in a different way, that inclusion-minimal common antisummands in the plane are unique up to translations.

2. *Christoffel's problem*. It is clear from the introduction in Christoffel [429] that the surfaces that he studied are assumed to be convex ('allenthalben gewölbt'), but in the existence part of his investigation no such property is discussed (an erroneous statement to the contrary was made in Bonnesen and Fenchel [284], p. 123). Favard [554] and Süss [1831, 1832] claimed to prove that the condition $\int u \, \mathrm{d}\varphi(u) = o$ is sufficient for the existence of a convex body K with $S_1(K, \cdot) = \varphi$ (under smoothness assumptions). However, Aleksandrov [14, 15] gave examples of positive measures on $\mathcal{B}(\mathbb{S}^{n-1})$, some even with analytic densities, which satisfy that necessary condition and are not the first area measure $S_1(K, \cdot)$ of any convex body K.

   The proof of the uniqueness result, Theorem 8.3.6, by means of spherical harmonics goes back essentially to Hurwitz [1024]. His proof for the three-dimensional case is reproduced in Blaschke [249], §95, and Pogorelov [1538] and was extended to higher dimensions by Kubota [1154].

For $n = 3$, a formula of type (8.35) with a Green function already appears in Blaschke [249], p. 232, though without proof and possibly with a wrong sign, compared to Berg [198], pp. 61 and 26.

A brief review of the various treatments of Christoffel's problem was given by Firey [609]. An application of Firey's [593] existence result to the study of surfaces of constant width appears in Fillmore [582].

In the centrally symmetric case, Goodey and Weil [747] found a connection between Christoffel's problem and spherical Radon transforms.

Goodey, Yaskin and Yaskina [755] used Koldobsky's [1136] Fourier transform techniques for a new approach to Berg's solution of the Christoffel problem. They obtained a more explicit representation of Berg's functions g_n, and also the following regularity result. Let $f \in C^{m,\alpha}(S^{n-1})$, $m \geq 0$, $0 < \alpha < 1$, and

$$h(u) := \int_{\mathbb{S}^{n-1}} g_n(\langle u, v\rangle) f(v)\,\mathrm{d}\sigma(v), \qquad u \in S^{n-1}.$$

Then $h \in C^{m+2,\beta}(\mathbb{S}^{n-1})$ for all $0 < \beta < \alpha$.

3. *Stability results*. Theorem 8.3.7, for the special case of three-dimensional convex bodies with twice continuously differentiable support functions, is due to Pogorelov [1538], p. 502; the general case is in Schneider [1693], Theorem (9.8).

   For convex bodies $K, K' \in \mathcal{K}^3$, Sen'kin [1768, 1769] proved the estimate

   $$|w(K,u) - w(K',u)| \leq \frac{1}{\pi} \sup_{\omega \subset \mathbb{S}^2} |S_1(K,\omega) - S_1(K',\omega)|$$

   for the width $w(\cdot, u)$ in any direction $u \in \mathbb{S}^2$. He used an elementary argument for strongly isomorphic polytopes and then Aleksandrov's approximation theorem 2.4.15.
4. Weil [1932] (Satz 4.7) showed the following. If the convex body $K \in \mathcal{K}^n$ satisfies $S_1(K,\cdot) \leq c\sigma$ with some constant c, where σ is the spherical Lebesgue measure on $\mathbb{S}^{n-1}$, then K is a summand of some ball (the converse assertion is trivial); in particular, K is strictly convex.
5. Campi and Gronchi [392] investigated the first-order projection body of a convex body $K \in \mathcal{K}^n$, that is, the zonoid with generating measure $S_1(K,\cdot)$ (up to a factor). They showed, in particular, that its support function is strictly convex if $n \geq 3$ and $\dim K \geq 3$.

8.4 The intermediate area measures

Minkowski's existence theorem shows, in particular, that for any two n-dimensional convex bodies K and M there exists another one, the Blaschke sum $K\#M$, whose surface area measure is the sum of the surface area measures of K and M. In this way, a new addition for convex bodies is defined. The Minkowski sum $K + M$ of K and M has the property that its first-order area measure is the sum of the first-order area measures of K and M. Hence, for $i = 1$ and $i = n-1$, the sum of two ith-order area measures is always an ith-order area measure. This is no longer true for $1 < i < n-1$. In fact, it is false in the following strong sense.

Theorem 8.4.1 *Let $1 < i < n-1$. For most pairs $(K, M) \in \mathcal{K}^n_n \times \mathcal{K}^n_n$ (in the Baire category sense), the sum $S_i(K,\cdot) + S_i(M,\cdot)$ is not the ith area measure of a convex body.*

Although this is a negative result, we give a proof (following [1719]), since it is one of the rare general results about the rather mysterious nature of the intermediate area measures. As for Theorem 8.1.7, the polytopal case is treated first. Polytopes $P, Q \in \mathcal{P}^n$ are said to be *in general relative position* if, for any two faces F of P and G of Q lying in parallel supporting hyperplanes of P and Q, respectively, their direction spaces $L(F)$ and $L(G)$ satisfy $L(F) \cap L(G) = \{o\}$. Here and in the following, $L(F)$ is the linear subspace parallel to the affine hull of the face F.

Theorem 8.4.2 *Let $P, Q \in \mathcal{P}^n_n$ be polytopes in general relative position. If $1 < i < n-1$, then $S_i(P, \cdot) + S_i(Q, \cdot)$ is not the ith area measure of a convex body.*

Proof We use the same notations as in the proof of Theorem 8.1.8 and in particular make use of

$$\operatorname{supp} S_i(P, \cdot) = \sigma_{n-1-i}(P). \tag{8.39}$$

Let $i \in \{2, \dots, n-2\}$, and let $P, Q \in \mathcal{P}^n_n$ be polytopes in general relative position. We assume, to the contrary, that there exists a convex body $K \in \mathcal{K}^n$ such that

$$S_i(P, \cdot) + S_i(Q, \cdot) = S_i(K, \cdot). \tag{8.40}$$

By Theorem 4.5.4, the body K is a polytope. Suppose, first, that $\sigma_{n-1-i}(P) \cap \sigma_{n-1-i}(Q) = \emptyset$. Then $\sigma_{n-1-i}(P) \cup \sigma_{n-1-i}(Q)$ is not connected. By (8.40), this set is the support of $S_i(K, \cdot)$ and hence equal to $\sigma_{n-1-i}(K)$. For $i < n-1$ and polytopes, this set is always connected, a contradiction. Therefore, we can assume in the following that there exists a vector

$$u \in \sigma_{n-1-i}(P) \cap \sigma_{n-1-i}(Q). \tag{8.41}$$

Then $\dim F(P, u) \geq i$ and $\dim F(Q, u) \geq i$. Since P and Q are in general relative position, it follows that

$$i \leq \frac{n-1}{2}. \tag{8.42}$$

By (8.41), $u \in N(P, F) \cap N(Q, G)$ for some i-faces F of P and G of Q. Suppose that

$$\operatorname{relint} N(P, F) \cap \operatorname{relint} N(Q, G) = \emptyset. \tag{8.43}$$

By Theorem 1.3.8, there exists an $(n-1)$-dimensional linear subspace H of $\mathbb{R}^n$ that properly separates the cones $N(P, F)$ and $N(Q, G)$. Then $u \in H$. The intersection $N(P, F) \cap H$ is of positive dimension and hence is the normal cone $N(P, F')$ of a proper face F' of P containing F. Similarly, $N(Q, G) \cap H$ is the normal cone $N(Q, G')$ of a proper face G' of Q containing G. Then $L(F')^\perp \subset H$ and $L(G')^\perp \subset H$, hence $H^\perp \subset L(F') \cap L(G')$. Since u is a normal vector for F' and for G', this is a contradiction to the assumption that P and Q are in general relative position. Thus (8.43) was false, and we can assume in the following that there exists a vector

$$u \in \operatorname{relint} N(P, F) \cap \operatorname{relint} N(Q, G). \tag{8.44}$$

If the cones $N(P, F)$ and $N(Q, G)$ do not span $\mathbb{R}^n$, then there is a linear subspace H of dimension $n-1$ containing them, hence $H^\perp \subset L(F) \cap L(G)$. Again, this contradicts the assumption that P and Q are in general relative position. Thus $N(P,F)$ and $N(Q,G)$ span $\mathbb{R}^n$, and it follows that

$$\dim\,[N(P,F)\cap N(Q,G)] = n-2i. \tag{8.45}$$

The polytope K satisfies

$$\sigma_{n-1-i}(K) = \sigma_{n-1-i}(P)\cup\sigma_{n-1-i}(Q). \tag{8.46}$$

By (8.44), we can choose a neighbourhood U of u in $\mathbb{R}^n$ such that

$$U\cap N(P,\overline{F}) = \emptyset, \qquad U\cap N(Q,\overline{G}) = \emptyset$$

for all i-faces $\overline{F}\neq F$ of P and all i-faces $\overline{G}\neq G$ of Q. Then, in view of (8.46),

$$\sigma_{n-1-i}(K)\cap U = [\nu(P,F)\cup\nu(Q,G)]\cap U. \tag{8.47}$$

Let $Z := F(K,u)$. By (8.47), u is contained in the normal cones of at least two i-faces of K, where the normal cones have an intersection of dimension $n-2i$, by (8.45). We deduce that $\dim Z \geq 2i \geq i+2$. We choose an $(i+1)$-face X of Z and then two i-faces F', F'' of X such that

$$\operatorname{lin} N(K,F') \neq \operatorname{lin} N(K,F''). \tag{8.48}$$

We have

$$\nu(K,F')\cap U \subset \sigma_{n-1-i}(K),$$

hence by (8.47)

$$\nu(K,F')\cap U \subset \nu(P,F)\cup\nu(Q,G),$$

similarly

$$\nu(K,F'')\cap U \subset \nu(P,F)\cup\nu(Q,G).$$

Since $\nu(K,F')\cap U$ is of dimension $n-1-i$, it must be contained either in $\nu(P,F)$ or in $\nu(Q,G)$, say

$$\nu(K,F')\cap U \subset \nu(P,F).$$

Similarly, $\nu(K,F'')\cap U$ is either contained in $\nu(P,F)$ or in $\nu(Q,G)$. The first case is impossible by (8.48), and the second case is impossible since then

$$\begin{aligned}\nu(K,X)\cap U &\subset \nu(K,F')\cap U \subset \nu(P,F),\\ \nu(K,X)\cap U &\subset \nu(K,F'')\cap U \subset \nu(Q,G),\end{aligned}$$

which would imply

$$\dim N(K,X) \leq \dim\,[N(P,F)\cap N(Q,G)] = n-2i,$$

thus $i+1 = \dim X \geq 2i$ and hence $i=1$, a contradiction. This shows that K cannot exist, which completes the proof. $\square$

Proof of Theorem 8.4.1 Let $\mathcal{A}$ denote the set of pairs $(K, M) \in \mathcal{K}^n_n \times \mathcal{K}^n_n$ for which $S_i(K, \cdot) + S_i(M, \cdot)$ is not the ith area measure of any convex body. The set of pairs of n-polytopes which are in general relative position is dense in $\mathcal{K}^n_n \times \mathcal{K}^n_n$. This follows from the fact that the polytopes are dense in $\mathcal{K}^n$ and that to given polytopes P and Q there exists, by Lemma 4.4.1, a rotation ρ arbitrarily close to the identity such that P and ρQ are in general relative position. Hence, Theorem 8.4.2 implies that $\mathcal{A}$ is dense in $\mathcal{K}^n_n \times \mathcal{K}^n_n$.

For $k \in \mathbb{N}$, we denote by $\mathcal{A}_k$ the set of pairs $(K, M) \in \mathcal{K}^n_n \times \mathcal{K}^n_n$ for which there exists a convex body $L \in \mathcal{K}^n$ such that

$$S_i(K, \cdot) + S_i(M, \cdot) = S_i(L, \cdot), \qquad \delta(L, K) \leq k.$$

To show that $\mathcal{A}_k$ is closed, let $((K_j, M_j))_{j\in\mathbb{N}}$ be a sequence in $\mathcal{A}_k$ converging to a pair $(K, M) \in \mathcal{K}^n_n \times \mathcal{K}^n_n$. For each $j \in \mathbb{N}$ we can choose a convex body L_j satisfying

$$S_i(K_j, \cdot) + S_i(M_j, \cdot) = S_i(L_j, \cdot) \tag{8.49}$$

and

$$\delta(L_j, K_j) \leq k. \tag{8.50}$$

From $K_j \to K$ and (8.50) it follows that the sequence $(L_j)_{j\in\mathbb{N}}$ is bounded. By the Blaschke selection theorem it contains a convergent subsequence. Changing the notation, we may assume that the sequence $(L_j)_{j\in\mathbb{N}}$ converges to a convex body L. From (8.49) and the weak continuity of the ith area measures we deduce that $S_i(K, \cdot) + S_i(M, \cdot) = S_i(L, \cdot)$, and from (8.50) we get $\delta(L, K) \leq k$. Thus $(K, M) \in \mathcal{A}_k$. This shows that $\mathcal{A}_k$ is closed. Evidently,

$$\mathcal{A} = \mathcal{K}^n_n \times \mathcal{K}^n_n \setminus \bigcup_{k\in\mathbb{N}} \mathcal{A}_k,$$

hence $\mathcal{A}$ is a dense G_δ set in $\mathcal{K}^n_n \times \mathcal{K}^n_n$. This completes the proof of Theorem 8.4.1. □

Theorem 8.4.2 describes a situation where the sum of two ith area measures is not an ith area measure. The next theorem (also from [1719]) exhibits a class of ith area measures which cannot be decomposed into a sum of other ith area measures except in a trivial way.

Theorem 8.4.3 *Let $i \in \{1, \dots, n-2\}$, and let $P \in \mathcal{P}^n_n$ be a polytope with the property that all its $(i+1)$-faces are simplices. If $K, M \in \mathcal{K}^n_n$ are convex bodies satisfying*

$$S_i(K, \cdot) + S_i(M, \cdot) = S_i(P, \cdot), \tag{8.51}$$

then K and M are homothetic to P.

Proof The relation (8.51) implies $S_i(K, \cdot) \leq S_i(P, \cdot)$, hence K is homothetic to P by Theorem 8.1.8. Similarly, M is homothetic to P. □

The case $i = 1$ of Theorem 8.4.3 yields nothing new, since a polytope all of whose two-dimensional faces are triangles is known to be indecomposable, by Corollary 3.2.17. For $i = n - 1$ and P not a simplex, however, decomposition in the sense of (8.51) is often possible in a non-trivial way; see Firey and Grünbaum [610].

Notes for Section 8.4

1. *The existence problem for the area measure of order i.* Necessary and sufficient conditions for a measure φ on $\mathcal{B}(\mathbb{S}^{n-1})$ to be the ith area measure $S_i(K, \cdot)$ of some convex body $K \in \mathcal{K}^n$, where $1 < i < n - 1$, are not known. That the necessary condition
$$\int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\varphi(u) = o$$
is not sufficient even if φ has an analytic density was shown by Aleksandrov [15]. There is a complete solution of the existence problem for the special case of sufficiently smooth bodies of revolution, due to Firey [600], and in a special case to Nádeník [1460]. Some necessary conditions follow from results of Firey [598] and Weil [1943] and from Theorem 4.5.3. The latter, together with a result of Larman and Rogers [1174] (applied to the polar body), implies that the support of an ith area measure is arcwise connected if $i < n - 1$ (Fedotov [563]).
Whether the sum of two ith area measures is always an ith area measure, if $i \in \{2, \ldots, n - 2\}$, was asked, more or less explicitly, by Firey [592, 599, 600, 606] and Chakerian [400]. Negative answers were given by Fedotov [562, 563] and, in a weaker form, [560], and independently by Goodey and Schneider [742]. The latter authors showed that even suitably chosen parallelepipeds with parallel edges provide counterexamples. In [560] and [742], the following was also proved. Let $P, P_0, P_1 \in \mathcal{P}^n_n$ be convex polytopes, $i \in \{1, \ldots, n - 1\}$, and suppose that
$$S_i(P, \cdot) = S_i(P_0, \cdot) + S_i(P_1, \cdot).$$
If π denotes orthogonal projection onto any $(i + 1)$-dimensional plane, then πP is the Blaschke sum (relative to the carrying plane) of πP_0 and πP_1.
2. *A positive result on sums of ith area measures.* Let $i \in 2, \ldots, n - 2$. While Theorems 8.4.1 and 8.4.3 demonstrate that in many cases the sum of ith area measures of convex bodies is not an ith area measure, there are also positive results, obtainable under sufficiently high smoothness assumptions and using the continuity method in PDE. In the second part of the paper by Zhang [2056], the following is proved. Let $\mathcal{F}^{2,\alpha}_e$ be the class of convex bodies in $\mathcal{K}^n_n$ whose support functions are of class $C^{2,\alpha}$ over $\mathbb{S}^{n-1}$ and are even; let $C^\alpha_e(\mathbb{S}^{n-1})$ be the class of even functions in the Hölder class $C^\alpha(\mathbb{S}^{n-1})$. If $s_i(M, \cdot)$ is the ith curvature function (the ith elementary symmetric function of the principal radii of curvature as a function of the outer unit normal vector) of $M \in \mathcal{F}^{2,\alpha}_e$, then there exists a neighbourhood of $s_i(M, \cdot)$ in $C^\alpha_e(\mathbb{S}^{n-1})$ consisting of ith curvature functions of convex bodies in $\mathcal{F}^{2,\alpha}_e$.
3. Since no general characterizing criterion for ith area measures is known if $i \in \{2, \ldots, n - 2\}$, it is of interest to investigate qualitatively the set $\mathcal{S}_i$ of all ith-order area measures of convex bodies in $\mathcal{K}^n_n$. This was done by Weil [1943] and Goodey [732]. Weil showed for $i < n - 1$ that $\mathcal{S}_i$ is not dense in $\mathcal{M}$, the set of all finite measures on $\mathcal{B}(\mathbb{S}^{n-1})$ with barycentre o, endowed with the weak topology, but that $\mathcal{S}_i - \mathcal{S}_i$ is dense in $\mathcal{M} - \mathcal{M}$. Goodey proved for $i < n - 1$ that $\mathcal{S}_i - \mathcal{S}_i$ is of first Baire category in the Banach space $\mathcal{M} - \mathcal{M}$ with the total variation norm. Goodey also investigated weak limits of ith-order area measures.
4. For the kth mean section body of a convex body $K \in \mathcal{K}^n_n$, defined by
$$h(M_k(K), \cdot) = \int_{A(n,k)} h(K \cap E, \cdot) \, \mathrm{d}\mu_k(E)$$

(see §4.4, Note 16), Goodey and Weil [754] found the representation

$$h^*(M_k(K), u) = p_{n,k} \int_{\mathbb{S}^{n-1}} g_k(\langle u, v\rangle)\, S_{n+1-k}(K, \mathrm{d}v) \tag{8.52}$$

for $k = 2, \dots, n$. Here h^* denotes the centred support function (support function with respect to the Steiner point), $p_{n,k}$ is an explicitly given constant and g_k is Berg's function appearing in his solution of the Christoffel problem (see Theorem 8.3.8). For $k = n$, (8.52) reduces to (8.35). As Goodey and Weil remark, their result shows that the general Christoffel–Minkowski problem is equivalent to characterizing the support functions of mean section bodies.

5. *Existence theorems for intermediate area and curvature measures from the PDE viewpoint.* The differential-geometric version of Minkowski's existence theorem asks for the existence of a sufficiently smooth convex body for which the product of the principal radii of curvature is a given positive function of the outer unit normal vector. Similarly, one may ask for smooth convex bodies for which the kth elementary symmetric function of the principal radii of curvature, for some $k \in \{1, \dots, n-2\}$, is given on the spherical image. Thorough investigations of these and more general problems, with various sufficient conditions, were made by Guan and Ma [861, 862], Sheng, Trudinger and Wang [1773], Guan, Lin and Ma [859], Guan, Ma and Zhou [864]. For elementary symmetric functions of the curvatures (instead of the radii of curvature), existence theorems under suitable conditions were obtained by B. Guan and P. Guan [856]. The case of radial correspondence instead of the spherical image mapping was treated by Guan, Lin and Ma [860]. For this case, an existence theorem for star bodies with given kth curvature function was obtained by Guan, J. Li and Y. Li [858].

8.5 Stability and further uniqueness results

In the preceding sections, we have seen several uniqueness theorems for the determination of convex bodies by area measures. Along with any uniqueness assertion comes the question of stability: if the assumption enforcing uniqueness is only satisfied approximately, can one ascertain approximate uniqueness? One would want to have explicit estimates, implying the uniqueness assertion, and sharpening it in a quantitative way. In this section, we treat some results of this type.

The following stability version of Theorem 8.1.1 is due to Diskant [500]. Recall that $\mathcal{K}^n(r, R)$ denotes the set of convex bodies in $\mathbb{R}^n$ that contain a ball of radius r and are contained in a concentric ball of radius R.

Theorem 8.5.1 *Let $0 < r < R$. There exist numbers $\varepsilon_0 > 0$ and γ, depending only on n, r, R, with the following property. If $K, L \in \mathcal{K}^n(r, R)$ are convex bodies satisfying*

$$|S_{n-1}(K, \omega) - S_{n-1}(L, \omega)| \leq \varepsilon \tag{8.53}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ with some $\varepsilon \in [0, \varepsilon_0]$, then

$$\delta(K, L') \leq \gamma \varepsilon^{1/n} \tag{8.54}$$

for a suitable translate L' of L.

Since the exponent $1/n$ in (8.54) is probably not optimal, it is of little interest to look more closely at the size of the constant γ. Similar remarks refer to the stability assertions made below.

Part of the proof will be stated as a lemma, in a more general form since this will be needed later. We assume that a number $m \in \{2, \dots, n\}$ and convex bodies K, L are given and we use the notation (7.62) with an $(n-m)$-tuple of balls, that is,

$$V_{(i)} := V(K[m-i], L[i], B^n[n-m]).$$

Lemma 8.5.2 *If $K, L \in \mathcal{K}^n(r, R)$ and*

$$|S_{m-1}(K, \omega) - S_{m-1}(L, \omega)| \le \varepsilon \tag{8.55}$$

for all $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ with some $\varepsilon \ge 0$, then

$$|V_{(0)} - V_{(m-1)}| \le \frac{2R}{n}\varepsilon, \quad |V_{(m)} - V_{(1)}| \le \frac{2R}{n}\varepsilon, \tag{8.56}$$

$$0 \le V_{(1)} - V_{(0)}^{(m-1)/m} V_{(m)}^{1/m} \le \frac{2R}{n}\left(\frac{R}{r} + 1\right)\varepsilon. \tag{8.57}$$

Proof By (5.19),

$$V_{(0)} - V_{(m-1)} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K, u)\ [S_{m-1}(K, \mathrm{d}u) - S_{m-1}(L, \mathrm{d}u)]\,.$$

The Hahn–Jordan decomposition of the signed measure $S_{m-1}(K, \cdot) - S_{m-1}(L, \cdot)$ together with (8.55) yields the first inequality of (8.56), and the second one is obtained similarly. By (7.63) with $i = 0$, $j = m-1$, $k = m$ we have $V_{(m-1)}^m \ge V_{(m)}^{m-1} V_{(0)}$ (and similarly the left-hand inequality in (8.57)). We conclude that

$$\begin{aligned}
&V_{(1)} - V_{(0)}^{(m-1)/m} V_{(m)}^{1/m} \\
&= V_{(m)} - \left(\frac{V_{(m)}}{V_{(0)}}\right)^{1/m} V_{(m-1)} + \left(\frac{V_{(m)}}{V_{(0)}}\right)^{1/m} (V_{(m-1)} - V_{(0)}) + (V_{(1)} - V_{(m)}) \\
&\le \left(\frac{R}{r} + 1\right)\frac{2R}{n}\varepsilon.
\end{aligned}$$

□

Proof of Theorem 8.5.1 We use inequality (7.28), which states that

$$r(K, L) \ge \left(\frac{V_{(1)}}{V_{(n)}}\right)^{1/(n-1)} - \frac{\left(V_{(1)}^{n/(n-1)} - V_{(0)} V_{(n)}^{1/(n-1)}\right)^{1/n}}{V_{(n)}^{1/(n-1)}}, \tag{8.58}$$

and Lemma 8.5.2 for $m = n$. In the following, $c_1, \dots, c_6$ denote constants depending only on n, r, R. By (8.56), $V_{(n)} - V_{(1)} \le c_1\varepsilon$, hence

$$\left(\frac{V_{(1)}}{V_{(n)}}\right)^{1/(n-1)} \ge (1 - c_2\varepsilon)^{1/(n-1)} \ge 1 - c_2\varepsilon^{1/n},$$

provided that $\varepsilon \le \varepsilon_1 := \min\{1, 1/c_2\}$. By (8.57),

$$0 \le V_{(1)} - V_{(0)}^{(n-1)/n} V_{(n)}^{1/n} \le c_3\varepsilon.$$

Using $x^p - y^p \le px^{p-1}(x-y)$ for $x \ge y \ge 0$ and $p > 1$ (Hardy, Littlewood and Pólya [938], p. 39), we obtain

$$0 \le V_{(1)}^{n/(n-1)} - V_{(0)} V_{(n)}^{1/(n-1)} \le c_4 \varepsilon.$$

Now (8.58) yields

$$r(K, L) \ge 1 - c_5 \varepsilon^{1/n} =: \alpha.$$

Since K and L may be interchanged, we also have $r(L, K) \ge \alpha$. By the definition of $r(L, K)$, there is a translate L' of L such that $\alpha K \subset L'$. Similarly, there is a translate $K + t$ of K such that $L' \subset \alpha^{-1}(K + t)$. Thus $\alpha \le 1$, where the trivial case $\alpha = 1$ can be excluded. The bodies αK and $\alpha^{-1}(K + t)$ are homothetic. After a suitable translation, we can assume that $t = 0$, thus $\alpha K \subset L' \subset \alpha^{-1} K$, and hence $o \in K$. Now

$$K \subset \alpha^{-1} L' = L' + \frac{1-\alpha}{\alpha} L' \subset L' + \frac{1-\alpha}{\alpha} 2RB^n$$

and similarly $L' \subset K + [(1-\alpha)/\alpha]2RB^n$. This yields

$$\delta(K, L') \le 2R \frac{1-\alpha}{\alpha} \le c_6 \varepsilon^{1/n},$$

if $\varepsilon \le \varepsilon_0 := \min\{\varepsilon, (2c_5)^{-n}\}$, which implies $\alpha \ge 1/2$. □

Theorem 8.5.1 can be improved, by weakening the assumption (8.53). The latter can be viewed as an assumption on the total variation norm, $\|\cdot\|_{TV}$, of the difference of the surface area measures of K and L. In fact, (8.53) implies that

$$\|S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot)\|_{TV} \le 2\varepsilon,$$

and this inequality implies (8.53) with ε replaced by 2ε. The total variation norm of $S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot)$ can be large even if the bodies K and L are close together. This is shown by the example where K is a polytope and L is a slightly rotated copy of K. We can obtain a more useful stability result by working with the Lévy–Prokhorov metric.

For a set $\omega \subset \mathbb{S}^{n-1}$ and for $\varepsilon > 0$, define

$$\omega_\varepsilon := \{u \in \mathbb{S}^{n-1} : |u - v| < \varepsilon \text{ for some } v \in \omega\}.$$

For finite Borel measures φ, ψ on $\mathbb{S}^{n-1}$, their *Lévy–Prokhorov distance* $d_P(\varphi, \psi)$ is defined by

$$d_P(\varphi, \psi) := \inf\left\{\varepsilon > 0 : \varphi(\omega) \le \psi(\omega_\varepsilon) + \varepsilon,\ \psi(\omega) \le \varphi(\omega_\varepsilon) + \varepsilon\ \forall \omega \in \mathcal{B}(\mathbb{S}^{n-1})\right\}.$$

Theorem 8.5.3 *Let $0 < r < R$. There exists a number c, depending only on n, r, R, such that, for $K, L \in \mathcal{K}^n(r, R)$,*

$$\delta(K, L') \le c\, d_P(S_{n-1}(K, \cdot), S_{n-1}(L, \cdot))^{1/n} \tag{8.59}$$

for a suitable translate L' of L.

Proof In this proof, $c_1, c_2, \dots$ denote constants depending only on n, r, R. Let $K, L \in \mathcal{K}^n(r, R)$ be given. We abbreviate $\varphi := S_{n-1}(K, \cdot)$, $\psi := S_{n-1}(L, \cdot)$, $\varphi_1 := \varphi(\mathbb{S}^{n-1})$, $\psi_1 := \psi(\mathbb{S}^{n-1})$, $\varepsilon := d_P(\varphi, \psi)$. Then $|\varphi_1 - \psi_1| \le \varepsilon$ and $\varphi_1, \psi_1 \ge 1/c_1$, hence

$$\left|\frac{\varphi_1}{\psi_1} - 1\right| \le c_1\varepsilon, \qquad \left|\frac{\psi_1}{\varphi_1} - 1\right| \le c_1\varepsilon.$$

For any Borel set $\omega \subset \mathbb{S}^{n-1}$ we deduce that

$$\frac{\varphi(\omega)}{\varphi_1} \le \frac{\psi_1}{\varphi_1}\left(\frac{\psi(\omega_\varepsilon)}{\psi_1} + \frac{\varepsilon}{\psi_1}\right) \le (1 + c_1\varepsilon)\left(\frac{\psi(\omega_\varepsilon)}{\psi_1} + \frac{\varepsilon}{\psi_1}\right) \le \frac{\psi(\omega_\varepsilon)}{\psi_1} + c_2\varepsilon.$$

By symmetry, we find that

$$d_P\left(\frac{\varphi}{\varphi_1}, \frac{\psi}{\psi_1}\right) \le c_2\varepsilon.$$

For a function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ we put

$$\|f\|_{BL} := \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|} + \sup_x |f(x)|.$$

It follows from the proof of Corollary 11.6.5 in Dudley [519] that

$$\left|\int_{\mathbb{S}^{n-1}} f \, \mathrm{d}\left(\frac{\varphi}{\varphi_1} - \frac{\psi}{\psi_1}\right)\right| \le 2\|f\|_{BL}\, d_P\left(\frac{\varphi}{\varphi_1}, \frac{\psi}{\psi_1}\right).$$

From this, for any function f with $\|f\|_{BL} \le 1$ we get

$$\begin{aligned}\left|\int_{\mathbb{S}^{n-1}} f \, \mathrm{d}(\varphi - \psi)\right| &\le \varphi_1 \left\{\left|\int_{\mathbb{S}^{n-1}} f \, \mathrm{d}\left(\frac{\varphi}{\varphi_1} - \frac{\psi}{\psi_1}\right)\right| + \left|\frac{1}{\psi_1} - \frac{1}{\varphi_1}\right| \left|\int_{\mathbb{S}^{n-1}} f \, \mathrm{d}\psi\right|\right\} \\ &\le \varphi_1(2c_2\varepsilon + c_3\varepsilon) = c_4\varepsilon.\end{aligned}$$

We may assume that $K \subset RB^n$; then $\|h(K, \cdot)\|_{BL} \le 2R$, by Lemma 1.8.12. Therefore,

$$|V_{(0)} - V_{(n-1)}| = \left|\frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K, u) \, \mathrm{d}(\varphi - \psi)(u)\right| \le \frac{2R}{n} c_4\varepsilon = c_5\varepsilon,$$

and similarly $|V_{(n)} - V_{(1)}| \le c_5\varepsilon$. These estimates correspond to the inequalities (8.56) for $m = n$, and the proof can now be completed as the proof of Lemma 8.5.2 and of Theorem 8.5.1. Note that the latter proof gives $\delta(K, L') \le c\varepsilon^{1/n}$ if ε is smaller than a certain positive constant ε_1 depending only on n, r, R; if $\varepsilon \ge \varepsilon_1$, then the same inequality is achieved by a suitable choice of c. □

For the Aleksandrov–Fenchel–Jessen theorem (Corollary 8.1.4), we already know stability versions in the cases of S_1 and S_{n-1}. We shall now state a stability result for the general case, though of a weaker order.

Theorem 8.5.4 *Let $m \in \{2, \dots, n\}$ and positive numbers ε_0, r, R be given. If $K, L \in \mathcal{K}^n(r, R)$ are convex bodies satisfying*

$$|S_{m-1}(K, \omega) - S_{m-1}(L, \omega)| \le \varepsilon \tag{8.60}$$

for all $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ *with some* $\varepsilon \in [0, \varepsilon_0]$*, then*

$$\delta(K, L') \leq \gamma \varepsilon^q \quad \text{with } q = \frac{1}{(n+1)2^{m-2}}$$

for a suitable translate L' *of* L*, where the constant* γ *depends only on* n, ε_0, r, R*.*

The proof makes use of Lemma 8.5.2, the Aleksandrov–Fenchel inequalities, Theorem 7.6.6 and Lemma 7.6.5. The complete proof can be found in Schneider [1712].

We turn now to the extension of some classical differential-geometric characterizations of balls by curvature properties. First, let K be a convex body of class C^2_+ and assume that, for some $j \in \{1, \dots, n-1\}$, the function s_j, the jth (normalized) elementary symmetric function of the principal radii of curvature, is constant. Then K is a ball. This well-known result can be generalized as follows. The assumption $s_j = \alpha$ with a constant α gives, by integration using (4.26), that

$$S_j(K, \omega) = \alpha \mathcal{H}^{n-1}(\omega) \quad \text{for } \omega \in \mathcal{B}(\mathbb{S}^{n-1}). \tag{8.61}$$

Since $\mathcal{H}^{n-1}(\omega) = S_j(B^n, \omega)$ for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, condition (8.61) is equivalent to $S_j(K, \cdot) = S_j(\alpha^{1/j} B^n, \cdot)$. By the Aleksandrov–Fenchel–Jessen theorem, this assumption implies that K is a ball of radius $\alpha^{1/j}$. This conclusion, namely that a convex body $K \in \mathcal{K}^n_n$ satisfying (8.61) with a constant α must be a ball, holds for arbitrary K, and this fact generalizes the classical differential-geometric result.

There is also a stability version of the latter uniqueness result (where assumption and conclusion are stronger than in the corresponding case of Theorem 8.5.4).

Theorem 8.5.5 *Let* $j \in \{1, \dots, n-1\}$*. There exist numbers* ε_0 *and* γ*, depending only on n, with the following property. If* $K \in \mathcal{K}^n$ *is a convex body satisfying*

$$1 - \varepsilon \leq \frac{S_j(K, \omega)}{\mathcal{H}^{n-1}(\omega)} \leq 1 + \varepsilon$$

for each $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$ *with* $\mathcal{H}^{n-1}(\omega) > 0$ *and for some* $\varepsilon \in [0, \varepsilon_0]$*, then K is contained in a ball of radius* $1 + \gamma \varepsilon^{1/(n-1)}$ *and contains a ball of radius* $1 - \gamma \varepsilon^{1/(n-1)}$*.*

This was was proved by Diskant [498], using symmetrization. For $j = n-1$, the exponent $1/(n-1)$ of ε can be replaced by 1, which is optimal; this is due to Hug [1007].

The differential-geometric result mentioned above has a counterpart where radii of curvature are replaced by curvatures. If K is a convex body of class C^2 for which the function H_i, the ith (normalized) elementary symmetric function of the principal curvatures, is constant for some $i \in \{1, \dots, n-1\}$, then K is a ball. If we want to formulate the condition

$$H_i = \alpha \tag{8.62}$$

with a constant α in a way that makes sense for arbitrary convex bodies, we have (at least) two possibilities. By (4.25), condition (8.62) (for K of class C^2) is equivalent to

$$C_{n-1-i}(K,\cdot) = \alpha\mathcal{H}^{n-1}(\cdot \cap \operatorname{bd} K)$$

and therefore, in view of (4.31), to

$$C_{n-1-i}(K,\cdot) = \alpha C_{n-1}(K,\cdot). \tag{8.63}$$

But if K is of class C^2_+, then by (2.51) the condition (8.62) is equivalent to

$$s_{n-1-i} = \alpha s_{n-1},$$

and this, by (2.50), is equivalent to

$$S_{n-1-i}(K,\cdot) = \alpha S_{n-1}(K,\cdot). \tag{8.64}$$

Thus, aiming at an extension of the differential-geometric result concerning the condition (8.62) to general convex bodies, we may take either (8.63) or (8.64) as an assumption. As it turns out, the results are different. Let us first consider (8.64). This condition is not only fulfilled by balls. In fact, suppose that K is an $(n-1-i)$-tangential body of a ball, say of radius one. Then it follows from Theorem 7.6.17 that $W_0(K) = W_1(K) = \cdots = W_{i+1}(K)$, and by Theorem 7.4.2 this implies that

$$S_{n-1}(K,\cdot) = S_{n-2}(K,\cdot) = \cdots = S_{n-i-1}(K,\cdot).$$

The converse is also true, in a stronger form.

Theorem 8.5.6 *Let $j \in \{0,\dots,n-2\}$. If $K \in \mathcal{K}^n_n$ is a convex body satisfying*

$$S_j(K,\cdot) = \alpha S_{n-1}(K,\cdot) \tag{8.65}$$

with a constant α, then K is a j-tangential body of a ball of radius $\alpha^{-1/(n-j-1)}$.

Proof After replacing K by a suitable homothetic copy, we may assume that $\alpha = 1$. From (8.65) and (5.19), integrating the support function of either B^n or K, we obtain

$$W_{n-j} = W_1 \quad \text{and} \quad W_{n-j-1} = W_0 \tag{8.66}$$

for the quermassintegrals $W_i = W_i(K) = V(K[n-i], B^n[i])$. By the Aleksandrov–Fenchel inequalities,

$$\frac{W_1}{W_0} \geq \frac{W_2}{W_1} \geq \cdots \geq \frac{W_{n-j}}{W_{n-j-1}}. \tag{8.67}$$

By (8.66), equality holds here throughout; in particular $W_1^2 = W_0W_2$. By Theorem 7.6.19, K is an $(n-2)$-tangential body of a ball. If this ball has radius r, then Theorem 7.6.17, together with a suitable dilatation, implies $W_0 = rW_1$. Since equality holds in (8.67), we have $W_0 = r^{n-j}W_{n-j}$, and together with (8.66) this yields $r = 1$. Thus $W_0 = W_1 = \cdots = W_{n-j}$. By Theorem 7.6.17, K is a j-tangential body of B^n. □

If, however, condition (8.63) is accepted as a substitute for (8.62), then a proper generalization of the classical differential-geometric result is obtained.

Theorem 8.5.7 *Let $j \in \{0, \dots, n-2\}$. If $K \in \mathcal{K}_n^n$ is a convex body satisfying*

$$C_j(K, \cdot) = \alpha C_{n-1}(K, \cdot) \tag{8.68}$$

with some constant α, then K is a ball.

We sketch only the first steps of the proof. We may assume that $\alpha = 1$. Let $\omega \subset \mathbb{S}^{n-1}$ be a closed set; then

$$\begin{aligned} S_j(K, \omega) &\le C_j(K, \tau(K, \omega)) = C_{n-1}(K, \tau(K, \omega)) \\ &= C_{n-1}(K, \tau(K, \omega) \cap \operatorname{reg} K) = C_j(K, \tau(K, \omega) \cap \operatorname{reg} K) \\ &\le S_j(K, \omega), \end{aligned}$$

where we have used successively Lemma 4.2.4, the assumption (8.68) with $\alpha = 1$, (4.31) and Theorem 2.2.5, the assumption (8.68), and Lemma 4.2.4. Since the equality sign must hold, we deduce that

$$S_j(K, \omega) = C_{n-1}(K, \tau(K, \omega)) = \mathcal{H}^{n-1}(\tau(K, \omega)) = S_{n-1}(K, \omega)$$

by (4.32). The equality $S_j(K, \omega) = S_{n-1}(K, \omega)$ for all closed sets ω permits us to conclude that $S_j(K, \cdot) = S_{n-1}(K, \cdot)$. Now Theorem 8.5.6 shows that K is a j-tangential body of a ball. The rest of the proof consists in showing that among j-tangential bodies of balls, only the balls satisfy (8.68). We omit this part of the proof, since it is not particularly relevant to the theory of mixed volumes. We mention, however, that Theorem 2.6.1 and the result of Section 4.2, Note 2, are essential tools. The proof can be found in Schneider [1691].

For the case $j = 0$, and only for this case, a stability version of Theorem 8.5.7 is known. This is due to Diskant [497], and we quote it without proof.

Theorem 8.5.8 *Let $K \in \mathcal{K}_n^n$, $0 < \varepsilon < 1/2$, and suppose that*

$$1 - \varepsilon \le \frac{C_0(K, \beta)}{C_{n-1}(K, \beta)} \le 1 + \varepsilon$$

for each set $\beta \in \mathcal{B}(\mathbb{R}^n)$ with $C_{n-1}(K, \beta) > 0$. Then K lies in the $\gamma\varepsilon$-neighbourhood of a unit ball, where the constant γ depends only on n.

Now we explain an application of the Brunn–Minkowski theory to almost umbilical convex hypersurfaces. Let $K \in \mathcal{K}^n$ be of class C_+^2. A point $x \in \operatorname{bd} K$ is called an *umbilical point* if at x the principal curvatures of K are all the same. It is a classical fact of elementary differential geometry that K must be a ball if all its boundary points are umbilical points. We shall prove a stability version of this result. For this, it is convenient to measure the deviation of a point with normal vector $u \in \mathbb{S}^{n-1}$ from being umbilical by the quantity

$$\alpha(u) := \frac{r_{\max}(u)}{r_{\min}(u)} - 1,$$

where $r_{\max}(u)$ is the largest and $r_{\min}(u)$ is the smallest principal radius of curvature of K at u. The following theorem shows that it is sufficient to assume that the deviation from umbilicity, α, is small in the quadratic mean, to conclude that K must be close to some ball in the L_2-metric. One can then use Lemma 7.6.4 to obtain an estimate of the Hausdorff distance from a ball. In the following, σ denotes spherical Lebesgue measure.

Theorem 8.5.9 *Let $K \in \mathcal{K}^n$ be of class C^2_+ and normalized so that it has the same Steiner point and mean width as the unit ball B^n. Then*

$$\delta_2(K, B^n) \leq \frac{1}{2(n+1)} \int_{\mathbb{S}^{n-1}} \left(\frac{r_{\max}}{r_{\min}} - 1 \right)^2 \mathrm{d}\sigma.$$

Proof At a given normal vector, we number the principal radii of curvature such that $0 < r_1 \leq r_2 \leq \cdots \leq r_{n-1}$; then $r_{n-1} = (1+\alpha)r_1$. This gives $r_j \leq (1+\alpha)r_1 \leq (1+\alpha)r_i$ and hence $r_j - r_i \leq \alpha r_i$, thus $(r_i - r_j)^2 \leq \alpha^2 r_i^2 \leq \alpha^2 r_i r_j$ for $i < j$. Together with the identity

$$(n-2)(r_1 + \cdots + r_{n-1})^2 = \sum_{i<j} (r_i - r_j)^2 + 2(n-1) \sum_{i<j} r_i r_j$$

this yields

$$(n-2)(r_1 + \cdots + r_{n-1})^2 \leq (2n - 2 + \alpha^2) \sum_{i<j} r_i r_j.$$

Since

$$s_1 = \frac{1}{n-1} \sum r_i, \qquad s_2 = \frac{2}{(n-1)(n-2)} \sum_{i<j} r_i r_j,$$

we deduce that

$$s_1^2 \leq \left(1 + \frac{\alpha^2}{2(n-1)} \right) s_2.$$

Integration over $\mathbb{S}^{n-1}$ and an application of the Cauchy–Schwarz inequality yield

$$\begin{aligned} \left(\int s_1 \, \mathrm{d}\sigma \right)^2 &\leq \left(\int \left(1 + \frac{\alpha^2}{2(n-1)} \right)^{1/2} \sqrt{s_2} \, \mathrm{d}\sigma \right)^2 \\ &\leq \int \left(1 + \frac{\alpha^2}{2(n-1)} \right) \mathrm{d}\sigma \int s_2 \, \mathrm{d}\sigma. \end{aligned}$$

In view of (4.26) and $S_m(K, \mathbb{S}^{n-1}) = nW_{n-m}(K)$ this leads to

$$\frac{1}{\kappa_n} W_{n-1}^2 - W_{n-2} \leq \frac{1}{2(n-1)n\kappa_n} W_{n-2} \int \alpha^2 \mathrm{d}\sigma.$$

Here, $W_{n-2} \leq \kappa_n$ by (7.67), since K has the same mean width as the unit ball. Now inequality (7.123) completes the proof. □

Finally in this section, we remark that the uniqueness part of Aleksandrov's theorem formulated in Section 8.2, Note 10, can be extended to curvature measures of different orders.

Theorem 8.5.10 *Let $j \in \{0, \dots, n-1\}$. Let $K, L \in \mathcal{K}_n^n$ be convex bodies with o as an interior point, and let the radial projection $f : \mathrm{bd}\, K \to \mathrm{bd}\, L$ be defined by $f(x) = \lambda(x)x \in \mathrm{bd}\, L$ with $\lambda(x) > 0$ for $x \in \mathrm{bd}\, K$. If*

$$C_j(K, \beta) = C_j(L, f(\beta))$$

for each Borel set $\beta \subset \mathrm{bd}\, K$, then K and L differ only by a dilatation with centre o if $j = 0$, and they are identical if $j > 0$.

The case $j = 0$ is due to Aleksandrov [21], and the case $j > 0$ was proved by Schneider [1687].

Notes for Section 8.5

1. *Stability for Minkowski's uniqueness theorem.* The first stability result of the type of Theorem 8.5.1 was proved by Volkov [1898] (a sketch of his proof is given in Pogorelov [1538], Chapter VII, §10). His result was slightly weaker than that of Diskant [500], since it had an exponent of ε equal to $1/(n+2)$ instead of Diskant's $1/n$. Later Diskant [506] showed that the exponent can be improved to $1/(n-1)$ if one makes the stronger assumption that K and L have continuous radii of curvature and that $|s_{n-1}(K, u) - s_{n-1}(L, u)| < \varepsilon$ for all $u \in \mathbb{S}^{n-1}$. The optimal exponent in Theorem 8.5.1 is unknown, but it cannot be larger than $1/(n-1)$. Under an additional assumption on the area measures, the optimal exponent $1/(n-1)$ was achieved by Hug and Schneider [1014]. The assumption says that the singular parts of the area measures of K and L, with respect to $\mathcal{H}^{n-1}$, have only point masses. This is satisfied, for example, by polytopes and by bodies with support functions of class C^2.

   A short alternative proof of a weaker form of Theorem 8.5.1 was given by Bourgain and Lindenstrauss [316].

   The improvement involving the Lévy–Prokhorov metric, Theorem 8.5.3, was proved in Hug and Schneider [1014].
2. *Stability for the Aleksandrov–Fenchel–Jessen theorem.* The stability theorem 8.5.4 and its proof appear in Schneider [1712]. A weaker theorem of this type had previously been proved by Diskant [509] (and for the special case $m-1 = n-2$ also by Diskant [505]). His assumptions are stronger in several respects: he assumed K and L to be smooth and to satisfy an inequality of the form $|S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot)| \le \varepsilon \mathcal{H}^{n-1}(\omega)$ for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, and he had to impose the assumption that one of the bodies has a spherical projection on some hyperplane. Moreover, his exponent for ε is smaller than the one given in Theorem 8.5.4.

   Under stronger regularity assumptions, Oliker [1491], Theorem 4, proved a stability theorem involving s_{m-1} for convex hypersurfaces with boundaries and with common spherical image contained in an open hemisphere. Earlier, more specialized results of this type were obtained by Volkov and Oliker [1899] and by Oliker [1488].
3. *Stability results via symmetrization.* Extremal properties of balls and thus uniqueness theorems characterizing balls can often be proved by means of symmetrization procedures. Sometimes symmetrization is also useful for obtaining stability properties of balls. In this way, Diskant [498] obtained Theorem 8.5.5, and the proof of Diskant [497] for Theorem 8.5.8 also made use of symmetrizations. Earlier applications of such methods to similar results can be found in Fet [575] and Diskant [495, 496]. Symmetrization is also used in Schneider [1699] to obtain an estimate for convex hypersurfaces with unique projection to a hyperplane and satisfying an inequality $C_0 \ge \alpha C_{n-1}$ with $\alpha > 0$.

4. *Characterizations of balls and stability*. Theorem 8.5.6, which characterizes tangential bodies of balls, appears in Schneider [1689], and Theorem 8.5.7 was proved in Schneider [1691]. For the latter result, Kohlmann [1128, 1131] developed a new method of proof, employing generalized Minkowskian integral formulae and permitting extensions to spaces of constant curvature. In [1134], Kohlmann characterized the ball by the linear relation

$$C_{n-1}(K,\cdot) = \sum_{r=0}^{n-2} \lambda_r C_r(K,\cdot)$$

with fixed $\lambda_0,\dots,\lambda_{n-2} \geq 0$. Characterizations of the ball in terms of curvature measures in standard space forms were proved in Kohlmann [1133].

In the literature, there are several investigations, differential-geometric or convex-geometric, of the stability of the sphere within a class of convex hypersurfaces having some curvature restriction, such as almost constant Gauss–Kronecker curvature or mean curvature, or almost umbilical points. Besides the references in Note 3, we mention the following. Surfaces with almost constant mean curvature are treated in Diskant [499], Koutroufiotis [1144], Moore [1449], Treibergs [1851], Schneider [1714], Arnold [75]. The latter two papers use strengthened Aleksandrov–Fenchel inequalities to prove stability estimates for convex bodies with pinched mean curvature. Arnold's result reads as follows.

Theorem Let $K \subset \mathbb{R}^n$ $(n \geq 3)$ be a convex body of class C^2_+, and let $0 \leq \varepsilon < 1$. If the mean curvature H_1 of K satisfies

$$1 - \varepsilon \leq H_1 \leq 1 + \varepsilon,$$

then there is a ball B_K such that

$$\delta_2(K, B_K) \leq c_n V_n(K)\sqrt{\varepsilon},$$

where c_n is an explicit constant depending only on n.

Stability results for convex bodies with one curvature measure close to the boundary measure were proved by Kohlmann [1131], and in a stronger form in [1135]. In [1132], Kohlmann also obtained corresponding splitting results, under pinching assumptions, for noncompact closed convex sets.

Estimates for almost umbilical convex hypersurfaces appear in Pogorelov [1537], Rešetnjak [1575], Guggenheimer [865], Vodop'yanov [1895], Leichtweiss [1194]. The paper by Rešetnjak [1575] has stronger stability estimates than Theorem 8.5.9. The result and proof given here are, however, particularly simple.

5. *Further uniqueness results involving curvature measures*. Aleksandrov's [21] proof of the case $j = 0$ of Theorem 8.5.10 proceeds by direct geometric reasoning (see also Busemann [370], p. 30); the proof for $j > 0$ by Schneider [1687] has to use in addition the integral-geometric formula of Theorem 4.4.5. Using the latter method, the following more general version can be obtained. Let K, L, f be as in Theorem 8.5.10, and suppose that

$$\sum_{i=0}^{n-1} \alpha_i C_i(K,\beta) = \sum_{i=0}^{n-1} \alpha_i C_i(L, f(\beta))$$

for each Borel set $\beta \subset \operatorname{bd} K$, where $\alpha_0,\dots,\alpha_{n-1}$ are nonnegative real constants with $\alpha_1 + \cdots + \alpha_{n-1} > 0$. Then $K = L$.

In Schneider [1687], the following result was also deduced. Let $K \in \mathcal{K}^n$ be a convex body with $o \in \operatorname{int} K$. Suppose that $C_j(K, H^+_{u,0}) = C_j(K, H^-_{u,0})$ for some $j \in \{0,\dots,n-1\}$ and all $u \in \mathbb{S}^{n-1}$. Then K is centrally symmetric with respect to o.

The following characterization of the ball was proved for $n = 3$ by Blind [256] and for $n \geq 3$ by Schneider [1686]. Let $K \in \mathcal{K}^n_n$ be a smooth convex body with the property that every Jordan domain in the boundary of K that halves the curvature measure C_0 also halves the area C_{n-1}; then K is a ball.

6. *A characterization involving inequalities*. Aleksandrov [26] proved the following uniqueness theorem, which involves only inequalities. Suppose that $K, L \in \mathcal{K}_n^n$ are convex bodies for which

$$S_j(K, \cdot) \leq S_j(L, \cdot) \quad \text{and} \quad V_{j+1}(K) \geq V_{j+1}(L)$$

for some $j \in \{1, \dots, n-1\}$. Then K and L are translates of each other.

7. *A problem of Aleksandrov and its surprising solution* The following question from the differential geometry of closed convex surfaces in three-space has found considerable interest over many years. Let $K \in \mathcal{K}^3$ be a convex body of class C_+^2. If there is a constant c such that the principal radii of curvature r_1, r_2 of K satisfy $r_1 \leq c \leq r_2$ with suitable numeration, or

$$(r_1 - c)(r_2 - c) \leq 0 \tag{8.69}$$

in general, must K be a ball? For K with a real-analytic boundary, Aleksandrov [20] proved this under the additional assumption that the equality $(r_1 - c)(r_2 - c) = 0$ holds only if both factors are zero, and in [27] without this assumption. He conjectured that the answer is affirmative under differentiability assumptions alone. Another proof in the analytic case was given by Münzner, who also proved the conjecture for surfaces of revolution. This was extended by Koutroufiotis: it suffices to assume that the body has an enveloping circular cylinder. With this condition, the result can be generalized to arbitrary convex bodies $K \in \mathcal{K}_3^3$, if assumption (8.69) is extended to

$$S_2(K, \cdot) - 2cS_1(K, \cdot) + c^2 S_0(K, \cdot) \leq 0 \tag{8.70}$$

(which is equivalent in the C_+^2 case). It was proved by Schneider [1676] that a body K satisfying (8.70) and admitting some circular orthogonal projection must be the sum of a ball and a (possibly degenerate) segment. The assumption on the circular projection appeared rather crude at the time (a friend called it 'cheating'), but it turned out that some additional assumption cannot be avoided. It came as a great surprise when Martinez-Maure [1341] constructed a C^2 conterexample to Aleksandrov's conjecture. Later, Panina [1506, 1507] even obtained a series of different C^∞ counterexamples.

9

Extensions and analogues of the Brunn–Minkowski theory

The Brunn–Minkowski theory, developed in previous chapters, is based on the Minkowski addition of convex sets, combined with the notion of volume. One can replace Minkowski addition by other combinations of convex bodies or star bodies, and in this way generalize or dualize certain parts of the classical theory. The basic facts of such extensions and analogues will be described in the first four sections of this chapter. We point out that, in these variations of the Brunn–Minkowski theory, the position of the origin of the space plays an essential role; the classical theory remains the only one that is translation invariant.

The fifth section describes first steps on a different route of extension from convex bodies to more general objects: characteristic functions of convex bodies are replaced by log-concave or more general functions.

Some of the generalizations to be mentioned are fairly recent, and we can only try to give a preliminary sketch. In a few years, the picture may again look different.

9.1 The L_p Brunn–Minkowski theory

Since in this chapter the position of the origin must be carefully taken into account, we remind the reader that by $\mathcal{K}^n_o$ we denote the set of convex bodies in $\mathbb{R}^n$ containing the origin, whereas $\mathcal{K}^n_{(o)}$ denotes the subset of bodies with the origin as interior point.

Let $p \geq 1$. For convex bodies $K, L \in \mathcal{K}^n_o$, consider the function defined by

$$f(x) := \left[h_K(x)^p + h_L(x)^p\right]^{1/p} = M_p(h_K(x), h_L(x)), \qquad x \in \mathbb{R}^n,$$

where the p-mean $M_p(a,b)$ of numbers $a, b \geq 0$ is defined by $M_p(a,b) := [a^p + b^p]^{1/p}$. By Minkowski's inequality ([938], p. 30),

$$M_p(a+a', b+b') \leq M_p(a,b) + M_p(a',b') \quad \text{for } a, a', b, b' \geq 0.$$

Together with the sublinearity of support functions, for $x, y \in \mathbb{R}^n$ this gives

$$\begin{aligned} f(x+y) &= M_p(h_K(x+y), h_L(x+y)) \le M_p(h_K(x)+h_K(y), h_L(x)+h_L(y)) \\ &\le M_p(h_K(x), h_L(x)) + M_p(h_K(y), h_L(y)) = f(x) + f(y). \end{aligned}$$

Since the function f is obviously positively homogeneous and nonnegative, it is the support function of a convex body in $\mathcal{K}^n_o$, and this body is in $\mathcal{K}^n_{(o)}$ if at least one of the bodies K, L is in $\mathcal{K}^n_{(o)}$. The body with support function f is denoted by $K +_p L$ and called the *p-sum* of K and L. One also defines a *p-scalar multiplication* by

$$\lambda \cdot_p K := \lambda^{1/p} K \qquad \text{for } \lambda \ge 0. \tag{9.1}$$

In the following, if this scalar multiplication is used together with the p-sum, the index of $\cdot_p$ will often be omitted, but must be kept in mind. Thus, the *p-linear combination* is given by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p \tag{9.2}$$

for $\lambda, \mu \ge 0$. We supplement the definition by

$$h(K +_\infty L, \cdot) := \max\{h(K, \cdot), h(L, \cdot)\}, \qquad K +_\infty L = \operatorname{conv}(K \cup L).$$

This p-Minkowski combination of convex bodies containing the origin was introduced and studied by Firey [585]. Lutwak, beginning with [1283], investigated p-sums and their consequences more systematically. We essentially follow his approach in presenting the basic definitions and facts. While Lutwak coined the name 'Brunn–Minkowski–Firey theory', it has become more common to speak of the 'L_p Brunn–Minkowski theory'. The addition $+_p$ has been called *p-addition* as well as *L_p addition*.

In the classical Brunn–Minkowski theory, a fruitful principle for obtaining new functionals is to apply familiar functionals, such as the volume, to a Minkowski linear combination $K + \varepsilon L$. In particular, $V_n(K + \varepsilon L)$ is a polynomial in ε, and the derivative of the function $\varepsilon \mapsto V_n(K + \varepsilon L)$ at $\varepsilon = 0$ is, up to a factor, the mixed volume $V_1(K, L)$. This latter approach is also possible for p-linear combinations: one considers first variations to obtain new functionals.

We assume that a number $m \in \{1, \dots, n\}$ and an $(n-m)$-tuple $\mathcal{C}$ of convex bodies $K_{m+1}, \dots, K_n \in \mathcal{K}^n_n$ are given; these are fixed in the first three theorems to follow. For convex bodies $K, L \in \mathcal{K}^n$ we use the abbreviations (cf. (7.62))

$$V_{(0)}(K) := V(K[m], \mathcal{C}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \, \mathrm{d}S(K[m-1], \mathcal{C}, \cdot), \tag{9.3}$$

$$V_{(1)}(K, L) := V(K[m-1], L, \mathcal{C}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L \, \mathrm{d}S(K[m-1], \mathcal{C}, \cdot). \tag{9.4}$$

Theorem 9.1.1 *Let $p \ge 1$, let m and $\mathcal{C}$ be as above and let $K, L \in \mathcal{K}^n_{(o)}$. Then*

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \frac{V_{(0)}(K +_p \varepsilon \cdot L) - V_{(0)}(K)}{\varepsilon} \\ &= \frac{m}{p} \cdot \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p h_K^{1-p} \, \mathrm{d}S(K[m-1], \mathcal{C}, \cdot) =: \frac{m}{p} V_{(p,m,1)}(K, L, \mathcal{C}). \end{aligned} \tag{9.5}$$

Proof The following simple proof is essentially the one given by Gardner, Hug and Weil [680] in the Orlicz case. We write $K_\varepsilon := K +_p \varepsilon \cdot L$. From

$$h_{K_\varepsilon} - h_K = (h_K^p + \varepsilon h_L^p)^{1/p} - h_K$$

we get

$$\lim_{\varepsilon \downarrow 0} \frac{h_{K_\varepsilon} - h_K}{\varepsilon} = \frac{1}{p} h_L^p h_K^{1-p} \quad \text{uniformly on } \mathbb{S}^{n-1}. \tag{9.6}$$

For $\varepsilon > 0$ we have

$$\begin{aligned}
&\frac{V_{(0)}(K +_p \varepsilon \cdot L) - V_{(0)}(K)}{\varepsilon} = \frac{V(K_\varepsilon[m], \mathscr{C}) - V(K[m], \mathscr{C})}{\varepsilon} \\
&= \sum_{i=0}^{m-1} \frac{V(K_\varepsilon[i+1], K[m-i-1], \mathscr{C}) - V(K_\varepsilon[i], K[m-i], \mathscr{C})}{\varepsilon}.
\end{aligned}$$

For each $i \in \{0, \dots, m-1\}$, using (5.19),

$$\begin{aligned}
&\frac{V(K_\varepsilon[i+1], K[m-i-1], \mathscr{C}) - V(K_\varepsilon[i], K[m-i], \mathscr{C})}{\varepsilon} \\
&= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \frac{h_{K_\varepsilon} - h_K}{\varepsilon} \, \mathrm{d}S(K_\varepsilon[i], K[m-i-1], \cdot) \\
&= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(\frac{h_{K_\varepsilon} - h_K}{\varepsilon} - \frac{1}{p} h_L^p h_K^{1-p} \right) \mathrm{d}S(K_\varepsilon[i], K[m-i-1], \cdot) \\
&\quad + \frac{1}{n} \int_{\mathbb{S}^{n-1}} \frac{1}{p} h_L^p h_K^{1-p} \, \mathrm{d}S(K_\varepsilon[i], K[m-i-1], \cdot).
\end{aligned}$$

From (9.6) and the fact that

$$\lim_{\varepsilon \downarrow 0} S(K_\varepsilon[i], K[m-i-1], \mathscr{C}, \cdot) \xrightarrow{w} S(K[m-1], \mathscr{C}, \cdot),$$

we get the assertion. □

The new functional $V_{(p,m,1)}(K, L, \mathscr{C})$ defined by (9.5) satisfies inequalities reminiscent of those for mixed volumes. This is not surprising, since they follow from the latter, after an application of Hölder's inequality.

In the following two theorems, m and $\mathscr{C}$ are given as in Theorem 9.1.1.

