Pace and Critical Gradient for Hill Runners: An Analysis of Race Records

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Abstract

Route choice through mountainous terrain requires a knowledge of how pace (the reciprocal of speed) varies with gradient of ascent or descent. To model this variation for runners, we analyse record times for 82 uphill and 14 downhill races or race stages. The pace is modelled as a polynomial function of gradient and a linear function of race duration, using ordinary least squares to obtain a best fit. For the gradient-dependence, a quartic function is derived which fits the data well; however, at steep gradients the quartic model is unrealistic and it may be argued that a linear model is more appropriate. Critical gradients, at which a runner's vertical speed (uphill or downhill) is maximised, may be calculated from the quartic. However, there remains considerable uncertainty over the true value, and even the existence, of these critical gradients.

Keywords: Gradient; Hill running; Nonlinear regression; Pace; Race records

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1 Introduction

Aside from navigational considerations, route choice in orienteering depends on the *runnability* and slope of the terrain. Runnability is the effect of vegetation or uneven ground on running speed, and is typically classified into four categories (indicated by different shadings) on orienteering maps. Slope principally affects running speed through the work required for vertical displacement, although there must also be a contribution independent of direction, given that traversing horizontally across a steep slope is slower than running across flat ground (Arnet, 2009). Whereas the runnability shading on an orienteering map gives clear guidance to competitors on the effect of vegetation on running speed, there is no direct indication as to how their speed will depend on steepness of ascent or descent. Ideally, a competitor should know his/her *pace function*, a formula yielding pace p as a function of gradient m in the direction of travel. *Pace* is defined here as time per unit horizontal distance (as shown on the map), and is a more convenient variable for route choice studies than speed. In an earlier study of the route choice problem (Kay, 2012) we proposed several pace functions, derived in a rather *ad hoc* manner from various data sources including race results. The present work is a more careful statistical study of results from hill races over a wide range of uphill and downhill gradients, with the objective of deriving a pace function $p(m)$ that should at least be valid for a "typical" elite male hill runner.

The only comparable previous analysis that we are aware of is that by Scarf (1998, 2007). However, that author used record finishing times for British fell races, which typically start and finish at the same location, and developed a model for record time as a function of total distance and total ascent. He also discussed Naismith's Rule, a formula which gives time as a linear function of ascent and distance (Langmuir, 1984), again under the assumption of finishing at the same altitude as the start. If we make the further assumption that the pace at all downhill gradients is equal to that on the flat, Naismith's Rule would give pace as a function of ascent gradient, with the latter calculated as total ascent divided by total distance. However, allowing for nonlinearity of the pace–gradient relationship and relaxing the other assumptions, it is not possible to determine the pace for either ascent or descent when only the finishing time of an up-and-down hill race is given.

An alternative to examining race results is to do controlled experiments. Townshend *et al.* (2010) measured speed and oxygen uptake for eight participants running a hilly course, although the mean gradients of the uphill and downhill sections were only 0.082. Minetti *et al.* (2002) measured the performance of ten elite mountain runners on an inclined treadmill with gradients up to a maximum of 0.45, both up and down, and derived the theoretical maximum running speeds at each gradient from their physiological measurements. They compared these speeds with actual speeds in small samples of uphill and downhill races and found good agreement uphill, whereas downhill race speeds were considerably slower than would be expected from physiological considerations alone. Both Minetti *et al.* (2002) and Townshend *et al.* (2010) refer to biomechanical constraints on downhill running, and on steep downhill slopes there may also be psychological constraints (concern over personal safety). Even for uphill running, Norman (2004) and Scarf (2007) argue that treadmill experiments may not give a true representation of overground running speeds. Hence we consider that an analysis of data from races will be the most reliable basis for route choice decisions.

