Optimization models for the force and energy in competitive running

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Abstract. In [2] the author has developed an optimization model for the force and energy in competitive running. In this paper the energy processes in the muscle were described by a three-compartment hydraulic model. Here this is reviewed briefly and applied to the current world records in order to determine the key parameters, maximal force, energy reserves and oxygen uptake. These values agree well with those given in the literature and those obtained by other means. The velocity profiles for 100 m sprints are described equally well. The model is then applied to older world records to deduce a relation between the force and energy by linear regression. Finally the fully parameterized model is used to compute the effects of adverse wind and altitude. Inasmuch as there are data available, there is a good agreement.

Key words: Competitive running – Force – Energy

Introduction

In [2] a simple mathematical model for competitive running has been developed. Its key variables are the forces and the energy reserves. It is made up of three submodels, one for the biomechanics of running, one for the energetics and an optimal control model of the running strategies. The model for the energetics is an extension of the hydraulic model of Margaria [11] and Morton [12].

Here this model will be applied to the world records in running. Thus the validity of the model will be checked, and basic physiological parameters and their relationships can be determined. The model with parameters determined from the world records is then applied to study the influence of adverse or favourable wind and altitude. Since these effects are in general smaller than the individual variations in running performance, we will speak of small effects henceforth. Comparison with real data shows the model to be fairly accurate.

This in turn should be of considerable concern to the track-and-field community, because the influence of these small effects is often in the order of 1 per cent or 0.1 sec, one order of magnitude higher than the present accuracy in the electronic measurement of running time. It is obvious that such results should have an impact on track-and-field regulations or the definition of world records.

This paper is divided into five sections: 1 The model, 2 World records, 3 Adverse wind, 4 Altitude, 5 Discussion.

Throughout we use the m, kg, sec system. However, for reasons of normalization and easier comparison, many quantities such as force, energy reserves or power will be defined per kilogram of bodyweight. This rescaling of the mass effectively makes the mass dimensionless.

1 The model

We describe the motion of the athlete as a one-dimensional motion, where the coordinate axis coincides with the track. Thus $x(t)$, $v(t)$ will denote the position and velocity of the centre of mass at time t. Newton's law of motion in the mass rescaled form is then

$$v' = f - r_{\text{int}}(v) - cv^2, \qquad v(0) = 0 \tag{1}$$

where f is the force per kilogram of body weight. $r_{\text{int}}(v)$ describes the internal dissipation of energy due to acceleration and deceleration within each step cycle. It is given by

$$r_{\text{int}}(v) = \begin{cases} r_0 & v \leqq 6 \text{ m/sec} \\ r_0 + r_1(v - 6) + r_2(v - 6)^2 & v \geqq 6 \text{ m/sec} \end{cases} \tag{2}$$

This form of r_{int} is dictated by the fact that the oxygen consumption at moderate velocities for men and animals is proportional to v. For higher velocities, $v \geqq 6$ m/sec, (2) is just the Taylor expansion to the second order of a resistance function. The air resistance term cv^2 is derived from

$$\tfrac{1}{2} AC \cdot \rho M^{-1} v^2 = cv^2 \ .$$

Here A measures the frontal area of the athlete and M is his mass. ρ is the density of air and C a geometric shape factor, which is found to be about 0.87 for men [5]. This gives approximately $c = 0.00375 \text{ m}^{-1}$. Since muscles and tendons have a limited strength, one has

$$0 \leqq f \leqq F \tag{3}$$

In the muscles, chemical energy is transformed into mechanical energy. This conversion process is governed by an efficiency factor η, which is a monotonically increasing function of v. From empirical data the piecewise linear approximation

$$\eta(v) = \begin{cases} 0.6 & v \leqq 6 \text{ m/sec} \\ 0.6 + 0.04(v - 6) & v \geqq 6 \text{ m/sec} \end{cases} . \tag{4}$$

is derived [2]. Since the cost of transport of 1 kilogram per metre is about 1 cal $= 4.184$ J [11], one finds

$$r_0 = 2.3754 \text{ m/sec}^2 \qquad (5)$$

Sprints are almost entirely run with maximal force F. With the separate modelling of a starting phase, which includes a reaction time of $\tau = 0.15$ sec and a higher initial acceleration, (1), (2) and (3) allow sprints to be described. Analysing the data of Ballreich [1] shows $r_2 = 0$. This is to be expected, because in a run with almost constant velocity v, the kinetic energy dissipated per step cycle is $\frac{1}{2}(v + \Delta)^2 - \frac{1}{2}(v - \Delta)^2 = 2v\Delta$ if the velocity varies by $\pm \Delta$. Thus the dissipated power is $2v\Delta v$, where v is the step frequency. Even though there is a slight indication that r_1, which is a measure of the efficiency in running, is slightly higher for men than for women, we will use a common parameter for both. With

$$F = 6.93 \text{ m/s}^2 \text{ (men)}, \qquad F = 6.22 \text{ m/sec}^2 \text{ (women)}, \qquad r_1 = 0.705 \text{ sec}^{-1} \quad (6)$$

the following is obtained for the current world record sprints.

