Mathematical Aspects of Public Transportation Networks
Problem Set 8

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Exercise 1

(a) Let C be a cycle basis of G, with cycle matrix Γ. Then $\Gamma \in \{-1, 0, 1\}^{C \times E}$. Each entry of $\Gamma \cdot A^T \in \mathbb{Q}^{C \times V}$ equals to $c_i^T \cdot p_j$, where c_i is the oriented edge-incidence vector of the i-th cycle in C and p_j is the oriented edge-incidence vector of the j-th vertex. Thus, the only edges that affect the inner product $c_i^T p_j$ are those edges of the cycle i, which are incident to the vertex j. Since j is a cycle, i.e. an Eulerian subgraph of G, these edges come in pairs, every time that the associated Eulerian cycle enters and exits the vertex j. If both of them are in-edges or if both of them are out-edges of vertex j, then they both have the same value in p_j (1 or -1 respectively) and their sign differs in c_i. If the one is an in-edge and the other an out-edge of vertex j, the sign of their respective entries differs in p_j, but they have the same value in c_i. In either case, the sum of the pairwise product of these entries vanishes, proving that $c_i^T p_j = 0$ and thus:

$$\Gamma \cdot A^T = 0_{C \times V}$$

(b) ($\Leftarrow$) Let $G' = G[e_1, \ldots, e_k]$ be the directed subgraph induced by this edge set. Then the incident matrix A' of G' has the collumns $a_1, \ldots, a_k$ (up to a permutation). Let C' be a cycle basis of G' and

$$\Gamma' = \begin{pmatrix} c_1 \\ \hline \vdots \\ \hline c_{\mu'} \end{pmatrix} \in \{-1, 0, 1\}^{[\mu'] \times [k]}$$

be the respective cycle matrix, where $\mu' = |C'|$ and the above is a block notation of Γ'. Let c be a cycle in G'. This means that there exists a vector $(\lambda_1, \ldots, \lambda_{\mu'}) \in \mathbb{Q}^{1 \times \mu'}$, such that:

$$c = (\lambda_1, \ldots, \lambda_{\mu'}) \cdot \Gamma' \in \mathbb{Q}^k$$

This means that

$$c \cdot \begin{pmatrix} a_1 \\ \hline \vdots \\ \hline a_k \end{pmatrix} = (\lambda_1, \ldots, \lambda_{\mu'}) \cdot \Gamma' \cdot A'^T \overset{\text{(a)}}{=} 0_{1 \times V(G')}$$

which is a vanishing non-zero linear combination of $a_1, \dots a_k$, making this set $\mathbb{Q}$-linear dependent.
($\Rightarrow$) Let $e_1, \dots, e_k \in E$ be such that $G' = G[e_1, \dots, e_k]$ contains no cycles and let $\lambda_1, \dots, \lambda_k \in \mathbb{Q}$, s.t.

$$(\lambda_1, \dots, \lambda_k) \cdot A'^T = (\lambda_1, \dots, \lambda_k) \cdot \begin{pmatrix} a_1 \\ \hline \vdots \\ \hline a_k \end{pmatrix} = \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k = 0_{1 \times V(G')}$$

We will show that $\lambda_1 = \dots = \lambda_k = 0$. Since G' contains no cycle, it is a forest and thus it has at least one leaf, i.e. a vertex $v \in V(G')$ of degree 1. Let e_i be the adjacent edge to v. Then, the v-th entry of $(\lambda_1, \dots, \lambda_k) \cdot A'^T$ equals λ_i (since e_i is the only adjacent edge of v). This means that $\lambda_i = 0$. Removing the vertex v from G', we get another cycle-free graph $G'' = G' \setminus v$, which contains every other edge, except for λ_i. Let A'' be A' without the collumn a_i. Then the equation $(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_k) \cdot A''^T = 0$ still holds in G''. After $V(G') - 1$ iterations there will be no edge left in the graph and thus everyone among $\lambda_1, \dots, \lambda_k$ is proven eventually to be zero. This proves that there does not exist a $\mathbb{Q}$-linear dependance of $a_1, \dots, a_k$, finishing the proof.