There are certain constraints that the pace function $p(m)$ should satisfy. It will be fitted to data from uphill and downhill races, but should yield a reasonable pace on level ground: we quantify this by demanding that $p(0)$ should not be faster than the current (2011) 10,000 m world record pace, 0.1577 s.m^{-1}. Next, the formulation of the route choice problem by Kay (2012), based on a continuous representation of space, requires a twice-differentiable pace function (although this would not be necessary for a route choice algorithm such as that of Collischonn and Pilar (2000) which uses a discretisation of space). Furthermore, Kay (2012) reasoned that $\mathrm{d}^2p/\mathrm{d}m^2$ should be non-negative: a route at constant gradient should never be slower than a route with the same distance and total ascent but involving some steeper and some less steep terrain. If hill runners used the same gait at all gradients, a pace function satisfying these constraints would be a reasonable expectation. However, at some uphill gradient, athletes will spontaneously make the transition from running to walking. We do not have any data on the gradient at which this occurs, and there will certainly be considerable variation between athletes, but we can expect that participants in a race will choose their gait to optimise their performance. This implies that $p(m)$ will be continuous, but not necessarily differentiable, at a gradient where the gait changes. In summary, we expect $p(m)$ to be a smooth function with non-negative second derivative except possibly at gait-change gradients.

A runner's vertical speed (rate of height gain or loss) is m/p , which is maximised at a *critical gradient* m_c where the pace function satisfies

$$\frac{\mathrm{d}p}{\mathrm{d}m} = \frac{p}{m} \tag{1}$$

with $\mathrm{d}^2p/\mathrm{d}m^2 > 0$ (if a solution of (1) exists). It was first noted by Davey *et al.* (1994) that an optimal route would never ascend or descend more steeply than a critical gradient; the implications for route choice were discussed in more detail by Llobera & Sluckin (2007) (who sought routes minimising metabolic cost rather than time taken) and by Kay (2012). Hence we shall take a particular interest in the existence and value of uphill and downhill critical gradients for any pace function derived from our data.

2 Data sources

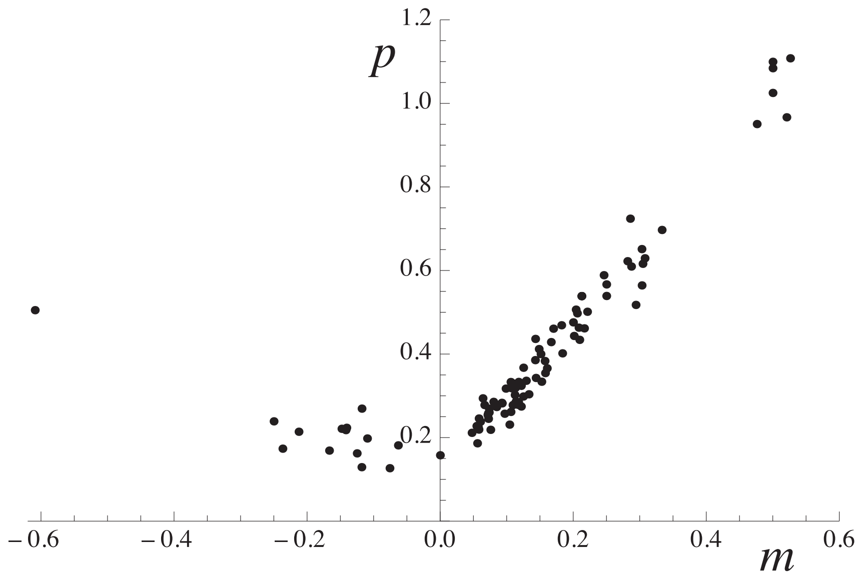

Figure 1: Record pace p (s.m^{-1}) vs. gradient m for 82 uphill and 14 downhill races or race stages and the world 10,000m track record.

All data were obtained from a search of publicly available websites conducted during 2010/11. Results from three types of race were obtained: uphill races, downhill races, and races including both ascent and descent, for which times for an uphill stage and/or a downhill stage were published. Altogether we used record times for 82 uphill and 14 downhill races or race stages, the difference in numbers reflecting the fact that uphill races are popular, especially in continental Europe, while downhill races are rare. The mean gradient and record pace was calculated for each race or race stage, and these are plotted in Figure 1 (which also shows the 10,000m world record at $m = 0$). Mean gradients of races ranged from 0.0476 to 0.526 uphill and from -0.0631 to -0.609 downhill (observing the convention that gradients are positive uphill). The extreme downhill slope of -0.609 (from the Tryfan Downhill Dash) must be considered an influential data point, not an outlier to be ignored. There is then a gap in the data to the second steepest downhill gradient of -0.25 at the Scafell Pike Race. A smaller but still significant gap also exists in the uphill data: there are six data points from Vertical Kilometre races at gradients between 0.526 and 0.476, but the gradient of the next steepest race (also a Vertical Kilometre) is 0.333. Data are also lacking at gentle gradients, both uphill and downhill. The fastest pace in the dataset is $0.127\,\mathrm{s.m^{-1}}$, recorded at the Meltham Maniac Mile ($m = -0.076$), while the fastest vertical speeds are $0.568\,\mathrm{m.s^{-1}}$ uphill at a gradient $m = 0.294$ at the Grouse Grind, and $-1.365\,\mathrm{m.s^{-1}}$ downhill at $m = -0.237$ at the Pen y Fan Race.

