Cohomology ring of the real Grassmannians with coefficients from $\mathbb{F}_2$

Contents

Introduction

In this thesis our ultimate goal is to study the $\mathbb{F}_2$ Cohomology ring of the real Grassmannians $\mathrm{Gr}_{\mathbb{R}}(k, n)$ and to examine a fascinating connection between this algebraic structure and the set of all real vector bundles of $\mathrm{Gr}_{\mathbb{R}}(k, n)$, which is a purely geometric structure. We mainly follow the book [JJ74] but we try to present the theory in a more detailed and concise way. The thesis is divided in three chapters. The first two chapters are going to build the language we need. They are dedicated, respectively, to the two sides of the coin, namely the topology and the geometry involved. Each one of these two chapters is going to be self-contained and thus a completely independent and sufficient read. This means in particular, that these two chapters do not aim at all towards the proof of the main theorem (this is stated and proved in the third chapter), but their goal is instead for the reader to build a stable entry point to the theory on Grassmannians and Vector bundles, respectively. Most importantly, this includes forming a sufficiently good intuition on the topics involved. Thus, we have enriched the theory with plenty of examples and helpful comments. On the other side, we have tried to remove hand-waving as much as possible of our proofs. This has of course made some of the proofs more precise than in the literature and thus lengthier. Our advice to the reader would be to skip some of them in the first read, since we always enclose the proofs inside more intuitive comments about the main arguments in the proof. The reader can either choose to start at Chapter 1 or Chapter 2, in accordance with her interests and her level of comfort.

In the first chapter we define the Grassmannians and immediately prove that they are indeed Manifolds. We also prove some first topological properties of them, in order to establish that they are "good" topological spaces which deserve our attention. Next, we introduce a very natural cell structure making them CW-manifolds. These cells are going to be named "Schubert cells". This cell decomposition plays a very important role in the study of the (co)homology of the Grassmannians. Namely, one can define a very interesting product on the set of the Schubert classes, which -you could already guess- are very close related to Schubert cells. One then speaks for a whole area called "Schubert calculus". We get involved with Schubert calculus just enough to convince the reader that this is a really deep and interesting area, but our approach of this topic is very introductory and not even close to be characterized as complete.

In the second chapter we define the vector bundles in general and prove some basic first results, mainly, in order to understand how different bundles can get combined to form new ones. The highlight of this chapter is going to be the notion of the "characteristic classes", already giving the reader an idea how one can bridge the gap between the set of all vector bundles of a space and its topology.

In the last chapter we are going to unleash the combined power we have been gathering in the first two chapters of this thesis in order to prove an amazing result, fully characterising the cohomology algebra of any Grassmannian. At this point we first examine and fully prove the infinite case and then we prove the general result involving the finite Grassmannians.

As far as we know, the results of the third chapter are well known and widely used in the community, but we have yet to see some written proof of the finite case, i.e. the last part of the thesis.

We hope that this over-analytic approach is going to help future students introduce themselves to such beautiful and deep mathematical concepts.

1 Grassmannians

Since the topology of projective spaces has been thoroughly studied and characterized, a logical generalization is imposing a natural topology on the set of k-dimensional subspaces of a vector space for any $k \geq 1$, which we are going call a *Grassmannian*.

So, the goal of this chapter is to present the basic properties of the Grassmann spaces, or Grassmannians. This chapter will be self-contained and used as a point of reference for the rest of the thesis. Our aim is for the reader to get to know the combinatorial structure of a Grassmannian and be able to eventually do basic computations using the Schubert decomposition of these manifolds. A good point to start would be with their definition.

Recall the definition of the real projective spaces as topological spaces:

$$P\mathbb{R}^n \cong \mathbb{R}^{n+1} \setminus \{0\} \big/ \sim$$

where two vectors are equivalent, if they span the same line. Notice that in this definition we need to take a proper subset of the whole vector space, in order for the quotient to be well defined. Namely we need the set of all vectors, which span a line. In accordance with that, for the case of the Grassmannians, we need $\mathrm{V}\left(k,n\right) \subsetneq \left(\mathbb{R}^n\right)^k$, the space of all k-tuples of vectors in $\mathbb{R}^n$, spanning a k-dimensional vector space (i.e. k-frames), as defined and discussed in Appendix A.

Definition 1.1. Let $0 < k < n$ be some natural numbers. Then, the real *Grassmann space* $\mathrm{Gr}_{\mathbb{R}}\left(k,n\right) = \mathrm{Gr}\left(k,n\right)$ is the set of all linear k-dimensional subspaces of $\mathbb{R}^n$, equipped with the following quotient topology:

$$\mathrm{Gr}_{\mathbb{R}}\left(k,n\right) := \mathrm{V}_{\mathbb{R}}\left(k,n\right) \big/ \sim$$

where $\left(\vec{v}_1, \ldots, \vec{v}_k\right) \sim \left(\vec{u}_1, \ldots, \vec{u}_k\right)$, if $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle = \langle \vec{u}_1, \ldots, \vec{u}_k \rangle$.

If the reader has still in mind the case of the projective spaces, it would be of no surprise that we are about to give an alternative definition of the Grassmannians. We know that the projective space of some dimension could alternatively be defined as the quotient over the unit sphere, rather than over the set of every nonzero vector. The analog of the set of all unit vectors would be here the Stiefel manifold $\mathrm{V}_0(k,n)$, which is the space of all *orthonormal* k-frames, as defined and discussed again in Appendix A.

Proposition 1.2. *Let $0 < k < n$ be some natural numbers. Let $q : \mathrm{V}(k,n) \to \mathrm{Gr}(k,n)$ be the quotient map used in the definition of the Grassmannians and let $q_0 := q|_{\mathrm{V}_0(k,n)} : \mathrm{V}_0(k,n) \to \mathrm{Gr}(k,n)$ be its restriction to the Stiefel manifold. Then q_0 is surjective and continuous. In other words, q_0 and q induce the same quotient topology on the set of all k-planes in $\mathbb{R}^n$.*

Proof. Let $i : \mathrm{V}_0(k,n) \to \mathrm{V}(k,n)$ be the inclusion and $\mathfrak{gs} : \mathrm{V}(k,n) \to \mathrm{V}_0(k,n)$ be the Gram-Schmidt process. Then, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{V}_0(k,n) \arrow[r, hook, "i"] \arrow[dr, "q_0"'] & \mathrm{V}(k,n) \arrow[r, two heads, "\mathfrak{gs}"] \arrow[d, "q"] & \mathrm{V}_0(k,n) \arrow[dl, "q_0"] \\
& \mathrm{Gr}(k,n) &
\end{tikzcd}$$

The left part of this diagram implies $q_0 = q \circ i$, which means in particular that q_0 is continuous. Moreover, the right part gives $q = q_0 \circ \mathfrak{gs}$, which gives us that q_0 is a surjective map as well, since q is surjective. Thus, the map q_0 is a quotient map. In other words, the maps q and q_0 induce the same quotient topology on the set of all k-dim subspaces of $\mathbb{R}^n$. □

1.1 Grassmannians are Manifolds

Our first real goal is to prove that the Grassmannians are in fact compact topological manifolds.

Lemma 1.3. *For each pair of natural numbers k, n, with $0 < k < n$, the space $\mathrm{Gr}(k,n)$ is compact, Hausdorff and locally homeomorphic to $\mathbb{R}^{k(n-k)}$.*

Proof.

- The Stiefel manifold $\mathrm{V}_0(k,n)$ is a compact topological space, as proven in the Appendix in Lemma A.7 and q_0 is a continuous map. Thus, $\mathrm{Gr}(k,n) = q_0(\mathrm{V}_0(k,n))$ is also compact.

- In order to show that $\mathrm{Gr}(k,n)$ is Hausdorff, it suffices to show that it is completely Hausdorff, i.e. that any two distinct points in $\mathrm{Gr}(k,n)$ can be separated by a continuous function $\mathrm{Gr}(k,n) \to \mathbb{R}$. First, for every vector $v \in \mathbb{R}^n$, we define the map $\phi_v : \mathrm{V}_0(k,n) \to \mathbb{R}$ which takes each orthonormal k-frame $(v_1,\ldots,v_k)$ to the square of the distance between v and the linear space spanned by $(v_1,\ldots,v_k)$, i.e.:

$$\phi_v(v_1,\ldots,v_k) = v \cdot v - \sum_{i=1}^{k} (v \cdot v_i)^2$$

  This is a continuous map, which depends only on the spanned k-plane, i.e. if two k-frames have the same image under q_0, they also have the same image under ϕ_v. The universal property of the quotient map q_0 implies that there exists a unique continuous map ψ_v making the following diagram commute:

$$\begin{tikzcd} \mathrm{V}_0(k,n) \arrow[d, "q_0"'] \arrow[dr, "\phi_v"] & \\ \mathrm{Gr}(k,n) \arrow[r, dotted, "\psi_v"'] & \mathbb{R} \end{tikzcd}$$

  Moreover, since $\phi_v(v_1,\ldots,v_k) = 0$ iff $v \in \langle v_1,\ldots,v_k\rangle$, we have that $\psi_v(H) = 0$ iff $v \in H$. Let now $H_1, H_2 \in \mathrm{Gr}(k,n)$ be two distinct k-planes and $v \in H_1 \setminus H_2$. Then, we get $\psi_v(H_1) = 0 \neq \psi_v(H_2)$, proving that $\mathrm{Gr}(k,n)$ is completely Hausdorff.

