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A two-runners model: optimization of running strategies according to the physiological parameters

Amandine Aftalion* and Camilla Fiorini†

CNRS UMR 8100, Laboratoire de Mathématiques de Versailles,
Université de Versailles Saint-Quentin,
45 avenue des États-Unis, 78035 Versailles Cedex, France.

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Abstract

In order to describe the velocity and the anaerobic energy of two runners competing against each other for middle-distance races, we present a mathematical model relying on an optimal control problem for a system of ordinary differential equations. The model is based on energy conservation and on Newton's second law: resistive forces, propulsive forces and variations in the maximal oxygen uptake are taken into account. The interaction between the runners provides a minimum for staying one meter behind one's competitor. We perform numerical simulations and show how a runner can win a race against someone stronger by taking advantage of staying behind, or how he can improve his personal record by running behind someone else. Our simulations show when it is the best time to overtake, depending on the difference between the athletes. Finally, we compare our numerical results with real data from the men's $1500-m$ finals of different competitions.

1 Introduction

The running strategy to win an Olympic medal is quite complex. It relies on outstanding physiology, good preparation, psychological factors and the optimal way to compete with the others to beat them. Quite a few mathematical works starting with Keller's [10], have analysed running strategies for a single runner

*amandine.aftalion@uvsq.fr
†camilla.fiorini@ens.uvsq.fr

[1, 2, 12, 13, 22], but very few take into account the competition situation where the point is to beat the others [11, 16].

The point of view of Keller [10] is to write the equations governing the energy and the velocity of a single runner, starting from Newton's second law and energy conservation. He considers a simple problem, in which the athlete runs alone on a straight path of length D, and the aim is to minimize the time T when the runner reaches the final distance D. This model matches the final times of world records. However some hypotheses are not physiologically reasonable, and this leads to a non-realistic velocity profile. Some authors have tried to improve Keller's model: Woodside in [22] and Mathis in [12] introduce a correction by adding a fatigue term for long races. Ward-Smith [21] and Morton [13, 14, 15] follow a different approach. Morton has introduced a three component model to take into account the variations in the oxygen uptake ($\dot{V}O2$) but the full optimal control problem is not solved. Behncke [2] incorporates the hydraulic model of Morton to a biomechanical model that extends the ones of Keller and Ward-Smith: it is more detailed in terms of resistive forces and takes into account the reaction time of the athlete. Aftalion and Bonnans [1] improve the models of Keller, Behncke and Morton and assume that maximal oxygen uptake is a function of the anaerobic energy of the athlete. The aim is to match experimental measurements, in particular, in [8, 9, 20], where Hanon et al. show how the oxygen uptake varies during races of $400-$, $800-$ and $1500-m$. They solve the full numerical control problem using an optimal control solver Bocop.

As pointed out by Pitcher [16], in middle distance running, it is common practice to try to position oneself behind but within striking distance of the leader for most of the race and then overtake him near the finish line. Pitcher explains that the runner behind can take advantage of the slipstream of the runner in front and relies on analyses of Pugh [17] and Kyle [11]. We believe that it is a combination of slipstream and psychological factors which explain why it is better to stay behind, and the equations can incorporate all this. The weakness of Pitcher's paper is that she imposes a strategy for one of the runners and allows only the second runner to have a free strategy. Therefore, in this paper, based on the recent work of Aftalion-Bonnans [1], we extend the model of Pitcher [16] to include slipstream and psychological factors in a two runners race and we set a realistic optimal control problem with each runner having a free strategy.

1.1 Mathematical model for a single runner

The system of Keller couples together the velocity of the runner at instant t, $v(t)$, the energy of the runner at instant t, $e(t)$, and the propulsive force of the runner at instant t, $f(t)$. The first equation is Newton's second law. It involves the propulsive force and the friction. Here, τ is a constant coefficient which gathers together all the friction effects, supposed to be linear in v. The friction term can be modified to include air resistance [11] which adds a term in $-cv^2$ to the first equation.

The second equation is an energy balance incorporating the oxygen uptake, σ, considered constant in Keller's paper, while the second term is the work of the propulsive force f. Both equations are normalized with respect to the runner's mass:

$$\begin{cases} \dot{v}(t) = f(t) - \frac{v(t)}{\tau} & v(0) = 0,\ x(0) = 0,\ x(T) = D \\ \dot{e}(t) = \sigma - f(t)v(t) & e(0) = e^0. \end{cases} \quad (1)$$

We use the dot notation to indicate the derivative with respect to time, i.e. $\dot{v} = \frac{dv}{dt}$ and $\dot{e} = \frac{de}{dt}$. We have set $x(t)$ to be the position so that $\dot{x} = v$. The final time T is defined as the time to reach the distance D. Moreover, e^0 is the initial energy. It is necessary to add some physiological constraints to the system (1):

· the energy must be positive:

$$e(t) \geq 0 \quad \forall t \geq 0; \quad (2)$$

· the propulsive force has an upper bound which depends on the runner's physiology, and a positive lower bound due to the fact that he is moving forwards:

$$0 \leq f(t) \leq f_M \quad \forall t \geq 0. \quad (3)$$

Therefore, in this model, the athlete is identified by four parameters: e^0, his initial energy, τ, the friction coefficient, σ, the oxygen uptake, and f_M the maximal propulsive force. The aim is to solve (1)-(2)-(3) in such a way that given a distance D, the time T to reach it is minimal. From a mathematical point of view, it is a problem of optimal control: the propulsive force f is the control variable and the time T is the cost functional to be minimized, which depends on f through the states variables v and e. Therefore, the problem is:

$$\min_{f \in \mathcal{F}} T(f) \quad \text{s.t (1)-(2)}, \quad (4)$$

where $\mathcal{F}$ is the set of admissible controls:

$$\mathcal{F} = \{f : \ 0 \leq f(t) \leq f_M \quad \forall t \geq 0\}.$$

In his work, Keller [10] claims that his energy balance takes into account only the aerobic energy (i.e. energy provided by oxygen consumption). However, Aftalion and Bonnans [1] remark that what he encompasses in the balance is the accumulated oxygen deficit: $e^0 - e(t)$. Therefore, $e(t)$ in equation (1) is in fact the anaerobic energy (i.e. energy provided by glycogen and lactate). In order to reproduce the results of [7, 8, 9], the oxygen uptake σ introduced in [1] is piecewise defined: in the most part of the race, σ is constant equal to its maximal value $\sigma_{\max}$, but it is increasing at the beginning of the race, and decreasing at the end (see Figure 1). In fact, σ depends on five parameters: the initial value at rest σ_r, the