Table 1. World record sprints

Men			Women		
D [m]	T [sec]	D_{comp} [m]	D [m]	T [sec]	D_{comp} [m]
50	5.61	50.39	50	6.00	50.50
60	6.41	59.69	60	6.92	60.39
100	9.85	99.95	100	10.49	99.00
$F = 6.93 \text{ m/s}^2$		$r_1 = 0.705 \text{ sec}^{-1}$		$F = 6.22 \text{ m/s}^2$	

The energy for running and general muscular activity is mainly derived from the following sources, listed in their usual order of recruitment and power:
   i) ATP and creatine phosphate, the alactic anaerobic store
   ii) glycolysis, the anaerobic transformation of glucose or glycogen to pyruvic or lactic acid
   iii) the oxydation of carbohydrates, mostly glucose or glycogen, and lipids.
The release of energy from these sources and their interplay is regulated enzymatically and to a lesser extent hormonally in a rather complex way. In particular, these processes are rate and capacity limited.

ATP is the only immediate source of energy, because the contraction of the muscle involves the hydrolysis of ATP to drive the sliding of the actin and myosin filaments in the muscle fibres. Thus all other processes operate via ATP production and the system tries to keep the ATP concentration in the muscle at a constant level. Since the ATP stores would be depleted within 3 seconds in intensive muscular work, they are replenished within milliseconds from the creatine phosphate stores. Only when this source is depleted does

a decrease of power set in. In intensive muscular work this happens after approximately 7 seconds or when the phosphagen stores, i.e. the combined ATP-creatine phosphate stores are reduced to about 60%.

Glycolysis is the second most important source of energy. It sets in when the phosphagen stores are low, because it is triggered by ADP in the muscle. In order to provide sufficient power for a longer time, about 20–40 sec, its onset has to be rather rapid.

Oxydation of carbohydrates and fats is the most important source of energy for longer lasting activities. Increased oxygen uptake is a delayed process with a characteristic time of about 30 seconds, because it is triggered by ADP in the muscle. Thus the main function of the two anaerobic processes is to bridge this time interval. Hence we expect a relation of the form

Oxygen deficit $\approx 30 \times$ maximal oxygen uptake

$\approx$ Capacity of the anaerobic processes/Safety factor.

This is indeed the case with a safety factor of about 3, which is quite common in nature. Glycolysis is a rather inefficient process because it only produces 2 ATP-molecules from one molecule of glucose compared to 38 ATP molecules in the oxydative process. Since the interplay of these processes is rather complicated, Margaria [11] has proposed a simplified description in terms of a hydraulic model. This was later refined and extended by Morton [12]. Basically, it is a three-component compartment model in which the flow rates are given by the hydraulic pressure of a liquid representing the energy. The differential equations governing this system are thus piecewise linear.

The hydraulic model consists of three vessels, one for each source of energy in the muscle. The central vessel (P) represents the phosphagen stores. It is connected to the glycolytic vessel (L), because ATP is formed from ADP via glycolysis. (P) is also connected to the glucose-lipid vessel $(\mathcal{O})$, which corresponds to the oxydation of glucose or fats. The basic description of this model is given in [2].

The volumes of these vessels are denoted by V_P, V_L and V_0. They correspond to the capacities of the relevant sources. Since all energy-providing processes are rate limited, the flow rates from the vessels are bounded. The maximal flow of energy out of (P), which corresponds to the maximal muscular power is thus M_P, while the maximal flow from (L) or $(\mathcal{O})$ to (P) is denoted by M_L or $M_{\mathcal{O}}$. Since glycolysis uses glucose, there is also a flow from $(\mathcal{O})$ to (L). In order to model fatigue, the exhaustion of ATP and an increasing acidosis in connection with glycolysis, further dynamic limits on the flows have to be imposed. The geometric parameters θ, λ, ϕ and μ reflect the dynamics of this system. In [2] they were chosen so that the hydraulic model corresponds most closely to the real muscle system,

$$\theta = 0.6; \qquad \phi = 0.25; \qquad \lambda = 0.33 . \tag{7}$$

The state of this three-compartment hydraulic system at time t is given by $(h(t), l(t), k(t))$, the levels in the vessels $(P), (L)$ and $(\mathcal{O})$, respectively. The

dynamics are then determined by three coupled piecewise linear differential equations in h, l and k [2].