Theorem 9.1.2 *Let $p > 1$ and $K, L \in \mathcal{K}^n_{(o)}$. Then*

$$V_{(p,m,1)}(K, L, \mathscr{C})^m \geq V_{(0)}(K)^{m-p} V_{(0)}(L)^p. \tag{9.7}$$

Proof Using Hölder's inequality ([938], (6.9.3) with $k = 1/(1-p) < 0$), we get

$$V_{(p,m,1)}(K, L, \mathscr{C}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p h_K^{1-p} \, \mathrm{d}S(K[m-1], \mathscr{C}, \cdot) \geq V_{(1)}(K, L)^p V_{(0)}(K)^{1-p}.$$

If $m = 1$, this is already inequality (9.7), since in that case we have $V_{(1)}(K, L) = V_{(0)}(L)$.

By (7.63) we have

$$V_{(1)}(K,L)^m \geq V_{(0)}(K)^{m-1} V_{(0)}(L). \tag{9.8}$$

Now let $m > 1$. Then (9.8) gives

$$V_{(1)}(K,L)^p \geq V_{(0)}(K)^{p(m-1)/m} V_{(0)}(L)^{p/m}.$$

Together with the preceding inequality, this yields (9.7). □

The following extends the general Brunn–Minkowski theorem.

Theorem 9.1.3 *Let $p > 1$ and $K, L \in \mathcal{K}^n_{(o)}$. Then*

$$V_{(0)}(K +_p L)^{p/m} \geq V_{(0)}(K)^{p/m} + V_{(0)}(L)^{p/m}. \tag{9.9}$$

Proof For $K, L, M \in \mathcal{K}^n_{(o)}$, we see from (9.5) that

$$V_{(p,m,1)}(M, K +_p L, \mathcal{C}) = V_{(p,m,1)}(M, K, \mathcal{C}) + V_{(p,m,1)}(M, L, \mathcal{C}).$$

Using (9.7), we get

$$V_{(p,m,1)}(M, K +_p L, \mathcal{C}) \geq V_{(0)}(M)^{(m-p)/m} \left[V_{(0)}(K)^{p/m} + V_{(0)}(L)^{p/m} \right].$$

Inserting $M = K +_p L$ and observing that $V_{(p,m,1)}(M, M, \mathcal{C}) = V_{(0)}(M)$, we get the assertion. □

About the equality cases in (9.7) and (9.9), not much can be said in the general situation considered so far. This is different in the following specialization.

From now on, we consider the case where the $(n-m)$-tuple $\mathcal{C}$ consists only of unit balls, and we write $m = n - i$, $V_{(0)}(K) = W_i(K)$, $V_{(1)}(K,L) = W_i(K,L)$, $V_{(p,m,1)}(K, L, \mathcal{C}) = W_{p,i}(K,L)$, thus

$$W_i(K) = V(K[n-i], B^n[i]) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \, \mathrm{d}S_{n-i-1}(K, \cdot),$$

$$W_i(K,L) = V(K[n-i-1], L, B^n[i]) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L \, \mathrm{d}S_{n-i-1}(K, \cdot),$$

$$W_{p,i}(K,L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p h_K^{1-p} \, \mathrm{d}S_{n-i-1}(K, \cdot).$$

We also write $W_{p,0}(K,L) =: V_p(K,L)$, thus

$$V_p(K,L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p h_K^{1-p} \, \mathrm{d}S_{n-1}(K, \cdot). \tag{9.10}$$

We recall that, according to Theorem 9.1.1, this is the first variation given by

$$V_p(K,L) = \frac{p}{n} \lim_{\varepsilon \downarrow 0} \frac{V_n(K +_p \varepsilon^{1/p} L) - V_n(K)}{\varepsilon}. \tag{9.11}$$

The special cases of the inequalities obtained above can now be stated with equality conditions.

Theorem 9.1.4 *Let $p > 1$, $K, L \in \mathcal{K}^n_{(o)}$ and $i \in \{0, \dots, n-1\}$. Then*

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \tag{9.12}$$

with equality if and only if K and L are dilatates. The case $i = 0$ of inequality (9.12) reads

$$V_p(K, L)^n \geq V_n(K)^{n-p} V_n(L)^p. \tag{9.13}$$

Proof Using Hölder's inequality as above,

$$W_{p,i}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p h_K^{1-p} \, \mathrm{d}S_{n-i-1}(K, \cdot) \geq W_i(K, L)^p W_i(K)^{1-p}.$$

If $i = n-1$, this is inequality (9.12). Since $S_0(K, \cdot) = \sigma$, equality in the employed Hölder inequality holds if and only if h_K and h_L are proportional.

Now let $i < n-1$. As above, we use

$$W_i(K, L)^p \geq W_i(K)^{p(n-i-1)/(n-i)} W_i(L)^{p/(n-i)}. \tag{9.14}$$

Together with the preceding inequality, this yields (9.12). By Corollary 7.6.11, equality in (9.14) holds if and only if K and L are homothetic, which is equivalent to $h_K = \lambda h_L + \langle x, \cdot \rangle$ with some $\lambda > 0$ and some vector $x \in \mathbb{R}^n$. Equality in the employed Hölder inequality holds if and only if $h_K = \mu h_L$ holds $S_{n-i-1}(K, \cdot)$-almost everywhere on $\mathbb{S}^{n-1}$, with some $\mu > 0$. Necessarily, $\lambda = \mu$. Then $x = 0$, since otherwise the measure $S_{n-i-1}(K, \cdot)$ would be concentrated on $x^\perp \cap \mathbb{S}^{n-1}$, a contradiction, since K has interior points. Thus, equality in (9.12) holds if and only if K and L are dilatates. □

Corollary 9.1.5 *Let $p > 1$, $K, L \in \mathcal{K}^n_{(o)}$, and $i \in \{0, \dots, n-1\}$. Then*

$$W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{9.15}$$

with equality if and only if K and L are dilatates.

It seems natural to define, for $K \in \mathcal{K}^n_{(o)}$ and for all $p \geq 0$, a measure $S_{p,i}(K, \cdot)$ by

$$S_{p,i}(K, \omega) := \int_\omega h_K^{1-p} \, \mathrm{d}S_{n-i-1}(K, \cdot) \qquad \text{for } \omega \in \mathcal{B}(\mathbb{S}^{n-1}), \tag{9.16}$$

which, as usual, we abbreviate by writing

$$\mathrm{d}S_{p,i}(K, \cdot) = h_K^{1-p} \, \mathrm{d}S_{n-i-1}(K, \cdot).$$

Then we can write

$$W_{p,i}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p \, \mathrm{d}S_{p,i}(K, \cdot), \tag{9.17}$$

in particular

$$V_p(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p \, \mathrm{d}S_{p,0}(K, \cdot). \tag{9.18}$$

Since (9.17) parallels the classical formula

$$W_i(K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L \,\mathrm{d}S_{n-i-1}(K,\cdot),$$

the measure $S_{p,i}(K,\cdot)$ is the L_p generalization of the area measure $S_{n-i-1}(K,\cdot)$. The measure $S_{p,0}(K,\cdot)$ is called the L_p *surface area measure* of K.

The following is a counterpart to the Aleksandrov–Fenchel–Jessen theorem (Corollary 8.1.4), showing how far the measure $S_{p,i}(K,\cdot)$ determines the convex body K.

Theorem 9.1.6 *Let $p > 1$, $K, L \in \mathcal{K}^n_{(o)}$ and $i \in \{0,\dots,n-1\}$. If $p \neq n-i$ and if*

$$S_{p,i}(K,\cdot) = S_{p,i}(L,\cdot), \tag{9.19}$$

then $K = L$.

If

$$S_{n-i,i}(K,\cdot) \leq S_{n-i,i}(L,\cdot), \tag{9.20}$$

then K and L are dilatates (and hence $S_{n-i,i}(K,\cdot) = S_{n-i,i}(L,\cdot)$).

Proof Let $p \neq n-i$. Integrating the function h_L^p with (9.19), we obtain $W_{p,i}(K,L) = W_i(L)$, by (9.17). Together with (9.12) this gives

$$W_i(L)^{n-i} = W_{p,i}(K,L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p,$$

hence $W_i(L) = W_i(K)$, by symmetry. Therefore, (9.12) must hold with equality, which implies that K and L are dilatates, and by (9.19) they are equal (note that $S_{p,i}(K,\cdot)$ is homogeneous of degree $n-i-p \neq 0$).

If $p = n-i$, then integration of h_L^p with (9.20) yields $W_i(L) \geq W_{p,i}(K,L)$, and from

$$W_i(L)^{n-i} \geq W_{p,i}(K,L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p = W_i(L)^p$$

we obtain again that (9.12) holds with equality, hence K and L are dilatates. □

Interesting problems arise if the assumption $p \geq 1$ is relaxed. For $p \geq 1$, the p-linear combination can be written in the form

$$(1-\lambda)\cdot K +_p \lambda \cdot L = \bigcap_{u\in\mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x,u\rangle \leq \left[(1-\lambda)h_K(u)^p + \lambda h_L(u)^p\right]^{1/p}\right\}, \tag{9.21}$$

for $K, L \in \mathcal{K}^n_{(o)}$ and $\lambda \in [0,1]$. This definition can also be used for $0 < p < 1$. The right side is now the Wulff shape of the function $[(1-\lambda)h_K^p + \lambda h_L^p]^{1/p}$, which is in general not a support function. Since $\lim_{p\downarrow 0}[(1-\lambda)a^p + \lambda b^p]^{1/p} = a^{1-\lambda}b^\lambda$ for $a, b > 0$, it is natural to further define a *geometric Minkowski combination* by

$$(1-\lambda)\cdot K +_0 \lambda \cdot L := \bigcap_{u\in\mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x,u\rangle \leq h_K(u)^{1-\lambda} h_L(u)^\lambda\right\}, \tag{9.22}$$

which is the Wulff shape associated with the function $h_K^{1-\lambda}h_L^\lambda$.

Introducing the *cone-volume probability measure* $\overline{V}_K$ of $K \in \mathcal{K}^n_{(o)}$ on $\mathbb{S}^{n-1}$ by

$$\mathrm{d}\overline{V}_K = \frac{1}{nV_n(K)} h_K \,\mathrm{d}S_{n-1}(K,\cdot),$$

we can write inequality (9.13) in the form

$$\left(\int_{\mathbb{S}^{n-1}} \left(\frac{h_L}{h_K}\right)^p \mathrm{d}\overline{V}_K\right)^{1/p} \geq \left(\frac{V_n(L)}{V_n(K)}\right)^{1/n} \tag{9.23}$$

for $K, L \in \mathcal{K}^n_{(o)}$ and $p \geq 1$. For o-symmetric $K, L \in \mathcal{K}^n_{(o)}$, it was conjectured by Böröczky, Lutwak, Yang and Zhang [295] that (9.23) holds also for $0 < p < 1$. They also conjectured that, for o-symmetric $K, L \in \mathcal{K}^n_{(o)}$,

$$\int_{\mathbb{S}^{n-1}} \log \frac{h_L}{h_K} \,\mathrm{d}\overline{V}_K \geq \frac{1}{n} \log \frac{V_n(L)}{V_n(K)}, \tag{9.24}$$

and that

$$V_n((1-\lambda)\cdot K +_p \lambda \cdot L) \geq V_n(K)^{1-\lambda} V_n(L)^{\lambda} \tag{9.25}$$

for $0 \leq p < 1$ and $\lambda \in [0,1]$. They proved all these conjectures for $n = 2$, together with complete equality conditions. For $p = 0$, the latter are particularly interesting. For example, for o-symmetric convex bodies $K, L \in \mathcal{K}^2_{(o)}$, equality in the inequality

$$V_2((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V_2(K)^{1-\lambda} V_2(L)^{\lambda}$$

for some $\lambda \in (0,1)$ holds if and only if K and L are either dilatates or are parallelograms with parallel sides.

The following extensions of the L_p Brunn–Minkowski theory are described only briefly, since they are very recent at the time of writing. The p-sum is only a special case of very general constructions. Given an arbitrary subset M of $\mathbb{R}^2$, the *M-sum* of sets K and L in $\mathbb{R}^n$ is defined by

$$K \oplus_M L := \{ax + by : x \in K,\ y \in L,\ (a,b) \in M\}. \tag{9.26}$$

It was first introduced by Protasov [1548], for special sets M. In the more general version of the M-combination of m sets in $\mathbb{R}^n$, with a given set $M \subset \mathbb{R}^m$, it was studied by Gardner, Hug and Weil [679]. Definition (9.26) yields the Minkowski sum if $M = \{(1,1)\}$, and for $p > 1$ its restriction to $(\mathcal{K}^n_o)^2$ yields the L_p sum if $M = \{(a,b) \in [0,1]^2 : a^q + b^q = 1\}$, where $1/p + 1/q = 1$. The operation $\oplus_M$ maps $(\mathcal{K}^n)^2$ to $\mathcal{K}^n$ if and only if $M \in \mathcal{K}^2$ and M is contained in one of the four closed quadrants of $\mathbb{R}^2$. The M-addition on the set $\mathcal{K}^n_{os}$ of o-symmetric convex bodies in $\mathcal{K}^n_o$ is characterized in [679] (Theorem 7.6) in the following way.

Theorem 9.1.7 *Let $n \geq 2$. The projection covariant operations $* : (\mathcal{K}^n_{os})^2 \to \mathcal{K}^n$ are precisely those defined for all $K, L \in \mathcal{K}^n_{os}$ by*

$$K * L = K \oplus_M L,$$

where $M \in \mathcal{K}^2$ is 1-unconditional (that is, symmetric with respect to the standard coordinate axes). The set M is uniquely determined by $*$.

The fact that L_p spaces have a natural generalization, known as Orlicz spaces, motivated Lutwak, Yang and Zhang [1303, 1304] to initiate an extension of the L_p Brunn–Minkowski theory to an Orlicz–Brunn–Minkowski theory. The definition of a corresponding addition came later, in work of Gardner, Hug and Weil [680]. They developed a very general and comprehensive Orlicz–Brunn–Minkowski theory, of which we can only describe here basic principles in special cases. For $m \in \mathbb{N}$, let Φ_m be the set of all convex functions $\varphi : [0,\infty)^m \to [0,\infty)$ that are increasing in each variable and satisfy $\varphi(0,\dots,0) = 0$ and $\varphi(0,\dots,1,\dots,0) = 1$, where the argument 1 can be at any of the m places. Now let $\varphi \in \Phi_2$. The *Orlicz sum* $K +_\varphi L$ of convex bodies $K, L \in \mathcal{K}^n_o$ is defined implicitly by

$$\varphi\left(\frac{h_K(x)}{h_{K+_\varphi L}(x)}, \frac{h_L(x)}{h_{K+_\varphi L}(x)}\right) = 1$$

for $x \in \mathbb{R}^n$, if $h_K(x) + h_L(x) > 0$, and by $h_{K+_\varphi L}(x) = 0$ otherwise. The operation $+_\varphi$ is continuous, $\mathrm{GL}(n)$ covariant and preserves the o-symmetry. It reduces to L_p addition, $1 \le p < \infty$, if $\varphi(x_1, x_2) = x_1^p + x_2^p$, and to L_∞ addition if $\varphi(x_1, x_2) = \max\{x_1, x_2\}$. These are the only cases where the Orlicz addition is associative. It is commutative if and only if it is L_∞ addition or $\varphi(x_1, x_2) = \psi(x_1) + \psi(x_2)$ with $\psi \in \Phi_1$. Relations between Orlicz addition and M-addition are studied in [680]. Also treated there are an extension of Orlicz addition to arbitrary sets and a corresponding version of the Brunn–Minkowski theorem. This says that for compact sets K, L in $\mathbb{R}^n$ with $V_n(K)V_n(L) > 0$ the inequality

$$1 \ge \varphi\left(\left(\frac{V_n(K)}{V_n(K +_\varphi L)}\right)^{1/n}, \left(\frac{V_n(L)}{V_n(K +_\varphi L)}\right)^{1/n}\right)$$

holds. When φ is strictly convex, equality holds if and only if K and L are convex, contain the origin and are dilatates of each other. When $\varphi(x_1, x_2) = x_1^p + x_2^p$ with $p > 1$, this result reduces to the extension of Firey's L_p Brunn–Minkowski inequality to compact sets that was proved by Lutwak, Yang and Zhang [1305].

Let $K, L \in \mathcal{K}^n_o$. In the special case where $\varphi(x_1, x_2) = \varphi_1(x_1) + \varepsilon\varphi_2(x_2)$ with $\varphi_1, \varphi_2 \in \Phi_1$ and $\varepsilon > 0$, one writes $K +_{\varphi,\varepsilon} L$ instead of $K +_\varphi L$. Extending the variational formula (9.11), it is shown in [680] that

$$V_{\varphi_2}(K, L) := \frac{(\varphi_1)'_l(1)}{n} \lim_{\varepsilon \downarrow 0} \frac{V_n(K +_{\varphi,\varepsilon} L) - V_n(K)}{\varepsilon} \tag{9.27}$$

exists for $K \in \mathcal{K}^n_{(o)}$ and $L \in \mathcal{K}^n_o$ and is given by

$$V_{\varphi_2}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \varphi_2\left(\frac{h_L}{h_K}\right) h_K \, \mathrm{d}S_{n-1}(K, \cdot). \tag{9.28}$$

Further in [680], among other inequalities, for convex bodies $K \in \mathcal{K}^n_{(o)}$ and $L \in \mathcal{K}^n_o$ an extension of the L_p Minkowski inequality (9.13) is obtained in the form

$$V_\varphi(K, L) \ge V_n(K)\,\varphi\left(\left(\frac{V_n(L)}{V_n(K)}\right)^{1/n}\right),$$

where $\varphi \in \Phi_1$. When φ is strictly convex, equality holds if and only if K and L are dilatates or $L = \{o\}$.

Motivated by the definition of the Orlicz norm, the following general construction is performed in [680]. Let $m \in \mathbb{N}$ and let $\varphi \in \Phi_m$. Let μ be a Borel measure on $(\mathcal{K}_o^n)^m$ with

$$\int_{(\mathcal{K}_o^n)^m} \varphi(h_{K_1}(x), \dots, h_{K_m}(x))\, \mathrm{d}\mu(K_1, \dots, K_m) < \infty$$

for all $x \in \mathbb{R}^n$. Then

$$h_{C_{\varphi,\mu}}(x) := \inf \left\{ \lambda > 0 : \int_{(\mathcal{K}_o^n)^m} \varphi\left(\frac{h_{K_1}(x)}{\lambda}, \dots, \frac{h_{K_m}(x)}{\lambda} \right) \mathrm{d}\mu(K_1, \dots, K_m) \leq 1 \right\}$$

defines the support function of a convex body $C_{\varphi,\mu} \in \mathcal{K}_o^n$. This construction comprises an Orlicz sum of m convex bodies with positive real coefficients, as well as, for suitable choices of m and μ, the Orlicz projection body and Orlicz centroid body of [1303, 1304] and their asymmetric versions.

Notes for Section 9.1

1. A first systematic study of various means of convex bodies was made by Firey [583, 584, 585, 588, 595], among them p-means in [585]. Except for a note of Fedotov [558], it took 30 years until this topic was taken up and elaborated, in Lutwak's papers [1283, 1287], which mark the beginning of a remarkable development.

   Firey [585] already obtained inequality (9.15), which was proved in a different way by Lutwak [1283].
2. *Axiomatic characterization.* An axiomatic characterization of L_p addition for centrally symmetric convex bodies was given by Gardner, Hug and Weil [679]. For $n \geq 2$, they showed that an operation between o-symmetric convex bodies is continuous, $\mathrm{GL}(n)$ covariant and associative if and only if it is L_p addition for some $1 \leq p \leq \infty$, or one of three trivial exceptions. See [679], Theorem 7.9, for a precise formulation.
3. Lutwak, Yang and Zhang [1305] extended the p-addition for $p \geq 1$ to arbitrary sets and proved an L_p Brunn–Minkowski theorem for compact sets.
4. The case $i = 0$ of inequalities (9.12) and (9.15) may be extended to Wulff shapes. With the definition

   $$V_p(K, f) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} f^p h_K^{1-p}\, \mathrm{d}S_{n-1}(K, \cdot)$$

   for $K \in \mathcal{K}_{(o)}^n$ and $f \in C^+(\mathbb{S}^{n-1})$, the inequality

   $$V_p(K, f)^n \geq V_n(K)^{n-p} V(f)^p$$

   holds for $p \geq 1$ (recall that $V(f)$ denotes the volume of the Wulff shape associated with f). Also for $p \geq 1$, if $f, g \in C^+(\mathbb{S}^{n-1})$, then

   $$V((f^p + g^p)^{1/p})^{p/n} \geq V(f)^{p/n} + V(g)^{p/n}.$$

   These inequalities, together with equality conditions, were proved by Hu and Jiang [996].
5. A stability result for the Aleksandrov–Fenchel–Jessen type theorem 9.1.6 was proved by Hug and Schneider [1014], Theorem 4.1.

9.2 The L_p Minkowski problem and generalizations

Minkowski's existence theorem for convex bodies with given surface area measure (Theorem 8.2.2) motivates the following *L_p Minkowski problem*. For given $p \geq 1$, characterize the measures $S_{p,0}(K, \cdot)$ of convex bodies $K \in \mathcal{K}^n_{(o)}$. In other words, find necessary and sufficient conditions for a Borel measure φ on $\mathbb{S}^{n-1}$ in order that there is a convex body $K \in \mathcal{K}^n_{(o)}$ with

$$h_K^{1-p} \mathrm{d}S_{n-1}(K, \cdot) = \mathrm{d}\varphi. \tag{9.29}$$

The appearance of the special function h_K^{1-p} with $p \geq 1$ on the left side of (9.29) is caused by the use of Firey's p-sum for convex bodies. Independent of this, generalized Minkowski problems of the form

$$G(h_K)\, \mathrm{d}S_{n-1}(K, \cdot) = \mathrm{d}\varphi \tag{9.30}$$

with suitable functions G deserve interest. In some cases, it may be appropriate to also admit bodies $K \in \mathcal{K}^n_o$ as solutions, for which the support function may vanish somewhere.

The L_p Minkowski problem (9.29) with $p > 0$ is accessible to the methods developed for the case $p = 1$ if one assumes central symmetry. For this case, we follow Lutwak [1283] and give a proof which is modelled after the one given by Aleksandrov [15] (§3) for the classical case. It minimizes a suitable functional and then uses Aleksandrov's variational formula. Since the L_p theory for $p \neq 1$ is not translation invariant, one must make sure (for the variation argument) that the minimizing body has the origin in its interior; this is the reason for the assumption of central symmetry.

Theorem 9.2.1 *Let $p > 0$. Let φ be an even finite Borel measure on $\mathbb{S}^{n-1}$ which is positive on each open hemisphere of $\mathbb{S}^{n-1}$.*

If $p \neq n$, then there exists an o-symmetric convex body $K \in \mathcal{K}^n_{(o)}$ such that $S_{p,0}(K, \cdot) = \varphi$.

If $p = n$, then there are a number $\lambda > 0$ and an o-symmetric convex body $K \in \mathcal{K}^n_{(o)}$ such that $\lambda S_{p,0}(K, \cdot) = \varphi$.

Proof Let $C_e^+(\mathbb{S}^{n-1})$ be the set of even, positive, continuous functions on the unit sphere. For $f \in C_e^+(\mathbb{S}^{n-1})$, let A be the Wulff shape associated with f (see (7.97) and the explanations given there), and let $V(f)$ be its volume. Define $\Phi : C_e^+(\mathbb{S}^{n-1}) \to (0, \infty)$ by

$$\Phi(f) := V(f)^{-p/n} \int_{\mathbb{S}^{n-1}} f(u)^p \, \mathrm{d}\varphi(u).$$

Since $h_A \leq f$ and $V(h_A) = V(f)$, we have $\Phi(h_A) \leq \Phi(f)$. Therefore, if Φ attains a minimum on the set $\mathscr{L} := \{h_L : L \in \mathcal{K}^n_{(o)}, L = -L\}$, then this is at the same time the minimum of Φ over all functions in $C_e^+(\mathbb{S}^{n-1})$. We show that Φ indeed attains a minimum on $\mathscr{L}$. Since Φ is homogeneous of degree zero, it suffices to find a minimum on the set

$$\mathscr{L}' := \{h_L : L \in \mathcal{K}^n_{(o)},\ L = -L,\ V_n(L) = 1,\ \Phi(h_L) \le b\},$$

where b is the value of Φ at the o-centred ball of volume one.

Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of convex bodies with $h_{K_i} \in \mathscr{L}'$ such that

$$\lim_{i\to\infty} \Phi(h_{K_i}) = \inf\{\Phi(f) : f \in \mathscr{L}'\}.$$

To show the boundedness of this sequence, we note that there is a number $a > 0$ with

$$\int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|^p \mathrm{d}\varphi(v) \ge a \qquad \text{for } u \in \mathbb{S}^{n-1}.$$

This is true since the integral is a continuous function of u which is positive, by the properties of φ. Now for given i, let $x \in K_i$ and write $x = |x|v$ with $v \in \mathbb{S}^{n-1}$. Then $-x \in K_i$, hence $h(K_i, u) \ge |x| \cdot |\langle u, v\rangle|$ for $u \in \mathbb{S}^{n-1}$, which gives

$$a|x|^p \le |x|^p \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|^p \mathrm{d}\varphi(u) \le \int_{\mathbb{S}^{n-1}} h_{K_i}(u)^p \,\mathrm{d}\varphi(u) = \Phi(h_{K_i}) \le b,$$

thus $|x|^p \le b/a$. It follows that the sequence $(K_i)_{i\in\mathbb{N}}$ is bounded. By the Blaschke selection theorem, it has a subsequence converging to some convex body M. This body is o-symmetric and has volume one; therefore o is an interior point of M and thus M belongs to $\mathscr{L}'$.

Let $f \in C_e^+(\mathbb{S}^{n-1})$. Then $(h_M^p + \varepsilon f)^{1/p} \in C_e^+(\mathbb{S}^{n-1})$ for sufficiently small $|\varepsilon|$, hence the function

$$\begin{aligned}\varepsilon \mapsto &\Phi((h_M^p + \varepsilon f)^{1/p}) \\ &= V((h_M^p + \varepsilon f)^{1/p})^{-p/n} \left(\int_{\mathbb{S}^{n-1}} h_M^p \,\mathrm{d}\varphi + \varepsilon \int_{\mathbb{S}^{n-1}} f \,\mathrm{d}\varphi \right)\end{aligned} \tag{9.31}$$

attains a minimum at $\varepsilon = 0$. Lemma 7.5.3 with $G(\varepsilon, \cdot) := (h_M^p + \varepsilon f)^{1/p} - h_M$ yields

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} V((h_M^p + \varepsilon f)^{1/p})\Big|_{\varepsilon=0} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} f h_M^{1-p} \,\mathrm{d}S_{n-1}(M, \cdot).$$

Hence, the derivative of the function (9.31) at $\varepsilon = 0$ is given by

$$-\int_{\mathbb{S}^{n-1}} f h_M^{1-p} \,\mathrm{d}S_{n-1}(M, \cdot) \cdot \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_M^p \,\mathrm{d}\varphi + \int_{\mathbb{S}^{n-1}} f \,\mathrm{d}\varphi$$

(recall that $V(h_M) = 1$), and this is equal to zero. Defining

$$\lambda := \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_M^p \,\mathrm{d}\varphi, \tag{9.32}$$

we have $\lambda > 0$ and

$$\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}\varphi = \lambda \int_{\mathbb{S}^{n-1}} f \,\mathrm{d}S_{p,0}(M, \cdot).$$

Since this holds for all functions $f \in C_e^+(\mathbb{S}^{n-1})$, it holds for all even functions $f \in C(\mathbb{S}^{n-1})$, and since φ and $S_{p,0}(M,\cdot)$ are even, we conclude that $\varphi = \lambda S_{p,0}(M,\cdot)$.

If $p \neq n$, we can define $c > 0$ by $c^{n-p} = \lambda$ and then have $\varphi = S_{p,0}(cM,\cdot)$. □

Some uniqueness results corresponding to Theorem 9.2.1 follow from Theorem 9.1.6. If $p \geq 1$ and $p \neq n$, then the o-symmetric convex body that exists by Theorem 9.2.1 is unique. If $p = n$ and if two pairs $(K_1,\lambda_1),(K_2,\lambda_2)$ satisfy $\lambda_i S_{p,0}(K_i,\cdot) = \varphi$, $i = 1,2$, then the second part of Theorem 9.1.6 yields that $\lambda_1 = \lambda_2$ and that K_1 and K_2 are dilatates. Uniqueness results for $0 < p < 1$ seem to be unknown.

The following 'volume normalized version' of the L_p Minkowski problem does not require a distinction between $p \neq n$ and $p = n$.

Corollary 9.2.2 *Let $p > 0$. Let φ be an even finite Borel measure on $\mathbb{S}^{n-1}$ which is positive on each open hemisphere of $\mathbb{S}^{n-1}$. There exists an o-symmetric convex body $K \in \mathcal{K}^n_{(o)}$ such that*

$$\varphi = \frac{S_{p,0}(K,\cdot)}{V_n(K)}.$$

Proof With M and λ as found at the end of the proof of Theorem 9.2.1, we have $\varphi = \lambda S_{p,0}(M,\cdot)$. Defining $a > 0$ by $a^{-p}/V_n(M) = \lambda$, we obtain $\varphi = S_{p,0}(aM,\cdot)/V_n(aM)$. □

Obviously, in the previous corollary the volume functional can be replaced by any functional on convex bodies that is homogeneous of some degree different from $n-p$.

In the general, non-even case, we quote the following result without proof.

Theorem 9.2.3 *Let $p > 1$, $p \neq n$. Let φ be a finite Borel measure on $\mathbb{S}^{n-1}$ which is positive on each open hemisphere of $\mathbb{S}^{n-1}$. There exists a unique convex body $K \in \mathcal{K}^n_o$ such that*

$$h_K^{p-1}\,\mathrm{d}\varphi = \mathrm{d}S_{n-1}(K,\cdot).$$

Moreover, if φ is discrete, then K is a polytope in $\mathcal{K}^n_{(o)}$, and, for general φ, if $p > n$ then $K \in \mathcal{K}^n_{(o)}$.

This formulation combines results that were obtained by two different approaches. With PDE methods and approximation, the theorem, without the discrete case, was proved by Chou and Wang [428]. Their methods also yield regularity results. They further obtained the following result for $p = n$. There are a number $\lambda > 0$ and a convex body $K \in \mathcal{K}^n_o$ such that $\lambda h_K^{p-1}\,\mathrm{d}\varphi = \mathrm{d}S_{n-1}(K,\cdot)$. An elementary approach to Theorem 9.2.3 was followed by Hug, Lutwak, Yang and Zhang [1012]. They gave two proofs for the discrete case, one by using Aleksandrov's mapping lemma and the other by a variational argument, and then employed approximation. An additional argument (extending one by Chou and Wang) shows for $p > n$ that the solution K contains the origin in its interior. The authors also showed by an example that here the assumption $p > n$ is necessary, but that it can be avoided in the discrete case.

In the generalized Minkowski problem (9.30), the case $G = \mathrm{id}$, which is (9.29) with $p = 0$, is of particular interest, since the involved measure V_K, which for $K \in \mathcal{K}^n_{(o)}$ is defined by

$$V_K(\omega) := \frac{1}{n} S_{0,0}(K, \omega) = \frac{1}{n} \int_\omega h_K \, \mathrm{d}S_{n-1}(K, \cdot), \quad \omega \in \mathcal{B}(\mathbb{S}^{n-1}),$$

has a particular geometric meaning.

Lemma 9.2.4 *For $K \in \mathcal{K}^n_{(o)}$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$,*

$$V_K(\omega) = \mathcal{H}^n \left(\bigcup_{x \in \tau(K,\omega)} [o, x] \right). \tag{9.33}$$

The measure V_K depends weakly continuously on K.

If K is a polytope, (9.33) is clear, since $V_K(\omega)$ is the sum of the volumes of the cones with apex at the origin and bases the facets of K with outer normal vectors in ω. For this reason, V_K is called the *cone-volume measure* of K, also for general convex bodies K.

Proof of Lemma 9.2.4 First we assume that $K_j \to K_0$ in $\mathcal{K}^n_{(o)}$ and show that this implies $V_{K_j} \xrightarrow{w} V_{K_0}$. For $j \in \mathbb{N}_0$, let η_j be the measure defined by

$$\eta_j(\omega) := \frac{1}{n} \int_\omega h_{K_0} \, \mathrm{d}S_{n-1}(K_j, \cdot), \quad \omega \in \mathcal{B}(\mathbb{S}^{n-1}).$$

Since h_{K_0} is continuous and S_{n-1} is weakly continuous, we have $\eta_j \xrightarrow{w} \eta_0$ for $j \to \infty$.

By Theorem 1.8.15, the sequence $(h_{K_j})_{j\in\mathbb{N}}$ converges uniformly on $\mathbb{S}^{n-1}$ to h_{K_0}. Hence, for each $\varepsilon > 0$ we have $h_{K_0}(u) \le h_{K_j}(u) + \varepsilon$ for all $u \in \mathbb{S}^{n-1}$ and hence $\eta_j(\omega) \le V_{K_j}(\omega) + c\varepsilon$, if j is sufficiently large; here c is a constant independent of j. Since this holds for all $\varepsilon > 0$ and since $\eta_j \xrightarrow{w} \eta_0$, we deduce that for open sets ω we have

$$V_{K_0}(\omega) = \eta_0(\omega) \le \liminf_{j\to\infty} \eta_j(\omega) \le \liminf_{j\to\infty} V_{K_j}(\omega).$$

Since $V_{K_0}(\mathbb{S}^{n-1}) = \lim_{j\to\infty} V_{K_j}(\mathbb{S}^{n-1})$ by the continuity of the volume, it follows that $V_{K_j} \xrightarrow{w} V_{K_0}$, as claimed.

Let $\varphi(x) := x/|x|$ for $x \in \mathbb{R}^n \setminus \{o\}$ and note that the use of polar coordinates gives

$$\begin{aligned} \zeta(K, \omega) := \mathcal{H}^n \left(\bigcup_{x \in \tau(K,\omega)} [o, x] \right) &= \int_{\varphi(\tau(K,\omega))} \int_0^{\rho(K,u)} s^{n-1} \, \mathrm{d}s \, \mathrm{d}\mathcal{H}^{n-1}(u) \\ &= \frac{1}{n} \int_{\varphi(\tau(K,\omega))} \rho(K, u)^n \, \mathrm{d}\mathcal{H}^{n-1}(u). \end{aligned}$$

Suppose that $K_j \to K_0$ in $\mathcal{K}^n_{(o)}$. For $\mathcal{H}^{n-1}$-almost all $u \in \mathbb{S}^{n-1}$, the outer unit normal vector $n(K_j, u)$ of K at $\rho(K_j, u)u$ is uniquely determined ($j \in \mathbb{N}_0$), as follows from Theorem 2.2.5, applied to the countably many bodies K_j. Hence, for almost all u we have $n(K_j, u) \to n(K_0, u)$ for $j \to \infty$. If ω is open, this implies that for almost all $u \in \mathbb{S}^{n-1}$ we have

$$\mathbf{1}_{\varphi(\tau(K_0,\omega))}(u) \le \liminf_{j\to\infty} \mathbf{1}_{\varphi(\tau(K_j,\omega))}(u).$$

Therefore, Fatou's lemma and the continuous dependence of $\rho(K,\cdot)$ on K give

$$\zeta(K_0, \omega) \le \liminf_{j\to\infty} \zeta(K_j, \omega).$$

Together with $\zeta(K_0, \mathbb{S}^{n-1}) = \lim_{j\to\infty} \zeta(K_j, \mathbb{S}^{n-1})$ this shows that $\zeta(K_j, \cdot) \xrightarrow{w} \zeta(K_0, \cdot)$ for $j \to \infty$.

Since the assertion $\zeta(K,\cdot) = V_K$ holds for polytopes, approximation now yields it for general convex bodies $K \in \mathcal{K}_{(o)}$. □

The cone-volume measure obviously has the following invariance property. If $\phi \in \mathrm{SL}(n)$, then

$$V_{\phi K}(\omega) = V_K(\langle \phi^t \omega \rangle),$$

where $\langle \phi^t \omega \rangle = \{\phi^t u/|\phi^t u| : u \in \omega\}$.

Just as, in the case of the classical Minkowski problem, the translation invariance of the volume imposes a nontrivial necessary condition on the surface area measures of convex bodies, so the geometric meaning (9.33) of the cone-volume measure imposes a nontrivial necessary condition on the cone-volume measures of o-symmetric convex bodies. We derive this condition now.

Let $K \in \mathcal{K}^n_{(o)}$ be o-symmetric. Let L be a q-dimensional linear subspace of $\mathbb{R}^n$, where $q \in \{1, \dots, n-1\}$. Let $K_L := K|L$ be the orthogonal projection of K to L, and let $\mathbb{S}_L := \mathbb{S}^{n-1} \cap L$ be the unit sphere in L. We want to estimate the cone-volume measure $V_K(\mathbb{S}_L)$. Writing $M := \bigcup_{x\in\tau(K,\mathbb{S}_L)}[o, x]$, decomposing $\mathbb{R}^n = L \oplus L^\perp$ and using polar coordinates in L, we have

$$V_K(\mathbb{S}_L) = \mathcal{H}^n(M) = \int_{\mathbb{S}_L} \int_0^{\rho(u)} \mathcal{H}^{n-q}((L^\perp + su) \cap M) s^{q-1} \,\mathrm{d}s \,\mathrm{d}\sigma_{q-1}(u).$$

Here ρ denotes the radial function of K_L and σ_{q-1} is spherical Lebesgue measure on $\mathbb{S}_L$.

Let $u \in \mathbb{S}_L$, $0 < s \le \rho(u)$ and $y \in (L^\perp + su) \cap M$. Then y is contained in a segment $[o, x]$, where $x \in \tau(K, \mathbb{S}_L)$. Define

$$F_u := (L^\perp + \rho(u)u) \cap K.$$

Then $x \in F_u$. Conversely, every point $x \in F_u$ gives rise to a point $y \in [o, x] \cap (L^\perp + su) \cap M$, and a simple proportionality argument shows that

$$(L^\perp + su) \cap M = \frac{s}{\rho(u)} F_u.$$

This gives

$$V_K(\mathbb{S}_L) = \int_{\mathbb{S}_L} \int_0^{\rho(u)} \left(\frac{s}{\rho(u)}\right)^{n-q} V_{n-q}(F_u) s^{q-1}\, \mathrm{d}s\, \mathrm{d}\sigma_{q-1}(u)$$
$$= \frac{1}{n} \int_{\mathbb{S}_L} \rho(u)^q V_{n-q}(F_u)\, \mathrm{d}\sigma_{q-1}(u).$$

For the volume of K we obtain

$$V_n(K) = \int_{\mathbb{S}_L} \int_0^{\rho(u)} V_{n-q}((L^\perp + su) \cap K) s^{q-1}\, \mathrm{d}s\, \mathrm{d}\sigma_{q-1}(u).$$

For fixed $u \in \mathbb{S}_L$, it follows from the Brunn–Minkowski theorem and the central symmetry of K that the function $s \mapsto V_{n-q}((L^\perp + su) \cap K)^{1/(n-q)}$, $s \in [-\rho(u), \rho(u)]$, is concave and even, hence

$$V_{n-q}((L^\perp + su) \cap K) \geq V_{n-q}(F_u)$$

and thus

$$V_n(K) \geq \int_{\mathbb{S}_L} \int_0^{\rho(u)} V_{n-q}(F_u) s^{q-1}\, \mathrm{d}s\, \mathrm{d}\sigma_{q-1}(u)$$
$$= \frac{1}{q} \int_{\mathbb{S}_L} \rho(u)^q V_{n-q}(F_u)\, \mathrm{d}\sigma_{q-1}(u).$$

We conclude that

$$V_K(\mathbb{S}_L) \leq \frac{q}{n} V_n(K). \tag{9.34}$$

Suppose that equality holds here. Then by the equality condition of the Brunn–Minkowski theorem and the central symmetry of K, for each $z \in K_L$ the intersection $(L^\perp + z) \cap K$ is a translate of $L^\perp \cap K$, say $(L^\perp \cap K) + t(z)$. We assert that, for $z_1, z_2 \in K_L$ and $\lambda \in (0, 1)$, we have $t((1-\lambda)z_1 + \lambda z_2) = (1-\lambda)t(z_1) + \lambda t(z_2)$. Suppose this is false for some z_1, z_2, λ. Then

$$v := (1-\lambda)t(z_1) + \lambda t(z_2) - t((1-\lambda)z_1 + \lambda z_2) \neq o$$

(note that $v \in L^\perp$). Let $x \in L^\perp \cap K$ be such that $h(L^\perp \cap K, v) = \langle x, v \rangle$. Then $x + t(z_i) \in K$ for $i = 1, 2$, but

$$(1-\lambda)(x + t(z_1)) + \lambda(x + t(z_2)) = x + t((1-\lambda)z_1 + \lambda z_2) + v \notin K,$$

which contradicts the convexity of K.

The assertion thus proved shows that the points $t(z)$, $z \in K_L$, lie in a subspace E of $\mathbb{R}^n$, necessarily complementary to $L^\perp$, and that K is the direct sum of $K \cap L^\perp$ and $K \cap E$. A computation corresponding to the one above now shows that

$$V_K(E^\perp \cap \mathbb{S}^{n-1}) = \frac{\dim E^\perp}{n} V_n(K).$$

We are ready for a definition.

Definition Let φ be a finite Borel measure on $\mathbb{S}^{n-1}$. The measure φ satisfies the *subspace concentration condition* if

$$\varphi(L \cap \mathbb{S}^{n-1}) \leq \frac{\dim L}{n} \varphi(\mathbb{S}^{n-1}) \tag{9.35}$$

holds for every subspace L of $\mathbb{R}^n$, and if equality holds here, then there exists a subspace L' complementary to L such that

$$\varphi(L' \cap \mathbb{S}^{n-1}) = \frac{\dim L'}{n} \varphi(\mathbb{S}^{n-1}).$$

The measure φ is said to satisfy the *strict subspace concentration inequality* if (9.35) holds with strict inequality whenever $0 < \dim L < n$.

We have shown above that the cone-volume measure of an o-symmetric convex body satisfies the subspace concentration condition. This was proved (see also Note 5 below) by Böröczky, Lutwak, Yang and Zhang [294]. Remarkably, these authors have shown that this necessary condition is also sufficient.

Theorem 9.2.5 (Böröczky, Lutwak, Yang, Zhang) *A non-zero even finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$ is the cone-volume measure of an o-symmetric convex body in $\mathcal{K}^n_{(o)}$ if and only if it satisfies the subspace concentration condition.*

Uniqueness of the solution in Theorem 9.2.5 has not been established in full generality. Under smoothness assumptions, the following is known. If the cone-volume measure of an o-symmetric convex body is proportional to spherical Lebesgue measure, then the body is a ball. This was proved by Firey [605] in the course of his investigation of asymptotic shapes of worn stones. Without the symmetry assumption, Andrews [71] showed the uniqueness in $\mathbb{R}^3$. A complete solution is known in the planar case. Böröczky, Lutwak, Yang and Zhang [295] proved the following. If $K, L \in \mathcal{K}^2_{(o)}$ are o-symmetric and have the same cone-volume measure, then either $K = L$ or else K and L are parallelograms with parallel sides. The special case of smooth planar convex bodies with positive curvature was proved earlier by Gage [658], and the case of polygons by Stancu [1806].

In the case of the generalized Minkowski problem (9.30), one cannot expect a solution without a multiplicative constant. This is already clear from the case $p = n$ of Theorem 9.2.1, where suitable homogeneity properties to remove the constant are lacking. Under appropriate assumptions on the function G, however, other conditions can be satisfied. The following result was obtained by Haberl, Lutwak, Yang and Zhang [878].

Theorem 9.2.6 *Let $G : (0, \infty) \to (0, \infty)$ be a continuous decreasing function. If φ is an even finite Borel measure on $\mathbb{S}^{n-1}$ which is positive on each open hemisphere of $\mathbb{S}^{n-1}$, then for each $\alpha \in (0, 1)$ there exists an o-symmetric convex body $K \in \mathcal{K}^n_{(o)}$ such that*

$$cG(h_K)\, \mathrm{d}S_{n-1}(K, \cdot) = \mathrm{d}\varphi,$$

with $c = V_n(K)^{(\alpha/n)-1}$.

Another achievable condition is suggested by Theorem 9.2.1. The proof of the latter yields a convex body $M \in \mathcal{K}^n_{(o)}$ with $\varphi = \lambda S_{p,0}(M, \cdot)$, where the constant λ is given by (9.32), that is,

$$\lambda = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_M^p \, \mathrm{d}\varphi.$$

For $p \neq n$, one can use homogeneity to find a dilatate aM such that $\varphi = cS_{p,0}(aM, \cdot)$ with a constant c and $\|h_{aM}\|_p = 1$, where $\|\cdot\|_p$ denotes the L_p norm with respect to φ. The latter condition can be extended by using a suitable Orlicz norm.

Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function, continuously differentiable on $(0, \infty)$ with positive derivative, and satisfying $\lim_{t\to\infty} \phi(t) = \infty$. For a finite, non-zero Borel measure φ and a continuous, nonnegative function f on $\mathbb{S}^{n-1}$, the *Orlicz norm* $\|f\|_\phi$ is defined by

$$\|f\|_\phi := \inf \left\{ \lambda > 0 : \frac{1}{\varphi(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \phi\left(\frac{f}{\lambda}\right) \mathrm{d}\varphi \le \phi(1) \right\}.$$

The following was also proved in [878].

Theorem 9.2.7 *Let $G : (0, \infty) \to (0, \infty)$ be a continuous function such that $\phi(t) = \int_0^t 1/G(s)\, \mathrm{d}s$ exists for every positive t and is unbounded as $t \to \infty$. If φ is an even finite Borel measure on $\mathbb{S}^{n-1}$ which is positive on each open hemisphere, then there exist an o-symmetric convex body $K \in \mathcal{K}^n_{(o)}$ and a constant $c > 0$ such that*

$$cG(h_K)\, \mathrm{d}S_{n-1}(K, \cdot) = \mathrm{d}\varphi$$

and $\|h_K\|_\phi = 1$.

By further developing the methods of [1012] and [878], Huang and He [997] extended the latter result to not necessarily even measures φ, under the additional assumption that $\lim_{s\downarrow 0} G(s) = \infty$.

Notes for Section 9.2

1. The proof of Theorem 9.2.1 given here follows Lutwak [1283] (where $p \ge 1$ is assumed, but only $p > 0$ is used in the proof). As mentioned, it is an adaptation of the one given by Aleksandrov [15] (§3) for the case $p = 1$. For Corollary 9.2.2, Lutwak, Yang and Zhang [1296] have given a different proof, which first treats the case of discrete data and polytopes and then uses approximation.
2. *L_p Blaschke addition.* Let $p > 0$. For o-symmetric convex bodies $K, L \in \mathcal{K}^n_n$, Theorem 9.2.1 allows one to define a unique o-symmetric convex body $K \#_p L$ by

$$S_{p,0}(K \#_p L, \cdot) = S_{p,0}(K, \cdot) + S_{p,0}(L, \cdot).$$

For $n \neq p > 1$ one has (Lutwak [1283])

$$V_n(K \#_p L)^{(n-p)/n} \geq V_n(K)^{(n-p)/n} + V_n(L)^{(n-p)/n},$$

with equality if and only if K and L are dilatates. This extends the Kneser–Süss inequality of Theorem 8.2.3.

Lu and Leng [1234] have various Brunn–Minkowski type inequalities involving L_p Blaschke addition, for L_p projection bodies, L_p centroid bodies, L_p curvature images and L_p polar projection bodies.

3. *Regular solutions of the L_p Minkowski problem.* If the given measure φ in the L_p Minkowski problem (9.29) has a density g with respect to spherical Lebesgue measure σ, then the problem can be formulated as follows. For $p \geq 1$, find necessary and sufficient conditions on g in order that there is a sufficiently smooth convex body $K \in \mathcal{K}_{(0)}$ (or perhaps $\mathcal{K}_0$) with

$$s_{n-1}(K, \cdot) = h_K^{p-1} g, \tag{9.36}$$

where $s_{n-1}(K, \cdot)$ denotes the product of the principal radii of curvature of K (the reciprocal Gauss curvature as a function of the outer unit normal vector).

The following was proved by Lutwak and Oliker [1290]. If $g \in C^m(\mathbb{S}^{n-1})$, $m \geq 3$, is positive and even, and if $p \neq n$, then there exists a solution K of (9.36) with a support function of class $C^{m+1,\alpha}(\mathbb{S}^{n-1})$, for any $\alpha \in (0, 1)$. If g is analytic, then h_K is analytic.

Strong regularity results for the Monge–Ampère equation equivalent to (9.36) were obtained by Chou and Wang [428]. They had to distinguish between the cases $p > n$, $p = n$ and $1 < p < n$. They also studied the corresponding Monge–Ampère equation for $p = -n$.

Further contributions to the L_p Minkowski problem from the PDE point of view are due to Huang and Lu [999], Lu and Wang [1241].

4. *The planar L_p Minkowski problem and relatives.* In the plane, several variants of L_p Minkowski problems have been considered. Some relevant results appear as special cases of more general investigations about curvature flows. Under smoothness assumptions, Gage and Li [659] obtained a solution of the L_0 Minkowski problem. Solutions of the L_p Minkowski problem appear as homothetic solutions of curvature flows in Andrews [72]. The discrete planar L_0 Minkowski problem was investigated by Stancu [1805, 1806, 1808]. Further studies of the planar L_p Minkowski problem for various p were made by Umanskiy [1863], Chen [416], Jiang [1040], Dou and Zhu [512], Ivaki [1027].
5. *The cone-volume measure.* For o-symmetric polytopes, the inequality (9.34) was apparently first proved in Henk, Schürmann and Wills [958]. Special cases of this inequality also appear in Stancu [1808], He, Leng and Li [948], Xiong [1995]. The general subspace concentration condition was proved in Böröczky, Lutwak, Yang and Zhang [294].
6. *Intermediate L_p Christoffel–Minkowski problems.* Let $p \geq 1$. For $K \in \mathcal{K}^n_{(o)}$ of class C^2_+, the function

$$h_K^{1-p} s_j(K, \cdot)$$

(recall that $s_j(K, \cdot)$ denotes the jth normalized elementary symmetric function of the principal radii of curvature of K) has been called the jth p-area function of K by Hu, Ma and Shen [994]. These authors proved the following theorem. Here a positive function f of class C^2 on $\mathbb{S}^{n-1}$ is called *spherical convex* if the spherical Hessian $(f_{ij} + \delta_{ij} f)_{i,j=1}^{n-1}$ is positive semi-definite on $\mathbb{S}^{n-1}$ (the subscripts of f denote covariant derivatives with respect to a local orthonormal frame field on $\mathbb{S}^{n-1}$).

Theorem Let $j \in \{1, \dots, n-2\}$ and $p \geq j+1$. For any positive function f of class C^m ($m \geq 2$) on $\mathbb{S}^{n-1}$ for which $f^{-1/(p+j+1)}$ is spherical convex on $\mathbb{S}^{n-1}$, there exists a unique convex body $K \in \mathcal{K}^n_{(o)}$ with a boundary of class $C^{m+1,\alpha}$ (for some $0 < \alpha < 1$) such that its jth p-area function is equal to f.

The authors also have a corresponding result for $p = j + 1$.

9.3 The dual Brunn–Minkowski theory

What is known as the dual Brunn–Minkowski theory is not based on an exact duality. To mention some instances of true dualities, we have already seen polar convex bodies, dual convex cones and conjugate convex functions, and we may think of dual vector spaces or of the duality between points and lines in projective planes. In these and in many other examples from various parts of mathematics, duality is a precise notion, with well-defined rules for the transfer from one situation to its dual. In contrast, the dual Brunn–Minkowski theory is 'dual' to the classical Brunn–Minkowski theory only in a vague and heuristic sense. But for this very reason, it is all the more intriguing and fruitful. The formal similarities in the appearances of objects, formulae and inequalities between the Brunn–Minkowski theory and its dual are often striking. Particular interest comes, for example, from the fact that missing dual objects are not found by predetermined constructions, but must be discovered with a feeling for the 'duality', rather than translated mechanically.

A first step in a dual Brunn–Minkowski theory for star bodies, which to a great part is the creation of Lutwak [1261, 1262], consists in replacing support functions by radial functions, and area measures by suitable powers of radial functions.

Recall from Section 1.7 that a set $S \subset \mathbb{R}^n$ is *starshaped* (with respect to o) if $S \neq \emptyset$ and $[o, x] \subset S$ for all $x \in S$, and that for a compact starshaped set K the *radial function* $\rho(K, \cdot) = \rho_K$ is defined by

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{o\}.$$

Since ρ_K is homogeneous of degree -1, it is determined by its restriction $\bar{\rho}_K$ to the unit sphere $\mathbb{S}^{n-1}$. A *star body* is a compact starshaped set with a positive continuous radial function, and the set of all star bodies in $\mathbb{R}^n$ is denoted by $\mathcal{S}^n_o$. The *radial metric* on $\mathcal{S}^n_o$ is defined by

$$\tilde{\delta}(K, L) := \sup_{u \in \mathbb{S}^{n-1}} |\rho(K, u) - \rho(L, u)| = \|\bar{\rho}_K - \bar{\rho}_L\|$$

for $K, L \in \mathcal{S}^n_o$, where $\|\cdot\|$ is the maximum norm for continuous functions on the unit sphere. On $\mathcal{S}^n_o$, a duality $K \mapsto K^\star$, called *star duality*, can be defined by $\rho(K^\star, \cdot) := 1/\rho(K, \cdot)$ (Moszyńska [1451]).

A *radial addition* $\widetilde{+}$ can be defined by letting $x \widetilde{+} y$ be $x + y$ if x and y are on a line through o, and o otherwise ($x, y \in \mathbb{R}^n$). For $K, L \in \mathcal{S}^n_o$, the *radial sum*

$$K \widetilde{+} L := \{x \widetilde{+} y : x \in K,\ y \in L\}$$

then has the property that $\rho(K \widetilde{+} L, \cdot) = \rho(K, \cdot) + \rho(L, \cdot)$. More generally, for $K_1, \dots, K_m \in \mathcal{S}^n_o$ and $\lambda_1, \dots, \lambda_m \geq 0$, the *radial linear combination*

$$\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m$$

is defined by

$$\rho(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m, \cdot) := \lambda_1 \rho(K_1, \cdot) + \cdots + \lambda_m \rho(K_m, \cdot)$$

for $K_1, \dots, K_m \in \mathcal{S}^n_o$ and $\lambda_1, \dots, \lambda_m > 0$.

Also for star bodies K we use the notation $V_n(K) = \mathcal{H}^n(K)$ to denote the volume.

From (1.53) one immediately has the polynomial expansion

$$V_n(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \widetilde{V}(K_{i_1}, \dots, K_{i_n})$$

with

$$\widetilde{V}(K_1, \dots, K_n) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u)\, \mathrm{d}u.$$

The function $\widetilde{V} : (\mathcal{S}^n_o)^n \to \mathbb{R}$ is known as the *dual mixed volume*. One abbreviates

$$\widetilde{V}_i(K, L) := \widetilde{V}(K[n-i], L[i])$$

for $K, L \in \mathcal{S}^n_o$, where the argument K appears $n-i$ times, etc. Also the functions $\widetilde{V}_i$ have been called *dual mixed volumes*. More generally, and consistent with this, one defines

$$\widetilde{V}_r(K, L) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(u)^{n-r} \rho_L(u)^r \, \mathrm{d}u \qquad \text{for } r \in \mathbb{R}. \tag{9.37}$$

The special case

$$\widetilde{W}_i(K) := V_i(K, B^n), \qquad i = 0, \dots, n,$$

yields the *dual quermassintegral* $\widetilde{W}_i$ of order i.

For the dual quermassintegrals $\widetilde{W}_i$, a representation (or inductive definition) in analogy to Kubota's integral recursion is possible. As a counterpart to (5.72) (case $j = k$) one has

$$\widetilde{W}_{n-k}(K) = \frac{\kappa_n}{\kappa_k} \int_{G(n,k)} V_k(K \cap E)\, \mathrm{d}\nu_k(E) \tag{9.38}$$

for $k = 1, \dots, n-1$. This was pointed out by Lutwak [1267]. It follows from the definition of the dual quermassintegrals and properties of the invariant measures ν on $\mathrm{SO}(n)$ and ν_k on $G(n, k)$.

Generalizing the radial linear combination, one can define, for arbitrary $p \in \mathbb{R}\setminus\{0\}$, the *radial p-combination* by

$$\rho(\lambda \cdot K \widetilde{+}_p \mu \cdot L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p,$$

for $K, L \in \mathcal{S}^n_o$ and $\lambda, \mu > 0$. Here $\lambda \cdot K$ stands for

$$\lambda \cdot_p K := \lambda^{1/p} K,$$

which extends (9.1) to star bodies; the index of $\cdot_p$ is omitted if this symbol occurs together with $\widetilde{+}_p$.

Note that for convex bodies $K, L \in \mathcal{K}^n_{(o)}$ it follows from (1.52) that

$$K \,\widetilde{+}_{-p}\, L = (K^\circ +_p L^\circ)^\circ \qquad \text{for } p \geq 1. \tag{9.39}$$

The radial p-addition can be supplemented by

$$K \,\widetilde{+}_{-\infty}\, L := K \cap L, \qquad K \,\widetilde{+}_{\infty}\, L := K \cup L.$$

The special case of radial p-combination where $p = n - 1$ has been called *radial Blaschke linear combination* in [1277]. The radial Blaschke sum is also denoted by $\widetilde{\#}$, thus

$$\rho(\lambda \cdot K \,\widetilde{\#}\, \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}.$$

Since the functionals defined above, and several others, are defined by integrating (products of) powers of continuous functions over the unit sphere, classical integral inequalities for such powers of functions can be used to obtain various inequalities of the dual Brunn–Minkowski theory. We state here appropriate forms of Hölder's and Minkowski's inequalities for integrals as lemmas. Both are only needed for continuous functions on the unit sphere and for measures with a continuous density with respect to the spherical Lebesgue measure σ. In the following, we say that positive functions f, g are *proportional* if $f = \lambda g$ with a constant $\lambda > 0$, and star bodies K, L are said to be *dilatates* if $K = \lambda L$ with a constant $\lambda > 0$.

Lemma 9.3.1 (Hölder's inequality for integrals) *Let $f_1, \dots, f_m, g : \mathbb{S}^{n-1} \to \mathbb{R}$ be positive continuous functions and let $p_1, \dots, p_m$ be positive numbers with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$. Then*

$$\int_{\mathbb{S}^{n-1}} f_1 \cdots f_m \, g \, \mathrm{d}\sigma \leq \prod_{i=1}^m \left(\int_{\mathbb{S}^{n-1}} f_i^{p_i} g \, \mathrm{d}\sigma \right)^{1/p_i}.$$

Equality holds if and only if $f_1^{p_1}, \dots, f_m^{p_m}$ are proportional.

Lemma 9.3.2 (Minkowski's inequality for integrals) *Let $f, g : \mathbb{S}^{n-1} \to \mathbb{R}$ be positive continuous functions and let $1 < p < \infty$. Then*

$$\left(\int_{\mathbb{S}^{n-1}} (f + g)^p \mathrm{d}\sigma \right)^{1/p} \leq \left(\int_{\mathbb{S}^{n-1}} f^p \mathrm{d}\sigma \right)^{1/p} + \left(\int_{\mathbb{S}^{n-1}} g^p \mathrm{d}\sigma \right)^{1/p}.$$

Equality holds if and only if f and g are proportional.

For related inequalities, with other ranges of the exponents or involving quotients of integrals (Dresher's inequality, for example), we refer the reader to the books by Hardy, Littlewood and Pólya [938] and by Beckenbach and Bellman [187]. Here we give only a few examples of how these inequalities can be used in the dual Brunn–Minkowski theory.

As a counterpart to the general Aleksandrov–Fenchel inequality (7.64) for mixed volumes, we have

$$\widetilde{V}(K_1,\dots,K_n)^m \le \prod_{i=1}^m \widetilde{V}(K_i[m],K_{m+1},\dots,K_n)$$

for $m \in \{2,\dots,n\}$ and $K_1,\dots,K_n \in \mathcal{S}_o^n$. This follows from Lemma 9.3.1 by choosing $p_i = m$, $f_i = \rho_{K_i}$ for $i = 1,\dots,m$ and $g = \rho_{K_{m+1}}\cdots\rho_{K_n}$. In contrast to (7.64), the equality case is easy to decide: equality holds if and only if $K_1,\dots,K_m$ are dilatates.

The inequality

$$\widetilde{V}_j^{k-i}(K,L) \le \widetilde{V}_i^{k-j}(K,L)\widetilde{V}_k^{j-i}(K,L), \qquad i<j<k, \tag{9.40}$$

for $K,L \in \mathcal{S}_o^n$ is a counterpart to (7.63), but here $i,j,k \in \mathbb{R}$ is allowed. The inequality is obtained from Lemma 9.3.1 by choosing

$$m=2,\quad f_1=\rho_K^{i-j}\rho_L^{j-i},\quad f_2=1,\quad g=\rho_K^{n-i}\rho_L^i,\quad p_1=\frac{k-i}{j-i},\quad p_2=\frac{k-i}{k-j}.$$

The special case $i=0$, $j=1$, $k=n$ gives a counterpart to Minkowski's first inequality (7.18).

Lemma 9.3.2 immediately yields the Brunn–Minkowski type inequality

$$V_n(K \widetilde{+}_p L)^{p/n} \le V_n(K)^{p/n} + V_n(L)^{p/n} \tag{9.41}$$

for $K,L \in \mathcal{S}_o^n$ and $0<p\le n$. If $p>n$ or $p<0$, then (9.41) holds with the inequality sign reversed. For $p \ne n$, equality holds if and only if K and L are dilatates.

An important concept, which is properly subsumed under the dual Brunn–Minkowski theory, is the intersection body; it will be considered in Section 10.10.