- We will prove that for every $H \in \mathrm{Gr}(k,n)$, the following subset of $\mathbb{R}^n$ is a neighborhood of H, homeomorphic to $\mathbb{R}^{k(n-k)}$:

$$\mathcal{U}_H := \left\{ K \in \mathrm{Gr}(k,n) : K \cap H^\perp = \{0\} \right\}$$

  First of all, one can fix an orthonormal basis $\{u_1,\ldots,u_k\}$ of H and an orthonormal basis $\{\bar{u}_1,\ldots,\bar{u}_{n-k}\}$ of $H^\perp$. It will be also convenient to regard $\mathbb{R}^n$ as the direct sum $H \oplus H^\perp$ for this proof. We also define the orthogonal projections $p_H, p_{H^\perp}$ from $H \oplus H^\perp$ to each component.

One can now consider $\mathcal{U}_H$ to be the set of all k-planes K in $H \oplus H^\perp$ for which the map $p_H|_K$ is a homeomorphism. Indeed, the subspace $K \cap H^\perp$ is at least one dimensional, if and only if $p_H|_K$ is not injective, if and only if $p_H|_K$ is not surjective.

Each $K \in \mathcal{U}_H$ can now be considered as the graph of some linear transformation T_K inside $\mathrm{Hom}_{\mathbb{R}}\left(H, H^\perp\right)$ (which has the desired dimension as a vector space over $\mathbb{R}$). Our goal is to define rigorously the function taking K to T_K and prove that this is bicontinuous. We urge the reader to convince herself at this point that T_K depends continuously on K and vice versa, because our approach towards proving this fact is going to be rather technical.

In particular, we are going to construct the desired function using the universal property of the quotient map $q : \mathrm{V}(k,n) \to \mathrm{Gr}(k,n)$. Moreover, since we need the topology of $\mathrm{Hom}_{\mathbb{R}}\left(H, H^\perp\right)$, we are going to use the fact that its topology is also an induced topology by an isomorphism to $\mathbb{R}^{k(n-k)}$. Finally, this last space, being a direct product, is also topologized through the initial topology with respect to the orthogonal projections on each coordinate. One can summarize these steps in the following commutative diagram:

$$
\begin{tikzcd}
 & & & (a_1,\dots,a_k) \arrow[rr, mapsto] & & \bar{u}_j^t \cdot p_{H^\perp} \circ \left(p_H|_{\langle a_1,\dots,a_k\rangle}\right)^{-1}(u_i) \\
(\mathbb{R}^n)^k \supseteq & \mathrm{V}(k,n) \arrow[d, "q"] & \supseteq & q^{-1}(\mathcal{U}_H) \arrow[rr, "f_{i,j}"'] \arrow[d, "q|"] \arrow[dr, dotted, "\tilde{T}"] \arrow[drr, dotted, "f"] & & \mathbb{R} \\
 & \mathrm{Gr}(k,n) & \supseteq & \mathcal{U}_H \arrow[r, dotted, "T"] & \mathrm{Hom}_{\mathbb{R}}\left(H, H^\perp\right) \arrow[r, "\phi", "\cong"'] & \mathbb{R}^{[k]\times[n-k]} \arrow[u, "\pi_{i,j}"'] \\
 & & & K \arrow[r, dotted, mapsto] & p_{H^\perp} \circ \left(p_H|_K\right)^{-1} &
\end{tikzcd}
$$

Remember that we are trying to define the map T taking each k-space K to the function with graph K. Thus, we first should define the map $\tilde{T}$ which is later going to be equal with $T \circ q|$. For now we just define it to be:

$$\tilde{T}(a_1,\dots,a_k) = p_{H^\perp} \circ \left(p_H|_{\langle a_1,\dots,a_k\rangle}\right)^{-1}$$

Notice that $\tilde{T}$ is well defined, since $q^{-1}(\mathcal{U}_H)$ is exactly the set of all k-frames, for which the map $p_H|_{\langle a_1,\dots,a_k\rangle}$ is a homeomorphism. Also, notice that $\tilde{T}$ is going to have the same value for any two k-frames that span the same k-space, since on input $(a_1, \ldots, a_k)$ the output only depends on $\langle a_1, \ldots, a_k \rangle$.

Next, we want to argue that $\tilde{T}$ is a continuous map. In order to do this, we need to remember how $\mathrm{Hom}_{\mathbb{R}}\left(H, H^{\perp}\right)$ is topologized. Having fixed the bases $\{u_i\}_{i\in[k]}$ and $\{\bar{u}_j\}_{j\in[n-k]}$, we can define an isomorphism of vector spaces ϕ, which takes a linear map L to

$$\phi(L) = \left(\bar{u}_j^t \cdot L(u_i)\right)_{(i,j)\in[k]\times[n-k]} \in \mathbb{R}^{[k]\times[n-k]}$$

Then, the natural topology on $\mathrm{Hom}_{\mathbb{R}}\left(H, H^{\perp}\right)$ is the induced topology by ϕ. This means in particular, that our $\tilde{T}$ is continuous if and only if $f := \phi \circ \tilde{T}$ is continuous.

Finally, since $\mathbb{R}^{[k]\times[n-k]} = \prod_{(i,j)\in[k]\times[n-k]} \mathbb{R}$ is a categorical product, $\mathbb{R}^{[k]\times[n-k]}$ is equipped with the initial topology, with respect to all orthogonal projections $(\pi_{i,j})_{(i,j)\in[k]\times[n-k]}$. This means in particular, that the function f that interests us is continuous if and only if every $f_{i,j} := \pi_{i,j} \circ f$ is continuous. In order to show that every $f_{i,j}$ is a continuous map, we have to write it down in the language of linear algebra:

First, define A to be the matrix of the linear function taking u_i to a_i for each $i \in [k]$, expressed in the bases $\{u_i\}$ and $\{\bar{u}_j\}$:

$$A = \left(\begin{array}{ccc} a_1^t \cdot u_1 & \cdots & a_k^t \cdot u_1 \\ \vdots & \ddots & \vdots \\ a_1^t \cdot u_k & \cdots & a_k^t \cdot u_k \\ \hline a_1^t \cdot \bar{u}_1 & \cdots & a_k^t \cdot \bar{u}_1 \\ \vdots & \ddots & \vdots \\ a_1^t \cdot \bar{u}_{n-k} & \cdots & a_k^t \cdot \bar{u}_{n-k} \end{array}\right) =: \left(\begin{array}{c} A_H \\ \hline A_{H^{\perp}} \end{array}\right) \in \mathbb{R}^{n\times k}$$

The two blocks $A_H \in \mathbb{R}^{k\times k}$ and $A_{H^{\perp}} \in \mathbb{R}^{(n-k)\times k}$ of A correspond to the maps taking u_i to the projections of a_i inside H and inside $H^{\perp}$ respectively.

Notice that the matrix corresponding to the linear map $p_{H^{\perp}}$, with regard to the bases we have fixed is

$$\left(0_{(n-k)\times k} | I_{n-k}\right) \in \mathbb{R}^{(n-k)\times n}$$

and the matrix corresponding to the linear map $\left(p_H|_{\langle a_1,\ldots,a_k\rangle}\right)^{-1}$, with regard to the same bases is

$$A \cdot A_H^{-1} = \left(\frac{I_k}{A_{H^\perp} \cdot A_H^{-1}} \right) \in \mathbb{R}^{n \times k}$$

Indeed, since $\langle a_1, \ldots, a_k \rangle \in \mathcal{U}_H$, we know that $\{p_H(a_1), \ldots, p_H(a_k)\}$ is a basis of K, which means that A_H is invertible and thus $A \cdot A_H^{-1}$ is well defined. Moreover, we can easily compute that the application of this matrix to any $p_H(a_r)$ gives us a_r which is exactly what the map $\left(p_H|_{\langle a_1, \ldots, a_k \rangle} \right)^{-1}$ does to the same basis, which proves our assertion.