With respect to the models of Margaria and Morton the following changes have been made:

- i) ($\mathcal{O}$) has a finite capacity, modelling fatigue in longer lasting events.
- ii) The transition to an increased burning of lipids is modelled by a change in the cross-section of the ($\mathcal{O}$) vessel, because there is a nearly unlimited supply of lipids and because the oxygen equivalent of fat is lower than that of glucose.
- (iii) There is a connection from the ($\mathcal{O}$) vessel to (L), reflecting the fact that glycolysis uses glucose heavily. This connection is not governed by the hydraulic pressure, but determined by the glycolytic power.
- iv) There are additional dynamic rate limitations of energy production in (P) and (L).
- v) The geometric parameters correspond more closely to physiological principles.

Since we are mainly dealing with distances $D \leqq 5000$ m, we will neglect the change in the cross-section of the $A_{\mathcal{O}}$ vessel, which was proposed in [2] to model the increased burning and almost unlimited supply of lipids.

In the combined model the power output $A_P h'$ of the energy model is equated to $f \cdot v / \eta$, the power demand of the runner. Here f, the force of the legs, can be chosen freely subject to (3). Thus f can be considered as the control variable in an optimal control system, which is defined with (1), (2), (3) and the equations for the energy model. The corresponding optimal control model can be solved qualitatively. The optimal solutions for v are approximately of the following form:

- i) On some initial time interval $0 < \tau \leqq t \leqq t_1$, the runner accelerates with maximal force F.
- ii) On $[t_1, t_2]$ the velocity v is approximately constant $v = v_d$. t_2 is determined by the fact that v_d cannot be kept up any more, i.e. one of the constraints from the energy model becomes active.
- iii) On $[t_2, T]$ the run is completed with active power constraints.

Functions v satisfying these conditions will be called approximate solutions. With all parameters fixed, they depend essentially on t_1, the acceleration time. Thus the optimization amounts to the determination of an optimal t_1, given T. This is a considerable simplification, because the initial conditions for the model are given at $t = 0$, while the initial conditions for the adjoint system are given at $t = T$.

The optimal control models also show that longer distance runs should involve the glycolytic pathway less and less. In nature this is realized by the Pasteur effect, which cuts off glycolysis when the oxygen uptake has reached its maximal value. In the model this could only be realized by a further dynamic restriction on W_L, the power of the glycolytic channel. A long-distance model for distances $D \geqq 5000$ m should therefore neglect glycolysis completely.

2 World records

The model still contains a number of parameters which have to be determined. Some of these are known a priori, e.g. c, and some are almost the same for all athletes, because they reflect a common biological process or principle, while others depend on the individual athlete.

The maximal force F is an individual quantity, while r_0, r_1 and c are almost the same for all good runners, since they are of similar build and running style. We will thus speak of universal or individual parameters. In the energy model the geometric parameters θ, λ, ϕ and μ are almost the same for all runners, because these parameters reflect the underlying dynamics. Since the anaerobic reserves are stored in the muscle and because of similar dynamics in the energy providing process we can expect that A_L/A_P and M_L/M_P are almost universal, too. From optimal design in biology one expects that

$$M_P \approx \frac{F \cdot v_\infty}{\eta(v_\infty)} \tag{8}$$

where v_∞ is the maximal velocity in a sprint, $F = r_{\text{int}}(v_\infty) - c\, v_\infty^2$.

In [11] Margaria states $V_L \approx 1.4\, V_P$ and $M_L \approx 0.4\, M_P$. Here we use

$$V_L = 1.25\, V_P \quad \text{and} \quad M_L = 0.35\, M_P \tag{9}$$

in order to de-emphasize the role of glycolysis. In [2] we stated $A_0 \approx A_P \cdot 90$ because the glycogen stores can support about 90 minutes of marathon running. Here we employ

$$A_0 = 100\, A_P \ . \tag{10}$$

The remaining parameters are thus F, r_1, V_P and M_0. F and r_1 can be determined from the sprint data (Sect. 1). With F and r_1 fixed, $A_P = V_P$ and M_0 are the only remaining individual parameters. They will be computed from the records for distances D less than 5000 m. This is done in the following steps. First the acceleration times $t_1 = t_1(D)$ are determined for approximate values of V_P and M_0. With these values V_P and M_0 are computed such that the sum of the squares of the relative errors is minimal. This involves the integration of all model equations for the power constraint phase by the standard fourth-order Runge–Kutta method.

With these improved values for V_P and M_0 the above procedure is repeated several times. Actually, it turns out that the performance of the athlete for a given running time T does not depend critically on t_1, so that the convergence of this procedure is rather rapid.

The remaining parameters A_P and M_v will now be used to provide an optimal fit of the short-and middle-distance runs based on the class of approximate solutions. The results are summarized in Table 2.