Notes for Section 9.3

1. The dual Brunn–Minkowski theory was initiated by Lutwak [1261, 1262], as already mentioned, and it was elaborated by him in [1267, 1277, 1279]. Inequality (9.41) for convex bodies and $p \le -1$ appears already in Firey [583].
2. *Axiomatic characterization.* An axiomatic characterization of radial p-addition for centrally symmetric star bodies was given by Gardner, Hug and Weil [679]. For $n \ge 2$, they showed that an operation between o-symmetric star bodies is section covariant, continuous in the radial metric, rotation covariant, homogeneous of degree 1 and associative if and only if it is radial p-addition for some $-\infty \le p \le \infty$, $p \ne 0$, or one of three trivial exceptions. See [679], Theorem 7.17, for a precise formulation.
3. Zhang [2058] proved an integral-geometric formula involving dual quermassintegrals, which is in striking formal analogy to the principal kinematic formula (4.54). For $K,K' \in \mathcal{S}_o^n$ he showed that

$$\int_{G_n} \chi(K \cap gK' \cap [o,go])\,\mathrm{d}\mu(g) = \frac{1}{\kappa_n}\sum_{i=0}^n \binom{n}{i}\widetilde{W}_i(K)\,\widetilde{W}_{n-i}(K'),$$

and further formulae of integral geometry.

4. *Star valuations.* In order to study valuations on starshaped sets appropriately, Klain [1082, 1083] extended the setting of the dual theory as follows. In his terminology, a set $A \subset \mathbb{R}^n$ is starshaped if $o \in A$ and, for each line L through the origin, the intersection $A \cap L$ is a closed segment. The radial function of such a set is defined by $\rho_A(u) := \max\{\lambda \geq 0 : \lambda u \in A\}$, for $u \in \mathbb{S}^{n-1}$. A starshaped set $K \subset \mathbb{R}^n$ is an L^n-*star* if ρ_K is an L^n function on $\mathbb{S}^{n-1}$. Let $\mathscr{S}^n$ denote the set of all L^n-stars in $\mathbb{R}^n$ (or rather of their equivalence classes, identifying two L^n-stars if their radial functions differ only on a set of spherical Lebesgue measure zero). On $\mathscr{S}^n$, the star topology is introduced by demanding that a sequence in $\mathscr{S}^n$ converges if the corresponding sequence of radial functions converges in the L^n norm. Aiming at a counterpart to Hadwiger's characterization theorem, Klain [1083] proved the following result.

   Theorem (Klain) There is a bijective correspondence between the continuous valuations φ on $\mathscr{S}^n$ that are invariant under rotations and the continuous functions $G : [0, \infty) \to \mathbb{R}$ satisfying $|G(x)| \leq ax^n + b$ for some $a, b \geq 0$. The correspondence is given by the equations

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} G \circ \rho_K \, \mathrm{d}\sigma$$

   for all $K \in \mathscr{S}^n$ and

$$G(\alpha) = \frac{1}{\omega_{n-1}} \varphi(\alpha B^n)$$

   for all $\alpha \geq 0$.

   In particular, if the valuation φ is, in addition, homogeneous of some degree $i \in \{0, \dots, n\}$, then φ is a constant multiple of the dual quermassintegral $\widetilde{W}_{n-i}$. This was already proved in [1082]. In [1083], Klain also proved that an $\mathrm{SL}(n)$ invariant continuous valuation on $\mathscr{S}^n$ is a linear combination, with constant coefficients, of the Euler characteristic and the volume.
5. Gardner and Vassallo [682] initiated a systematic study of stability versions of inequalities from the dual Brunn–Minkowski theory. They proved, for example, counterparts to (7.52), (7.124) and dual Bonnesen-type and Favard-type inequalities, providing, respectively, lower and upper bounds for dual isoperimetric deficits. Stability versions of the dual Aleksandrov–Fenchel inequality, the dual Brunn–Minkowski inequality and the dual isoperimetric inequality were obtained by Gardner and Vassallo in [683]. In [684], they continued this line of research with quantitative versions of the equivalence of the Brunn–Minkowski inequality and Minkowski's first inequality in the dual (and also in the classical) theory.
6. *Extensions of the dual Brunn–Minkowski theory.* An extension of the notion of sets starshaped at o to sets not necessarily containing o, with corresponding extensions of radial functions and dual mixed volumes, was promoted by Gardner and Volčič [686]. They made extensive applications to the geometric tomography of sections.

   A further extension of parts of the dual Brunn–Minkowski theory, to bounded Borel sets, appears in Gardner, Jensen and Volčič [685]. This extension is motivated by, and has applications to, local stereology, a collection of stereological designs based on sections through a fixed reference point.

   To this general setting of bounded Borel sets, Gardner [676] extended notions like chord-power integrals, random simplex integrals and dual affine quermassintegrals, and he generalized several inequalities, together with equality conditions.
7. Li and Leng [1215] proved counterparts to the inequalities (7.96) for dual mixed volumes of star bodies.
8. Inequalities for volume differences in the style of Note 13 of §7.1, but pertaining to the dual theory, were obtained by Lv [1307].
9. Gardner and Zvavitch [688] made a thorough study of Brunn–Minkowski type inequalities for the Gauss measure of star sets under radial addition, with a view also to some analogues and ramifications.

For the $(n-1)$-capacity of radial sums of star bodies, a dual capacitary Brunn–Minkowski inequality was established by Gardner and Hartenstine [678].

10. *Generalizations of the Busemann–Petty problem involving dual mixed volumes.* Although for problems concerning sections and projections of convex bodies we generally refer to the books by Gardner [675], Koldobsky [1136], Koldobsky and Yaskin [1142], we mention here that for o-symmetric convex bodies, Rubin and Zhang [1600] have made a thorough investigation of the Busemann–Petty problem for dual quermassintegrals of central sections of intermediate dimensions.
11. *Radial Blaschke–Minkowski homomorphisms.* In analogy to the Blaschke–Minkowski homomorphisms (see Subsection 8.2.2), Schuster [1746] defined a *radial Blaschke–Minkowski homomorphism* as a mapping $\Psi : \mathcal{S}_o^n \to \mathcal{S}_o^n$ that is continuous and rotation equivariant and satisfies

$$\Psi(K \mathbin{\widetilde{\#}} L) = \Psi(K) \mathbin{\widetilde{+}} \Psi(L)$$

for all $K, L \in \mathcal{S}_o^n$. He proved that Ψ is such a mapping if and only if there is a positive zonal measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1}, p)$ such that

$$\rho(\Psi K, \cdot) = \rho^{n-1}(K, \cdot) * \mu.$$

As a consequence, he obtained various inequalities for images under radial Blaschke–Minkowski homomorphisms. This work was continued by Zhao and Cheung [2069].

9.4 Further combinations and functionals

In addition to the analogues of notions from the Brunn–Minkowski theory treated in the previous sections of this chapter, we describe here further constructions and functionals which parallel corresponding ones in the classical theory. There are some open questions around them, and interesting connections with the affine geometry of convex bodies; the latter will be considered in Chapter 10.

If in definition (9.2) one uses gauge functions instead of support functions, then another combination of convex bodies is obtained. It was introduced by Firey [583]. One can use radial functions instead and then, following Lutwak [1287], define for star bodies $K, L \in \mathcal{S}_o^n$ and numbers $\lambda, \mu > 0$ and $p \geq 1$ the *harmonic p-combination* $\lambda \diamond K \mathbin{\widehat{+}}_p \mu \diamond L$ by

$$\rho(\lambda \diamond K \mathbin{\widehat{+}}_p \mu \diamond L, \cdot)^{-p} := \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}.$$

In particular,

$$\lambda \diamond K = \lambda^{-1/p} K.$$

A connection with the radial p-combination is given by

$$\lambda \diamond K \mathbin{\widehat{+}}_p \mu \diamond L = \lambda \cdot K \mathbin{\widetilde{+}}_{-p} \mu \cdot L.$$

For convex bodies $K, L \in \mathcal{K}_{(o)}^n$ and for $p \geq 1$, we have

$$K \mathbin{\widehat{+}}_p L = K \mathbin{\widetilde{+}}_{-p} L = (K^\circ +_p L^\circ)^\circ,$$

by (9.39). Therefore, the operation $(K, L) \mapsto K \mathbin{\widehat{+}}_p L$ is also called *polar L_p addition*.

For $p \geq 1$ and $K, L \in \mathcal{S}^n_o$, it follows immediately from (1.53) and the definition that

$$\lim_{\varepsilon \downarrow 0} \frac{V_n(K \widehat{+}_p \varepsilon \diamond L) - V_n(K)}{\varepsilon} = \frac{n}{-p} \widetilde{V}_{-p}(K, L) \tag{9.42}$$

with

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^{n+p} \rho_L^{-p} \, \mathrm{d}\sigma, \tag{9.43}$$

which is (9.37) for $r = -p$. The special case $(i, j, k) = (-p, 0, n)$ of (9.40) gives

$$\widetilde{V}_{-p}(K, L)^n \geq V_n(K)^{n+p} V_n(L)^{-p}, \tag{9.44}$$

with equality if and only if K and L are dilatates.

For the volume of a harmonic p-combination one has the inequality ([1287], Proposition 1.12)

$$V_n(\lambda \diamond K \widehat{+}_p \mu \diamond L)^{-p/n} \geq \lambda V_n(K)^{-p/n} + \mu V_n(L)^{-p/n}, \tag{9.45}$$

with equality if and only if K and L are dilatates. For convex bodies, the special case $p = 1$ of (9.45) was studied further by Firey [584], who showed that

$$W_i(K \widehat{+}_1 L)^{-1/(n-i)} \geq W_i(K)^{-1/(n-i)} + W_i(L)^{-1/(n-i)} \tag{9.46}$$

for $K, L \in \mathcal{K}^n_{(o)}$ and $i = 0, \dots, n-1$, with the same cases of equality as in (9.45).

Another addition was introduced by Lutwak [1279] for $p = 1$ and by Lu and Leng [1234] for $p \geq 1$, in the following way. Let $K, L \in \mathcal{S}^n_o$. Let ξ be the number with

$$\xi^{p/(n+p)} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left[V_n(K)^{-1} \rho_K^{n+p} + V_n(L)^{-1} \rho_L^{n+p} \right]^{n/(n+p)} \mathrm{d}\sigma.$$

The L_p *harmonic Blaschke* addition, $K \widehat{\#}_p L$, is defined by

$$\xi^{-1} \rho(K \widehat{\#}_p L, \cdot)^{n+p} = V_n(K)^{-1} \rho(K, \cdot)^{n+p} + V_n(L)^{-1} \rho(L, \cdot)^{n+p}.$$

Then $\xi = V_n(K \widehat{\#}_p L)$, and hence

$$\frac{\rho(K \widehat{\#}_p L, \cdot)^{n+p}}{V_n(K \widehat{\#}_p L)} = \frac{\rho(K, \cdot)^{n+p}}{V_n(K)} + \frac{\rho(L, \cdot)^{n+p}}{V_n(L)}.$$

One has (extending the proof of [1279], Theorem (5.14))

$$V_n(K \widehat{\#}_p L)^{p/n} \geq V_n(K)^{p/n} + V_n(L)^{p/n}, \tag{9.47}$$

with equality if and only if K and L are dilatates.

We turn to some functionals. Many of the formal properties of the mixed volume (multilinearity, translation invariance, continuity and monotonicity) are shared by the *mixed width integral* introduced by Lutwak [1266]. It is defined by

$$A(K_1, \dots, K_n) := \frac{1}{n 2^n} \int_{\mathbb{S}^{n-1}} w(K_1, u) \cdots w(K_n, u) \, \mathrm{d}u$$

for $K_1, \dots, K_n \in \mathcal{K}^n$, where $w(K, u)$ denotes the width of K in direction $u \in \mathbb{S}^{n-1}$. It follows from Hölder's inequality that

$$A(K_1, \dots, K_n)^m \leq \prod_{i=1}^m A(K_i[m], K_{m+1}, \dots, K_n)$$

for $1 < m \leq n$. By specialization, one obtains the width integrals

$$B_i(K) := A(K[n-i], B^n[i])$$

of Lutwak [1263]. They satisfy

$$B_j(K)^{k-i} \leq B_i(K)^{k-j} B_k(K)^{j-i}$$

for $0 \leq i < j < k \leq n$, with equality if and only if K is of constant width; they also satisfy

$$B_i(K+L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)}$$

for $i \in \{0, \dots, n-1\}$, and other inequalities.

Hadwiger [911] (p. 267) introduced *harmonic quermassintegrals*, defining $\widehat{W}_0(K) := V_n(K)$, $\widehat{W}_n(K) := \kappa_n$ and

$$\widehat{W}_{n-k}(K) = \frac{\kappa_n}{\kappa_k} \left(\int_{G(n,k)} V_k(K \,|\, E)^{-1} \mathrm{d}\nu_k(E) \right)^{-1} \tag{9.48}$$

for $K \in \mathcal{K}^n_n$ and $k = 1, \dots, n-1$, and showed that

$$\widehat{W}_i(K+L)^{1/(n-i)} \geq \widehat{W}_i(K)^{1/(n-i)} + \widehat{W}_i(L)^{1/(n-i)}. \tag{9.49}$$

Lutwak [1276] proved the inequality

$$\kappa_n^j \widehat{W}_i(K)^{n-j} \leq \kappa_n^i \widehat{W}_j(K)^{n-i}, \tag{9.50}$$

for $0 \leq i < j < n$ and $K \in \mathcal{K}^n_n$; equality holds if and only if K is a ball. A special case is

$$\kappa_n^i V_n(K)^{n-i} \leq \widehat{W}_i(K)^n. \tag{9.51}$$

For $i = n-1$, this is the harmonic Urysohn inequality proved by Lutwak [1261], and for $i = 1$ it is the harmonic isepiphanic inequality of Lutwak [1269].

As a counterpart to (9.48), J. Yuan, S. Yuan and Leng [2011] introduced *dual harmonic quermassintegrals*, defining $\breve{W}_0(K) := V_n(K)$, $\breve{W}_n(K) := \kappa_n$ and

$$\breve{W}_{n-k}(K) = \frac{\kappa_n}{\kappa_k} \left(\int_{G(n,k)} V_k(K \cap E)^{-1} \mathrm{d}\nu_k(E) \right)^{-1} \tag{9.52}$$

for $K \in \mathcal{S}^n_o$ and $k = 1, \dots, n-1$. They proved, among other inequalities, that

$$\breve{W}_i(K+L)^{1/(n-i)} \geq \breve{W}_i(K)^{1/(n-i)} + \breve{W}_i(L)^{1/(n-i)} \tag{9.53}$$

for $K, L \in \mathcal{K}^n_{(o)}$ and $0 \leq i \leq n$, with equality if and only if K and L are dilatates.

As special cases of more general averaging processes, Lutwak [1269] proposed to define *affine quermassintegrals* $\Phi_0(K), \Phi_1(K), \dots, \Phi_n(K)$ for $K \in \mathcal{K}_n^n$ by taking $\Phi_0(K) := V_n(K)$, $\Phi_n(K) := \kappa_n$ and, for $0 < k < n$,

$$\Phi_{n-k}(K) := \frac{\kappa_n}{\kappa_k}\left(\int_{G(n,k)} V_k(K\,|\,E)^{-n}\,\mathrm{d}\nu_k(E)\right)^{-1/n}. \tag{9.54}$$

Then

$$\Phi_i(K) \le \widehat{W}_i(K) \le W_i(K),$$

by Jensen's inequality. Lutwak showed that

$$\Phi_i(K+L)^{1/(n-i)} \ge \Phi_i(K)^{1/(n-i)} + \Phi_i(L)^{1/(n-i)}. \tag{9.55}$$

In analogy to (7.67), Lutwak [1276] conjectured that

$$\kappa_n^j \Phi_i(K)^{n-j} \le \kappa_n^i \Phi_j(K)^{n-i} \tag{9.56}$$

for $0 \le i < j < n$ and $K \in \mathcal{K}_n^n$. A special case would be the inequality

$$\kappa_n^i V_n(K)^{n-i} \le \Phi_i(K)^n. \tag{9.57}$$

The cases $i = n-1$ and $i = 1$ are true; they follow, respectively, from the Blaschke–Santaló inequality (10.28) and the Petty projection inequality (10.86), as noted by Lutwak.

The *dual affine quermassintegrals*, also proposed by Lutwak (orally, in the 1980s), are defined, for $K \in \mathcal{K}_n^n$, by letting $\widetilde{\Phi}_0(K) := V_n(K)$, $\widetilde{\Phi}_n(K) := \kappa_n$ and, for $0 < k < n$,

$$\widetilde{\Phi}_{n-k}(K) := \frac{\kappa_n}{\kappa_k}\left(\int_{G(n,k)} V_k(K \cap E)^n\,\mathrm{d}\nu_k(E)\right)^{1/n}. \tag{9.58}$$

From Jensen's inequality,

$$\widetilde{W}_i(K) \le \widetilde{\Phi}_i(K).$$

Gardner [676] showed that

$$\widetilde{\Phi}_i(K \,\widetilde{+}\, L)^{1/(n-i)} \le \widetilde{\Phi}_i(K)^{1/(n-i)} + \widetilde{\Phi}_i(L)^{1/(n-i)}.$$

The inequality

$$\widetilde{\Phi}_i(K)^n \le \kappa_n^i V_n(K)^{n-i} \tag{9.59}$$

for $0 < i < n-1$ follows from work of Busemann and Straus ([377], p. 70) and was rediscovered by Grinberg [773]. Equality in (9.59) holds if and only if K is a centred ellipsoid. For $i = 1$, this is the Busemann intersection inequality (10.103). Grinberg [773] also proved that the affine quermassintegrals and the dual affine quermassintegrals are, as the names suggest, invariant under volume-preserving linear transformations (and the affine quermassintegrals, of course, under translations).

In analogy to (9.58), one may also define *mean dual affine quermassintegrals* by

$$\overline{\Phi}_{n-k}(K) := \frac{\kappa_n}{\kappa_k}\left(\int_{A(n,k)} V_k(K\cap E)^{n+1}\,\mathrm{d}\mu_k(E)\right)^{1/(n+1)}$$

for $0<k<n$ and $K\in\mathcal{K}_n^n$. They are related to the dual affine quermassintegrals by

$$\overline{\Phi}_{n-k}(K) = \frac{\kappa_n}{\kappa_k}\left(\int_K \widetilde{\Phi}_{n-k}(K-x)^n\,\mathrm{d}x\right)^{1/(n+1)}$$

(see Schneider and Weil [1740], p. 373). Therefore, $\overline{\Phi}_{n-k}(K)$ is invariant under unimodular affine transformations of K. Among convex bodies of given positive volume, precisely the ellipsoids attain the maximal value of $\overline{\Phi}_{n-k}$. This was proved by Schneider [1702] (Theorem 1, second part). For a more general inequality, with details of the proof, see Schneider and Weil [1740], Theorem 8.6.4.

Notes for Section 9.4

1. Extensions of the inequalities (9.41) and (9.47), of Brunn–Minkowski type and involving radial p-addition and radial and harmonic Blaschke addition, were proved by Zhao and Leng [2073]. Chai and Lee [396] obtained some inequalities involving the harmonic p-combination. Wang and Leng [1913] extended (9.42) and (9.43), the latter to

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K^{n+p-i}\rho_L^{-p}\,\mathrm{d}\sigma,$$

and proved various inequalities involving these functionals.
2. In analogy to the mixed width integrals, Lu [1233] introduced mixed chord integrals and proved a number of inequalities for them.
3. *Inequalities for integral means*. Inequalities for power means of the width of a convex body were treated by Lutwak [1264], and similar inequalities for the brightness by Lutwak [1269]. Lutwak [1275] also obtained various inequalities for the power means of quermassintegrals of projections of convex bodies.
4. Further inequalities for the dual harmonic quermassintegrals (9.52) were obtained by Yuan, Zhao and Duan [2013]. Zhao and Leng [2072], introducing additional parameters, extended inequalities of Lutwak for volumes and dual mixed volumes involving radial addition, radial Blaschke addition and harmonic Blaschke addition of star bodies.
5. An analogue of the conjectured inequalities (9.56) for dual affine quermassintegrals would be the inequalities

$$\kappa_n^j\widetilde{\Phi}_i(K)^{n-j} \ge \kappa_n^i\widetilde{\Phi}_j(K)^{n-i}$$

for $0\le i<j\le n$. They were formulated as Problem 9.6 in the first edition of Gardner's book [675], but Gardner himself gave counterexamples for $i\ge 1$ ([676], Theorem 7.7).

9.5 Log-concave functions and generalizations

The Brunn–Minkowski inequality in the form (7.6) says that the Lebesgue measure μ on $\mathbb{R}^n$ satisfies

$$\mu((1-\lambda)A+\lambda B) \ge \mu(A)^{1-\lambda}\mu(B)^{\lambda} \tag{9.60}$$

for $0 < \lambda < 1$ and bounded measurable sets A, B for which $(1-\lambda)A + \lambda B$ is measurable. A locally finite Borel measure μ on $\mathbb{R}^n$ with this property is called *log-concave*. According to Borell [286], any log-concave measure on $\mathbb{R}^n$ that is not concentrated on a hyperplane has a density with respect to Lebesgue measure which is a log-concave function. A function $f : \mathbb{R}^n \to [0,\infty)$ is *log-concave* if $\log f$ is concave, hence if

$$f = e^{-\varphi}$$

with a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. The interplay between log-concave functions and the geometry of convex sets becomes increasingly important, a development which can be traced back to the work of Ball [116, 118].

An example of a log-concave function is given by the characteristic function $\mathbf{1}_K$ of a convex body $K \in \mathcal{K}^n$. In fact,

$$\mathbf{1}_K = e^{-I_K^\infty},$$

and the indicator function I_K^∞ is a closed (lower semi-continuous) convex function. In this way, upper semi-continuous log-concave functions can be considered as a generalization of convex bodies. This is one of the reasons to give here first examples showing how notions and results from classical convex geometry can be extended to log-concave functions. In fact, some of these generalizations extend even further, to α-concave or quasi-concave functions. At the time of writing, these developments have probably not yet reached their final state, so we mainly restrict ourselves to some hints to ongoing work.

In the following, we denote by $\mathrm{LC}(\mathbb{R}^n)$ the set of upper semi-continuous log-concave functions on $\mathbb{R}^n$ that are not identically zero. Thus, the elements of $\mathrm{LC}(\mathbb{R}^n)$ are precisely the functions $e^{-\varphi}$ where φ is a closed convex function on $\mathbb{R}^n$.

The basic notions of the classical Brunn–Minkowski theory are Minkowski addition and volume, so these are the first we ought to extend to log-concave functions. For $f, g \in \mathrm{LC}(\mathbb{R}^n)$ and for $\lambda > 0$, let

$$(f \star g)(x) := \sup_{x_1+x_2=x} f(x_1)g(x_2), \qquad (\lambda \cdot f)(x) := f\left(\frac{x}{\lambda}\right)^\lambda. \tag{9.61}$$

One calls $f \star g$ the *Asplund sum* (or the *sup-convolution*) of f and g. Immediately from the definitions we have

$$\mathbf{1}_K \star \mathbf{1}_M = \mathbf{1}_{K+M}, \qquad \lambda \cdot \mathbf{1}_K = \mathbf{1}_{\lambda K}$$

for $K, M \in \mathcal{K}^n$ and $\lambda > 0$. If $f = e^{-\varphi}$ and $g = e^{-\psi}$, then

$$f \star g = e^{-(\varphi \,\square\, \psi)}, \tag{9.62}$$

by the definition of the infimal convolution $\square$. Note that $f \star g$ is not always a log-concave function, since $\varphi \,\square\, \psi$ can attain the value $-\infty$. Even if $f \star g$ is log-concave, it need not be in $\mathrm{LC}(\mathbb{R}^n)$, since $\varphi \,\square\, \psi$ need not be closed.

In passing from $\mathbf{1}_K$ with $K \in \mathcal{K}^n$ to general log-concave functions f, it seems natural to generalize the volume of K by the integral of f with respect to Lebesgue measure (though the integral may be infinite). As a first result on log-concave functions, we note that the extension of the Brunn–Minkowski inequality (in the multiplicative form (9.45)) is provided by the Prékopa–Leindler inequality. In the rest of this section, we repeatedly write $\int f$ for $\int_{\mathbb{R}^n} f \, d\mathcal{H}^n$.

Theorem 9.5.1 *If* $f, g \in \mathrm{LC}(\mathbb{R}^n)$ *and* $0 < \lambda < 1$*, then*

$$\int [(1-\lambda) \cdot f \star \lambda \cdot g] \geq \left(\int f \right)^{1-\lambda} \left(\int g \right)^{\lambda}.$$

Proof For the function $h := (1-\lambda) \cdot f \star \lambda \cdot g$, the definitions (9.61) immediately give

$$h((1-\lambda)x + \lambda y) = \sup_{x_1 + x_2 = (1-\lambda)x + \lambda y} f\left(\frac{x_1}{1-\lambda} \right)^{1-\lambda} g\left(\frac{x_2}{\lambda} \right)^{\lambda} \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for $x, y \in \mathbb{R}^n$, hence the Prékopa–Leindler inequality (Theorem 7.1.2) yields the assertion. □

Next, we care about extending the essential 'linear' notions of support function and mean width. Guided by the fact (see (1.43)) that $h_K = \mathcal{L} I_K^\infty = \mathcal{L}(-\log \mathbf{1}_K)$ for $K \in \mathcal{K}^n$, where $\mathcal{L}$ denotes the Legendre transform, Artstein-Avidan and Milman [91] suggested defining the *support function* of the log-concave function f by

$$h_f := \mathcal{L}(-\log f). \tag{9.63}$$

This is a closed convex function but, of course, in general not homogeneous.

If $f, g \in \mathrm{LC}(\mathbb{R}^n)$ and $f \star g < \infty$, then for the support functions we obtain

$$h_{f \star g} = \mathcal{L}(-\log(f \star g)) = \mathcal{L}(\varphi \,\square\, \psi) = \mathcal{L}\varphi + \mathcal{L}\psi$$

by Theorem 1.6.17, thus

$$h_{f \star g} = h_f + h_g.$$

Further, using (1.19),

$$h_{\lambda \cdot f} = \lambda h_f$$

for $\lambda > 0$.

The classical mean width $w(K)$ of a convex body $K \in \mathcal{K}^n$ is proportional to

$$V(K, B^n, \dots, B^n) = \lim_{\varepsilon \downarrow 0} \frac{V_n(B^n + \varepsilon K) - V_n(K)}{\varepsilon}$$

and also to the integral of the support function over the unit sphere. It has been suggested (arguably) that in the class of log-concave functions the role that the unit ball plays in $\mathcal{K}^n$ should be played by the standard Gaussian, $G(x) = e^{-|x|^2/2}$. Natural definitions of the *mean width* of the log-concave function f are then given by

$$M^*(f) = \frac{2}{n(2\pi)^{n/2}} \lim_{\varepsilon \downarrow 0} \frac{\int G \star (\varepsilon \cdot f) - \int G}{\varepsilon} \tag{9.64}$$

$$= \frac{2}{n(2\pi)^{n/2}} \int h_f G. \tag{9.65}$$

(The normalization constant is chosen such that $M^*(G) = 1$.) Definition (9.64) was used by Klartag and Milman [1102]. Rotem [1591] proved that it is equivalent to (9.65). An Urysohn inequality for this mean width was proved by Klartag and Milman and by Rotem; the following version, which includes equality conditions, appears in Rotem [1591].

Theorem 9.5.2 *For any $f \in \mathrm{LC}(\mathbb{R}^n)$,*

$$M^*(f) \geq 2 \log \left(\frac{\int f}{\int G} \right)^{1/n} + 1,$$

with equality if and only if $\int f = \infty$ or $f(x) = Ce^{-|x-a|^2/2}$ for some $C > 0$ and some $a \in \mathbb{R}^n$.

It should be observed that $M^*(f)$ can be negative. If, however, there exists a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \geq 1$, then $M^*(f) \geq 0$. This and the following properties were proved by Rotem [1591]. We have $M^*(f) > -\infty$ for every $f \in \mathrm{LC}(\mathbb{R}^n)$. For $f, g \in \mathrm{LC}(\mathbb{R}^n)$ and for $\lambda, \mu > 0$,

$$M^*((\lambda \cdot f) \star (\mu \cdot g)) = \lambda M^*(f) + \mu M^*(g).$$

M^* is rotation and translation invariant.

A systematic investigation of the transfer of further basic notions from the Brunn–Minkowski theory to log-concave functions was undertaken by Colesanti and Fragalà [441]. Denoting the integral $\int f$ by $I(f)$, they show that for $f, g \in \mathrm{LC}(\mathbb{R}^n)$ with $I(f) > 0$, the first variation

$$\delta I(f, g) := \lim_{\varepsilon \downarrow 0} \frac{I(f \star \varepsilon \cdot g) - I(f)}{\varepsilon}$$

exists. Minkowski's first inequality takes the functional form

$$\delta I(f, g) \geq I(f)[\log I(g) + n] + \mathrm{Ent}(f)$$

with

$$\mathrm{Ent}(f) := \int f \log f - I(f) \log I(f),$$

and equality holds if and only if there exists some $x_0 \in \mathbb{R}^n$ such that $g(x) = f(x - x_0)$ for all $x \in \mathbb{R}^n$. A main goal of [441] is the study of a functional counterpart to the surface area measure of convex bodies. Under suitable assumptions on the log-concave functions f and g, for example that they are positive on the whole space and that $h_f - ch_g$ is convex for suitable $c > 0$, they show the existence of a measure μ_f on $\mathbb{R}^n$, depending only on f, such that

$$\delta I(f, g) = \int_{\mathbb{R}^n} h_g \, \mathrm{d}\mu_f.$$

The measure μ_f can play the role of the surface area measure of f. Counterparts to Minkowski's existence theorem are then also discussed; as the authors say, they deserve further investigation.

A more general class than that of log-concave functions is provided by the class of α-concave functions. Let $-\infty \le \alpha \le \infty$. For $\alpha \notin \{0, \pm\infty\}$, a function $f : \mathbb{R}^n \to [0, \infty)$ is called *α-concave* if f is supported on some convex set Ω and, for all $x, y \in \Omega$ and $0 \le \lambda \le 1$,

$$f((1-\lambda)x + \lambda y) \ge [(1-\lambda)f(x)^\alpha + \lambda f(y)^\alpha]^{1/\alpha}.$$

For $\alpha \in \{0, \pm\infty\}$, the definition is supplemented in the limit sense. Thus, 0-concave functions are the same as log-concave functions. The $(-\infty)$-concave functions are called quasi-concave. A function $f : \mathbb{R}^n \to [0, \infty)$ is *quasi-concave* if

$$f((1-\lambda)x + \lambda y) \ge \min\{f(x), f(y)\}$$

for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$, equivalently if the *upper level sets*

$$\{f \ge t\} := \{x \in \mathbb{R}^n : f(x) \ge t\}, \qquad t > 0,$$

are convex. An ∞-concave function is constant where it is non-zero. If $\alpha_1 < \alpha_2$, then each α_2-concave function is α_1-concave. Early appearances of α-concave functions are in the papers by Avriel [103] and Borell [286].

The extension of notions and results from the classical Brunn–Minkowski theory to α-concave or quasi-concave functions was investigated, at the same time, by Milman and Rotem [1430, 1431], Rotem [1593], Bobkov, Colesanti and Fragalà [260].

We describe here, in slightly more detail, an extension of the mixed volume to the class of quasi-concave functions. It was announced by Milman and Rotem in [1431] and developed by them in [1430], to which we refer for the proofs. Let $f : \mathbb{R}^n \to [0, \infty)$ be a quasi-concave function. As mentioned, its upper level sets $\{f \ge t\}$ are convex, and if f is upper semi-continuous, then they are closed. We denote by $\mathrm{QC}(\mathbb{R}^n)$ the set of all upper semi-continuous, quasi-concave functions on $\mathbb{R}^n$ with compact upper level sets.

On $\mathrm{QC}(\mathbb{R}^n)$, an addition $\oplus$ is defined by

$$(f \oplus g)(x) := \sup_{x_1 + x_2 = x} \min\{f(x_1), g(x_2)\}$$

and a scalar multiplication $\odot$ by

$$(\lambda \odot f)(x) := f\left(\frac{x}{\lambda}\right)$$

for $\lambda > 0$. Then

$$\{f \oplus g \geq t\} = \{f \geq t\} + \{g \geq t\}, \qquad \{\lambda \odot f \geq t\} = \lambda\{f \geq t\}. \tag{9.66}$$

Thus, $f \oplus g \in \mathrm{QC}(\mathbb{R}^n)$, and if f, g are α-concave for some $\alpha \leq 0$, then it can be shown that $f \oplus g$ is α-concave. The operation $\oplus$ extends Minkowski addition, since

$$\mathbf{1}_K \oplus \mathbf{1}_L = \mathbf{1}_{K+L}$$

for $K, L \in \mathcal{K}^n$.

With (9.66), Fubini's theorem together with the classical results for convex bodies can be used to obtain the following extension of the mixed volume. For $f_1, \dots, f_m \in \mathrm{QC}(\mathbb{R}^n)$, one defines the *mixed integral*

$$V(f_1, \dots, f_n) := \int_0^\infty V(\{f_1 \geq t\}, \dots, \{f_n \geq t\})\, \mathrm{d}t.$$

Then (5.17) extends to

$$\int_{\mathbb{R}^n} [(\lambda_1 \odot f_1) \oplus \cdots \oplus (\lambda_m \odot f_m)]\, \mathrm{d}\mathcal{H}^n = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(f_{i_1}, \dots, f_{i_n})$$

for $\lambda_1, \dots, \lambda_m > 0$. Here V can attain infinite values, a possibility which can be excluded by considering only functions $f \in \mathrm{QC}(\mathbb{R}^n)$ with $\int f\, \mathrm{d}\mathcal{H}^n < \infty$. The mixed volume of convex bodies $K_1, \dots, K_n$ is retrieved from

$$V(\mathbf{1}_{K_1}, \dots, \mathbf{1}_{K_n}) = V(K_1, \dots, K_n).$$

Extending (5.31), the kth *quermassintegral* of $f \in \mathrm{QC}(\mathbb{R}^n)$ is defined by

$$W_k(f) := V(\underbrace{f, \dots, f}_{n-k}, \underbrace{\mathbf{1}_{B^n}, \dots, \mathbf{1}_{B^n}}_{k})$$

for $k = 0, \dots, n$, and in particular the *surface area* by

$$S(f) := nW_1(f).$$

Classical inequalities for convex bodies can be extended in the form of rearrangement inequalities. The *symmetric decreasing rearrangement* f^* of $f \in \mathrm{QC}(\mathbb{R}^n)$ is defined by

$$\{f^* \geq t\} = \left(\frac{V_n(\{f \geq t\})}{V_n(B^n)}\right)^{1/n} B_n, \qquad t > 0.$$

Thus, f^* is the rotationally symmetric function in $\mathrm{QC}(\mathbb{R}^n)$ with the property that the upper level sets of f and f^* with the same parameter have the same volume.

It is convenient (though not always necessary) to consider only the so-called 'geometric' functions $f \in \mathrm{QC}(\mathbb{R}^n)$, that is, those satisfying

$$\max_{x \in \mathbb{R}^n} f(x) = f(o) = 1.$$

The set of these functions is denoted by $\mathrm{QC}_0(\mathbb{R}^n)$; let $\mathrm{LC}_0(\mathbb{R}^n) := \mathrm{QC}_0(\mathbb{R}^n) \cap \mathrm{LC}(\mathbb{R}^n)$.

We give the following examples of rearrangement inequalities that extend classical inequalities for convex bodies. For $f \in \mathrm{QC}_0(\mathbb{R}^n)$, the isoperimetric inequality

$$S(f) \geq S(f^*)$$

holds, with equality if and only if f is rotation invariant. For $f, g \in \mathrm{QC}_0(\mathbb{R}^n)$, the Brunn–Minkowski inequality is valid in the form

$$(f \oplus g)^* \geq f^* \oplus g^*.$$

An extended version of the Aleksandrov–Fenchel inequality states that

$$V(f_1, \dots, f_n) \geq V(f_1^*, \dots, f_n^*)$$

for $f_1, \dots, f_n \in \mathrm{QC}_0(\mathbb{R}^n)$.

For log-concave functions, sharper inequalities, namely with restricted equality cases, are possible. Let

$$g(x) := e^{-|x|}, \qquad x \in \mathbb{R}.$$

For $f \in \mathrm{LC}_0(\mathbb{R}^n)$ and $0 \leq k < m < n$ the inequality

$$\left(\frac{W_m(f)}{W_m(g)}\right)^{\frac{1}{n-m}} \geq \left(\frac{W_k(f)}{W_k(g)}\right)^{\frac{1}{n-k}} \tag{9.67}$$

holds, with equality if and only if $f(x) = e^{-c|x|}$ with some $c > 0$. The special case $k = 0$, $m = 1$ gives the isoperimetric inequality in the form

$$\left(\frac{S(f)}{S(g)}\right)^n \geq \left(\frac{\int f}{\int g}\right)^{n-1}$$

for $f \in \mathrm{LC}_0(\mathbb{R}^n)$, with equality as before. For this inequality, the assumptions that f is log-concave and geometric are crucial.

Several constructions for α-concave functions were introduced and studied by Rotem [1593] and by Bobkov, Colesanti and Fragalà [260]. For $-\infty \leq \alpha \leq \infty$ let $C_\alpha(\mathbb{R}^n)$ denote the set of all functions $f : \mathbb{R}^n \to [0, \infty)$ that are α-concave, upper semi-continuous and not identically zero. For $f \in C_\alpha(\mathbb{R}^n)$ and $-\infty < \alpha < 0$, Rotem [1593] defined the convex function

$$\mathrm{base}_\alpha f := \frac{1 - f^\alpha}{\alpha},$$

inspired by the work of Bobkov [259]. Note that $\lim_{\alpha \uparrow 0} \mathrm{base}_\alpha f = -\log f$ and that

$$f = (1 - \alpha\varphi)^{1/\alpha} \quad \text{if} \quad \varphi = \mathrm{base}_\alpha f.$$

If $K \in \mathcal{K}^n$, then $\mathrm{base}_\alpha \mathbf{1}_K = I_K^\infty$, independent of α. For given α, Rotem defined the support function of $f \in C_\alpha(\mathbb{R}^n)$ by

$$h_f^{(\alpha)} := \mathcal{L}(\mathrm{base}_\alpha f).$$

He then defined a corresponding mean width and extended the Urysohn inequality from log-concave to α-concave functions. In [1431], inequality (9.67) was also extended to certain α-concave functions.

For α-concave functions, suitable notions of addition and multiplication by positive reals (both operations depending on α), which extend Minkowski addition and multiplication by scalars for convex bodies, were introduced in slightly different ways by Bobkov, Colesanti and Fragalà [260], Rotem [1593], Milman and Rotem [1431].

Independent of [1430], Bobkov, Colesanti and Fragalà [260] introduced and studied quermassintegrals of quasi-concave functions. They consider the class $\mathcal{Q}^n$ of upper semi-continuous quasi-concave functions $f : \mathbb{R}^n \to [0, \infty]$ (thus, infinite values are allowed) with the properties $\lim_{|x|\to\infty} f(x) = 0$ and $f \not\equiv 0$, as well as the subclasses of α-concave functions for $-\infty \leq \alpha \leq \infty$. For $f \in \mathcal{Q}^n$ and $i \in \{0, \dots, n\}$ they define the ith quermassintegral of f by

$$W_i(f) := \int_0^\infty W_i(\{f \geq t\})\, \mathrm{d}t.$$

With

$$f_\rho(x) := \sup_{y \in B(x,\rho)} f(y) \quad \text{for } \rho > 0,$$

the Steiner-type formula

$$I(f_\rho) = \sum_{i=0}^{n} \binom{n}{i} W_i(f)\rho^i$$

is valid. The main results of [260] on quermassintegrals are concavity inequalities, deduced from more general Prékopa–Leindler type inequalities, isoperimetric type inequalities, the valuation property with respect to the lattice operations $\vee$ (maximum) and $\wedge$ (minimum) and integral-geometric formulae of Cauchy–Kubota type.

Finally in this section, we anticipate the Blaschke–Santaló inequality (10.28) and prove the simplest version of a functional Blaschke–Santaló type inequality, for even log-concave functions. For $f \in \mathrm{LC}(\mathbb{R}^n)$, one defines a dual function f° by

$$f^\circ := e^{-\mathcal{L}(-\log f)}.$$

Note that it follows from (9.62) and (1.17) that $(f \star g)^\circ = f^\circ g^\circ$ for $f, g \in \mathrm{LC}(\mathbb{R}^n)$. Note also that, for $K \in \mathcal{K}^n_{(o)}$, the function $f = \exp\left(-\frac{1}{2}\|\cdot\|_K^2\right)$ satisfies the relation $f^\circ = \exp\left(-\frac{1}{2}\|\cdot\|_{K^\circ}^2\right)$, by (1.49).

We assume that $f \in \mathrm{LC}(\mathbb{R}^n)$ is even with $0 < \int f < \infty$ and show that

$$\int f \int f^\circ \leq (2\pi)^n. \tag{9.68}$$

In a more general version, this is due to Ball [116]. We reproduce here the version of the proof given in [85]. The function f is of the form $f = e^{-\varphi}$ with a convex function φ. Writing $\{\varphi < t\}$ for $\{x \in \mathbb{R}^n : \varphi(x) < t\}$, we see from

$$\int e^{-\varphi} = \int_{\mathbb{R}} e^{-t} V_n(\{\varphi < t\})\,\mathrm{d}t$$

and the assumption on f that $V_n(\{\varphi < t\})$ is finite for all t and positive for some t. Let t be such that $V_n(\{\varphi < t\}) > 0$. By definition (1.13) of the Legendre transform,

$$\varphi(x) + (\mathcal{L}\varphi)(y) \geq \langle x, y\rangle \tag{9.69}$$

for $x, y \in \mathbb{R}^n$. If $(\mathcal{L}\varphi)(y) < s$, then (9.69) gives $\langle x, y\rangle \leq s + t$ for all $x \in \{\varphi < t\}$, hence $y/(s+t) \in \{\varphi < t\}^\circ$ and thus $\{\mathcal{L}\varphi < s\} \subset (s+t)\{\varphi < t\}^\circ$. The set $\{\varphi < t\}$ is an o-symmetric bounded convex set, hence the Blaschke–Santaló inequality (10.28) gives

$$V_n(\{\varphi < t\})V_n(\{\mathcal{L}\varphi < s\}) \leq (s+t)^n V_n(\{\varphi < t\})V_n(\{\varphi < t\}^\circ) \leq (s+t)^n \kappa_n^2.$$

With $F(t) := e^{-t}V_n(\{\varphi < t\})$ and $G(s) := e^{-s}V_n(\{\mathcal{L}\varphi < s\})$ this reads

$$F(t)G(s) \leq e^{-(t+s)}(s+t)^n \kappa_n^2,$$

and this holds for all $s, t \in \mathbb{R}$. With $H(u) := e^{-u}(2u)^{n/2}\kappa_n$ and using the notation (9.61), this can be written as $\left(\frac{1}{2}\cdot F\right) \star \left(\frac{1}{2}\cdot G\right) \leq H$. By the one-dimensional case of Theorem 9.5.1, this gives

$$\begin{aligned}\int e^{-\varphi} \int e^{-\mathcal{L}\varphi} &= \int_{\mathbb{R}} e^{-t}V_n(\{\varphi < t\})\,\mathrm{d}t \int_{\mathbb{R}} e^{-s}V_n(\{\mathcal{L}\varphi < s\})\,\mathrm{d}s \\ &= \int_{\mathbb{R}} F(t)\,\mathrm{d}t \int_{\mathbb{R}} G(s)\,\mathrm{d}s \leq \left(\int_{\mathbb{R}} H(u)\,\mathrm{d}u\right)^2 = (2\pi)^n.\end{aligned}$$

This completes the proof of (9.68).

From (9.68), the Blaschke–Santaló inequality for o-symmetric $K \in \mathcal{K}_n^n$ is retrieved by choosing $f = \exp\left(-\frac{1}{2}\|\cdot\|_K^2\right)$, as follows from (1.54).

Notes for Section 9.5

1. *Characterization theorems for the support function.* In order to confirm that h_f is the 'right' support function for log-concave functions f, some characterization theorems have been proved. Recall that $\mathrm{LC}(\mathbb{R}^n) = \{e^{-\varphi} : \varphi \in \mathrm{Cvx}(\mathbb{R}^n)\}$. The mapping $\mathcal{T} : f \mapsto h_f$ from $\mathrm{LC}(\mathbb{R}^n)$ to $\mathrm{Cvx}(\mathbb{R}^n)$ has the following properties.

   (Q1) $\mathcal{T}$ is bijective.
   (Q2) $\mathcal{T}$ is order-preserving: $f \leq g$ if and only if $\mathcal{T}f \leq \mathcal{T}g$.
   (Q3) $\mathcal{T}$ is additive: $\mathcal{T}(f \star g) = \mathcal{T}f + \mathcal{T}g$ for all f, g.

   The following two theorems follow from the work of Artstein-Avidan and Milman [90, 91]. The formulations are taken from Rotem [1593].

   Theorem If a mapping $\mathcal{T} : \mathrm{LC}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ satisfies (Q1) and (Q2), then there are an invertible affine map $B : \mathbb{R}^n \to \mathbb{R}^n$, constants C_1, C_2 and a vector $v \in \mathbb{R}^n$ such that

   $$(\mathcal{T}f)(x) = C_1 h_f(Bx) + \langle x, v\rangle + C_2$$

   for all $f \in \mathrm{LC}(\mathbb{R}^n)$.

Theorem If a mapping $\mathcal{T} : \mathrm{LC}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ satisfies (Q1) and (Q3), then there are an invertible affine map $B : \mathbb{R}^n \to \mathbb{R}^n$ and a constant C such that

$$(\mathcal{T} f)(x) = C h_f(Bx)$$

for all $f \in \mathrm{LC}(\mathbb{R}^n)$.

Rotem [1593] proved a similar result without the assumption of surjectivity, which at the same time includes a characterization of the Asplund sum.

Theorem Suppose a mapping $\mathcal{S} : \mathrm{LC}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ and an operation $\oplus : \mathrm{LC}(\mathbb{R}^n) \times \mathrm{LC}(\mathbb{R}^n) \to \mathrm{LC}(\mathbb{R}^n)$ have the following properties.

(1) $\mathcal{S}$ is order-preserving: $f \leq g$ if and only if $\mathcal{S}f \leq \mathcal{S}g$.
(2) $\mathcal{S}$ extends the usual support function: if K is a nonempty, closed, convex set, then $\mathcal{S}\mathbf{1_K} = h_K$.
(3) $\mathcal{S}(f \oplus g) = \mathcal{S}f + \mathcal{S}g$.

Then there is a constant $C > 0$ with

$$(\mathcal{S}f)(X) = \frac{1}{C} h_f(Cx)$$

and $f \oplus g = f \star g$ for all $f, g \in \mathrm{LC}(\mathbb{R}^n)$.

2. The following functional Brunn–Minkowski type inequality was proved by Bobkov [258]. Let $0 < t < 1$ and let u, v, w be nonnegative functions on $\mathbb{R}^n$ which are either log-concave or smooth and quasi-concave, and such that $w(x) \to 0$ if $|x| \to \infty$ and

$$w(tx + (1-t)y) \geq u(x)^t v(y)^{1-t}$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int |\nabla w(z)| \, \mathrm{d}z \geq \left(\int |\nabla u(x)| \, \mathrm{d}x \right)^t \left(\int |\nabla v(y)| \, \mathrm{d}y \right)^{1-t}.$$

3. For more general functional versions of the Blaschke–Santaló inequality, we refer to Artstein-Avidan, Klartag and Milman [85], Fradelizi and Meyer [629], and for other proofs to Lehec [1180, 1181]. Various functional forms of the reverse Blaschke–Santaló inequality were investigated by Klartag and Milman [1102], Fradelizi and Meyer [630, 631, 632], Fradelizi, Gordon, Meyer and Reisner [628]. Stability versions of the functional Blaschke–Santaló inequality were proved by Barthe, Böröczky and Fradelizi [162].
4. Further geometric inequalities have found functional counterparts. Klartag and Milman [1102] extended the reverse Brunn–Minkowski inequality (see Note 12 of §7.1) to log-concave functions. A functional version of the Rogers–Shephard inequality (see §10.1) was proved by Colesanti [436].

   Artstein-Avidan, Klartag, Schütt and Werner [86] investigated a functional version of the affine isoperimetric inequality (see §10.5) for log-concave functions and its consequences.

   For a survey on functional inequalities for log-concave functions and a description of the general programme of extension from convex bodies to log-concave functions, we refer to Milman [1428].

9.6 A glimpse of other ramifications

The principal idea of Brunn–Minkowski theory, to study the interrelations between combinations of convex bodies and functionals measuring their size, has been so successful that it is tempting to try to save at least rudiments of this approach in other situations. The constructions of convex geometry have proved so powerful that it seems worthwhile to find and study suitable modifications when leaving the realm of convexity or the Euclidean space. In this short section, we have a very brief look at some ramifications of this kind.

If $K \in \mathcal{K}^n$ is a convex body with a support function h of class C^1, then it follows from Corollary 1.7.3 that

$$\mathrm{bd}\, K = \{\nabla h(u) : u \in \mathbb{S}^{n-1}\}.$$

Equivalently, bd K is the envelope of the hyperplane system

$$\left\{\{x \in \mathbb{R}^n : \langle x, u\rangle = h(u)\} : u \in \mathbb{S}^{n-1}\right\}. \tag{9.70}$$

If h is any C^1 function on $\mathbb{S}^{n-1}$, one can still consider the envelope (9.70), denoted by F_h. If h is extended homogeneously of degree one to $\mathbb{R}^n$ (and the extension is denoted by the same symbol), then

$$F_h = \left\{\nabla h(u) : u \in \mathbb{S}^{n-1}\right\}.$$

The smooth parts of the hypersurface F_h have a natural orientation, and the corresponding normal vector at $\nabla h(u)$ is the vector u. Thus, the mapping ∇h can be interpreted as the inverse Gauss map of F_h. In two and three dimensions, such closed curves and surfaces with a bijective Gauss map were first studied by Geppert [694], who called them 'stützbare Bereiche' and 'stützbare Flächen' (supportable surfaces), and from different aspects by Langevin, Levitt and Rosenberg [1168], who called them *hedgehogs* (also *hérissons* is in use).

In a long series of papers, hedgehogs and their extensions were studied by Martinez-Maure, under various aspects; see [1332]–[1349]. In particular, geometric inequalities for hedgehogs are treated in [1336, 1337]. Hedgehogs can also be defined for differences of arbitrary support functions; for the planar case, see [1338, 1342, 1347], and for polytopes, see [1344]. Special polyhedral hedgehogs appear also in Rodríguez and Rosenberg [1584]. Uniqueness problems for the Minkowski problem extended to hedgehogs are the subject of Martinez-Maure [1349].

The counterexamples to Aleksandrov's conjecture mentioned in Note 7 of Section 8.5 that were constructed by Panina were based on so-called hyperbolic virtual polytopes. The theory of virtual polytopes, which includes polyhedral hedgehogs, gives geometric interpretations to the differences of support functions of convex polytopes in $\mathbb{R}^3$, in the form of closed polyhedral surfaces, generally non-convex and with self-intersections, but with a well-defined normal fan and a (non-convex) piecewise linear support function. The reader is referred to the work of Panina [1504, 1505, 1506, 1507]; illustrations are found in Knyazeva and Panina [1127].

Different routes have been followed to extend fundamental notions of the Brunn–Minkowski theory to hyperbolic spaces. Leichtweiß [1195] started from the kinematic interpretation $K + L = \bigcup_{x\in K}(L + x)$ of the Minkowski sum in Euclidean space, to model after this a certain (non-commutative) addition of smooth convex sets in the hyperbolic plane. Under additional assumptions, convexity of the sum can be shown and an analogue of the mixed area can be defined. A version of the support function in the hyperbolic plane, with some applications, appears in Leichtweiß [1196].

Gallego, Solanes and Teufel [664] introduced and investigated linear combinations of hypersurfaces in n-dimensional hyperbolic space which are envelopes of horospheres. They made use of a linear structure in the space of horospheres. A different approach was followed by Leichtweiß [1197].

An elegant theory was developed by Fillastre [579]. He considers the hyperbolic space $\mathbb{H}^n$ as a pseudosphere in the Minkowski space-time $\mathbb{R}^{n,1}$. In his work, a Fuchsian group Γ is a group Γ of linear isometries of $\mathbb{R}^{n,1}$ such that $\mathbb{H}^n/\Gamma$ is a compact manifold. Given such a group Γ, Fuchsian convex bodies are defined as closed convex subsets of the future cone in $\mathbb{R}^{n,1}$ that are invariant under Γ. For Fuchsian convex bodies, Minkowski addition and volume can be defined, they behave well and there are analogues of the Brunn–Minkowski and Aleksandrov–Fenchel inequalities (with reversed inequality signs).

10

Affine constructions and inequalities

The mixed volume $V(K_1, \ldots, K_n)$, which is a central notion of the Brunn–Minkowski theory, remains unchanged if the same volume-preserving affine transformation of $\mathbb{R}^n$ is applied to each of the convex bodies $K_1, \ldots, K_n$. The general theory of mixed volumes thus belongs to the affine geometry of convex bodies.

This affine geometry of convex bodies has much more to offer. In fact, affine-invariant constructions, functionals and extremum problems for convex bodies are a rich source of questions and results of considerable geometric beauty. Moreover, surprising relations to some other fields and unexpected applications have surfaced. In some parts of this field, the Brunn–Minkowski theory may be of help, and its extensions considered in the previous chapter play a prominent role and have had considerable impact, while other parts require various tools and methods of their own, and some new approaches still need to be discovered.

This last chapter is meant as an outlook. We collect and present various aspects of the affine geometry of convex bodies, but give very few proofs. We hope that this survey will be helpful for interested readers to find their own way into this fascinating field and its original literature.

10.1 Covariogram and difference body

The *covariogram* g_K of a convex body $K \in \mathcal{K}^n_n$ is the function $g_K : \mathbb{R}^n \to \mathbb{R}$ defined by

$$g_K(x) := V_n(K \cap (K + x)) = (\mathbf{1}_K * \mathbf{1}_{-K})(x), \qquad x \in \mathbb{R}^n,$$

where $*$ denotes the usual convolution. Trivially, it is an even function satisfying $g_K(o) = V_n(K)$. Highly non-trivial is the question of whether it determines K. Here we say that K is determined by its covariogram if any convex body $L \in \mathcal{K}^n$ with $g_K = g_L$ arises from K by a translation or a reflection in a point. What the covariogram obviously does determine is its support, which is the convex body

$$\mathrm{D}K := \{x \in \mathbb{R}^n : K \cap (K + x) \neq \emptyset\} = K - K.$$

This is called the *difference body* of K. Its support function is the width function of K, and the value $\rho(\mathrm{D}K, u)$ of its radial function at $u \in \mathbb{S}^{n-1}$ is the length of a longest chord of K in direction u. In particular, the covariogram of a convex body determines its width function. A centrally symmetric convex body is determined (among all convex bodies) by its covariogram, since the latter determines the volume and the difference body, and, among all convex bodies with given difference body, the o-symmetric one is, up to translations, the unique one with largest volume (see Theorem 10.1.4).

Also determined by g_K is the *brightness function* of K, which is the function $u \mapsto V_{n-1}(K \mid u^\perp)$, $u \in \mathbb{S}^{n-1}$. In fact, for $u \in \mathbb{S}^{n-1}$,

$$\frac{\mathrm{d}}{\mathrm{d}r} g_K(ru)\Big|_{r=0} = -V_{n-1}(K \mid u^\perp). \tag{10.1}$$

This follows from the obvious fact that

$$rV_{n-1}((K \cap (K + ru)) \mid u^\perp) \leq V_n(K \setminus (K + ru)) \leq rV_{n-1}(K \mid u^\perp)$$

and $\lim_{r \to 0+} K \cap (K + ru) = K$ (by Theorem 1.8.10).

The observation that the covariogram determines the width and brightness functions already yields many non-symmetric convex bodies that are determined by their covariogram. The following was proved by Goodey, Schneider and Weil [743].

Theorem 10.1.1 *Let $n \geq 3$. Let $P \in \mathcal{K}_n^n$ be a simplicial polytope with the property that P and $-P$ are in general relative position. Then P is determined by its covariogram.*

Most convex bodies in $\mathbb{R}^n$ (in the sense of Baire category) are determined by their covariogram.

The question of whether every two-dimensional convex body is determined by its covariogram was first asked by Matheron [1359]. The following theorem collects the main (and considerably deep) known results on the determination of a convex body by its covariogram.

Theorem 10.1.2 *A two-dimensional convex body is determined by its covariogram. A three-dimensional convex polytope is determined by its covariogram. In $\mathbb{R}^n$ with $n \geq 4$, there exist convex bodies (certain direct sums) that are not determined by their covariogram.*

The two-dimensional case is due to Bianchi [221] and to Averkov and Bianchi [102]. The first of these papers deals with the case of planar convex bodies that are either not smooth or not strictly convex, and the second paper settles the remaining cases. The case of three-dimensional polytopes was treated by Bianchi [222], and the counterexample in four dimensions is found in Bianchi [221]. It is still not known whether every three-dimensional convex body is determined by its covariogram.

More general than the covariogram, the *cross covariogram* of two convex bodies $K, L \in \mathcal{K}_n^n$ is the function defined by

$$g_{K,L}(x) := V_n(K \cap (L + x)) = (\mathbf{1}_K * \mathbf{1}_{-L})(x), \qquad x \in \mathbb{R}^n.$$

Bianchi [223] was able to show that two convex polygons in the plane are determined by their cross covariogram, up to a small, explicitly described family of exceptions.

The following theorem on the covariogram was proved by Meyer, Reisner and Schmuckenschläger [1415].

Theorem 10.1.3 *If $K \in \mathcal{K}_n^n$ is o-symmetric and has the property that $g_K(x)$ depends only on $\|x\|_K$, then K is an ellipsoid.*

We turn to the difference body. Here we can give some proofs, which are closer to the Brunn–Minkowski theory. Given the volume of a convex body K, the following theorem provides sharp bounds for the volume of its difference body.

Theorem 10.1.4 *Let $K \in \mathcal{K}_n^n$. Then*

$$2^n \leq \frac{V_n(\mathrm{D}K)}{V_n(K)} \leq \binom{2n}{n}. \tag{10.2}$$

Equality holds on the left precisely if K is centrally symmetric and on the right precisely if K is a simplex.

We note that the left-hand inequality of (10.2), together with the characterization of the equality case, is an immediate consequence of the Brunn–Minkowski theorem. The right-hand inequality is due to Rogers and Shephard [1586] and is known as the *difference body inequality* or the *Rogers–Shephard inequality*. For its proof, we first prove a lemma, following Chakerian [398].

Lemma 10.1.5 *Let $B \in \mathcal{K}_n^n$, $x_0 \in B$, and let $f : B \to \mathbb{R}$ be a nonnegative concave function on B. If $h : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then*

$$\int_B h(f(x))\,\mathrm{d}x \geq nV_n(B) \int_0^1 h(tf(x_0))(1-t)^{n-1}\,\mathrm{d}t.$$

Equality holds if and only if for each $y \in \mathrm{bd}\, B$ the function f is linear on $[x_0, y]$ and $f(y) = 0$.

Proof We may assume that $x_0 = o$. For $x \in B \setminus \{o\}$, let $x \in [o, y] \subset B$ be such that $[o, y]$ is maximal, and put $g(x) := f(o)(1 - |x|/|y|)$ and, further, $g(o) := f(o)$. Then $f \geq g$, since f is concave and nonnegative. It follows that

$$\begin{aligned}\int_B h(f(x))\,\mathrm{d}x \geq \int_B h(g(x))\,\mathrm{d}x \\ &= \int_{\mathbb{S}^{n-1}} \int_0^{\rho(B,u)} h(f(o)(1 - r/\rho(B,u)))r^{n-1}\,\mathrm{d}r\,\mathrm{d}u \\ &= \int_{\mathbb{S}^{n-1}} \rho(B,u)^n \int_0^1 h(tf(o))(1-t)^{n-1}\,\mathrm{d}t\,\mathrm{d}u \\ &= nV_n(B) \int_0^1 h(tf(o))(1-t)^{n-1}\,\mathrm{d}t.\end{aligned}$$

Equality holds if and only if $f = g$. □

Proof of Theorem 10.1.4 Let $K \in \mathcal{K}_n^n$ and define $D(K,x) := K \cap (K+x)$ for $x \in \mathrm{D}K$. Let $x_1, x_2 \in \mathrm{D}K$, $\lambda \in [0,1]$, and $a \in (1-\lambda)D(K,x_1) + \lambda D(K,x_2)$. Then $a = (1-\lambda)a_1 + \lambda a_2$ with $a_i \in D(K,x_i)$, hence $a_i \in K$ and $a_i = b_i + x_i$ with $b_i \in K$ $(i = 1,2)$. It follows that $a \in K$ and $a \in K + (1-\lambda)x_1 + \lambda x_2$, thus $a \in D(K,(1-\lambda)x_1 + \lambda x_2)$. We have proved that

$$(1-\lambda)D(K,x_1) + \lambda D(K,x_2) \subset D(K,(1-\lambda)x_1 + \lambda x_2).$$

From the Brunn–Minkowski theorem we conclude that the function defined by

$$f(x) := V_n(K \cap (K+x))^{1/n}, \quad x \in \mathrm{D}K,$$

is concave. Now Lemma 10.1.5, with $B = \mathrm{D}K$, $x_0 = o$, $h(\xi) = \xi^n$ and this function f, together with the identity

$$\int_{\mathrm{D}K} V_n(K \cap (K+x))\,\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_K(y)\mathbf{1}_K(y-x)\,\mathrm{d}y\,\mathrm{d}x = V_n(K)^2, \tag{10.3}$$

yields the right-hand inequality of (10.2).

Equality holds if and only if f is linear on each segment joining o to a boundary point of $\mathrm{D}K$, and this is true if and only if $K \cap (K+x)$ is homothetic to K, for each $x \in \mathrm{D}K$. Simplices have this property, and it remains to show that they are characterized by it.

Assume, therefore, that $K \cap (K+x)$ is homothetic to K for each $x \in \mathrm{D}K$. Let $p \in \exp K$. Let $u \in \mathbb{S}^{n-1}$ be such that the ray $\{p + \lambda u : \lambda \geq 0\}$ meets K in a non-degenerate segment $[p,q]$. Let $\lambda \in (0,1)$ and $K_\lambda := K \cap [K + \lambda(p-q)]$; then $K_\lambda = h_\lambda K$ with a homothety h_λ. Since $p \in \exp K$, there is a hyperplane H such that $K \cap H = \{p\}$. The point $h_\lambda p$ must be an exposed point of K_λ, lying in the supporting hyperplane $h_\lambda H$. It follows that $h_\lambda p = p$ and hence that p is the centre of homothety of h_λ.

