Pergamon

0021-9290(95)00041-0

J. Biomechanics, Vol. 29, No. 2, pp. 245-249, 1996

0021-9290/96 $15.00 + .00

TECHNICAL NOTE

MINIMUM-TIME RUNNING AND SWIMMING: AN OPTIMAL CONTROL APPROACH

Ryszard Maroński

Institute of Aeronautics and Applied Mechanics, ul. Nowowiejska 24, 00-665 Warsaw, Poland

Abstract—During analysis of the competitor's velocity in a run, strong assumptions are imposed upon the runner's tactic. It is assumed that the competitor uses his/her maximal propulsive force in short-distance events. The runner's velocity is assumed constant in long-distance races. None of these assumptions is satisfied during middle-distance races. In this study, the competitor's velocity, minimizing the time taken to cover the distance, is determined by means of extremization of linear integrals using Green's theorem (Miele's method). The model of the competitor's motion is based on two differential equations: the first one derives from Newton's second law, the second one is the equation for power balance. The theory is illustrated with two examples referring to competitive running and swimming. The minimum-time competitive run can be broken into three phases:

—the acceleration,
—the cruise with the constant velocity, and
—the negative kick at the end of the race.

The problem has a similar solution in competitive swimming, however, the acceleration is replaced by the gliding phase.

NOMENCLATURE

A	initial point
B	final point
D	given distance
e	available energy per unit mass
e_0	energy at the rest per unit mass
F	actual propulsive force
f_{max}	maximal propulsive force per unit mass
g	gravitational acceleration
m	mass of the competitor
N	normal reaction of the ground
R	overall resistance of motion, $R = R^* + mg \sin \alpha$
R^*	resistive force
r	overall resistance of motion per unit mass
T	time of covering the distance
t	time
v	velocity
v_c	cruising velocity
v_{cr}	critical velocity
v_0	velocity at the rest
x	coordinate (covered distance)
α	slope angle of the track
η	propulsive force setting
λ	Lagrange's multiplier
μ	efficiency of transforming the chemical energy into the mechanical one
σ	rate of energy production per unit mass
τ	damping coefficient
ω_*	augmented fundamental function

INTRODUCTION

Mathematical analysis of running is especially focused on relating distance to time. Such analysis depends on the creation of a reasonable model of the competitor's motion. The model is usually represented by a set of ordinary differential equations supplemented with inequality constraints. The time of the run may be determined after integrating equations of the competitor's motion. Integration methods are well described in the literature, but integration may be performed only when the right-hand sides of equations are determined. And here we encounter a difficulty. In middle- and long-distance events the runner may use his/her energy in different ways, therefore some functions appearing in the equations of motion are not determined and the integration cannot be performed. These unknown functions represent the tactic of the race and they do not follow from the model at this stage of reasoning.

Such difficulty does not appear in short-distance races where the competitor moves using his maximal abilities (Dapena and Feltner, 1987; Ward-Smith, 1985a, b). The distance is too short to deplete all his/her sources of energy. In long-distance events this difficulty may be overcome by assuming that the runner's velocity is constant (Lloyd, 1966; Peronnet and Thibault, 1989; Ward-Smith, 1985a, b). However, this assumption lacks theoretical support. Moreover, none of these assumptions is satisfied for middle-distance races where the initial and final stages of the race cannot be neglected.

All these facts induce one to formulate the following question: how should the competitor vary his/her speed with distance to minimize the time of the event? In this study, the optimal velocity profile is obtained by formulating and solving a mathematical problem in optimal control theory.

The optimal control approach is seldom used in the extensive bibliography referring to competitive races. In the work of Keller a similar problem (maximization of the distance of the run for the given time) has been solved by equating to zero the first variation of the functional (Keller, 1973, 1974). The same problem as in Keller has been considered by Behncke (1987) for running and swimming, and by Cooper (1990) for wheelchair athletics. More general models have been applied. Proofs presented in both papers are very complicated – they are based on Pontryagin's maximum principle. In the approach presented in this study, the model of the competitor's motion is more general than Keller's and is similar to Behncke's. It has been shown that

Received in final form 13 *March* 1995.

the optimal race consists of three phases (not four like in Behncke and Cooper). Miele's method of problem solution is relatively simple and gives a clear graphical interpretation.

MODEL

The main assumptions of the model of the competitor's motion during the race are as follows. The racer is regarded as a particle with mass m. The vertical displacements of his/her body associated with the cyclic nature of the stride pattern, and the vertical displacement of the centre of mass at the start of the race are neglected. The competitor moves on a linear track which may be inclined to the horizontal. Problems relating to technique, assistance of other athletes, motivation or sport psychology are neglected. Forces exerted on the athlete are shown in Fig. 1.

Applying Newton's second law, the first equation of motion is given in the form

$$\frac{dv}{dt} = f_{max}(v)\eta - r(v), \tag{1}$$

where all quantities refer to the unit mass of the competitor. The actual propulsive force, $F = mf_{max}(v)\eta$, is the product of the mass m, the maximal propulsive force per unit mass $f_{max}(v)$, which may be a function of the running velocity v, and the propulsive force setting η. The athlete may adjust his/her actual propulsive force setting to the actual conditions, therefore the propulsive force setting varies with the distance within the given interval

$$0 \leqslant \eta \leqslant 1. \tag{2}$$

The upper limit of η shows that the actual propulsive force per unit mass cannot surpass its maximal value $f_{max}(v)$, the lower limit does not occur in running but it does in swimming since we have the gliding phase there. The overall resistive force $R = mr(v)$ may be any given function of the velocity v. It may include the steady wind assistance and the steady slope of the track. Symbol t denotes the time.