The overall fit of this model is relatively good. The deviations for the middle-distance records might be due to the fact that some of the records are older and that these distances are run more rarely. In addition, it is likely that

Table 2. World records

Men				Women			
D [m]	T [sec]	D_{comp}	Rel.error [%]	D [m]	T [sec]	D_{comp}	Rel.error [%]
100.00	9.85	99.91	− 0.09	100.00	10.49	98.95	− 1.05
200.00	19.72	198.12	− 0.94	200.00	21.34	197.52	− 1.24
400.00	43.29	400.62	0.15	400.00	47.60	399.61	− 0.10
800.00	101.73	815.34	1.92	800.00	113.28	811.89	1.49
1 000.00	132.18	1 011.64	1.16	1 000.00	150.67	1 024.10	2.41
1 500.00	208.86	1 485.09	− 0.99	1 500.00	230.46	1 471.44	− 1.90
1 609.35	224.39	1 573.87	− 2.20	1 609.35	255.61	1 606.31	− 0.19
2 000.00	290.81	1 987.73	− 0.61	2 000.00	325.36	1 993.07	− 0.35
3 000.00	445.11	2 958.87	− 1.37	3 000.00	486.11	2 892.87	− 3.57
5 000.00	776.96	5 060.05	1.20	5 000.00	877.33	5 098.12	1.96

$F = 6.93$ m/sec² $\quad A_P = 1220\ J$
$M_0 = 29.5\ W$

$F = 6.22$ m/sec² $\quad A_P = 1140\ J$
$M_0 = 22.2\ W$

the hydraulic model does not provide a completely adequate description of glycolysis. Besides the fit of the world records, there are further independent checks on the model.

The first would be the comparison of the parameters with values determined by other means. The value for F clearly lies in the range $4\ \mathrm{m\,sec^{-2}} \leqq F \leqq 7\ \mathrm{m\,sec^{-2}}$ expected for good athletes [1]. Margaria [11] gives $V_P \approx 850\ J$ and $V_L \approx 1250\ J$. These values hold for well-trained people, but should be higher for world record athletes. The highest value of $V0_{2\,max} = M_0$ measured so far is 28.2 W. This compares well with our value of 29.5 W (men) and 22.2 W (women). When comparing these results with other data it should be kept in mind that our results depend critically on η and r_0, i.e. the energetic cost of transport of one kilogram by one metre. While we have adopted Margaria's value of 1 cal/m, Ward-Smith [15], for example, uses 3.9 J for the cost of transport of 1 kg per metre. This would result in a reduction of our results by about 7 per cent.

As a further check one might compare the velocity profiles. As an example, we consider the 9.92 sec sprint of Carl Lewis in the finals of the 1988 Olympic Games in Seoul [4], Fig. 1. The squares □ give the actual data, while the crosses + mark the average velocity for the 10 m intervals according to the model. The curve represents the solution of our model equation.

The 10.54 sec sprint of Florence Griffith Joyner in the finals allows a similarly good fit.

For long distances, $D \geqq 5000$ m, the above results led to the following simple model, which is a refinement of that in [2]. We assume that the velocity profile for the first T_0 seconds, T_0 the time to run 5000 m, is the same. For $t \geqq T_0$ one has

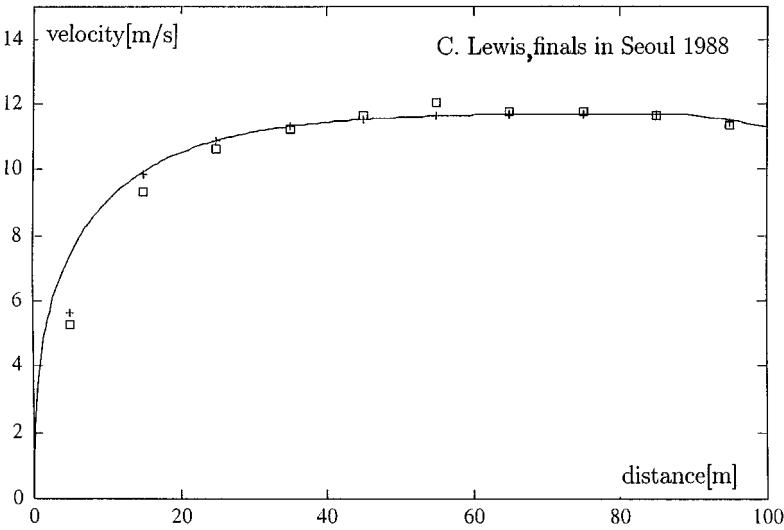$$W(t) = W(T_0)\mathrm{e}^{-(t - T_0)/\tau}\ .$$

Fig. 1. Average velocity of the model compared to the winner of the olympics in Seoul

where

$$\tau^{-1} = \lambda_F \approx \frac{M_0}{A_F((1 - \phi) + M_0/M_P(1 - \phi))} \,. \tag{11}$$

Since the velocity and the oxygen uptake change only very little in this last phase, (1) can be simplified to

$$v' = \frac{W(t) \cdot \eta}{v} - r_0 - c\bar{v}^2$$

where $\bar{v}$ is the average velocity which can be computed from a first-order model [2]. Since v is proportional in first order to $\exp(-t/\tau)$ this can even be simplified to

$$v(t) = \frac{W(t)\eta}{(r_0 - \bar{v}/\tau + c\bar{v}^2)} \tag{12}$$

and

$$D(T) = 5000 + \frac{W(T_0)\eta \cdot \tau}{(r_0 - \bar{v}/\tau + c\bar{v}^2)} (1 - \exp(T_0 - T)/\tau) \tag{13}$$

This leads to the following results (Table 3).

