A CHILD'S GUIDE TO OPTIMAL CONTROL THEORY

1. Introduction

This is a simple guide to optimal control theory. In what follows, I borrow freely from Kamien and Shwartz (1981) and King (1986).

2. The Finite Horizon Problem

Optimal control theory is useful to solve continuous time optimization problems of the following form:

$$\max \int_0^T F\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt \tag{P}$$

subject to

$$\dot{x}_i = Q_i(\mathbf{x}(t), \mathbf{u}(t), t), \quad i = 1, \ldots, n; \tag{1}$$

$$x_i(0) = x_{i0}, \quad i = 1, \ldots, n; \tag{2}$$

$$x_i(T) \geq 0, \quad i = 1, \ldots, n; \tag{3}$$

$$\mathbf{u}(t) \in \Upsilon, \quad \Upsilon \in I\!R^m. \tag{4}$$

where,

$\mathbf{x}(t)$ is a n-vector of *state* variables ($x_i(t)$). These describes the state of the system at any point in time. Also, note that $dx_i/dt = \dot{x}_i$.

$\mathbf{u}(t)$ is a m-vector of *control* variables. These are the choice variables in the optimization problem.

$F(\cdot)$ is a twice continuously differentiable objective function. Throughout, we assume that this function is time additive.

$Q_i(\cdot)$ are twice continuously differentiable *transition* functions for each state variable.

The different constraints are:

i. Equation (1) defines the transition equations for each state variables. It describes how the state variables evolve over time.

ii. Equation (2) show the initial conditions for each state variable. In this, x_{i0} are constants.

iii. Equation (3) show the terminal conditions for each state variable.

iv. Equation (4) defines the feasible set for control variables.

We can now discuss the Pontryagin maximum principle. This principle says that we can solve the optimization problem P using a *Hamiltonian* function H over one period. A version of the Theorem is (see Kamien and Schwartz):

Theorem *In order that* $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$ *be optimal for the problem* P, *it is necessary that there exist a constant* λ_0 *and continuous functions* $\Lambda(t) = (\,\lambda_1(t), \ldots, \lambda_n(t)\,)$, *where for all* $0 \le t \le T$ *we have* $\lambda_0 \neq 0$ *and* $\Lambda(t) \neq 0$ *such that for every* $0 \le t \le T$

$$H\left(\mathbf{x}^*(t), \mathbf{u}, \Lambda(t), t\right) \le H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \Lambda(t), t\right),$$

where the Hamiltonian function H *is defined by*

$$H\left(\mathbf{x}, \mathbf{u}, \Lambda, t\right) = \lambda_0 F(\mathbf{x}, \mathbf{u}, t) + \sum_{i=1}^{n} \lambda_i Q_i(\mathbf{x}, \mathbf{u}, t).$$

Except at points of discontinuity of $\mathbf{u}_t^*$,

$$\frac{\partial H(\cdot)}{\partial x_i} = -\dot{\lambda}_i(t), \quad i = 1, \ldots, n.$$

Furthermore, $\lambda_0 = 0$ *or* $\lambda_0 = 1$ *and, finally, the following transversality conditions are satisfied:*

$$\lambda_i(T) \ge 0, \quad \lambda_i(T) x_i^*(T) = 0, \quad i = 1, \ldots, n.$$

3. The Present Value Hamiltonian

The previous theorem suggests that one can solve problem P by simply setting up the following *present value* Hamiltonian:

$$H\left(\mathbf{x}, \mathbf{u}, \Lambda, t\right) = F(\mathbf{x}, \mathbf{u}, t) + \sum_{i=1}^{n} \lambda_i Q_i(\mathbf{x}, \mathbf{u}, t).$$

In the above, we call $\lambda_i(t)$ *co-state* variables. They are analogous to Lagrange multipliers.

The necessary conditions for a maximum are:

$$\begin{aligned}
&\frac{\partial H(\cdot)}{\partial u_k} = 0, \quad k = 1, \ldots, m; \\
&\frac{\partial H(\cdot)}{\partial x_i} = -\dot{\lambda}_i(t), \quad i = 1, \ldots, n; \\
&\lambda_i(T) \geq 0, \quad \lambda_i(T) x_i^*(T) = 0, \quad i = 1, \ldots, n.
\end{aligned}$$

The sufficient conditions for a maximum are that the Hamiltonian function be concave in $\mathbf{x}$ and $\mathbf{u}$.