This means, that the map f_{ij} takes the element $(a_1, \ldots, a_k)$ to the real number

$$\begin{aligned} f_{ij}(a_1, \ldots, a_k) &= \bar{u}_j^t \cdot p_{H^\perp} \circ \left(p_H|_{\langle a_1, \ldots, a_k \rangle} \right)^{-1} (u_i) \\ &= \bar{u}_j^t \cdot \left(0_{(n-k) \times k} \middle| I_{n-k} \right) \cdot \left(\frac{I_k}{A_{H^\perp} \cdot A_H^{-1}} \right) \cdot u_i \\ &= \bar{u}_j^t \cdot A_{H^\perp} \cdot A_H^{-1} \cdot u_i \end{aligned}$$

Since inner product, matrix multiplication and inversion are continuous, f_{ij} is a continuous function for every $i, j \in [k] \times [n-k]$. This proves that f is continuous as well, which proves that $\tilde{T}$ is also continuous. Since $\tilde{T}$ is a continuous function which depends only on the k-plane spanned by its input, the universal property of the quotient spaces ensures the existence of a continuous map $T : \mathcal{U}_H \to \mathrm{Hom}_{\mathbb{R}} \left(H, H^\perp \right)$ such that $T \circ q| = \tilde{T}$. The uniqueness part of this property ensures that the map T is the one taking K to $p_{H^\perp} \circ \left(p_H|_K \right)^{-1}$, as we wanted.

If we think again this function as taking K to the linear function, whose graph is K, we can easily see that this function is both one to one and onto, since two linear maps are different if and only if they have different graphs. This makes T a homeomorphism, proving that finally there exists the homeomorphism

$$\mathcal{U}_H \overset{\phi \circ T}{\cong} \mathbb{R}^{n(n-k)}$$

which finishes the proof of the lemma. □

In the next section we are going to further examine these spaces topologically and build the appropriate language in order to tackle problems regarding their Homology and Cohomology structures. Before we dive in into this topic though, it would be useful to notice a first duality between these spaces, arising from the duality between a k-space inside $\mathbb{R}^n$ and its $n-k$ complement.

We urge the reader now to get convinced that the space $K^\perp \in \mathrm{Gr}\,(n-k,n)$ depends continuously on $K \in \mathrm{Gr}\,(k,n)$, because in order to show this fact, we are going to use again the open sets $\mathcal{U}_H$ defined in the proof of the previous lemma.

Lemma 1.4. *For each pair of natural numbers k, n, with $0 < k < n$, the space $\mathrm{Gr}\,(k,n)$ is homeomorphic to $\mathrm{Gr}\,(n-k,n)$, with the homeomorphism taking some k-space to its orthogonal complement inside $\mathbb{R}^n$.*

Proof. The orthogonal-complement function $\perp\colon \mathrm{Gr}\,(k,n) \to \mathrm{Gr}\,(n-k,n)$ is obviously one to one and onto. Thus, it suffices to show that it is continuous. (Since $\left(K^\perp\right)^\perp = K$, for all spaces, continuity for every $0 < k < n$ implies bicontinuity.) We are going to prove first that for any $H \in \mathrm{Gr}\,(k,n)$ the restriction of this function in $\mathcal{U}_H$ is continuous. For this proof we are going to use the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{R}^n)^k \supseteq q_0^{-1}(\mathcal{U}_H) \arrow[r, "\mathfrak{gs}'"] \arrow[d, "q_0|"] \arrow[drr, dotted, "\tilde{\perp}|"] & \mathrm{V}_0(n,n) \arrow[r, "\pi_{[k+1,n]}"] & \mathrm{V}_0(n-k,n) \arrow[d, "q_0"] \\
\mathcal{U}_H \arrow[rr, dotted, "\perp|"] & & \mathrm{Gr}\,(n-k,n) \\
K \arrow[rr, mapsto] & & K^\perp
\end{tikzcd}$$

In this diagram, $\mathfrak{gs}'$ is the function that takes an orthonormal k-frame $(a_1, \ldots, a_k)$ to the orthonormal n-frame constructed after applying the Gram-Schmidt process to the n-basis $(a_1, \ldots, a_k, \bar{u}_1, \ldots, \bar{u}_{n-k})$ where $(\bar{u}_j)$ is an orthonormal basis of $H^\perp$, just like in the previous Lemma. The next map $\pi_{[k+1,n]}$ is just the orthogonal projection in the last $n-k$ coordinates. Both of these maps are continuous and well defined, and thus we get a continuous map $\tilde{\perp}\Big|$, like in the diagram. This map depends only on the plane spanned by the input, and thus the universal property of the quotient spaces ensures the continuity of the map $\perp|$.

After establishing the continuity of $\perp|_{\mathcal{U}_H}$ for every H, notice that

$$\mathrm{Gr}\,(k,n) = \bigcup_{H \in \left\{\langle B\rangle : B \in \binom{\{e_1,\ldots,e_n\}}{k}\right\}} \mathcal{U}_H$$

The union is over all k-planes spanned by any k vectors among $\{e_1, \ldots, e_n\}$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. This means that $\#\binom{\{e_1,\ldots,e_n\}}{k} = \binom{n}{k} < \infty$ sets are participating in the union and thus one can use the pasting lemma, proving that $\perp$ is continuous as a function from the whole space $\mathrm{Gr}\,(k,n)$ to $\mathrm{Gr}\,(n-k,n)$. $\square$

It would be helpful at this point to mention what are the "small" examples of Grassmannians. We already know that of course $\mathrm{Gr}\,(1,n) \cong \mathbb{P}^{n-1}$ and because of the last lemma we also know that $\mathrm{Gr}\,(n-1,n) \cong \mathbb{P}^{n-1}$. This already takes care of the cases $n = 2, 3$:

$$\mathrm{Gr}\,(1,2) \cong \mathbb{P}^1 \qquad \mathrm{Gr}\,(1,3) \cong \mathrm{Gr}\,(2,3) \cong \mathbb{P}^2$$

This forces us to always consider $\mathrm{Gr}\,(2,4)$ as the smallest non-trivial case in our further discussion.

1.2 They are also CW-Complexes

In the previous section we proved that the finite Grassmannians are compact topological manifolds. Our goal now is to prove that they are in fact finite CW complexes. For this, we need to define a cell decomposition of each Grassmannian, which we are going to do next. Before we start laying out the formal definition, it would be best for the reader to have in mind the analogous cell decomposition of the projective space $\mathbb{P}^{n-1} \cong \mathrm{Gr}\,(1,n)$, consisting of the following n cells:

$$\left\{l \subseteq \mathbb{R}^1\right\} \cong \mathbb{R}^0 \ , \ \left\{l \subseteq \mathbb{R}^2 \setminus \mathbb{R}^1\right\} \cong \mathbb{R}^1 \ , \ \ldots \ , \ \left\{l \subseteq \mathbb{R}^n \setminus \mathbb{R}^{n-1}\right\} \cong \mathbb{R}^{n-1}$$

This cell decomposition seems natural, but it depends heavily on our basis choice for $\mathbb{R}^n$. This fact does not bother us, since for a different choice we get essentially the same decomposition, in terms of the homology classes we are going to eventually compute. This freedom of choice is going to play an important role though, towards the end of this chapter, when we are going to use different decompositions (i.e. depending on different bases) in order to understand the multiplicative structure of the cohomology ring of the Grassmannians. Thus, we first need to define what flags are in an n-dimensional vector space.

Definition 1.5. Let V be an n-dimensional vector space over a field k. A *flag* $\mathbb{F}_\bullet$ for V is a sequence $(\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_n)$, such that $\dim_k \mathbb{F}_i = i$ for all $i \in \{0, 1, \ldots, n\}$ and:

$$0 = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{F}_n = V$$

Given a flag $\mathbb{F}_\bullet$ of V, denote an orthonormal basis $f_\bullet = (f_1, \dots, f_n)$ of V to be *compatible* with $\mathbb{F}_\bullet$, if

$$f_i \in \mathbb{F}_i$$

for every $i \in [n]$.

In the future we are going to use the non-standard notation $[n]_0$ to denote the set $[n] \cup \{0\} = \{0, 1, \dots, n\}$.

An obvious example of flag is the one we used above, namely the flag with $\mathbb{F}_i = \mathbb{R}_i = \langle e_1, \dots, e_i \rangle$. This is sometimes referred to as *standard flag*, but since we avoid to fix some basis of $\mathbb{R}^n$ in this section, we are going to treat every flag equally.

An obvious remark is that given a flag, one can always find a compatible basis with this flag. In fact, there always exist 2^n different compatible orthonormal bases and fixing one is like fixing an "orientation" of the flag.

Notice, that the role of the flags on our example above is to distinguish between all the different ways a line can intersect this flag. This is exactly the role a flag is going to play in the general definition of Schubert cells.

Definition 1.6. Let $k, n \in \mathbb{N}$ with $0 < k < n$. Moreover let $\mathbb{F}_\bullet$ be a flag of $\mathbb{R}^n$. Then, for each k-element subset $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\}$ of $[n]$ the *Schubert cell* $\mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)$ is defined to be the following subset of $\mathrm{Gr}(k, n)$:

$$\mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet) := \left\{ H \in \mathrm{Gr}(k,n) : \ \dim(H \cap \mathbb{F}_i) = \max\{l \in [k]_0 : j_l \le i\} \ \ \forall i \in [n]_0 \right\}$$

where we define j_0 to be 0.