Now we use the cones $P(K,p)$ and $S(K,p) = \mathrm{cl}\,P(K,p)$ defined in Section 2.2. Suppose that $x \in [P(K,p)+p] \cap [P(K,q)+q]$. Then there are points $y,z \in K$ such that $y - p = \alpha(x-p)$ and $z - q = \beta(x-q)$ with $\alpha,\beta > 0$. For $\lambda < 1$ and sufficiently close to 1, the segments $[p,y]$ and $[q,z] + \lambda(p-q)$ will intersect in a point x_λ. Since $x_\lambda \in K_\lambda$ and, by similarity, $x_\lambda = h_\lambda x$, we deduce that $x \in K$. It follows that $[P(K,p)+p] \cap [P(K,q)+q] \subset K$ and hence

$$K' := [S(K,p)+p] \cap [S(K,q)+q] \subset K.$$

Since $K \subset K'$ holds trivially, we conclude that $K = K'$. This implies that each two-dimensional halfplane bounded by aff $\{p, q\}$ intersects K in a (possibly degenerate) triangle with side $[p, q]$. This holds for any point q for which $[p, q]$ is the (non-degenerate) intersection of K with a line. It is now easy to see that all possible points q make up a facet of K and that K is the convex hull of p and this facet. Since p was an arbitrary exposed point of K, induction with respect to the dimension now shows that K is a simplex. □

Extending the definition of the convex body $K + (-K)$ to p-addition, one may ask, for given $p > 1$, for the best constant $c_{n,p}$ such that

$$V_n\left(K +_p (-K)\right) \le c_{n,p} V_n(K) \qquad \text{for all } K \in \mathcal{K}^n_o.$$

The answer for $n = 2$ was given by Bianchini and Colesanti [229]. Triangles with one vertex at the origin are extremal bodies.

Notes for Section 10.1

1. For more references on the covariogram problem and its history, we refer to Note 5 of §8.6 in [1740], and in particular to the quoted articles by Bianchi. The work of Bianchi, Gardner and Kiderlen [226] succeeds in providing algorithms, for example, for the construction of an approximation to K from a finite number of noisy measurements of g_K, if K is uniquely determined by its covariogram g_K.
2. *An axiomatic characterization of the difference body operator* was given by Gardner, Hug and Weil [679] (§8), in the following way. Let $n \ge 2$. A mapping $\diamond$ from $\mathcal{K}^n$ into the space of o-symmetric convex bodies in $\mathbb{R}^n$ is GL(n) covariant, translation invariant and continuous if and only if there is a $\lambda \ge 0$ such that $\diamond K = \lambda \mathrm{D}K$ for all $K \in \mathcal{K}^n$.

   Another characterization is the following one due to Ludwig [1249] (Corollary 1.2). If $Z : \mathcal{P}^n \to \mathcal{K}^n$, $n \ge 2$, is a translation invariant, SL(n) covariant Minkowski valuation, then there is a constant $c \ge 0$ such that $Z = c\mathrm{D}$.
3. Special cases of the difference body inequality

   $$V_n(\mathrm{D}K) \le \binom{2n}{n} V_n(K) \tag{10.4}$$

   are treated in §53 of Bonnesen and Fenchel [284], where references to earlier work are given, and in the thesis of Godbersen [724]. The general case was settled by Rogers and Shephard [1586]. Chakerian [398] simplified and extended their argument; we have followed his approach. Lemma 10.1.5 appears in his paper. In the articles [1587, 1588] of Rogers and Shephard one finds a variant of their proof and similar and related inequalities. Another proof of the difference body inequality appears in Ball [116] (Theorem. 5.15).
4. *Intersections of translates*. The characterization of simplices by equality in (10.4) depends on the following result. A convex body K with the property that each nonempty intersection $K \cap (K + x)$ is homothetic to K is necessarily a simplex. (For compact sets in infinite-dimensional vector spaces, this property has been used in the definition of Choquet simplices.) The proof given above is a modified version of a proof by Martini [1352]. The original proof of Rogers and Shephard was rather long. A shorter proof for polytopes was found by Eggleston, Grünbaum and Klee [534]. A survey on more recent related results with many references is given by Soltan [1799].

   The Rogers–Shephard characterization of simplices can be considered as a special case of a rather general type of problem concerning characterizations of specific sets by intersection properties of their translates. Let $\mathcal{M}$ be a family of subsets of $\mathbb{R}^n$, and let $\simeq$ be a binary relation on $\mathcal{M}$. The problem is to determine the subfamily $\mathcal{M}(\simeq) \subset \mathcal{M}$ defined by

$$\mathcal{M}(\simeq) := \{K \in \mathcal{M} : (K + x) \cap (K + y) \in \mathcal{M},\ x, y \in \mathbb{R}^n \Rightarrow (K + x) \cap (K + y) \simeq K\}.$$

One may also restrict the translation vectors x, y to smaller sets. Contributions to this problem, of various kinds, are due to Schneider [1654, 1659], Gruber [808, 809, 810, 811], McMullen, Schneider and Shephard [1397], Fourneau [626], Soltan [1798].

5. *A conjectured strengthening of the difference body inequality*. Godbersen [724] and independently Makai, Jr [1321] conjectured that

$$V(K[i], -K[n-i]) \le \binom{n}{i} V_n(K) \tag{10.5}$$

for $K \in \mathcal{K}_n^n$ and $i \in \{1, \dots, n-1\}$, with equality only for simplices. For $i = 1$ and $i = n-1$, this follows from the fact that $-K \subset nK$ if K has centroid o (see Bonnesen and Fenchel [284], p. 53), but the general case is unknown. If (10.5) is true, it implies the inequality (10.4), by

$$V_n(\mathrm{D}K) = V_n(K - K) = \sum_{i=0}^{n} \binom{n}{i} V(K[i], -K[n-i])$$
$$\le \sum_{i=0}^{n} \binom{n}{i}^2 V_n(K) = \binom{2n}{n} V_n(K).$$

It would also imply the inequality

$$\int_0^1 V_n((1-\lambda)K + \lambda(-K))\, \mathrm{d}\lambda \le \frac{2^n}{n+1} V_n(K),$$

which was established in a different way by Rogers and Shephard [1587].

6. *A generalization of the difference body*. For $K \in \mathcal{K}_n^n$ and a number $p \in \mathbb{N}$, define

$$\mathrm{D}_p K := \{(x_1, \dots, x_p) \in (\mathbb{R}^n)^p : K \cap (K + x_1) \cap \dots \cap (K + x_p) \ne \emptyset\},$$

so that $\mathrm{D}_1 K$ is the difference body of K. It was shown by Schneider [1661] that the pn-dimensional volume of $\mathrm{D}_p K$ satisfies

$$V_{pn}(\mathrm{D}_p K) \le \binom{pn+n}{n} V_n(K)^p,$$

with equality if and only if K is a simplex.

Write $\delta_p(K) := V_{pn}(\mathrm{D}_p K) V_n(K)^{-p}$. For $n = 2$,

$$\delta_p(K) = \frac{1}{2} p(p+1)\delta_1(K) + 1 - p^2,$$

hence δ_p attains its minimum precisely for centrally symmetric convex bodies. This is not true for $n \ge 3$ and $p \ge 2$. For these cases, one may conjecture that the minimum of δ_p is attained by ellipsoids.

More generally than in [1661], for convex bodies $K_0, K_1, \dots, K_p \in \mathcal{K}_n^n$ one can consider the translative integral-geometric quantity

$$\mathrm{D}(K_0, K_1, \dots, K_p) := \int \dots \int \chi(K_0 \cap (K_1 + x_1) \cap \dots \cap (K_p + x_p))\, \mathrm{d}x_1 \cdots \mathrm{d}x_p.$$

Then an obvious extension of the argument used in Schneider [1661] (essentially an application of Lemma 10.1.5) yields the inequality

$$\mathrm{D}(K_0, K_1, \dots, K_p) \le \binom{pn+n}{n} \frac{V_n(K_0)V_n(K_1)\cdots V_n(K_p)}{V_n(K_0 \cap K_1 \cap \dots \cap K_p)}.$$

Here each K_i may be replaced by any of its translates K_i'. The best estimate is obtained if $V_n(K_0 \cap K_1' \cap \cdots \cap K_p')$ is maximal.

7. *A stability version.* The following stability version of the difference body inequality was proved by Böröczky [288]. If the convex body $K \in \mathcal{K}_n^n$ satisfies

$$\frac{V_n(\mathrm{D}K)}{V_n(K)} \geq (1-\varepsilon)\binom{2n}{n}$$

for some sufficiently small $\varepsilon > 0$, then the Banach–Mazur distance of K from the n-dimensional simplex T^n satisfies $d_{BM}(K, T^n) \leq 1 + n^{50n^2}\varepsilon$. The order of ε is optimal.

8. For convex bodies $K, L \in \mathcal{K}_n^n$, the inequality

$$\frac{V_n(K+L)}{V_n(K-L)} \leq \frac{1}{2^n}\binom{2n}{n} \tag{10.6}$$

holds (according to an indirect private communication by A. Litvak). In fact, $\mathrm{D}(K+L) = (K+L)-(K+L) = (K-L)-(K-L) = \mathrm{D}(K-L)$, hence (10.2) gives

$$2^n V_n(K+L) \leq V_n(\mathrm{D}(K+L)) = V_n(\mathrm{D}(K-L)) \leq \binom{2n}{n} V_n(K-L)$$

and thus (10.6). Equality in (10.6) holds if and only if K is a simplex and L is a translate of $-K$.

9. In the plane, Jonasson [1046] obtained the following remarkable strengthening of the difference body inequality (motivated by an application to continuum percolation). To any convex body $K \in \mathcal{K}_2^2$, there exists a triangle T such that

$$\mathrm{D}K \subset \mathrm{D}T \quad \text{and} \quad V_2(K) = V_2(T).$$

Because of $V_2(\mathrm{D}K) \leq V_2(\mathrm{D}T) = 6V_2(T) = 6V_2(K)$, this implies the planar difference body inequality.

It seems to be unknown whether the result of Jonasson extends to higher dimensions; probably it does not.

10. *Chord power integrals.* Let $\alpha \geq 1$ and $K \in \mathcal{K}_n^n$. The *chord power integral* $I_\alpha(K)$ of K is defined by

$$I_\alpha(K) := \frac{\omega_n}{2}\int_{A(n,1)} V_1(K \cap E)^\alpha \, \mathrm{d}\mu_1(E).$$

It was proved by Chakerian [398] that

$$I_{\alpha+1}(K) \geq \frac{\alpha(\alpha+1)}{2}\mathrm{B}(\alpha, n+1)V_n(K)\int_{\mathbb{S}^{n-1}} \rho(\mathrm{D}K, u)^\alpha \, \mathrm{d}u \tag{10.7}$$

(where B denotes Euler's beta function), with equality if and only if K is a simplex. The case $\alpha = n$ gives the difference body inequality, since $I_{n+1}(K) = \frac{1}{2}n(n+1)V_n(K)^2$ (see [1740], pp. 363, 374, for more information about chord power integrals).

The inequality

$$I_{\alpha+1}(K) \leq \frac{\alpha(\alpha+1)}{2}n^\alpha \mathrm{B}(\alpha, n+1)V_n(K)^{\alpha+1}\int_{\mathbb{S}^{n-1}} \rho(\Pi^\circ K, u)^\alpha \mathrm{d}u, \tag{10.8}$$

where $\Pi^\circ K$ denotes the polar projection body of K (see §10.9), gives a sharp upper bound; equality holds if and only if K is a simplex. The case $\alpha = n$ gives Zhang's projection inequality (10.91).

More general inclusion inequalities, of which (10.7) and (10.8) are special cases, were proved by Zhang [2055] (Theorems 8 and 9). For the chord power integral inequalities, see also Ren [1570] (§7.6, Theorems 2 and 3).

Inequality (10.8) with equality condition follows also from the work of Gardner and Zhang [687] (namely, from Lemma 2.1 and Theorem 5.5, which is (10.93) below, for $p = \alpha$). Inequality (10.8) also appears in Xiong and Song [1998].

(Note that the normalization of the chord power integrals used here is that of Santaló [1630] and of [1740], whereas the normalizations in [398] and [1998] are different from this and from each other.)

11. *An analogue of the difference body inequality.* In analogy to the definition of the volume of the difference body, namely

$$V_n(\mathrm{D}K) = \int_{\mathbb{R}^n} \mathbf{1}\{K \cap (K + x) \neq \emptyset\}\, \mathrm{d}x,$$

one can define a convex body HK by

$$h(\mathrm{H}K, \cdot) := \int_{\mathbb{R}^n} h(K \cap (K + x), \cdot)\, \mathrm{d}x.$$

Then, with a suitable translation vector t, the inclusion

$$V(\mathrm{D}K)K \subset (n + 1)\mathrm{H}K + t$$

holds. Here equality holds if and only if K is a simplex. A more general version of this inequality was proved in Schneider [1725].

10.2 Qualitative characterizations of ellipsoids

Subsequent sections will collect several characterizations of ellipsoids by extremal or other quantitative properties. In contrast, we state here very briefly some classical characterizations of ellipsoids by qualitative properties of a simple intuitive nature. Such characterizations are important for the identification of ellipsoids when symmetrization procedures are used to establish extremal properties. The first two theorems go back to Brunn [350].

Theorem 10.2.1 *If the convex body $K \in \mathcal{K}^n_n$ has the property that the midpoints of any family of parallel chords of K lie in a hyperplane, then K is an ellipsoid.*

According to Thompson [1845], p. 86, the two-dimensional case of the preceding theorem is older and due to Bertrand.

Theorem 10.2.2 *Let $n \geq 3$. If any nonempty intersection of the convex body $K \in \mathcal{K}^n_n$ with a hyperplane is centrally symmetric, then K is an ellipsoid.*

The celebrated characterization of ellipsoids by 'planar shadow boundaries' goes back to Blaschke. The following stronger version is due to Marchaud [1327].

Theorem 10.2.3 *Let $n \geq 3$ and $K \in \mathcal{K}^n_n$. If for any line G there exists a hyperplane H such that*

$$\mathrm{bd}\, K \cap \mathrm{bd}\,(K + G) \supset H \cap \mathrm{bd}\,(K + G),$$

then K is an ellipsoid.

The following 'simple but frequently useful lemma' (Busemann [369], p. 91) also plays a role.

Lemma 10.2.4 *Let $K \in \mathcal{K}^n_o$ have the property that each k-dimensional linear subspace, for some $k \in \{2, \dots, n-1\}$, intersects K in a k-dimensional ellipsoid. Then K is an ellipsoid.*

For more information, and for proofs, we refer to Gruber and Höbinger [837]. The survey of Petty [1530] and Section 3.4 of the book by Thompson [1845] are also recommended.

Notes for Section 10.2

1. Gruber [834] has proved a stability version of Theorem 10.2.3.
2. The following theorem was proved by Meyer and Reisner [1409]. Let $K \in \mathcal{K}^n_n$ have the property that for any hyperplane H, the centroids of all nonempty hyperplane sections of K parallel to H lie on a line. Then K is an ellipsoid.

10.3 Steiner symmetrization

Solutions to extremal problems for convex bodies often exhibit a high degree of symmetry. For this reason, symmetrization procedures are a useful and often indispensable tool for the identification of extremal bodies. Steiner symmetrization is probably the most powerful of these procedures. Its typical applications in the affine geometry of convex bodies establish that certain functionals on convex bodies attain one of their extreme values precisely at ellipsoids. Before explaining the symmetrization, we wish to point out that extremum problems for continuous, affine-invariant functions on convex bodies with interior points have the property that both of their extrema are attained.

For the proof, let $K \in \mathcal{K}^n_n$. There exists a simplex $T \subset K$ of maximal volume. Let F be a facet of T, let v be the opposite vertex and let H be the hyperplane through v parallel to F. Then H supports K, since otherwise the maximality of T would be contradicted. Since F was an arbitrary facet of T, we see that K is contained in the simplex $-n(T - c) + c$, where c is the centroid of T. Let Δ be a fixed n-dimensional simplex with centroid o, and let $\Delta' := -n\Delta$. There exists an affine transformation α of $\mathbb{R}^n$ with $\alpha T = \Delta$. Then $\Delta \subset \alpha K \subset \Delta'$. Hence, every convex body $K \in \mathcal{K}^n_n$ has an affine transform in the set $\{M \in \mathcal{K}^n_n : \Delta \subset M \subset \Delta'\}$. The latter set is compact, by the Blaschke selection theorem. From this, it follows that every continuous, affine-invariant function on convex bodies attains a maximum and a minimum.

Now we define the Steiner symmetrization, but we prove only very basic facts. Let $H \subset \mathbb{R}^n$ be a hyperplane. Let $C \subset \mathbb{R}^n$ be a nonempty compact set. The *Steiner symmetral* of C with respect to H is the set $S_H C$ with the property that, for each line G orthogonal to H and meeting C, the set $G \cap S_H C$ is a closed segment with midpoint on H and length equal to that of the set $G \cap C$. The mapping $S_H : C \mapsto S_H C$ is the *Steiner symmetrization* with respect to H. (We remark that the family of lines orthogonal to H could often be replaced by any family of parallel lines not parallel to H.)

For convex bodies K, we describe $S_H K$ more explicitly. Let u be a unit vector orthogonal to H. There are two functions $f, g : K \,|\, H \to \mathbb{R}$ with

$$K \cap (G + x) = \{x + \lambda u : g(x) \le \lambda \le f(x)\}, \quad x \in K \,|\, H. \tag{10.9}$$

Since K is convex, the function g is convex and f is concave. On relint $(K \,|\, H)$, the functions f, g are continuous, and for $x \in K \,|\, H$ we have

$$g(x) \le \liminf_{y \to x} g(y), \qquad f(x) \ge \limsup_{y \to x} f(y).$$

In fact, suppose that, say, the first relation is false for some $x \in K \,|\, H$. Then there are a number $\varepsilon > 0$ and a sequence $(y_i)_{i \in \mathbb{N}}$ in $K \,|\, H$ such that $y_i \to x$ and $g(y_i) \le g(x) - \varepsilon$ for all i. Without loss of generality, we can assume that $g(y_i) \to \alpha$ for some real number α. Then $\alpha \le g(x) - \varepsilon$, hence $x + \alpha u \notin K$. On the other hand, $y_i + g(y_i)u \in K$ and hence $x + \alpha u \in K$, a contradiction.

We have

$$S_H K = \{x + \lambda u : x \in K \,|\, H, \ -[f(x) - g(x)]/2 \le \lambda \le [f(x) - g(x)]/2\}.$$

This shows that $S_H K$ is convex, since the function $f - g$ is concave. Clearly, $S_H K$ is symmetric with respect to H, and by Fubini's theorem it has the same volume as K. To show that $S_H K$ is closed, let $(y_i)_{i \in \mathbb{N}}$ be a sequence in $S_H K$ with $y_i \to y$. Then $y_i = x_i + \lambda_i u$ with $x_i \in K \,|\, H$ and $\lambda_i \in \mathbb{R}$, and we have $x_i \to x \in K \,|\, H$ and $\lambda_i \to \lambda$ with $y = x + \lambda u$. From $2|\lambda_i| \le f(x_i) - g(x_i)$ it follows that

$$2|\lambda| = \lim_{i \to \infty} 2|\lambda_i| \le \limsup_{i \to \infty} f(x_i) - \liminf_{i \to \infty} g(x_i) \le f(x) - g(x)$$

and hence $x + \lambda u \in S_H K$. Thus, $S_H K \in \mathcal{K}^n_n$.

In the following, H is a hyperplane through o, G is the line through o orthogonal to H and σ_H denotes the reflection at the hyperplane H.

Lemma 10.3.1 *If $(K_j)_{j \in \mathbb{N}}$ is a sequence in $\mathcal{K}^n_n$ with $K_j \to K$ and $S_H K_j \to K'$, then $K' \subset S_H K$.*

Proof Let $x \in K'$. There are points $x_j \in S_H K_j$ with $x_j \to x$. Further, there are points $y_j, z_j \in K_j \cap (G + x_j)$ with $|y_j - z_j| \ge |x_j - \sigma_H x_j|$. Without loss of generality, we can assume that $y_j \to y$ and $z_j \to z$. Then it follows that $y, z \in K \cap (G + x)$ and $|y - z| \ge |x - \sigma_H x|$, hence $x \in S_H K$. □

Let $H_1, \dots, H_k$ be hyperplanes through o. The mapping $S_{H_k} \circ \cdots \circ S_{H_1}$ is called an *iterated Steiner symmetrization*.

Theorem 10.3.2 *For $K \in \mathcal{K}^n_n$, let $\mathcal{S}(K)$ be the set of convex bodies that arise from K by applying iterated Steiner symmetrizations. Then $\mathcal{S}(K)$ contains a sequence that converges to a ball.*

Proof For $L \in \mathcal{K}^n$, let $R(L)$ denote the smallest radius of a ball with centre o containing L. Let $R_0 := \inf\{R(L) : L \in \mathcal{S}(K)\}$. Since $\mathcal{S}(K)$ is bounded, there is a sequence $(K_j)_{j\in\mathbb{N}}$ in $\mathcal{S}(K)$ with $\lim_{j\to\infty} R(K_j) = R_0$ and $\lim_{j\to\infty} K_j = K_0$ for some convex body $K_0 \in \mathcal{K}^n$, and by the continuity of R we have $R(K_0) = R_0$.

Assume that K_0 is not the ball B_0 with centre o and radius R_0. Then there is a point $z \in \mathrm{bd}\, B_0$ with $z \notin K_0$. There is a (nondegenerate) ball C with centre z such that $\mathrm{bd}\, B_0 \cap C \cap K_0 = \emptyset$. If H is an arbitrary hyperplane through o, then

$$\mathrm{bd}\, B_0 \cap C \cap S_H K_0 = \emptyset, \qquad \mathrm{bd}\, B_0 \cap \sigma_H C \cap S_H K_0 = \emptyset.$$

We can cover $\mathrm{bd}\, B_0$ by finitely many balls $C_1, \dots, C_m$ congruent to C with centres in $\mathrm{bd}\, B_0$. Let H_i be the hyperplane through o with $\sigma_H(C) = C_i$, and let $S^\star := S_{H_m} \circ \cdots \circ S_{H_1}$. By the preceding observation, $S^\star K_0 \cap \mathrm{bd}\, B_0 = \emptyset$. Since $S^\star K_0$ is compact, this implies that $R(S^\star K_0) < R_0$. Now, $K_j \in \mathcal{S}(K)$ and $\lim_{j\to\infty} K_j = K_0$. Without loss of generality (compactness, choice of subsequences, change of notation), we can assume that all sequences $((S_{H_r} \circ \cdots \circ S_{H_1})K_j)_{j\in\mathbb{N}}$, $r = 1, \dots, m$, are convergent and hence, in particular, that $S^\star K_j \to K'$ for some K'. From Lemma 10.3.1 we see that $K' \subset S^\star K_0$, thus $R(K') < R_0$ and hence $R(S^\star K_j) < R_0$ for sufficiently large j. Since $S^\star K_j \in \mathcal{S}(K)$, this is a contradiction. Therefore, $K_0 = B_0$, and $\lim_{j\to\infty} K_j = B_0$. □

The preceding theorem can be helpful in establishing extremal properties of balls or ellipsoids, particularly in cases where essential uniqueness of the extremal bodies is not achievable. In other cases, the typical proof scheme for the application of Steiner symmetrization is as follows. Suppose, for example, that one wants to establish that a certain real functional f on $\mathcal{K}^n_n$ attains its maximum precisely at ellipsoids. In the interesting cases (but not always), f is often continuous, and it may be invariant under affine transformations (the latter possibly after multiplication by a suitable power of the volume). In such a case, as remarked initially, there exists a convex body K at which the maximum of f is attained. Under the assumption that K is not an ellipsoid, one may then be able to show that some Steiner symmetral of K yields a strictly larger value of f, which would be a contradiction. It is clear that such a proof, if it works, needs some suitable characteristic property of ellipsoids, and here one of the characterization theorems of the previous section may provide the key.

We mention some classical success stories of this type, referring to the original literature for the proofs.

Theorem 10.3.3 *For $K \in \mathcal{K}^n_n$ and $m \geq n+1$, let P_m be a polytope with at most m vertices and maximal volume contained in K. Then $V_n(P_m)/V_n(K)$ is minimal if K is an ellipsoid.*

In the cases $n = 2$, $m = 3$ and $n = 3$, $m = 4$, this result is due to Blaschke [242, 245]. He also showed the uniqueness of the ellipsoid in these cases, under a restriction which was later removed by Gross [807] (see also Blaschke [248], §72). These papers of Blaschke, together with Blaschke [246], were the pioneering work for the application of Steiner symmetrization in the affine geometry of convex bodies. Theorem 10.3.3 in its generalized version is due to Macbeath [1314].

For $K \in \mathcal{K}_n^n$, $m \in \mathbb{N}$ with $m \geq n+1$, and $p \geq 1$, let

$$S(K, m, p) := \frac{1}{V_n(K)^{m+p}} \int_K \cdots \int_K V_n(\mathrm{conv}\{x_1, \ldots, x_m\})^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_m. \tag{10.10}$$

The case $n = 2$, $m = 3$, $p = 1$ of the following theorem is due to Blaschke [246]. Groemer [784, 785] proved several extensions, and the result is now known as the *Blaschke–Groemer inequality*.

Theorem 10.3.4 *For $K \in \mathcal{K}_n^n$, $m \geq n+1$, $p \geq 1$, the minimum of $S(K, m, p)$ is attained if and only if K is an ellipsoid.*

We refer to [1740], Section 8.6, for the role of $S(K, n+1, 1)$ in Sylvester's problem on geometric probabilities, and for the explicit value of $S(K, m, p)$ when K is an ellipsoid.

For $K \in \mathcal{K}_n^n$, $m \in \mathbb{N}$ with $m \geq n+1$, and $p \geq 1$, let

$$B(K, m, p) := \frac{1}{V_n(K)^{m+p}} \int_K \cdots \int_K V_n(\mathrm{conv}\{o, x_1, \ldots, x_m\})^p \, \mathrm{d}x_1 \cdots \mathrm{d}x_m. \tag{10.11}$$

Also for the next theorem, we refer to [1740], Section 8.6, for explicit values in the case of ellipsoids. The theorem was proved by Busemann [368] (for $p = 1$, but the proof extends immediately) and is known as the *Busemann random simplex inequality*.

Theorem 10.3.5 *For $K \in \mathcal{K}_{(o)}^n$ and $p \geq 1$, the minimum of $B(K, n, p)$ is attained if and only if K is an o-symmetric ellipsoid.*

Theorems 10.3.4 and 10.3.5 can be extended to non-convex sets. This was done by Pfiefer [1532], for compact sets. The equality conditions have to be modified, of course; we refer to the very general formulation for bounded Borel sets in Gardner [676] (Proposition 4.1).

Recalling that $D_n(x_1, \ldots, x_n)$ denotes the volume of the parallelepiped spanned by the vectors $x_1, \ldots, x_n$ (and referring to [1740] (Theorem 8.6.1) for the explicit constants), we can write the case $p = 1$ of the Busemann random simplex inequality in the form

$$\int_K \cdots \int_K D_n(x_1, \ldots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \geq \frac{2\kappa_{n+1}^{n-1}}{(n+1)\kappa_n^{n+1}} V_n(K)^{n+1}. \tag{10.12}$$

We cannot give here more than this brief presentation, without technical details, of the best known classical applications of Steiner symmetrization. The later uses of this method for proving extremal properties of ellipsoids are often more sophisticated. As examples, we mention Meyer and Pajor [1408], Hug [1002], Lutwak and Zhang [1306], Lutwak, Yang and Zhang [1292, 1303, 1304].

Steiner's method of symmetrization has a counterpart of 'antisymmetrization', called 'shaking'. Let H^+ be a closed halfspace, bounded by the hyperplane H. For a nonempty compact set C, the set $S_{H^+}C$ is defined as the set contained in H^+ with the property that, for each line G orthogonal to H and meeting C, the set $G \cap S_{H^+}C$ is a closed segment with one endpoint on H and length equal to that of the set $G \cap C$. The mapping $S_{H^+} : C \mapsto S_{H^+}C$ is the *shaking* to H within H^+.

The method of shaking ('Schüttelung') was used by Blaschke ([246] and [248], §25) to show that, in the plane and for $m = 3$, the functional (10.10) attains its maximum precisely at triangles. In higher dimensions, the method of shaking suffers from the drawback that it does not always transform simplices into simplices. An isolated application of shaking in higher dimensions appears in Schneider [1669], where it is shown that, among all convex bodies $K \in \mathcal{K}_n^n$, the simplices are characterized by the property that the volume of every circumscribed cylinder is not less than $nV_n(K)$.

The following was proved by Campi, Colesanti and Gronchi [385] (extending a planar result of Biehl [230]). There exist $n + 1$ hyperplanes such that every compact set can be transformed into a simplex by a sequence of iterated shakings at these hyperplanes. The authors remark that this can be used to prove the Brunn–Minkowski theorem for compact sets.

Notes for Section 10.3

1. Falconer [544] showed that a compact set with the property that all its Steiner symmetrals are convex must itself be convex.
2. Theorem 10.3.2 goes back to Carathéodory and Study [393] and to Gross [806]. Its proof can be found in many places, of which we mention Eggleston [532], p. 98, Hadwiger [911], §4.5.3, Webster [1928], p. 313, Gruber [834], p. 172.
3. *Refinements of* Theorem 10.3.2. Although not required for the applications to extremal problems, it is of intrinsic interest to improve Theorem 10.3.2 in several directions, in particular to estimate the number of Steiner symmetrizations that are necessary to approximate a ball up to given precision. A first (exponential) estimate was given by Hadwiger [902]. Isomorphic estimates (dependence on the dimension, for fixed degree of approximation) were established by Bourgain, Lindenstrauss and Milman [322] and improved by Klartag and Milman [1101]. The following was proved by Klartag [1097]. Let $K \in \mathcal{K}$, $V_n(K) = \kappa_n$, and $0 < \varepsilon < 1/2$. With some constant $c > 0$, there are $cn^4 \log^2 1/\varepsilon$ Steiner symmetrizations that transform K into a convex body $\tilde{K}$ satisfying $(1-\varepsilon)B^n \subset \tilde{K} \subset (1+\varepsilon)B^n$.

   A connection of rapid Steiner symmetrization with the slicing problem was found by Klartag and Milman [1102].

   Lower bounds for the generally sufficient number of Steiner symmetrizations to reach a given degree of approximation are established by Bianchi and Gronchi [227].
4. Bianchi, Klain, Lutwak, Yang and Zhang [228] have investigated whether Theorem 10.3.2 remains true if the directions of the Steiner symmetrizations have to be chosen from a countable, dense set. They found that the answer is affirmative, but that the order matters. Klain [1092] investigated the convergence of sequences of Steiner symmetrizations with directions taken from a finite set, each chosen infinitely often.

   A thorough study of the convergence of iterated Steiner symmetrizations of compact sets, to various shapes, was made by Bianchi, Burchard, Gronchi and Volčič [224].
5. *Random Steiner symmetrizations.* Let $(u_i)_{i\in\mathbb{N}}$ be a stochastically independent sequence of random vectors in $\mathbb{S}^{n-1}$ with uniform distribution. For $K \in \mathcal{K}_n^n$, define $K_1 := K$ and $K_{i+1} := S_{u_i^\perp} K_i$ for $i \in \mathbb{N}$. Mani–Levitska [1324] proved that the sequence $(K_i)_{i\in\mathbb{N}}$ converges (in the Hausdorff metric) almost surely to a ball. His question, of whether this holds also for compact sets, was answered affirmatively by van Schaftingen [1869]. An analogous result for bounded measurable sets (with suitably extended definition of

Steiner symmetrization) and the symmetric-difference distance was proved by Volčič [1897].

Strong asymptotice results on random Steiner and Minkowski symmetrizations were obtained by Coupier and Davydov [452].

6. For $n = 2$, the only extremal bodies in Theorem 10.3.3 are ellipses. This result is due to Blaschke [248], §22, for $m = 3$, and to Sas [1637], for arbitrary $m \geq 3$ (not using Steiner symmetrization, but a mean value argument and Fourier series; see also Fejes Tóth [566], chap. II, §4). For the case $n = 3$, $m = 5$, Bianchi [220] showed that the maximum in Theorem 10.3.3 is attained only by ellipsoids.
7. To determine the maximum of the functional (10.10) for $n \geq 3$ and $m = n + 1$ is one of the major open problems of convex geometry. Some restrictions on possible maximizers were obtained by Campi, Colesanti and Gronchi [384], who made use of linear parameter systems (see §10.4). If it could be shown that simplices are extremal bodies, this would have significant consequences; see Milman and Pajor [1429].

   For $n = 2$, it was shown by Dalla and Larman [463] that the maximum of (10.10) is attained by triangles, and Giannopoulos [701] showed that it is only attained by triangles.

   Stability results for the minimum of the functional (10.10) and for its maximum if $n = 2$ were proved by Ambrus and Böröczky [68].
8. A counterpart to Theorem 10.3.4, with the volume replaced by a strictly increasing function of an intrinsic volume and with balls as minimizers, was proved by Hartzoulaki and Paouris [940].
9. One possible extension of a special case of Theorem 10.3.5 reads as follows. For $K_1, \dots, K_m \in \mathcal{K}_o^n$ with $m \geq n$ and for $p \geq 0$, let

$$I_p(K_1, \dots, K_m) := \int_{K_1} \cdots \int_{K_m} \left(V_n \left(\sum_{i=1}^{m} [o, x_i] \right) \right)^p \mathrm{d}x_1 \cdots \mathrm{d}x_m.$$

   For balls $B_1, \dots, B_m$ with $V_n(B_i) = V_n(K_i)$, Bourgain, Meyer, Milman and Pajor [323] proved that

$$I_p(K_1, \dots, K_m) \geq I_p(B_1, \dots, B_m),$$

   using Steiner symmetrization. This was extended, and ellipsoids were identified as the only minimizers, by Makai and Martini [1322] and by Campi and Gronchi [391]. The latter authors also showed for $n = 2$ and $K_1 = \cdots = K_m$ that triangles with one vertex at the origin are maximizers, and, under the assumption of o-symmetry, that parallelograms are maximizers. Corresponding generalizations and uniqueness results were obtained by Saroglou [1634].
10. The functionals $S(K, m, p)$ of (10.10), $B(K, m, p)$ of (10.11) and I_p of Note 9 can be generalized by replacing the pth power of the volume in the integrand by a strictly increasing function h of the volume, choosing the integration variables $x_1, \dots, x_m$ from different convex bodies and replacing the volume in the integrand by an intrinsic volume, as in Note 8. All the mentioned results on minimizers have been extended in this way by Saroglou [1634].
11. The method of shaking in the plane was applied by Bisztriczky and Böröczky [231].

10.4 Shadow systems

Steiner symmetrization can be viewed as a special case of more general constructions, which were introduced by Rogers and Shephard [1588] and by Shephard [1778].

We view $\mathbb{R}^n$ as a subspace of $\mathbb{R}^n \times \mathbb{R}$, and for a vector $y \in \mathbb{R}^n \times \mathbb{R}$ not parallel to $\mathbb{R}^n$, we denote by $\pi_y : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the projection along the line spanned by y.

The vector $e := (o, 1) \in \mathbb{R}^n \times \mathbb{R}$ is orthogonal to $\mathbb{R}^n$ (with respect to the standard scalar product). Let $K \subset \mathbb{R}^n \times \mathbb{R}$ be a convex body. Then the system of convex bodies in $\mathbb{R}^n$ defined by

$$K(u) := \pi_{e+u} K = \pi_{(u,1)} K, \quad u \in \mathbb{R}^n,$$

is called the *shadow system* induced by K. This was introduced by Shephard [1778].

Let $V^\bullet$ denote the mixed volume in $\mathbb{R}^n \times \mathbb{R}$. If $S_{e+u} \subset \mathbb{R}^n \times \mathbb{R}$ denotes the segment with endpoints $(o, 0)$ and $(u, 1)$, then from (5.77) and the fact that e is a unit vector orthogonal to $\mathbb{R}^n$, it follows that

$$(n+1)V^\bullet(K, \dots, K, S_{e+u}) = V_n(K(u)).$$

This result extends to n shadow systems. If $K_1, \dots, K_n$ are convex bodies in $\mathbb{R}^n \times \mathbb{R}$ and $K_i(u)$ is the shadow system induced by K_i, then

$$(n+1)V^\bullet(K_1, \dots, K_n, S_{e+u}) = V(K_1(u), \dots, K_n(u)).$$

By (5.77) and (5.78), the function $y \mapsto V^\bullet(K_1, \dots, K_n, S_y)$ is convex. This immediately implies the following.

Theorem 10.4.1 *If $u \mapsto K_1(u), \dots, K_n(u)$ are shadow systems, then the function $u \mapsto V(K_1(u), \dots, K_n(u))$ is convex.*

If K_i is the unit ball B^n in $\mathbb{R}^n$, then $K_i(u) = B^n$ for all u. Therefore, in particular, each intrinsic volume function $u \mapsto V_r(K(u))$ is convex, if $K(u)$ is a shadow system.

Rogers and Shephard [1588] introduced a *linear parameter system* along a given vector v in $\mathbb{R}^n$ as a family $t \mapsto C(t)$ of convex sets of the form

$$C(t) = \operatorname{conv}\{a_j + \lambda_j t v : j \in J\}, \quad t \in I,$$

where $I \subset \mathbb{R}$ is an interval, J is an index set and $\{a_j\}_{j \in J} \subset \mathbb{R}^n$ and $\{\lambda_j\}_{j \in J} \subset \mathbb{R}$ are bounded sets.

With the notations used above, we define

$$\widetilde{K} := \operatorname{cl} \operatorname{conv}\{(a_j, \lambda_j) \in \mathbb{R}^n \times \mathbb{R} : j \in J\}.$$

For $j \in J$ we have

$$(a_j, \lambda_j) + \lambda_j(tv, -1) = (a_j + \lambda_j tv, 0),$$

hence $a_j + \lambda_j tv = \pi_{(tv,-1)}(a_j, \lambda_j) = \pi_{(-tv,1)}(a_j, \lambda_j)$. It follows that

$$\mathrm{cl}\, C(t) = \pi_{(-tv,1)}\widetilde{K} = \widetilde{K}(-tv).$$

In this sense, linear parameter systems are special shadow systems.

A simple example is given by two convex bodies $K_0, K_1 \in \mathcal{K}^n$, $I = \mathbb{R}$, $J = J_0 \cup J_1$ with disjoint sets J_0, J_1, and a mapping $j \mapsto a_j$ that maps J_ν bijectively to K_ν, $\nu = 0, 1$. If we define $\lambda_j = \nu$ for $j \in J_\nu$, then $C(t) = \mathrm{conv}\,(K_0 \cup (K_1 + tv))$. From Theorem 10.4.1 it follows, in particular, that $V_n(\mathrm{conv}\,(K_0 \cup (K_1 + tv)))$ is a convex function of t. This was first proved by Fáry and Rédei [550].

A still more special case is a *parallel chord movement* of a convex body $K \in \mathcal{K}^n_n$. It is defined as a family

$$K(t) = \{x + \beta(x \,|\, v^\perp)tv : x \in K\}, \quad t \in I,$$

where I is an interval, v is a unit vector and $\beta : K \,|\, v^\perp \to \mathbb{R}$ is a continuous function with the property that $K(t)$ is convex for all $t \in I$.

If we represent a convex body $K \in \mathcal{K}^n_n$ by

$$K = \{x + \lambda v : x \in K \,|\, v^\perp,\ g(x) \le \lambda \le f(x)\}$$

and define $\beta(x) := -[f(x) + g(x)]$ for $x \in K \,|\, v^\perp$, then $K(0) = K$, $K(1) = \sigma_H K$ with $H = v^\perp$, and $K(1/2)$ is the Steiner symmetral of K at the hyperplane H. Thus, it follows from Theorem 10.4.1 that each intrinsic volume is not increased under Steiner symmetrization.

Note for Section 10.4

1. For the use of special shadow systems in the treatment of affine extremal problems, we refer to Campi and Gronchi [387, 388, 389, 390], Fradelizi, Meyer and Zvavitch [633], Meyer and Reisner [1412, 1413], Weberndorfer [1925], to which we will come back in later sections, and to Bianchini and Colesanti [229], Saroglou [1634], which were already mentioned.

10.5 Curvature images and affine surface areas

The affine differential geometry of convex hypersurfaces (see the books of Blaschke [248], Salkowski [1607], P. and A. Schirokow [1643], Li, Simon and Zhao [1214] and Section 1.4 of Leichtweiß [1193]) has given rise to some basic notions, which have then been extended to general convex bodies and which now play an important role in the affine geometry of convex bodies. In this section, we describe two of these notions, curvature images and affine surface area. First we sketch briefly how they arose in differential geometry, and then we describe their stepwise extension to general convex bodies.

Let $K \in \mathcal{K}^n_n$ be a convex body of class C^2_+ and F its boundary hypersurface. Let $X : U \to \mathbb{R}^n$, where U is open in $\mathbb{R}^{n-1}$, be a local parametrization of F. We use classical tensor notation, with indices attached to functions on U indicating partial differentiation with respect to local coordinates. Let $L_{ij} := \det(X_{ij}, X_1, \dots, X_{n-1})$, $i, j = 1, \dots, n-1$. Since K is of class C^2_+, we have $L := \det(L_{ij}) > 0$, under a suitable orientation. The positive definite tensor $G_{ij} := L^{-1/(n+1)} L_{ij}$ defines a Riemannian metric on F, which is invariant under volume-preserving affine transformations applied to F. It serves as the first fundamental form of the equiaffine differential geometry of F. The *affine surface area* of F and of K, denoted by $\Omega(K)$, is defined as the total Riemannian volume of F with respect to this Riemannian metric. Expressed in terms of the Euclidean structure, one finds that

$$\Omega(K) = \int_{\mathbb{S}^{n-1}} f(K, u)^{n/(n+1)}\, \mathrm{d}u \tag{10.13}$$

(recall that $\mathrm{d}u$ is short for $\mathrm{d}\sigma(u)$ and σ is spherical Lebesgue measure), where $f(K, u) = s_{n-1}(K, u)$ is the product of the Euclidean principal radii of curvature of K at the point with normal vector u.

The *affine normal vector* of F is

$$Y := \frac{1}{n-1} \Delta X,$$

where Δ is the Laplace–Beltrami operator of the given Riemannian metric. By

$$\xi := \det(G_{ij})^{-1/2} [X_1, \dots, X_{n-1}],$$

where $[\cdot, \dots, \cdot]$ is the $(n-1)$-fold vector product, the *covector of the tangent plane* is defined. One has

$$\langle Y, \xi \rangle = 1, \quad \langle Y_i, \xi \rangle = 0, \quad \langle Y, \xi_i \rangle = 0, \quad i = 1, \dots, n-1. \tag{10.14}$$

The hypersurface $\overline{F}$ traced out by the affine normal vector was called by Blaschke the ‘affines Krümmungsbild’ (*affine curvature image*) of F. In general, it is neither convex nor regular. Expressed in terms of Euclidean quantities, we have

$$\xi = -f(K, u)^{1/(n+1)} u, \tag{10.15}$$

where u is the outer unit normal vector of F at the point where ξ is taken. Thus, the vector function ξ traces out a hypersurface $F^\star$, which is the boundary of a star body $K^\star$ with radial function given by

$$\rho(K^\star, u) = f(K, -u)^{1/(n+1)}. \tag{10.16}$$

Since $\int f(K, u) u \, \mathrm{d}u = o$, it follows (by using polar coordinates) that $K^\star$ has its centroid at o.

The function $\langle X, -\xi \rangle$ is known as the *affine distance* from the origin (also known as the *affine support function*). If it is constant for F, which in Euclidean terms means that

$$f(K, \cdot)^{1/(n+1)} h(K, \cdot) = \text{const.}, \tag{10.17}$$

then F is a *proper affine hypersphere*, and under sufficiently strong differentiability assumptions it is known to be an o-symmetric ellipsoid. This was proved by Blaschke

[248] Sections 74, 77, for $n = 3$, and extended to higher dimensions by Deicke [474]. Later proofs were given by Brickell [334] and Schneider [1657], Section 6.

Under stronger differentiability assumptions, an affine curvature theory can be based on the tensor defined by $B_{ij} := \langle Y_i, -\xi_j \rangle$, which is symmetric. If $\det(B_{ij}) \neq 0$ throughout, in which case the hypersurface F is called *elliptically curved*, then it turns out that the 'affine curvature image' $\overline{F}$ is a convex hypersurface and hence bounds a convex body $\overline{K}$. In this case, also $K^\star$ is convex, and it follows from (10.14) that $\overline{K}$ and $K^\star$ are polar to each other.

In a first step of extension without differentiability assumptions, we consider a special class of convex bodies, comprising those of class C^2_+. One says that the convex body $K \in \mathcal{K}^n_n$ has the *curvature function* $f(K, \cdot) : \mathbb{S}^{n-1} \to \mathbb{R}$ if its surface area measure $S_{n-1}(K, \cdot)$ has $f(K, \cdot)$ as a density with respect to spherical Lebesgue measure or, equivalently, if

$$V(L, K, \dots, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u) f(K, u)\, \mathrm{d}u$$

for all $L \in \mathcal{K}^n$. If K is of class C^2_+, then K has the curvature function $f(K, \cdot) = s_{n-1}(K, \cdot)$, the reciprocal Gauss curvature, viewed as a function of the unit normal vector. By $\mathcal{F}^n$ we denote the set of all convex bodies in $\mathcal{K}^n_n$ that have a positive continuous curvature function. We also write $\mathcal{F}^n_{(o)} := \mathcal{F}^n \cap \mathcal{K}^n_{(o)}$. Instead of $f(K, \cdot)$, we often write f_K.

It is convenient to extend the curvature function to all of $\mathbb{R}^n \setminus \{o\}$, by positive homogeneity of degree $-(n+1)$, that is, to define

$$f(K, \lambda u) := \lambda^{-(n+1)} f(K, u) \qquad \text{for } u \in \mathbb{S}^{n-1} \text{ and } \lambda > 0.$$

With this definition,

$$f(\phi K, u) = f(K, \phi^t u) \qquad \text{for } u \in \mathbb{R}^n \setminus \{o\}$$

for all $\phi \in \mathrm{SL}(n)$ (Lutwak [1281], Proposition (2.9)).

For a convex body $K \in \mathcal{F}^n$, the *affine surface area* $\Omega(K)$ (in the plane also called *affine perimeter*) is defined by

$$\Omega(K) := \int_{\mathbb{S}^{n-1}} f(K, u)^{n/(n+1)}\, \mathrm{d}u, \tag{10.18}$$

which is consistent with (10.13). The functional Ω is invariant under volume-preserving affine transformations.

A classical result of affine differential geometry is the *affine isoperimetric inequality*, which says that

$$\Omega(K)^{n+1} \leq n^{n+1} \kappa_n^2 V_n(K)^{n-1}, \tag{10.19}$$

or equivalently

$$\left(\frac{\Omega(K)}{\Omega(B^n)} \right)^{n+1} \leq \left(\frac{V_n(K)}{V_n(B^n)} \right)^{n-1} \tag{10.20}$$

for $K \in \mathcal{F}^n$, with equality precisely if K is an ellipsoid. Under suitable differentiability assumptions, this was proved by Blaschke [239, 248] for $n = 2, 3$. That his result and proof extend to higher dimensions was pointed out by Nakajima [1467], Santaló [1626] and Deicke [474]. The proof is by means of Steiner symmetrization. The extension to $\mathcal{F}^n$, including the equality condition, was achieved by Petty [1531]. His discussion of the equality case, where a version of Theorem 10.5.1 below played a role, was rather intricate.

As mentioned in the differentiable case, the covector of the tangent plane always describes the boundary of a star body, with centroid o, and with radial function given by (10.16). This is the starting point for introducing a first new type of curvature image. Let $\mathcal{S}^n_c$ (respectively, $\mathcal{K}^n_c$, $\mathcal{F}^n_c$) denote the set of star bodies (n-dimensional convex bodies, bodies in $\mathcal{F}^n$) in $\mathbb{R}^n$ with centroid o. For $K \in \mathcal{S}^n_c$, Lutwak [1281] defined the *polar curvature image* ΛK by

$$f(\Lambda K, \cdot) = \frac{\kappa_n}{V_n(K)}\rho(K, \cdot)^{n+1}, \qquad c(\Lambda K) = o. \tag{10.21}$$

(The normalization factor simplifies some formulae.) This does indeed define a convex body ΛK: since K has centroid o, the indefinite σ-integral of the function $\rho(K, \cdot)^{n+1}$ satisfies the sufficiency condition of Minkowski's existence theorem (Theorem 8.2.2). Hence, there exists a convex body $\Lambda K \in \mathcal{K}^n_n$ with the curvature function given by (10.21). It is uniquely determined up to a translation, hence the requirement that $c(\Lambda K) = o$ makes it unique. It is clear that $\Lambda : \mathcal{S}^n_c \to \mathcal{F}^n_c$ is a bijective mapping.

For the definition of a second type of curvature image (introduced by Petty [1531] and, under this name, by Lutwak [1279], with different notation), let $K \in \mathcal{K}^n_n$ be given. For $z \in \operatorname{int} K$,

$$K^z := (K - z)^\circ,$$

is the polar body of K with respect to z. As noted by Santaló [1626], there is a unique point $\mathbf{s} = \mathbf{s}(K) \in \operatorname{int} K$ (to be distinguished from the Steiner point $s(K)$) such that

$$V_n(K^{\mathbf{s}}) \le V_n(K^z) \quad \text{for all } z \in \operatorname{int} K.$$

This point is called the *Santaló point* of K. The minimum property of $\mathbf{s}(K)$ implies that

$$\int_{\mathbb{S}^{n-1}} h(K - \mathbf{s}(K), u)^{-(n+1)} u \, \mathrm{d}u = o \tag{10.22}$$

(see Santaló [1626]), or, equivalently, that

$$\int_{\mathbb{S}^{n-1}} \rho(K^{\mathbf{s}}, u)^{n+1} u \, \mathrm{d}u = o. \tag{10.23}$$

Hence, $K^{\mathbf{s}}$ has centroid o. Conversely, if $(K - z)^\circ$ has its centroid at o, then z is the Santaló point of K.

Equation (10.22) says that the indefinite σ-integral of $h(K - \mathbf{s}(K), \cdot)^{-(n+1)}$ satisfies the sufficiency condition of Minkowski's existence theorem. Hence, there exists a unique convex body $\mathrm{C}K \in \mathcal{F}_c^n$ with curvature function

$$f(\mathrm{C}K, \cdot) = h(K - \mathbf{s}(K), \cdot)^{-(n+1)}. \tag{10.24}$$

The body $\mathrm{C}K$ is called the *curvature image* of K. It satisfies

$$\mathrm{C}\phi K = \phi\, \mathrm{C}K \quad \text{for } \phi \in \mathrm{SL}(n)$$

(see Lutwak [1279], (7.12)). We notice the following relation to Blaschke's curvature image: if $\mathrm{C}K$ is sufficiently smooth, then $-(K - \mathbf{s}(K))$ is the Blaschke affine curvature image of $\mathrm{C}K$. We also note that for $K \in \mathcal{K}_n^n$ with $\mathbf{s}(K) = o$ we have

$$\mathrm{C}K = \left(\frac{V_n(K^\circ)}{\kappa_n}\right)^{1/(n-1)} \Lambda K^\circ. \tag{10.25}$$

(Thus, the reader should be alert to the fact that the so-called 'polar curvature image' Λ is, up to dilatation, the curvature image of the polar, if restricted to convex bodies with Santaló point at the origin.)

Suppose that a convex body $K \in \mathcal{K}_n^n$ has the property that it is homothetic to its curvature image $\mathrm{C}K$. Then $K \in \mathcal{F}_{(o)}^n$ and $f(K, \cdot)^{1/(n+1)} h(K, \cdot) = \text{const}$. In the differentiable case, this was equation (10.17).

Theorem 10.5.1 *If the convex body $K \in \mathcal{F}_{(o)}^n$ satisfies*

$$f(K, \cdot)^{1/(n+1)} h(K, \cdot) = \text{const.}, \tag{10.26}$$

then K is an o-symmetric ellipsoid.

Petty [1531] proved this for $n = 2$, and for $n \geq 3$ he proved it if K is either of class C^2 or a body of revolution. His proof was based on a regularity result for Monge–Ampère equations, from which he deduced that a body of class C^2 satisfying (10.26) must have a boundary which is so smooth that the classical differential-geometric proof is effective. For arbitrary convex bodies, this can now be deduced from the regularity results of Caffarelli (see Note 2 of Section 8.2). Unfortunately, no simpler proof seems to be known.

The range of C, denoted by $\mathcal{V}_c^n$, is a very special subset of $\mathcal{F}_c^n$. It belongs to $\mathcal{V}^n$, the set of convex bodies $M \in \mathcal{F}^n$ for which $f(M, \cdot)^{-1/(n+1)}$ is a support function, and thus comprises the convex bodies of class C_+^3 with centroid o and with an elliptically curved boundary hypersurface. The bodies of $\mathcal{V}^n$ are called of *elliptic type* (for these bodies, see also Leichtweiß [1188]). For such bodies, the affine surface area is a relative surface area. In fact, for $K \in \mathcal{K}_n^n$, we have

$$\begin{aligned} V_n(K^{\mathbf{s}}) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho(K^{\mathbf{s}}, u)^n \, \mathrm{d}u = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K - \mathbf{s}(K), u)^{-n} \, \mathrm{d}u \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(\mathrm{C}K, u)^{n/(n+1)} \, \mathrm{d}u = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K - \mathbf{s}(K), u) f(\mathrm{C}K, u) \, \mathrm{d}u \end{aligned}$$

and hence

$$\Omega(\mathrm{C}K) = nV(\mathrm{C}K, \dots, \mathrm{C}K, K) = nV_n(K^{\mathrm{s}}). \tag{10.27}$$

In particular, if M is a convex body of elliptic type, then there exists a convex body L such that

$$\Omega(M) = nV(M, \dots, M, L).$$

Leichtweiß [1188] has drawn a number of conclusions from this.

Using (10.27), Minkowski's inequality (7.18), the affine isoperimetric inequality (10.20) for convex bodies of elliptic type, and (10.27) again, we get

$$\begin{aligned} V_n(K^{\mathrm{s}})^n &= V(\mathrm{C}K, \dots, \mathrm{C}K, K)^n \ge V_n(K)V_n(\mathrm{C}K)^{n-1} \\ &\ge V_n(K)(n^{n+1}\kappa_n^2)^{-1}\Omega(\mathrm{C}K)^{n+1} = \kappa_n^{-2}V_n(K)V_n(K^{\mathrm{s}})^{n+1}, \end{aligned}$$

thus

$$V_n(K)V_n(K^{\mathrm{s}}) \le \kappa_n^2. \tag{10.28}$$

Equality holds if and only if K is an ellipsoid. This is the *Blaschke–Santaló inequality*, to which we will come back in Section 10.7. The approach to this inequality as given here is the original one, inspired by Blaschke's treatment within affine differential geometry. Blaschke [243] (see also [248], p. 208) considered the three-dimensional case and assumed that either K or K° has its centroid at the origin. In its present form, the proof was carried out by Santaló [1626], except that the discussion of the equality case was only completed by Petty's [1531] treatment of the equality condition for the affine isoperimetric inequality on $\mathcal{F}^n$.

Conversely, if inequality (10.28) is known with equality condition, we immediately obtain the affine isoperimetric inequality (10.20) with equality condition (see Petty [1529], Remark (3.16), or Leichtweiß [1188]). Let $K \in \mathcal{F}^n$, without loss of generality with Santaló point o, have curvature function f_K. Then, using Hölder's inequality with a negative exponent (integrations are over $\mathbb{S}^{n-1}$ with spherical Lebesgue measure),

$$\int h_K f_K \ge \left(\int h_K^{-n}\right)^{-1/n} \left(\int f_K^{n/(n+1)}\right)^{(n+1)/n} = [nV_n(K^{\mathrm{s}})]^{-1/n}\Omega(K)^{(n+1)/n},$$

hence

$$\Omega(K)^{n+1} \le [nV_n(K)]^n nV_n(K^{\mathrm{s}}) \le n^{n+1}\kappa_n^2 V_n(K)^{n-1}.$$

We turn to various extensions and generalizations of the affine surface area. The definition (10.18) immediately suggests a generalization: for $K_1, \dots, K_n \in \mathcal{F}^n$, a *mixed affine surface area* can be defined by

$$\Omega(K_1, \dots, K_n) := \int_{\mathbb{S}^{n-1}} [f(K_1, u) \cdots f(K_n, u)]^{1/(n+1)} \, \mathrm{d}u.$$

This notion was studied by Lutwak [1274], who obtained a number of inequalities, for example

$$\Omega(K_1,\dots,K_n)^{n+1} \leq \kappa_n^2 n^{n+1} V(K_1,\dots,K_n)^{n-1},$$

with equality if and only the K_i are homothetic ellipsoids.

Petty [1528, 1529] introduced the interesting concept of the geominimal surface area, which serves, as he explained, as a connecting link between Minkowski geometry, relative differential geometry and affine differential geometry. He defined

$$\mathcal{T}^n := \{T \in \mathcal{K}_n^n : \mathbf{s}(T) = o,\ V_n(T^\circ) = \kappa_n\},$$

$$A(K,T) := nV(K,\dots,K,T),$$

and the *geominimal surface area* of $K \in \mathcal{K}_n^n$ by

$$G(K) := \inf\{A(K,T) : T \in \mathcal{T}^n\}.$$

Equivalently,

$$G(K) = \inf\{nV(K,\dots,K,Q^\circ) : Q \in \mathcal{K}_c^n,\ V_n(Q) = \kappa_n\}. \tag{10.29}$$

The geominimal surface area is invariant under volume-preserving affine transformations; it is homogeneous of degree $n-1$, monotonic under set inclusion and continuous, and $G^{1/(n-1)}$ is concave under Minkowski addition. The following lemma collects Petty's results [1529], ((2.5), (2.8), (3.13)) on the convex bodies attaining the infimum in the definition.

Lemma 10.5.2 *To each $K \in \mathcal{K}_n^n$, there exists a unique $T \in \mathcal{T}^n$ with $G(K) = A(K,T)$. It is denoted by $T = T(K)$.*

Let $K \in \mathcal{V}_c^n$ and $T \in \mathcal{T}^n$. Then K is a dilatate of the curvature image of T if and only if $T = T(K)$. If this holds, then $A(K,T)^n = G(K)^n = (n\kappa_n)^{-1}\Omega(K)^{n+1}$.

Besides many other results, Petty [1529] proved (using the Blaschke–Santaló inequality) the *geominimal surface area inequality,*

$$G(K)^n \leq n^n \kappa_n V_n(K)^{n-1}, \tag{10.30}$$

with equality if and only if K is an ellipsoid (see Petty [1531] for the equality case), and (using Hölder's inequality)

$$\Omega(K)^{n+1} \leq n\kappa_n G(K)^n \tag{10.31}$$

for $K \in \mathcal{F}^n$, where equality holds if and only if K is of elliptic type, that is, is a translate of the curvature image of some convex body. Note that (10.30) and (10.31) yield the affine isoperimetric inequality (for bodies in $\mathcal{F}^n$) again.

Definition (10.29) of the geominimal surface area can equivalently be written in the form

$$\begin{aligned} \kappa_n^{1/n} G(K) &= \inf\left\{nV_1(K,Q^\circ)V_n(Q)^{1/n} : Q \in \mathcal{K}_c^n\right\} \\ &= \inf\left\{nV_1(K,Q)V_n(Q^\circ)^{1/n} : Q \in \mathcal{K}_{(o)}^n,\ \mathbf{s}(Q) = o\right\}, \end{aligned} \tag{10.32}$$

since $c(Q^\circ) = o \Leftrightarrow \mathbf{s}(Q) = o$. For given $Q \in \mathcal{K}_{(o)}^n$, let $Q_\mathbf{s} := Q - \mathbf{s}(Q)$; then $\mathbf{s}(Q_\mathbf{s}) = o$, $V_1(K,Q_\mathbf{s}) = V_1(K,Q)$ and $V_n(Q_\mathbf{s}^\circ) \le V_n(Q^\circ)$. It follows also that

$$\kappa_n^{1/n} G(K) = \inf\left\{nV_1(K,Q)V_n(Q^\circ)^{1/n} : Q \in \mathcal{K}_{(o)}^n\right\}, \tag{10.33}$$

as noted by Lutwak [1287].

Variants of the geominimal surface area are obtained if the infimum in its definition extends only over bodies Q restricted to a $\mathrm{GL}(n)$ invariant subclass of $\mathcal{K}_{(o)}^n$, such as o-symmetric zonoids, polar zonoids or ellipsoids. Note that, in the latter case, the geominimal surface area of K is nothing but the minimum of the Euclidean surface areas of all images of K under $\mathrm{SL}(n)$ (we come back to this in Section 10.13).

The extension of the affine surface area to all of $\mathcal{K}_n^n$ has a slightly complicated but interesting history. The first definition of an extended affine surface area for general n-dimensional convex bodies was suggested by Leichtweiß [1187]. His definition was motivated by an intuitive interpretation of the affine surface area of very smooth three-dimensional bodies due to Blaschke [248], §47, and is given by formula (10.60) in the next section, in connection with (10.59). Leichtweiß showed that his affine surface area can also be expressed by

$$\widetilde{\Omega}(K) = \int_{\mathbb{S}^{n-1}} [D_{n-1}(h_K)]^{n/(n+1)}\, \mathrm{d}\sigma, \tag{10.34}$$

where $D_{n-1}(h_K)$ denotes the sum of the principal minors of the (almost everywhere existing) Hessian matrix of the (homogeneous) support function of K.