Finally, in some cases, our optimization problem will also include other constraints:

$$\max \int_0^T F\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt \tag{$P1$}$$

subject to

$$\begin{aligned}
&\dot{x}_i = Q_i(\mathbf{x}(t), \mathbf{u}(t), t), \quad i = 1, \ldots, n; \\
&G_j(\mathbf{x}(t), \mathbf{u}(t), t) \geq 0, \quad j = 1, \ldots, q; \\
&x_i(0) = x_{i0}, \quad i = 1, \ldots, n; \\
&x_i(T) \geq 0, \quad i = 1, \ldots, n; \\
&\mathbf{u}(t) \in \Upsilon, \quad \Upsilon \in I\!R^m.
\end{aligned}$$

In that case, we can write problem $P1$ as a Lagrangian:

$$\begin{aligned}
L(\mathbf{x}, \mathbf{u}, \Lambda, \Phi, t) &= H(\mathbf{x}, \mathbf{u}, \Lambda, t) + \sum_{j=1}^{q} \phi_j G_j(\mathbf{x}, \mathbf{u}, t) \\
&= F(\mathbf{x}, \mathbf{u}, t) + \sum_{i=1}^{n} \lambda_i Q_i(\mathbf{x}, \mathbf{u}, t) + \sum_{j=1}^{q} \phi_j G_j(\mathbf{x}, \mathbf{u}, t),
\end{aligned}$$

where $\Phi = (\,\phi_1, \ldots, \phi_q\,)$. In this case, the necessary conditions for a maximum are:

$$\begin{aligned}
&\frac{\partial L(\cdot)}{\partial u_k} = 0, \quad k = 1, \ldots, m; \\
&\frac{\partial L(\cdot)}{\partial x_i} = -\dot{\lambda}_i(t), \quad i = 1, \ldots, n; \\
&\lambda_i(T) \geq 0, \quad \lambda_i(T) x_i^*(T) = 0, \quad i = 1, \ldots, n. \\
&\phi_j(t) \geq 0, \quad \phi_j(t) Q(\mathbf{x}(t), \mathbf{u}(t), t) = 0, \quad j = 1, \ldots, q.
\end{aligned}$$

The sufficient conditions are as before.

4. An Economic Example

Consider the following problem:

$$\max \int_0^T e^{-\rho t} u\left(c(t)\right) dt$$

subject to

$$\begin{aligned} &\dot{a}(t) = ra(t) - c(t); \\ &a(0) = a_0; \\ &a(T) \geq a_T. \end{aligned}$$

The present value Hamiltonian is

$$H(a, c, \lambda, t) = e^{-\rho t} u\left(c\right) + \lambda\left[ra - c\right].$$

The first-order conditions are

$$\begin{aligned} &\frac{\partial H}{\partial c} = 0 \quad \Longrightarrow \quad e^{-\rho t} u'\left(c(t)\right) - \lambda(t) = 0; \\ &\frac{\partial H}{\partial a} = -\dot{\lambda}(t) \quad \Longrightarrow \quad \lambda(t) r = -\dot{\lambda}(t); \\ &\lambda(T) \geq 0, \quad \lambda(T) a(T) = 0. \end{aligned}$$

A simple interpretation of these first-order conditions is to relate the growth rate of consumption, $g_c(t)$, to the elasticity of intertemporal substitution, $\sigma(c(t))$, and the market and subjective discount rates, r and ρ. To do this, first note that

$$\dot{\lambda}(t) = -r\lambda(t) = -re^{-\rho t} u'(c(t)).$$

Then, totally differentiating the first first-order condition yields

$$-\rho e^{-\rho t} u'\left(c(t)\right) + e^{-\rho t} u''\left(c(t)\right) \dot{c}(t) - \dot{\lambda}(t) = 0.$$

Define the growth rate of consumption as $g_c = (1/c) dc/dt$:

$$g_c(t) = \frac{\dot{c}(t)}{c(t)} = -(r - \rho)\frac{u'(c(t))}{c(t) u''(c(t))},$$

such that

$$g_c(t) = \sigma(c(t))(r - \rho),$$

where $\sigma(c(t)) = -u'(c(t))/[c(t)u''(c(t))] > 0$. Thus, consumption grows whenever the market discount rate is larger than the subjective discount rate. In that case, the consumer is less impatient than the market.

5. The Infinite Horizon Problem

In this section we consider an extension of the finite horizon problem to the case of infinite horizon. To simplify the exposition, we will focus our attention to stationary problems.