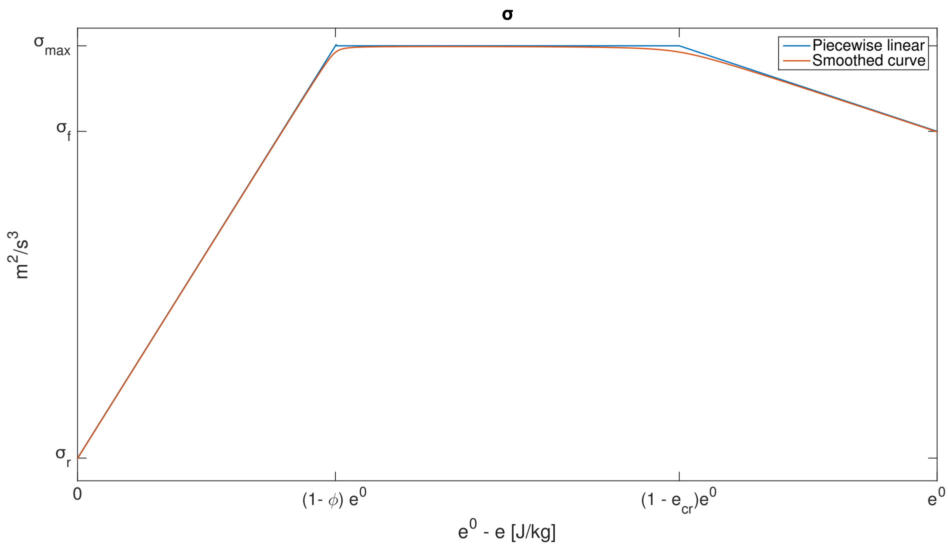

Figure 1: Typical curve $\sigma(e^0 - e)$ for a $1500 - m$ race.

maximal value $\sigma_{\max}$, the final value σ_f and two parameters φ and e_{cr} which denote the transition point from one zone to another, $\varphi,\ e_{cr} \in (0,1)$:

$$\sigma(e;\sigma_{\max},\sigma_f,\sigma_r,\varphi,e_{cr}) = \begin{cases} \sigma_{\max}\frac{e}{e^0 e_{cr}} + \sigma_f\left(1 - \frac{e}{e^0 e_{cr}}\right) & \text{if } e < e^0 e_{cr} \\ \sigma_{\max} & \text{if } e^0 e_{cr} \le e \le e^0\varphi \\ \sigma_r + \frac{(\sigma_{\max}-\sigma_r)(e^0-e)}{e^0(1-\varphi)} & \text{if } e \ge e^0\varphi, \end{cases} \tag{5}$$

Let us observe that the σ used in (5) is continuous but not C^1. In the numerical simulations, it has been smoothed, since from the physiology, it is clear that the passage from one zone to the other occurs smoothly. The sigma used is shown in Figure 1.

Moreover, in [1], a second modification is introduced to the energy equation: for sufficiently long races (longer than $1000 - m$), it has been observed that slowing down recreates some of the anaerobic energy. Therefore, the energy equation results in:

$$\dot{e} = \sigma(e) + \eta(\dot{v}) - fv,$$

where η depends on the acceleration $\dot{v}$ and has the following form:

$$\eta(\dot{v}) = \begin{cases} 0 & \text{if } \dot{v} > 0 \\ c_\eta|\dot{v}|^2 & \text{if } \dot{v} \le 0, \end{cases} \tag{6}$$

where c_η is a constant to be tuned. This leads to oscillations in the velocity profile [1], Figure 2.4. In this paper, we will use both forms of the energy equation.

1.2 Equations for two runners

In real races, the competition between runners has a fundamental impact on the strategy. Starting from the works of Keller [10], Quinn [18] and Kyle [11], Pitcher introduces a two-runners model in [16] based on the slipstream. The observation is that running behind someone can save 1 or 2 seconds per lap in middle distance races. Therefore, in the equations, the friction gets reduced when a runner is just behind his competitor.

Some of the quantities in this model have a subscript i, which refers to the runner i: therefore x_i, v_i and e_i are, respectively, the position, the velocity and the energy of runner-i. All the physiological parameters which depend on the runner have the subscript i, too. Finally, the state variable x_D represents the distance between the runners, defined as $x_D := x_2 - x_1$. The energy balance for each runner is the same as in (1), with a constant value of σ, while the dynamics equation incorporates an aerodynamical term. Because it is the relative position which is important, instead of using x_i, the position of each runner as parameters, we use x_1 and the relative position x_D. The resulting model is the following: for $i = 1, 2$

$$\begin{cases} \dot{x}_1 = v_1 & x_1(0) = 0 \\ \dot{x}_D = v_2 - v_1 & x_D(0) = 0 \\ \dot{v}_1 = f_1 - \frac{v_1}{\tau_1} - c_1 v_1^2 (1 - \gamma(e^{-\alpha(x_D - \beta)^2})) & v_1(0) = 0 \\ \dot{v}_2 = f_2 - \frac{v_2}{\tau_2} - c_2 v_2^2 (1 - \gamma(e^{-\alpha(x_D + \beta)^2})) & v_2(0) = 0 \\ \dot{e}_i = \sigma_i - f_i v_i & e_i(0) = e_i^0. \end{cases} \tag{7}$$

As in Keller's model, the velocities and energies equations are normalized with respect to the mass of the runner. The term $-c_i v_i^2$ is the friction with the air. It is necessary to highlight and separate the effect of the friction with the air from the other ones, because it is the only one that is reduced while running in the slipstream of someone else. This frictional term is modulated by $1 - \gamma(e^{-\alpha(x_D \pm \beta)^2})$, which is shown in Figure 2.

The parameter β represents the optimal distance a runner should keep from the other in order to obtain the maximal reduction of the air friction, while $\gamma \in (0, 1)$ is the percentage reduction at the optimal distance. The parameter α is an index of the variance of the phenomenon and is chosen large enough so that the exponential term becomes negligible as we deviate by half a meter from β. These parameters do not depend on the runner, however the parameter c is related to the drag coefficient and depends on the shape and surface properties of the athlete's body (see [18] for further details). For simplicity, in this work, we consider $c_1 = c_2 = c$. If ever the athletes have different masses, then we would have $m_1 c_1 = m_2 c_2$. The problem has physiological constraints:

$$e_i(t) \geq 0 \quad \forall t \geq 0, \; i = 1, 2. \tag{8}$$

In order to have the state equations for both runners defined on the same time interval, Pitcher chooses a fixed final time T. Moreover, she fixes the running

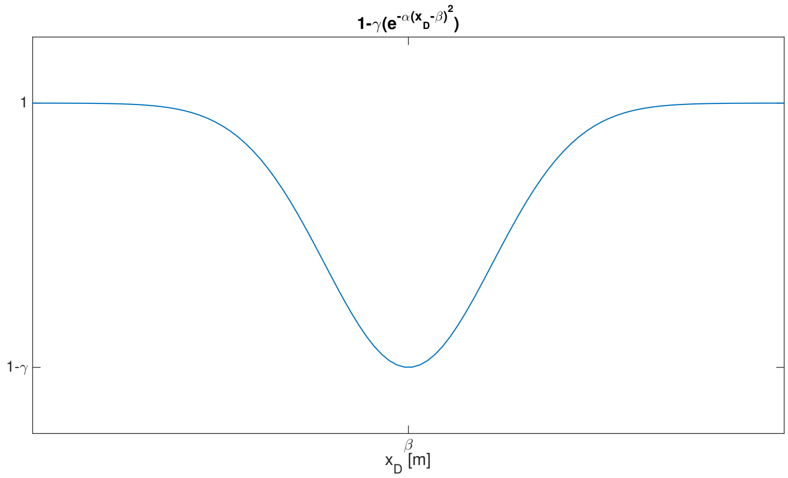

Figure 2: Term that modulates the friction with the air in the two-runners model.

strategy of runner 1 (therefore f_1 is given), as the optimal strategy he would adopt if running alone, therefore the only control is $f_2(t)$. Formally, the resulting optimization problem is:

$$\max_{f_2 \in \mathcal{F}} x_D(T) \quad \text{s.t. (7)-(8).} \tag{9}$$

It is clear that this strategy is not realistic, because every runner adapts his strategy according to the performances of his opponents. In fact, one of the crucial point in a two-runners problem is how to model competition between them: some possible ways are game theory or multi-objective optimization. However, in this work, as explained below, we model the competition by leaving both strategies free, therefore having two controls, f_1 and f_2, and by having a proper cost functional.