A different approach was followed by Lutwak [1281]. He noted that a seemingly slight, but in fact essential, modification of definition (10.29) of the geominimal surface area serves well for an extension of the affine surface area to general convex bodies. Recall that volume $V_n(Q)$ and centroid $c(Q)$ of a star body $Q \in \mathcal{S}_o^n$ are given, respectively, by

$$V_n(Q) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho(Q,u)^n\, \mathrm{d}u, \quad V_n(Q)c(Q) = \frac{1}{n+1}\int_{\mathbb{S}^{n-1}} \rho(Q,u)^{n+1} u\, \mathrm{d}u.$$

Without defining Q°, we may also define

$$V_1(K,Q^\circ) := \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho(Q,u)^{-1}\, S_{n-1}(K,\mathrm{d}u) \tag{10.35}$$

for $K \in \mathcal{K}^n$. With this convention, Lutwak's [1281] definition of the affine surface area $\Omega(K)$ of a convex body $K \in \mathcal{K}_n^n$ reads

$$n^{-1/n}\Omega(K)^{(n+1)/n} := \inf\left\{nV_1(K,Q^\circ)V_n(Q)^{1/n} : Q \in \mathcal{S}_c^n\right\}. \tag{10.36}$$

We show that this definition is consistent, that is, gives the right value on $\mathcal{F}^n$. Let $K \in \mathcal{F}^n_c$, without loss of generality. We have to show that

$$\alpha := \inf\left\{V_1(K, Q^\circ)V_n(Q)^{1/n} : Q \in \mathcal{S}^n_c\right\} = \left(\frac{1}{n}\Omega(K)\right)^{(n+1)/n} =: \beta.$$

Since the polar curvature image map $\Lambda : \mathcal{S}^n_c \to \mathcal{F}^n_c$ is surjective, there exists a star body $K^\star \in \mathcal{S}^n_c$ with $\Lambda K^\star = K$. Inserting (10.21) into the definition of $\Omega(K)$, we obtain

$$\beta = \kappa_n V_n(K^\star)^{1/n}.$$

From (10.35), (10.21) and (9.37) we get

$$V_1(K, Q^\circ) = V_1(\Lambda K^\star, Q^\circ) = \frac{\kappa_n}{V_n(K^\star)}\widetilde{V}_{-1}(K^\star, Q),$$

hence

$$\alpha = \frac{\kappa_n}{V_n(K^\star)} \inf\left\{\widetilde{V}_{-1}(K^\star, Q)V_n(Q)^{1/n} : Q \in \mathcal{S}^n_c\right\}.$$

Since $K^\star \in \mathcal{S}^n_c$, trivially

$$\alpha \leq \frac{\kappa_n}{V_n(K^\star)}\widetilde{V}_{-1}(K^\star, K^\star)V_n(K^\star)^{1/n} = \beta.$$

Inequality (9.40) for $(i, j, k) = (-1, 0, n)$ says that

$$\widetilde{V}_{-1}(K^\star, Q)^n \geq V_n(K^\star)^{n+1}V_n(Q)^{-1}$$

for $Q \in \mathcal{S}^n_o$, which yields $\alpha \geq \beta$. Thus, definition (10.36) is, in fact, consistent and can be used to define the affine surface area for arbitrary convex bodies $K \in \mathcal{K}^n_n$.

It was pointed out by Lutwak that his definition of the generalized affine surface area allows easy proofs of its essential properties. We collect these properties in the following theorem and refer to Lutwak [1281] for the proofs.

Theorem 10.5.3 *The affine surface area on $\mathcal{K}^n_n$ has the following properties. It is invariant under volume-preserving affine transformations, homogeneous of degree $n(n-1)/(n+1)$ and upper semi-continuous. It vanishes on polytopes. It satisfies the affine isoperimetric inequality*

$$\Omega(K)^{n+1} \leq \kappa_n^2 n^{n+1} V_n(K), \tag{10.37}$$

with equality if and only if K is an ellipsoid and, under Blaschke addition,

$$\Omega(K \# L)^{(n+1)/n} \geq \Omega(K)^{(n+1)/n} + \Omega(L)^{(n+1)/n}.$$

After this approach, Leichtweiß [1189] showed that his extension of the affine surface area can also be obtained from a modified version of Lutwak's definition (10.36), namely

$$n^{-1/n}\widetilde{\Omega}(K)^{(n+1)/n} := \inf\left\{nV_1(K, Q^\circ)V_n(Q)^{1/n} : Q \in \mathcal{S}^n_o\right\} \tag{10.38}$$

for $K \in \mathcal{K}_n^n$ (note that $c(Q) = o$ is no longer required). Clearly, $\widetilde{\Omega} \le \Omega$. Moreover, Leichtweiß pointed out that, as a result of this definition, his extended affine surface area has all the properties that Lutwak established for Ω (see also Leichtweiß [1190]).

Schütt and Werner [1757] used convex floating bodies to define a generalized affine surface area by means of (10.62) in the next section, and they showed that it can be represented by

$$\widetilde{\Omega}(K) = \int_{\mathrm{bd}\,K} H_{n-1}(K,x)^{1/(n+1)}\,\mathrm{d}\mathcal{H}^{n-1}(x), \tag{10.39}$$

where $H_{n-1}(K,x)$ denotes the (for $\mathcal{H}^{n-1}$-almost all $x \in \mathrm{bd}\,K$ existing) Gauss–Kronecker curvature of $\mathrm{bd}\,K$ at x. Schütt [1754] proved that, in fact, this is the same functional as the one defined by Leichtweiß. Schütt also proved that the generalized affine surface area $\widetilde{\Omega}$ is a valuation. Finally, Dolzmann and Hug [510] proved that $\Omega = \widetilde{\Omega}$, so that all the proposed generalizations of the affine surface area do, in fact, coincide. As a consequence, the affine surface area of $K \in \mathcal{K}_n^n$ can now be represented by

$$\begin{aligned} &\Omega(K) \\ &= \inf\left\{\left(\int_{\mathbb{S}^{n-1}} g\,\mathrm{d}S_{n-1}(K,\cdot)\right)^{n/(n+1)}\left(\int_{\mathbb{S}^{n-1}} g^{-n}\,\mathrm{d}\sigma\right)^{1/(n+1)} : g \in C^+(\mathbb{S}^{n-1})\right\}, \end{aligned} \tag{10.40}$$

where $C^+(\mathbb{S}^{n-1})$ denotes the space of positive continuous functions on $\mathbb{S}^{n-1}$.

The extension of the affine surface area allows us to further generalize previous inequalities. An example is inequality (10.31).

Theorem 10.5.4 *For $K \in \mathcal{K}_n^n$,*

$$\Omega(K)^{n+1} \le n\kappa_n G(K)^n, \tag{10.41}$$

with equality if and only if K is of elliptic type.

Proof Let $K \in \mathcal{K}_n^n$. By Lemma 10.5.2, there is a unique convex body $T \in \mathcal{T}^n$ with

$$G(K) = A(K,T) = \int_{\mathbb{S}^{n-1}} h_T\,\mathrm{d}S_{n-1}(K,\cdot). \tag{10.42}$$

With respect to spherical Lebesgue measure σ, the measure $S_{n-1}(K,\cdot)$ has a Lebesgue decomposition into the sum of an absolutely continuous measure S_K^a and a singular measure S_K^s. It is known that

$$S_K^a(\omega) = \int_\omega D_{n-1}(h_K)\,\mathrm{d}\sigma \qquad \text{for Borel sets } \omega \subset S^{n-1}. \tag{10.43}$$

A proof is given, for example, in Hug [1005], Section 3. With (10.42) this gives

$$G(K) = \int_{\mathbb{S}^{n-1}} h_T\,\mathrm{d}S_K^a + \int_{\mathbb{S}^{n-1}} h_T\,\mathrm{d}S_K^s \ge \int_{\mathbb{S}^{n-1}} h_T D_{n-1}(h_K)\,\mathrm{d}\sigma. \tag{10.44}$$

Hölder's inequality with the negative exponent $-n$ gives (integrations are over the unit sphere)

$$\begin{aligned}\int h_T D_{n-1}(h_K)\,\mathrm{d}\sigma &\geq \left(\int h_T^{-n}\,\mathrm{d}\sigma\right)^{-1/n}\left(\int [D_{n-1}(h_K)]^{n/(n+1)}\,\mathrm{d}\sigma\right)^{(n+1)/(n)}\\ &= [nV_n(T^\circ)]^{-1/n}\Omega(K)^{(n+1)/(n)}\\ &= (n\kappa_n)^{-1/n}\Omega(K)^{(n+1)/(n)},\end{aligned}$$

where (10.34) (together with $\widetilde{\Omega} = \Omega$) was used. The inequality (10.41) follows. Suppose that equality holds here. Then equality holds in Hölder's inequality. This implies (see [938], p. 140) that h_T^{-n} and $D_{n-1}(h_K)^{n/(n+1)}$ are proportional outside a set of measure zero, hence there is a positive constant A with

$$Ah_T^{-(n+1)} = D_{n-1}(h_K) \quad \sigma\text{-almost everywhere on } \mathbb{S}^{n-1}. \tag{10.45}$$

Since equality holds also in (10.44), and $h_T > 0$ everywhere on $\mathbb{S}^{n-1}$, the singular part S_K^s is the zero measure. Now it follows from (10.43) and (10.45) that $Ah_T^{-(n+1)}$ is a density for $S_{n-1}(K,\cdot)$. Thus, K has the curvature function $f_K = Ah_T^{-(n+1)}$, and $f_K^{-1/(n+1)}$ is a support function. Therefore, $K \in \mathcal{V}^n$, that is, K is of elliptic type, which completes the proof. □

The preceding proof is taken from [1732]. As a consequence, a result of Winternitz [1986] can be extended. He proved that a convex body K that is properly contained in an ellipsoid E satisfies $\Omega(K) < \Omega(E)$. The equality condition in the following result was previously only known under the additional assumption that K is also of elliptic type (Leichtweiß [1188], Satz 1(d)).

Theorem 10.5.5 *If $K, L \in \mathcal{K}_n^n$, $K \subset L$ and L is of elliptic type, then*

$$\Omega(K) \leq \Omega(L), \tag{10.46}$$

with equality if and only if $K = L$.

Proof Let $K \in \mathcal{K}^n$, $L \in \mathcal{V}^n$ and $K \subset L$. From Theorem 10.5.4, and since the geominimal surface area is monotonic under set inclusion, we get

$$\Omega(K)^{n+1} \leq n\kappa_n G(K)^n \leq n\kappa_n G(L)^n = \Omega(L)^{n+1}.$$

This proves inequality (10.46). If equality holds here, then the equality condition of Theorem 10.5.4 shows that also K is of elliptic type; moreover,

$$G(L) = G(K) \leq A(K, T(L)) \leq A(L, T(L)) = G(L).$$

The first inequality follows from the inf-definition of the geominimal surface area, the second from the monotonicity of mixed volumes. It follows that $A(K, T(K)) = G(K) = A(K, T(L))$. By the uniqueness result of Lemma 10.5.2, we have $T(K) = T(L)$. By Lemma 10.5.2, $K - c(K)$ and $L - c(L)$ are dilatates of the curvature image of the same set, thus they are homothetic and then, necessarily, identical. □

After a unique notion of generalized affine surface area on $\mathcal{K}^n_n$ and its essential properties had been established, the way was open for the following deep characterization theorem due to Ludwig and Reitzner [1257]. (The case $n = 2$ had been proved before by Ludwig [1243], and a more general planar result by Ludwig [1244].)

Theorem 10.5.6 (Ludwig, Reitzner) *If $\varphi : \mathcal{K}^n_n \to \mathbb{R}$ is an upper semi-continuous valuation that is invariant under volume-preserving affine transformations, then there are constants $c_0, c_1 \in \mathbb{R}$ and $c_2 \geq 0$ such that*

$$\varphi(K) = c_0 V_0(K) + c_1 V_n(K) + c_2 \Omega(K)$$

for all $K \in \mathcal{K}^n_n$.

In the L_p Brunn–Minkowski theory, most of the notions introduced in this section find their natural extension. This was carried out by Lutwak [1287]. We refer to this paper for the proofs of the following statements, as well as for the affine invariance properties of the functionals introduced.

Recall that $\mathcal{F}^n_{(o)} := \mathcal{F}^n \cap \mathcal{K}^n_{(o)}$ is the set of all convex bodies in $\mathbb{R}^n$ having a positive continuous curvature function and containing the origin in the interior. Let $p \geq 1$ and $K \in \mathcal{F}^n_{(o)}$. The *p-curvature function* of K is defined by

$$f_p(K, \cdot) := h(K, \cdot)^{1-p} f(K, \cdot),$$

where $f(K, \cdot)$ is the curvature function of K. Thus, according to (9.10),

$$V_p(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u)^p f_p(K, u)\, \mathrm{d}u$$

for all $L \in \mathcal{K}^n$. Lutwak introduced an L_p curvature image $\Lambda_p K$ of $K \in \mathcal{F}^n_{(o)}$ by

$$\rho(\Lambda_p K, \cdot)^{n+p} = \frac{V_n(\Lambda_p K)}{\kappa_n} f_p(K, \cdot).$$

Note that $\Lambda_1 : \mathcal{F}^n_{(o)} \to \mathcal{S}^n_o$ is essentially different from the polar curvature image mapping $\Lambda : \mathcal{S}^n_c \to \mathcal{F}^n_c$ defined by (10.21).

Modifying the definition (10.18), one defines the *p-affine surface area* by

$$\Omega_p(K) := \int_{\mathbb{S}^{n-1}} f_p(K, u)^{n/(n+p)}\, \mathrm{d}u = \int_{\mathbb{S}^{n-1}} \frac{f(K, u)^{n/(n+p)}}{h(K, u)^{n(p-1)/(n+p)}}\, \mathrm{d}u. \tag{10.47}$$

The definition can be extended to all of $K \in \mathcal{K}^n_{(o)}$ by extending (10.38), namely

$$n^{-p/n} \Omega_p(K)^{(n+p)/n} := \inf \left\{ n V_p(K, Q^\circ) V_n(Q)^{p/n} : Q \in \mathcal{S}^n_o \right\}, \tag{10.48}$$

where

$$V_p(K, Q^\circ) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho(Q, u)^{-p} h(K, u)^{1-p}\, S_{n-1}(K, \mathrm{d}u).$$

Then, for $K \in \mathcal{K}^n_c$ (note the condition $c(K) = o$) the inequality

$$\Omega_p(K)^{n+p} \leq n^{n+p} \kappa_n^{2p} V_n(K)^{n-p} \tag{10.49}$$

holds, with equality if and only if K is an ellipsoid. For $p = n$, this inequality reduces to

$$\Omega_n(K) \le \kappa_n.$$

Note that Ω_n is homogeneous of degree zero and is, therefore, invariant under all nondegenerate linear transformations. It is an extension of the classical *centroaffine surface area*.

Definition (10.33) extends to the definition of the *p-geominimal surface area* $G_p(K)$, by

$$\kappa_n^{p/n} G_p(K) := \inf\left\{ nV_p(K,Q)V_n(Q^\circ)^{p/n} : Q \in \mathcal{K}^n_{(o)} \right\}. \tag{10.50}$$

Petty's inequalities (10.30) and (10.31) extend to

$$G_p(K)^n \le n^n \kappa_n^p V_n(K)^{n-p}$$

for $K \in \mathcal{K}^n_c$, with equality if and only if K is an ellipsoid, and

$$\Omega_p(K)^{n+p} \le (n\kappa_n)^p G_p(K)^n$$

for $K \in \mathcal{F}^n_{(o)}$, with equality if and only if K is of p-elliptic type, which means that $f_p(K,\cdot)^{-1/(n+p)}$ is a support function.

The two representations (10.34) and (10.39) of the extended affine surface area can be generalized to a p-affine surface area for all $p > 0$, and at the same time localized. This was done by Hug [1002]. For $K \in \mathcal{K}^n_{(o)}$ and $p > 0$ ($K \in \mathcal{K}^n_n$ if $p = 1$), he defined

$$\Omega_p(K,\beta) := \int_{\beta\cap \mathrm{bd}\, K} \left\{ \frac{H_{n-1}(K,x)}{\langle x, u_K(x)\rangle^{(p-1)n/p}} \right\}^{p/(n+p)} \mathrm{d}\mathcal{H}^{n-1}(x) \tag{10.51}$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$ and

$$\widetilde{\Omega}_p(K,\omega) := \int_\omega \left\{ \frac{D_{n-1}h_K(u)}{h_K(u)^{p-1}} \right\}^{n/(n+p)} \mathrm{d}\mathcal{H}^{n-1}(u) \tag{10.52}$$

for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. He proved that

$$\Omega_p(K,\beta) = \widetilde{\Omega}_p(K,\sigma(K,\beta)), \qquad \widetilde{\Omega}_p(K,\omega) = \Omega_p(K,\tau(K,\omega))$$

and also indicated a proof that Lutwak's definition of p-affine surface area can be localized for all $p > 0$ and then also leads to (10.52). Hug also gave necessary and sufficient geometric criteria for $\Omega_p(K,\beta) > 0$ and for $\widetilde{\Omega}_p(K,\omega) > 0$. Moreover, he gave, by Steiner symmetrization, a proof of the general affine isoperimetric inequality (10.37) (with equality condition) which is independent of a previous proof of the Blaschke–Santaló inequality (with equality condition); rather, the latter is obtained as a consequence.

In [1003], Hug proved that

$$\Omega_p(K) = \Omega_{n^2/p}(K^\circ) \tag{10.53}$$

for $K \in \mathcal{K}^n_{(o)}$ and $p > 0$. For K of class C^2_+ and $p = n$, this reduces to a classical formula of centroaffine differential geometry.

The definition (10.47) can further be used to define a p-affine surface area for $K \in \mathcal{F}^n_{(o)}$ and for $-n < p < \infty$. This was pointed out by Meyer and Werner [1417], who gave a geometric interpretation for this p-affine surface area (cf. Note 8 of Section 10.6). For all $K \in \mathcal{K}^n_n$ and all $p \neq -n$, Schütt and Werner [1760] defined

$$\Omega_p(K) = \int_{\operatorname{bd} K} \frac{H_{n-1}(K,x)^{p/(n+p)}}{\langle x, u_K(x)\rangle^{n(p-1)/(n+p)}} \, \mathrm{d}\mathcal{H}^{n-1}(x) \tag{10.54}$$

(note that $\Omega_0(K) = nV_n(K)$). Under the condition that a ball rolls freely in K and K rolls freely in a ball, they showed that this is finite and can be interpreted geometrically in terms of approximation by random polytopes, whose vertices are suitably distributed on the boundary of K. General properties of Ω_p are discussed in Werner [1962], Schütt and Werner [1761]. Extensions of the inequalities (10.49) and the identity (10.53) to all $p \neq -n$, under suitable assumptions, are proved by Werner and Ye [1967].

The L_p affine surface areas for $p > 0$ appear in the following strong classification result for homogeneous $\mathrm{SL}(n)$ invariant valuations. It was proved by Ludwig and Reitzner [1258].

Theorem 10.5.7 (Ludwig, Reitzner) *A function $\Phi : \mathcal{K}^n_{(o)} \to \mathbb{R}$ is an upper semi-continuous and* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree q if and only if there are real constants c_0 and $c_1 \geq 0$ such that*

$$\Phi(K) = \begin{cases} c_0 V_0(K) + c_1 \Omega_n(K) & \text{for } q = 0, \\ c_1 \Omega_{n(n-q)/(n+q)}(K) & \text{for } -n < q < n \text{ and } q \neq 0, \\ c_0 V_n(K) & \text{for } q = n, \\ c_0 V_n(K^\circ) & \text{for } q = -n, \\ 0 & \text{for } q < -n \text{ or } q > n, \end{cases}$$

for every $K \in \mathcal{K}^n_{(o)}$.

A natural final step of extension are the general affine surface areas, which were introduced by Ludwig and Reitzner [1258] and further studied by Ludwig [1253]. Let $\mathrm{Conc}(0,\infty)$ denote the set of concave functions $\phi : (0,\infty) \to (0,\infty)$ with the properties that $\lim_{t\to 0} \phi(t) = 0$ and $\lim_{t\to\infty} \phi(t)/t = 0$; set $\phi(0) = 0$. For $\phi \in \mathrm{Conc}(0,\infty)$, the L_ϕ *affine surface area* of the convex body $K \in \mathcal{K}^n_{(o)}$ (respectively, $K \in \mathcal{K}^n_n$ if $\phi(t) = t^{1/(n+1)}$) is defined by

$$\Omega_\phi(K) := \int_{\operatorname{bd} K} \phi\left(\frac{H_{n-1}(K,x)}{\langle x, u_K(x)\rangle^{n+1}} \right) \langle x, u_K(x)\rangle \, \mathrm{d}\mathcal{H}^{n-1}(x). \tag{10.55}$$

For $\phi(t) = t^{p/(n+p)}$ one obtains the L_p affine surface area, if $p > 0$. It was proved in [1258] that the L_ϕ affine surface area Ω_ϕ is finite, upper semi-continuous, $\mathrm{SL}(n)$ invariant and a valuation. It vanishes on polytopes. The affine isoperimetric inequality now extends as follows. If $K \in \mathcal{K}^n_{(o)}$ has centroid o and if B_K is the ball with centre o and the same volume as K, then

$$\Omega_\phi(K) \le \Omega_\phi(B_K).$$

If ϕ is strictly increasing, then equality holds if and only if K is an ellipsoid.

The following two theorems provide strong reasons why this can be considered as the natural final extension of affine surface area. The first of the theorems was established in [1258].

Theorem 10.5.8 (Ludwig, Reitzner) *If* $\Phi : \mathcal{K}^n_{(o)} \to \mathbb{R}$ *is an upper semi-continuous and* $\mathrm{SL}(n)$ *invariant valuation that vanishes on polytopes, then there exists a function* $\phi \in \mathrm{Conc}(0, \infty)$ *such that*

$$\Phi(K) = \Omega_\phi(K)$$

for every $K \in \mathcal{K}^n_{(o)}$.

Finally, the following theorem of Haberl and Parapatits [880] achieves a characterization comprising Theorems 10.5.6, 10.5.7 and 10.5.8, without an additional assumption of translation invariance, homogeneity or vanishing on polytopes.

Theorem 10.5.9 (Haberl, Parapatits) *A map* $\Phi : \mathcal{K}^n_{(o)} \to \mathbb{R}$ *is an upper semi-continuous and* $\mathrm{SL}(n)$ *invariant valuation if and only if there exist constants* $c_0, c_1, c_2 \in \mathbb{R}$ *and a function* $\phi \in \mathrm{Conc}(0, \infty)$ *such that*

$$\Phi(K) = c_0 V_0(K) + c_1 V_n(K) + c_2 V_n(K^\circ) + \Omega_\phi(K)$$

for every $K \in \mathcal{K}^n_{(o)}$.

Ludwig [1253] extended relation (10.53) to

$$\Omega_\phi(K^\circ) = \Omega_{\phi_*}(K)$$

for $\phi \in \mathrm{Conc}(0, \infty)$ and $K \in \mathcal{K}^n_{(o)}$, where $\phi_*(s) := s\phi(1/s)$. She also gave a representation of Ω_ϕ corresponding to (10.52).

The L_p affine surface areas for $-n < p < 0$ are generalized by Ludwig as follows. Let $\mathrm{Conv}(0, \infty)$ denote the set of convex functions $\psi : (0, \infty) \to (0, \infty)$ with the properties that $\lim_{t\to 0} \psi(t) = \infty$ and $\lim_{t\to\infty} \psi(t) = 0$; set $\psi(0) = \infty$. For $\psi \in \mathrm{Conv}(0, \infty)$, the L_ψ *affine surface area* of the convex body $K \in \mathcal{K}^n_{(o)}$ is defined by (10.55), with ϕ replaced by ψ. The L_ψ affine surface area Ω_ψ is positive, lower semi-continuous, $\mathrm{SL}(n)$ invariant and a valuation. It is equal to infinity on polytopes. The affine isoperimetric inequality reads as follows. If $K \in \mathcal{K}^n_{(o)}$ has centroid o and if B_K is the ball with centre o and the same volume as K, then

$$\Omega_\psi(K) \ge \Omega_\psi(B_K).$$

If ψ is strictly decreasing, then equality holds if and only if K is an ellipsoid.

Further, for $\psi \in \mathrm{Conc}(0, \infty)$, Ludwig defined $\Omega^*_\psi(K) := \Omega_\psi(K^\circ)$ for $K \in \mathcal{K}^n_{(o)}$. Properties of Ω^*_ψ follow from those of Ω_ψ. For $p < -n$, one has $\Omega_p = \Omega^*_\psi$ with $\psi(t) = t^{n/(n+p)}$.

Notes for Section 10.5

1. *The Santaló point.* See Petty [1529], Lemma (2.2), for a proof that the Santaló point mapping is affinely equivariant and continuous.
2. On the interplay between Blaschke–Santaló inequality, affine isoperimetric inequality and Minkowski's inequality, see also Lutwak [1270]. For $K \in \mathcal{K}_n^n$ and $A \in \mathcal{F}^n$, he states the *general affine isoperimetric inequality*,

$$n^{n+1}\kappa_n^2 V_1(A,K)^n \geq V_n(K)\Omega(A)^{n+1}, \tag{10.56}$$

with equality if and only if A and K are homothetic ellipsoids. It follows from the affine isoperimetric inequality and Minkowski's inequality, but also from the Blaschke–Santaló and Hölder inequalities. Of course, for $A = K$, (10.56) reduces to the affine isoperimetric inequality. With $A = \mathrm{C}K$, one can deduce from (10.56) the Blaschke–Santaló inequality (see [1270]).

   For the two-dimensional version of (10.56), a more general form, which does not need convexity assumptions, was given by Chen, Howard, Lutwak, Yang and Zhang [417]. A different proof was provided by Ni and Zhu [1472] (who, on p. 160, point to a problem with the proof of Lemma 6.2 in [417]).
3. Böröczky [289] obtained stability results (for $n \geq 3$), involving the Banach–Mazur distance, for the affine isoperimetric inequality (10.19), the Blaschke–Santaló inequality (10.28) and Petty's geominimal surface area inequality (10.30).
4. Andrews [70] studied affine-invariant evolution equations for sufficiently smooth convex hypersurfaces. He showed that these also lead to affine inequalities, in particular the affine isoperimetric inequality and the Blaschke–Santaló inequality.
5. *Affine surface area and polytopal approximation.* The affine surface area arises naturally in asymptotic results on best or random approximation of convex bodies by polytopes, if the deviation is measured in a way that has suitable invariance properties. This was apparently first pointed out, heuristically, by Fejes Tóth [566], p. 152. Later contributions, for sufficiently smooth convex bodies, are, in chronological order, by Rényi and Sulanke [1573, 1574], McClure and Vitale [1371] (all in the plane), Schneider [1703], Gruber [821, 825], Bárány [148], Gruber [830] (here the centroaffine surface area appears, in a problem of approximation with respect to the Banach–Mazur distance), Gruber [831], Ludwig [1243], Reitzner [1568]. All these results assume the convex bodies that are approximated to be at least of class C^2. Schütt [1755] was the first to prove a result on random approximation of general convex bodies, where the generalized affine surface area then shows up. Interpretations of extended p-affine surface areas in terms of random approximation are investigated by Schütt and Werner [1760].
6. *Maximal affine surface area or perimeter.* Given a convex body $K \in \mathcal{K}_n^n$, it follows from the upper semi-continuity of affine surface area that among the convex bodies contained in K there exists one, K_a, with maximal affine surface area. For $n = 2$, it was proved by Bárány [141] that K_a is uniquely determined, and by Bárány and Prodromou [150] that K_a is (up to translation) a curvature image and that $K_a = K$ if K was already a curvature image.

   In higher dimensions, uniqueness of K_a is unknown.

   Regularity properties of K_a are studied by Sheng, Trudinger and Wang [1774].
7. *Maximal affine perimeter and limit shape.* The unique convex body of maximal affine perimeter, K_a, contained in a given two-dimensional convex body K (see the previous note), and its affine perimeter $\Omega(K_a)$ play an important role in remarkable work of Bárány [141, 142]. He has similar results about lattice points (see also Bárány and Prodromou [150]) and about random points in K, and we describe here only the latter. Assume that $V_2(K) = 1$, and let X_m be a set of m stochastically independent, uniformly distributed random points in K. Let $Q(X_m)$ denote the set of all convex polygons with vertices in X_m. Denoting by $\mathbb{E}$ mathematical expectation and by $|\cdot|$ the number of elements, one result of Bárány [142] says that

$$\lim_{m\to\infty} m^{-1/3} \log \mathbb{E}\,|Q(X_m)| = 3 \cdot 2^{-2/3}\Omega(K_a).$$

Another of his results states that, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{\mathbb{E}\,|\{P \in Q(X_m) : \delta(P, K_a) > \varepsilon\}|}{\mathbb{E}\,|Q(X_m)|} = 0.$$

Further, let $p(K, m)$ denote the probability that the points of X_m are in convex position (i.e., none of them is in the convex hull of the others). Then

$$\lim_{m\to\infty} m^2 \sqrt[m]{p(K, m)} = \frac{e^2}{4}\Omega(K_a)^3.$$

8. Various inequalities connecting the mixed affine surface area with other mixtures were proved by Lu, Wang and Leng [1240] and by Cheung and Zhao [422].
9. *Mixed p-affine surface area.* For $K_1, \dots, K_n \in \mathcal{K}^n$ of class C_+^2 with centroids at the origin, the *mixed p-affine surface area* was introduced by Lutwak [1287] as

$$\Omega_p(K_1, \dots, K_n) := \int_{\mathbb{S}^{n-1}} \left[h_{K_1}^{1-p} f_{K_1} \cdots h_{K_n}^{1-p} f_{K_n}\right]^{1/(n+p)} \mathrm{d}u$$

for $p \geq 1$, and Werner and Ye [1968] suggested studying it for all $p \neq -n$. They proved various inequalities for it, as well as interpretations by means of a generalization of illumination bodies (which are defined in the next section). This was further extended by Ye [2002] to mixed L_ϕ affine surfaces areas.
10. For $K, L \in \mathcal{F}_{(o)}^n$, $p \geq 1$ and any real $i \in \mathbb{R}$, Wang and Leng [1914] introduced the ith L_p mixed affine surface area by

$$\Omega_{p,i}(K, L) := \int_{\mathbb{S}^{n-1}} f_p(K, u)^{(n-i)/(n+p)} f_p(L, u)^{i/(n+p)} \,\mathrm{d}u$$

and extended a number of previously known inequalities.
11. *L_p dual affine surface area.* As a counterpart to (10.48), Wang and He [1912] defined, for $K \in \mathcal{S}_o^n$ and for $1 \leq p < n$, the *L_p dual affine surface area* $\widetilde{\Omega}_{-p}(K)$ of K by

$$n^{p/n}\widetilde{\Omega}_{-p}(K)^{(n-p)/p} := \inf\left\{n\widetilde{V}_{-p}(K, Q^\circ)V_n(Q)^{-p/n} : Q \in \mathcal{K}_c^n\right\}.$$

They proved various inequalities for this functional. A geometric interpretation of the L_p dual affine surface area was not given. We remark, however, that the argument after (10.36) can be dualized, to show that for $K \in \mathcal{S}_c^n$ one has

$$\widetilde{\Omega}_{-1}(K) = n\left(\frac{V_n(K)}{\kappa_n}\right)^{n/(n-1)} V_n(\Lambda K),$$

where ΛK is the polar curvature image of K. This corresponds to part of formula (10.27), which says that $\Omega(\mathrm{C}\,K) = nV_n(K^{\mathrm{s}})$.
12. Various inequalities involving the L_p curvature image $\Lambda_p K$ of $K \in \mathcal{F}_{(o)}^n$ were proved by Wang, Wei and Xiang [1921]. The notion of L_p curvature image was extended by Lu and Wang [1239] to that of L_p mixed curvature image. They correspondingly extended some inequalities.
13. L_p polar curvature images were introduced and used by Zhu, Lv and Leng [2077].
14. Zhu, Li and Zhou [2074] proved various inequalities for the L_p geominimal surface area, also in connection with other functionals.
15. As a counterpart to (10.50), Wang and Qi [1919] defined, for o-symmetric star bodies $K \in \mathcal{S}^n$ and for $p \geq 1$, the *L_p dual geominimal surface area* by

$$\kappa_n^{-p/n}\widetilde{G}_{-p}(K) := \inf\left\{n\widetilde{V}_{-p}(K, Q)V_n(K^\circ)^{-p/n} : Q \in \mathcal{K}_{(os)}^n\right\},$$

where $\mathcal{K}^n_{(os)}$ is the set of o-symmetric convex bodies in $\mathcal{K}^n_n$. They proved various inequalities for this functional.

16. For sufficiently smooth convex bodies, Stancu [1810] used centro-affine flows to generate new equicentro-affine differential invariants and to obtain corresponding inequalities. She also gave a geometric interpretation, in terms of flows, of the L_ϕ affine surface areas of Ludwig and Reitzner.

10.6 Floating bodies and similar constructions

Floating bodies of a given convex body are a kind of affine-equivariant inner parallel body, which allow a particular approach to the affine surface area and have also proved useful in other circumstances.

Let $K \in \mathcal{K}^n_n$. For $u \in \mathbb{S}^{n-1}$ and $0 < \delta < V_n(K)$, there is a unique number $t(u,\delta)$ such that

$$V_n(K \cap H^+_{u,h(K,u)-t(u,\delta)}) = \delta. \tag{10.57}$$

In general, the function $u \mapsto h(K,u) - t(u,\delta)$ will not be the restriction of a support function, but, if it is, then it defines a convex body $K_{[\delta]}$ contained in K, with the property that each support plane of $K_{[\delta]}$ cuts off from K a cap of volume δ. For obvious mechanical reasons, the body $K_{[\delta]}$ is then called the *flotation body* of K, with parameter δ.

There are two remarkable instances where the flotation body does exist. The following was proved by Leichtweiß [1186]. Let $K \in \mathcal{K}^n_n$ be of class C^2_+, and let $\delta_0 := \kappa_n r_0^n/2$, where r_0 denotes the minimum of the radii of curvature of $\mathrm{bd}\,K$. Then $K_{[\delta]}$ exists for $0 < \delta < \delta_0$, and it is of class C^2_+. (In fact, this already holds under weaker assumptions; see Leichtweiss [1187], p. 435.) Meyer and Reisner [1410] established the following result. Let $K \in \mathcal{K}^n_n$ be centrally symmetric, and let $0 < \delta < V_n(K)/2$. Then $K_{[\delta]}$ exists, and it is centrally symmetric and strictly convex. If, moreover, K is smooth and strictly convex, then $K_{[\delta]}$ is of class C^2. The proofs of these results are also outlined in the survey article by Leichtweiß [1191], where the boundary of $K_{[\delta]}$ is called an *equiaffine inner parallel hypersurface* of $\mathrm{bd}\,K$.

If the flotation body $K_{[\delta]}$ exists, then for each point $x \in \mathrm{bd}\,K_{[\delta]}$ and each support hyperplane H of $K_{[\delta]}$ through x, the point x is the centroid of the section $H \cap K$.

The relation of flotation bodies to affine surface area originates in a formula of Blaschke [248], §47, for three-dimensional convex bodies with an analytic boundary. It was extended by Leichtweiß [1186] to convex bodies $K \in \mathcal{K}^n_n$ of class C^2_+, and then it says that

$$\Omega(K) = \lim_{\delta\to 0} c_n \frac{V_n(K) - V_n(K_{[\delta]})}{\delta^{2/(n+1)}}, \qquad c_n = 2\left(\frac{\kappa_{n-1}}{n+1}\right)^{2/(n+1)}. \tag{10.58}$$

If K is an outer parallel body of some convex body (equivalently, a ball of positive radius rolls freely in K), then Leichtweiß [1187] showed that $K_{[\delta]}$ exists for sufficiently small $\delta > 0$ and that the limit

$$\widetilde{\Omega}(K) := \lim_{\delta\to 0} nc_n \frac{V_n(K) - V_1(K, K_{[\delta]})}{\delta^{2/(n+1)}} \tag{10.59}$$

exists. Leichtweiß then defined the affine surface area of general convex bodies K by

$$\widetilde{\Omega}(K) := \inf_{\rho>0} \widetilde{\Omega}(K + \rho E), \tag{10.60}$$

where E is an arbitrary ellipsoid of volume κ_n. He showed that this definition is independent of E and yields the classical affine surface area for bodies of class C^2_+.

Whereas for $K \in \mathcal{K}^n_n$ the flotation body $K_{[\delta]}$ need not exist, the intersection

$$K_\delta := \bigcap_{u\in\mathbb{S}^{n-1}} H^-_{u,h(K,u)-t(u,\delta)}, \tag{10.61}$$

where $t(u, \delta)$ is defined by (10.57), is always convex, and if it is not empty, it is called the *convex floating body* (or briefly the *floating body*) of K for the parameter δ. Of course, if $K_{[\delta]}$ exists, it is equal to K_δ. Schütt and Werner [1757] coined the name 'convex floating body' and showed that

$$\lim_{\delta\to 0} c_n \frac{V_n(K) - V_n(K_\delta)}{\delta^{2/(n+1)}} = \int_{\mathrm{bd}\, K} \lim_{\delta\to 0} c_n \frac{\Delta(x, \delta)}{\delta^{2/(n+1)}} \, \mathrm{d}\mathcal{H}^{n-1}(x), \tag{10.62}$$

where $\Delta(x, \delta)$ denotes the height of a slice of volume δ cut off from K by a hyperplane parallel to the ($\mathcal{H}^{n-1}$-almost everywhere unique) tangent hyperplane of K at x. As shown by Leichtweiß [1187], Hilfssatz 2, the right side of (10.62) is equal to the right side of (10.39) and thus to $\Omega(K)$.

Some similar constructions can also be used to represent the affine surface area of a convex body. For $K \in \mathcal{K}^n_n$ and $\delta > 0$, Werner [1959] introduced the *illumination body* of K by

$$K^\delta := \{x \in \mathbb{R}^n : V_n(\mathrm{conv}\,(K \cup \{x\})) - V_n(K) \le \delta\}.$$

Its convexity follows from a more general result of Fáry and Rédei [550] (Satz 4). Werner proved that

$$\Omega(K) = \lim_{\delta\to 0} d_n \frac{V_n(K^\delta) - V_n(K)}{\delta^{2/(n+1)}}, \qquad d_n := 2\left(\frac{\kappa_n}{n(n+1)}\right)^{2/(n+1)}.$$

For $K \in \mathcal{K}^n_n$, the Macbeath region $M_K(x) = K \cap (2x - K)$ can be used to define the *convolution body*

$$C(K, t) := \{x \in K : V_n(M_K(x)) \ge tV_n(K)\}$$

(there are variants of this definition in the literature). Schmuckenschläger [1649] proved that

$$\Omega(K) = c_n \left(\frac{2}{V_n(K)}\right)^{2/(n+1)} \lim_{t\to 0} \frac{V_n(K) - V_n(C(K, t))}{t^{2/(n+1)}},$$

where c_n is given by (10.58). (The symmetry assumption made in [1649] is not necessary, according to Reisner. In [1650], the preceding relation was used to give another proof of the affine isoperimetric inequality.)

Meyer and Werner [1416] investigated the *Santaló regions* of a convex body $K \in \mathcal{K}_n^n$, defined by

$$S(K,t) := \{x \in K : V_n(K)V_n((K-x)^\circ) \le t\kappa_n^2\},$$

and proved that

$$\Omega(K) = 2\left(\frac{\kappa_n}{V_n(K)}\right)^{2/(n+1)} \lim_{t\to\infty} t^{2/(n+1)}[V_n(K) - V_n(S(K,t))].$$

Finally, Werner [1961] found an axiomatic scheme comprising all these four constructions, and proved a corresponding general representation of the affine surface area.

For the extended p-affine surface area, defined by (10.54) for all $p \neq -n$, Werner [1962] gave a geometric interpretation after introducing *weighted floating bodies*, and Schütt and Werner [1761] used the surface bodies, which they had introduced before in [1760], for the same purpose. In [1964], Werner extended the previously mentioned axiomatic scheme to obtain, under some restrictions, geometric interpretations of the extended p-affine surface area for all $p \neq -n$.

Returning to the floating body, we note that it was introduced shortly before [1757], though not under this name, and with different applications in mind. For given $K \in \mathcal{K}_n^n$, Bárány and Larman [147] defined

$$v(x) := \min\{V_n(K \cap H^+) : x \in H^+,\ H^+ \text{ a closed halfspace}\}$$

for $x \in K$ and

$$K(v \ge \delta) := \{x \in K : v(x) \ge \delta\}$$

for $\delta > 0$. If $x \in K_\delta$, then x is not contained in any cap $K \cap H^+$ of volume less than δ, hence $v(x) \ge \delta$ and thus $x \in K(v \ge \delta)$. If $x \in K \setminus K_\delta$, there is a closed halfspace H^+ with $V_n(K \cap H^+) = \delta$ and $x \in \operatorname{int} H^+$. A suitable translate H_0^+ of H^+ satisfies $x \in H_0^+$ and $V_n(H_0^+ \cap K) < \delta$, hence $x \notin K(v \ge \delta)$. Thus, $K(v \ge \delta) = K_\delta$.

The applications of floating bodies in [147] and subsequent work are to stochastic geometry and are based on the discovery of Bárány and Larman that the convex hull of k stochastically independent, uniformly distributed random points in a convex body K of volume 1 is, in a sense which can be made precise in terms of expectations, close to the floating body $K(v \ge 1/k)$. (See [1740], Note 3 on page 321, for a very brief description of some applications.) A powerful tool in these applications is the 'economic cap covering theorem' of Bárány and Larman. It is in the spirit of Theorem 2.3.2, but concerns coverings of the 'wet part' $K(v \le \delta)$ of K and involves only volumes. We refer to the introduction [144] and the surveys [143], [145] given by Bárány, where further references are found.

Notes for Section 10.6

1. *Equiaffine inner parallel curves.* In the plane, non-convex 'flotation curves' were successfully applied by Buchta and Reitzner [352]. For $K \in \mathcal{K}_2^2$ and $\delta > 0$, let M_δ be the locus

of the midpoints of all chords of K that divide its area in the ratio δ to $1-\delta$. Then M_δ is a closed (in general non-simple) curve, called the *equiaffine inner parallel curve* of K for the parameter δ. For Buchta and Reitzner, this was a powerful tool to obtain some results on geometric probabilities connected with independent random points in K.

2. For the floating body of a polytope $P \in \mathcal{P}^n_n$, Schütt [1753] proved that

$$\lim_{\delta\to 0} \frac{V_n(P)-V_n(P_\delta)}{\delta(-\log\delta)^{n-1}} = \frac{1}{n!}\frac{1}{n^{n-1}}\,\phi_n(P),$$

where $\phi_n(P)$ denotes the number of all towers $F_0 \subset F_1 \subset \cdots \subset F_{n-1}$, where F_i is an i-face of P. He deduced some estimates for the volume difference of a convex body K and a polytope contained in it.

Schütt and Werner [1758] gave some qualitative conditions on a function h that are sufficient for the existence of a convex body $K \in \mathcal{K}^2_2$ with $c^{-1}h(\delta) \le V_2(K \setminus K_\delta) \le ch(\delta)$ for $0 < \delta < \delta_0$.

3. For floating bodies K_δ with sufficiently small $\delta > 0$, Fresen [635] gave upper and lower bounds for $d_{\mathcal{L}}(K, K_\delta)$, where

$$d_{\mathcal{L}}(K,L) := \inf\{\lambda \ge 1 : \exists\, x \in \operatorname{int} K,\ \lambda^{-1}(K-x)+x \subset L \subset \lambda(K-x)+x\}.$$

4. Formulae for the volumes of illumination bodies of simplices were given by Werner [1960].
5. A brief introduction to floating bodies and illumination bodies and their relation to affine surface area is presented by Werner [1963].
6. *Polytopal approximation.* Schütt [1756] proved the following results on polytopal approximation in connection with floating bodies K_δ and illumination bodies K^δ. Let $K \in \mathcal{K}^n_n$. In the following, the c_i are explicitly known positive constants. If $0 < \delta \le c_1(K)$, there exists a polytope P_k with at most $k \le c_2(n)V_n(K \setminus K_\delta)/\delta$ vertices such that $K_\delta \subset P_k \subset K$. If $0 < \delta \le c_3(n,K)$ and if $c_4(n) \le k \le c_5(n)V_n(K^\delta \setminus K)$, then every polytope P_k containing K and having at most k facets satisfies $V_n(P_k \setminus K) \ge c_6(n,K)V_n(K^\delta \setminus K)$.
7. *Homothetic floating bodies.* If, for some $K \in \mathcal{K}^n_n$ and some $\delta > 0$, the body K and its floating body K_δ are homothetic, must K be an ellipsoid? This question was asked by Schütt and Werner [1759]. They gave a positive answer under the stronger assumption that K and K_{δ_k} are homothetic with the same centre of homothety, for a sequence $(\delta_k)_{k\in\mathbb{N}}$ tending to 0. Their proof uses Theorem 10.5.1 for bodies of class C^2, due to Petty. Under stronger smoothness assumptions, partial solutions were obtained by Stancu [1807, 1809]. Finally, Werner and Ye [1969] proved that for each $K \in \mathcal{K}^n_n$ there exists a number $\delta(K) > 0$ such that K_δ is homothetic to K for some $0 < \delta < \delta(K)$ if and only if K is an ellipsoid.
8. Meyer and Werner [1417] extended the notion of Santaló body and used it to give a limit representation of the p-affine surface area for sufficiently smooth convex bodies with positive Gauss curvature.
9. Stancu and Werner [1811] used convex floating bodies and an iterative procedure, modelled after the left side of (10.62), to define higher-order equiaffine invariants of convex bodies. They obtained integral representations under smoothness assumptions and proved sharp inequalities.

10.7 The volume product

The *volume product* is the functional $K \mapsto V_n(K)V_n(K^{\mathbf{s}})$, $K \in \mathcal{K}^n_n$, where $K^{\mathbf{s}} = (K - \mathbf{s}(K))^\circ$ and $\mathbf{s}(K)$ is the Santaló point of K, as explained in Section 10.5. The volume product is continuous and affine-invariant. The Blaschke–Santaló inequality (10.28) says that

$$V_n(K)V_n(K^{\mathbf{s}}) \le \kappa_n^2,$$

with equality if and only if K is an ellipsoid. As described in Section 10.5, the classical proof of this inequality deduced it from the affine isoperimetric inequality, with the consequence that the equality case was also based on the equality case for the affine isoperimetric inequality on $\mathcal{F}^n$. The latter was finally settled by Petty [1531], who had to use a regularity result from PDE. In the two first proofs of the general affine isoperimetric inequality, by Leichtweiß [1188] and Lutwak [1281], the equality discussion was also based on Petty's result.

Fortunately, and somewhat surprisingly, more direct proofs of the Blaschke–Santaló inequality, including identification of the equality case, were found later, which use only classical tools of convexity, namely Steiner symmetrization and the Brunn–Minkowski theorem. For centrally symmetric bodies, such proofs were given by Saint Raymond [1605] and Meyer and Pajor [1407]. For arbitrary convex bodies, Meyer and Pajor [1408] obtained the following result, which is more general than the Blaschke–Santaló inequality and from which they deduced that equality in the latter holds only for ellipsoids.

Theorem 10.7.1 (Meyer, Pajor) *Let $K \in \mathcal{K}^n_n$ and let H be a hyperplane with $H \cap \operatorname{int} K \neq \emptyset$. Then there exists a point $z \in H \cap \operatorname{int} K$ such that*

$$V_n(K)V_n(K^z) \leq \frac{\kappa_n^2}{4\lambda(1-\lambda)},$$

where $\lambda \in (0,1)$ is defined by $V_n(K \cap H^+) = \lambda V_n(K)$.

Simplified versions of the proof for the equality case in the Blaschke–Santaló inequality appear in Meyer and Reisner [1412] and in Fradelizi and Meyer [629], Remark 4. As mentioned in Section 10.5, an independent proof of the Blaschke–Santaló inequality with equality case was also given by Hug [1002].

The minimum of the volume product on $\mathcal{K}^n_{(o)}$ for $n \geq 3$ is still unknown. It has repeatedly been conjectured (though not in Mahler [1318], contrary to some statements in the literature) that

$$\frac{(n+1)^{n+1}}{(n!)^2} \leq V_n(K)V_n(K^\circ), \tag{10.63}$$

with equality precisely for simplices with centroid o, and that

$$\frac{4^n}{n!} \leq V_n(K)V_n(K^\circ) \tag{10.64}$$

for o-symmetric bodies $K \in \mathcal{K}^n_n$ (the latter was conjectured by Mahler [1318]). Equality holds in (10.64) for affine transforms of cubes and crosspolytopes, but for $n \geq 4$ not only for these. The latter fact was pointed out by Makai (see Guggenheimer [866]) and independently by Saint Raymond [1605]. The conjectured extremal bodies are the *Hanner polytopes*. The Hanner polytopes with centre at o are the unit balls of the so-called Hansen–Lima spaces. Hanner polytopes can be defined recursively. Every closed segment is a Hanner polytope. An o-symmetric n-polytope, $n \geq 2$, is a

Hanner polytope if it is either the convex hull of the union, or the sum, of two Hanner polytopes with centre o lying in complementary subspaces.

Non-sharp lower bounds for the volume product were obtained by Mahler [1318], Dvoretzky and Rogers [523], Bambah [128], Gordon and Reisner [764], Kuperberg [1156]. Bourgain and Milman [324, 326] (announced in [325]) were the first to obtain an estimate of the form

$$\left(\frac{V_n(K)V_n(K^\circ)}{\kappa_n^2}\right)^{1/n} \geq c$$

for $K \in \mathcal{K}^n_{(o)}$, with $c > 0$ independent of the dimension. Different proofs were given by Pisier [1535] and, with explicit constants, by Kuperberg [1157] and Nazarov [1468], as well as by Giannopoulos, Paouris and Vritsiou [709].

More is known in the two-dimensional case. First, a different proof for the Blaschke–Santaló inequality for $n = 2$, not using the affine isoperimetric inequality, was found by Heil [950]. Inequalities (10.63) and (10.64) for $n = 2$ were proved by Mahler [1319]. The equality case, however, was settled by him only under the assumption that K is a polygon. An educational presentation of Mahler's proofs is given by Thompson [1846]. A new proof of (10.63) for $n = 2$, together with equality condition, was given by Meyer [1406]. Further proofs are due to Lin and Leng [1217] (symmetric case) and to Meyer and Reisner [1413]. The latter authors have also treated the maximum of the volume product on polygons with a given number of vertices.

For the special case where K is a zonoid, inequality (10.64) was proved by Reisner [1563], and Reisner [1564] showed that equality holds only for parallelepipeds. A shorter proof was later given by Gordon, Meyer and Reisner [763]. The case $n = 2$ also settled the equality case in Mahler's inequality (10.64) for $n = 2$. A stability version of Reisner's result for zonoids was obtained by Böröczky and Hug [293].

Another case where (10.64) has been proved is when K is the unit ball of a normed space with a 1-unconditional basis. Geometrically, this means that some affine transform of K is symmetric with respect to each of the coordinate hyperplanes. For such K, inequality (10.64) was proved by Saint Raymond [1605]. A simpler proof, including the equality case, was given by Meyer [1405]; the equality case was also determined by Reisner [1565]. Equality is attained precisely by the Hanner polytopes. Meyer's method of proof was later extended by Barthe and Fradelizi [163], who determined the minimum of the volume product for convex bodies with an orthogonal symmetry group which is sufficiently large (this is interpreted in different ways). In particular, if K has the symmetry group of the regular simplex, then (10.63) holds. Fradelizi, Meyer and Zvavitch [633] found the minimum of the volume product for three-dimensional convex bodies (symmetric or not) which can be represented as the convex hull of two of their two-dimensional faces.

Concerning the inequality (10.64), Nazarov, Petrov, Ryabogin and Zvavitch [1469] showed that the cube is a strict local minimizer in the class of o-symmetric convex bodies with the Banach–Mazur distance. An analogous result for inequality (10.63) and the simplex was proved by Kim and Reisner [1077].

Reisner, Schütt and Werner [1566] showed that a convex body with minimal volume product cannot have a boundary point where the generalized Gauss curvature is positive. Another proof, and an analogous result for a functional Mahler inequality, was given by Gordon and Meyer [762].

Notes for Section 10.7

1. Applications of the Blaschke–Santaló inequality to other problems from the geometry of convex bodies can be found in Lutwak [1268, 1285], Schneider [1658, 1698, 1705], Wieacker [1973]. See also [1740], pp. 155, 505, for its applications in stochastic geometry.
2. Lopez and Reisner [1232] proved inequality (10.64) for o-symmetric polytopes $P \in \mathcal{P}^n_n$, $2 \le n \le 8$, with at most $2n + 2$ vertices (or facets), and characterized the equality cases.
3. For a shadow system K_t, $t \in [0, 1]$, of o-symmetric convex bodies in $\mathbb{R}^n$, Campi and Gronchi [389] showed that $V_n(K_t^\circ)^{-1}$ is a convex function of t. From this, they obtained new proofs of the Blaschke–Santaló inequality and of Mahler's inequality (10.64) for $n = 2$. The convexity result was extended to shadow systems of arbitrary convex bodies by Meyer and Reisner [1412]. One of their conclusions is an exact reverse Blaschke–Santaló inequality for n-dimensional polytopes with at most $n + 3$ vertices. Other applications are new proofs of the Blaschke–Santaló inequality in $\mathbb{R}^n$ and of its reverse in the plane, with smooth treatments of the equality cases.
4. Böröczky, Makai, Meyer and Reisner [297] made a thorough study of the volume product and of the product $V_2(K)V_2((\mathrm{D}K)^\circ)$ for planar convex bodies K with m-fold symmetry. They determined the exact minima and obtained stability results.
5. Hu [995] determined the minimum of the volume product for some special three-dimensional convex bodies.
6. For a convex body $K \in \mathcal{K}^n_{(o)}$ of class C^2_+, Stancu [1809] showed the existence of a number $\delta(K) > 0$ such that for $0 < \delta < \delta(K)$ one has $V_n(K_\delta)V_n((K_\delta)^\circ) \ge V_n(K)V_n(K^\circ) \ge V_n(K^\delta)V_n((K^\delta)^\circ)$, where K_δ is the floating body and K^δ is the illumination body with parameter δ.

10.8 Moment bodies and centroid bodies

The *moment body* $\mathrm{M}K$ of a star body $K \in \mathcal{S}^n_o$ is defined by its support function

$$h(\mathrm{M}K, x) := \int_K |\langle x, y\rangle| \,\mathrm{d}y \quad \text{for } x \in \mathbb{R}^n. \tag{10.65}$$

Using polar coordinates,

$$h(\mathrm{M}K, x) = \frac{1}{n+1} \int_{\mathbb{S}^{n-1}} |\langle x, u\rangle| \, \rho(K, u)^{n+1} \,\mathrm{d}u, \tag{10.66}$$

which shows that $\mathrm{M}K$ is a zonoid with centre o and with a generating measure that has a continuous density.

The *centroid body* ΓK of $K \in \mathcal{S}^n_o$ is its volume normalized moment body, thus

$$h(\Gamma K, x) = \frac{1}{V_n(K)} \int_K |\langle x, y\rangle| \,\mathrm{d}y \quad \text{for } x \in \mathbb{R}^n. \tag{10.67}$$

One has

$$\Gamma\phi K = \phi\Gamma K \quad \text{for } \phi \in \mathrm{GL}(n)$$

(see Lutwak [1279], Section 7, for a systematic study of transformation rules for associated bodies). If $K \in \mathcal{K}_n^n$ is centrally symmetric with respect to o, then $\mathrm{bd}\,\Gamma K$ is the set of all centroids of the intersections $K \cap H_{u,0}^+$, $u \in \mathbb{S}^{n-1}$. For this case (and $n = 3$), centroid bodies already appear (though not under this name) in the paper of Blaschke [244], §5, where they are attributed to Dupin. Blaschke also posed the problem that is solved by inequality (10.70).

From the representation (10.66), which exhibits the generating measure of the zonoid ΓK, and formula (5.82) for the mixed volume of zonoids, we obtain a formula for the mixed volume of centroid bodies,

$$\begin{aligned} &V(\Gamma K_1, \dots, \Gamma K_n) \\ &= \frac{2^n}{n!} \frac{1}{V_n(K_1) \cdots V_n(K_n)} \int_{K_1} \cdots \int_{K_n} D_n(x_1, \dots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \end{aligned} \tag{10.68}$$

which in a special case already appears in Blaschke [244], and in the general case in Petty [1525]. (Recall that $D_n(x_1, \dots, x_n)$ denotes the volume of the parallelotope spanned by the vectors $x_1, \dots, x_n$.) In particular,

$$V_n(\Gamma K) = \frac{2^n}{n! V_n(K)^n} \int_K \cdots \int_K D_n(x_1, \dots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{10.69}$$

If K is a convex body, we can apply the Busemann random simplex inequality (10.12) (and use the identity $n!\kappa_n\kappa_{n-1} = 2^n\pi^{n-1}$) to obtain the *Busemann–Petty centroid inequality*,

$$V_n(\Gamma K) \geq \left(\frac{2\kappa_{n-1}}{(n+1)\kappa_n}\right)^n V_n(K), \tag{10.70}$$

which was first obtained by Petty [1525]. Here, equality holds if and only if K is an o-symmetric ellipsoid. The Busemann–Petty centroid inequality can be extended to star bodies, with the same equality condition; this will be done in the next section.

For star bodies K and for $p \geq 1$, the *p-centroid body* (or L_p centroid body) $\Gamma_p K$ is defined by

$$h(\Gamma_p K, x)^p := \frac{1}{c_{n,p} V_n(K)} \int_K |\langle x, y\rangle|^p\, \mathrm{d}y \quad \text{for } x \in \mathbb{R}^n, \tag{10.71}$$

where

$$c_{n,p} := \frac{\kappa_{n+p}}{\kappa_2\kappa_n\kappa_{p-1}}. \tag{10.72}$$

It was introduced by Lutwak and Zhang [1306]. The normalization is made in such a way that $\Gamma_p B^n = B^n$; therefore, $\Gamma_1 \neq \Gamma$.

The reader should be aware that normalizations of associated bodies are rather flexible in the literature. Some authors prefer to define the L_p centroid body $Z_p(K)$ of K by

$$h(Z_pK, x)^p := \int_K |\langle x, y\rangle|^p \, \mathrm{d}y \quad \text{for } x \in \mathbb{R}^n. \tag{10.73}$$

Definition (10.71) can also be written in the form

$$h(\Gamma_p K, x)^p = \frac{1}{nc_{n-2,p} V_n(K)} \int_{\mathbb{S}^{n-1}} |\langle x, u\rangle|^p \rho(K, u)^{n+p} \, \mathrm{d}u.$$

The body $\Gamma_2 K$ is an ellipsoid, a dilatate of the Legendre ellipsoid of K. This will be treated in Section 10.12.

For $K \in \mathcal{S}^n_o$ and $1 \leq p \leq \infty$, Lutwak and Zhang [1306] proved the inequality

$$V_n(K)V_n(\Gamma^\circ_p K) \leq \kappa^2_n, \tag{10.74}$$

with equality if and only if K is an o-symmetric ellipsoid. Here $\Gamma^\circ_p K := (\Gamma_p K)^\circ$. The body $\Gamma^\circ_\infty K$ has to be defined as the limit of $\Gamma^\circ_p K$ for $p \to \infty$, which exists, and in the case of an o-symmetric convex body K it coincides with K°. For that reason, (10.74) has been called the L_p *Blaschke–Santaló inequality*. The proof of (10.74) involves Steiner symmetrization.

Also using Steiner symmetrization, together with dual mixed volumes, Lutwak, Yang and Zhang [1292] established the L_p *Busemann–Petty centroid inequality*, namely

$$V_n(\Gamma_p K) \geq V_n(K) \tag{10.75}$$

for $K \in \mathcal{S}^n_0$ and $1 \leq p < \infty$. It strengthens (10.74) (which is obtained from (10.75) by applying the Blaschke–Santaló inequality to $\Gamma_p K$). Equality holds if and only if K is an o-symmetric ellipsoid. An alternative proof of (10.75) was given by Campi and Gronchi [387]. They used shadow systems and also derived some related inequalities in [387] and [388].

L_p centroid bodies and the inequalities pertaining to them have been generalized further. Aiming at a characterization of L_p moment bodies by their essential properties (see Section 10.16), Ludwig [1249] was necessarily led to introduce asymmetric versions, which have been further investigated by Haberl and Schuster [881]. For $K \in \mathcal{S}^n_o$ and $p > 1$, the *asymmetric L_p moment bodies* $\mathrm{M}^+_p K$, $\mathrm{M}^-_p K$ of K are defined by

$$h(\mathrm{M}^\pm_p K, x)^p := a_{n,p}(n + p) \int_K \langle x, y\rangle^p_\pm \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{10.76}$$

where (following [881]) the constant $a_{n,p}$ is chosen such that $\mathrm{M}^+_p B^n = B^n$. Here $\langle x, y\rangle_+ = \max\{\langle x, y\rangle, o\}$ and $\langle x, y\rangle_- = \min\{\langle x, y\rangle, o\}$. The following was proved by Haberl and Schuster [881]. Let $K \in \mathcal{K}_{(o)}$ and $p > 1$. If $\Psi_p K$ is the convex body defined by $\Psi_p K = c_1 \mathrm{M}^+_p K +_p c_2 \mathrm{M}^-_p K$, where $c_1, c_2 \geq 0$ are not both zero, then

$$V_n(K)^{-n/p-1} V_n(\Psi_p K) \geq V_n(B^n)^{-n/p-1} V_n(\Psi_p B^n),$$

with equality if and only if K is an o-symmetric ellipsoid.

A further extension leads from L_p to Orlicz. Lutwak, Yang and Zhang [1304] introduce the *Orlicz centroid body* $\Gamma_\phi K$ of $K \in \mathcal{S}_o$ in the following way. Let

$\phi : \mathbb{R} \to [0, \infty)$ be a convex function with $\phi(0) = 0$ and such that ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$. Then

$$h(\Gamma_\phi K, x) := \inf\left\{\lambda > 0 : \frac{1}{V_n(K)} \int_K \phi\left(\frac{\langle x, y\rangle}{\lambda}\right) \mathrm{d}y \leq 1\right\}, \quad x \in \mathbb{R}^n.$$

For $\phi(t) = |t|^p$ with $p \geq 1$, $\Gamma_\phi K$ is a dilatate of $\Gamma_p K$. The following *Orlicz–Busemann–Petty centroid inequality* is proved in [1304].

Theorem 10.8.1 (Lutwak, Yang, Zhang) *If $K \in \mathcal{K}^n_{(o)}$, then*

$$V_n(\Gamma_\phi K)/V_n(K)$$

is minimized if and only if K is an o-symmetric ellipsoid.