The discrepancies for the distances 20 000 to 30 000 arise from the fact that these records are old and because these distances are rarely run. The 3 600 sec 20 944 m result for men, for example, is the one-hour race, which is rarely run. This is rather close to the half marathon for women. The fact that τ (men) $= 32 000 > \tau$ (women) $= 28 000$ contradicts the often heard claim that women are more enduring than men.

Table 3. Long-distance world records

Men ($T_0 = 778.39$ sec, $W(T_0) = 24.8\,W$, $\tau = 32\,000$ sec)

T (sec)	1 628.23	3 444.2	3 600	4 435.8	5 358.8	7 610	22 220
D (m)	10 000	20 000	20 944	25 000	30 000	42 195	100 000
D_{comp} (m)	9 982.3	20 235.9	21 091.2	25 615.6	30 488.3	41 846.1	99 899.6
Rel. error %	0.2	1.2	0.7	2.5	1.6	0.8	0.1

Women ($T_0 = 877.33$ sec, $W(T_0) = 22.9\,W$, $\tau = 28\,000$ sec)

T (sec)	1 813.74	2 837	4 000	8 466	26 337
D (m)	10 000	15 000	21 097.5	42 195	100 000
D_{comp} (m)	1 090.5	15 476.9	21 382.2	42 053.0	99 494.3
Rel. error %	0.9	3.2	1.4	0.3	0.5

Since the muscles generate the force and serve as a store for the anaerobic energy reserves, one should expect a monotonic relationship between F and A_P and A_L. In the narrow range of masses considered here, this relationship might well be approximated by a linear equation. Linear regression applied to the older world records since 1920 results in

$$A_P = 350\,F - 1150 \text{ (men)}; \qquad A_P = 315\,F - 775 \text{ (women)} . \qquad (14)$$

This relation has to be taken with a pinch of salt, however, because the older records are based on hand-stopped times. Moreover, the track surface has been altered dramatically since those days. The correlation coefficients, however, are $r = 0.93$ (men) and $r = 0.99$ (women). Because of this we have not pursued a common linear relation as one might expect. Even though there is no direct relation between F and M_0 from a physiological point of view, the same method yields

$$\begin{aligned} M_0 &= 12.2\,F - 55.3 \quad (r = 0.97) \quad \text{for men} \\ M_0 &= 8.4\,F - 30.3 \quad (r = 0.93) \quad \text{for women} . \end{aligned} \qquad (15)$$

With (14) and (15) one finally obtains a model which depends only on F and which is therefore also applicable to weaker athletes. As an example we apply this model, which has only F as a parameter, to the Junior world records (Table 4).

It is now also possible to determine the velocity profiles for non-world-class athletes (Fig. 2). The results are curves, which are well known from experimental studies.

The relations (14) and (15) should be interpreted with some care. First of all they represent an average law for fictitious world-class athletes. Secondly, there will always be deviations from these relations for individual athletes. For example, $M_0 < 12.2\,F - 30.3$ will hold for sprinters while for long-distance runners A_P and F are relatively smaller.