Before we start examining mathematically this definition, let us write down the Schubert cells of the first non-trivial example we have, $\mathrm{Gr}(2, 4)$, with respect to the standard flag of $\mathbb{R}^4$:

$$\begin{aligned}
\mathcal{C}_{\{1,2\}} &= \big\{H:\ \dim(H\cap\mathbb{R}^0)=0,\ \dim(H\cap\mathbb{R}^1)=1,\ \dim(H\cap\mathbb{R}^{2,3,4})=2\big\}\\
&= \big\{\mathbb{R}^2\big\}\\
\mathcal{C}_{\{1,3\}} &= \big\{H:\ \dim(H\cap\mathbb{R}^0)=0,\ \dim(H\cap\mathbb{R}^{1,2})=1,\ \dim(H\cap\mathbb{R}^{3,4})=2\big\}\\
&= \big\{H:\ \mathbb{R}^1\subseteq H\subseteq\mathbb{R}^3,\ H\neq\mathbb{R}^2\big\}\\
\mathcal{C}_{\{1,4\}} &= \big\{H:\ \dim(H\cap\mathbb{R}^0)=0,\ \dim(H\cap\mathbb{R}^{1,2,3})=1,\ \dim(H\cap\mathbb{R}^4)=2\big\}\\
&= \big\{H:\ \mathbb{R}^1\subseteq H,\ H\not\subseteq\mathbb{R}^3\big\}\\
\mathcal{C}_{\{2,3\}} &= \big\{H:\ \dim(H\cap\mathbb{R}^{0,1})=0,\ \dim(H\cap\mathbb{R}^2)=1,\ \dim(H\cap\mathbb{R}^{3,4})=2\big\}\\
&= \big\{H:\ H\subseteq\mathbb{R}^3,\ \mathbb{R}^1\not\subseteq H\big\}\\
\mathcal{C}_{\{2,4\}} &= \big\{H:\ \dim(H\cap\mathbb{R}^{0,1})=0,\ \dim(H\cap\mathbb{R}^{2,3})=1,\ \dim(H\cap\mathbb{R}^4)=2\big\}\\
&= \big\{H:\ \dim(H\cap\mathbb{R}^2)=1,\ \mathbb{R}^1\not\subseteq H,\ H\not\subseteq\mathbb{R}^3\big\}\\
\mathcal{C}_{\{3,4\}} &= \big\{H:\ \dim(H\cap\mathbb{R}^{0,1,2})=0,\ \dim(H\cap\mathbb{R}^3)=1,\ \dim(H\cap\mathbb{R}^4)=2\big\}\\
&= \big\{H:\ H\cap\mathbb{R}^2=\{0\}\big\}
\end{aligned}$$

Although we see that there exist dimension restrictions in the definitions of the cells which can be omitted (for example that $H\cap\mathbb{R}^0=0$), the final form doesn't feel intuitive either. Let us take a step back for a moment and see what the Schubert cell decomposition of the (well-known) projective space is. Take for example $\mathrm{Gr}\,(1,4)\cong\mathbb{P}^2$:

$$\begin{aligned}
\mathcal{C}_{\{1\}} &= \big\{l:\ \dim(l\cap\mathbb{R}^0)=0,\ \dim(l\cap\mathbb{R}^{1,2,3,4})=1\big\}\\
&= \big\{\mathbb{R}^1\big\}\\
\mathcal{C}_{\{2\}} &= \big\{l:\ \dim(l\cap\mathbb{R}^{0,1})=0,\ \dim(l\cap\mathbb{R}^{2,3,4})=1\big\}\\
&= \big\{l:\ l\subseteq\mathbb{R}^2,\ l\neq\mathbb{R}^1\big\}\\
\mathcal{C}_{\{3\}} &= \big\{l:\ \dim(l\cap\mathbb{R}^{0,1,2})=0,\ \dim(l\cap\mathbb{R}^{3,4})=1\big\}\\
&= \big\{l:\ l\subseteq\mathbb{R}^3,\ l\not\subseteq\mathbb{R}^2\big\}\\
\mathcal{C}_{\{4\}} &= \big\{l:\ \dim(l\cap\mathbb{R}^{0,1,2,3})=0,\ \dim(l\cap\mathbb{R}^4)=1\big\}\\
&= \big\{l:\ l\not\subseteq\mathbb{R}^3\big\}
\end{aligned}$$

We can easily predict how the general Schubert cell (with respect to some flag $\mathbb{F}_\bullet$) of any projective space looks like: It will be the set of all lines contained in $\mathbb{F}_k\setminus\mathbb{F}_{k-1}$. This gives us a serial way to think of the cells of a particular projective space, which is the result of the total order that the set $\binom{[n]}{1}$ naturally has. Since $\binom{[n]}{k}$ is in general naturally a poset (inheriting the coordinate-wise ordering on the set of k element sequences $[n]^k$), it is now of no surprise that the same poset structure lies behind the Schubert decomposition. We are going to more precisely investigate into this structure, when we examine the closure of these cells we just defined.

Our goal now is to convince the reader that this is a meaningful decomposition of the Grassmannians, i.e. to prove eventually that this makes every $\mathrm{Gr}\,(k,n)$ into a CW complex. Let us start with proving that $\{\mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)\}_{\mathbf{j}\in\binom{[n]}{k}}$ is indeed a decomposition. The following proof makes sense, if one conceptualizes a k-subset of $[n]$, as the k "jump points" of the dimension of the intersections $H \cap \mathbb{F}_i$, for the various i.

Lemma 1.7. *For any integers $0 < k < n$ and for every flag $\mathbb{F}_\bullet$ for $\mathbb{R}^n$, the set of all Schubert cells $\{\mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)\}_{\mathbf{j}\in\binom{[n]}{k}}$ is a partition of the Grassmannian $\mathrm{Gr}\,(k,n)$.*

Proof. It is rather obvious that two cells are disjoint, since each set of k elements in $[n]$ describes uniquely k jump points of the dimensions of $H\cap\mathbb{F}_0, H\cap\mathbb{F}_1,\dots,H\cap\mathbb{F}_n$. Moreover, for a k-plane H the dimensions in this sequence of intersections can increase at most by 1 in each step. Indeed, because of the short exact sequence

$$0 \to H\cap\mathbb{F}_{i-1} \to H\cap\mathbb{F}_i \to {}^{H\cap\mathbb{F}_i}\!\big/_{H\cap\mathbb{F}_{i-1}} \to 0$$

we get, for every $i\in[n]$:

$$\dim_{\mathbb{R}}(H\cap\mathbb{F}_i) - \dim_{\mathbb{R}}(H\cap\mathbb{F}_{i-1}) = \dim_{\mathbb{R}} {}^{H\cap\mathbb{F}_i}\!\big/_{H\cap\mathbb{F}_{i-1}}$$

Using the second isomorphism theorem for vector spaces, we get:

$$\begin{aligned} {}^{H\cap\mathbb{F}_i}\!\big/_{H\cap\mathbb{F}_{i-1}} &= {}^{H\cap\mathbb{F}_i}\!\big/_{H\cap\mathbb{F}_i\cap\mathbb{F}_{i-1}} \cong {}^{(H\cap\mathbb{F}_i)+\mathbb{F}_{i-1}}\!\big/_{\mathbb{F}_{i-1}} \\ &\cong {}^{(H+\mathbb{F}_{i-1})\cap(\mathbb{F}_i+\mathbb{F}_{i-1})}\!\big/_{\mathbb{F}_{i-1}} \cong {}^{(H+\mathbb{F}_{i-1})\cap\mathbb{F}_i}\!\big/_{\mathbb{F}_{i-1}} \end{aligned}$$

with the last vector space being obviously a subspace of ${}^{\mathbb{F}_i}\!\big/_{\mathbb{F}_{i-1}}$, which gives finally:

$$\dim_{\mathbb{R}} {}^{H\cap\mathbb{F}_i}\!\big/_{H\cap\mathbb{F}_{i-1}} \le \dim_{\mathbb{R}} {}^{\mathbb{F}_i}\!\big/_{\mathbb{F}_{i-1}} = 1$$

Which means that there exist exactly k jump points in the sequence of the dimensions of $H\cap\mathbb{F}_0,\dots,H\cap\mathbb{F}_n$, putting H in some Schubert cell. □

We now want to prove that these cells are indeed building blocks we can use, i.e. that they are in fact homeomorphic to open balls of various dimensions. Before writing down the formula for the dimension of a general Schubert cell, let us revisit the case of $\mathrm{Gr}\,(2,4)$ and try to compute the dimension of the cells by hand-waving:

Example 1.8. *In* $\mathrm{Gr}(2,4)$ *we have the Schubert decomposition we discussed earlier, w.r.t. the standard flag. Let us represent each plane* H *in* $\mathbb{R}^4$ *by the unique corresponding* 2×4 *matrix whose rows span* H *and the matrix is in its reduced echelon form. Then, we get the following picture if we look at the form of the matrices for each cell. Remember, the index* $\mathbf{j} \subseteq \binom{[4]}{2}$ *refers to the dimension jump points:*

$$\mathcal{C}_{1,2} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathcal{C}_{1,3} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}$$

$$\mathcal{C}_{1,4} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix} \qquad \mathcal{C}_{2,3} \longleftrightarrow \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{C}_{2,4} \longleftrightarrow \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix} \qquad \mathcal{C}_{3,4} \longleftrightarrow \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$

In fact, every matrix of a given form gives a unique plane in the according cell. Thus, we get a bijection and the dimension of the cells equals the number of the choices we have in each matrix, i.e. the number of stars. Notice how the jumps in the dimensions now correspond to pivot elements of the rows. Moreover, notice that the first row has always $j_1 - 1$ *stars and the second* $j_2 - 2$*.*

We are now going to compute the dimension in general. Given a set $\mathbf{j} = \{j_1 < \cdots < j_k\} \in \binom{[n]}{k}$, define the number

$$d(\mathbf{j}) = (j_1 - 1) + (j_2 - 2) + \cdots + (j_k - k)$$

Using the reduced echelon form for a matrix written in a suitable base of $\mathbb{R}^n$, depending on the flag $\mathbb{F}_\bullet$, one can easily argue that $\mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)$ is an open cell of dimension $d(\mathbf{j})$, for any $\mathbf{j} \in \binom{[n]}{k}$, but our goal is to eventually prove that these open cells also give the Grassmannian a CW structure. For this we need to find maps from the closed cells of the appropriate dimensions into the Grassmannian and unfortunately it is not easy to work with the compactification of the matrices defined above. Thus, the approach in the bibliography may seem more artificial and it is also the one we employ here:

Our main goal is to prove Lemma 1.12. The approach is:

(i) First to define appropriate sets $\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet)$ living in the Stiefel manifold, each one "above" the matching Schubert cell. (Definition 1.9)

(ii) Then to prove that each closure $\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet)^-$ is homeomorphic with a closed disk of the right dimension. (Lemma 1.11)

(iii) Finally, to prove that q_0 maps $\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet)$ homeomorphically onto $\mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)$. (Lemma ??)

Let us begin with the definition:

Definition 1.9. Let $k, n \in \mathbb{N}$, with $0 < k < n$. Moreover, let $\mathbb{F}_\bullet$ be a flag of $\mathbb{R}^n$ and $f_\bullet$ an orthonormal basis of $\mathbb{R}^n$, compatible with $\mathbb{F}_\bullet$. Then, for each $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\} \in \binom{[n]}{k}$ define $\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet) = \tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet, f_\bullet)$ to be the following subset of $\mathrm{V}_0\,(k,n)$:

$$\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet, f_\bullet) := \{(v_1, v_2, \ldots, v_k) \in \mathrm{V}_0\,(k,n) : v_l \in \mathbb{H}_{j_l}(\mathbb{F}_\bullet, f_\bullet)\ \forall l \in [k]\}$$

where $\mathbb{H}_i(\mathbb{F}_\bullet, f_\bullet)$ is defined to be the "positive open halfspace" of $\mathbb{F}_i$, w.r.t. the orientation defined by f_i:

$$\mathbb{H}_i(\mathbb{F}_\bullet, f_\bullet) := \{v \in \mathbb{F}_i : v \cdot f_i > 0\}$$

for every $i \in [n]$.

Although we are going to prove that the images of these subspaces of $\mathrm{V}_0\,(k,n)$ are the Schubert decomposition of $\mathrm{Gr}\,(k,n)$, notice that for a fixed basis $f_\bullet$, they do not even cover $\mathrm{V}_0\,(k,n)$ and if we regard all possible compatible bases, they do cover the whole space, but we take most of these sets multiple times.

Let us prove now that these sets have the right dimension and that their closure (now much easier to handle than the matrices in echelon form) is homeomorphic to a closed cell of this dimension. In order to do so in Lemma 1.11, we first need the following trivial (but lengthy) assertion:

Lemma 1.10. *For any integers $0 < k < n$, for any flag $\mathbb{F}_\bullet$ of $\mathbb{R}^n$, for any orthonormal basis $f_\bullet$ of $\mathbb{R}^n$, compatible with $\mathbb{F}_\bullet$ and for any set $\mathbf{j} \in \binom{[n]}{k}$ the closure of $\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet, f_\bullet)$ inside $(\mathbb{R}^n)^k$ is:*

$$\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet, f_\bullet)^- = \left\{(v_1, v_2, \ldots, v_k) \in \mathrm{V}_0\,(k,n) : v_l \in \mathbb{H}_{j_l}(\mathbb{F}_\bullet, f_\bullet)^-\ \forall l \in [k]\right\}$$

where:

$$\mathbb{H}_i(\mathbb{F}_\bullet, f_\bullet)^- := \{v \in \mathbb{F}_i : v \cdot f_i \geq 0\}$$

Proof. Using simple point-set topology and the fact that $\mathrm{V}_0\,(k,n)$ is closed inside $(\mathbb{R}^n)^k$ as proven in Proposition A.5, we have:

$$\begin{aligned}
\tilde{\mathcal{C}}_{\mathbf{j}}(\mathbb{F}_\bullet, f_\bullet)^- &= \{(v_1, v_2, \ldots, v_k) \in \mathrm{V}_0\,(k,n) : v_l \in \mathbb{H}_{j_l}(\mathbb{F}_\bullet, f_\bullet)\ \forall l \in [k]\}^- \\
&= \big(\mathrm{V}_0\,(k,n) \cap \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)\big)^- \\
&\subseteq \mathrm{V}_0\,(k,n) \cap \big(\mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)\big)^- \\
&= \mathrm{V}_0\,(k,n) \cap \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet)^- \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)^- \\
&= \big\{(v_1, v_2, \ldots, v_k) \in \mathrm{V}_0\,(k,n) : v_l \in \mathbb{H}_{j_l}(\mathbb{F}_\bullet, f_\bullet)^-\ \forall l \in [k]\big\}
\end{aligned}$$

The inclusion in the third line is actually an equality, as we are going to prove. Indeed, let $(v_1, \ldots, v_k) \in \mathrm{V}_0\,(k,n) \cap \big(\mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)\big)^-$. This means that there exists a sequence $(v_1^m, \ldots, v_k^m)_m \in \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$ converging to $(v_1, \ldots, v_k) \in \mathrm{V}_0\,(k,n) \subseteq \mathrm{V}\,(k,n)$, inside $(\mathbb{R}^n)^k$. Since $\mathrm{V}\,(k,n)$ is open, as proven in Proposition A.2, there exists some $m_0 \in \mathbb{N}$, such that $(v_1^m, \ldots, v_k^m)_m \in \mathrm{V}\,(k,n)$ for all $m \geq m_0$. For each such tuple, define now $(w_1^m, \ldots, w_k^m)_m \in \mathrm{V}_0\,(k,n)$ to be the result of the Gram-Schmidt process on input $(v_1^m, \ldots, v_k^m)_m$ for every $m \geq m_0$:

$$(w_1^m, \ldots, w_k^m)_m := \mathfrak{gs}\big((v_1^m, \ldots, v_k^m)_m\big)$$

First of all, $\mathfrak{gs} : \mathrm{V}\,(k,n) \to \mathrm{V}_0\,(k,n)$ is a continuous map, which means that

$$(w_1^m, \ldots, w_k^m)_m = \mathfrak{gs}\big((v_1^m, \ldots, v_k^m)_m\big) \overset{m\to\infty}{\to} \mathfrak{gs}(v_1, \ldots, v_k) = (v_1, \ldots, v_k)$$

It now suffices to show that $\mathfrak{gs}(v_1', \ldots, v_k') \in \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$ for any $(v_1', \ldots, v_k') \in \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$. We are going to show this recursively, using the recursive nature of the Gram-Schmidt process. For $k = 1$, let $v_1' \in \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet)$. Then:

$$\mathfrak{gs}(v_1') = \frac{1}{\|v_1'\|} v_1' \in \mathbb{F}_{j_1}$$

Moreover:

$$\mathfrak{gs}(v_1') \cdot f_{j_1} = \frac{1}{\|v_1'\|} v_1' \cdot f_{j_1} > 0$$

which proves the base case. Assume now the assertion holds for $k-1$, let $(v_1', \ldots, v_k') \in \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$ and let also $(w_1', \ldots, w_{k-1}') = \mathfrak{gs}(v_1', \ldots, v_{k-1}')$. Then:

$$\mathfrak{gs}(v_1', \ldots, v_k') = \left(w_1', \ldots, w_{k-1}', \frac{1}{\left\| v_k' - \sum_{l=1}^{k-1} (w_l' \cdot v_k') w_l' \right\|} \left(v_k' - \sum_{l=1}^{k-1} (w_l' \cdot v_k') w_l' \right) \right)$$