An alternative proof, using shadow systems, was given by Li and Leng [1211], and an extension to star bodies by Zhu [2075]. A partial extension of the theorem, with a new approach, was obtained by Paouris and Pivovarov [1511].

Notes for Section 10.8

1. Petty [1525] proved that centroid bodies are of class C^2_+. This fact, and its extension, played an important role in the proofs of Lutwak, Yang and Zhang [1292].
2. Bisztriczky and Böröczky [231] conjecture that the maximum of the functional $V_n(\Gamma K)/V_n(K)$ on o-symmetric convex bodies in $\mathcal{K}^n_n$ is attained by parallelotopes. For $n = 2$, they prove that $V_2(\Gamma K)/V_2(K) \leq 5/27$ for $K \in \mathcal{K}^2$ with centre o, with equality precisely if K is a parallelogram, and $V_2(\Gamma K)/V_2(K) \leq 16/27$ for $K \in \mathcal{K}^2$ with $o \in K$, with equality precisely if K is a triangle with one vertex at o. This was extended to p-centroid bodies by Campi and Gronchi [388]. The reverse form of the Orlicz–Busemann–Petty centroid inequality in $\mathbb{R}^2$ was proved by Chen, Zhou and Yang [415].
3. L_q centroid bodies have turned out to be a useful tool in the study of asymptotic properties of convex bodies, beginning with the work of Paouris [1510] and followed by Fleury, Guédon and Paouris [618], Klartag and Milman [1100], Giannopoulos, Paouris and Vritsiou [708].

   Also in [618], a stability result is obtained for a convex body whose L_p centroid body is close to that of the unit ball.

10.9 Projection bodies

The *projection body* ΠK of a convex body $K \in \mathcal{K}^n$ has already been defined by equation (5.80), namely by

$$h(\Pi K, u) = V_{n-1}(K \,|\, u^\perp) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, S_{n-1}(K, \mathrm{d}v) \quad \text{for } u \in \mathbb{S}^{n-1}. \tag{10.77}$$

Every projection body is a zonoid, and it follows from Minkowski's existence and uniqueness theorems (Theorems 8.2.2, 8.1.1) that every o-symmetric zonoid in $\mathcal{K}^n_n$ is the projection body of a unique o-symmetric convex body.

More generally, the *mixed projection body* $\Pi(K_1, \dots, K_{n-1})$ of the convex bodies $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ is defined by

$$\begin{aligned} h(\Pi(K_1, \dots, K_{n-1}), u) &:= v(K_1 \,|\, u^\perp, \dots, K_{n-1} \,|\, u^\perp) \\ &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, S(K_1, \dots, K_{n-1}, \mathrm{d}v) \quad \text{for } u \in \mathbb{S}^{n-1}, \end{aligned} \tag{10.78}$$

where $v(\cdot, \dots, \cdot)$ denotes the $(n-1)$-dimensional mixed volume in $u^\perp$.

For $K \in \mathcal{K}^n_n$, the origin is an interior point of ΠK, hence the *polar projection body* $\Pi^\circ K = (\Pi K)^\circ$ is defined.

It is not immediately obvious from the definition, but not difficult to show (see Petty [1527] or Lutwak [1279]), that

$$\Pi \phi K = \phi^{-t} \Pi K \quad \text{for } \phi \in \mathrm{SL}(n).$$

Since also

$$(\phi K)^\circ = \phi^{-t} K^\circ \quad \text{for } \phi \in \mathrm{SL}(n),$$

this implies that

$$\Pi^\circ \phi K = \phi \Pi^\circ K \quad \text{for } \phi \in \mathrm{SL}(n),$$

and, further, that the second projection body, $\Pi^2 K := \Pi\Pi K$, satisfies

$$\Pi^2 \phi K = \phi \Pi^2 K \quad \text{for } \phi \in \mathrm{SL}(n).$$

For the volume of the projection body, an integral representation is obtained from (10.77) and Theorem 5.3.2, namely

$$V_n(\Pi K) = \frac{1}{n!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} D_n(u_1, \dots, u_n) \, S_{n-1}(K, \mathrm{d}u_1) \cdots S_{n-1}(K, \mathrm{d}u_n). \tag{10.79}$$

The sharp bounds for the affine-invariant functional

$$V_n(K)^{1-n} V_n(\Pi K) \tag{10.80}$$

are unknown. Petty [1528] has conjectured that the minimum of (10.80) is attained precisely by ellipsoids. Some preliminary information on the minimum is available. Let $K \in \mathcal{K}^n_n$ be a convex body for which $V_n(K)^{1-n} V_n(\Pi K)$ is minimal. For $L \in \mathcal{K}^n_n$ we have

$$V(\Pi K, L, \dots, L) = V(\Pi L, K, \dots, K), \tag{10.81}$$

by (10.83) below. With $L := \Pi K$, Minkowski's inequality (7.18) yields

$$V_n(\Pi K) V_n(K)^{1-n} \geq V_n(\Pi^2 K) V_n(\Pi K)^{1-n}. \tag{10.82}$$

By the assumed minimum property of K, the equality sign must hold here, hence $\Pi^2 K$ and K are homothetic. In particular, K must be a zonoid. Thus, to determine the convex bodies minimizing (10.80), one need only determine the zonoids K minimizing (10.80), and having the additional property that $\Pi^2 K$ and K are homothetic.

This fact (observed in [1705], p. 182) is an instance of ‘class reduction’; see Note 3. If K has centre o, then $\Pi^2 K = \lambda K$, and since (10.82) holds with equality, necessarily $\lambda = V_n(\Pi K)/V_n(K)$. Ellipsoids have this property, but they are not the only convex bodies K for which $\Pi^2 K$ is homothetic to K (so this approach does not succeed; see, however, Lutwak [1278] for a case where a similar argument does work). Weil [1931] has determined the polytopes K for which $\Pi^2 K$ and K are homothetic; these are the direct sums of centrally symmetric polygons or segments.

The symmetry relation used above holds more generally for mixed projection bodies, namely

$$V(\Pi(K_1, \dots, K_{n-1}), L_1, \dots, L_{n-1}) = V(\Pi(L_1, \dots, L_{n-1}), K_1, \dots, K_{n-1}) \tag{10.83}$$

for $K_1, \dots, K_{n-1}, L_1, \dots, L_{n-1} \in \mathcal{K}^n$. This follows from

$$\begin{aligned} &V(\Pi(K_1, \dots, K_{n-1}), L_1, \dots, L_{n-1}) \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \, S(K_1, \dots, K_{n-1}, \mathrm{d}v) \, S(L_1, \dots, L_{n-1}, \mathrm{d}u) \end{aligned}$$

and Fubini’s theorem.

The preceding symmetry relation can be used, together with classical inequalities, to obtain various inequalities of Brunn–Minkowski, Minkowski and Aleksandrov–Fenchel type for projection bodies and mixed projection bodies. The principle of proof already becomes clear from a special example, for which we choose the inequality

$$V_n(\Pi(K+L))^{1/n(n-1)} \geq V_n(\Pi K)^{1/n(n-1)} + V_n(\Pi L)^{1/n(n-1)} \tag{10.84}$$

for $K, L \in \mathcal{K}^n$. If K, L have interior points, equality holds if and only if K and L are homothetic. For the proof of (10.84), let $M \in \mathcal{K}^n$ be arbitrary. In the following, we use in this order the symmetry relation (10.83), the general Brunn–Minkowski theorem 7.4.5, then (10.83) again, and the Minkowski inequality (7.18), to obtain

$$\begin{aligned} &V(M[n-1], \Pi(K+L))^{1/(n-1)} = V((K+L)[n-1], \Pi M)^{1/(n-1)} \\ &\geq V(K[n-1], \Pi M)^{1/(n-1)} + V(L[n-1], \Pi M)^{1/(n-1)} \\ &= V(M[n-1], \Pi K)^{1/(n-1)} + V(M[n-1], \Pi L)^{1/(n-1)} \\ &\geq V_n(M)^{1/n} V_n(\Pi K)^{1/n(n-1)} + V_n(M)^{1/n} V_n(\Pi L)^{1/n(n-1)}. \end{aligned}$$

If we now choose $M = \Pi(K+L)$, we obtain (10.84), together with the assertion on the equality sign. We refer to Lutwak [1284] for this and further inequalities for mixed projection bodies and also for mixed volumes involving a number of balls.

The volume of the polar projection body is, according to (10.77), (1.52) and (1.53), given by

$$V_n(\Pi^\circ K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} V_{n-1}(K \,|\, u^\perp)^{-n} \, \mathrm{d}u. \tag{10.85}$$

Sharp bounds for this volume, given the volume of K, are known. The inequality

$$V_n(K)^{n-1}V_n(\Pi^\circ K) \le \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n \tag{10.86}$$

for $K \in \mathcal{K}^n_n$, together with the equality condition characterizing ellipsoids, was proved by Petty [1528]. This inequality is now known as the *Petty projection inequality*. Here we follow Lutwak's [1286] elegant presentation of the relationship between the Busemann–Petty centroid inequality and the Petty projection inequality. Let $K \in \mathcal{K}^n_n$, and let $L \in \mathcal{S}^n_o$ be a star body. From (10.66) and Fubini's theorem, we immediately obtain

$$V_1(K, \Gamma L) = \frac{2}{(n+1)} \frac{\widetilde{V}_{-1}(L, \Pi^\circ K)}{V_n(L)}.$$

Special cases are

$$V_1(K, \Gamma\Pi^\circ K) = \frac{2}{n+1} \tag{10.87}$$

and

$$V_n(\Gamma L) = \frac{2}{n+1} \frac{\widetilde{V}_{-1}(L, \Pi^\circ \Gamma L)}{V_n(L)}. \tag{10.88}$$

Let

$$\mathrm{pp}(K) := \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n \frac{1}{V_n(K)^{n-1}V_n(\Pi^\circ K)}, \qquad \mathrm{bpc}(L) := \left(\frac{(n+1)\kappa_n}{2\kappa_{n-1}}\right)^n \frac{V_n(\Gamma L)}{V_n(L)}.$$

From (10.87) and Minkowski's inequality (7.18) we get

$$\mathrm{pp}(K) \ge \mathrm{bpc}(\Pi^\circ K). \tag{10.89}$$

Here $\mathrm{bpc}(\Pi^\circ K) \ge 1$, since the Busemann–Petty centroid inequality (10.70) holds for convex bodies, thus $\mathrm{pp}(K) \ge 1$. Equality in the latter inequality holds if and only if $\Pi^\circ K$ is an ellipsoid, which holds if and only if K is an ellipsoid. This is the inequality (10.86), with equality condition. Further, from (10.88) and the dual mixed volume inequality (9.40) (with $(i, j, k) = (-1, 0, n)$) we get

$$\mathrm{bpc}(L) \ge \mathrm{pp}(\Gamma L). \tag{10.90}$$

Here $\mathrm{pp}(\Gamma L) \ge 1$, since the Petty projection inequality holds for convex bodies. This gives $\mathrm{bpc}(L) \ge 1$, with equality if and only if ΓL is an ellipsoid and L and ΓL are dilatates, hence if and only if L is an o-symmetric ellipsoid. In this way, the Busemann–Petty centroid inequality with equality condition is extended to star bodies.

Different presentations of the proof for the Petty projection inequality are found in Lutwak [1271] and Gardner [675], Theorem 9.2.9.

Remark 10.9.1 The true nature of the functional $V_n(\Pi^\circ K)$ becomes clearer if instead we consider $V_n(\Pi^\circ K)^{-1/n}$, which has the character of a surface area. Like the

Euclidean surface area $S(K)$, it is homogeneous of degree $n-1$ and strictly increasing under set inclusion (on $\mathcal{K}_n^n$), but in contrast it is invariant under volume-preserving affine transformations. Defining the p-mean of the projection function of K by

$$S_p(K) := \left(\frac{1}{n\kappa_n}\int_{\mathbb{S}^{n-1}} V_{n-1}(K\,|\,u^\perp)^p\,\mathrm{d}u\right)^{1/p}$$

for $p \neq 0$, we have $S_{-n}(K) = \kappa_n^{1/n} V_n(\Pi^\circ K)^{-1/n}$. By Cauchy's surface area formula (5.73), the surface area of K is given by $S(K) = (n\kappa_n/\kappa_{n-1})S_1(K)$. By the monotonicity of the means and the Petty projection inequality (10.86), we get

$$\frac{\kappa_{n-1}}{n\kappa_n} S(K) = S_1(K) \geq S_{-n}(K) \geq \kappa_n \kappa_n^{(1-n)/n} V_n(K)^{(n-1)/n}.$$

Thus, the Petty projection inequality is stronger than the Euclidean isoperimetric inequality (for convex bodies).

We can rewrite (10.85) in the form

$$V_n(\Pi^\circ K)^{-1/n} = \left(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \left(\frac{1}{2}\int_{\mathrm{bd}\,K} |\langle u, u_K(x)\rangle|\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)^{-n} \mathrm{d}u\right)^{-1/n},$$

where $u_K(x)$ denotes the outer unit normal vector of K at x, which exists for $\mathcal{H}^{n-1}$-almost all $x \in \mathrm{bd}\,K$. This shows that $V_n(\Pi^\circ K)^{-1/n}$ can be expressed by integration over the boundary of K. As a consequence, this functional can be extended to suitable non-convex sets. This was noted by Zhang [2060], who generalized Petty's projection inequality (10.86) to compact sets K which are the closure of their interior and have piecewise C^1 boundary. This extension makes use of a 'convexification', employing Minkowski's existence theorem, and is applied to obtain an affine Sobolev inequality (see Section 10.15).

For convex bodies $K \in \mathcal{K}_n^n$, the opposite inequality

$$\frac{1}{n^n}\binom{2n}{n} \leq V_n(K)^{n-1} V_n(\Pi^\circ K), \tag{10.91}$$

where equality characterizes simplices, was conjectured by Ball [120] and was first proved by Zhang [2054]; it is known as the *Zhang projection inequality*. The minimum of $V_n(K)^{n-1}V_n(\Pi^\circ K)$ on centrally symmetric convex bodies K is unknown for $n \geq 3$.

The Zhang projection inequality can be proved by a variation of the proof of the difference body inequality. For given $u \in \mathbb{S}^{n-1}$, it follows from (10.1) that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{g_K(ru)}{V_n(K)}\right)^{1/n}\Bigg|_{r=0} = -\frac{h_{\Pi K}(u)}{nV_n(K)}.$$

Since the function $r \mapsto g_K(ru)^{1/n}$ is concave on its support, as shown in the proof of Theorem 10.1.4, it follows that

$$g_K(ru) \leq V_n(K)\left(1 - \frac{h_{\Pi K}(u)}{nV_n(K)} r\right)^n$$

for $0 \le r \le \rho(\mathrm{D}K, u)$, and that

$$1 - \frac{h_{\Pi K}(u)}{nV_n(K)} \rho(\mathrm{D}K, u) \ge 0.$$

We deduce that

$$\begin{aligned}\int_0^{\rho(\mathrm{D}K,u)} g_K(ru) r^{n-1}\, \mathrm{d}r &\le V_n(K) \int_0^{\rho(\mathrm{D}K,u)} \left(1 - \frac{h_{\Pi K}(u)}{nV_n(K)} r\right)^n r^{n-1}\, \mathrm{d}r \\ &\le V_n(K) \left(\frac{nV_n(K)}{h_{\Pi K}(u)}\right)^n \int_0^1 (1-t)^n t^{n-1}\, \mathrm{d}t = \mathrm{B}(n+1, n) V_n(K) \left(\frac{nV_n(K)}{h_{\Pi K}(u)}\right)^n,\end{aligned}$$

where B denotes Euler's beta function. By (10.3), and using polar coordinates,

$$\begin{aligned}V_n(K)^2 &= \int_{\mathrm{D}K} g_K(x)\, \mathrm{d}x = \int_{\mathbb{S}^{n-1}} \int_0^{\rho(\mathrm{D}K,u)} g_K(ru) r^{n-1}\, \mathrm{d}r\, \mathrm{d}u \\ &\le \mathrm{B}(n+1, n) n^n V_n(K)^{n+1} \int_{\mathbb{S}^{n-1}} h_{\Pi K}(u)^{-n}\, \mathrm{d}u \\ &= \mathrm{B}(n, n+1) n^{n+1} V_n(K)^{n+1} V_n(\Pi^\circ K),\end{aligned}$$

thus

$$V_n(K)^{n-1} V_n(\Pi^\circ K) \ge \frac{1}{n^n} \binom{2n}{n}. \tag{10.92}$$

Here, equality holds if and only if K is a simplex. This proof is taken from Gardner and Zhang [687].

Gardner and Zhang [687] have also introduced a remarkable array of convex bodies connecting the difference body with the polar projection body. Let $K \in \mathcal{K}_n^n$ and $p > -1$, $p \neq 0$. The *radial pth mean body R_pK* of K is defined by

$$\rho_{R_pK}(u) := \left(\frac{1}{V_n(K)} \int_K \rho_K(x, u)^p\, \mathrm{d}x\right)^{1/p},$$

where $\rho_K(x, \cdot)$ is the radial function of K with respect to x as origin. With the constant $c_{n,p} := (n\mathrm{B}(p+1, n))^{-1/p}$, Gardner and Zhang proved that

$$\mathrm{D}K \subset c_{n,q} R_q K \subset c_{n,p} R_p K \subset nV_n(K) \Pi^\circ K \tag{10.93}$$

for $-1 < p < q$. In each inclusion, equality holds if and only if K is a simplex. Consequently,

$$\frac{V_n(\mathrm{D}K)}{V_n(K)} \le c_{n,q}^n \frac{V_n(R_qK)}{V_n(K)} \le c_{n,p}^n \frac{V_n(R_pK)}{V_n(K)} \le n^n V_n(K)^{n-1} V_n(\Pi^\circ K),$$

where, in each inequality, equality holds if and only if K is a simplex. Since

$$\frac{V_n(R_nK)}{V_n(K)} = 1,$$

these inequalities include the difference body inequality and the Zhang projection inequality.

We turn to L_p extensions. For $1 \le p < \infty$ and $K \in \mathcal{K}^n_{(o)}$, Lutwak, Yang and Zhang [1292] introduced the L_p *projection body* $\Pi_p K$ by means of

$$h(\Pi_p K, u)^p := \frac{1}{c_{n-2,p} n\kappa_n} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|^p \, S_{p,0}(K, \mathrm{d}v) \quad \text{for } u \in \mathbb{S}^{n-1}, \tag{10.94}$$

where the constant $c_{n-2,p}$ is defined by (10.72) and the measure $S_{p,0}(K, \cdot)$ by (9.16). The normalization of $\Pi_p K$ is such that $\Pi_p B^n = B^n$; therefore, $\Pi_1 = \kappa_{n-1}^{-1}\Pi \neq \Pi$. Lutwak, Yang and Zhang established the following L_p analogue of the Petty projection inequality. For $1 < p < \infty$ and for $K \in \mathcal{K}^n_{(o)}$, with $\Pi_p^\circ K := (\Pi_p K)^\circ$,

$$V_n(K)^{(n-p)/p} V_n(\Pi_p^\circ K) \le \kappa_n^{n/p}, \tag{10.95}$$

with equality if and only if K is an o-symmetric ellipsoid.

For the study of the polars of L_p projection bodies, a different normalization has turned out to be convenient; therefore, different notation is used. For $K \in \mathcal{K}^n_{(o)}$ and (more generally) for $p > 0$, the L_p *polar projection body* $\Gamma_{-p} K$ is defined by

$$\rho(\Gamma_{-p} K, u)^{-p} = \frac{1}{V_n(K)} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|^p \, S_{p,0}(K, \mathrm{d}v), \quad u \in \mathbb{S}^{n-1}. \tag{10.96}$$

For $p \ge 1$, the body $\Gamma_{-p} K$ is convex. More on these bodies is found in [1299].

For L_p projection bodies, there are similar asymmetric extensions as for L_p moment bodies. For $K \in \mathcal{K}^n_{(o)}$ and $p > 1$, the *asymmetric L_p projection bodies* $\Pi_p^+ K$ and $\Pi_p^- K$ of K, first introduced by Ludwig [1249], are defined by

$$h(\Pi_p^\pm K, u)^p := a_{n,p} \int_{\mathbb{S}^{n-1}} \langle u, v\rangle_\pm^p \, S_{p,0}(K, \mathrm{d}v), \quad u \in \mathbb{S}^{n-1}, \tag{10.97}$$

where the normalizing constant $a_{n,p}$ is such that $\Pi_p^+ B^n = B^n$. The following was proved by Haberl and Schuster [881]. Let $K \in \mathcal{K}_{(o)}$ and $p > 1$. If $\Phi_p K$ is the convex body defined by $\Phi_p K = c_1 \Pi_p^+ K +_p c_2 \Pi_p^- K$, where $c_1, c_2 \ge 0$ are not both zero, then

$$V_n(K)^{n/p-1} V_n(\Phi_p^\circ K) \ge V_n(B^n)^{n/p-1} V_n(\Phi_p B^n),$$

with equality if and only if K is an o-symmetric ellipsoid.

Lutwak, Yang and Zhang [1303] introduced the *Orlicz projection body* $\Pi_\phi K$ of $K \in \mathcal{K}_{(o)}$ in the following way. Let $\phi : \mathbb{R} \to [0, \infty)$ be a convex function such that ϕ is either strictly decreasing on $(-\infty, 0]$ or strictly increasing on $[0, \infty)$. Then $\Pi_\phi K$ can be defined by

$$h(\Pi_\phi K, x) := \inf \left\{ \lambda > 0 : \frac{1}{nV_n(K)} \int_{\mathbb{S}^{n-1}} \phi\left(\frac{\langle x, u\rangle}{\lambda h_K(u)} \right) h_K(u) \, S_{n-1}(K, \mathrm{d}u) \le 1 \right\}$$

for $x \in \mathbb{R}^n$. For $\phi(t) = |t|^p$ with $p \ge 1$, $\Pi_\phi K$ is a dilatate of $\Pi_p K$. The following *Orlicz–Petty projection inequality* holds.

Theorem 10.9.2 (Lutwak, Yang, Zhang) *If $K \in \mathcal{K}^n_{(o)}$, then*

$$V_n(\Pi_\phi^\circ K)/V_n(K)$$

is maximal if and only if K is an o-symmetric ellipsoid.

This was proved in [1303], except that the uniqueness of the ellipsoids as maximizers was proved for strictly convex ϕ only, and conjectured in general. This conjecture was proved by Böröczky [290]. He also obtained corresponding stability results.

The extension of the Petty projection inequality by Haberl and Schuster quoted above is obtained when $\phi(t) = (|t| + \alpha t)^p$ with suitable $\alpha \in [-1, 1]$.

Notes for Section 10.9

1. Projections of convex bodies, and therefore also projection bodies, belong to the field of geometric tomography. We refer to the monograph of Gardner [675], and for the analytic approach also to the books by Koldobsky [1136] and Koldobsky and Yaskin [1142], for a much more comprehensive treatment than we are able to give here.
2. In relation to the functional (10.80), Lutwak, Yang and Zhang [1293] studied the following function on polytopes. Let $P \in \mathcal{K}^n_{(o)}$ be a polytope, with outer facet unit normal vectors $u_1, \dots, u_N$ and corresponding $(n-1)$-volumes of the facets given by $a_1, \dots, a_N$. For the functional U defined by

$$U(P)^n := \frac{1}{n^n} \sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} h_{i_1} \cdots h_{i_N} a_{i_n} \cdots a_{i_N},$$

where $h_i := h(P, u_i)$, it was proved in [1293] that

$$\frac{V_n(\Pi P)}{U(P)^{n/2} V_n(P)^{n/2-1}} \leq 2^n \left(\frac{n^n}{n!}\right)^{1/2},$$

with equality if and only if P is a parallelotope. Since $U(P) < V_n(P)$, it follows that

$$\left[V_n(K)^{1-n} V_n(\Pi K)\right]^{1/n} \leq 2\sqrt{e}.$$

A problem posed in [1293], namely whether

$$U(P) \geq n^{-1}(n!)^{1/n} V_n(P)$$

for o-symmetric P, with equality if and only if P is a parallelotope, was answered affirmatively by He, Leng and Li [948]. The analogous problem for asymmetric P, also posed in [1293], was solved by Xiong [1995] for $n = 3$.
3. *Class reduction.* The explained relationship between the Busemann–Petty centroid inequality and the Petty projection inequality shows more than was used above. If one only knows the Busemann–Petty centroid inequality, with equality condition, for polar zonoids, then this can be used to deduce the Petty projection inequality, with equality condition, for convex bodies. If the latter is known (it would be sufficient to know it for zonoids), then the Busemann–Petty centroid inequality, with equality condition, can be deduced for general star bodies. Thus, a certain inequality, together with equality conditions, for a restricted class of convex bodies yields (together with some other identities and inequalities) the same inequality for a more general class of sets, along with equality conditions.

   The following aspect is also important. From (10.89) and (10.90) we have

$$\mathrm{pp}(K) \geq \mathrm{bpc}(\Pi^\circ K) \geq \mathrm{pp}(\Gamma \Pi^\circ K).$$

Hence, if pp attains its minimum at the convex body K, it also attains its minimum at $\Gamma\Pi^\circ K$, and K is homothetic to this body. As Petty [1525] has proved, centroid bodies are of class C^2_+. Thus, to determine the minimizer of pp in the class of all convex bodies, it suffices to determine the minimizer in the class of centrally symmetric convex bodies of class C^2_+.

The described phenomenon was emphasized in Lutwak [1273] and was later named 'class reduction'; it was fully exploited, for example, in Lutwak, Yang and Zhang [1292].

The fact that the Busemann–Petty centroid inequality and the Petty projection inequality imply one another was extended to the L_p versions of these functionals and their dual counterparts by Yu [2004].

4. Schmuckenschläger [1650] used the distribution function of the covariogram to give another proof of the Petty projection inequality.
5. In the plane, the Zhang projection inequality (10.91) can be written as

$$V_2(K)V_2((\mathrm{D}\,K)^\circ) \geq \frac{3}{2},$$

with equality precisely if K is a triangle. This inequality was proved earlier by Eggleston [533]. He also proved that $V_2(K)V_2((\mathrm{D}\,K)^\circ) \geq 2$ if K is o-symmetric.

6. *The volume of the projection body and related inequalities*. Petty's conjectured inequality

$$V_n(K)^{1-n}V_n(\Pi K) \geq \kappa_{n-1}^n \kappa_n^{2-n}$$

can be put into a more symmetric form. Lutwak [1280] showed that this inequality is equivalent to the following conjecture. If $K, L \in \mathcal{K}_n^n$, then

$$\int_{\mathrm{bd}\,K}\int_{\mathrm{bd}\,L} |\langle u, v\rangle| \,\mathrm{d}u\,\mathrm{d}v \geq \frac{2n\kappa_{n-1}}{\kappa_n^{1-2/n}}[V_n(K)V_n(L)]^{(n-1)/n},$$

with equality if and only if K, L are homothetic to o-symmetric polar ellipsoids. Here $\mathrm{d}u, \mathrm{d}v$ denote the area elements on $\mathrm{bd}\,L, \mathrm{bd}\,K$ whose outer unit normal vectors are u, v and it is assumed that K, L are of class C_+^2.

Lutwak proved the following formal analogue of the above inequality. If $K, L \in \mathcal{K}_{(o)}^n$, then

$$\int_K\int_L |\langle x, y\rangle| \,\mathrm{d}x\,\mathrm{d}y \geq \frac{2n\kappa_{n-1}}{(n+1)^2\kappa_n^{1+2/n}}[V_n(K)V_n(L)]^{(n+1)/n},$$

with equality if and only if K, L are dilatates of o-symmetric polar ellipsoids. Here $\mathrm{d}x, \mathrm{d}y$ denote volume elements. Furthermore, Lutwak proved that

$$\int_K\int_L \langle x, y\rangle^2 \,\mathrm{d}x\,\mathrm{d}y \geq \frac{n}{(n+2)^2\kappa_n^{4/n}}[V_n(K)V_n(L)]^{(n+2)/n}, \tag{10.98}$$

with equality if and only if K, L are dilatates of o-symmetric polar ellipsoids.

Let $K \in \mathcal{K}_n^n$ be o-symmetric. Inequality (10.98) with $L = K^\circ$ gives

$$\int_K\int_{K^\circ} \langle x, y\rangle^2 \,\mathrm{d}x\,\mathrm{d}y \geq \frac{n}{(n+2)^2\kappa_n^{4/n}}[V_n(K)V_n(K^\circ)]^{(n+2)/n}, \tag{10.99}$$

which was also proved by Ball [117]. Ball conjectured that

$$\frac{n\kappa_n^2}{(n+2)^2} \geq \int_K\int_{K^\circ} \langle x, y\rangle^2 \,\mathrm{d}x\,\mathrm{d}y \tag{10.100}$$

for centrally symmetric K, with equality only for o-symmetric ellipsoids. He proved this for K with the property that an affine image of K is symmetric with respect to the coordinate hyperplanes. In view of (10.99), inequality (10.100) would be stronger than the Blaschke–Santaló inequality (for symmetric bodies).

In Lutwak, Yang and Zhang [1298], inequality (10.98) was extended to

$$\int_K\int_L |\langle x, y\rangle|^p \,\mathrm{d}x\,\mathrm{d}y \geq \frac{n}{n+p}c_2^{-p/n}[V_n(K)V_n(L)]^{(n+p)/n} \tag{10.101}$$

for compact sets $K, L \subset \mathbb{R}^n$ and $p \geq 1$, with an explicitly given constant c_2. Equality holds if and only if K and L are, up to sets of measure zero, dilatates of polar o-symmetric ellipsoids. For $p \to \infty$, one obtains

$$\max_{x \in K, y \in L} |\langle x, y\rangle| \geq \kappa_n^{-2/n} [V_n(K)V_n(L)]^{1/n}.$$

If K is an o-symmetric convex body and L is its polar, this is the Blaschke–Santaló inequality again.

For o-symmetric convex bodies $K \in \mathcal{K}_n^n$, it was conjectured by Kuperberg [1157] that

$$\phi(K) := \frac{1}{V_n(K)V_n(K^\circ)} \int_K \int_{K^\circ} \langle x, y\rangle^2 \,\mathrm{d}y \,\mathrm{d}x$$

attains its maximum when K is an ellipsoid. Alonso-Gutiérrez [61] noted that this would imply the hyperplane conjecture (and the Blaschke–Santaló inequality), and he proved the conjecture for the unit balls of the ℓ_p^n spaces.

7. It was conjectured by Schneider [1698] that the maximum of (10.80) on centrally symmetric convex bodies is equal to 2^n and is attained by direct sums of two-dimensional symmetric bodies and segments. This was disproved by Brannen [329], who mentioned in passing that the conjecture might still be true for zonoids. Brannen conjectured that the maximum of (10.80) is attained by simplices. The value of the functional (10.80) for some special three-dimensional convex bodies was calculated by Brannen [331]. In three dimensions, Saroglou [1635] determined the maximum of (10.80) on zonoids (confirming the value 2^3, but finding more extremal bodies), cones and double cones; he also has some more results on special convex bodies.
8. Martini [1353] proved that an n-polytope P in $\mathbb{R}^n$, $n \geq 3$, is a simplex if and only if its polar projection body $\Pi^\circ P$ is homothetic to its difference body $\mathrm{D}P$.
9. From the Petty projection inequality, Lutwak [1269] derived inequalities between the volume of a convex body and power means of its brightness function.
10. *Mixed projection bodies* appeared first in the work of Süss [1827]. They were thoroughly studied by Lutwak [1271, 1272, 1278, 1284], who obtained a large number of inequalities, also for polars of mixed projection bodies and for quermassintegrals of specialized mixed projection bodies.

    Cheung and Zhao [422] proved Brunn–Minkowski type inequalities for width integrals of mixed projection bodies.

    Leng, Zhao, He and Li [1200] proved inequalities for polars of mixed projection bodies, denoted by $\Pi^\circ(K_1, \dots, K_{n-1})$ for $K_1, \dots, K_{n-1} \in \mathcal{K}_n^n$, for example the Aleksandrov–Fenchel type inequality

$$V_n(\Pi^\circ(K_1, \dots, K_{n-1}))^r \leq \prod_{j=1}^r V_n(\Pi^\circ(K_j[r], K_{r+1}, \dots, K_{n-1}))$$

    for $r \in \{2, \dots, n-1\}$, and a Brunn–Minkowski type inequality.
11. As explained in §8.3, the projection body mapping has a natural generalization, in the form of Blaschke–Minkowski homomorphisms. As observed by Schuster [1746] and continued by Zhao [2065], many of the inequalities for mixed projection bodies mentioned in the previous note (and also for their polars, using the dual theory) can be extended to this setting.
12. Xiong and Cheung [1997] related dual mixed volumes of the radial pth mean body of a convex body to its chord power integrals.

    The symmetry relation (10.81) has an abstract (and more difficult) generalization to Minkowski valuations (which are explained in §6.4, Note 12). Let $0 \leq i \leq n-1$. If $\Phi_i \in \mathbf{MVal}_i$ is $\mathrm{O}(n)$ equivariant, then Alesker, Bernig and Schuster [52] showed that

$$V(\Phi_i(K), L[i], B^n[n-i-1]) = V(K[i], \Phi_i(L), B^n[n-i-1]).$$

This was used to prove that

$$V_{i+1}(\Phi_i(K+L))^{1/i(i+1)} \geq V_{i+1}(\Phi_i(K))^{1/i(i+1)} + V_{i+1}(\Phi_i(L))^{1/i(i+1)}.$$

It was also used by Parapatits and Schuster [1518] to further generalize the latter inequality.

13. *Petty's affine projection inequality.* For $K \in \mathcal{F}^n$, let M be the star body with radial function $\rho(M,\cdot) = f_K^{1/(n+1)}$. Then, by (10.66),

$$h(\Gamma M, x) = \frac{1}{(n+1)V_n(M)} \int_{\mathbb{S}^{n-1}} |\langle x, u\rangle| f_K(u)\,\mathrm{d}u = \frac{2}{(n+1)V_n(M)} h(\Pi K, x),$$

thus $(n+1)V_n(M)\Gamma M = 2\Pi K$, and hence the Busemann–Petty centroid inequality (10.70) gives

$$V_n(\Pi K) \geq \frac{\kappa_{n-1}^n}{\kappa_n^n} V_n(M)^{n+1}.$$

Since

$$V_n(M) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} f_K(u)^{n/(n+1)}\mathrm{d}u = \frac{1}{n}\Omega(K)$$

by (10.18), it results that

$$V_n(\Pi K) \geq \frac{\kappa_{n-1}^n}{n^{n+1}\kappa_n^n} \Omega(K)^{n+1},$$

with equality if and only if K is an ellipsoid. This is *Petty's affine projection inequality*, proved by Petty [1527].

L_p versions of Petty's affine projection inequality appear in Yuan, Lv and Leng [2009] and in Wang and Leng [1917]. Further extensions to asymmetric L_p projection bodies, and also inequalities involving the L_p geominimal surface area, were treated by Wang and Feng [1911].

14. Wang and Leng [1917] further combine and extend various inequalities, to obtain inequalities connecting p-affine surface area with volumes and p-mixed volumes of p-centroid bodies and L_p projection bodies. For example, they extend (10.56). Further inequalities in this style, also involving p-curvature images and mixed p-affine surface areas, are proved by Lv and Leng [1309].

15. It follows from (10.66), (1.52), (10.24) and (10.77) that

$$\Gamma K^{\mathbf{s}} = \frac{2}{(n+1)V_n(K)} \Pi C K$$

for $K \in \mathcal{K}_n^n$. This relation between various operators is due to Petty [1529], Theorem (3.11).

16. From the right inclusion of (10.93), Wang [1910] deduced that for $K \in \mathcal{K}_n^n$ there exists a point $x_0 \in K$ such that the ith dual quermassintegral satisfies

$$\widetilde{W}_i(\Pi^\circ K) \geq \left(\frac{c_{n,n-i}}{nV_n(K)}\right)^{n-i} \widetilde{W}_i(K - x_0).$$

Equality holds if and only if K is a simplex.

17. Wang and Leng [1915] extended the L_p Petty projection inequality (10.95) to what they call *L_p mixed projection bodies*. In [1916], Wang and Leng gave lower bounds for $V_n(K)^{(p-n)/p} V_n(\Pi_p K)$ (for $K \in \mathcal{K}_{(o)}^n$), which for $n \geq 2$ are sharp precisely for $p = 2$.

18. Zhao and Leng [2072] and Zhao and Cheung [2067] proved inequalities for dual quermassintegrals and L_p dual volumes of *mixed projection bodies*, which are in analogy to Aleksandrov–Fenchel and Brunn–Minkowski type inequalities.

19. Volume inequalities involving the body $\Gamma_{-p}K$ were proved by Si, Xiong and Yu [1790].
20. Abardia and Bernig [2] studied a version of the projection body in complex vector spaces. They obtained a characterization theorem and various inequalities of Brunn–Minkowski, Aleksandrov–Fenchel and Minkowski type.
21. The following L_p version of Aleksandrov's projection theorem (Corollary 8.1.5) was proved by Ryabogin and Zvavitch [1601]. Let $p > 1$, $p \neq n$, where p is not an even integer, and let $K, L \in \mathcal{K}_n^n$ be o-symmetric. If $\Pi_p K = \Pi_p L$, then $K = L$.

10.10 Intersection bodies

We begin this section with a volume formula, namely

$$V_n(K)^{n-1} = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{K \cap u^\perp} \cdots \int_{K \cap u^\perp} D_{n-1}(x_1, \ldots, x_{n-1}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \, \mathrm{d}u \tag{10.102}$$

for bounded measurable sets K. It follows immediately by applying a general integral-geometric transformation, the linear Blaschke–Petkantschin formula (see, e.g., [1740], Theorem 7.2.1), to the function

$$(x_1, \ldots, x_{n-1}) \mapsto \mathbf{1}_K(x_1) \cdots \mathbf{1}_K(x_{n-1}).$$

For $K \in \mathcal{K}_n^n$, we apply the Busemann random simplex inequality (10.10) to the inner integrals of (10.102) and obtain the inequality

$$\frac{1}{n} \int_{\mathbb{S}^{n-1}} V_{n-1}(K \cap u^\perp)^n \, \mathrm{d}u \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-2}} V_n(K)^{n-1}, \tag{10.103}$$

which is commonly called the *Busemann intersection inequality*. Equality holds for $n = 2$ if and only if K is o-symmetric, and for $n \geq 3$ by Lemma 10.2.4 if and only if K is an o-symmetric ellipsoid.

The left side of (10.103) can be interpreted as the volume of a star body associated with K. More generally, for a star body $K \in \mathcal{S}_o^n$, there is a unique star body $\mathrm{I}K$ with radial function given by

$$\rho(\mathrm{I}K, u) = V_{n-1}(K \cap u^\perp) \quad \text{for } u \in \mathbb{S}^{n-1}.$$

This star body was introduced by Lutwak [1277] and called the *intersection body* of K. Intersection bodies of o-symmetric convex bodies had appeared earlier (though not under this name) as solutions of the isoperimetric problem in Busemann's theory of area in Minkowski spaces (see Busemann [367] and the monograph of Thompson [1845]). The intersection body operator satisfies

$$\mathrm{I}\phi K = \phi^{-t}\mathrm{I}K \qquad \text{for } \phi \in \mathrm{SL}(n)$$

(see Lutwak [1282]). If $K \in \mathcal{K}_n^n$ is o-symmetric, then the intersection body $\mathrm{I}K$ is convex. This follows from *Busemann's convexity theorem*. The proof is found in Busemann [365] and, for example, in Gardner ([672], Theorem 8.1.10); strengthenings and extensions are in Barthel [164], Barthel and Franz [168], Milman and Pajor [1407].

For $K \in \mathcal{K}_n^n$, the Busemann intersection inequality now says that

$$V_n(\mathrm{I}K) \le \frac{\kappa_{n-1}^n}{\kappa_n^{n-1}} V_n(K)^{n-1}, \tag{10.104}$$

with equality for $n = 2$ if and only if K is o-symmetric, and for $n \ge 3$ if and only if K is an o-symmetric ellipsoid. This inequality can be extended to star bodies. Let $K \in \mathcal{S}_o^n$. In the volume formula (10.102) for $K \in \mathcal{S}_o^n$, we insert formula (10.69) for dimension $n-1$, denoting by Γ_u the centroid body operator in $u^\perp$, and obtain

$$V_n(K)^{n-1} = \frac{(n-1)!}{2^n} \int_{\mathbb{S}^{n-1}} V_n(K \cap u^\perp)^{n-1} V_{n-1}(\Gamma_u(K \cap u^\perp))\,\mathrm{d}u.$$

Introducing a star body $\mathrm{P}K$ by

$$\rho(\mathrm{P}K, u) := V_{n-1}(\Gamma_u(K \cap u^\perp)) \quad \text{for } u \in \mathbb{S}^{n-1},$$

we can write this in the form

$$V_n(K)^{n-1} = \frac{n!}{2^n} \widetilde{V}_1(\mathrm{I}K, \mathrm{P}K), \tag{10.105}$$

a formula of Petty [1525], in a notation suggested by Lutwak [1286]. The Busemann–Petty centroid inequality (10.70) in dimension $n-1$ gives $\mathrm{P}K \supset c_n \mathrm{I}K$ with $c_n = (2\kappa_{n-2}/n\kappa_{n-1})^{n-1}$, which yields

$$V_n(K)^{n-1} = \frac{n!}{2^n} \widetilde{V}_1(\mathrm{I}K, \mathrm{P}K) \ge \frac{n!}{2^n} c_n \widetilde{V}_1(\mathrm{I}K, \mathrm{I}K) = \frac{n!}{2^n} c_n V_n(\mathrm{I}K)$$

and thus inequality (10.104) for the star body K, with the same equality conditions. This is due to Petty [1525].

The L_p *intersection body* of a star body $K \in \mathcal{S}_o^n$, for $p < 1$, $p \ne 0$, has been defined by

$$\rho(\mathrm{I}_p K, u)^p = \frac{1}{\Gamma(1-p)} \int_K |\langle u, x\rangle|^{-p}\,\mathrm{d}x, \qquad u \in \mathbb{S}^{n-1}.$$

(Different normalizations are found; we use that of Haberl [873].) This can be reduced to already established constructions, since up to a power of the volume of K and a numerical constant, $\mathrm{I}_p K$ is the polar of the $(-p)$-centroid body of K. That the L_p intersection body is, in fact, an extension of the ordinary intersection body is indicated, for example, by the limit relation

$$\lim_{p \uparrow 1} \rho(\mathrm{I}_p K, u)^p = 2\rho(\mathrm{I}K, u)$$

for $u \in \mathbb{S}^{n-1}$ (see Grinberg and Zhang [775], Corollary 8.3, Gardner and Giannopoulos [677], Proposition 3.1, Koldobsky [1136], p. 9, Haberl [873], Theorem 1).

Berck [196] has extended Busemann's convexity theorem, showing that the L_p intersection body of an o-symmetric convex body is convex, for $p \ge -1$, $p \ne 0$.

An asymmetric version was introduced in Haberl and Ludwig [877] and investigated by Haberl [873]. For $0 < p < 1$ and $K \in \mathcal{S}^n_o$, define $\mathrm{I}^+_p K$ by

$$\rho(\mathrm{I}^+_p, u)^p := \frac{1}{\Gamma(1-p)} \int_{K \cap u^+} |\langle u, x\rangle|^{-p} \, \mathrm{d}x, \qquad u \in \mathbb{S}^{n-1}, \tag{10.106}$$

where $u^+ := \{x \in \mathbb{R}^n : \langle u, x\rangle \geq o\}$. Further, $\mathrm{I}^-_p K := \mathrm{I}^+_p(-K)$, thus $\mathrm{I}_p K = \mathrm{I}^+_p K \,\widetilde{+}_p\, \mathrm{I}^-_p K$. As Haberl suggests, these are the appropriate extensions of the intersection body to the dual L_p Brunn–Minkowski theory. Among the results of Haberl is the fact that $\mathrm{I}^+_p K$ uniquely determines the star body K. For convex bodies, Haberl also proved corresponding stability results.

Notes for Section 10.10

1. As for projections, also for sections of convex bodies and for intersection bodies, we refer to the much more comprehensive treatments given in the books by Gardner [675], Koldobsky [1136], Koldobsky and Yaskin [1142].
2. The intersection bodies of star bodies defined above have been generalized in various directions. In one generalization, an *intersection body* is a star body whose radial function is the spherical Radon transform of an even positive finite Borel measure on $\mathbb{S}^{n-1}$. We refer to Koldobsky [1136] and Gardner [675]. See also [675, p. 341] for the notions of intersection bodies of order i and of (i, p)-intersection bodies, introduced and investigated by Zhang [2056, 2057] and Grinberg and Zhang [775].

   Several isomorphic estimates for intersection bodies and their generalizations were proved by Koldobsky, Paouris and Zymonopoulou [1140].
3. Intersection bodies and projection bodies of o-symmetric convex bodies play an important role in the theory of areas in Minkowski spaces. We refer to the book of Thompson [1845]. This occurrence of projection and intersection bodies in Minkowski geometry has led to some interesting questions, which are still open. For example, which centrally symmetric convex bodies B yield the minimum or maximum for the affine-invariant functional $V(B, \dots, B, \mathrm{I}^\circ B)$ or for $V(B, \dots, B, \Pi B^\circ)$, the 'self-surface area' of the unit ball in the respective theory? Moreover: if $\mathrm{I}B$ is homothetic to B°, must B be an ellipsoid? If ΠB is homothetic to B°, must B be an ellipsoid? These questions were posed by Busemann and Petty [376] and by Holmes and Thompson [986], respectively.

   The investigation of the Busemann area in Minkowski spaces in Schneider [1724] required finding special o-symmetric convex bodies for which the polar intersection body is, or is not, a zonoid.
4. *Projection and intersection bodies*. For $K \in \mathcal{K}^n_{(o)}$ $(n \geq 3)$, Martini [1351] observed that $\mathrm{I}K \subset \Pi K$. Here equality holds if and only if K is an o-symmetric ellipsoid.
5. Alfonseca [58] proved that, for $n \geq 5$, a direct sum $A + B$ of convex bodies A, B with $\dim A \geq 1$ and $\dim B \geq 4$ cannot be an intersection body. This was motivated by a result of Lonke [1231] on projection bodies.
6. Earlier characterizations of projection bodies by inequalities between mixed volumes (see the references in [741]) were unified and extended by Goodey, Lutwak and Weil [741]. They obtained analogous characterizations of intersection bodies, generalized as in Note 2, and of polar projection bodies, in terms of inequalities between dual mixed volumes. Using the same general notion of intersection bodies, Goodey and Weil [750] proved that a star body is an intersection body if and only if it is the limit (in the topology induced by the radial metric) of finite radial sums of ellipsoids.
7. Yaskina [2001] constructed centrally symmetric convex bodies in dimensions $n \geq 5$ that are not intersection bodies but have the property that all their central hyperplane sections are intersection bodies.
8. S. Yuan, J. Yuan and Leng [2012] proved dual Brunn–Minkowski inequalities for intersection bodies that involve harmonic Blaschke addition or p-radial addition.

9. *Mixed intersection bodies.* For star bodies $K_1, \dots, K_{n-1} \in \mathcal{S}_o^n$, the *mixed intersection body* is the star body with radial function

$$\rho(\mathrm{I}(K_1, \dots, K_{n-1}), u) := \frac{1}{n-1} \int_{\mathbb{S}^{n-1} \cap u^\perp} \rho(K_1, v) \cdots \rho(K_{n-1}, v) \, \mathrm{d}\sigma_{n-2}(v).$$

Mixed intersection bodies were suggested by Leichtweiß [1193], p. 251, and were first intensively studied by Yuan and Leng [2016]. Various inequalities for intersection bodies of radial sums and for mixed intersection bodies, and partially for their star duals or p-extensions, were proved by Zhao and Leng [2070, 2071], Lu, Mao and Leng [1238], Yuan, Zhu and Leng [2015], Lu and Leng [1236], Zhao [2063, 2064], Zhao and Cheung [2068], Zhu and Shen [2078].
10. *Variants of intersection bodies.* For variants of intersection bodies, the cross-section bodies and p-cross-section bodies, we refer to Gardner's monograph [672] and the references given there. Another variant, the 'convex cross-section body', was introduced by Meyer and Reisner [1414].
11. Fish, Nazarov, Ryabogin and Zvavitch [612] showed that the iterations of the intersection body operator applied to any o-symmetric star body sufficiently close to the unit ball B^n in the Banach–Mazur distance converge to B^n in the Banach–Mazur distance.
12. Busemann's convexity theorem was extended by Kim, Yaskin and Zvavitch [1076] to certain nonconvex sets. Let $0 < p \leq 1$. A compact set $K \subset \mathbb{R}^n$ with nonempty interior is *p-convex* if $t^{1/p}x + (1-t)^{1/p}y \in K$ whenever $x, y \in K$ and $t \in (0,1)$. If K is p-convex and o-symmetric, the authors showed that the intersection body $\mathrm{I}K$ is q-convex for all $q \leq [((1/p) - 1)(n-1) + 1]^{-1}$.

    A complex analogue of this result was proved by Huang, He and Wang [998].
13. Inequalities for L_p intersection bodies were proved by Zhu and Leng [2076], Yuan [2007] and Lu and Mao [1237] and inequalities for L_p mixed intersection bodies by Zhao [2062].
14. Complex intersection bodies are introduced and studied by Koldobsky, Paouris and Zymonopoulou [1141].

10.11 Volume comparison

The following innocent-looking question was posed by Busemann and Petty [376] (motivated by the theory of area in Minkowski spaces) and has become known as the *Busemann–Petty problem*. If $K, L \in \mathcal{K}_n^n$ are o-symmetric convex bodies with the property that

$$V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp) \qquad \text{for all } u \in \mathbb{S}^{n-1}, \tag{10.107}$$

does it follow that $V_n(K) \leq V_n(L)$? A 'dual' question regarding projections was asked by Shephard [1778]. If $K, L \in \mathcal{K}_n^n$ are centrally symmetric convex bodies with the property that

$$V_{n-1}(K \,|\, u^\perp) \leq V_{n-1}(L \,|\, u^\perp) \qquad \text{for all } u \in \mathbb{S}^{n-1}, \tag{10.108}$$

does it follow that $V_n(K) \leq V_n(L)$? In either case, examples show that without the assumption of central symmetry the answers would be negative.

As it turned out, Shephard's question was much easier, and was answered independently by Petty [1527] and Schneider [1656]. Both used inequalities for mixed volumes to prove the first part of the following theorem; the second part is from [1656] (Petty [1527] gave a special counter example).

Theorem 10.11.1 *Let $K, L \in \mathcal{K}_n^n$. If L is a projection body, then $\Pi K \subset \Pi L$ implies that $V_n(K) \le V_n(L)$, with equality only if K and L are translates.*

If $K \in \mathcal{K}_n^n$ is centrally symmetric, sufficiently smooth and of positive Gauss curvature, but not a projection body, then there exists a centrally symmetric convex body $L \in \mathcal{K}_n^n$ such that $\Pi K \subset \Pi L$, but $V_n(K) > V_n(L)$.

The following was proved by Lutwak [1277]. 'Intersection body' here means intersection body of a star body.

Theorem 10.11.2 *Let $K, L \in \mathcal{S}_o^n$. If K is an intersection body, then $\mathrm{I}K \subset \mathrm{I}L$ implies that $V_n(K) \le V_n(L)$, with equality only if $K = L$.*

If $L \in \mathcal{S}_o^n$ is o-symmetric, with a sufficiently smooth radial function, but not an intersection body, then there exists an o-symmetric star body $K \in \mathcal{S}_o^n$ such that $\mathrm{I}K \subset \mathrm{I}L$, but $V_n(K) > V_n(L)$.

In the second part of Theorem 10.11.1, the existence of a convex body K with the required property is easy to confirm for each $n \ge 3$. Hence, Shephard's question has a negative answer for all $n \ge 3$. In contrast, the use of Theorem 10.11.2 for a solution of the Busemann–Petty problem is much harder. After counterexamples were noted by several authors for dimensions $n \ge 5$, a positive answer to the Busemann–Petty problem in dimension three was given by Gardner [671] and in dimension four by Zhang [2059]. A far-reaching generalization of the full solution to the Busemann–Petty problem, including sections of intermediate dimensions and dual quermassintegrals, was presented by Rubin and Zhang [1600]. As already mentioned in the preface, questions on sections and projections of convex bodies are generally not considered in this book, since they are thoroughly treated in the monographs by Gardner [675], Koldobsky [1136], Koldobsky and Yaskin [1142]. Therefore, to these books we refer for comprehensive information about the history of the Busemann–Petty problem, the details of its solution and its later generalizations and ramifications.

The reason for treating the inequalities (10.107) and (10.108) and some analogous assumptions here lies in the fact that they can be formulated as inclusion relations for some of the associated bodies considered in the previous sections. As already used, (10.107) is equivalent to $\mathrm{I}K \subset \mathrm{I}L$, and (10.108) is equivalent to $\Pi K \subset \Pi L$. Whether such inclusions imply inequalities for the volume (or other functionals) can be asked for other associated bodies, too, and has, in fact, been considered from several different points of view.

For $j \in \{1, \dots, n-1\}$, the jth projection body $\Pi_j K$ of the convex body K is defined by

$$h_{\Pi_j K}(u) = \kappa_{n-j-1} \binom{n-1}{j}^{-1} V_j(K \,|\, u^\perp) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \, S_j(K, \mathrm{d}v)$$

for $u \in \mathbb{S}^{n-1}$. The following much stronger version of the second part of Theorem 10.11.1 was proved by Goodey and Zhang [756].

Theorem 10.11.3 *Let $j \in \{1, \ldots, n-1\}$, let $K \in \mathcal{K}^n$ be a smooth, centrally symmetric convex body and let $\mathcal{P}$ be any cone dense in the set of centrally symmetric convex bodies. If K is not a zonoid, then there is a body $L \in \mathcal{P}$, which is not a zonoid, such that*

$$\Pi_j K \subset \Pi_j L \quad \textit{but} \quad V_{j+1}(K) > V_{j+1}(L).$$

To formulate a comparison result for lower-dimensional projections, let $j \in \{1, \ldots, n-1\}$ and let $\mathcal{K}^n(j)$ denote the set of centrally symmetric convex bodies $K \in \mathcal{K}^n$ for which there is a positive Borel measure ρ on the Grassmannian $G(n, j)$ such that

$$V_j(K \,|\, E) = \int_{G(n,j)} |\langle E, F \rangle| \, \mathrm{d}\rho(F) \qquad \text{for } E \in G(n, j),$$

where $\langle E, F \rangle := [E, F^\perp]$ (see Weil [1949] for more information about the classes $\mathcal{K}^n(j)$). Thus, $\mathcal{K}^n(1)$ is the class of zonoids and $\mathcal{K}^n(n-1)$ is the class of centrally symmetric convex bodies. The following result of Goodey and Zhang [757] extends the first part of Theorem 10.11.1.

Theorem 10.11.4 *Let $j \in \{1, \ldots, n-1\}$ and let $K, L \in \mathcal{K}^n$ be centrally symmetric. If $L \in \mathcal{K}^n(n-j)$ and $V_j(K \,|\, E) \le V_j(L \,|\, E)$ for all $E \in G(n, j)$, then $V_n(K) \le V_n(L)$.*

Concerning sections of intermediate dimensions, the following result was proved by Bourgain and Zhang [327]. There exist o-symmetric convex bodies of revolution $K, L \in \mathcal{K}^n$ such that the inequality $V_j(K \cap E) < V_j(L \cap E)$ holds for all $E \in G(n, j)$ and for all $j \in \{4, \ldots, n-1\}$, but $V_n(K) > V_n(L)$.

We continue with various analogues of the Theorems 10.11.1 and 10.11.2 or parts thereof. The following counterpart for centroid bodies is due to Lutwak [1279]. A 'polar zonoid' means the polar of an n-dimensional zonoid with centre o.

Theorem 10.11.5 *Let $K \in \mathcal{S}^n_o$. If L is a polar zonoid , then $\Gamma K \subset \Gamma L$ implies that $V_n(K) \le V_n(L)$, with equality only if $K = L$.*

If $K \in \mathcal{S}^n_o$ is o-symmetric, with a sufficiently smooth radial function, but not a polar zonoid, then there exists an o-symmetric star body $L \in \mathcal{S}^n_o$ such that $\Gamma K \subset \Gamma L$, but $V_n(K) > V_n(L)$.

Extensions of the first parts of Theorems 10.11.1 and 10.11.2 to L_p projection bodies and L_p centroid bodies appear in Wang and Leng [1918]. By Fourier-analytic methods, Ryabogin and Zvavitch [1601] established a generalization of Theorem 10.11.1 to L_p projection bodies.

Part of Theorem 10.11.2 was extended by Yuan [2007] to L_p intersection bodies. If K is an L_p intersection body and $L \in \mathcal{S}^n_o$, he showed that $\mathrm{I}_p K \subset \mathrm{I}_p K$ implies $V_n(K) \le V_n(L)$ for $0 < p < 1$ and $V_n(K) \ge V_n(L)$ for $p < 0$, with equality in either case only if $K = L$. For $p \le -1$, the latter appears also in Lu and Mao [1237]. For the asymmetric L_p intersection bodies, the following was proved by Haberl [873]. Let $K, L \in \mathcal{S}^n_o$. If K is in the range of I^+_p, then $\mathrm{I}^+_p K \subset \mathrm{I}^+_p L$ implies that $V_n(K) \le V_n(L)$, with equality only if $K = L$. If $L \in \mathcal{S}_o^n$ is sufficiently smooth and is not in the range of I_p^+, then there exists a body $K \in \mathcal{S}_o^n$ with $\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L$, but $V_n(K) > V_n(L)$. Note that Haberl's result allows one to compare volumes of asymmetric bodies.

An extension of Theorem 10.11.5 to p-centroid bodies for $p \geq 1$ was obtained by Grinberg and Zhang [775] (Corollaries 4.12, 4.13). For $-1 < p < 1$, the p-centroid body is no longer convex, but the polar p-centroid body can still be defined (as a star body), and for this Yaskin and Yaskina [2000] proved an analogue of Theorem 10.11.5.

Theorems 10.11.1 and 10.11.2 were put by Schuster [1748] into the general framework of body-valued valuations on $\mathcal{K}_n^n$ or $\mathcal{S}_o^n$. Recall that a valuation on $\mathcal{K}^n$ or a subset is a Minkowski valuation if it has its values in the semigroup $(\mathcal{K}^n, +)$, where $+$ is Minkowski addition. The projection body operator is an example, and so the following theorem of Schuster [1748] is an abstract generalization of the first part of Theorem 10.11.1.

Theorem 10.11.6 *Let $\Phi : \mathcal{K}_n^n \to \mathcal{K}^n$ be a continuous and translation invariant Minkowski valuation which is homogeneous of degree $n-1$ and $\mathrm{SO}(n)$ equivariant. If $K \in \mathcal{K}_n^n$ and $L \in \Phi\mathcal{K}_n^n$, then $\Phi K \subset \Phi L$ implies $V_n(K) \leq V_n(L)$, with equality if and only if K and L are translates.*

There is also a counterpart to the second half of Theorem 10.11.1. It involves a certain subset of $\mathcal{K}_n^n$. Whether a convex body K belongs to this subset is determined by Φ and the spherical harmonics occurring in the surface area measure $S_{n-1}(K, \cdot)$.

Schuster's analogous generalization of Theorem 10.11.2 involves valuations on star bodies with values in the semigroup of star bodies with radial addition.

Notes for Section 10.11

1. For the Busemann–Petty problem and its ramifications we refer, as already mentioned, to the books of Koldobsky [1136], Gardner [675], Koldobsky and Yaskin [1142]. In addition, we note that the complex Busemann–Petty problem was solved by Koldobsky, König and Zymonopoulou [1139]. This was continued by investigations of Zymonopoulou [2083].
2. Stability results for the volume comparison in the Busemann–Petty and Shephard problems were obtained by Koldobsky [1137] and for the complex Busemann–Petty problem by Koldobsky [1138].
3. For $K, L \in \mathcal{K}_n^n$, Chakerian and Lutwak [404] showed that $\Pi K \subset \Pi L$ (i.e., the assumption in Theorem 10.11.1) is equivalent to $V_{n-1}(\phi K) \leq V_{n-1}(\phi L)$ for all $\phi \in \mathrm{GL}(n)$.
4. The following counterpart to Theorem 10.11.1, with volume replaced by affine surface area, was proved by Lutwak [1279]. Let $K \in \mathcal{F}^n$. If L is the curvature image of a zonoid, then $\Pi K \subset \Pi L$ implies that $\Omega(K) \leq \Omega(L)$, with equality only if K and L are translates. If $K \in \mathcal{F}^n$ is centrally symmetric, has a sufficiently smooth curvature function but is not the curvature image of a zonoid, then there exists a centrally symmetric convex body $L \in \mathcal{F}^n$ such that $\Pi K \subset \Pi L$, but $\Omega(K) > \Omega(L)$.

   Further results of this kind, for L_p projection bodies and L_p polar projection bodies, were proved by Ma and Wang [1312], Wan and Wang [1905], Wang and Wan [1920]. Similar results involving L_p affine surface areas are due to Zhu, Lv and Leng [2077].
5. There are several refinements and extensions of Theorem 10.11.2. We refer to the papers of Zhang [2056, 2057] and Lü and Leng [1242], and again recommend the book of Gardner [672] for more information.