(c) Let G be a directed graph with incident matrix A, and let Γ be a cycle matrix of some cycle basis of G. Let $a_1, \dots, a_n \in \mathbb{Q}^E$ be the rows of A and $c_1, \dots, c_{m-n+c} \in \mathbb{Q}^E$ the rows of Γ, where n, m and c are the number of vertices, edges and weakly connected components of G, respectively (we do not need that $c = 1$). Then, almost by definition we have:

$$\text{rank}\Gamma = \dim_{\mathbb{Q}} \mathcal{C}_{\mathbb{Q}}(G) = \mu(G) = m - n + c$$

and from (b) we have:

$$\text{rank}A = \max\{k : \text{ there exist } k \text{ } \mathbb{Q}\text{-lin. ind. rows of } A\} = n - c$$

because every maximal set of edges in G, which do not include a cycle forms a spanning forest, which has $n - c$ edges. Moreover, from (a) we get:

$$a_i^T \cdot c_j = 0 \ \forall (i, j) \in [m - n + c] \times [n - c]$$

Thus, the rows of Γ and A span two perpendicular subspaces of $\mathbb{Q}^m$ of dimensions $m - n + c$ and $n - c$ respectively, which proves that:

$$\langle a_1, \dots, a_{n-c}, c_1, \dots, c_{m-n+c} \rangle \cong \mathbb{Q}^{n-c+m-n+c} = \mathbb{Q}^m$$

Exercise 2 Let $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{R}^m$.

(a) Let moreover $x \in Q(A,b)$. We will show that $x \in P(A,b)$: Let us first define for each $i \in [m]$ the positive integer r_i as the least common factor of all denominators of every entry in row i of $A = (a_{ij})_{i\in[m],j\in[n]}$:

$$r_i := \min\{t \in \mathbb{Z}_+ : ta_{ij} \in \mathbb{Z}\ \forall j \in [n]\}$$

Fix now some $s \in [m]$ and define the vector $\lambda^{(s)} = \left(\lambda_i^{(s)}\right)_{i\in[m]}$, with $\lambda_i^{(s)} := r_s e_i$, i.e. $\lambda^{(s)}$ has every coordinate equal to zero, except for the s-th one, which is equal to r_s. For our choice x, we know in particular that:

$$\left(\lambda^{(s)}\right)^T Ax \leq \left\lfloor \left(\lambda^{(s)}\right)^T b \right\rfloor \quad \forall s \in [m]$$

Define the integer matrix $A' := (r_i a_{ij})_{i\in[m],j\in[n]}$ and the vector $b' = (r_i b_i)_{i\in[m]} \in \mathbb{R}^m$. Notice that $P(A,b) = P(A',b')$, since we just multiplied each inequality with some positive number. Notice now that the above m inequalities for x can be summarized to:

$$A'x \leq \lfloor b' \rfloor \leq b'$$

which means that $x \in P(A',b') = P(A,b)$, proving the assertion.

(b) Let now $x \in P(A,b) \cap \mathbb{Z}^n$. This means that $Ax \leq b$, and thus we also have $\lambda^T Ax \leq \lambda^T b$, for every $\lambda \geq 0$. If $\lambda^T A$ has integer entries, since x is also an integer vector, the LHS is an integer, for every inequality $1,\dots,m$. Thus, $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, i.e. $x \in Q(A,b)$.

(c) Let $\lambda = (\gamma_+^T, \gamma_-^T)^T \in \{0,1\}^{2m}$. Obviously $\lambda \geq 0$ and $\lambda^T A \in \mathbb{Z}^{n+m}$, since A has also integer entries. This means, that for every $(\pi,p) \in Q(A,b)$ we get (in block-matrix notation):