2 Mathematical model

In this work, we use together Pitcher's two-runners model [16] with the single runner model of Aftalion-Bonnans [1] to build a new model for two runners, which incorporate psychological factors.

In order to simplify the notation, we substitute the expression $\sigma(e_i; \sigma_{\max,i}, \sigma_{f,i}, \sigma_{r,i}, \varphi_i, e_{cr,i})$ from (5) with $\sigma_i(e_i)$. We recall that the friction in Pitcher's paper is in $-cv_i^2(1-\gamma e^{-\alpha(x_D \pm \beta)^2})$ and is supposed to model the slipstream. Though shielding behind someone has a strong impact when there is wind or when the velocity is high, we do not believe that it has a strong impact on middle distance races. Moreover, in real races, the second runner is often slightly to the right of the leader, to avoid the feet of his opponent and in order to be able to overtake more easily. This position does not come from slipstream but from strategic factors, for which the position one meter behind is the best. Therefore, the potential $1-\gamma e^{-\alpha(x_D \pm \beta)^2}$

can also be considered to model a psychological factor which consists in trying to follow one's competitor, in order to be able to overtake. Indeed, it is a potential which has a minimum at distance β behind and decreases global friction because it increases the will to follow. On the other hand, when the other runner is too far, there is no benefit. Let us observe that this model does not take into account the lateral displacement, and therefore the additional propulsive force, that is necessary to overtake: however, we estimated this term numerically and it is negligible. One can model the fact that overtaking requires some additional energy, by possibly using a non symmetric well potential $1-\gamma e^{-\alpha(x_D\pm\beta)^2}$, which is not what we have done in the simulations presented to reduce the number of parameters involved.

We obtain the following equations: for $i=1,2$

$$\begin{cases} \dot{x}_1 = v_1 & x_1(0)=0 \\ \dot{x}_D = v_2 - v_1 & x_D(0)=0 \\ \dot{v_1} = f_1 - \frac{v_1}{\tau_1} - cv_1^2(1-\gamma(e^{-\alpha(x_D-\beta)^2})) & v_1(0)=0 \\ \dot{v_2} = f_2 - \frac{v_2}{\tau_2} - cv_2^2(1-\gamma(e^{-\alpha(x_D+\beta)^2})) & v_2(0)=0 \\ \dot{e}_i = \sigma_i(e_i) + \eta_i(\dot{v_i}) - f_i v_i & e_i(0)=e_i^0. \end{cases} \tag{10}$$

The equations in (10) are defined for $t\in(0,T)$, where T is the time at which the first of the two runners reaches the final distance D; to model this, it is necessary to add the following boundary condition to the system:

$$(x_1(T)-D)(x_2(T)-D)=0. \tag{11}$$

As in the previous models, the energy has a lower bound:

$$e_i(t)\geq 0 \quad \forall t\in(0,T),\ i=1,2. \tag{12}$$

The choice of the cost functional, i.e. the quantity to be minimized, is a key point. As said before, in contrast with Pitcher's choice, in this case none of the strategies is fixed, therefore there are two controls: $f_1(t)$ and $f_2(t)$.Here, we propose to minimize the following quantity, given a proper constant weight $c_w>0$:

$$J(f_1,f_2)=T+c_w|x_D(T)|. \tag{13}$$

The aim of this choice is to minimize the final time of the winner, and the term $c_w|x_D(T)|$ models the fact that the loser has tried to win as well. The resulting problem is:

$$\min_{f_i\in\mathfrak{F}_i} J \quad \text{s.t. (10)-(11)-(12)}, \tag{14}$$

where $\mathfrak{F}_i$ is the set of the admissible controls, and depends on the athlete. For physiological reasons, it is necessary to impose a bound to the variations of $\dot{f}$, in addition to the bounds on f already introduced, related to the fact that an athlete cannot vary his propulsive force too quickly (see more details in [1]). This leads to the following definition of $\mathfrak{F}_i$:

$$\mathfrak{F}_i := \{f : 0 \leq f(t) \leq f_{M,i},\ |\dot{f}(t)| \leq K_i\ \forall t \in (0,T)\}, \tag{15}$$

where K_i and $f_{M,i}$ are constants depending on the athlete. The rest of the paper consists in providing numerical simulations of (14).

3 Numerical results

All the results presented in this section are obtained with the free software BOCOP [3]. The equations are solved with a finite difference scheme (implicit Euler), while the optimization problem is solved with an iterative method.

The aim of these simulations is to find out if a runner can win a race against someone stronger, by running behind, and to quantify in term of variation of some parameters how much weaker he can be and when the best time to overtake is.

The single runner strategy, computed numerically in [1] and found experimentally in [8], consists in three parts:

- a first part of maximal force with a strong acceleration during which the peak velocity is achieved,
- a second part in which the propulsive force first decreases smoothly and then increases again, with the corresponding decrease and increase of the velocity,
- a final part at maximal force and maximal velocity again, until zero energy level is reached, where the velocity drops.

From the two-runners model, we expect an overall similar strategy: however, the additional term in the velocities equations encourages one of the runners to start slightly slower and to position himself at distance β from the other. This allows him to keep the same velocity as the other runner while using a smaller propulsive force, which, in turn, leads to a lower energy consumption. We expect that, at a certain point during the race, the runner who is behind, will overtake the other by using the energy he has saved throughout the race and will be able to perform a longer final sprint. Moreover, it is reasonable to think that this moment occurs sooner if the runner who starts behind is stronger. Let us observe that this running strategy could also lead, in some favourable situations, to an improvement of the personal record. Nonetheless, it is important to underline that the decrease in the final time is not the main goal for a runner in some occasions, such as the Olympic finals, in which the final position is definitely more important than the final time: in [19], Thiel et al. study, starting from the Beijing 2008 Olympic Games data and the world records data, how the difference in the goal affects the pacing strategies: win the race versus minimizing the final time.