The assumption of stationarity implies that the main functions of the problem are time invariant:

$$F(\mathbf{x}(t),\mathbf{u}(t),t) = F(\mathbf{x}(t),\mathbf{u}(t));$$
$$G_i(\mathbf{x}(t),\mathbf{u}(t),t) = G_i(\mathbf{x}(t),\mathbf{u}(t));$$
$$Q_j(\mathbf{x}(t),\mathbf{u}(t),t) = Q_j(\mathbf{x}(t),\mathbf{u}(t)).$$

In general, to obtain an interior solution to our optimization problem, we require the objective function to be bounded away from infinity. One way to achieve boundedness is to assume discounting. However, the existence of a discount factor is neither necessary nor sufficient to ensure boundedness. Thus, we define

$$F(\mathbf{x}(t),\mathbf{u}(t)) = e^{-\rho t} f(\mathbf{x}(t),\mathbf{u}(t)).$$

Our general problem becomes:

$$\max \int_0^\infty e^{-\rho t} f\left(\mathbf{x}(t),\mathbf{u}(t)\right) dt \tag{P2}$$

subject to

$$\dot{x}_i = Q_i(\mathbf{x}(t),\mathbf{u}(t)), \quad i = 1,\ldots,n; \tag{1}$$
$$G_j(\mathbf{x}(t),\mathbf{u}(t)) \geq 0, \quad j = 1,\ldots,q;$$
$$x_i(0) = x_{i0}, \quad i = 1,\ldots,n; \tag{2}$$

The present value Hamiltonian is

$$H\left(\mathbf{x},\mathbf{u},\Lambda\right) = e^{-\rho t} f(\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} \lambda_i Q_i(\mathbf{x},\mathbf{u}).$$

The Lagrangian is

$$L(\mathbf{x},\mathbf{u},\Lambda,\Phi) = e^{-\rho t} f(\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} \lambda_i Q_i(\mathbf{x},\mathbf{u}) + \sum_{j=1}^{q} \phi_j G_j(\mathbf{x},\mathbf{u}).$$

The first-order conditions are:

$$\frac{\partial L(\cdot)}{\partial u_k} = 0, \quad k = 1,\ldots,m;$$
$$\frac{\partial L(\cdot)}{\partial x_i} = -\dot{\lambda}_i(t), \quad i = 1,\ldots,n;$$
$$\lim_{T\to\infty} \lambda_i(T) \geq 0, \quad \lim_{T\to\infty} \lambda_i(T) x_i^*(T) = 0, \quad i = 1,\ldots,n.$$
$$\phi_j(t) \geq 0, \quad \phi_j(t) Q(\mathbf{x}(t),\mathbf{u}(t)) = 0, \quad j = 1,\ldots,q.$$

The sufficient conditions are as before.

6. The Current Value Hamiltonian

Any discounted optimal control problem can be solved using both a present value Hamiltonian or a current value Hamiltonian. We sometime prefer the current value Hamiltonian for its simplicity.

We obtain the current value Hamiltonian as follows:

$$\mathbf{H}(\mathbf{x}, \mathbf{u}, \Psi) = e^{\rho t} H(\mathbf{x}, \mathbf{u}, \Lambda),$$

where $\psi_i = e^{\rho t}\lambda_i$. The current value Hamiltonian is thus

$$\mathbf{H}\left(\mathbf{x}, \mathbf{u}, \Psi\right) = f(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} \psi_i Q_i(\mathbf{x}, \mathbf{u}).$$

The transformation from present value to current value affects the first-order conditions. To see this, let us restate our problem $P2$ using the current value Hamiltonian. The problem is

$$\max \int_0^\infty e^{-\rho t} f\left(\mathbf{x}(t), \mathbf{u}(t)\right) dt \tag{$P2$}$$

subject to

$$\dot{x}_i = Q_i(\mathbf{x}(t), \mathbf{u}(t)), \quad i = 1, \ldots, n; \tag{1}$$

$$G_j(\mathbf{x}, \mathbf{u}) \geq 0, \quad j = 1, \ldots, q;$$

$$x_i(0) = x_{i0}, \quad i = 1, \ldots, n; \tag{2}$$

The current value Hamiltonian is

$$\mathbf{H}\left(\mathbf{x}, \mathbf{u}, \Psi\right) = f(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} \psi_i Q_i(\mathbf{x}, \mathbf{u}).$$

The Lagrangian is

$$\mathbf{L}(\mathbf{x}, \mathbf{u}, \Psi, \Phi) = f(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} \psi_i Q_i(\mathbf{x}, \mathbf{u}) + \sum_{j=1}^{q} \phi_j G_j(\mathbf{x}, \mathbf{u}).$$

The first-order conditions are:

$$\begin{aligned}
&\frac{\partial \mathbf{L}(\cdot)}{\partial u_k} = 0, \quad k = 1, \ldots, m;\\
&\frac{\partial \mathbf{L}(\cdot)}{\partial x_i} = -\left[\dot{\psi}_i(t) - \rho\psi_i(t)\right], \quad i = 1, \ldots, n;\\
&\lim_{T\to\infty} e^{-\rho T}\psi_i(T) \geq 0, \quad \lim_{T\to\infty} e^{-\rho T}\psi_i(T)x_i^*(T) = 0, \quad i = 1, \ldots, n.\\
&\phi_j(t) \geq 0, \quad \phi_j(t)Q(\mathbf{x}(t), \mathbf{u}(t)) = 0, \quad j = 1, \ldots, q.
\end{aligned}$$

The sufficient conditions are as before.