6. *L_p mixed projection bodies.* If in (10.94) one replaces the measure $S_{p,0}(K,\cdot)$ by the measure $S_{p,i}(K,\cdot)$ (defined by (9.16)), for $i \in \{0,\dots,n-1\}$, one obtains the *L_p mixed projection body* $\Pi_{p,i}K$. Extending the method of Ryabogin and Zvavitch [1601], Liu, Wang and He [1229] further extended Theorem 10.11.1 to L_p mixed projection bodies $\Pi_{p,i}K$, with the volume replaced by the quermassintegral W_i.
   Similarly, Liu and Wang [1228] extended definition (10.96) by
$$\rho(\Gamma_{-p,i}K,u)^{-p} := \frac{1}{W_i(K)}\int_{\mathbb{S}^{n-1}} |\langle u,v\rangle|^p \, \mathrm{d}S_{p,i}(K,v), \quad u \in \mathbb{S}^{n-1},$$
   and treated the question as to how far $\Gamma_{-p,i}K \subset \Gamma_{-p,i}L$ implies an inequality between $W_i(K)$ and $W_i(L)$.
7. Yu and Wu [2006] extended the first parts of Theorems 10.11.1, 10.11.2 and 10.11.5 to ith projection and intersection bodies, defined by
$$h_{\Pi_i K}(u) = W_i(K\,|\,u^\perp), \qquad \rho_{\mathrm{I}_i K}(u) = \widetilde{W}_i(K\cap u^\perp),$$
   respectively, and to a newly introduced ith centroid body. Here, the volume was replaced either by the ith quermassintegral or by the ith dual quermassintegral.
8. Schuster's [1748] result mentioned after Theorem 10.11.6, on star body valuations with respect to radial addition and corresponding volume comparison, was extended by Wang, Liu and He [1909] to star body valuations with respect to p-radial addition.

10.12 Associated ellipsoids

With a convex or star body, or even with a more general set, one can associate a number of interesting ellipsoids, some of them known for a longer time, such as the inertia ellipsoid from classical mechanics or the often used John ellipsoid, and others of more recent vintage. In this section, we give a brief description of the most important of them.

Let $K \in \mathcal{K}^n_n$. Among all ellipsoids containing K, there is a unique one of minimal volume, usually called the *Loewner ellipsoid* of K and denoted by $\mathcal{E}^L(K)$. Similarly, among the ellipsoids contained in K, there is a unique one of maximal volume, known as the *John ellipsoid* of K, denoted by $\mathcal{E}_J(K)$. (The distinction may not be quite correct historically, and the use of names in the literature is not uniform.) In either case, existence follows by standard arguments. Simple uniqueness proofs were given by Danzer, Laugwitz and Lenz [465]. Of great use is the characterization of the John ellipsoid (from which a characterization of the Loewner ellipsoid can be obtained by duality). This characterization can be expressed in a convenient way if, by an affine transformation, the John ellipsoid of K has been mapped to the Euclidean unit ball B^n. The following is known as *John's theorem*.

Theorem 10.12.1 *Let $K \in \mathcal{K}^n_n$ and $B^n \subset K$. The unit ball B^n is the John ellipsoid of K if and only if there are vectors $u_1,\dots,u_m \in \mathrm{bd}\, B^n \cap \mathrm{bd}\, K$ (for some $m \in \mathbb{N}$) and positive numbers $c_1,\dots,c_m$ such that*
$$\sum_{i=1}^m c_i u_i = o \tag{10.109}$$

and

$$\sum_{i=1}^{m} c_i \langle x, u_i \rangle^2 = |x|^2 \qquad \textit{for all } x \in \mathbb{R}^n. \tag{10.110}$$

The necessity of the conditions was proved by John [1044]. Proofs of the sufficiency part were given by Pelczyński [1522] and Ball [122]. Alternative proofs of John's theorem are found in Lewis [1207] (symmetric case), Juhnke [1055], Gruber and Schuster [842]; see also Ball [123], Theorem 3.1, and Gruber [834], Theorem 11.2. For more information on Loewner and John ellipsoids and their applications, and also for short biographies of Loewner and John, we refer to the survey article by Henk [955].

Condition (10.110) is equivalent to

$$\sum_{i=1}^{m} c_i \langle x, u_i \rangle u_i = x \qquad \text{for all } x \in \mathbb{R}^n, \tag{10.111}$$

which is often written as

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n$$

('decomposition of the identity'), where $u \otimes u$ denotes the orthogonal projection to $\operatorname{lin} u$ and I_n is the identity map of $\mathbb{R}^n$. Applying (10.110) to the vectors of an orthonormal basis and summing, we obtain

$$\sum_{i=1}^{m} c_i = n. \tag{10.112}$$

The number m appearing in John's theorem can be restricted by $n + 1 \le m \le \frac{1}{2}n(n+3)$ in the general case and $n \le m \le \frac{1}{2}n(n+1)$ in the case of centrally symmetric K (see [842], for example).

A first useful consequence of Theorem 10.12.1, already drawn by John, is the following result.

Theorem 10.12.2 *Let $K \in \mathcal{K}_n^n$, let E be the John ellipsoid of K and let c be its centre. Then $E \subset K \subset n(E - c) + c$.*

If K is o-symmetric, then $E \subset K \subset \sqrt{n}E$.

Proof After applying an affine transformation to K, we may assume that $E = B^n$. Let u_i, c_i be as in Theorem 10.12.1. Since $u_i \in \operatorname{bd} B^n \cap \operatorname{bd} K$, we have $K \subset \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le 1\}$. If $x \in K$ and $|x| = r$, then $-r \le \langle x, u_i \rangle \le 1$ and hence

$$\begin{aligned} 0 &\le \sum_{i=1}^{m} c_i (1 - \langle x, u_i \rangle)(r + \langle x, u_i \rangle) \\ &= r \sum_{i=1}^{m} c_i + (1 - r) \sum_{i=1}^{m} c_i \langle x, u_i \rangle - \sum_{i=1}^{m} c_i \langle x, u_i \rangle^2 = rn - r^2, \end{aligned}$$

where (10.112), (10.109) and (10.110) were used. This shows that $r \le n$ and hence $K \subset nB^n$. (Here we have followed Ball [122].)

Now let K be centrally symmetric. Since B^n is the John ellipsoid of K and this is unique, the symmetry centre of K is o. As above, each $x \in K$ satisfies $\langle x, u_i\rangle \le 1$, and since $-x \in K$, also $|\langle x, u_i\rangle| \le 1$. This gives $|x|^2 = \sum_i c_i \langle x, u_i\rangle^2 \le \sum_i c_i = n$ and hence $|x| \le \sqrt{n}$, thus $K \subset \sqrt{n}B^n$. □

The factors n and $\sqrt{n}$ are best possible, as shown by the examples of a simplex and a cube, respectively.

If for $K, L \in \mathcal{K}^n_n$ the (generalized) *Banach–Mazur distance* is defined by

$$\begin{aligned} &d_{\mathrm{BM}}(K, L) \\ &:= \min\{\lambda \ge 1 : K + x \subset \phi(L + y) \subset \lambda(K + x),\ \phi \in \mathrm{GL}(n),\ x, y \in \mathbb{R}^n\}, \end{aligned}$$

then Theorem 10.12.2 obviously implies that

$$d_{\mathrm{BM}}(K, B^n) \le n \qquad \text{for } K \in \mathcal{K}^n_n \tag{10.113}$$

and

$$d_{\mathrm{BM}}(K, B^n) \le \sqrt{n} \qquad \text{for } o\text{-symmetric } K \in \mathcal{K}^n_n. \tag{10.114}$$

It was proved by Leichtweiß [1182] and independently by Palmon [1499] (and generalized by Jiménez and Naszódi [1041]) that equality in (10.113) holds if and only if K is a simplex. Equality in (10.114) holds if K is an affine image of a cube or a crosspolytope. Equality holds only in these cases if $n = 2$ (Behrend [190]) or $n = 3$ (Ader [5]), but not in higher dimensions (Leichtweiß [1182]).

Important applications of John's theorem will be considered in the next section, although already in the present section we shall mention some consequences of these applications.

We turn to various ellipsoids that can be associated with a convex body. More generally, with a compact set $K \subset \mathbb{R}^n$ with positive volume, we can associate the rank-two tensor (bilinear form) defined by (5.103) for $r = 2$, which we now denote by $M_K/2$, thus

$$M_K(a, b) := \int_K \langle a, x\rangle\langle b, x\rangle\, \mathrm{d}x, \qquad a, b \in \mathbb{R}^n. \tag{10.115}$$

Its matrix with respect to the standard orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ is denoted by $M_2(K) = (m_{ij}(K))^n_{i,j=1}$, thus

$$m_{ij}(K) = \int_K \langle e_i, x\rangle\langle e_j, x\rangle\, \mathrm{d}x. \tag{10.116}$$

The matrix $M_2(K)$ is called the *moment matrix* of K. We have

$$M_K(a, a) = \int_K \langle a, x\rangle^2\, \mathrm{d}x$$

for $a \in \mathbb{R}^n$, hence $M_2(K)$ is positive definite, and the set

$$\mathcal{B}(K) := \{a \in \mathbb{R}^n : \langle a, M_2(K)a\rangle \le 1\}$$

(where a is identified with its coordinate row vector) is an o-symmetric ellipsoid with

$$\rho(\mathcal{B}(K), a)^{-2} = \|a\|^2_{\mathcal{B}(K)} = \int_K \langle a, x\rangle^2 \, \mathrm{d}x.$$

In mechanics, the ellipsoid $\mathcal{B}(K)$ (in dimension three) is known as the *fundamental ellipsoid* of inertia; see Webster [1926], p. 231, where it is stated (without reference) that it was discovered by Binet. We follow Milman and Pajor [1429] in calling $\mathcal{B}(K)$ the *Binet ellipsoid* of K. (Petty [1525] calls the dilatate $V_n(K)^{1/2}\mathcal{B}(K)$ 'Fenchel's momental ellipsoid' of K. In [1220] and [705], the latter ellipsoid is called the Binet ellipsoid of K, and denoted by $\mathcal{E}_B$ and $E_B(K)$, respectively.)

The Binet ellipsoid is related to the 2-centroid body $\Gamma_2 K$, since by (10.72) and (10.71) we have

$$h(\Gamma_2 K, x)^2 = \frac{n+2}{V_n(K)} \int_K \langle x, y\rangle^2 \, \mathrm{d}y, \qquad x \in \mathbb{R}^n,$$

thus

$$\Gamma_2 K = \left(\frac{n+2}{V_n(K)}\right)^{1/2} \mathcal{B}^\circ(K). \tag{10.117}$$

For every o-symmetric ellipsoid E it follows from $\Gamma_2 E = E$ (which holds by the normalization and affine covariance of Γ_2) and (10.117) that

$$\mathcal{B}(E) = \left(\frac{n+2}{V_n(E)}\right)^{1/2} E^\circ \tag{10.118}$$

(this is (1.5) in [1429]). We can define an ellipsoid $\mathcal{L}(K)$ (there is no danger of confusion with the Legendre transform of a function) such that

$$\mathcal{L}(K) = \left(\frac{n+2}{V_n(\mathcal{L}(K))}\right)^{1/2} \mathcal{B}^\circ(K); \tag{10.119}$$

then $\mathcal{B}(\mathcal{L}(K)) = \mathcal{B}(K)$ by (10.118). This means that

$$\int_K \langle x, y\rangle^2 \, \mathrm{d}y = \int_{\mathcal{L}(K)} \langle x, y\rangle^2 \, \mathrm{d}y \qquad \text{for all } x \in \mathbb{R}^n.$$

Thus, $\mathcal{L}(K)$ is Legendre's ellipsoid of inertia of K, which is characterized by the property that it has the same moments of inertia with respect to all axes through o as K. We call $\mathcal{L}(K)$ the *Legendre ellipsoid* of K, thus following [1429]. (There is some confusion with the terminology. Note that what Petty [1525] calls 'Fenchel's ellipsoid' (the polar of his 'Fenchel's momental ellipsoid') is $(n+2)^{-1/2}\Gamma_2 K$. This ellipsoid and also $\Gamma_2 K$ have been called the 'Legendre ellipsoid' in the literature, but both are different from $\mathcal{L}(K)$, though they are dilatates. According to Routh [1596], §19, the Legendre ellipsoid was suggested by Legendre in his *Traité des fonctions elliptiques* of 1825.)

From (10.117) and (10.119), the volumes are related by

$$V_n(\Gamma_2 K)V_n(K)^{n/2} = (n+2)^{n/2}V_n(\mathcal{B}^\circ(K)) = V_n(\mathcal{L}(K))^{(n+2)/2}. \tag{10.120}$$

Together with the definition of $\mathcal{B}$ and $V_n(\mathcal{B})V_n(\mathcal{B}^\circ) = \kappa_n^2$, this gives

$$V_n(\mathcal{L}(K))^{n+2} = (n+2)^n \kappa_n^2 \det M_2(K) \tag{10.121}$$

(this is (1.10) in [1429]).

We have

$$V_n(\Gamma_2 K) \geq V_n(K) \tag{10.122}$$

or, equivalently,

$$V_n(\mathcal{L}(K)) \geq V_n(K), \tag{10.123}$$

with equality if and only if K is, up to a set of measure zero, an o-symmetric ellipsoid. For star bodies, (10.122) is a special case of (10.75). Quick proofs of (10.122) or (10.123), which in dimension three goes back to Blaschke [247], are found in John [1043], Petty [1525], Milman and Pajor [1429], Lutwak [1280].

The determination of the maximum of $V_n(\Gamma_2 K)/V_n(K)$ on the set of o-symmetric convex bodies $K \in \mathcal{K}_n^n$ is a major open problem. For its possible implications, see Milman and Pajor [1429], Lindenstrauss and Milman [1220].

Lutwak, Yang and Zhang [1291] have discovered a 'dual' counterpart to the ellipsoid $\Gamma_2 K$, which we call (following Ludwig [1248]) the *LYZ ellipsoid*. For a convex body $K \in \mathcal{K}_{(o)}^n$, the LYZ ellipsoid $\Gamma_{-2}K$ is defined by

$$\rho(\Gamma_{-2}K, u)^{-2} := \frac{1}{V_n(K)} \int_{\mathbb{S}^{n-1}} \langle u, v\rangle^2 \, S_{2,0}(K, \mathrm{d}v) \tag{10.124}$$

for $u \in \mathbb{S}^{n-1}$, where $S_{2,0}$ denotes the L_2 surface area measure defined by (9.16) for $p = 2$. Note that the definition (10.124) is consistent with the definition (10.96) of the polar projection bodies. We have $\Gamma_{-2}B^n = B^n$. Defining (with respect to the standard orthonormal basis $(e_1, \dots, e_n)$) the matrix $M_{-2}(K) = (\bar{m}_{ij}(K))_{i,j=1}^n$ with

$$\bar{m}_{ij}(K) := \int_{\mathbb{S}^{n-1}} \langle v, e_i\rangle\langle v, e_j\rangle h(K, v)^{-1} \, S_{n-1}(K, \mathrm{d}v) \tag{10.125}$$

and the ellipsoid

$$E_{M_{-2}(K)} := \{a \in \mathbb{R}^n : \langle a, M_{-2}(K)a\rangle \leq 1\},$$

we have

$$\Gamma_{-2}K = V_n(K)^{1/2} E_{M_{-2}(K)}.$$

The duality is nicely put into evidence by the formula

$$\frac{V_2(L, \Gamma_2 K)}{V_n(L)} = \frac{V_{-2}(K, \Gamma_{-2}L)}{V_n(K)},$$

valid for $L \in \mathcal{K}^n_{(o)}$ and $K \in \mathcal{S}^n_o$. It was proved in [1291] and used to show that $\Gamma_{-2}\phi K = \phi\Gamma_{-2}K$ for $K \in \mathcal{K}^n_{(o)}$ and $\phi \in \mathrm{GL}(n)$.

As a consequence of (9.13), for $K \in \mathcal{K}^n_{(o)}$ the inequality

$$V_n(\Gamma_{-2}K) \le V_n(K) \tag{10.126}$$

was obtained in [1291]. Equality holds if and only if K is an o-symmetric ellipsoid.

For o-symmetric convex bodies $K \in \mathcal{K}^n_n$, Lutwak, Yang and Zhang [1291] deduced from Ball's volume ratio inequality (10.138) and its equality condition that

$$V_n(\Gamma_{-2}K) \ge \frac{\kappa_n}{2^n} V_n(K), \tag{10.127}$$

with equality if and only if K is a parallelotope. Similarly, they derived from (10.139) and its equality condition that, for any $K \in \mathcal{K}^n_n$ with John point (centre of the John ellipsoid) at the origin,

$$V_n(\Gamma_{-2}K) \ge \frac{n!\kappa_n}{n^{n/2}(n+1)^{(n+1)/2}} V_n(K), \tag{10.128}$$

with equality if and only if K is a simplex.

An interesting common generalization of the LYZ ellipsoid and the John ellipsoid was introduced by Lutwak, Yang and Zhang [1299], under the name 'L_p John ellipsoid'. For its definition, it is convenient to normalize the L_p mixed volume (9.10), for $K, L \in \mathcal{K}^n_{(o)}$ and $p > 0$, by

$$\overline{V}_p(K,L) := \left(\frac{V_p(K,L)}{V_n(K)}\right)^{1/p} = \left(\frac{1}{nV_n(K)}\int_{\mathbb{S}^{n-1}}\left(\frac{h_L}{h_K}\right)^p h_K \,\mathrm{d}S_{n-1}(K,\cdot)\right)^{1/p},$$

and to supplement this for $p = \infty$ by its limit,

$$\overline{V}_\infty(K,L) := \max\left\{\frac{h_L(u)}{h_K(u)} : u \in \operatorname{supp} S_{n-1}(K,\cdot)\right\}.$$

For $K \in \mathcal{K}^n_{(o)}$ and for $0 < p \le \infty$, consider the optimization problem

$$\begin{aligned}(S_p):\qquad &\text{maximize}\quad (V_n(E)/\kappa_n)^{1/n}\quad \text{subject to}\quad \overline{V}_p(K,E) \le 1\\ &\text{over all } o\text{-symmetric ellipsoids } E.\end{aligned}$$

This optimization problem was suggested in [1299], together with an essentially equivalent one. It was proved that (S_p) has a unique solution, and that for real $p > 0$ the ellipsoid E is this solution if and only if

$$h(E^\circ,x)^2 = \frac{1}{V_n(K)}\int_{\mathbb{S}^{n-1}} \langle x,u\rangle^2 h(E,u)^{p-2}(u)\, S_{p,0}(K,\mathrm{d}u) \qquad \text{for all } x \in \mathbb{R}^n. \tag{10.129}$$

The solution is denoted by E_pK and is called the L_p *John ellipsoid* of K. It satisfies $E_p\phi K = \phi E_pK$ for $\phi \in \mathrm{GL}(n)$. For o-symmetric K, the ellipsoid $E_\infty K$ is the John ellipsoid of K, and from (10.129) it follows that $E_2K = \Gamma_{-2}K$. From the many results obtained in [1299], we select here the following. The inclusion relation for the John ellipsoid (part of Theorem 10.12.2) for o-symmetric $K \in \mathcal{K}^n_{(o)}$ is extended by

$$K \subset \sqrt{n} E_p K \qquad \text{for } 2 \le p \le \infty. \tag{10.130}$$

The inequality (10.126) for $K \in \mathcal{K}^n_{(o)}$ is generalized to

$$V_n(E_p K) \le V_n(K) \qquad \text{for } 1 \le p \le \infty, \tag{10.131}$$

with equality for $p > 1$ if and only if K is an o-symmetric ellipsoid, and equality for $p = 1$ if and only if K is an ellipsoid. Using Ball's volume ratio inequality (10.138), the latter is extended for o-symmetric $K \in \mathcal{K}^n_{(o)}$ to

$$V_n(K) \le \frac{2^n}{\kappa_n} V_n(E_p K) \qquad \text{for } 0 \le p \le \infty, \tag{10.132}$$

with equality if and only if K is a parallelotope.

A perfect dualization of this investigation was carried out by Yu, Leng and Wu [2005]. They normalize the L_p dual mixed volume (9.37), for $K, L \in \mathcal{K}^n_{(o)}$ and $p > 0$, by

$$\overline{V}_{-p}(K,L) := \left(\frac{\widetilde{V}_{-p}(K,L)}{V_n(K)} \right)^{1/p} = \left(\frac{1}{nV_n(K)} \int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K}{\rho_L} \right)^p \rho_K^n \, \mathrm{d}\sigma \right)^{1/p},$$

and supplement this for $p = \infty$ by

$$\overline{V}_{-\infty}(K,L) := \max \left\{ \frac{\rho_K(u)}{\rho_L(u)} : u \in \mathbb{S}^{n-1} \right\}.$$

For $K \in \mathcal{K}^n_{(o)}$ and for $0 < p \le \infty$, they consider the optimization problem

$$(\widetilde{S}_p): \qquad \text{Maximize} \quad (\kappa_n / V_n(E))^{1/n} \quad \text{subject to} \quad \overline{V}_{-p}(K,E) \le 1$$
$$\text{over all } o\text{-symmetric ellipsoids } E.$$

They also formulate an essentially equivalent optimization problem. They show that $(\widetilde{S}_p)$ has a unique solution, and a representation of the solution for $p < \infty$ is given. The solution is denoted by $\widetilde{E}_p K$ and is called the *dual L_p John ellipsoid* of K. For o-symmetric K, the ellipsoid $\widetilde{E}_\infty K$ is the Loewner ellipsoid of K, and $\widetilde{E}_2 K = \Gamma_2 K$. For most of the inclusion and inequality results of [1299], dual counterparts are proved in [2005].

Notes for Section 10.12

1. The uniqueness of the (with respect to volume) largest ellipsoid contained in a convex body K, and of the smallest ellipsoid containing K, was proved for $n = 2$ earlier by Behrend [191]. John [1044] deduced the characterization (which includes the uniqueness) of the ellipsoid of smallest volume containing a given convex body from general results on extremum problems with inequalities as constraints. The name 'Loewner ellipsoid' was apparently first used by Busemann [367], but refers there to the ellipsoid with given centre and smallest volume containing a convex body. Busemann mentions that Loewner did not publish his result. Proofs for the uniqueness of the Loewner and John ellipsoids

were given by Danzer, Laugwitz and Lenz [465] and by Zaguskin [2017], independently of each other and apparently also of John's paper.

Uniqueness of the smallest containing ellipsoid has also been proved for some functions different from the volume, for example for the intrinsic volumes; see Gruber [835] and Schröcker [1744], who used different methods.

2. Gruber [822] proved that most convex bodies in $\mathbb{R}^n$, in the sense of Baire category, touch the boundaries of their John ellipsoid and their Loewner ellipsoid in precisely $n(n+3)/2$ points.
3. The main tool in the proof of John's theorem given by Gruber and Schuster [842] is an idea of Voronoi, to represent o-symmetric ellipsoids in $\mathbb{R}^n$ by points in $\mathbb{R}^{n(n+1)/2}$. This idea was thoroughly exploited and applied to various John type problems by Gruber [821, 835, 836].
4. Extensions of John's theorem, where ellipsoids are replaced by affine images of a fixed convex body (possibly centrally symmetric), were indicated by Lewis ([1207], Theorem 1.3) and Milman (Theorem 14.5 in [1850]) and investigated and applied by Giannopoulos, Perissinaki and Tsolomitis [711], Bastero and Romance [173], Gordon, Litvak, Meyer and Pajor [761], Gruber and Schuster [842], Gruber [836].
5. Barvinok [169], Theorem (2.4), gives a direct proof (i.e., without using the conditions of John's theorem) for the first part of Theorem 10.12.2.
6. Given an o-symmetric convex body $K \in \mathcal{K}_n^n$, Klartag [1096] defines a continuous map u_K from o-symmetric ellipsoids to o-symmetric ellipsoids, which has interesting properties, among them that the John ellipsoid of K is the unique fixed point of u_K.
7. A stability result for inequality (10.113) seems to be unknown. From (10.113) one immediately obtains (as pointed out by Leichtweiß [1182], Korollar) the inequality

$$\mathrm{vq}(K) := \left(\frac{V_n(\mathcal{E}_L(K))}{V_n(\mathcal{E}_J(K))}\right)^{1/n} \leq d_{\mathrm{BM}}(K, B^n) \leq n,$$

where $\mathcal{E}_L(K), \mathcal{E}_J(K)$ are, respectively, the Loewner and John ellipsoids of K. For this inequality, the following stability result was proved by Hug and Schneider [1016]. There exist constants $c_0(n), \varepsilon_0(n) > 0$, depending only on the dimension n, such that the following holds. If $0 \leq \varepsilon \leq \varepsilon_0$ and $\mathrm{vq}(K) \geq (1-\varepsilon)n$, then $d_{\mathrm{BM}}(K, T^n) \leq 1 + c_0(n)\varepsilon^{1/4}$, where T^n is an n-dimensional simplex.
8. *Mixed discriminants of moment matrices.* With a probability measure μ on $\mathbb{R}^n$, associate the moment matrix $M(\mu)$ with entries

$$[M(\mu)]_{ij} = \int_{\mathbb{R}^n} \langle x, e_i\rangle\langle x, e_j\rangle \,\mathrm{d}\mu(x)$$

with respect to an orthonormal basis $(e_1, \dots, e_n)$. If $\mu_1, \dots, \mu_n$ are probability measures on $\mathbb{R}^n$ and $D(M(\mu_1), \dots, M(\mu_n))$ denotes the mixed discriminant (as defined by (5.117)) of their moment matrices, then

$$D(M(\mu_1), \dots, M(\mu_n)) = \frac{1}{n!}\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} D_n(x_1, \dots, x_n)^2 \,\mathrm{d}\mu_1(x_1)\cdots \mathrm{d}\mu_n(x_n).$$

Almost in this generality (for special measures, but the proof extends), the formula was proved by Petty [1525]. Compare also the proof indicated by Lutwak, Yang and Zhang [1297] and that for a special case in Giannopoulos [702], Proposition 1.3.3.

As a consequence (taking for μ_i the Lebesgue measure restricted to K and normalized), the volume of the ellipsoid $\Gamma_2 K$, a dilatate of the Legendre ellipsoid, is given by

$$V_n(\Gamma_2 K)^2 = \frac{\kappa_n^2}{n!}\frac{1}{V_n(K)^n}\int_K \cdots \int_K D_n(x_1, \dots, x_n)^2 \,\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

This formula is also proved in Leichtweiß [1193], Proposition 11.7, together with the formula

$$V_n(\Gamma_2 K)^2 = \frac{n!\kappa_n^2}{n+1}\frac{1}{V_n(K)^{n+1}}\int_K \cdots \int_K V_n(\mathrm{conv}\{x_0,\dots,x_n\})^2 \, \mathrm{d}x_0 \cdots \mathrm{d}x_n$$

for K with centroid o, which for $n = 3$ goes back to Blaschke [237]. Blaschke used it for the first proof of the volume inequality (10.122) (for $n = 3$, with a remark that the proof extends to all dimensions $n \geq 2$).

9. Matveev and Troyanov [1365] established the Legendre ellipsoid as a useful tool in Finsler geometry.
10. Yuan, Si and Leng [2010] observed that the inequalities (10.126), (10.127) and (10.128) allow the following reformulation. If $K \in \mathcal{K}_n^n$ is o-symmetric, then one can find a parallelotope P and an ellipsoid E, both of the same volume as K, such that $\Gamma_{-2}P \subset \Gamma_{-2}K \subset \Gamma_{-2}E$. Equality in the left inclusion holds if and only if K is a parallelotope, and in the right inclusion if and only if K is an ellipsoid. Further, if $K \in \mathcal{K}_n^n$ is a convex body with John point o, then there is a simplex T with the same volume as K and such that $\Gamma_{-2}T \subset \Gamma_{-2}K$. Equality holds if and only if K is a simplex.

    This was extended by Ma [1311] who, by combining various known inequalities, obtained inclusion relations for the L_p polar projection bodies $\Gamma_{-p}K$ and L_q John ellipsoids of special bodies as above.
11. Lu and Leng [1235] proved several inequalities between the volumes of L_p John ellipsoids, dual L_p John ellipsoids, L_p centroid bodies and L_p polar projection bodies.
12. *Brunn–Minkowski theory and information theory.* Lutwak, Yang and Zhang [1294] extended the operator Γ_{-2} to star bodies and proved that for any star body $K \in \mathcal{S}_o^n$ the inclusion

$$\Gamma_{-2}K \subset \Gamma_2 K$$

holds, with equality if and only if K is an o-symmetric ellipsoid. This is interpreted as a geometric analogue of the Cramér–Rao inequality of information theory.

    The interrelations between geometric inequalities for the ellipsoids $\Gamma_2 K$ and $\Gamma_{-2}K$ and information-theoretic inequalities were deepened in Guleryuz, Lutwak, Yang and Zhang [869], and were applied, in both directions, to contoured probability distributions. A probabilty distribution is called *contoured* if there is a set such that any level set of the probability density function is a dilatate of this set.

    Subsequent investigations have widely expanded and deepened the connections between information theory and convex geometric analysis. We refer to Lutwak, Yang and Zhang [1298], Lutwak, Lv, Yang and Zhang [1289], Paouris and Werner [1512, 1513], Jenkinson and Werner [1033], Werner [1965, 1966].

10.13 Isotropic measures, special positions, reverse inequalities

The contact points and corresponding weights in John's theorem (Theorem 10.12.1) provide an example of an isotropic measure. Let $\mu \neq 0$ be a finite Borel measure on $\mathbb{S}^{n-1}$. It is called *isotropic* if

$$\frac{n}{\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \langle x,u\rangle\langle y,u\rangle \, \mathrm{d}\mu(u) = \langle x,y\rangle \qquad \text{for all } x,y \in \mathbb{R}^n,$$

or, equivalently, if

$$\frac{n}{\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \langle x,u\rangle^2 \, \mathrm{d}\mu(u) = |x|^2 \qquad \text{for all } x \in \mathbb{R}^n. \tag{10.133}$$

Often in the literature, an isotropic measure is defined by the requirement that

$$\int_{\mathbb{S}^{n-1}} \langle x, u\rangle^2 \, \mathrm{d}\mu(u) = |x|^2 \qquad \text{for all } x \in \mathbb{R}^n. \tag{10.134}$$

If (10.134) holds, then its application to the vectors of an orthonormal basis and subsequent summation yields $\mu(\mathbb{S}^{n-1}) = n$. Thus, (10.133) together with the normalization $\mu(\mathbb{S}^{n-1}) = n$ is equivalent to (10.134). To make a distinction, we say that a measure μ that satisfies (10.134) is *normalized isotropic*.

The *centroid* $c(\mu)$ of μ is the point defined by

$$c(\mu) := \frac{1}{\mu(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\mu(u).$$

If u_i, c_i, m are the John data provided by Theorem 10.12.1, then the measure $\sum_{i=1}^m c_i \delta_{u_i}$, where δ_u denotes the Dirac measure at u, is normalized isotropic with centroid o.

Isotropic measures often show up if some suitable functional is maximized or minimized over all positions of a convex body. A *position* of the convex body K is the image of K under a nondegenerate affine transformation (in some contexts, only volume-preserving affine transformations are considered). One says that a convex body $K \in \mathcal{K}_n^n$ is in *John position* (*Loewner position*) if its John ellipsoid (Loewner ellipsoid) is the unit ball B^n.

Special positions can be useful to obtain reverse inequalities. This notion is understood here in the following restricted sense. In Euclidean convex geometry, there are many examples of functionals $f : \mathcal{K}_n^n \to \mathbb{R}$ which are continuous, translation invariant, positively homogeneous of degree zero (possibly after multiplication by a suitable power of the volume), and which attain a known minimum (trivially, or by some established theorem) but are unbounded on $\mathcal{K}_n^n$. One can then define the affine-invariant functional f_{GL} by

$$f_{\mathrm{GL}}(K) := \min_{\phi \in \mathrm{GL}(n)} f(\phi K)$$

(the minimum exists; see Section 10.5), and one may be able to determine its maximum and the extremal bodies. In the plane, this type of affine extremal problem was first suggested and investigated systematically by Behrend [190]. He determined the maxima and the extremal bodies of f_{GL}, mostly for centrally symmetric convex domains, for the functionals

$$f \in \{D/\Delta, L/\Delta, A/\Delta^2, D/L, D^2/A, L^2/A\},$$

where D, Δ, L, A denote, respectively, diameter, thickness (minimal width), perimeter, area.

The use of special positions for obtaining reverse inequalities is evident. If f has the properties as described, one may be able to find, for each $K \in \mathcal{K}_n^n$, a special position $\widetilde{K}$ of K for which $f(\widetilde{K}) \le C$, with a constant C independent of K. Then, a fortiori, $f_{\mathrm{GL}}(K) \le C$, which is a reverse inequality.

As an example in $\mathbb{R}^n$, consider the functional $f = R/r$, with R the circumradius and r the inradius. Trivially, $f(K) \geq 1$, with equality if and only if K is a ball. If $\phi \in \mathrm{GL}(n)$ is such that ϕK is in John position, then $r(\phi K) = 1$ and $R(\phi K) \leq n$, by Theorem 10.12.2, thus $f(\phi K) \leq n$, therefore

$$\left(\frac{R}{r}\right)_{\mathrm{GL}}(K) \leq n \tag{10.135}$$

for $K \in \mathcal{K}_n^n$. Equality in (10.135) holds if and only if K is a simplex.

We present some interesting examples of isotropic measures, special positions and reverse inequalities, first considering those related to the John position.

It was discovered by K. Ball that a general analytic inequality due to Brascamp and Lieb, when specialized to data provided by the conditions in John's theorem, takes a simple form and becomes a powerful tool for geometric conclusions. A dual counterpart of this inequality was conjectured by Ball and proved by Barthe. We follow Gardner [674] in calling (10.136) the *Geometric Brascamp–Lieb inequality* and (10.137) the *Geometric Barthe inequality*.

Theorem 10.13.1 *Let $u_1, \dots, u_m \in \mathbb{S}^{n-1}$ and $c_1, \dots, c_m \in \mathbb{R}$ be such that $\sum_{i=1}^m c_i \delta_{u_i}$ is a normalized isotropic measure. Let $f_i \in L^1(\mathbb{R})$ be nonnegative, $i = 1, \dots, m$. Then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, u_i\rangle)^{c_i}\,\mathrm{d}x \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i}. \tag{10.136}$$

If $h : \mathbb{R}^n \to \mathbb{R}$ is measurable and

$$h\left(\sum_{i=1}^m c_i\theta_i u_i\right) \geq \prod_{i=1}^m f_i(\theta_i)^{c_i}$$

for all $\theta_1, \dots, \theta_m \in \mathbb{R}$, then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i}. \tag{10.137}$$

For a proof and for references to the origins, we refer to Barthe [156, 158]. Section 1 of Ball's [124] and Section 17 of Gardner's [674] survey articles put this topic in its wider context. (The extremizers of the general Brascamp–Lieb inequalities were characterized by Valdimarsson [1865].) A continuous extension of Theorem 10.13.1, that is, one in which the discrete measure $\sum_{i=1}^m c_i\delta_{u_i}$ is replaced by a general normalized isotropic measure, was proved by Barthe [161].

Among Ball's striking applications of the inequality (10.136) are his volume ratio inequality and, as a consequence, his reverse isoperimetric inequality. For $K \in \mathcal{K}_n^n$, the *inner volume ratio* is defined by

$$\mathrm{vr_i}(K) := \left(\frac{V_n(K)}{V_n(\mathcal{E}_J(K))}\right)^{1/n}.$$

The following inequality and its proof are found in Ball [119].

Theorem 10.13.2 *If $K \in \mathcal{K}_n^n$ is centrally symmetric, then*

$$\mathrm{vr_i}(K) \le 2\kappa_n^{-1/n}, \tag{10.138}$$

with equality if K is a parallelotope.

Proof We can assume that $\mathcal{E}_J(K) = B^n$ and that u_i, c_i, m are as in Theorem 10.12.1. Since K is o-symmetric, we see as in the proof of Theorem 10.12.2 that $K \subset W := \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \le 1,\ i = 1, \dots, m\}$. With $f_i := \mathbf{1}_{[-1,1]}$ we have

$$\mathbf{1}_W(x) = \prod_{i=1}^m f_i(\langle x, u_i\rangle)^{c_i},$$

hence the Geometric Brascamp–Lieb inequality (10.136) gives

$$V_n(K) \le V_n(W) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, u_i\rangle)^{c_i} \mathrm{d}x \le \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right)^{c_i} = \prod_{i=1}^m 2^{c_i} = 2^n.$$

For a cube, the John ellipsoid is evidently the inscribed ball. □

It was proved by Barthe [158] that equality in (10.138) holds only if K is a parallelotope.

The following counterpart for general convex bodies was proved by Ball [120]; the necessity of the equality condition is again due to Barthe [158].

Theorem 10.13.3 *If $K \in \mathcal{K}_n^n$, then*

$$\mathrm{vr_i}(K) \le \mathrm{vr_i}(T^n), \tag{10.139}$$

where T^n is an n-simplex and equality holds if and only if K is a simplex.

We describe the essential ingredients of Ball's proof. With a finite Borel measure μ on the sphere $\mathbb{S}^{n-1}$ one can associate the closed convex set

$$W(\mu) := \{x \in \mathbb{R}^n : \langle x, u\rangle \le 1 \text{ for all } u \in \operatorname{supp}\mu\}. \tag{10.140}$$

This is the Wulff shape (see Section 7.5) associated with the pair $(\operatorname{supp}\mu, 1)$. Let K be in John position, let u_i, c_i, m be as in Theorem 10.12.1 and set $\mu := \sum_{i=1}^m c_i\delta_{u_i}$. Then $B^n \subset K \subset W(\mu)$, since the points of $\operatorname{supp}\mu$ are common boundary points of B^n and K. To find a sharp estimate for the volume of $W(\mu)$, Ball's proof uses the mapping $\eta : \mathbb{S}^{n-1} \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}$ defined by

$$\eta(u) := \left(-\frac{\sqrt{n}}{\sqrt{n+1}}u, \frac{1}{\sqrt{n+1}} \right), \qquad u \in \mathbb{S}^{n-1}, \tag{10.141}$$

which has two useful properties. If $u_1, \dots, u_{n+1} \in \mathbb{S}^{n-1}$ are the outer unit normal vectors of a regular simplex, then $\eta(u_1), \dots, \eta(u_{n+1})$ are orthonormal in $\mathbb{R}^n \times \mathbb{R}$, and conversely. Also, if μ is a normalized isotropic measure on $\mathbb{S}^{n-1}$ with centroid o, then the image measure $\eta(\mu)$ is a normalized isotropic measure on $\mathbb{S}^n$. The Geometric Brascamp–Lieb inequality is then applied to a finite family of functions on $\mathbb{R}^n \times \mathbb{R}$

with the property that their product is supported on a cone in $\mathbb{R}^n \times \mathbb{R}$ whose cross-sections parallel to $\mathbb{R}^n$ are similar to $W(\mu)$.

Dual counterparts to Theorems 10.13.2 and 10.13.3 for the *outer volume ratio* $\mathrm{vr_o}$, defined by

$$\mathrm{vr_o}(K) := \left(\frac{V_n(K)}{V_n(\mathcal{E}^L(K))} \right)^{1/n},$$

were proved by Barthe [155, 158], who used his inequality (10.137).

Theorem 10.13.3 implies that

$$\left(\frac{V_n}{r^n} \right)_{\mathrm{GL}} (K) \le \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!},$$

with equality if and only if K is a simplex. For centrally symmetric bodies $K \in \mathcal{K}_n^n$, Theorem 10.13.2 yields that

$$\left(\frac{V_n}{r^n} \right)_{\mathrm{GL}} (K) \le 2^n,$$

with equality if and only if K is a parallelotope. Both are reverse inequalities to the trivial inequality $(V_n/r^n)(K) \ge \kappa_n$.

The *reverse isoperimetric inequality*, first proved by Ball [120] (with equality conditions later supplied by Barthe [158]), concerns the affine-invariant isoperimetric quotient

$$I(K) := \left(\frac{S^n}{V_n^{n-1}} \right)_{\mathrm{GL}} (K)$$

for $K \in \mathcal{K}_n^n$, where S denotes the surface area.

Theorem 10.13.4 (Ball's reverse isoperimetric inequality) *For $K \in \mathcal{K}_n^n$,*

$$\left(\frac{S^n}{V_n^{n-1}} \right)_{\mathrm{GL}} (K) \le \frac{n^{(n+2)/2}(n+1)^{(n+1)/2}}{n!}, \tag{10.142}$$

with equality if and only if K is a simplex.

If K is centrally symmetric, then

$$\left(\frac{S^n}{V_n^{n-1}} \right)_{\mathrm{GL}} (K) \le (2n)^n, \tag{10.143}$$

with equality if and only if K is a parallelotope.

Proof Let K be in John position, and let Δ^n be a regular simplex circumscribed to B^n. By Ball's volume ratio inequality (10.139), $V_n(K) \le V_n(\Delta^n)$. From $B^n \subset K$ it follows that $S(K) \le nV_n(K)$ (e.g., using the volume formula (5.3)), and similarly $S(\Delta^n) = nV_n(\Delta^n)$. This gives

$$I(K) \le \frac{S(K)^n}{V_n(K)^{n-1}} \le n^n V_n(K) \le n^n V_n(\Delta^n) = \frac{S(\Delta^n)^n}{V_n(\Delta^n)^{n-1}} = I(\Delta^n),$$

where the last equality is elementary (e.g., Hadwiger [911], Section 6.5.4). By Theorem 10.13.3, equality holds if and only if K is a simplex.

The proof for centrally symmetric K is similar. □

In the plane, the reverse isoperimetric inequality was proved earlier by Behrend [190] and by Gustin [872].

Ball's Handbook article [124] gives a very illuminating presentation of the Geometric Brascamp–Lieb inequality and its application to the reverse isoperimetric inequality.

In Ball's proof of his reverse isoperimetric inequality, one can distinguish three main parts. First, the given convex body is affinely transformed into John position, which yields a normalized isotropic measure $\mu = \sum_{i=1}^{m} c_i \delta_{u_i}$ on $\mathbb{S}^{n-1}$ with centroid o. The second step starts with the Wulff shape $W(\mu)$ of $\operatorname{supp} \mu$, defined by (10.140)). The crucial task here is to obtain a sharp upper bound for the volume of $W(\mu)$. The third step compares volumes and surface areas of K and $W(\mu)$ and leads to an optimal result, since cube and simplex both have the property that their John ellipsoid touches each of the facets.

The middle step in the preceding proof scheme is the most delicate. This step can be considered independently of the fact that the isotropic measure μ comes from a John position. It has been generalized, and supplemented by a dual counterpart, as follows.

Theorem 10.13.5 *Let μ be a normalized isotropic measure on $\mathbb{S}^{n-1}$ with centroid o. Let $W(\mu)$ be defined by (10.140). Then*

$$V_n(W(\mu)) \leq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!}, \tag{10.144}$$

with equality if and only if $W(\mu)$ is a regular simplex circumscribed to $\mathbb{S}^{n-1}$.

For the polar convex body, $W(\mu)^\circ = \operatorname{conv} \operatorname{supp} \mu$, the inequality

$$V_n(W(\mu)^\circ) \geq \frac{(n+1)^{(n+1)/2}}{n! n^{n/2}} \tag{10.145}$$

holds, with equality if and only if $W(\mu)^\circ$ is a regular simplex inscribed to $\mathbb{S}^{n-1}$.

For discrete measures μ, inequality (10.144), as mentioned, is due to Ball [120], and the equality condition was provided by Barthe [158]. Also for discrete μ, inequality (10.145) was suggested by Ball [120] and proved by Barthe [158], with equality condition. The general Theorem 10.13.5 was proved by Lutwak, Yang and Zhang [1301]. They emphasize that they do not use the Brascamp–Lieb inequality, but their proof profits much from the measure transport techniques used by Barthe in his proof of the Geometric Brascamp–Lieb inequality and its dual. The proof follows Ball in using the mapping (10.141). Another ingredient is what the authors call the

Ball–Barthe inequality for isotropic measures. For a finite Borel measure ν on $\mathbb{S}^{n-1}$, the bilinear form $F = \int_{\mathbb{S}^{n-1}} u \otimes u \, d\nu(u)$ is defined by

$$F(a,b) = \int_{\mathbb{S}^{n-1}} \langle u,a\rangle\langle u,b\rangle \, d\nu(u), \qquad a,b \in \mathbb{R}^n.$$

Theorem 10.13.6 (Ball–Barthe inequality) *If μ is a normalized isotropic measure on $\mathbb{S}^{n-1}$ and $f : \mathbb{S}^{n-1} \to (0,\infty)$ is a continuous function, then*

$$\det \int_{\mathbb{S}^{n-1}} f(u)\, u \otimes u \, d\mu(u) \geq \exp \int_{\mathbb{S}^{n-1}} \log f(u) \, d\mu(u),$$

with equality if and only if $f(u_1)\cdots f(u_n)$ is constant for linearly independent $u_1,\dots,u_n$ in $\operatorname{supp}\mu$.

For isotropic measures coming from John's theorem, this inequality was first used (though not formulated explicitly) by Ball [119] when he determined the constant in the Brascamp–Lieb inequality in this special case. For discrete μ, a short proof of the inequality appears in Barthe [158]. The general case, with equality condition, was proved by Lutwak, Yang and Zhang [1297].

A counterpart to Theorem 10.13.5 for even isotropic measures is a special case of the more general Theorem 10.14.1, also due to Lutwak, Yang and Zhang.

Techniques similar to those described above and a version of the Ball–Barthe inequality are also used by Lutwak, Yang and Zhang [1302]. They put a convex body $K \in \mathcal{K}^n_{(o)}$ in a position such that its LYZ ellipsoid $\Gamma_{-2}K$ is the unit ball and use this to prove the inequality

$$V_n(\Gamma_{-2}(K))V_n(K^\circ) \geq \frac{\kappa_n (n+1)^{(n+1)/2}}{n! n^{n/2}}. \tag{10.146}$$

Here equality holds if and only if K is a simplex with centroid at the origin. The proof also employs the concept of 'isotropic embedding', which is motivated by Ball's use of the mapping (10.141).

Theorem 10.13.5 has been generalized further. The set $W(\mu)$ appearing there is the Wulff shape associated with the pair $(\operatorname{supp}\mu, 1)$. More generally, let ν be a finite Borel measure on $\mathbb{S}^{n-1}$ and f a positive continuous function on $\mathbb{S}^{n-1}$. Then

$$W_{\nu,f} := \{x \in \mathbb{R}^n : \langle x,u\rangle \leq f(u) \text{ for all } u \in \operatorname{supp}\nu\}$$

is the Wulff shape associated with the pair $(\operatorname{supp}\nu, f)$. Schuster and Weberndorfer [1751] have extended the inequalities (10.144) and (10.145), using similar techniques of measure transport and isotropic embedding, and adding appropriate assumptions on ν and f. Let ν be a normalized isotropic measure and f a positive continuous function on $\mathbb{S}^{n-1}$, satisfying

$$\int_{\mathbb{S}^{n-1}} f(u) u \, d\nu(u) = o \tag{10.147}$$

and

$$\left\langle c(W_{\nu,f}), \int_{\mathbb{S}^{n-1}} f(u)^{-1} u \, \mathrm{d}\nu(u) \right\rangle = 0. \tag{10.148}$$

If (ν, f) satisfies (10.147) and (10.148) (the latter can be relaxed), then

$$V_n(W_{\nu,f}) \leq \frac{(n+1)^{(n+1)/2}}{n!} \|f\|^n_{L^2(\nu)}.$$

Further, if (ν, f) satisfies (10.147), then

$$V_n(W^\circ_{\nu,f}) \geq \frac{(n+1)^{(n+1)/2}}{n!} \|f\|^{-n}_{L^2(\nu)}.$$

In either case, equality holds if and only if conv supp ν is a regular simplex inscribed in $\mathbb{S}^{n-1}$ and f is constant on supp ν.

We mention further consequences of special positions.

For a convex body $K \in \mathcal{K}^n_n$, one can assume, after applying an affine transformation, that its projection body ΠK is in John position. If this holds, then the projection volumes of K satisfy

$$V_{n-1}(K \,|\, u^\perp) \geq V_n(K)^{(n-1)/n} \qquad \text{for all } u \in \mathbb{S}^{n-1}. \tag{10.149}$$

This was proved by Ball [121]. The inequality is sharp, as shown by the example of a cube.

Recalling that w, R, r denote, respectively, mean width, circumradius and inradius, we have the trivial inequalities $R/w \geq 1/2$ and $w/r \geq 2$. Reverse inequalities, namely the maxima of $(R/w)_{\mathrm{GL}}$ and $(w/r)_{\mathrm{GL}}$, follow from work of Schmuckenschläger and Barthe. For this, it is convenient to use the ℓ-norm instead of the mean width. The *ℓ-norm* of a convex body $K \in \mathcal{K}^n_{(o)}$ is defined by

$$\ell(K) := \int_{\mathbb{R}^n} \|x\|_K \, \mathrm{d}\gamma_n(x),$$

where γ_n is the Gaussian probability measure on $\mathbb{R}^n$, with density at x given by $(2\pi)^{-n/2} e^{-|x|^2/2}$. Since $\|x\|_K = h(K^\circ, x)$, a calculation in polar coordinates gives

$$2\ell(K) = \ell(B^n) w(K^\circ).$$

Therefore, polarity can be used to translate results on the ℓ-norm into those about the mean width. The following results have been proved. Among all symmetric convex bodies in John position, the cubes have minimal ℓ-norm (Schechtman and Schmuckenschläger [1642], Proposition 4.11). Among all symmetric convex bodies in Loewner position, the crosspolytopes have maximal ℓ-norm (noted in [1642], with equality condition proved in Barthe [157]). Among all convex bodies in Loewner position, precisely the simplices have maximal ℓ-norm (Barthe [157]). Among all convex bodies in John position, the simplices have minimal ℓ-norm (Schmuckenschläger [1651]).

Just as Theorem 10.13.5 generalizes previous results on discrete isotropic measures, coming from John positions, to general isotropic measures, so the preceding results on ℓ-norms can be extended. Let μ be a normalized isotropic measure on $\mathbb{S}^{n-1}$ with centroid o; let $W(\mu)$ be the Wulff shape defined by (10.140). It was proved by Li and Leng [1213] that

$$\ell(W(\mu)) \geq \ell(\Delta_n), \qquad \ell(W(\mu)^\circ) \leq \ell(\Delta_n^\circ),$$

where Δ_n is a regular simplex circumscribed to B^n. Equality in either case holds if and only if $W(\mu)$ is congruent to Δ_n. If μ is, in addition, an even measure, then

$$\ell(W(\mu)) \geq \ell(C_n), \qquad \ell(W(\mu)^\circ) \leq \ell(C_n^\circ),$$

where C_n is a cube circumscribed to B^n. Equality in either case holds if and only if $W(\mu)$ is congruent to C_n.

We turn to some other positions. A convex body $K \in \mathcal{K}_n^n$ is said to be in *minimal surface area position* if it has smallest surface area among its affine images of the same volume. Petty [1526] proved that K is in minimal surface area position if and only if its surface area measure $S_{n-1}(K, \cdot)$ is isotropic. (The planar case was treated earlier by Behrend [190] and Green [771].) Giannopoulos and Papadimitrakis [710] gave a short new proof and showed that the minimal surface area position is unique up to orthogonal transformations. They made various applications to projection volumes, from which we select the following. If $K \in \mathcal{K}_n^n$ is in minimal surface area position, then (S denotes the surface area)

$$\frac{S(K)}{2n} \leq V_{n-1}(K \,|\, u^\perp) \leq \frac{S(K)}{2\sqrt{n}} \qquad \text{for all } u \in \mathbb{S}^{n-1}, \tag{10.150}$$

and, as a counterpart to (10.149)) for centrally symmetric K,

$$V_{n-1}(K \,|\, u^\perp) \leq \sqrt{n} V_n(K)^{(n-1)/n} \qquad \text{for all } u \in \mathbb{S}^{n-1}. \tag{10.151}$$

All these inequalities are sharp, as shown by the cube.

For o-symmetric $K \in \mathcal{K}_n^n$, the inequality (10.151) is also true if an L_p John ellipsoid E_pK with $1 \leq p \leq 2$ is a ball, as shown by Lutwak, Yang and Zhang [1299] (Theorem 6.1). The L_1 John ellipsoid E_1K is a ball if and only if K is in minimal surface area position.

A systematic study of the relationships between positions with certain extremal properties and isotropic measures was made by Giannopoulos and Milman [704]. In particular, they minimized quermassintegrals over volume-preserving affine transformations and characterized the minimizing positions in different ways. For example, the convex body K has minimal mean width among its affine transforms of the same volume if and only if the measure μ_K defined by $\mathrm{d}\mu_K = h_K \,\mathrm{d}\sigma$ is isotropic. Applications of the minimal mean width position were made by Giannopoulos, Milman and Rudelson [707].

A counterpart to the isoperimetric inequality, for which the reverse inequality was stated above, is the Urysohn inequality (7.21) between volume and mean width. More generally, for $k \in \{2, \dots, n\}$, the functional V_1^k/V_k, where V_k is the kth intrinsic volume, attains its minimum at balls. A reverse inequality seems to be unknown, but for the special case of zonoids $K \in \mathcal{K}_n^n$, it was proved by Hug and Schneider [1019] that

$$\left(\frac{V_1^k}{V_k}\right)_{\mathrm{GL}}(K) \leq n!\binom{n}{k}^{-1}, \tag{10.152}$$

with equality if and only if K is a parallelotope. For $k = n$, this inequality for zonoids had already been proved by Giannopoulos, Milman and Rudelson [707], though without determination of the equality case. In [1019], inequality (10.152) was deduced from the following result on mixed volumes. If $k \in \{2, \dots, n\}$ and if $Z_1, \dots, Z_k \in \mathcal{K}_n^n$ are zonoids with normalized isotropic generating measures (which can always be achieved by affine transformations), then

$$V(Z_1, \dots, Z_k, B^n[n-k]) \geq 2^k \kappa_{n-k}, \tag{10.153}$$

with equality if and only if $Z_1, \dots, Z_k$ are translates of a cube of side length 2. For $k = n$, $Z_1 = \cdots = Z_n$ and discrete generating measures, the inequality (10.153) was proved (in a more general form) by Ball [121] (Lemma 4). The proof in [1019] took up his ideas.

We conclude this section with an important special position. The convex body $K \in \mathcal{K}_n^n$ is said to be *in isotropic position*, or briefly *isotropic*, if it has volume one, centroid at the origin, and its Legendre ellipsoid is a ball, thus if

$$\int_K \mathrm{d}x = 1, \qquad \int_K x \, \mathrm{d}x = o$$

and

$$\int_K \langle x, y \rangle^2 \, \mathrm{d}y = L_K^2 |x|^2 \quad \text{for all } x \in \mathbb{R}^n, \tag{10.154}$$

with some constant L_K. For arbitrary $K \in \mathcal{K}_n^n$,

$$Tx := \int_K \langle x, y \rangle y \, \mathrm{d}y \qquad \text{for } x \in \mathbb{R}^n$$

defines a positive definite, symmetric linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$. It has a positive and hence invertible square root S, and

$$\int_{S^{-1}K} \langle x, y \rangle \, \mathrm{d}y = (\det S)^{-1} |x|^2 \qquad \text{for } x \in \mathbb{R}^n$$

(the calculation is found in Ball [118]). It follows that K always has an affine image that is in isotropic position.

This position is also characterized by an extremal property. The body K with $V_n(K) = 1$ and $c(K) = o$ is isotropic if and only if

$$\int_K |x|^2 \, \mathrm{d}x \leq \int_{\phi K} |x|^2 \, \mathrm{d}x \quad \text{for all } \phi \in \mathrm{SL}(n)$$

(see Milman and Pajor [1429], Giannopoulos and Milman [704]). This fact shows again that every convex body $K \in \mathcal{K}_n^n$ has an isotropic affine transform $\widetilde{K}$; it is uniquely determined up to a rotation. The number $L_K := L_{\widetilde{K}}$ is an affine invariant of K and is called the *isotropic constant*. A more explicit representation of the isotropic constant of K with centroid o is given by

$$L_K^2 = \frac{1}{n} \min \left\{ \frac{1}{V_n(K)^{(n+2)/n}} \int_{\phi K} |x|^2 \, \mathrm{d}x : \phi \in \mathrm{GL}(n) \right\}.$$

If K is in isotropic position, then its moment matrix $M_2(K) = (m_{ij}(K))_{i,j=1}^n$ (see (10.116)) with respect to the standard orthonormal basis is given by

$$m_{ij}(K) = L_K^2 \delta_{ij}$$

and it follows that

$$\int_K |x|^2 \, \mathrm{d}x = n L_K^2,$$

and also that

$$L_K^{2n} = \det M_2(K).$$

Since $V_n(K)^{-(n+2)} \det M_2(K)$ is $\mathrm{GL}(n)$ invariant, we conclude that

$$L_K^{2n} = V_n(K)^{-(n+2)} \det M_2(K)$$

for general $K \in \mathcal{K}_n^n$ with centroid o (Milman and Pajor [1429], (1.9)). Together with (10.120) and (10.121) the latter gives

$$\begin{aligned} L_K^{2n} &= (n+2)^{-n} \kappa_n^{-2} V_n(\mathcal{L}(K))^{n+2} V_n(K)^{-(n+2)} \\ &= (n+2)^{-n} \kappa_n^{-2} V_n(\Gamma_2 K)^2 V_n(K)^{-2}. \end{aligned}$$

It is an open problem, going back to Bourgain [312], whether L_K is bounded from above by a constant independent of the dimension. We refer to Milman and Pajor [1429], Giannopoulos and Milman [705, 706] and in particular to Giannopoulos [702] for further discussion and information on how this question is related to some other major open problems on convex bodies, and for surveys on the known results.

Notes for Section 10.13

1. *Measures with isotropic affine images.* Let μ be a finite and non-zero Borel measure on the unit sphere $\mathbb{S}^{n-1}$. For $\phi \in \mathrm{GL}(n)$, the image $\phi\mu$ under ϕ is the measure on $\mathbb{S}^{n-1}$ defined by $\phi\mu(\omega) := \mu(\langle \phi^{-1}\omega \rangle)$ for $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, where $\langle x \rangle := x/|x|$ for $x \in \mathbb{R}^n \setminus \{o\}$. The following was proved by Böröczky, Lutwak, Yang and Zhang [296]. (Recall that the subspace concentration condition (see (9.35) plays a role in the log-Minkowski problem.)

   Theorem The measure μ has an isotropic affine image if and only if μ satisfies the subspace concentration condition.

   As mentioned in [296], for the case of discrete measures this is due to Carlen and Cordero-Erausquin [394], and for a general measure μ, Klartag [1099] established that if it satisfies the strict subspace concentration inequality, then it has an affine isotropic image.

2. Li and Leng [1210] extended the type of data appearing in John's theorem to the notion of a 'double John basis' and proved a corresponding extension of Theorem 10.13.1. To this setting, in [1212] they also extended a generalized Loomis–Whitney inequality (due to Ball [121], and generalized to arbitrary normalized isotropic measures by Giannopoulos and Papadimitrakis [710], (1.7)).
3. For Petty's [1526] minimal surface area position, Clack [432] studied an analogue for Minkowski areas.
4. The relationship between minimizing positions and isotropic measures, which is prevalent in the paper by Giannopoulos and Milman [704], was carried over (with linear instead of affine maps) to the dual Brunn–Minkowski theory by Bastero and Romance [174]. For example, under smoothness assumptions on the convex body K, the condition $\widetilde{W}_i(K) = \min\{\widetilde{W}_i(\phi K) : \phi \in \mathrm{SL}(n)\}$ for the dual quermassintegral $\widetilde{W}_i$ and for $i \in (-\infty, 0)$, or for $i \in [n+1, \infty)$ and symmetric K, is equivalent to the fact that the measure μ given by $\mathrm{d}\mu = \rho_K^{n-i}\,\mathrm{d}\sigma$ is isotropic. As applications, the authors obtained reverse inequalities of the dual Brunn–Minkowski theory. These investigations were continued by Bastero, Bernués and Romance [171, 172]. Much of the latter work is devoted to the extrema of $\{\widetilde{W}_i(\phi K) : o \in \phi K \subset B^n\}$, where ϕK runs through the positions of K.
5. Markessinis and Valettas [1329] considered the minimal surface area position, the minimal width position, John's and Loewner's position and the isotropic position of a centrally symmetric convex body in $\mathbb{R}^n$ and derived upper bounds for the distance, suitably measured, between any two of them. The comparison of different positions, in particular with a view to the M-position studied in the asymptotic theory, is further investigated by Saroglou [1636] and by Markessinis, Paouris and Saroglou [1328].
6. Parts of the investigation of minimal mean width and minimal surface area positions in Giannopoulos and Milman [704] were extended by Yuan, Leng and Cheung [2008] to the L_p Brunn–Minkowski theory.
7. For a convex body K satisfying $(1/2)B^n \subset K \subset (\sqrt{n}/2)B^n$, He and Leng [947] proved that the isotropic constant satisfies $L_K \leq 1/(2\sqrt{3})$, with equality if and only if K is a unit cube.
8. For the isotropic constant of a convex polytope K in $\mathbb{R}^n$ with N vertices, Alonso-Gutiérrez, Bastero, Bernués and Wolff [63] showed that $L_K \leq C\sqrt{N/n}$, with a constant C.
9. *Isotropic constant and slicing problem.* The already mentioned paper by Milman and Pajor [1429] is highly recommended as a first introduction to some of the central open problems of asymptotic geometric analysis. It explains, for example, how the question of whether the isotropic constant is bounded from above by a constant independent of the dimension is equivalent to other problems, such as the *slicing problem*. This asks whether there is an absolute constant $c > 0$ such that every n-dimensional convex body of volume 1 has a hyperplane section through its centroid of $(n-1)$-dimensional volume at least c. For recent contributions, we refer to Giannopoulos, Paouris and Vritsiou [708] and to Vritsiou [1901].
10. Isotropic positions were used by Giannopoulos, Paouris and Vritsiou [709] to give new and more elementary proofs for the reverse Blaschke–Santaló inequality and the reverse Brunn–Minkowski inequality.

10.14 L_p zonoids

For $1 \leq p < \infty$, a convex body $K \in \mathcal{K}_n^n$ is called an L_p zonoid if its support function has a representation

$$h(K, x) = \|x\|_{K^\circ} = \left(\int_{\mathbb{S}^{n-1}} |\langle x, u \rangle|^p \, \mathrm{d}\mu(u) \right)^{1/p} \qquad \text{for } x \in \mathbb{R}^n,$$

with some even finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$. The measure μ is called a *generating measure* of K. (The case $p = 2$ shows that it need not be uniquely determined.)

An L_1 zonoid and any of its translates is called a zonoid, as defined in Section 3.5. Interest in L_p zonoids for $p > 1$ comes from functional analysis, since an n-dimensional real normed space is isometric to a subspace of L_p if and only if the polar of its unit ball is an L_p zonoid. We refer to the references given after Theorem (5.3) in [1738]. Interest in L_p zonoids also comes from the L_p Brunn–Minkowski theory, since L_p zonoids are precisely the convex bodies in $\mathcal{K}^n_{(o)}$ which can be approximated by finite p-sums of o-symmetric segments.