The reference values for the parameters used in the simulations are reported in Table 1. The initial, maximal and final value of sigma (respectively σ_r, σ_{max} and σ_f) are taken from the $\dot{V}O_2$ values reported in [8]: let us observe that these values are given in $ml\ kg^{-1}min^{-1}$. In order to convert them in the unity of measurement needed (i.e. $m^2\ s^{-3} = J\ kg^{-1}s^{-1}$), we consider that the uptake of $1ml$ of oxygen is often converted into an energy expenditure estimate of $21J$. It is then necessary to convert the minutes in seconds, obtaining in this way the conversion factor $21/60$. The values chosen for φ and e_{cr} aim at fitting the $\dot{V}O_2$ profile reported in Figure 2 of [8]. The values for f_M and τ are strongly related: in fact, a first order approximation of the maximal velocity v_{peak} a runner can reach is τf_M. Therefore, starting from the velocity profile plotted in Figure 1 of [8], one can chose a couple of reasonable values. The value of the constant c is taken from [18], c_η from [1], α and β from [16] and, finally, γ from [17].

Table 1: Parameters values for 1500-m

Parameter	Unit of Measurement	Value
τ	s	1.33
c	m^{-1}	0.0028
e^0	J/kg	1400
σ_r	m^2/s^3	6
σ_{max}	m^2/s^3	24.22
σ_f	m^2/s^3	20.44
φ	-	0.5
e_{cr}	-	0.3
c_η	s	4
α	m^{-2}	10
β	m	1
γ	-	0.8
f_M	N/kg	5

First of all, let us consider the perfectly symmetric situation, in which the two runners have the same parameters. In this case it is very influential which runner is starting behind: this can be mathematically modelled by choosing, in a proper way, the initial guess for the variable x_D given to the iterative method that solves the optimization problem. In fact, giving as initial guess β (or $-\beta$) forces runner-1 (or runner-2) to start the race behind. The results of this simulation are shown in Figures 3-4: one can observe that the runner who stays behind can keep the same velocity as the other runner, while using a significantly lower propulsive force and therefore having a much lower energy consumption. All the graphs in Figure 3 have the position on the x-axis, which means that the velocity of runner-i is plotted with respect to the position of runner-i (and it is the same for the propulsive force, the energy and sigma). The choice of plotting with respect to position, and

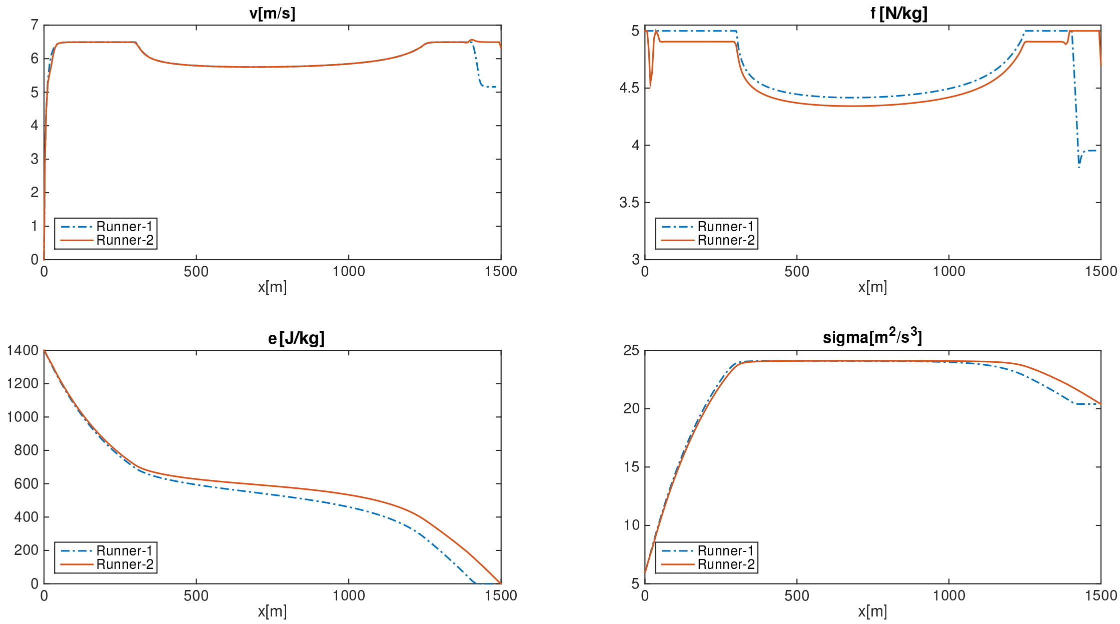

Figure 3: Competition between two runners with the same parameters.

not time, has been made because it is the most common in the sports literature. Figure 4 shows the distance between the runners: the overtaking occurs at about 94% of the race, corresponding to $1416m$. This variable is plotted with respect to a normalized time (i.e. t/T), because it does not concern only one runner, but both of them, therefore it does not make sense to plot it with respect to the position of any of them.

From this first result, it is clear that running behind someone allows a runner to win a race against an athlete as strong as himself. We now want to investigate how the strategy of a runner changes if he is running alone or behind someone else. In Figure 5, two performances of the same runner are compared: the blue line represents the optimal strategy of the runner running alone (he completes the race in $249.681s$), the red line represents the strategy adopted when running behind someone as strong as himself (he completes the race in $247.822s$). This difference in the final time (almost $2s$ of improvement) is equivalent to a difference of about $0.05m/s$ in the mean velocity. However, what we have simulated is not the most favourable situation to reduce the final time, and therefore to improve the mean velocity. If the aim is exclusively to improve the personal best performance and not to win the race, the best scenario possible for a runner is to run behind someone slightly stronger for the whole race. However, let us observe that the opponent must not be too much stronger: in this case, the runner would use too much energy to stay behind and would soon reach the zero energy level, which would cause a drop in the propulsive force too soon in the race. Nevertheless, from Figure 5 we can still observe how running behind someone else allows an athlete to have a higher velocity in spite of keeping a smaller force throughout the

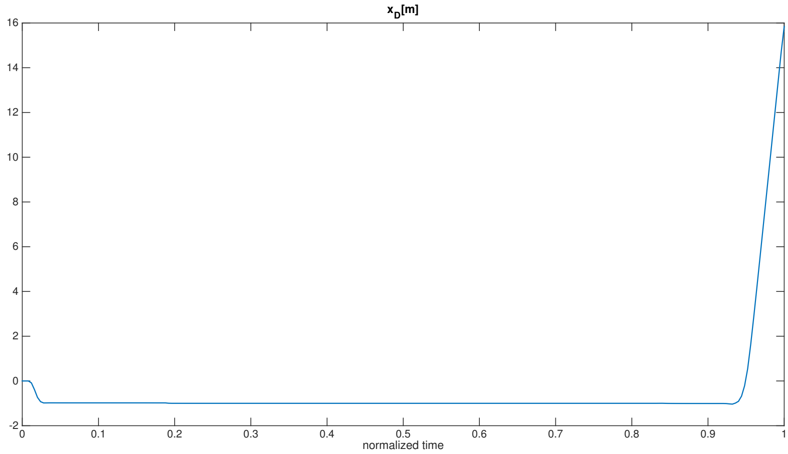

Figure 4: Competition between two runners with the same parameters - distance between the runners.

race.