Table 4. Junior world records

| Men				Women			
D [m]	T [sec]	D_{comp}	Rel. error [%]	D [m]	T [sec]	D_{comp}	Rel.error [%]
100.00	10.08	100.57	0.57	100.00	10.89	100.51	0.51
200.00	20.13	198.42	− 0.79	200.00	22.19	200.06	0.03
400.00	43.87	393.10	− 1.73	400.00	49.42	402.15	0.54
800.00	104.30	814.52	1.82	800.00	117.18	815.50	1.94
1 000.00	138.31	1 028.78	2.88	1 500.00	239.81	1 486.66	− 0.89
1 500.00	214.92	1 489.21	− 0.72	1 609.35	257.17	1 575.32	− 2.11
3 000.00	459.82	2 973.90	− 0.87	3 000.00	516.45	2 978.59	− 0.71
5 000.00	782.72	4 951.74	− 0.97	5 000.00	888.07	4 999.28	− 0.01
Men: $F = 6.75\,\mathrm{m/sec^2}$				Women: $F = 6.00\,\mathrm{m/sec^2}$			
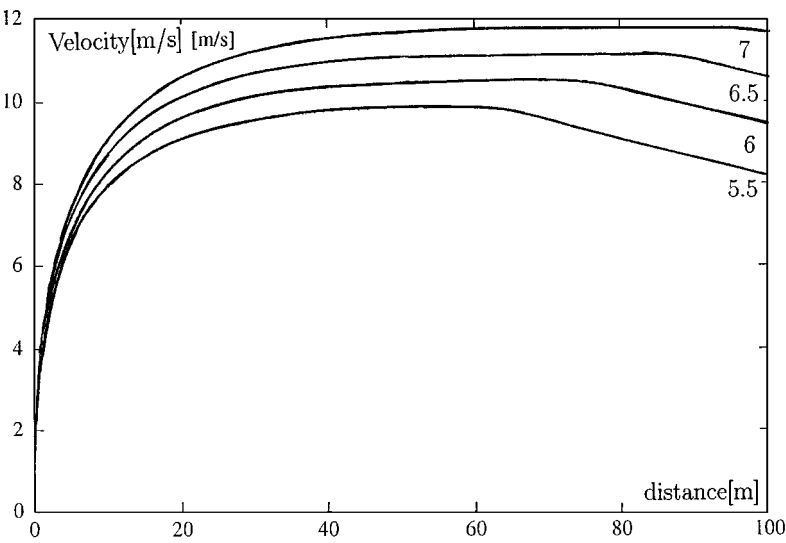
Fig. 2. Velocity profile for sprinters of various strength

This F-dependent model can also be used to derive the time series for F. A regression fit to the list of $F's$, $F(1948)$, $F(1960)$, $F(1964)$, . . . , $F(1992)$ with a formula $F(t) = F(\infty) - a\exp((t - 1992)\,b)$ shows then that the ultimate 100 m (200 m, 400 m) world records are 9.80 (19.49, 42.61) sec for men and 10.35 (20.70, 45.29) sec for women.

3 Adverse wind

A mathematical model like this is particularly well suited to study small effects, because in general it is difficult to derive these from actual data, since the background noise is in general much larger than these effects.

An athlete running with velocity v against a wind with velocity u and angle α with the track, experiences an opposing force of $c \cdot (v^2 + 2uv \cos \alpha + u^2)$. Its component in the direction of the run is

$$- c \cdot (v^2 + 2uv \cos \alpha + u^2)^{1/2} (v + u \cos \alpha) .$$

Its perpendicular component is

$$- c \cdot (v^2 + 2uv \cos \alpha + u^2)^{1/2} u \sin \alpha .$$

If one assumes $u \leqq 5$ m/sec then the perpendicular component F_1 is in general less than 0.25 m/sec². Simple biomechanics and comparison with the forces in running on a curved track shows that this has a negligible effect. For this reason it suffices to consider the action in the direction of the track only and we may set $\alpha = 0$. Then the effect of the wind with velocity u results in

$$cv^2 \to c \cdot (v + u)^2$$

$$b \to b + 2cu$$

$$F \to F - 12cu - cu^2 .$$

Since $v_\infty = 6 + w_1 \approx 6 + \frac{F - 2.5104}{b}$ this implies

$$\Delta v_\infty = \frac{\Delta F}{b} - \frac{w_1}{b} \Delta b = - 2cu(v_\infty + \tfrac{1}{2} u)/b .$$

If one assumes that the limit velocity in a sprint is attained after about 3 seconds, the net effect of the wind with $\alpha = 0$ in a 100 m sprint is roughly given by $\Delta s = (T_{100} - 3) \cdot \Delta v_\infty$. With our parameters this gives

$$\Delta s = - u(v_\infty + \tfrac{1}{2}u) \cdot 10^{-2}(T_{100} - 3) \approx - 0.8u.$$

This holds for men and women alike. These estimates agree fairly well with actual numerical computations.

For 200 m sprints estimates are more complicated, because a change in F and b results in a change of the energy demand and t_2. Since longer distances are run on curved tracks at varying velocities matters become even more complicated. The wind effect is approximately linear in F. Thus the 200 m curved track corrections can be obtained from the straight track corrections by a factor k, which depends on the relative position of the curved parts to the general wind direction. Since half of the track is curved, the factor ranges from $\frac{1}{2}(1 + \frac{2}{\pi})$ (start at the tip) to $\frac{1}{2}$ (start on the straight part). For this reason we choose an average correction factor $k = \frac{1}{2}(1 + \frac{1}{\pi}) = 0.66$. The gains in units of 10^{-2} seconds are given in Table 5. In brackets the gains are given for $c = 0.0032$.

The T_{100} dependent advantage for a 100 m sprint with a 2 m tailwind is plotted in Fig. 3.