Because of the inductive hypothesis, we only need to take care of the last vector, which we will denote as w'_k. Because $\mathbb{F}_\bullet$ is a flag, we have that $\mathbb{F}_{j_1} \subseteq \mathbb{F}_{j_2} \subseteq \cdots \subseteq \mathbb{F}_{j_k}$ and thus: $w'_1, w'_2, \ldots, w'_{k-1} \in \mathbb{F}_{j_k}$. Since $v'_k \in \mathbb{F}_{j_k}$ as well, we have that:

$$w'_k = \frac{1}{\left\| v'_k - \sum_{l=1}^{k-1}(w'_l \cdot v'_k)w'_l \right\|} \left(v'_k - \sum_{l=1}^{k-1}(w'_l \cdot v'_k)w'_l \right) \in \mathbb{F}_{j_k}$$

Moreover, it is true that $v \cdot f_i = 0$, if $v \in \mathbb{F}_{i-1}$ for some $i \in [n]$. Indeed, $f_\bullet$ is compatible with $\mathbb{F}_\bullet$, which means that $v \perp f_i$ for every $v \in \mathbb{F}_{i-1}$. Since $w'_1, \ldots, w'_{k-1} \in \mathbb{F}_{j_{k-1}} \subseteq \mathbb{F}_{j_k-1}$, we have:

$$w'_k \cdot f_{j_k} = \frac{1}{\left\| v'_k - \sum_{l=1}^{k-1}(w'_l \cdot v'_k)w'_l \right\|} v'_k \cdot f_{j_k} > 0$$

This proves that $w'_k \in \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$, which proves in turn:

$$\mathfrak{gs}(v'_1, \ldots, v'_k) \in \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$$

Thus, the sequence $(w_1^m, \ldots, w_k^m)_m$ is a sequence inside $\mathrm{V}_0(k,n) \cap \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)$ converging to $(v_1, \ldots, v_k)$, which means, finally that:

$$(v_1, \ldots, v_k) \in \left(\mathrm{V}_0(k,n) \cap \mathbb{H}_{j_1}(\mathbb{F}_\bullet, f_\bullet) \times \cdots \times \mathbb{H}_{j_k}(\mathbb{F}_\bullet, f_\bullet)\right)^-$$

which proves the desired inclusion. □

Lemma 1.11. *For any integers $0 < k < n$, for any flag $\mathbb{F}_\bullet$ of $\mathbb{R}^n$, for any orthonormal basis $f_\bullet$ of $\mathbb{R}^n$, compatible with $\mathbb{F}_\bullet$ and for any set $\mathbf{j} \in \binom{[n]}{k}$, there exists a homeomorphism*

$$\tilde{\Phi}_{\mathbf{j}} : D^{d(\mathbf{j})} \to \tilde{C}_{\mathbf{j}}(\mathbb{F}_\bullet, f_\bullet)^-$$

Proof. For this proof we are going to use the approach of Hatcher [Hat17] (p.37), in order to already familiarize ourselves with the notion of a trivial fiber bundle.

We will construct the desired homeomorphism inductively in k. For $k = 1$ we have:

$$\tilde{\mathcal{C}}_{\{j_1\}}(\mathbb{F}_\bullet, f_\bullet)^- = \{v \in \mathbb{F}_{j_1} : v \cdot v = 1 \ , \ v \cdot f_{j_1} \geq 0\}$$

It is known that the closed unit semisphere of $\mathbb{R}^p$ is homeomorphic to the closed disc D^{p-1} for every $p \in \mathbb{N}$. Let us fix now once and for all, for every $p \in \mathbb{N}$, a homeomorphism

$$\psi_p : D^{p-1} \to \{v \in \mathbb{R}^p : v \cdot v = 1 \ , \ v \cdot e_p \geq 0\}$$

One such homeomorphism would be the inverse of the restriction of the usual projection $\mathbb{R}^p \to \mathbb{R}^{p-1}$.

$$\mathrm{V}\,(k,n) \cap \cdots = \{(v_1, \ldots, v_k) : (v_1, \ldots, v_{k-1}) \in \mathrm{V}_F\,(k,n) \ \ v_l \in F_{j_l} \ v_k \cdot v_l = \delta_{k,l}\}$$

$\square$

Lemma 1.12. *For any integers $0 < k < n$, for any flag $\mathbb{F}_\bullet$ of $\mathbb{R}^n$ and for any set $\mathbf{j} \in \binom{[n]}{k}$ there exists a map*

$$\Phi_{\mathbf{j}} : D^{d(\mathbf{j})} \to \mathrm{Gr}\,(k,n)$$

such that:

(i) $\Phi_{\mathbf{j}}\left(\left(D^{d(\mathbf{j})}\right)^\circ\right) \subseteq \mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)$ *and*

(ii) $\Phi_{\mathbf{j}}|_{\left(D^{d(\mathbf{j})}\right)^\circ} \to \mathcal{C}_{\mathbf{j}}(\mathbb{F}_\bullet)$ *is a homeomorphism.*

2 Vector Bundles

Appendix A
Stiefel Manifolds

This appendix is devoted to the study of the well-known *Stiefel Spaces*. Our discussion begins at the space of all k-frames inside the $\mathbb{R}$-vector space $\mathbb{R}^n$:

Definition A.1. Let $0 < k \leq n$ be some natural numbers. Then, define $\mathrm{V}_{\mathbb{R}}(k, n) = \mathrm{V}(k, n)$ as:

$$\mathrm{V}_{\mathbb{R}}(k, n) := \left\{ (\vec{v}_1, \ldots, \vec{v}_k) \in (\mathbb{R}^n)^k : \dim \langle v_1, \ldots, v_k \rangle = k \right\}$$

equipped with the subspace topology. Every point in this set is called a *k-frame* of $\mathbb{R}^n$.

This is actually an open subspace of the space of all k-tuples of vectors in $\mathbb{R}^n$. Indeed, imagine slightly perturbing any one of the vectors in a k-frame. Then the resulting k-tuple is still going to be a k-frame. Let us also prove this fact formally.

Proposition A.2. *Let $0 < k \leq n$ be some natural numbers. Then $\mathrm{V}(k, n)$ is an open submanifold of $(\mathbb{R}^n)^k$.*

Proof. Define the map Φ taking a k-tuple of vectors in $\mathbb{R}^n$ to the matrix in $\mathbb{R}^{n \times k}$ having these vectors as columns:

$$\Phi : (\mathbb{R}^n)^k \to \mathbb{R}^{n \times k}$$

$$(v_1, \ldots, v_k) \mapsto \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}$$

Since the topology on both spaces is the product topology and Φ respects it, Φ is a homeomorphism.

Define now the subset D of all $n \times k$ matrices having at least one non-zero $k \times k$ minor:

$$D := \left\{ A \in \mathbb{R}^{n\times k} : \exists I \in \binom{[n]}{k} \text{ s.t. } \det(A_I) \neq 0 \right\}$$

where, given some $I \in \binom{[n]}{k}$, $A_I \in \mathbb{R}^{k\times k}$ is the matrix formed from the k rows of A indexed by I. Then, we have:

$$\mathrm{V}\left(k,n\right) = \Phi^{-1}(D) \cong D$$

and since D is an open submanifold of $\mathbb{R}^{n\times k}$, we also get that $\mathrm{V}\left(k,n\right)$ is an open submanifold of $\left(\mathbb{R}^n\right)^k$. □

Remark A.3. In particular, this proves that $\mathrm{V}\left(k,n\right)$ is a real manifold of dimension kn.

We can now define the Stiefel manifold as the space of all *orthonormal* k-frames:

Definition A.4. Let $0 < k \leq n$ be some natural numbers. Then, define the *Stiefel* space $\mathrm{V}_{0,\mathbb{R}}\left(k,n\right) = \mathrm{V}_0\left(k,n\right)$ as:

$$\mathrm{V}_{0,\mathbb{R}}\left(k,n\right) := \left\{ (\vec{v}_1, \ldots, \vec{v}_k) \in \left(\mathbb{R}^n\right)^k : \vec{v}_i \cdot \vec{v}_j = \delta_{i,j} \right\}$$

equipped with the subspace topology. Every point in this set is called an *orthonormal k-frame* of $\mathbb{R}^n$.

Notice that we obviously have $\mathrm{V}_0\left(k,n\right) \subseteq \mathrm{V}\left(k,n\right)$.