Durations of races or race stages in the dataset ranged from 300 s to 9160 s uphill and from 204 s to 2615 s downhill; our model will take account of the expected decrease in speed with increasing duration due to fatigue. However, there are many sources of variability or error that we have not been able to eliminate from our analysis of record times. First, to model pace as a function of gradient, data should ideally be taken from races or race stages at uniform gradient: no such race exists! The extent to which variability of gradient within a race introduces errors into our results depends on how nonlinear the relation between pace and gradient is. Second, we have taken no account of differences in the terrain of the races, which ranges from roads to heathery, boulder-strewn hillsides (although most are along well-trodden paths on which runnability effects will be minimal). Third, we have in most cases accepted without question the data on length and ascent given on race websites. There were a few cases in which we suspected the data to be erroneous and amended the data following a study of a map of the course. Fourth, the races ranged from prestigious events attracting internationally leading athletes to small-scale events attracting only athletes from local clubs. We have not made any allowance for the slower times expected in the latter.

3 Pace function for elite hill runners

3.1 Derivation from race records

A variety of functional forms have been suggested for pace functions for runners and walkers. Some, such as the exponential functions of Davey *et al.* (1994), have been proposed to model uphill pace only. Naismith's Rule and the power-law rule devised by Scarf (2007) could be used to derive pace as a function of gradient, but these rules are only intended to apply to courses with equal amounts of ascent and descent. Nevertheless, Naismith's Rule has been used for route choice problems by supposing that pace at all downhill gradients is equal to that on level ground (Scarf, 2008; Verriest, 2008; Arnet, 2009). However, this produces a discontinuity in the first derivative of the pace function $p(m)$ at $m = 0$, which does not seem reasonable since it does not relate to a change of gait. Unrealistic discontinuities are also found in some other functions proposed as models for pace on both uphill and downhill gradients: the "Hiking function" proposed by Tobler (1993) has a sharp minimum of pace (discontinuous $\mathrm{d}p/\mathrm{d}m$) at a downhill gradient $m = -0.05$, while Hayes & Norman (1984) proposed a set of rules which render $p(m)$ itself discontinuous. The simplest function satisfying our expectation of smoothness, with pace increasing as the gradient becomes very steep both uphill and downhill, is a quadratic as suggested by Rees (2004). However, this enforces an unwarranted degree of symmetry between uphill and downhill running: specifically, uphill and downhill critical gradients will have the same magnitude. Hence we seek to fit our data with higher-order polynomials: cubics and quartics, as suggested respectively by Kay (2012) and Llobera & Sluckin (2007). Other smooth functions have also been tried, but we have not been able to improve on the results shown below for polynomials. However, for comparison with earlier work it is also of interest to consider "Naismith-type" pace functions: piecewise linear, with a discontinuity in $\mathrm{d}p/\mathrm{d}m$ at $m = 0$.

We also wish to take into account the effect of race length on pace. While some quite sophisticated modelling of this effect has been done (Grubb, 1998), we are only concerned to eliminate it from our consideration of gradient effects; thus we shall follow Minetti *et al.* (2002) in assuming a simple linear dependence of pace on race duration. Hence we model our data on pace as

$$p = \tilde{p}(m)(1 + kT) \tag{2}$$

where the *adjusted pace* $\tilde{p}(m)$ represents the dependence of pace on gradient (and is therefore the function to be used in route choice studies), and T is the race duration (so that where pace and gradient data have been taken from an uphill or downhill stage of a longer race, T is the duration of the entire race, not just the stage).