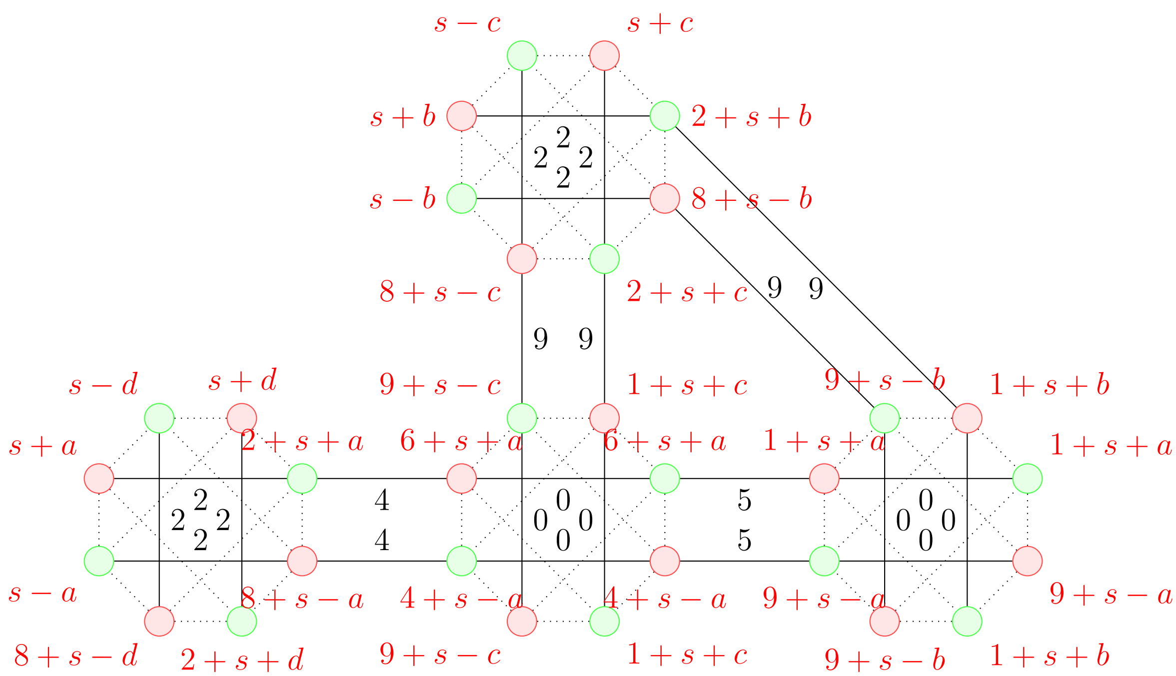
$$\begin{array}{lrcl}
 & \lambda^T A \begin{pmatrix} \pi \\ p \end{pmatrix} & \leq & \lfloor \lambda^T b \rfloor \\
\Rightarrow & (\gamma_+^T, \gamma_-^T) \left(\begin{array}{c|c} B & T\cdot I_m \\ \hline -B & -T\cdot I_m \end{array}\right) \begin{pmatrix} \pi \\ p \end{pmatrix} & \leq & \left\lfloor (\gamma_+^T, \gamma_-^T) \begin{pmatrix} u \\ -l \end{pmatrix} \right\rfloor \\
\Rightarrow & \left(\gamma_+^T B - \gamma_-^T B \mid T\cdot\gamma_+^T I_m - T\cdot\gamma_-^T I_m \right) \begin{pmatrix} \pi \\ p \end{pmatrix} & \leq & \lfloor \gamma_+^T u - \gamma_-^T l \rfloor \\
\Rightarrow & \left(\gamma^T B \mid T\cdot\gamma^T I_m \right) \begin{pmatrix} \pi \\ p \end{pmatrix} & \leq & \lfloor \gamma_+^T u - \gamma_-^T l \rfloor \\
\overset{(1)}{\Rightarrow} & T\cdot\gamma^T p & \leq & \lfloor \gamma_+^T u - \gamma_-^T l \rfloor \\
\overset{(2)}{\Rightarrow} & \gamma^T p & \leq & \left\lfloor \dfrac{\gamma_+^T u - \gamma_-^T l}{T} \right\rfloor
\end{array}$$

Where, (1) is true because $\gamma^T B = 0 \in \mathbb{R}^n$, like we showed in the first exercise and (2) is true because $\gamma^T p \in \mathbb{Z}$: Let $n \in \mathbb{Z}$, $a, x \in \mathbb{R}$. Then $n \leq a\lfloor x \rfloor$ gives $n \leq \lfloor ax \rfloor$. Indeed:

$$n \leq a\lfloor x \rfloor \overset{n\in\mathbb{Z}}{\Rightarrow} n \leq \lfloor a\lfloor x \rfloor \rfloor \overset{\lfloor x \rfloor \leq x}{\leq} n \leq \lfloor ax \rfloor$$

Exercise 3

(a) Let s be the symmetry axis. Then, we have the following EAN:



where the arrival events are red and the departure events are green. The red labels are the values of any timetable, depending on s and four parameters a, b, c, d, one for each line. The slack depends only on a, b, c, d, since each action has a value of the form s+sth. The minimum weighted periodic slack is 132 and the proof has the form of a C++ program, also sent to you.