The results presented in Figures 3-4-5 are obtained with the model (10) without the recreational term η introduced in (6). In Figures 6-7 we present the same scenario, this time with recreation: the two runners have the same parameters. One can observe, comparing Figure 3 with Figure 6 and Figure 4 with Figure 7, that the strategy does not change: runner-2 slows down at the beginning, in order to place himself behind runner-1; in the middle part of the race the propulsive forces are oscillating (this behaviour is caused by the additional term η and can be found also in the single-runner problem, see [1] for further details), and the mean value around which the propulsive force of runner-1 is oscillating is slightly bigger than the one around which the force of runner-2 is oscillating; the energy curve does not change significantly, compared to the previous results. From Figure 7, one can notice that runner-2 overtakes at about 94.85% of the race, which corresponds to $1417m$: one meter later, if compared with the case without oscillations. Let us observe that in this case the final time is smaller: in fact runner-2 completes the race in $247.75s$. This decrease in the final time, when adding the recreation η, is consistent with the results for the single-runner problem presented in [1].

Finally, we can say that the recreation term does not change the strategy of the race, however the results are more difficult to read, due to the oscillations. For this reason, from now on we will present only results obtained without η.

We now want to find the threshold values, i.e. how much a runner can be weaker than his opponent and still win the race by running behind. Therefore, we vary one parameter at a time, making runner-2 weaker. Figures 8-9 show the results for two runners who have a different initial energy. Given the reference value for $e_1^0 = 1400$ (as in Table 1), the lowest initial energy runner-2 can have,

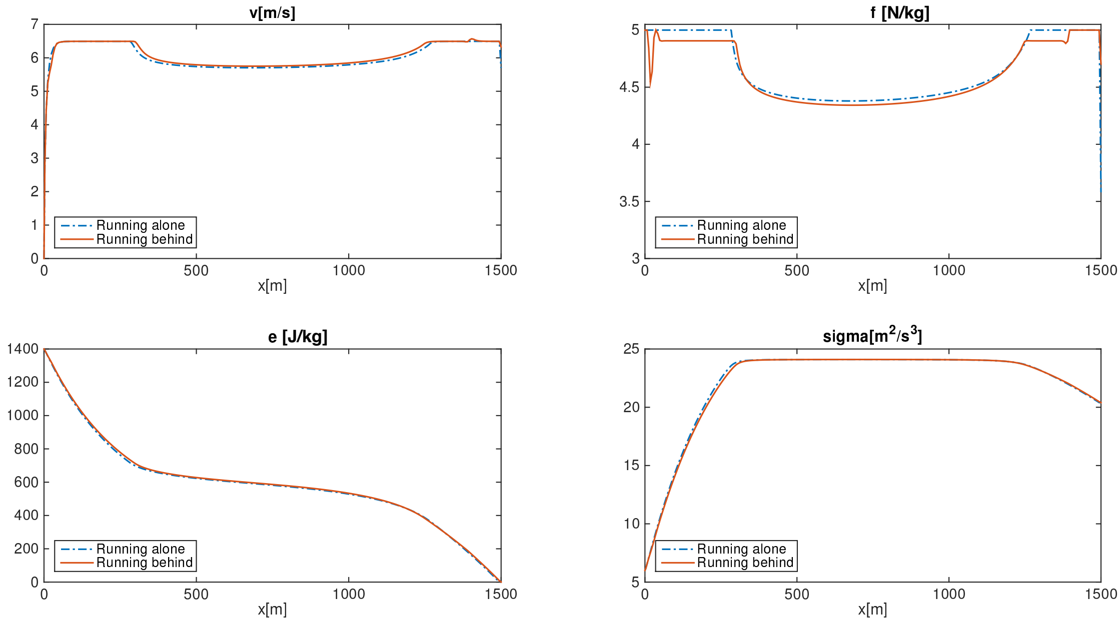

Figure 5: Running alone vs running behind.

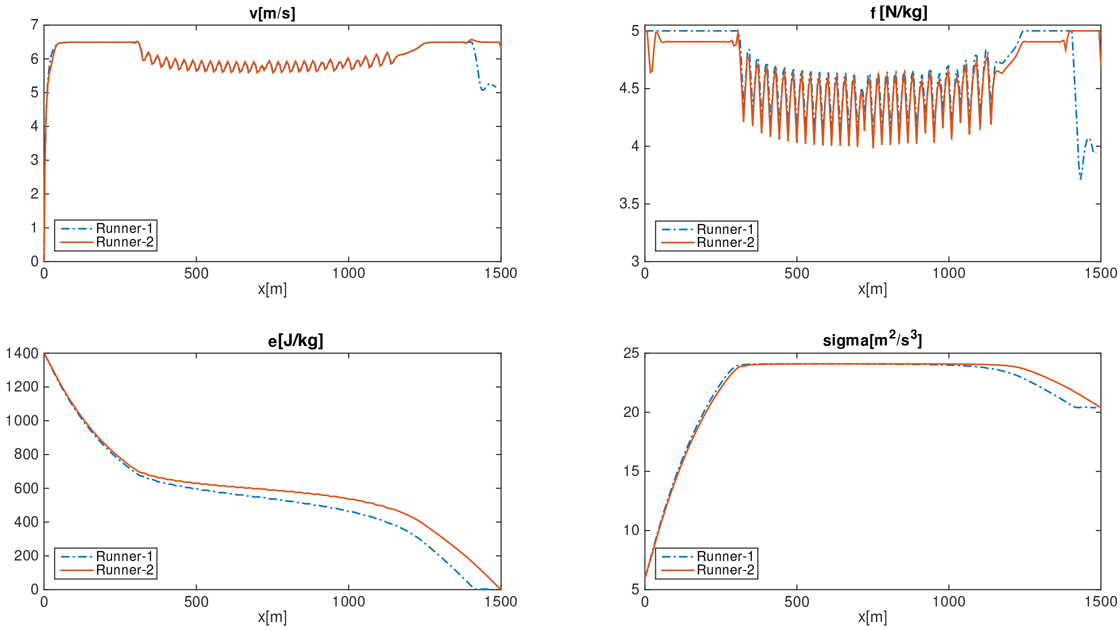

Figure 6: Competition between two runners with the same parameters, with the recreation term η.