The set of L_p zonoids is closed under linear transformations. By a result of Lewis [1206], each L_p zonoid has a linear image with an isotropic generating measure. For this result, Lutwak, Yang and Zhang [1299] (Theorem 8.2) have given a geometric proof, using the solution of the even L_p Minkowski problem and properties of the L_p polar projection body $\Gamma_{-p}K$.

For the volumes of L_p zonoids with normalized isotropic generating measures, and for their polars, there are a series of sharp inequalities, initiated by Ball and Barthe, and in their general forms obtained by Lutwak, Yang and Zhang [1297]. Let μ be an even normalized isotropic measure on the unit sphere $\mathbb{S}^{n-1}$. For $1 \le p < \infty$, the measure μ generates the L_p zonoid $Z_p = Z_p(\mu)$ with support function

$$h(Z_p, x) = \left(\int_{\mathbb{S}^{n-1}} |\langle x, u \rangle|^p \, \mathrm{d}\mu(u) \right)^{1/p}, \qquad x \in \mathbb{R}^n.$$

The definition is supplemented by the limit for $p \to \infty$,

$$h(Z_\infty, x) = \max_{u \in \operatorname{supp} \mu} \langle x, u \rangle,$$

thus

$$Z_\infty = \operatorname{conv} \operatorname{supp} \mu$$

and

$$Z_\infty^\circ = \{x \in \mathbb{R}^n : |\langle x, u \rangle| \le 1 \text{ for all } u \in \operatorname{supp} \mu\}.$$

(In the previous section, the latter was denoted by $W(\mu)$.)

For $p \in [1, \infty]$, let $p^* \in [1, \infty]$ be defined by $(1/p) + (1/p^*) = 1$, and for $p \in (0, \infty)$, let

$$\kappa_n(p) := 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)}, \qquad c_p := \left(\frac{\Gamma\left(1 + \frac{n}{2}\right) \Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(1 + \frac{1}{2}\right) \Gamma\left(\frac{n+p}{2}\right)} \right)^{n/p}.$$

Define also $\kappa_n(\infty) := 2^n$ and $c_\infty := 1$, and note that $\kappa_n(2) = \kappa_n$.

A normalized isotropic measure of the form $(1/2)\sum_{i=1}^n (\delta_{e_i} + \delta_{-e_i})$, where $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$, is called a *cross measure*.

The following theorem is taken from Lutwak, Yang and Zhang [1297].

Theorem 10.14.1 *Let $p \in [1, \infty]$, let μ be an even normalized isotropic measure on $\mathbb{S}^{n-1}$ and let $Z_p = Z_p(\mu)$. Then*

$$\kappa_n / c_p \leq V_n(Z_p^\circ) \leq \kappa_n(p). \tag{10.155}$$

If $p \in [1, \infty)$ is not an even integer, then there is equality in the left inequality if and only if μ is suitably normalized spherical Lebesgue measure. For $p \neq 2$, there is equality in the right inequality if and only if μ is a cross measure.

Further,

$$\kappa_n(p^*) \leq V_n(Z_p) \leq \kappa_n c_p. \tag{10.156}$$

For $p \neq 2$, there is equality in the left inequality if and only if μ is a cross measure. If $p \in [1, \infty)$ is not an even integer, then there is equality in the right inequality if and only if μ is suitably normalized spherical Lebesgue measure.

Tools for proving (10.155) (left) and (10.156) (right) in [1297] are Hölder's inequality, Urysohn's inequality and the injectivity of the L_p cosine transform if p is not an even integer. The right inequality (10.155), without the equality conditions, was proved by Ball [120], Proposition 8. For discrete measures, the equality conditions are due to Barthe [158], Corollary 3. The left inequality (10.155), without the equality conditions, was proved by Ball [121], Lemma 4, for $p = 1$, and for $p > 1$ by Barthe [158], Proposition 11. For discrete measures, the equality conditions again follow from Barthe [158]. The authors of [1297] point out that they do not use the Brascamp–Lieb inequality, but that the ideas and techniques of Ball and Barthe, in particular those of [158], play a critical role throughout their paper. One of their tools is the Ball–Barthe inequality of Theorem 10.13.6, which is proved here in full generality, with equality conditions. Later, Barthe [161] gave a simple approximation argument to get the results of [1297] from the previously known ones for discrete measures, though without equality conditions. Finally, he proved a version of the Brascamp–Lieb inequalities for arbitrary normalized isotropic measures (as mentioned after Theorem 10.13.1).

Let Λ be a set of finitely many vectors $v_1, \ldots, v_m$ spanning $\mathbb{R}^n$, no two of which are parallel. We write $\{v_1, \ldots, v_n\} = \Lambda_\perp$ if $v_1, \ldots, v_n$ form an orthonormal basis. For $1 \leq p \leq \infty$, the p-sum of the segments $[-v_i, v_i]$, $i = 1, \ldots, m$, is an L_p zonotope, denoted by $Z_p(\Lambda)$. Campi and Gronchi [390] have used shadow systems to prove the inequalities

$$\frac{V_n(Z_1(\Lambda))}{V_n(Z_p(\Lambda))} \geq \frac{V_n(Z_1(\Lambda_\perp))}{V_n(Z_p(\Lambda_\perp))} \tag{10.157}$$

and

$$V_n(Z_1(\Lambda)) V_n(Z_p^\circ(\Lambda)) \geq V_n(Z_1(\Lambda_\perp)) V_n(Z_p^\circ(\Lambda_\perp)). \tag{10.158}$$

Note that (10.158) comprises Reisner's reverse Blaschke–Santaló inequality for zonotopes. The equality cases of (10.157) and (10.158) have been settled by Weberndorfer [1925].

With Λ as above and a strictly increasing, convex function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(0) = 0$, one can define the *Orlicz zonotope* $Z_\phi(\Lambda)$ by

$$h(Z_\phi(\Lambda), x) := \inf\left\{\lambda > 0 : \sum_{i=1}^{m} \phi\left(\frac{|\langle x, v_i\rangle|}{\lambda}\right) \le 1\right\} \qquad \text{for } x \in \mathbb{R}^n.$$

Wang, Leng and Huang [1906] have extended (10.157) and (10.158) to Orlicz zonotopes.

For $1 < p < \infty$, the p-sum does not commute with translations. Therefore, p-sums of segments which are not o-symmetric generally lead to a new type of nonsymmetric convex body. Again with Λ as above, define the *asymmetric L_p zonotope* $Z_p^+(\Lambda)$ by

$$h(Z_p^+(\Lambda), x) := \left(\sum_{v\in\Lambda} \langle x, v\rangle_+^p\right)^{1/p} \qquad \text{for } x \in \mathbb{R}^n.$$

Thus, $Z_p^+(\Lambda)$ is the p-sum of finitely many segments with one endpoint at the origin. Such asymmetric L_p zonotopes were introduced and studied by Weberndorfer [1925]. He extended the method and results of Campi and Gronchi in the following way. For $p \ge 1$ (and Λ, $\Lambda_\perp$ as above),

$$V_n(Z_p^+(\Lambda)^{\mathrm{s}})V_n(Z_1^+(\Lambda)) \ge V_n(Z_p^+(\Lambda_\perp)^{\mathrm{s}})V_n(Z_1^+(\Lambda_\perp)),$$

where Z^{s} denotes the polar body of Z with respect to the Santaló point. Equality for $p > 1$ holds if and only if Λ is a $\mathrm{GL}(n)$ image of $\Lambda_\perp$. If $p = 1$, then equality holds if and only if $Z_1^+(\Lambda)$ is a parallelotope. The equality conditions for the counterpart to (10.157) are more unexpected. For $p > 1$,

$$\frac{V_n(Z_1^+(\Lambda))}{V_n(Z_p^+(\Lambda))} \ge \frac{V_n(Z_1^+(\Lambda_\perp))}{V_n(Z_p^+(\Lambda_\perp))},$$

with equality if and only if Λ is the $\mathrm{GL}(n)$ image of a set of vectors any two distinct elements u, v of which satisfy $\langle u, v\rangle \le 0$.

Notes for Section 10.14

1. Further volume estimates for L_p zonotopes were obtained and their consequences investigated by Alonso-Gutiérrez [62].
2. The case $p = 1$ of Theorem 10.14.1 concerns volume estimates for convex bodies which are defined by the cosine transform of an even normalized isotropic measure μ on the unit sphere. The *sine transform* $\mathcal{S}(\mu)$ of such a measure is defined by

$$(\mathcal{S}\mu)(x) := \int_{\mathbb{S}^{n-1}} \sqrt{1 - \langle x, u\rangle^2}\, \mathrm{d}\mu(u), \qquad x \in \mathbb{R}^n.$$

It is the support function of a convex body S_μ. Maresch and Schuster [1330] have proved a sharp lower bound for $V_n(S_\mu^\circ)$ and a sharp upper bound for $V_n(S_\mu)$, which are attained precisely if μ is a multiple of spherical Lebesgue measure. They have also obtained an upper bound for $V_n(S_\mu^\circ)$ and a lower bound for $V_n(S_\mu)$; both of them are asymptotically sharp, as $n \to \infty$.

10.15 From geometric to analytic inequalities

In many ways, geometric inequalities can lead to analytic inequalities (and conversely, of course). A classical case is the Sobolev inequality, saying that

$$\int_{\mathbb{R}^n} |\nabla f(x)| \, \mathrm{d}x \geq n\kappa_n^{1/n} \|f\|_{\frac{n}{n-1}} \tag{10.159}$$

for C^1 functions f on $\mathbb{R}^n$ with compact support. Here $\|\cdot\|_p$ denotes the usual L_p norm on $\mathbb{R}^n$. The constant in (10.159) is best possible. We refer to Gardner [674], Section 8, for a short demonstration of how the Sobolev inequality in this form (it can be generalized; see [538], for example) follows from the isoperimetric inequality for compact domains with C^1 boundaries, and vice versa.

It is a remarkable discovery of Zhang [2060] that the Petty projection inequality (10.86), extended to a suitable class of nonconvex sets, can replace the isoperimetric inequality in the previous argument and then leads to an affine Sobolev inequality which is stronger than (10.159). In the following, $\mathrm{D}_u f := \langle u, \nabla f \rangle$ denotes the directional derivative of f with respect to the vector u.

Theorem 10.15.1 (Zhang) *If f is a C^1 function with compact support on $\mathbb{R}^n$, then*

$$\left(\frac{1}{n} \int_{\mathbb{S}^{n-1}} \|\mathrm{D}_u f\|_1^{-n} \, \mathrm{d}u \right)^{-1/n} \geq \frac{2\kappa_{n-1}}{\kappa_n} \|f\|_{\frac{n}{n-1}}. \tag{10.160}$$

The constant in (10.160) is best possible. Another geometric aspect of the proof, besides using an extended version of the Petty projection inequality, is a 'convexification' procedure involving Minkowski's existence theorem. By Hölder's inequality, (10.160) is stronger than the classical Sobolev inequality (10.159).

Wang [1907] has generalized the Sobolev–Zhang inequality (10.160) to the space $BV(\mathbb{R}^n)$ of functions of bounded variation on $\mathbb{R}^n$. In that case, equality holds precisely for multiples of characteristic functions of ellipsoids. The generalization also comprises an extension of the Petty projection inequality to sets of finite perimeter, which implies the isoperimetric inequality for such sets.

All the geometric tools used in the proof of Theorem 10.15.1 have been extended to the L_p Brunn–Minkowski theory, in particular Petty's projection inequality to the inequality (10.95) and Minkowski's existence theorem in the even case to Theorem 9.2.1. This enabled Lutwak, Yang and Zhang [1295] to extend (10.160) to a sharp affine L_p Sobolev inequality. For this, it is convenient to define, for $1 \leq p < \infty$ and for each f in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ (for this, see Evans and Gariepy [538], for example), the L_p *affine energy* of f by

$$\mathcal{E}_p(f) := c_{n,p} \left(\int_{\mathbb{S}^{n-1}} \|\mathrm{D}_u f\|_p^{-n} \, \mathrm{d}u \right)^{-1/n}, \qquad c_{n,p} = (n\kappa_n)^{1/n} \left(\frac{n\kappa_n \kappa_{p-1}}{2\kappa_{n+p-2}} \right)^{1/p}.$$

Then $\mathcal{E}_p(f \circ \phi) = \mathcal{E}_p(f)$ for all $\phi \in \mathrm{SL}(n)$. Denoting by $\|\nabla f\|_p$ the L_p norm of $|\nabla f|$, an application of Hölder's and Fubini's theorems shows that

$$\|\nabla f\|_p \geq \mathcal{E}_p(f). \tag{10.161}$$

The following sharp *affine L_p Sobolev inequality* was proved in [1295].

Theorem 10.15.2 (Lutwak, Yang, Zhang) *Let $p \in (1, n)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{E}_p(f) \geq c\|f\|_{\frac{np}{n-p}}, \tag{10.162}$$

with an explicit optimal constant c.

Equality is attained when

$$f(x) = (a + |A(x - x_0)|^{p/(p-1)})^{1-(n/p)},$$

with $A \in \mathrm{GL}(n)$, $a > 0$ and $x_0 \in \mathbb{R}^n$. By (10.161), this affine-invariant inequality is stronger than the classical sharp L_p Sobolev inequality

$$\|\nabla f\|_p \geq c_0\|f\|_{\frac{np}{n-p}}, \tag{10.163}$$

where c_0 is a certain explicit constant.

A principle in this transfer from geometric to analytic inequalities, either in the background or used explicitly, is the association of convex bodies with quite general functions. The following approach was developed by Lutwak, Yang and Zhang [1300]. Let f be a function of the Sobolev space $W^{1,1}(\mathbb{R}^n)$. By the Riesz representation theorem, there exists a unique finite Borel measure $S(f, \cdot)$ on $\mathbb{S}^{n-1}$ such that

$$\int_{\mathbb{R}^n} g(\nabla f(x))\,\mathrm{d}x = \int_{\mathbb{S}^{n-1}} g(u)\,S(f, \mathrm{d}u)$$

for every nonnegative, continuous, 1-homogeneous function $g : \mathbb{R}^n \to \mathbb{R}$. If f is not almost everywhere equal to zero, then $S(f, \cdot)$ is not concentrated on a great subsphere. By Minkowski's existence theorem, the even part of $S(f, \cdot)$ is the surface area measure of a unique o-symmetric convex body, which is denoted by $\langle f \rangle$ and (following Ludwig [1255]) called the *LYZ body* of f. Thus, this body is characterized by

$$\int_{\mathbb{R}^n} g(\nabla f(x))\,\mathrm{d}x = \int_{\mathbb{S}^{n-1}} g(u)\,S_{n-1}(\langle f \rangle, \mathrm{d}u)$$

for every even, nonnegative, continuous, 1-homogeneous function $g : \mathbb{R}^n \to \mathbb{R}$. It follows that the projection body of $\langle f \rangle$ is given by

$$h(\Pi\langle f \rangle, u) = \frac{1}{2}\int_{\mathbb{R}^n} |\langle u, \nabla f(x) \rangle|\,\mathrm{d}x \qquad \text{for } u \in \mathbb{R}^n.$$

The definition of $\langle f \rangle$ has been extended by Wang [1908] to functions f in the space $BV(\mathbb{R}^n)$ (see, e.g., Evans and Gariepy [538]), and then $\Pi\langle \mathbf{1}_K \rangle = \Pi K$ for $K \in \mathcal{K}^n_n$.

The classical Sobolev inequality (10.159) can now be written as

$$w(\Pi\langle f \rangle) \geq c\|f\|_{\frac{n}{n-1}}$$

with a suitable constant c, where w is the mean width. The Sobolev–Zhang inequality (10.160) can be written as

$$V_n(\Pi^\circ\langle f\rangle)\,\|f\|^n_{\frac{n}{n-1}} \le \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n.$$

For $f = \mathbf{1}_K$, $K \in \mathcal{K}^n_n$, this becomes

$$V_n(\Pi^\circ K)V_n(K)^{n-1} \le \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n,$$

which is the Petty projection inequality (10.86) again.

In the Sobolev inequality (10.159), the Euclidean norm of the gradient can be replaced by any norm. Let $K \in \mathcal{K}^n_n$ be an o-symmetric convex body with $V_n(K) = \kappa_n$. Then, for $f \in W^{1,1}(\mathbb{R}^n)$, the inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^\circ}\,\mathrm{d}x \ge \kappa_n^{1/n}\|f\|_{\frac{n}{n-1}} \tag{10.164}$$

holds; see Gromov in [1432], Appendix I.2.1. Given a function $f \in W^{1,1}(\mathbb{R}^n)$, Lutwak, Yang and Zhang [1300] asked for the optimal norm, that is, for the norm ball K of volume κ_n that minimizes the left side of (10.164). They found that this is precisely a dilatate of the body $\langle f\rangle$.

Lutwak, Yang and Zhang [1300] also extended the construction of the convex body $\langle f\rangle$ to the L_p Brunn–Minkowski theory. Using the solution of the normalized even L_p Minkowski problem, they proved the following. If $1 \le p < \infty$ and $f \in W^{1,p}(\mathbb{R}^n)$, there exists a unique o-symmetric convex body $\langle f\rangle_p$ such that

$$\frac{1}{n}\int_{\mathbb{R}^n} g(\nabla f(x))^p \mathrm{d}x = \frac{1}{V_n(K)}\int_{\mathbb{S}^{n-1}} g(u)^p\, S_{p,0}(\langle f\rangle_p, \mathrm{d}u)$$

for every even, continuous, 1-homogeneous function $g : \mathbb{R}^n \to \mathbb{R}$. As a consequence,

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{K^\circ}\,\mathrm{d}x = \frac{V_p(\langle f\rangle_p, K)}{V_n(\langle f\rangle_p)} \tag{10.165}$$

for every o-symmetric convex body K. Among all such bodies K with $V_n(K) = \kappa_n$, precisely the dilatates of $\langle f\rangle_p$ minimize the left side of (10.165). The authors used this to obtain sharp affine versions of Gagliardo–Nirenberg inequalities, and also to get simplified proofs of Theorems 10.15.1 and 10.15.2.

Another ingredient, besides convexity methods, in the proofs of the sharp affine Sobolev-type inequalities is a strengthened Pólya–Szegö principle for rearrangements. For $f \in W^{1,p}(\mathbb{R}^n)$, let

$$\mu_f(t) := \mathcal{H}^n(\{x \in \mathbb{R}^n : |f(x)| > t\}), \qquad f^*(s) := \sup\{t > 0 : \mu_f(t) > 0\}$$

for $s < \mu_f(0)$, and $f^*(s) = 0$ for $s \ge \mu_f(0)$. The *symmetric rearrangement* of f is the function $f^\star : \mathbb{R}^n \to [0,\infty]$ with

$$f^\star(x) = f^*(\kappa_n|x|^n).$$

For $p \geq 1$, the Pólya–Szegö principle states that $f^\star \in W^{1,p}(\mathbb{R}^n)$ and

$$\|\nabla f^\star\|_p \leq \|\nabla f\|_p.$$

The choice of the normalization constant for the L_p affine energy implies that

$$\mathcal{E}_p(f^\star) = \|\nabla f^\star\|_p. \tag{10.166}$$

The *affine Pólya–Szegö principle* asserts that

$$\mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f). \tag{10.167}$$

It was proved by Lutwak, Yang and Zhang [1295] for $1 \leq p < n$ and by Cianchi, Lutwak, Yang and Zhang [430] for all $p \geq 1$. Since $\mathcal{E}_p(f) \leq \|\nabla f\|_p$, as shown in [1295], it follows that (10.167) is stronger than the classical Pólya–Szegö principle. The main aim of [430] was to complete the picture of affine Sobolev-type inequalities by establishing, for $p = n$, a stronger affine version of the Moser–Trudinger inequality, and for $p > n$ a stronger affine version of the Morrey–Sobolev inequality, in both cases involving the corresponding affine energy.

Further strengthenings of the preceding results were achieved by Haberl and Schuster [882] and by Haberl, Schuster and Xiao [883]. They introduced the *asymmetric L_p affine energy* by

$$\mathcal{E}_p^+(f) := 2^{1/p} c_{n,p} \left(\int_{\mathbb{S}^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, \mathrm{d}u \right)^{-1/n}$$

for $f \in W^{1,p}(\mathbb{R}^n)$ and $p \geq 1$, where $\mathrm{D}_u^+ f(x) := \max\{\mathrm{D}_u f(x), 0\}$. In [882], it was shown that

$$\mathcal{E}_p^+(f) \geq c\|f\|_{\frac{np}{n-p}}, \tag{10.168}$$

with an explicit optimal constant c, and that

$$\mathcal{E}_p^+(f) \leq \mathcal{E}_p(f).$$

Thus, (10.168) is stronger than (10.162). In [883], it was proved that

$$\mathcal{E}_p^+(f^\star) \leq \mathcal{E}_p^+(f),$$

which is stronger than (10.167). Applications were made to strengthenings of the previously mentioned inequalites and also of others, such as logarithmic Sobolev inequalities. All these affine stronger forms involve asymmetric affine energies.

Geometric methods used in the preceding works, such as convexification of level sets by means of the L_p Minkowski problem, are also used in further strengthenings of Sobolev type inequalities, due to Xiao [1994] and Ludwig, Xiao and Zhang [1259].

Notes for Section 10.15

1. Bobkov and Ledoux [261] have presented a direct derivation of the classical sharp L_p Sobolev inequality (10.163) from the Brunn–Minkowski inequality for measurable sets.
2. Alonso-Gutiérrez, Bastero and Bernués [64] proved another sharp inequality of the type (10.168), with a modified right-hand side.

10.16 Characterization theorems

Previous chapters of this book contain various examples of characterization and classification theorems for geometric objects of very different types. The present section presents characterization results of a more unified nature, pertaining to the affine geometry of convex bodies. Simple behaviour under linear transformations and valuation properties with respect to additive structures are always an essential part of the assumptions. Significant characterization theorems centring on extensions of the affine surface area have already been quoted: Theorems 10.5.6, 10.5.7, 10.5.8 by Ludwig and Reitzner, and Theorem 10.5.9 by Haberl and Parapatits. Much of the following theory owes its existence to the work of Ludwig.

There is no space here to sketch any proofs, which in general are intricate. Although the assumption of simple behaviour under the general linear group is strong enough to allow, together with the other assumptions, the characterization of very specific geometric objects, the identification of these objects requires hard work and original geometric ideas.

The first four theorems to be quoted characterize certain valuations on the set $\mathcal{P}^n_{(o)}$ of convex polytopes containing the origin in the interior. The formulation of the first one, due to Ludwig, is equivalent to that in [1246], although the present formulation is taken from [1247]. 'Measurable' for mappings on $\mathcal{P}^n_{(o)}$ means here that the pre-image of an open set is always a Borel set.

Theorem 10.16.1 (Ludwig) *A function $\zeta : \mathcal{P}^n_{(o)} \to \mathbb{R}$ is a measurable valuation with the property that*

$$\zeta(\phi P) = |\det \phi|^q \zeta(P)$$

for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$\zeta(P) = c \quad \text{or} \quad \zeta(P) = cV_n(P) \quad \text{or} \quad \zeta(P) = cV_n(P^\circ)$$

for every $P \in \mathcal{P}^n_{(o)}$.

The example of the affine surface area shows that $\mathcal{P}^n_{(o)}$ cannot be replaced by $\mathcal{K}^n_{(o)}$ in this theorem. In [1246] there is also a counterpart, where the assumption of measurability is replaced by nonnegativity.

The following theorem of Haberl and Parapatits [880] is in the same spirit.

Theorem 10.16.2 (Haberl, Parapatits) *A function $\zeta : \mathcal{P}^n_{(o)} \to \mathbb{R}$ is an upper semi-continuous and $\mathrm{SL}(n)$ invariant valuation if and only if there exist constants $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$\zeta(P) = c_0 V_0(P) + c_1 V_n(P) + c_2 V_n(P^\circ)$$

for every $P \in \mathcal{P}^n_{(o)}$.

The following result of Ludwig [1247] is an analogue of Theorem 10.16.1 for vector-valued valuations. The moment vector of K, defined in Subsection 5.4.1 and there denoted by $z_{n+1}(K)$, is now denoted by $m(K)$, thus $m(K) = \int_K x \, \mathrm{d}x$.

Theorem 10.16.3 (Ludwig) *A function $z : \mathcal{P}^n_{(o)} \to \mathbb{R}^n$ is a measurable valuation with the property that*

$$z(\phi P) = |\det \phi|^q \phi z(P)$$

for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = cm(P)$$

for every $P \in \mathcal{P}^n_{(o)}$.

A function $z : \mathcal{P}^n_{(o)} \to \mathbb{R}^n$ is a measurable valuation with the property that

$$z(\phi P) = |\det \phi^{-t}|^q \phi^{-t} z(P)$$

for every $\phi \in \mathrm{GL}(n)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = cm(P^\circ)$$

for every $P \in \mathcal{P}^n_{(o)}$.

The next step is an analogous characterization of certain matrix-valued valuations, namely the moment matrix operator M_2 and the LYZ matrix operator M_{-2}, defined by (10.116) and (10.125), respectively. According to Section 10.12, $M_2(K)$ is closely related to the ellipsoid $\Gamma_2 K$, a dilatate of the Legendre ellipsoid of K, and $M_{-2}(K)$ is closely related to the LYZ ellipsoid $\Gamma_{-2}K$. That M_2 is a valuation is clear from (10.116), and that M_{-2} is a valuation follows from (10.125) and Lemma 4.2.6. By $\mathscr{M}^n$ we denote the space of real symmetric $n \times n$ matrices. The operation of $\mathrm{GL}(n)$ on $\mathscr{M}^n$ is the usual one, after identifying $\phi \in \mathrm{GL}(n)$ with its matrix with respect to the standard orthonormal basis of $\mathbb{R}^n$.

We restrict ourselves in the following quotation from Ludwig [1248] to $n \geq 3$ and refer to [1248] for the modifications that are necessary for $n = 2$.

Theorem 10.16.4 (Ludwig) *A function $Z : \mathcal{P}^n_{(o)} \to \mathscr{M}^n$, $n \geq 3$, is a measurable valuation such that*

$$Z(\phi P) = |\det \phi|^q \phi Z(P) \phi^t$$

holds for every $\phi \in \mathrm{GL}(n)$ *with* $q \in \mathbb{R}$ *if and only if there is a constant* $c \in \mathbb{R}$ *such that*

$$Z(P) = cM_2(P) \quad or \quad Z(P) = cM_{-2}(P^\circ)$$

for every $P \in \mathcal{P}^n_{(o)}$.

A function $Z : \mathcal{P}^n_{(o)} \to \mathcal{M}^n$, $n \geq 3$, *is a measurable valuation such that*

$$Z(\phi P) = |\det \phi^{-t}|^q \phi^{-1} Z(P) \phi^{-1}$$

holds for every $\phi \in \mathrm{GL}(n)$ *with* $q \in \mathbb{R}$ *if and only if there is a constant* $c \in \mathbb{R}$ *such that*

$$Z(P) = cM_2(P^\circ) \quad or \quad Z(P) = cM_{-2}(P)$$

for every $P \in \mathcal{P}^n_{(o)}$.

With additional continuity assumptions, the results of the previous three theorems lead to characterization theorems for valuations on $\mathcal{K}^n_{(o)}$.

The body-valued operators introduced in previous sections all exhibit simple transformation behaviour under $\mathrm{GL}(n)$. A mapping Z from a suitable subset of $\mathcal{K}^n$ into such a subset is called $\mathrm{GL}(n)$ *covariant* if

$$Z(\phi K) = |\det \phi|^q \phi Z K,$$

and $\mathrm{GL}(n)$ *contravariant* if

$$Z(\phi K) = |\det \phi|^q \phi^{-t} Z K,$$

with some $q \in \mathbb{R}$, for every $\phi \in \mathrm{GL}(n)$ and all K in the domain of Z. If this holds for all $\phi \in \mathrm{SL}(n)$ (where $\det \phi = 1$) instead of $\phi \in \mathrm{GL}(n)$, the mapping Z is called $\mathrm{SL}(n)$ *covariant*, respectively $\mathrm{SL}(n)$ *contravariant*. Valuations with such a transformation behaviour are strong candidates for characterizations. The valuation property may refer to different additive structures. First we consider Minkowski valuations; recall that these are mappings into $\mathcal{K}^n$ which are valuations with respect to Minkowski addition.

We begin with characterizations of the projection body operator (see also [1733], where rotation equivariance instead of $\mathrm{SL}(n)$ contravariance is assumed; see Note 12 of Section 6.4). Confirming a conjecture of Lutwak, Ludwig [1245] obtained a characterization of the projection body operator by its affine invariance and valuation properties. In Ludwig [1249], this was strengthened to the following result.

Theorem 10.16.5 (Ludwig) *If* $Z : \mathcal{P}^n \to \mathcal{K}^n$, $n \geq 2$, *is a translation invariant,* $\mathrm{SL}(n)$ *contravariant Minkowski valuation, then there is a constant* $c \geq 0$ *such that* $Z = c\Pi$.

In [1249], a general classification result for homogeneous, $\mathrm{SL}(n)$ contravariant Minkowski valuations on $\mathcal{P}^n_o$ was obtained, from which the preceding result was deduced. Also deduced was a result on continuous valuations on $\mathcal{K}^n_o$. Haberl [876] was able to remove the homogeneity assumption from that theorem, so that now the following beautiful characterization can be stated.

Theorem 10.16.6 (Ludwig, Haberl) *If $Z : \mathcal{K}^n_o \to \mathcal{K}^n$, $n \geq 3$, is a continuous,* SL(n) *contravariant Minkowski valuation, then there is a constant $a \geq 0$ such that $ZK = a\Pi K$ for every $K \in \mathcal{K}^n_o$.*

Also proved in Haberl [876] are characterization theorems for valuations on $\mathcal{P}^n_o$, without continuity assumptions.

Turning to SL(n) covariant Minkowski valuations, we recall that the moment body operator M was defined by (10.65). The following is due to Ludwig [1249].

Theorem 10.16.7 (Ludwig) *Let $Z : \mathcal{P}^n_o \to \mathcal{K}^n$, $n \geq 3$, be a Minkowski valuation which is* SL(n) *covariant and homogeneous of degree r. If $r = n + 1$, then there are constants $a_0 \in \mathbb{R}$, $a_1 \geq 0$ such that*

$$ZP = a_0 m(P) + a_1 \mathrm{M}P$$

for every $P \in \mathcal{P}^n_o$. If $r = 1$, there are constants $a, b \geq 0$ such that

$$ZP = aP + b(-P)$$

for every $P \in \mathcal{P}^n_o$. In all other cases, $ZP = \{o\}$ for every $P \in \mathcal{P}^n_o$.

We refer to [1249] for the modified version of this theorem that holds for $n = 2$. As a corollary, the following result is obtained. If $Z : \mathcal{K}^n_o \to \mathcal{K}^n$ is a continuous, homogeneous, SL(n) covariant Minkowski valuation, then there are constants $a_0 \in \mathbb{R}$, $a_1, a_2 \geq 0$ such that

$$ZK = a_0 m(K) + a_1 \mathrm{M}K \quad \text{or} \quad ZK = a_1 K + a_2(-K)$$

for every $K \in \mathcal{K}^n_o$.

The following result of Haberl [876] does not require a homogeneity assumption. Recall that the asymmetric L_p moment bodies $\mathrm{M}^+_p K$, $\mathrm{M}^-_p K$ were defined by (10.76).

Theorem 10.16.8 (Haberl) *A map $Z : \mathcal{K}^n_o \to \mathcal{K}^n_o$ is a continuous,* SL(n) *covariant Minkowski valuation if and only if there exist nonnegative constants $c_1, \dots, c_4$ such that*

$$ZK = c_1 K + c_2(-K) + c_3 \mathrm{M}^+_1 K + c_4 \mathrm{M}^+_1(-K)$$

for every $K \in \mathcal{K}^n_o$.

The extension of some of these results to the L_p Brunn–Minkowski theory is also treated by Ludwig [1249]. For $p > 1$, an operator $Z : \mathcal{K}^n_o \to \mathcal{K}^n_o$ is called an L_p *Minkowski valuation* if

$$Z(K_1 \cup K_2) +_p Z(K_1 \cap K_2) = ZK_1 +_p ZK_2$$

whenever $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n_o$. Since the L_p Brunn–Minkowski theory for $p > 1$ is not translation invariant, the moment body and the projection body operator have to be extended not only to their L_p versions but also to asymmetric L_p versions. It was through the subsequent characterization results of Ludwig [1249] that the necessity of considering asymmetric versions of moment and projection bodies first arose; later they found their place in strengthenings of geometric and analytic inequalities, as described at appropriate places in previous sections.

Theorem 10.16.9 (Ludwig) *Let $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$, $n \geq 3$, be an L_p Minkowski valuation, $p > 1$, which is* SL(n) *covariant and homogeneous of degree r. If $r = (n/p) + 1$, then there are constants $c_1, c_2 \geq 0$ such that*

$$ZP = c_1 \mathrm{M}_p^+ P \ +_p \ c_2 \mathrm{M}_p^- P$$

for every $P \in \mathcal{P}_o^n$. If $r = 1$, then there are constants $a, b \geq 0$ such that

$$ZP = aP \ +_p \ b(-P)$$

for every $P \in \mathcal{P}_o^n$. In all other cases, $ZP = \{o\}$ for every $P \in \mathcal{P}_o^n$.

Again, there is a different two-dimensional version; see [1249].

Ludwig's counterpart to Theorem 10.16.9 for SL(n) contravariant L_p Minkowski valuations requires a further modification of the asymmetric L_p projection bodies $\Pi_p^+ K$, $\Pi_p^- K$ defined by (10.97). For this, let

$$h(\hat{\Pi}_p^+ K, u)^p := a_{n,p} \int_{\mathbb{S}^{n-1} \setminus \omega_0(K)} \langle u, v \rangle_+^p \, \mathrm{d}S_{p,0}(K, v), \quad u \in \mathbb{S}^{n-1},$$

where $\omega_0(K)$ is the set of outer unit normal vectors of facets of K that contain o. Similarly, $\hat{\Pi}_p^-$ is defined, with $\langle u, v \rangle_+$ replaced by $\langle u, v \rangle_-$ in the definition. We quote a result of Parapatits [1517], which strengthens a result of Ludwig [1249] by avoiding a homogeneity assumption.

Theorem 10.16.10 (Parapatits) *A mapping $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$, $n \geq 3$, is an* SL(n) *contravariant L_p Minkowski valuation, $p > 1$, if and only if there exist constants $c_1, c_2 \geq 0$ such that*

$$ZP = c_1 \hat{\Pi}_p^+ P \ +_p \ c_2 \hat{\Pi}_p^- P$$

for every $P \in \mathcal{P}_o^n$.

Restricting, in the preceding theorems, the domains of the Minkowski valuations to convex bodies containing the origin in the interior makes significant differences, as the following theorem of Ludwig [1252] shows. Here a mapping Z is called *trivial* if it is a linear combination of the identity and central reflection.

Theorem 10.16.11 (Ludwig) *A mapping $Z : \mathcal{K}_{(o)}^n \to \mathcal{K}^n$, $n \geq 3$, is a continuous, non-trivial* GL(n) *covariant Minkowski valuation if and only if either there are constants $c_0 \geq 0$ and $c_1 \in \mathbb{R}$ such that*

$$ZK = c_0 \mathrm{M}K + c_1 m(K)$$

for every $K \in \mathcal{K}^n_{(o)}$, *or there is a constant* $c_0 \geq 0$ *such that*

$$ZK = c_0 \Pi K^\circ$$

for every $K \in \mathcal{K}^n_{(o)}$.

As an application, the author obtained a characterization of the Holmes–Thompson area among all invariant areas in an n-dimensional Minkowski space (with a not necessarily symmetric norm). Here, invariant areas are considered as functions on $\mathcal{K}^n \times \mathcal{K}^n_{(o)}$ (where the elements of $\mathcal{K}^n_{(o)}$ play the role of the unit balls of norms), satisfying a natural set of conditions. The Holmes–Thompson area turns out to be the unique (up to a factor) invariant area that is a valuation in each of its arguments.

On the other hand, admitting all of $\mathcal{K}^n$ as the domain of a Minkowski valuation, without assuming any behaviour with respect to translations, changes the picture again. With the definition

$$K_o := \operatorname{conv}(K \cup \{o\}) \qquad \text{for } K \in \mathcal{K}^n,$$

the following result of Schuster and Wannerer [1750] holds for $\mathrm{GL}(n)$ contravariant Minkowski valuations.

Theorem 10.16.12 (Schuster, Wannerer) *A mapping* $Z : \mathcal{K}^n \to \mathcal{K}^n$ *is a* $\mathrm{GL}(n)$ *contravariant, continuous Minkowski valuation if and only if there exist constants* $c_1, c_2 \geq 0$ *such that*

$$ZK = c_1 \Pi K + c_2 \Pi K_o$$

for every $K \in \mathcal{K}^n$.

Parapatits [1517] proved a counterpart to this theorem for $\mathrm{SL}(n)$ contravariant L_p Minkowski valuations on $\mathcal{P}^n$.

To treat $\mathrm{GL}(n)$ covariant Minkowski valuations, define m_*, M_* by

$$m_*(K) := \int_{K_o \setminus K} x \, \mathrm{d}x, \qquad h(\mathrm{M}_* K, u) := \int_{K_o \setminus K} |\langle u, x \rangle| \, \mathrm{d}x$$

for $K \in \mathcal{K}^n$. The following is due to Wannerer [1923].

Theorem 10.16.13 (Wannerer) *A mapping* $Z : \mathcal{K}^n \to \mathcal{K}^n$ *is a* $\mathrm{GL}(n)$ *covariant, continuous Minkowski valuation if and only if either there are constants* $a_1, \dots, a_4 \geq 0$ *such that*

$$ZK = a_1 K + a_2(-K) + a_3 K_o + a_4(-K_o)$$

for every $K \in \mathcal{K}^n$, *or there are constants* $a_1, a_2 \in \mathbb{R}$ *and* $a_3, a_4 \geq 0$ *such that*

$$ZK = a_1 m(K) + a_2 m_*(K) + a_3 \mathrm{M} K + a_4 \mathrm{M}_* K$$

for every $K \in \mathcal{K}^n$.

In the L_p dual Brunn–Minkowski theory, Minkowski addition is replaced by L_p radial addition. Therefore, a mapping Z from some intersectional subclass of $\mathcal{S}_o^n$ into $\mathcal{S}_o^n$ is an L_p *radial valuation* (a *radial valuation* if $p = 1$) if

$$Z(K_1 \cup K_2) \widetilde{+}_p Z(K_1 \cap K_2) = Z(K_1) \widetilde{+}_p Z(K_2)$$

whenever $K_1, K_2, K_1 \cup K_2$ are in the domain of Z. The intersection body operator is such a radial valuation. In the following characterization, which is due to Ludwig [1250], an operator is called *trivial* if it is a radial linear combination of the identity and central reflection.

Theorem 10.16.14 (Ludwig) *A mapping $Z : \mathcal{P}^n_{(o)} \to \mathcal{S}^n_o$ is a non-trivial* $\mathrm{GL}(n)$ *covariant radial valuation if and only if there is a constant $c \geq 0$ such that*

$$ZP = c\,\mathrm{I}P^\circ$$

for every $P \in \mathcal{P}^n_{(o)}$.

The extension to the L_p dual theory for $0 < p < 1$ was carried out by Haberl and Ludwig [877]. Here the asymmetric L_p intersection bodies $\mathrm{I}_p^+, \mathrm{I}_p^-$, defined by (10.106), enter the scene.

Theorem 10.16.15 (Haberl, Ludwig) *For $0 < p < 1$, a mapping $Z : \mathcal{P}^n_{(o)} \to \mathcal{S}^n_o$, $n \geq 3$, is a non-trivial* $\mathrm{GL}(n)$ *covariant L_p radial valuation if and only if there are constants $c_1, c_2 \geq 0$ such that*

$$ZP = c_1 \mathrm{I}_p^+ P^\circ \widetilde{+}_p c_2 \mathrm{I}_p^- P^\circ$$

for every $P \in \mathcal{P}^n_{(o)}$. For $p > 1$, all $\mathrm{GL}(n)$ *covariant L_p radial valuations $Z : \mathcal{P}^n_{(o)} \to \mathcal{S}^n_o$ are trivial.*

Also proved in [877] is a counterpart to this theorem, in which the range of Z is replaced by the set of o-symmetric star bodies, and a characterization of the L_p intersection operator is obtained. For both cases, complete solutions for $n = 2$ are also contained in [877].

Valuations with respect to still other additive structures were classified by Haberl. In [874], such an addition is the harmonic p-sum for $p \neq 0$. For $n \geq 3$, the corresponding valuations from $\mathcal{P}_o^n$ to $\mathcal{S}_o^n$, which are linearly intertwining (that is, $\mathrm{SL}(n)$ covariant or contravariant and positively homogeneous) are determined. The non-trivial ones are related to the asymmetric L_p intersection body operators. In [875], Haberl considers mappings Z from $\mathcal{K}_n^n \cap \mathcal{K}_o^n$ to the o-symmetric bodies in this space which are valuations with respect to Blaschke addition. It is proved, for $n \geq 3$, that a continuous map of this kind which is linearly intertwining is either trivial (a multiple of Blaschke symmetrization) or of the form $K = c\Lambda_c K$, where $\Lambda_c K$ is the *symmetric curvature image* of K; this is the o-symmetric convex body with curvature function $f(K, \cdot) = \frac{1}{2}\rho(K, \cdot)^{n+1} + \frac{1}{2}\rho(-K, \cdot)^{n+1}$.

After body-valued valuations, we consider measure-valued valuations. The characterizations of curvature measures and area measures listed in Note 11 of Section 4.2 all belong to Euclidean geometry. In contrast, the surface area measure $S_{n-1}(K,\cdot)$ also has its place in affine geometry, as follows from the affine character of the mixed volume. According to (5.24), for $K, L \in \mathcal{K}^n$ and $\phi \in \mathrm{GL}(n)$ we have

$$\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}S_{n-1}(\phi K,\cdot) = |\det\phi| \int_{\mathbb{S}^{n-1}} f\circ\phi^{-t}\,\mathrm{d}S_{n-1}(K,\cdot)$$

for any continuous, positively homogeneous function $f : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}$. Generally, for $p \in \mathbb{R}$ a map μ from a subset of $\mathcal{K}^n$ to the set $\mathcal{M}^+(\mathbb{S}^{n-1})$ of finite (positive) Borel measures on $\mathbb{S}^{n-1}$ is called $\mathrm{SL}(n)$ *contravariant of degree* $p \in \mathbb{R}$ if

$$\int_{\mathbb{S}^{n-1}} f \,\mathrm{d}\mu(\phi K,\cdot) = \det\phi \int_{\mathbb{S}^{n-1}} f\circ\phi^{-t}\,\mathrm{d}\mu(K,\cdot)$$

holds for all $\phi \in \mathrm{SL}(n)$, every K in the domain of μ and every continuous p-homogeneous function $f : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}$. The L_p surface area measure $S_{p,o}$ is $\mathrm{SL}(n)$ contravariant of degree p. Haberl and Parapatits [879] made a thorough study of measure-valued valuations with this and similar properties. Of their many classification results, we quote the following two.

Theorem 10.16.16 (Haberl, Parapatits) *A map $\mu : \mathcal{K}^n \to \mathcal{M}^+(\mathbb{S}^{n-1})$ is a weakly continuous, translation invariant and* $\mathrm{SL}(n)$ *contravariant valuation of degree* 1 *if and only if there are constants $c_1, c_2 \geq 0$ such that*

$$\mu(K,\cdot) = c_1 S_{n-1}(K,\cdot) + c_2 S_{n-1}(-K,\cdot)$$

for every $K \in \mathcal{K}^n$.

Theorem 10.16.17 (Haberl, Parapatits) *Let $p > 0$, $p \neq 1$. A map $\mu : \mathcal{P}_o^n \to \mathcal{M}^+(\mathbb{S}^{n-1})$ is an* $\mathrm{SL}(n)$ *contravariant valuation of degree p if and only if there are constants $c_1, c_2 \geq 0$ such that*

$$\mu(P,\cdot) = c_1 S_{p,0}(P,\cdot) + c_2 S_{p,0}(-P,\cdot)$$

for every $P \in \mathcal{P}_o^n$.

Some further developments emerging from these characterizations are very promising, but as they take us farther away from the main theme of this book, we mention them only briefly. A mapping Z from a lattice $(\mathcal{L}, \vee, \wedge)$ of functions into an abelian semigroup with cancellation law is a valuation if

$$Z(f\vee g) + Z(f\wedge g) = Z(f) + Z(g)$$

for all $f, g \in \mathcal{L}$. In the following, $\vee$ denotes the pointwise maximum and $\wedge$ is the pointwise minimum. Ludwig [1254] characterized the Fisher information matrix on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ as the essentially unique continuous and affinely contravariant matrix-valued valuation. On the space $W^{1,1}(\mathbb{R}^n)$, Ludwig [1256] characterized the LYZ body operator $f \mapsto \langle f\rangle$ and its projection body operator $f \mapsto \Pi\langle f\rangle$

as essentially the only continuous maps into the space of o-symmetric convex bodies that have the correct behaviour under affine transformations and are valuations with respect to Blaschke addition, respectively Minkowski addition. This was extended by Wang [1924] to the space $BV(\mathbb{R}^n)$, where the notion of valuation has to be modified suitably. Real-valued and body-valued valuations on L^p spaces were investigated by Tsang [1854, 1855], and Kone [1143] studied valuations on Orlicz spaces.

Minkowski valuations on complex vector spaces, with specific transformation behaviour under the complex special linear group, were investigated by Abardia and Bernig [2] and by Abardia [1]. In particular, they obtained characterizations of complex counterparts of the projection body and the difference body.

Note for Section 10.16

1. A survey (up to 2006, roughly) on valuations in the affine geometry of convex bodies was given by Ludwig [1251], and a survey on valuations on function spaces by Ludwig [1255].

Appendix

Spherical harmonics

In several places in the geometry of convex bodies, spherical harmonics prove quite useful. In the present book, they are applied in Sections 3.3, 3.4, 3.5, 7.6 and 8.3, and some of the notes describe results that were obtained by means of this tool. In this appendix, we briefly collect, without proofs, the basic facts about spherical harmonics as they are needed in this book. For introductory material, including proofs, we refer to Müller [1454], Seeley [1764] and the literature quoted there; a more recent introduction is Atkinson and Han [98]. For the connections with representations of the rotation group we refer also to Coifman and Weiss [433]. For a more comprehensive view of the applications of Fourier series and spherical harmonics to the geometry of convex bodies, the reader is referred to the book by Groemer [800].

A *spherical harmonic of degree m* on the sphere $\mathbb{S}^{n-1}$ is, by definition, the restriction to $\mathbb{S}^{n-1}$ of a homogeneous polynomial f (with respect to Cartesian coordinates) of degree m (or identically zero) on $\mathbb{R}^n$ that satisfies $\Delta f = 0$; here Δ denotes the Laplace operator. The set $\mathscr{S}^m$ of spherical harmonics of degree m is a vector subspace of $C(\mathbb{S}^{n-1})$ of dimension

$$\dim \mathscr{S}^m = N(n,m) = \frac{(2m+n-2)\Gamma(n+m-2)}{\Gamma(m+1)\Gamma(n-1)}. \tag{A.1}$$

As a consequence of their definition, the spherical harmonics are eigenfunctions of the spherical Laplace operator Δ_S; each $Y_m \in \mathscr{S}_m$ satisfies

$$\Delta_S Y_m = -m(m+n-2)Y_m. \tag{A.2}$$

By $L_2(\mathbb{S}^{n-1})$ we denote the Hilbert space of square-integrable real functions on $\mathbb{S}^{n-1}$ (with the usual identification of functions coinciding almost everywhere), with scalar product $(\cdot,\cdot)$ defined by

$$(f,g) := \int_{\mathbb{S}^{n-1}} fg \, \mathrm{d}\sigma,$$

where σ denotes spherical Lebesgue measure on $\mathbb{S}^{n-1}$. The induced L_2-norm is denoted by $\|\cdot\|_2$. Spherical harmonics of different degrees are orthogonal, that is,

$$(f, g) = 0 \quad \text{for } f \in \mathscr{S}^k,\ g \in \mathscr{S}^j,\ k \neq j.$$

The system of spherical harmonics is complete, thus

$$(f, Y) = 0 \quad \text{for all } Y \in \mathscr{S}^m \text{ and all } m \in \mathbb{N}_0$$

for some $f \in L_2(\mathbb{S}^{n-1})$ implies $f = 0$ (almost everywhere). Every continuous real function on $\mathbb{S}^{n-1}$ can be approximated uniformly by finite linear combinations of spherical harmonics. Consequently, if

$$\int_{\mathbb{S}^{n-1}} Y \, \mathrm{d}\varphi = 0 \quad \text{for all } Y \in \mathscr{S}^m \text{ and all } m \in \mathbb{N}_0$$

for some finite signed Borel measure φ on $\mathbb{S}^{n-1}$, then $\varphi = 0$.

In each space $\mathscr{S}^m$ we choose an orthonormal basis $(Y_{m1}, \dots, Y_{mN(n,m)})$. Then $\{Y_{mj} : m \in \mathbb{N}_0,\ j = 1, \dots, N(n, m)\}$ is a complete orthonormal system in $L_2(\mathbb{S}^{n-1})$. For $f \in L_2(\mathbb{S}^{n-1})$ we write

$$\pi_m f := \sum_{j=1}^{N(n,m)} (f, Y_{mj}) Y_{mj}; \tag{A.3}$$

then $\pi_m f$ does not depend on the choice of the orthonormal basis, and the map $\pi_m : L_2(\mathbb{S}^{n-1}) \to \mathscr{S}^m$ is the orthogonal projection. If (A.3) holds for some $f \in L_2(\mathbb{S}^{n-1})$, we say that $\sum_{m=0}^{\infty} \pi_m f$ is the Fourier series of f, and we express this fact by writing

$$f \sim \sum_{m=0}^{\infty} \pi_m f.$$

The Fourier series of $f \in L_2(\mathbb{S}^{n-1})$ converges to f in the L_2-norm. The Parseval relation says that

$$\sum_{m=0}^{\infty} \|\pi_m f\|_2^2 = \|f\|_2^2, \tag{A.4}$$

or, more generally, that

$$\sum_{m=0}^{\infty} \sum_{j=1}^{N(n,m)} (f, Y_{mj})(g, Y_{mj}) = (f, g) \tag{A.5}$$

for $f, g \in L_2(\mathbb{S}^{n-1})$.

We remark that the projections π_0 and π_1 have a simple meaning. Since $\mathscr{S}^0$ contains only constant functions,

$$\pi_0 f = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} f d\sigma$$

is the mean value of f. Let $(e_1, \dots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$ and put

$$S_j(u) := \frac{1}{\sqrt{\kappa_n}} \langle e_j, u \rangle \quad \text{for } u \in \mathbb{S}^{n-1},\ j = 1, \dots, n.$$

Using (1.33), we get

$$(S_i, S_j) = \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} \langle e_i, u \rangle \langle e_j, u \rangle \, \mathrm{d}\sigma(u) = \langle e_i, e_j \rangle.$$

Since $\dim \mathscr{S}^1 = n$, we see that $(S_1, \dots, S_n)$ is an orthonormal basis of $\mathscr{S}^1$ and hence that

$$\begin{aligned}(\pi_1 f)(v) &= \sum_{j=1}^{n} (f, S_j) S_j(v) = \sum_{j=1}^{n} \int_{\mathbb{S}^{n-1}} f(u) \frac{1}{\sqrt{\kappa_n}} \langle e_j, u \rangle \, \mathrm{d}\sigma(u) \frac{1}{\sqrt{\kappa_n}} \langle e_j, v \rangle \\ &= \sum_{j=1}^{n} \langle e_j, s_f \rangle \langle e_j, v \rangle = \langle s_f, v \rangle\end{aligned}$$

with

$$s_f := \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} f(u) u \, \mathrm{d}\sigma(u).$$

Thus

$$\pi_1 f = \langle s_f, \cdot \rangle.$$

In particular, for the support function h_K of a convex body $K \in \mathcal{K}^n$ we have

$$\pi_0 h_K = \frac{1}{2} w(K), \tag{A.6}$$

$$\pi_1 h_K = \langle s(K), \cdot \rangle, \tag{A.7}$$

in view of (1.30) and (1.31).

If we require spherical harmonics in the form $Y(u) = f(\langle e, u \rangle)$ with fixed $e \in \mathbb{S}^{n-1}$, we are led to the Gegenbauer polynomials. Assuming that

$$\nu := \frac{n-2}{2} > 0,$$

the Gegenbauer polynomial C_m^ν of degree m and order ν can be defined by means of the generating function

$$(1 - tx + x^2)^{-\nu} = \sum_{m=0}^{\infty} C_m^\nu(t) x^m.$$

Explicitly,

$$C_m^\nu(t) = a_m^\nu (1 - t^2)^{-\nu + 1/2} \left(\frac{d}{dt} \right)^m (1 - t^2)^{m + \nu - 1/2}, \tag{A.8}$$

where

$$a_m^\nu = C_m^\nu(1)\left(-\frac{1}{2}\right)^m \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(m+\frac{n-1}{2}\right)}, \tag{A.9}$$

$$C_m^\nu(1) = \frac{\Gamma(m+n-2)}{\Gamma(n-2)\Gamma(m+1)}. \tag{A.10}$$

The function $C_m^\nu(\langle e,\cdot\rangle)$, where $e \in \mathbb{S}^{n-1}$ is fixed, is a spherical harmonic of degree m on $\mathbb{S}^{n-1}$.

The addition theorem says that

$$\sum_{j=1}^{N(n,m)} Y_{mj}(u)Y_{mj}(v) = \frac{N(n,m)}{\omega_n C_m^\nu(1)} C_m^\nu(\langle u,v\rangle) \tag{A.11}$$

for $u,v \in \mathbb{S}^{n-1}$ and $m \in \mathbb{N}_0$. As a consequence, one obtains that the maximum norm $\|Y_m\|$ of $Y_m \in \mathscr{S}^m$ can be estimated by

$$\|Y_m\| \le \left[\frac{N(n,m)}{\omega_n}\right]^{1/2} \|Y_m\|_2. \tag{A.12}$$

The following result is particularly useful.

Funk–Hecke theorem *Let $f \in L_2([-1,1])$ be a function satisfying*

$$\int_{-1}^1 |f(t)|(1-t^2)^{(n-3)/2}\,\mathrm{d}t < \infty.$$

If $Y_m \in \mathscr{S}^m$ is a spherical harmonic of degree m, then

$$\int_{\mathbb{S}^{n-1}} f(\langle u,v\rangle)Y_m(v)\,\mathrm{d}\sigma(v) = \lambda_m[f]Y_m(u) \tag{A.13}$$

for $u \in \mathbb{S}^{n-1}$, where

$$\lambda_m[f] = \omega_{n-1}[C_m^\nu(1)]^{-1} \int_{-1}^1 f(t)C_m^\nu(t)\left(1-t^2\right)^{(n-3)/2}\,\mathrm{d}t. \tag{A.14}$$

We add a remark on the development of sufficiently smooth support functions into series of spherical harmonics.

Let f be a real function of class C^∞ on $\mathbb{S}^{n-1}$ and let

$$f \sim \sum_{m=0}^\infty \sum_{j=1}^{N(n,m)} a_{mj}Y_{mj}, \tag{A.15}$$

with $a_{mj} = (f, Y_{mj})$, be its Fourier series with respect to the orthonormal system $\{Y_{mj}\}$. It follows from Theorems 4 and 5 of Seeley [1764] that

$$\mathrm{D}^\alpha f(x/|x|) = \sum_{m=0}^\infty \sum_{j=1}^{N(n,m)} a_{mj}\mathrm{D}^\alpha Y_{mj}(x/|x|), \tag{A.16}$$

where $\mathrm{D}^\alpha = \partial^{|\alpha|}/(\partial x_1)^{\alpha_1} \dots (\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and each of these series is uniformly convergent on $\mathbb{S}^{n-1}$.

Let g be a function of class C^2 on $\mathbb{S}^{n-1}$; let G be its positively homogeneous extension. By $\eta(g,x)$ we denote the smallest eigenvalue of the second differential of G at $x \in \mathbb{S}^{n-1}$, restricted to the orthogonal complement of x; and we define

$$\eta(g) := \min\left\{\eta(g,x) : x \in \mathbb{S}^{n-1}\right\}.$$

By Theorems 1.5.13 and 1.7.1, g is the restriction of a support function if and only if $\eta(g) \geq 0$.

Now let f be the support function of a convex body K of class C^∞_+. Then $\eta(f) > 0$ by Corollary 2.5.2. Consider the partial sum

$$f_k := \sum_{m=0}^{k} \sum_{j=1}^{N(n,m)} a_{mj} Y_{mj}$$

of (A.15). Using (A.16) and uniform convergence, we may show that $\eta(f_k) \to \eta(f)$ for $k \to \infty$, hence $\eta(f_k) > 0$ for almost all k. It follows that the partial sum f_k is the restriction of a support function for all sufficiently large k.

We conclude with a formula concerning integrations over the rotation group. The group $\mathrm{SO}(n)$ operates on the function space $C(\mathbb{S}^{n-1})$ by means of $(\vartheta f)(u) = f(\vartheta^{-1}u)$ for $u \in \mathbb{S}^{n-1}$, $\vartheta \in \mathrm{SO}(n)$. For $m \in \mathbb{N}_0$, we use the orthonormal basis $\{Y_{mj}\}$ of $\mathscr{S}^m$ introduced above. Since $\vartheta Y_{mj} \in \mathscr{S}^m$, it can be expressed in terms of the basis,

$$\vartheta Y_{mj} = \sum_{i=1}^{N(n,m)} t^m_{ij}(\vartheta) Y_{mi}, \tag{A.17}$$

with real coefficients $t^m_{ij}(\vartheta)$. This defines a matrix-valued mapping

$$\vartheta \mapsto \left(t^m_{ij}(\vartheta)\right)_{i,j=1}^{N(n,m)},$$

which is a continuous irreducible unitary representation of the rotation group. We refer, e.g., to Vilenkin [1871], also for the orthogonality relations used below, namely

$$N(n,m) \int_{\mathrm{SO}(n)} t^m_{ij} t^k_{rs} \, \mathrm{d}\nu = \delta_{mk}\delta_{ir}\delta_{js},$$

where δ is the Kronecker symbol.

We assert that for any function $f \in C(\mathbb{S}^{n-1})$ we have

$$\int_{\mathrm{SO}(n)} (\vartheta f)(u) t^m_{ij}(\vartheta) \, \mathrm{d}\nu(\vartheta) = \frac{1}{N(n,m)} (f, Y_{mj}) Y_{mi}(u) \tag{A.18}$$

for $u \in \mathbb{S}^{n-1}$, $m \in \mathbb{N}_0$ and $i, j = 1, \dots, N(n,m)$. For the proof of (A.18) (taken from Schneider [1720]), we assume first that f is a spherical harmonic, say $f = Y_{pq}$. Then

$$\begin{aligned}\int_{\mathrm{SO}(n)} (\vartheta Y_{pq})(u) t_{ij}^m(\vartheta)\, \mathrm{d}\nu(\vartheta) &= \int_{\mathrm{SO}(n)} \sum_{k=1}^{N(n,p)} t_{kq}^p(\vartheta) Y_{pk}(u) t_{ij}^m(\vartheta)\, \mathrm{d}\nu(\vartheta)\\ &= \sum_{k=1}^{N(n,p)} N(n,p)^{-1} \delta_{pm}\delta_{ki}\delta_{qj} Y_{pk}(u)\\ &= N(n,p)^{-1}\delta_{pm}\delta_{qj} Y_{mi}(u)\\ &= N(n,p)^{-1}(Y_{pq}, Y_{mj}) Y_{mi}(u).\end{aligned}$$

It follows that (A.18) holds if f is a finite sum of spherical harmonics. Since each $f \in C(\mathbb{S}^{n-1})$ can be approximated uniformly by finite sums of spherical harmonics, (A.18) follows.

Appendix Note

Applications to convex bodies. Spherical harmonics in space and Fourier series in the plane have been applied to a great variety of geometric problems on convex bodies. The first applications of this kind were Hurwitz's [1023] well-known proof of the isoperimetric inequality for closed (not necessarily convex) curves in the plane, the thorough investigation by Hurwitz [1024] of convex sets in the plane and in three-space and Minkowski's [1439] characterization of convex bodies of constant width. In the many different later contributions, two main principles of application are dominant: either the use of the Parseval relation to obtain quadratic inequalities, or the connection of spherical harmonics with irreducible representations of the rotation group. In particular, this often permits us to solve certain rotation invariant linear functional equations on the sphere by finding the spherical harmonics that satisfy the equation, and a number of geometric questions can be reduced to such equations.

Many authors have made geometric applications of Fourier series and spherical harmonics in the domain of convexity. Among them are Aleksandrov [13], Anikonov and Stepanov [73], Berg [198], Blaschke [237, 241, 240, 248, 251], Bol [267], Bourgain [313, 314], Bourgain and Lindenstrauss [315, 316, 318], Bourgain, Lindenstrauss and Milman [321], Campi [379, 380, 381, 382], Chernoff [421], Cieślak and Góźdź [431], Dinghas [478, 479], Falconer [545], Fillmore [582], Fisher [613], Focke [622], Fuglede [645, 646], Fujiwara [648, 650], Fujiwara and Kakeya [651], Fusco and Pratelli [656], Geppert [694], Gericke [696, 697, 698], Goodey and Groemer [737], Görtler [765, 766, 767], Groemer [795, 798], Hayashi [943, 944, 945, 946], Inzinger [1026], Kameneckii [1062], Knothe [1126], Kubota [1150, 1151, 1152, 1153, 1154], Matsumura [1363], Meissner [1400, 1401], Minoda [1442], Nakajima [1465], Oishi [1487], Petty [1525], Rešetnjak [1575], Sas [1637], Schneider [1656, 1663, 1664, 1665, 1666, 1670, 1677, 1684, 1700, 1712], Schneider and Schuster [1734], Shephard [1784], Su [1824, 1825].

Readers wanting a first introduction to the applications of Fourier series and spherical harmonics in convex geometry are referred to the survey articles by Groemer [801, 799]. Full details and much information are then found in Groemer's book [800].

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