In an extensive statistical analysis of 100 m sprint events Linthorne [10] determined the advantage due to a 2 m/sec tailwind. He found 0.10 ± 0.01 sec. Pritchard [14] used a modification of Kellers model [8] and determined

Table 5. Gains due to head- or tailwind. The units for the gains are $\sec \cdot 10^{-2}$

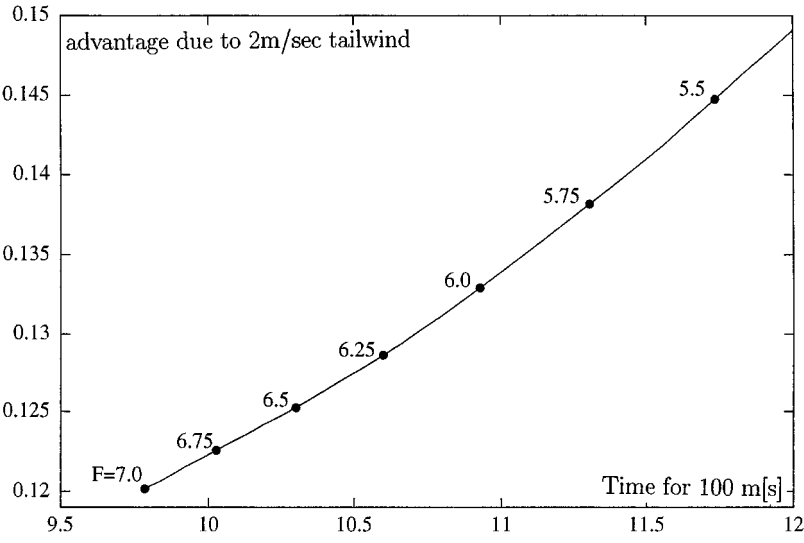| Velocity of wind [m/sec]	Men		Women	
	100	200	100	200
− 2	− 14.7 (− 12.5)	− 21.5 (− 18.4)	− 16.2 (− 13.8)	− 24.5 (− 21.1)
− 1	− 7.0 (− 6.0)	− 10.2 (− 8.7)	− 7.7 (− 6.6)	− 11.6 (− 9.9)
1	6.3 (5.4)	9.1 (7.6)	7.0 (5.9)	10.3 (8.8)
2	12.1 (10.3)	17.0 (14.7)	13.2 (11.3)	19.3 (16.5)

Fig. 3. Advantage due to tailwind of 2 m/sec for runners of different strength

a 1.1% advantage for men. Our values are only slightly above those data. This could be due to the fact that the frontal area of the average runner is not very well known. With $c = 0.0032$, i.e. a slightly reduced frontal area, we get extremely good agreement. For a 400 m sprint on a curved track the above results imply a disadvantage of about $3.7 \cdot 10^{-2}$ sec (men) or $4.6 \cdot 10^{-2}$ sec (women) for a 2 m headwind. Most likely the disadvantage will be higher, due to cross-wind effects.

4 Altitude

Altitude affects the running performance via a variation in gravitation and air pressure. Since the influence of the variation of gravitation lies in the order of 0.1 per cent, we will neglect it. The barometric pressure p at an altitude h is roughly given by

$$p(h) = p(0)\exp(- h/8435) \tag{16}$$

with h measured in m. Thus the resistance coefficient c changes to

$$c(h) = c(0)\frac{p(h)}{p(0)}.$$

The effect of the reduced partial pressure of O_2 is much more difficult to estimate, because at high altitude the ventilation rate increases and the body adapts partially to this [7]. In addition, the oxygen saturation coefficient $s(p)$ of blood is not proportional to the pressure. Moreover, there is a certain amount of oxygen A bound to the myoglobin of the muscle. Thus the oxygen uptake as a function of h can be written as

$$M_{\mathcal{C}}(h) = M_{\mathcal{C}}(0) \cdot \min[1, s(p(h)) + A/T] \tag{17}$$

where T is the running time. For $h = 2250$ m one obtains [13] $s(p(h)) = 0.93$.

Since A is of no importance for long-distance runs, this implies a disadvantage of about 6.5% for such runs. This is indeed well established.

If one assumes that the altitude of Mexico City has no effect on the 800 m records, one obtains

$$A = 120 \ J \tag{18}$$

This corresponds quite closely to the value given in the literature [9]. With (16) and (17) one obtains the following results for $h = 2250$ m, the altitude of Mexico City, Table 6. In order to compare these predictions with actual results, we compute for all distances D

Result in Mexico City $-\frac{1}{2}$ (Result in Tokyo + Result in Munich)

As a result we consider the average of all 6 finalists or 3 medalists. We denote this by F_{in} and M_e respectively. For the 100 m and 400 m distances we also take the hurdles and relays into account.

A closer analysis of all records shows that the data are not always consistent. This may be due to the fact that in Mexico City the running time was measured electronically for the first time and that a tartan track was used there.