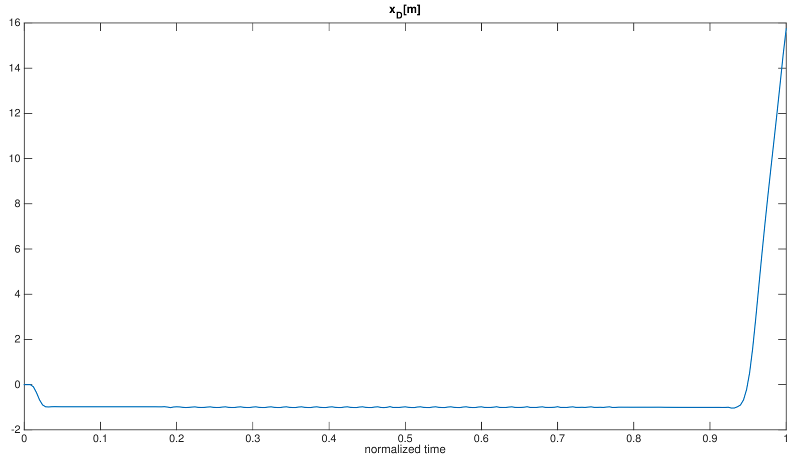

Figure 7: Competition between two runners with the same parameters, with the recreation term η - distance between the runners.

while still being able to win the race, is:

$$e_2^0 = 1275J/kg.$$

At the beginning of the race, runner-2 slows down in order to stay behind: in this way he manages to keep the same velocity as runner-1 using a smaller force; this leads to a smaller energy consumption, therefore at the end of the race he has enough energy to speed up and overtake runner one. For the boundary condition (11), the time stops as soon as the first runner finishes the race. Therefore, when runner-2 reaches the finish line (i.e. $1500m$), runner-1 has covered only $1498.13m$. The final time of runner-2 is $249.43s$, while his best performance running alone is $251.403s$, again an improvement of almost $2s$. As shown in Figure 9, the overtaking occurs later in the race, if compared with the case in which the runners were equally strong: here occurs at 99% of the race (i.e. about $1487m$). This is reasonable: in fact, being weaker, runner-2 has to exploit the advantage of staying behind as long as possible.

In Figures 10-11, the two runners have different oxygen uptake. The threshold values are the following:

$$\sigma_{max,1} = 24.22m^2/s^3,\ \sigma_{f,1} = 20.44m^2/s^3$$
$$\sigma_{max,2} = 23.75m^2/s^3,\ \sigma_{f,2} = 20.18m^2/s^3.$$

The initial strategy is the same as the previous case: runner-2 slows down in order to stay behind; this allows him to keep the same velocity as runner-1 using a smaller force and therefore compensating the smaller σ. The speed up in the final part is less evident than it was in the previous case, because the difference between the energies is smaller. The final distance covered by runner-1 in this case is $1499.72m$. The final time of runner-2 is $249.66s$, compared to $251.665s$ if

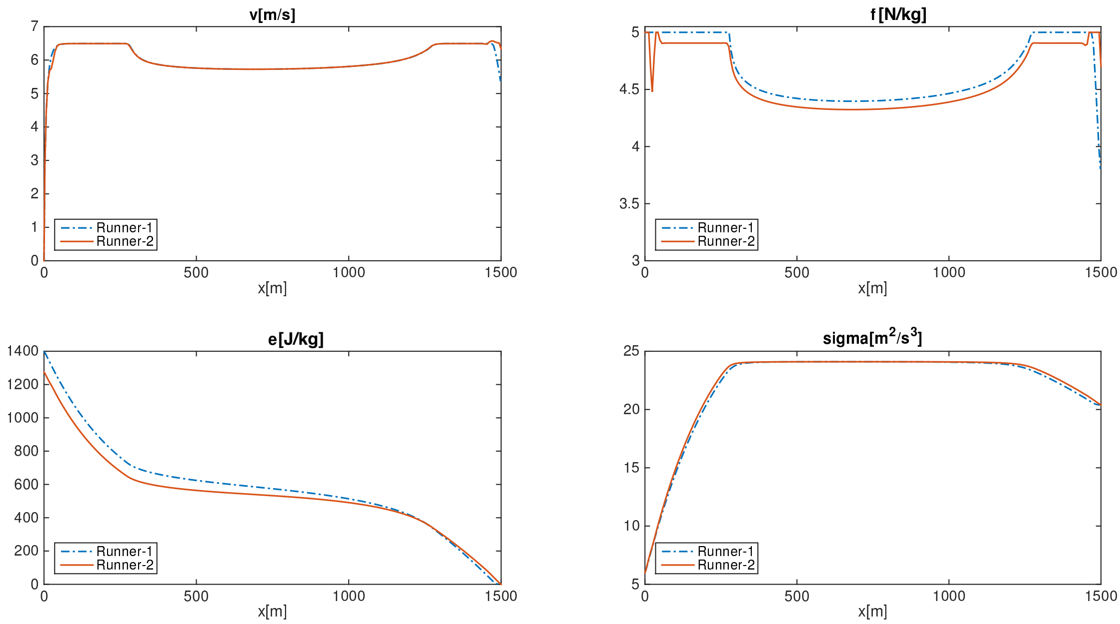

Figure 8: Competition between two runners with different e^0.

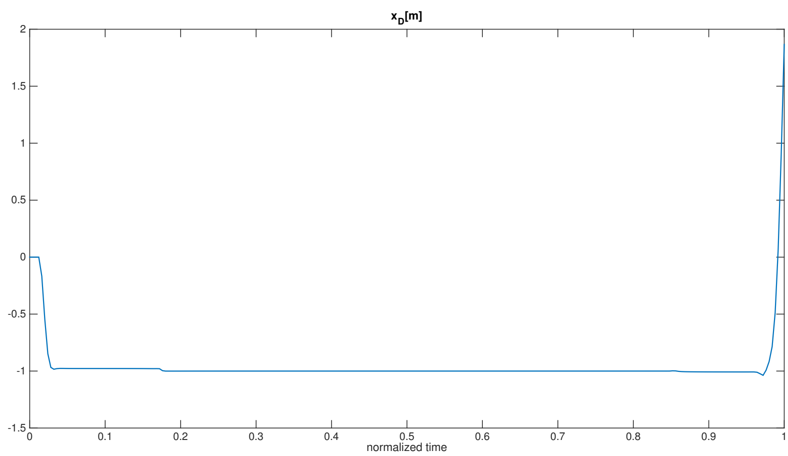

Figure 9: Competition between two runners with different e^0 - distance between the runners.

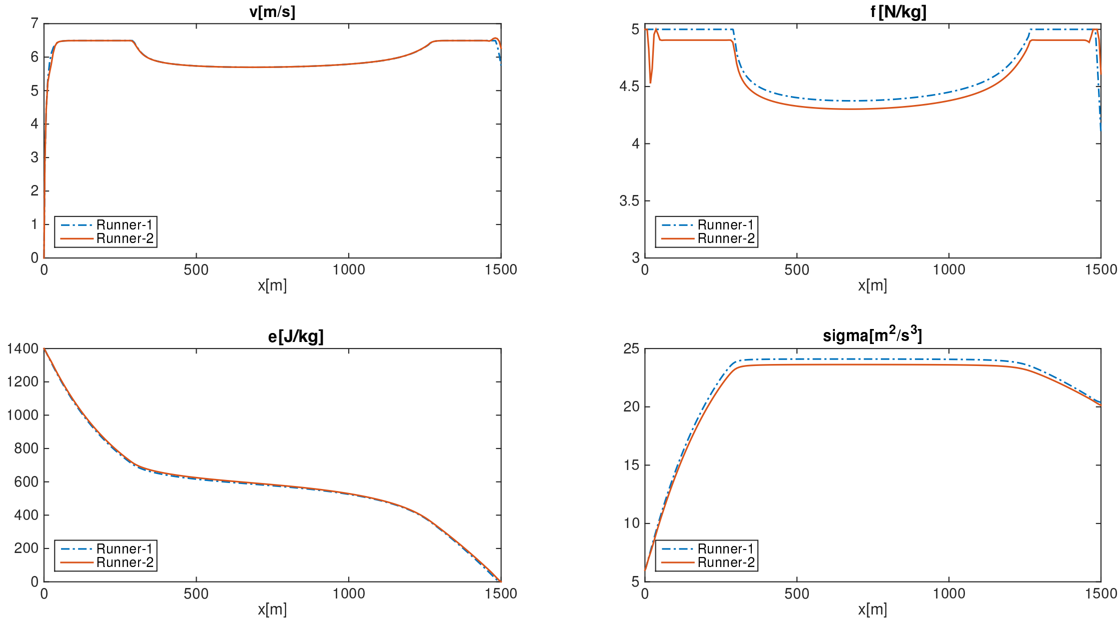

Figure 10: Competition between two runners with different σ.

running alone. Figure 11 shows the distance between the runners during the race: as in the previous case, the overtaking occurs late in the race.