Using a quartic form for $\tilde{p}(m)$, the best fit of (2) to our data, obtained by ordinary least squares, has

$$\begin{aligned} k &= 4.446 \times 10^{-5}\,\mathrm{s}^{-1}, && (3) \\ \tilde{p}(m) &= 0.1707 + 0.5656m + 3.2209m^2 - 0.3211m^3 - 4.3635m^4 && (4) \end{aligned}$$

(with units of pace being $\mathrm{s.m}^{-1}$). The quartic model (2) – (4) has the following features:-

- A minimum of $\tilde{p}(m)$ at $m = -0.0885$, i.e. the model estimates that the fastest horizontal speed is attainable at a gradient roughly midway between the -0.1

quoted by Minetti (1995) as being optimal in terms of metabolic energy expenditure and the gradient of -0.076 at which the fastest pace (and adjusted pace) in our dataset occurs;

- Critical gradients $m_{c+} = 0.3152$ and $m_{c-} = -0.2617$; these are the estimated gradients for fastest vertical speeds, uphill and downhill respectively;

- Inflection points of $\tilde{p}(m)$ at $m = 0.3328$ and $m = -0.3696$, so that $\mathrm{d}^2\tilde{p}/\mathrm{d}m^2 \geq 0$ only between these points. [A cubic form for $\tilde{p}(m)$ yielded a best-fit function with $\mathrm{d}^2\tilde{p}/\mathrm{d}m^2 \geq 0$ throughout the range of gradients encompassed by the data, but in every other respect that function was an inferior representation of the data compared to the quartic.]

- Adjusted-$R^2 = 0.9944$ – this is the appropriate measure of the goodness of fit of the model to the data.

The adjusted pace may be calculated for each data item as

$$\tilde{p}_i(m_i) = \frac{p_i}{1 + kT_i} \tag{5}$$

(in which subscript i refers to i'th data item). The quartic function is plotted together with these adjusted pace data in Figure 2. Note that $\tilde{p}(0)$ is about 16% slower than the adjusted pace of the world 10,000m record on the track, which is reasonable for a model based on data from races on mountain paths.

The uphill and downhill critical gradients are fairly close respectively to the gradients 0.294 and -0.237 at which the fastest positive and negative vertical speeds in our dataset were recorded. However, the value of m_{c+} is also rather close to an inflection point, which means that a slight "tweak" to the curve could make this critical gradient vanish. Clearly the uphill optimum, if it exists at all, is very shallow so that quite large variations in