Table 6. The relative gain in per cent due to the altitude of Mexico City for the medalists M_e, finalists F_{in} and from the model

Men				Women		
D [m]	M_e	F_{in}	Model	M_e	F_{in}	Model
100	1.4	1.7	0.99	1.7	1.6	0.93
200	1.9	1.4	0.91	0.5	0.9	0.84
400	1.6	1.2	0.77	− 0.3	0.8	0.75
800	1.0	0.7	− 0.10	− 1.3	− 1.4	0.08
1 500	− 0.3	− 1.0	− 0.60			
5 000	− 3.3	− 3.7	− 1.54			
10 000	− 5.1	− 4.9	− 4.24			
42 200	− 6.0	− 6.3	− 5.80			

The variation in temperature is another possible small effect. A change in temperature changes the pressure, but it is very likely that the physiological effects on the body are much larger. For this reason we will not pursue this here.

5 Discussion

In [2] we have developed several models to describe the force and energetics of competitive running. It is obvious that similar models can be developed for skating, swimming, rowing or cycling. Apart from the starting phase, the biomechanics of running is reduced to Newton's law and the internal resistance term $r_{\text{int}} = r_0 + r_1(v - 6) + r_2(v - 6)^2$. This ansatz thus extends the models of Keller [8] and Ward-Smith [15]. Not only does it provide a good fit of the world records, but it also describes the results of Ballreich [1] extremely well. In fact an analysis of these results shows $r_2 = 0$, as expected, and

$$0.7 \leqq r_1 + 12c = b \leqq 0.85 \ .$$

In addition, we note that the force F, as derived from the sprint data, takes on values expected from general considerations. The velocity profile of sprints also agrees well with actual data.

The complexity of the energy-providing process in the muscle makes it desirable to develop simple models which describe the essentials of these processes. As such the hydraulic model as a linear 3- or 4-compartment model is rather appealing. It is even more attractive since the absence of oscillatory solutions can be deduced from general considerations, so that all solutions are superpositions of real exponentials. Thus the dynamics can be deduced easily and the geometric parameters can be determined without difficulty, the sole exception being μ. Since θ marks the decline of the ATP concentration, its value lies between 0.5 and 0.65. The rapid onset of glycolysis leads to

$$0.05 \leqq 1 - \theta - \lambda \leqq 0.08 \ .$$

Though θ and λ are almost universal it could well be that smaller θ values are associated with weaker athletes. This would be in accordance with

$$0.15 \leqq \phi \leqq 0.3$$

and the fact that the anaerobic threshold $M_0 \cdot \frac{\theta}{1-\phi}$ lies between 0.6 M_0 and 0.85 M_0, where higher values correspond to better athletes. While the geometry of the system follows easily from general principles, the capacity and rates have to be determined from actual data on running. Even though the rates and capacities determined from the world records are higher than the values given in the literature, the agreement is nonetheless remarkable. Here it should be noted that these results do not apply to a single athlete, because no athlete is qualified for sprints, middle distances and long distances alike. Besides the geometric parameters θ, μ, λ and ϕ the quotients of capacities and rates should be nearly constant, at least for athletes of the same class. Above we have denoted this by the term universal parameter. Such considerations

are important when this model is scaled. Another important scaling relation is (14) even though it cannot be derived from general considerations. It has been derived from the time series of world records. It also agrees well with the data of Ballreich [1], since the size of A_P determines the onset of the power constraint phase.

Though at first the hydraulic model, as conceived by Margaria [11] or Morton [12], seems rather attractive, the constraints i.e. the maximal rates and dynamic constraints make explicit computations rather cumbersome. These constraints, however, are dictated by physiological conditions and require the consideration of several phases, in each of which the model is linear. We have seen that the full model describes the sprint data fairly well and that the reduced model provides a good fit for the long-distance data. The interpretation of $A_{\mathcal{C}}$ and A_F as capacities does not seem justified however, because glycogen and lipids are released from their depots on demand. Thus one should rather interpret the derived quantity λ_F as a parameter modelling physiological aspects of fatigue. A naive interpretation of the hydraulic model for long-distance running or endurance work is thus not justified. The status of this model is even worse when applied to middle-distance runs, as is shown by the results. This is due to the fact that the Pasteur effect, glycolysis and the dynamics of the glycogen providing processes are not modelled properly.

The analysis of the various optimization models shows that in general an optimal solution is almost of the form of an approximate solution. This is also born out in practice where the best athletes usually run with a constant speed during the middle phase of a run. In addition, the optimization models indicate the importance of the Pasteur effect and the negative effects of glycolysis in long-distance runs. For this reason the hydraulic model as developed here [2] would need another dynamic capacity constraint. Thus it might be desirable to abandon the hydraulic model and start with a general finite compartment model anew.

Since the model offers a fairly good fit of the world records, its extension to the study of small effects is justified, in particular since these results depend critically only on c and are rather insensitive to the other details. The considerable influence of altitude h and wind speed u should be of concern to athletic unions or track-and-field committees when comparing or defining records [6]. The influence of track curvature has been discussed elsewhere [3], hence it is not part of this model.

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