7. The Economic Example in Infinite Horizon

Consider the following problem:

$$\max \int_0^\infty e^{-\rho t} u\left(c(t)\right) dt$$

subject to

$$\begin{aligned}
&\dot{a}(t) = ra(t) - c(t);\\
&a(0) = a_0.
\end{aligned}$$

Abstracting from the time subscript, the current value Hamiltonian is

$$\mathbf{H}(a, c, \psi) = u(c) + \psi\left[ra - c\right].$$

Its first-order conditions are

$$\begin{aligned}
&\frac{\partial \mathbf{H}}{\partial c} = 0 \quad \Longrightarrow \quad u'(c) - \psi = 0;\\
&\frac{\partial \mathbf{H}}{\partial a} = -\left[\dot{\psi} - \rho\psi\right] \quad \Longrightarrow \quad \psi r = -\left[\dot{\psi} - \rho\psi\right];\\
&\lim_{T\to\infty} e^{-\rho t}\psi(T) \geq 0, \quad \lim_{T\to\infty} e^{-\rho t}\psi(T)a(T) = 0.
\end{aligned}$$

Once again, the interpretation of these first-order conditions relates consumption growth to the elasticity of intertemporal substitution, the market discount rate, and the subjective discount rate, r and ρ. To show this, first note that

$$\dot{\psi} = -(r - \rho)\psi = -(r - \rho)u'(c).$$

Then, totally differentiating the first first-order condition yields

$$u''(c)\dot{c} = \dot{\psi} = -(r - \rho)u'(c),$$

such that

$$g_c(t) = \frac{\dot{c}(t)}{c(t)} = -(r - \rho)\frac{u'(c(t))}{c(t)u''(c(t))}$$

or

$$g_c(t) = \sigma(c(t))(r - \rho),$$

where $\sigma(c(t)) = -u'(c(t))/[c(t)u''(c(t))] > 0$. Thus, consumption grows whenever the market discount rate is larger than the subjective discount factor. In that case, the consumer is less impatient than the market.

8. Summary

To summarize, here is the simple cookbook. Assume that you face the following problem:

$$\max \int_0^\infty e^{-\rho t} f\left(\mathbf{x}(t), \mathbf{u}(t)\right) dt$$

subject to

$$\begin{aligned}
&\dot{x}_i = Q_i(\mathbf{x}(t), \mathbf{u}(t)), \quad i = 1, \ldots, n; \\
&x_i(0) = x_{i0}, \quad i = 1, \ldots, n.
\end{aligned}$$

There are two ways to proceed:

1. Present Value Hamiltonian

Step 1 Write the present value Hamiltonian:

$$H\left(\mathbf{x}, \mathbf{u}, \Lambda\right) = e^{-\rho t} f(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} \lambda_i Q_i(\mathbf{x}, \mathbf{u}).$$

Step 2 Find the first-order conditions:

$$\begin{aligned}
&\frac{\partial H(\cdot)}{\partial u_k} = 0, \quad k = 1, \ldots, m; \\
&\frac{\partial H(\cdot)}{\partial x_i} = -\dot{\lambda}_i(t), \quad i = 1, \ldots, n; \\
&\lim_{T \to \infty} \lambda_i(T) \geq 0, \quad \lim_{T \to \infty} \lambda_i(T) x_i^*(T) = 0, \quad i = 1, \ldots, n.
\end{aligned}$$

2. Current Value Hamiltonian

Step 1 Write the current value Hamiltonian:

$$\mathbf{H}\left(\mathbf{x}, \mathbf{u}, \Psi\right) = f(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} \psi_i Q_i(\mathbf{x}, \mathbf{u}).$$

Step 2 Find the first-order conditions:

$$\begin{aligned}
&\frac{\partial \mathbf{H}(\cdot)}{\partial u_k} = 0, \quad k = 1, \ldots, m; \\
&\frac{\partial \mathbf{H}(\cdot)}{\partial x_i} = -\left[\dot{\psi}_i(t) - \rho \psi_i(t)\right], \quad i = 1, \ldots, n; \\
&\lim_{T \to \infty} e^{-\rho T} \psi_i(T) \geq 0, \quad \lim_{T \to \infty} e^{-\rho T} \psi_i(T) x_i^*(T) = 0, \quad i = 1, \ldots, n.
\end{aligned}$$

References

Kamien, Morton I. and Nancy L. Schwartz, 1981, *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, New York: North Holland.

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