Figures 12-13 show the results for two runners who have a different value for τ. The threshold values are:

$$\tau_1 = 1.33s \qquad \tau_2 = 1.31s.$$

A smaller τ indicates a bigger drop in velocity due to frictional effects, therefore, a greater force is necessary to keep the same velocity. Being behind another runner is a way to compensate this weakness: as shown in Figure 12, runner-2 manages to keep the same velocity as runner-1, using a slightly smaller force, in spite of having a smaller τ. In the final part, when the difference in the energies is sufficiently big, runner-2 overtakes runner-1. In this case, at the end of the race, runner-1 has covered a distance of $1498.82m$. The final time of runner-2 is $249.536s$, while his best performance running alone is $251.83s$, i.e. he has an improvement of more than $2s$.

Finally, let us consider a case in which the runner who starts behind is stronger. For this purpose, we use the following parameters:

$$\tau_1 = 1.29s \qquad \tau_2 = 1.33s.$$

All the other parameters remain unvaried (reference values from Table 1). Figures 14-15 show the results obtained: in this case, the difference between the athletes is much bigger than in the previous ones, and this is evident from all the curves in Figure 14. From Figure 15, one can notice that the overtaking occurs very early in the race: at about 87.1%, i.e. $1290m$. What is interesting in this case

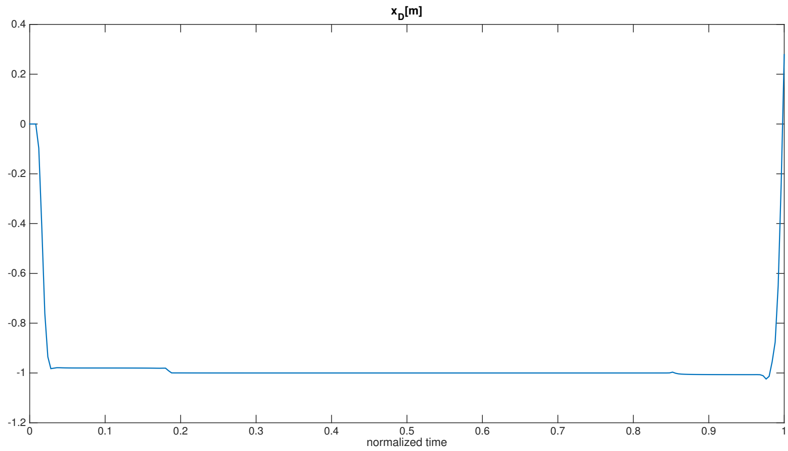

Figure 11: Competition between two runners with different σ - distance between the runners.

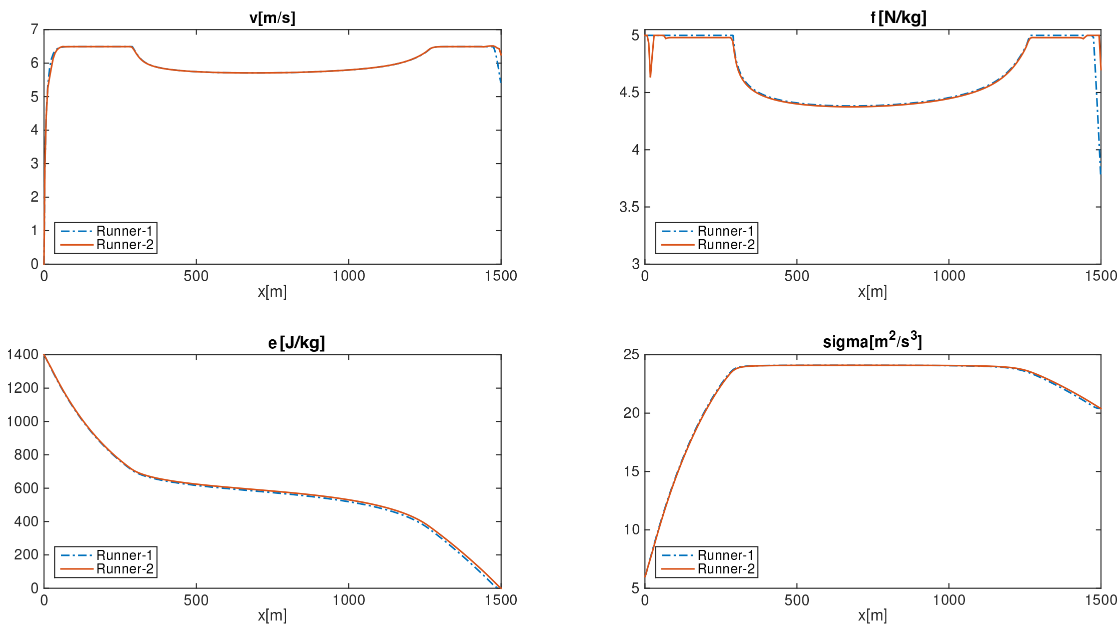

Figure 12: Competition between two runners with different τ.

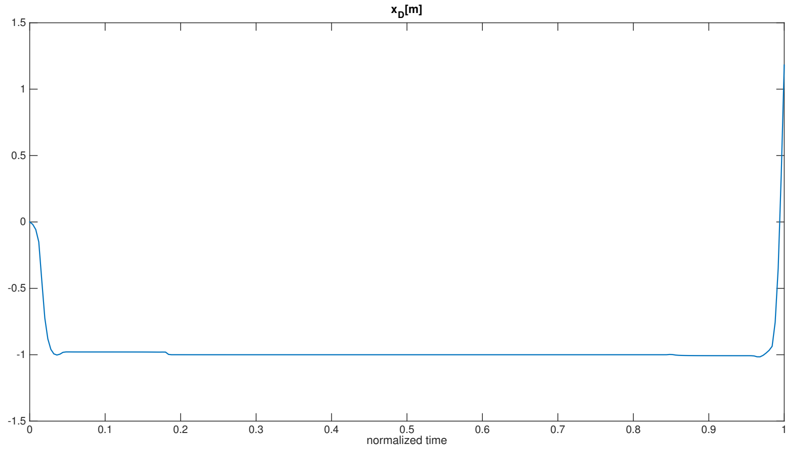

Figure 13: Competition between two runners with different τ - distance between the runners.

is that runner-2 completes the race in $248.726s$, which is almost $1s$ less than his best performance running alone: therefore, in order to improve a personal record, it is not necessary to run behind someone stronger.