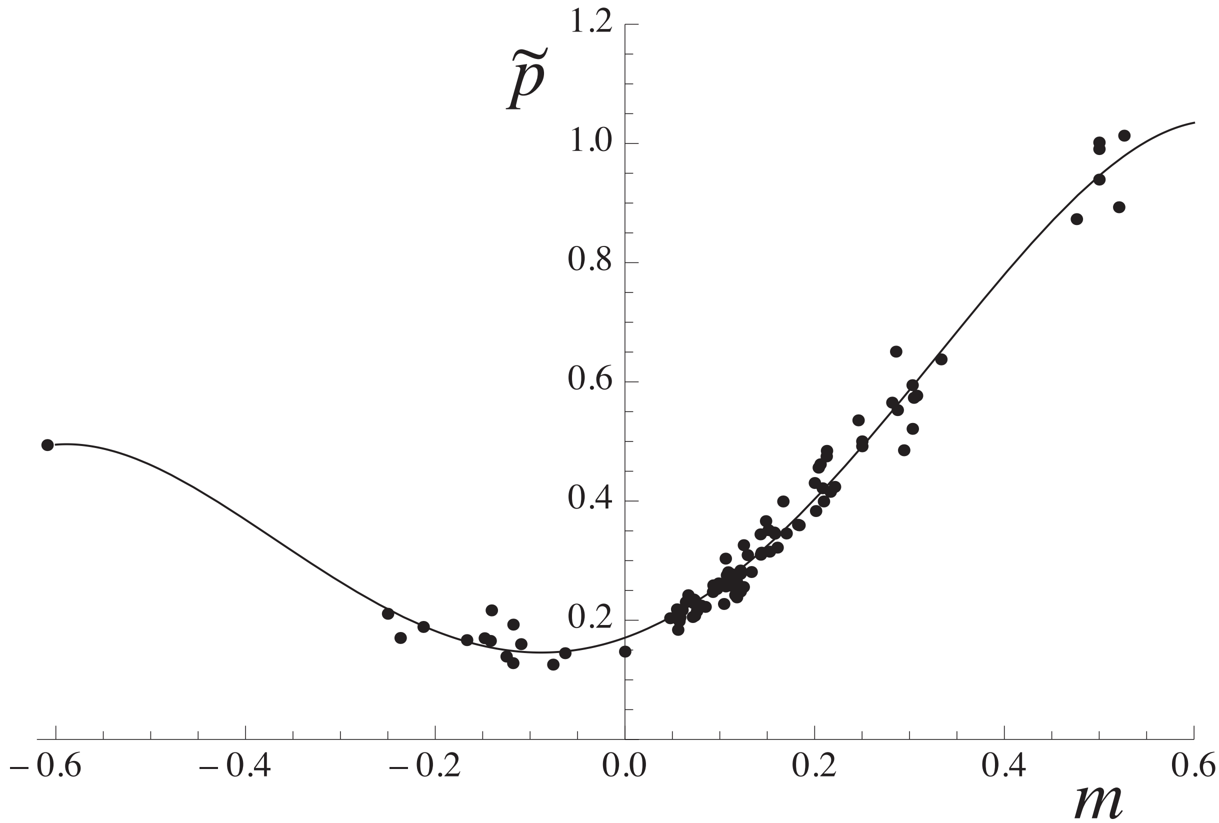

Figure 2: Adjusted pace data with quartic model; $\tilde{p}$ in s.m^{-1}.

gradient above 0.25 will make little difference to vertical speeds. Terrain is probably more important than steepness in determining vertical speed at these steep gradients. The fastest and third fastest ascent rates in our dataset are on courses on which substantial sections have steps cut into the mountainside, at mean gradients 0.294 and 0.303. The advantages of steps at steep gradients are discussed by Minetti (1995), while races up skyscraper stairs often yield ascent rates considerably greater than in any mountain race (Minetti *et al.*, 2011). Removing the two stepped courses from the dataset yields a best-fit quartic model with no uphill critical gradient, implying that ascent rate would be optimised by climbing as steeply as possible – this is consistent with the fastest ascent rate remaining in the dataset being at the very steep gradient of 0.521.

3.2 Comparison with Naismith's Rule

Naismith's Rule states that a horizontal distance x_1 and an ascent y_1 each require a time t_1, with $x_1 = 3$ miles, $y_1 = 2000$ feet and $t_1 = 1$ hour in Naismith's original formulation. Thus the time required for a journey of distance X and ascent Y may be calculated as $(X/x_1 + Y/y_1)t_1$. This is equivalent to a piecewise linear pace function,

$$p = \begin{cases} p_0 & (m < 0) \\ p_0(1 + \alpha m) & (m \geq 0) \end{cases} \tag{6}$$

with $p_0 = t_1/x_1$ and $\alpha = x_1/y_1$. Scarf (2007) refers to α, which gives the equivalence of distance and climb, as the *Naismith's number*. With Naismith's values of x_1 and y_1 we find $\alpha = 7.92$. However, Norman (2004) obtains the value $\alpha = 4.4$ from the results of a hilly road race, while an ordinary least squares fit of the Naismith function (6) to our dataset (using our procedure to account for fatigue) yields $\alpha = 11.7$, although the fit is much worse (adjusted-$R^2 = 0.9820$) than for our quartic model.

Defining Naismith's numbers for nonlinear pace functions is more problematic. Scarf (2007) continues to refer to the coefficient α as Naismith's number in his generalisation of Naismith's Rule which can be written in terms of gradient and pace as

$$p = p_0(1 + \alpha m)^\beta \,, \tag{7}$$

for which he obtains $\alpha = 8.6$ and $\beta = 1.14$ from British fell race data. Davey *et al.* (1994) propose an exponential pace function and regard the equivalence of distance and climb as being given by the quantity

$$N = \frac{[\mathrm{d}p/\mathrm{d}m]_{m=0}}{p_0} \,, \tag{8}$$

i.e. using the linear approximation about $m = 0$, although Norman (2004) warns that this can only be valid at gentle slopes; Davey *et al.* (1994) obtain $N = 2.8$ from data on treadmill experiments. According to (8), the pace function (7) would have $N = \alpha\beta = 9.8$, while our quartic model yields $N = 3.31$.