Proposition A.5. *Let $0 < k \leq n$ be some natural numbers. Then $\mathrm{V}_0\left(k,n\right)$ is a closed submanifold of $\left(\mathbb{R}^n\right)^k$ of dimension $nk - \frac{k(k+1)}{2}$.*

Proof. We need again the homeomorphism $\Phi : \left(\mathbb{R}^n\right)^k \to \mathbb{R}^{n\times k}$ taking a k-tuple in $\mathbb{R}^n$ to the matrix in $\mathbb{R}^{k\times n}$ having these vectors as columns. This time, define the subset S of all $n \times k$ semi-orthogonal matrices:

$$S := \left\{ A \in \mathbb{R}^{n\times k} : A^t A = I_k \right\}$$

Then, we have:

$$\mathrm{V}_0\left(k,n\right) = \Phi^{-1}(S) \cong S$$

S is a subset of $\mathbb{R}^{n\times}$, defined by $\binom{k+1}{2}$ (linearly independent) equations. This makes S a closed submanifold of $\mathbb{R}^{n\times k}$, of dimension $nk - \binom{k+1}{2}$. Thus, $\mathrm{V}_0\,(k,n)$ is a closed submanifold of $(\mathbb{R}^n)^k$ of the same dimension. □

In this thesis $\mathrm{V}\,(k,n)$ and $\mathrm{V}_0\,(k,n)$ are used in a similar way with $\mathrm{V}\,(k,n)$ having the advantage of being open and the Stiefel manifold having the advantage of being compact.

Remark A.6. Let $0 < k \leq n$ be some natural numbers. Then $\mathrm{V}_0\,(k,n)$ is a bounded subset of $(\mathbb{R}^n)^k$, with the metric induced by the isomorphism $(\mathbb{R}^n)^k \cong \mathbb{R}^{kn}$. Indeed, if $f = (v_1, \ldots, v_k) \in (\mathbb{R}^n)^k$ is an orthonormal k-frame of $\mathbb{R}^n$, then:

$$\|f\|_2^2 = \sum_{i=1}^{k} \|v_i\|_2^2 = k$$

Lemma A.7. *Let $0 < k \leq n$ be some natural numbers. Then $\mathrm{V}_0\,(k,n)$ is compact. Indeed the previous Proposition and Remark prove that $\mathrm{V}_0\,(k,n)$ is a closed and bounded subset of $(\mathbb{R}^n)^k \cong \mathbb{R}^{kn}$, which proves the assertion.*

Appendix B
CW Complexes

A CW structure on some space X is usually defined recursively, as an inductive "glueing" of cells of some dimension k to the previous, lower dimensional, skeleton of X, forming a new, k-dimensional, skeleton of X. A space X may exhibit many different CW structures, but the existence of one suffices in order for X to be characterized as CW complex. Here, we are going to use the following formal formulation of the above definition.

Definition B.1. A topological space X is a *CW-complex*, if there exists some filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$$

such that:

- $X = \varinjlim X_i$ with respect to all inclusion maps.
- For every $n \geq 0$ there exists a pushout diagram in the category of topological spaces:

$$\begin{tikzcd}
\coprod\limits_{e\in\pi_0(X_n\setminus X_{n-1})} S^{n-1} \arrow[r, "\coprod_e \phi_e"] \arrow[d, hook, "\coprod_e j_e"'] \arrow[dr, phantom, "\ulcorner"] & X_{n-1} \arrow[d, hook, "i_n"] \\
\coprod\limits_{e\in\pi_0(X_n\setminus X_{n-1})} D^n \arrow[r, "\coprod_e \Phi_e"'] & X_n
\end{tikzcd}$$

where $j_e : S^{n-1} \to D^n$ is the usual inclusion map and $i_n : X_{n-1} \to X_n$ is the inclusion map given by the filtration.

A filtration of a topological space X, making X a CW-complex is called a *CW-structure* of X. Moreover, given a filtration of X like in the above definition, the sets $\Phi_e((D^n)^\circ)$ (resp. $\Phi_e(D^n)$) are called the n-dimensional *open* (resp. *closed*) *cells* of this CW-structure. Recall the following known facts regarding the dependencies between CW-complexes, structures and cells.

Notes B.2. (i) A CW-complex X can have more than one CW-structures, even structures having different number of n-dimensional open cells each.

(ii) For a particular CW-structure of X, the maps ϕ_e and Φ_e are not predetermined by the structure, which means that there can be more than one choices for them. For example, one could always precompose Φ_e with a disc homeomorphism.

(iii) Even if the maps ϕ_e and Φ_e can vary, the open and closed cells of a CW-structure are part of the structure (i.e. independent of the choice of the maps)

Remarks B.3. (i) The property $X = \varinjlim X_i$ is equivalent to $X = \bigcup_{i \geq -1} X_i$ as a set, equipped with the final topology, with respect to all inclusion maps. In particular, a set A is open (closed) in X, iff $A \cap X_i$ is open (closed) in X_i for all $i \geq -1$, or equivalently, $A \cap \sigma$ is open (closed) in σ for every open cell σ of the CW structure. This property is what we usually refer to as the "weak topology" of X (the "W" part of the CW).

(ii)

Appendix C
Fiber Bundles

The Fiber Bundle is a topological object generalizing the notion of the product space. A fiber bundle E can be thought like a "twisted product" of a base topological space B and a fiber space F. This means that locally it always looks like the product of the two, but globally F may (homeomorphically) change, while one is going "around" B. Before we give some explicit examples, we need the formal definition of a fiber bundle.

Definition C.1. Let F, E and B be some topological spaces. A continuous map $p : E \to B$ is called a *fiber bundle with fiber* F, if for every $x \in B$ there exists a neighbourhood $x \in U \subseteq B$, and a function $f_U : F \times U \to p^{-1}(U)$ s.t.

(i) f_U is a homeomorphism.

(ii) The following diagram commutes:

$$\begin{tikzcd} F \times U \arrow[d, "f_U"'] \arrow[dr, "\pi_2"] & \\ E \supseteq p^{-1}(U) \arrow[r, "p|_{p^{-1}(U)}"'] & U \subseteq B \end{tikzcd}$$

i.e. if $p \circ f_U = \pi_2$

B is then called *base space* and E *total space* of the fibration. Moreover, for every $x \in B$, the space $p^{-1}(\{x\})$ is called *the fiber over* x and is denoted by F_x. Notice, that, sometimes, we refer to the fibration just by its total space, if the definition of p is clear.

Notation C.2. *It is very common to denote a fiber bundle as:*

$$F \to E \stackrel{p}{\to} B$$

which may remind to the reader the notion of a short exact sequence.

Notice that this is the smallest set of requirements one can demand of a fiber bundle. Most of the time though, people are interested in fiber bundles with additional structure imposed on the fiber space F, with the most noticable example being a group action on F. These fiber bundles are said to then be equipped with a "structure group" G. Another interesting structure one can require is discussed on Chapter **??**, where we define and examine the notion of "vector bundles", i.e. fiber bundles having vector spaces as bundles.

It is worth noting at this point that the definition of the "characteristic classes" one will see in **??** originated from the study of "sphere bundles", which is the particular subclass of fiber bundles where the fibers are topologically spheres of some dimension.

The reader should thus keep in mind that all the definitions she will encounter in this chapter are (in a more specific setting) usually enriched. So, questions of bundle isomorphisms should always be answered carefully, by determining the underlying setting. For example, we will encounter fiber bundles which will be proven isomorphic in this simple setting, whereas one can distinguise between them in the setting of fiber bundles equipped with a structure group G.

Proposition C.3. *Let $p : E \to B$ be a fiber bundle with fiber F, then $F_x \cong F$ for all $x \in B$.*

Proof. Indeed, let $x \in B$. Then, there exists a neighbourhood U of x inside B and a homeomorphism $f_U : F \times U \to p^{-1}(U)$, such that

$$p \circ f_U(a, x) = x \qquad \forall a \in F$$

Define now the map $f_x : F \to F_x = p^{-1}(\{x\})$ with:

$$f_x(a) := f_U(a, x)$$

this function is well defined, since $f_U(a, x) \in p^{-1}(\{x\})$ for every $a \in F$, as we just showed. Not only that, but in fact $f_x(F) = p^{-1}(\{x\})$. Indeed, let $\tilde{x} \in p^{-1}(\{x\})$. Then, $\pi_2 \circ f_U^{-1}(\tilde{x}) = p(\tilde{x}) = x$, i.e. $f_U^{-1}(\tilde{x}) = (a_x, x)$ for some $a_x \in F$. This gives:

$$f_x(a_x) = f_U(a_x, x) = \tilde{x}$$

which proves that $\tilde{x} \in f_x(F)$. This means that

$$f_x = i_1 \circ f_U|_{F\times\{x\}}$$

with $i_2 : F \to F \times \{x\}$ being trivially a homeomorphism. A restriction of a homeomorphism is also a homeomorphism, so f_x is a homeomorphism as well. $\square$

Now, one can see how bundles generalize products. Formally, one just needs to choose $U = B$ for every x and $\phi_B = id$. In fact, products are in this setup the "trivial" example.