Let us now compare the results obtained here with Pitcher's ones. In [16], Figure 5.2, when the weaker runner is the one with the fixed strategy, the stronger runner remains only slightly ahead of his opponent until nearly the end of the race. This is in order that the weaker runner does not gain the advantage of running in the slipstream of the stronger for a long part of the race. However, it is the runner who stays behind who should adjust his position with respect to the other one, and not the opposite. The advantage of having two strategies free allow us to avoid this unrealistic result, and in this case, we get that the weaker runner positions himself one meter behind, and either wins the race if the difference in energy is not too big or drops following if he does not have enough energy.

Finally, we want to analyse the strategy and to see when the overtaking occurs in real races and compare them with our results. For this purpose, we have considered the men's $1500-m$ finals of three different competitions: Beijing 2008 Olympic Games, Rome 2014 IAAF Diamond League and Singapore 2015 SEA Games. Videos of the races can be found on the internet [4, 5, 6]. We want to point out that the athlete who won the Beijing 2008 Olympics was disqualified one year later for doping and his gold medal was reassigned. Nonetheless, it is still interesting to analyse the race, knowing that doping increases the maximal value of σ but delays the time at which the peak velocity is reached: this should lead to a slower start, but it provides a capacity to keep a higher velocity for a longer part of the race. From the three videos [4, 5, 6], one can observe that the winner always starts behind, and this is consistent with our numerical results. The overtaking occurs at 84.6% of the race in Beijing 2008, at 96.9% in Rome 2014 and at 91.8% in Singapore 2015: these values are close to our numerical results,

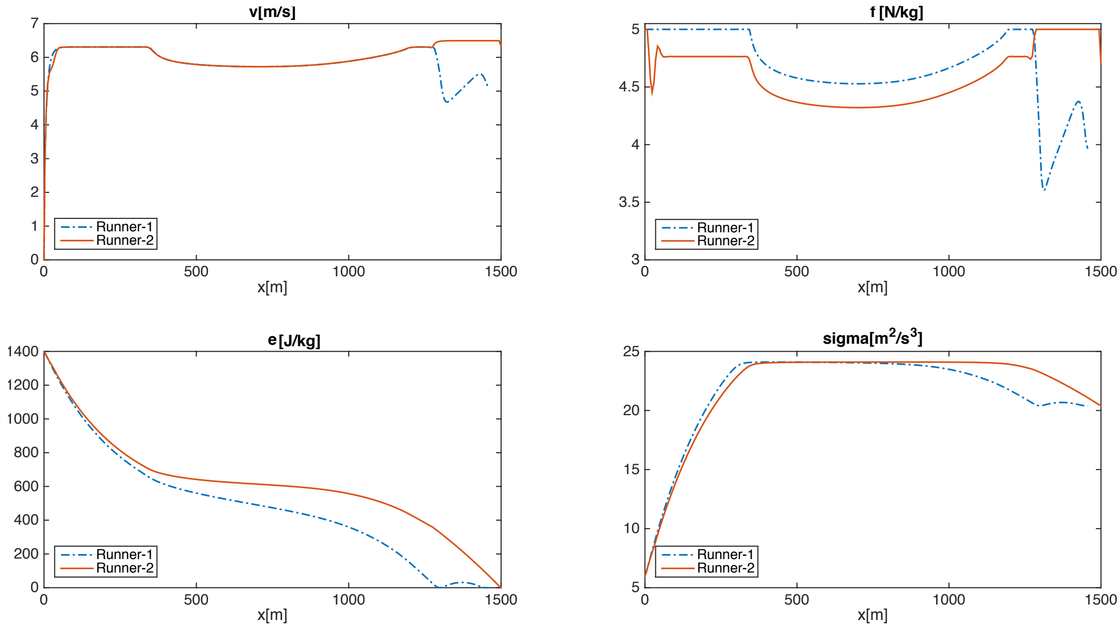

Figure 14: Competition between two runners with different τ - stronger starts behind.

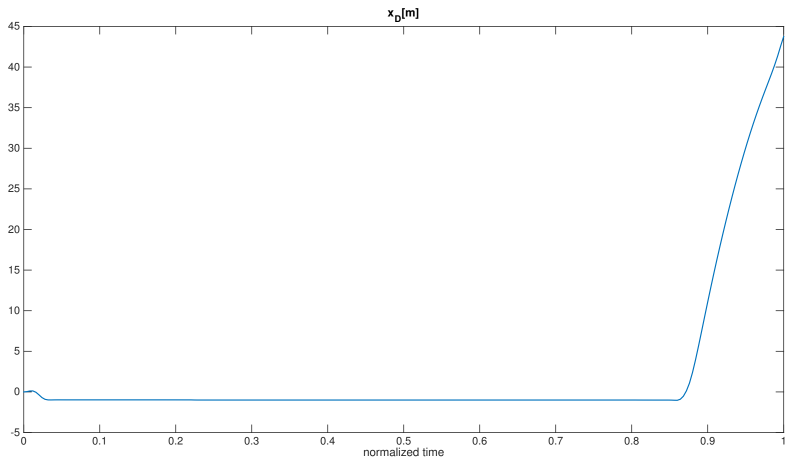

Figure 15: Competition between two runners with different τ - stronger starts behind - distance between the runners.

that vary between 87% and 99% of the race depending on the difference between the athletes.

4 Conclusion

In this work, we have presented a new model for a two-runners problem, starting from the single runner model of Aftalion and Bonnans [1] and from the two-runners model of Pitcher [16], changing the optimal control problem. The key of our simulations is that they quantify very precisely in terms of physiological parameters, optimal control problems and numerical simulations, phenomena which are only qualitatively understood. In this paper, we do not take into account the curvature of the track, which is the aim of an upcoming paper, since it requires more effort in the modelling.

As expected, going from one runner to two runners does not change the main characteristics of the velocity profile individuated already in [1]. We can still clearly distinguish the different phases of the race: the fast start, with maximal propulsive force and strong acceleration until the peak velocity is reached; an intermediate phase in which the propulsive force and the velocity first smoothly decrease and then increase; a final part at maximal force again, where the runner speeds up (final sprint), followed by a very short zero energy arc, in which there is a drop in force and velocity. Running behind someone allows to keep a velocity with a smaller propulsive force than the one needed when running alone; this leads to a smaller energy consumption, therefore the zero energy level occurs later and the speed up at the end of the race is more pronounced.

Our numerical results show that if a runner has a fast start and leads the race for the most part, even if he is slightly stronger than his opponent, at the end he is overtaken: in order to lead the race and win, the physiological difference between the athletes has to be significant. Furthermore, we have shown how a runner can improve his personal best performance by exploiting the advantage of running behind someone else, who can be stronger or weaker. The most significant improvements are obtained by running behind someone stronger.

This model suggests to use special runners to set the pace for others and help improve their racing times in training. The other major application for Olympic training could be for an athlete to estimate whether he should stay behind or lead, and when, given his physiology, and that of his opponents, is the best time to overtake.

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