The wide diversity of values of Naismith's number (α or N) has been attributed by Norman (2004) and Scarf (2007) principally to differences in the running environment (treadmill, road or rough mountain terrain), while Kay (2012) suggested that it was related to the derivation of some values from up-and-down races while others come from experiments on uphill running. These explanations all have some validity, but we contend here that the fundamental problem is that a linear model is being imposed on an inherently nonlinear phenomenon. A linear model can be justified only at steep gradients: Minetti (1995) observes that if $|m| > 0.15$ the work done in running or walking is purely positive (uphill) or purely negative (downhill), and that the efficiency is constant. Thus a constant work rate should imply a constant vertical speed, so that $p \propto m$. Fitting $\tilde{p} = b_{\pm}m$ to the 35 uphill data items with $m > 0.15$ and separately to the 5 downhill data items with $m < -0.15$, we obtain $b_+ = 1.837\,\mathrm{s.m}^{-1}$ and $b_- = -0.8012\,\mathrm{s.m}^{-1}$ with respective adjusted-R^2 values of 0.9939 and 0.9936. For $|m| < 0.15$, nonlinearity is certainly important, so a Naismith's number can only be a compromise between the true linear behaviour at steep uphill gradients and the nonlinear behaviour at gentler gradients. This compromise seems to work fairly well when predicting times taken in up-and-down fell races and mountain walks (Scarf, 2007), but yields the much smaller value of α found by Norman (2004) from a comparatively gently graded road race, and makes the use of (8) particularly inappropriate.

3.3 Implications for route choice

Our original motivation for this study was the orienteering route choice problem, for which we need to know pace as a function of gradient. Our quartic model provides the best fit to the data, but is problematic at steep gradients. It has critical gradients at which vertical speed is optimised, but they are rather shallow optima; this finding is consistent with the shallow minima of the vertical energy cost of walking and running found by Minetti *et al.* (2002), although our critical gradients are somewhat steeper. However, the quartic model also has unrealistic inflection points at gradients slightly steeper than the critical values, which would certainly lead to spurious results from route choice computations on terrain steeper than the inflection-point gradients.

We have argued on physiological grounds that a linear model is appropriate at gradients $|m| > 0.15$, in which case there would be no optimum gradient for vertical speed. The existence of optima would arise from biomechanical and/or psychological factors, which we would expect to be particularly important downhill. Nevertheless, the linear model does fit the data very well at gradients $|m| > 0.15$, both uphill and downhill; rather remarkably, removing the data item at the extreme downhill gradient of -0.609 makes less than 1% change to the slope of the line fitted to the remaining four steep downhill data points. However, the participants in the Tryfan Downhill Dash are probably a self-selecting group of expert descenders, so it is not valid to suggest that biomechanics and psychology are unimportant at very steep downhill gradients.

We thus conclude that the quartic model can be used unaltered in route choice calculations on terrain with gradients nowhere steeper than the lesser of m_{c+} and $|m_{c-}|$, but the correct model to use on steeper terrain is debatable. If an uphill or downhill critical gradient does exist, then an optimum route will never ascend or descend more steeply than the respective critical gradient; but if the pace function is linear at steep gradients, then the optimum route over steep terrain will always be a straight line (Kay, 2012). In the former case, the fastest route up or down a steep hill will be a zigzag, but in the latter case it will go straight up or down; yet the time taken on these very different routes may be very nearly the same. Taking the quartic model as valid where $m_{c-} \leq m \leq m_{c+}$, we seek to construct a model valid at all gradients by adjoining other functions at steeper gradients. If we believe that critical gradients do exist, we use quadratic functions, determined by a requirement to join smoothly to the quartic (with continuous first and second derivatives) at the critical gradients; this will enforce a route choice not steeper than the critical gradient. Alternatively, if we believe that the pace function must be linear at steep gradients, we can join linear functions, $\tilde{p} = b_{\pm} m$, to the quartic with a continuous first derivative at the critical gradient; this would allow direct routes up steep hills, but such a route would take the same time to run as the zigzag route at the critical gradient, or indeed any route of intermediate steepness. For completeness, we present below the quadratic and linear functions to be adjoined to the quartic model (4) following this procedure, on the understanding that these functions are merely devices to avoid the unrealistic features of the quartic model and to allow route choice computations to be made with or without critical gradients:-