Definition C.4. Let F, B be two topological spaces, then the projection

$$\pi_2 : F \times B \to B$$

is called the *trivial bundle over B with fiber F*, where π_2 is the projection on the second coordinate, i.e. $\pi_2(a, x) = x$.

Historically, the most famous non-trivial example of a fiber bundle has been the Möbius strip:

Example C.5. *Let $F = [0, 1]$, $B = S^1$ and $E = {}^{[0,1]^2}\!/\!_{(x,0)\,\sim\,(1-x,1)}$. Then $p : E \to B$, with:*

$$p([(x, y)]) := y, \qquad \forall [(x, y)] \in E$$

is a fiber bundle.

Notice how this fiber bundle is not the "same" as the trivial bundle over the circle with the same fiber. We are going to soon define stricter the notion of bundle maps and isomorphisms, but before going into that let us try the same Möbius like constructions, i.e. let us fix $B = S^1$ and examine how the bundles look like for different choices of fibers F and different equivallence relations $\sim$, which define $E = {}^{F\times B}\!/\!_{\sim}$. The bundle in these examples will always be $p : E \to B$ with $p([(x, y)]) = y$.

Examples C.6. *(i) For $F = \{0, 1\}$, and $(x, 0) \sim (1 - x, 1)$, E is obviously the boundary of the Möbius strip, which is topologically a circle. The image of p then traces the circle with twice the speed of the input variable. Notice, that this is different from the triviall bundle with the same fiber over B, since the trivial bundle is a disjoint union of two circles.*

(ii) $F = \mathbb{R}$ or $F = (0, 1)$, and again $(x, 0) \sim (1 - x, 1)$. This is the open version of the Möbius strip.

(iii) $F = S^1 \subseteq \mathbb{C}$, *and* $(z, 0) \sim (\overline{z}, 1)$. *This is maybe the second non-trivial example one usually sees, namely the Klein bottle.*

(iv) $F = S^1$ *again, but this time* $(z, 0) \sim (-z, 1)$. *Notice that this bundle contains the boundary of the Möbius strip (and in fact the boundary of every rotation of the Möbius strip around* B*). Moreover, notice that "turning" the circle* $S^1 \times \{1\}$ *by* π*, before glueing it back to* $S^1 \times \{0\}$*, seems to just produce a total space, which is topologically a torus, i.e.* E *seems to be the "same" as the trivial bundle (although it contains a non-trivial one).*

Notice how in the above discussion we first fixed a base space B, before starting to ask questions about similarity of bundles. This is so common, that the first definition of a maps between bundles assumes that both involved bundles have the same base space.

Definition C.7. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fiber bundles with fibers F_1 and F_2 respectively. A continuous map $\phi : E_1 \to E_2$ is a *bundle map from* p_1 *to* p_2 *over* B, if the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Proposition C.8. *Let* $p_1 : E_1 \to B$ *and* $p_2 : E_2 \to B$ *be two fiber bundles and* $\phi : E_1 \to E_2$ *be any continuous map, then* ϕ *is a bundle map, if and only if*

$$\phi\big((F_1)_x\big) \subseteq (F_2)_x, \qquad \forall x \in B$$

i.e. iff ϕ *maps every fiber over* x*, of the left fibration, to the fiber over the same* x*, of the right fibration.*

Proof. ($\Rightarrow$) Let $\phi : E_1 \to E_2$ be a bundle map, $x \in B$ and $v \in (F_1)_x$. Then $(p_2 \circ \phi)(v) = p_1(v) = x$, which proves that $\phi(v) \in (F_2)_x$.

($\Leftarrow$) Let $\phi : E_1 \to E_2$ be a continuous map, with $\phi\big((F_1)_x\big) \subseteq (F_2)_x$ for every $x \in B$. Moreover, let $v \in E_1$. Then, trivially, $v \in (F_1)_{p_1(v)}$, which gives $\phi(v) \in (F_2)_{p_1(v)}$, which means exactly that $(p_2 \circ \phi)(v) = p_1(v)$. □

Definition C.9. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fiber bundles with fibers F_1 and F_2 respectively and let $\phi : E_1 \to E_2$ be a bundle map from p_1 to p_2 over B. The map ϕ is called a *bundle isomorphism*, if ϕ is a homeomorphism.

Lemma C.10. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fiber bundles with fibers F_1 and F_2 respectively and let $\phi : E_1 \to E_2$ be a bundle map from p_1 to p_2 over B. Moreover, assume that for every $x \in B$ the map:*

$$\phi_x : (F_1)_x \to (F_2)_x$$

with $\phi_x(v) = \phi(v)$ for every $v \in E_1$ is an open map. Then ϕ is an open map as well.

Proof. Unfortunately

I DON'T KNOW IF THAT IS TRUE.

The End. □

Proposition C.11. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fiber bundles with fibers F_1 and F_2 respectively and let $\phi : E_1 \to E_2$ be a bundle map from p_1 to p_2 over B. Then ϕ is a bundle isomorphism, if and only if the map*

$$\phi_x : (F_1)_x \to (F_2)_x$$

with $\phi_x(v) = \phi(v)$ for every $v \in E_1$ is a homeomorphism for every $x \in B$.

Proof. ($\Rightarrow$) Let $\phi : E_1 \to E_2$ be a bundle isomorphism, and $x \in B$. Since $\phi : E_1 \to E_2$ is a homeomorphism, there exists a continuous map $\psi : E_2 \to E_1$, s.t.

$$\psi \circ \phi = id_{E_1} \quad \text{and} \quad \phi \circ \psi = id_{E_2}$$

It is trivial to see that such a ψ is also a bundle map. This means that, for every $x \in B$, $\psi\left((F_2)_x\right) \subseteq (F_1)_x$. Thus, the following map is well defined for every $x \in B$:

$$\psi_x : (F_2)_x \to (F_1)_x$$

with $\psi_x(v) = \psi(v)$ for every $v \in E_2$. Let us now fix a $x \in B$ and let $v_1 \in (F_1)_x$ and $v_2 \in (F_2)_x$. Then:

$$\left(\psi_x \circ \phi_x\right)(v_1) = \left(\psi \circ \phi\right)(v_1) = v_1$$
$$\left(\phi_x \circ \psi_x\right)(v_2) = \left(\phi \circ \psi\right)(v_2) = v_2$$

which proves that for every $x \in B$:

$$\psi_x \circ \phi_x = id_{(F_1)_x} \quad \text{and} \quad \phi_x \circ \psi_x = id_{(F_2)_x}$$

($\Leftarrow$) Let $\phi : E_1 \to E_2$ be a bundle map, with $\phi_x : (F_1)_x \to (F_2)_x$ being a homeomorphism, for every $x \in B$. This means that for every $x \in B$, there exists a continuous map $\psi_x : (F_2)_x \to (F_1)_x$ s.t.

$$\psi_x \circ \phi_x = id_{(F_1)_x} \quad \text{and} \quad \phi_x \circ \psi_x = id_{(F_2)_x}$$

Define now $\psi : E_2 \to E_1$ with $\psi|_{(F_2)_x} = \psi_x$ for every $x \in B$. Since the collection of all fibers is always a partition of the total space, ψ is well defined. In fact, we can write:

$$\psi(v_2) = \psi_{p_2(v_2)}(v_2) \qquad \forall v_2 \in E_2$$

Using the definition of ψ, it is easy to see that

$$\psi \circ \phi = id_{(F_1)_x} \quad \text{and} \quad \phi \circ \psi = id_{(F_2)_x}$$

i.e. that ϕ is a bijection and that

$$p_1 \circ \psi = p_2$$

i.e. that ψ is a good candidate to be a bundle map. We only have to prove now that ψ is also a continuous map. Since ϕ_x is a homeomorphism for every x, it is also, in particular, an open map for every $x \in B$. Thus, Lemma C.10 gives that ϕ is an open map as well. Since ϕ is an open bijection, it is also a homeomorphism. In particular, ψ is continuous.

□

Now that we have defined what an isomorphism of two fiber bundles is, let us examine what it means for a fiber bundle to be isomorphic to the trivial bundle.

Remark C.12. Let $p : E \to B$ be a fiber bundle with fiber F. Then, p is isomorphic to a trivial fiber bundle, if and only if there exists a map

$$\phi : F \times B \to E$$

such that, for every $x \in B$:

(i) $p \circ \phi(a, x) = x$ (i.e. ϕ is a bundle map), and

(ii) The restriction map $p_x : F \times \{x\} \to p^{-1}(\{x\})$ is an homeomorphism. (i.e. ϕ is a bundle isomorphism).

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