- Uphill ($m > m_{c+} = 0.3152$):

  Quadratic: $\tilde{p}(m) = 0.0314 + 1.7544m + 0.3162m^2$

  Linear: $\tilde{p}(m) = 1.9538m$

- Downhill ($m < m_{c-} = -0.2617$):

  Quadratic: $\tilde{p}(m) = 0.1151 + 0.0061m + 1.6802m^2$

  Linear: $\tilde{p}(m) = -0.8732m$

Once an optimal route has been computed between fixed points A and B, a notional time $\tilde{T}$ taken by a runner at the adjusted pace could then be computed by integration along the route,

$$\tilde{T} = \int_{\mathrm{A}}^{\mathrm{B}} \tilde{p}\,\mathrm{d}s\,. \tag{9}$$

Given that adjusted pace is a constant multiple of true pace along the route, the true time taken would then be computed as

$$T = \frac{\tilde{T}}{1 - k\tilde{T}}\,. \tag{10}$$

4 Conclusions

We have sought a formula to give runners' pace as a function of gradient of ascent or descent. Obviously the pace function will vary between individuals, and will also depend on the terrain. Since our calculations use record times for races, the results can be regarded as broadly applicable to elite male runners on typical hill race terrain, which consists of well made mountain footpaths in most of the continental European races which provide the majority of our data.

Our principal result is the quartic function (4), which is the best fit to the data after taking account of increasing fatigue in longer races. However, there is a shortage of data at very steep gradients (and also at very gentle gradients), with the consequence that the quartic model cannot be valid for gradients much steeper than 0.3. Furthermore, this data shortage leads to considerable uncertainty over the existence of critical gradients, at which vertical ascent and descent speeds are optimised. We have presented ways of "fixing" the quartic model to allow it to be used for route choice on all terrain, with or without a critical gradient according to the choice of the user. Of course, even a "fixed" model should not be used outside the range of gradients encompassed by our data. Optimal gradients for vertical speed must exist, since climbing or descending a near-vertical rock face is a slow process. Uphill, competitors nearly always take a direct line wherever this can be done without encountering a rock face, so m_{c+} may well be outside the range of our data – possibly at the gradient where walking gives way to scrambling, or scrambling to climbing. Given the requirements of agility and courage for steep downhill running, it is likely that most runners have a more moderate value of m_{c-}, with our estimate of -0.26 seeming very plausible.

The effect of fatigue is modelled by a linear dependence on race duration, with a coefficient $k = 4.446 \times 10^{-5}\,\mathrm{s}^{-1}$ for the quartic model. This contrasts with a value $k = 1.773 \times 10^{-5}\,\mathrm{s}^{-1}$ implied by the formula used by Minetti *et al.* (2002), and with values around $k = 7 \times 10^{-5}\,\mathrm{s}^{-1}$ obtained when we made linear fits to our steep uphill and steep downhill ($|m| > 0.15$) data. Since the effect of fatigue is not our primary concern, we have not investigated these differences, but they may be worthy of further study.

The transition from running to walking is not discernible in our data. Any effects of the gait change would be smoothed out because it would occur at different gradients for different runners, and would also depend on the terrain. Indeed, the terrain is a major source of variability in the data, and we have noted in particular that stepped paths are likely to facilitate fast uphill running.

Our results place severe limitations on the validity of Naismith's Rule or a Naismith's Number for competitive mountain running. If used as a pace function, the rule imposes a linear approximation on a nonlinear phenomenon, and cannot be valid at all gradients. Nevertheless, a formula such as (7) may still be valuable if used as intended, i.e. to model overall pace in races which start and finish at the same altitude.

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