Energy transformations in the competitor's body are described by the second equation of power balance

$$\frac{de}{dt} = \sigma(v) - \frac{vf_{max}(v)\eta}{\mu(v)}, \tag{3}$$

where e denotes the actual reserves of chemical energy per unit mass in excess of the non-running metabolism, $\sigma(v)$ is the recovery rate of chemical energy per unit mass, $vf_{max}(v)\eta$ is the actual mechanical power per unit mass used by the athlete, $\mu(v)$ is the efficiency of transforming the chemical energy into the mechanical one.

It is convenient to change the independent variable in equations (1) and (3) using the definition of velocity

$$v = \frac{dx}{dt}, \tag{4}$$

where x is the distance covered at the time t, then

$$\frac{dv}{dx} = \frac{f_{max}(v)\eta}{v} - \frac{r(v)}{v}, \qquad \frac{de}{dx} = \frac{\sigma(v)}{v} - \frac{f_{max}(v)\eta}{\mu(v)}. \tag{5}$$

Provided the following initial conditions hold

$$v(x_A) = v_A, \qquad e(x_A) = e_A, \tag{6}$$

equations (5) represent the so-called state equations. The first condition (6) gives the initial velocity, the second condition (6) represents the initial amount of energy. The amount of energy per unit mass cannot be lower than zero in the whole range of the independent variable x,

$$e(x) \geqslant 0. \tag{7}$$

The time of the run is given by the integral

$$T = \int_A^B dt, \tag{8a}$$

or, using equation (4), by the integral

$$T = \int_A^B \frac{1}{v}\, dx. \tag{8b}$$

Now, the problem may be formulated in the following manner.

The runner should vary his/her speed $v(x)$ during a race over a given distance D to minimize the time of the event T. It is possible due to variations of the propulsive force setting η which may be adjusted arbitrarily in every point of the distance.

In a formalized manner the problem is formulated as follows: Find $v(x)$, $\eta(x)$ and $e(x)$ satisfying equations (2), (5)–(7) so that T defined by equation (8b) is minimized. The functions: $f_{max}(v)$, $r(v)$, $\sigma(v)$, $\mu(v)$, the values of v_A and e_A, and the distance

$$D = \int_A^B dx, \tag{9}$$

are given.

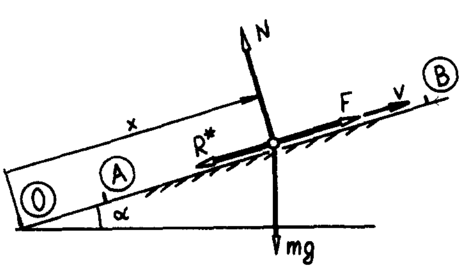

Fig. 1. Particle model of the competitor. Forces exerted on his/her body are: F – propulsive force, R^* – resistive force which does not contain the gravitational component onto direction of motion (in the text R is overall resistance of motion, therefore $R = R^* + mg \sin \alpha$), mg – runner's weight, N – normal reaction of the ground. Symbol v denotes the competitor's velocity, x is the coordinate describing the actual racer's position, α is the slope angle of the track.

SOLUTION

The method of extremization of linear integrals using Green's theorem (Miele's method) was worked out at the beginning of the 1950s and successfully applied to many problems in the dynamics of aircraft and rockets. It is still in use because it gives solutions involving singular arcs in a relatively simple manner (cf. Miele, 1962; Maroński, 1994).

The performance index to be minimized is given by the integral (8b), which may be expressed as a line integral in the (e, v)-plane after elimination of the propulsive force setting η from equations (5). We then have

$$T = \int_A^B \phi(e,v)\, de + \psi(e,v)\, dv, \tag{10}$$

where

$$\phi(e,v) = \frac{\mu(v)}{a(v)}, \qquad \psi(e,v) = \frac{v}{a(v)}, \tag{11}$$

and

$$a(v) = \sigma(v)\mu(v) - r(v)v.$$

The isoperimetric constraint is the given distance

$$D = \int_A^B dx = \int_A^B \phi_1(e, v)\, de + \psi_1(e, v)\, dv, \tag{12}$$

where

$$\phi_1(e, v) = \frac{\mu(v)v}{a(v)}, \qquad \psi_1(e, v) = \frac{v^2}{a(v)}. \tag{13}$$

In such a way, our minimization problem takes the form required by Miele's method.

The augmented fundamental function is of the form

$$\omega_*(e, v, \lambda) = \frac{v[1 + \lambda v]\left[\frac{d\mu}{dv}r(v) - \mu(v)\frac{dr}{dv} + \frac{\mu^2(v)}{v}\frac{d\sigma}{dv}\right] - \mu(v)[r(v) + \lambda\sigma(v)\mu(v)]}{[a(v)]^2}, \tag{14}$$

where λ is the constant Lagrange's multiplier. Expression (14), after equating to zero, yields the optimal condition on the so-called singular arc

$$v(x) = v_c = \text{constant}. \tag{15}$$

In fact, the equation $\omega_*(e, v, \lambda) = 0$ does not depend on the energy e. It depends only on the velocity v (which can vary with distance) and the constants, therefore, a constant value of the velocity v is the solution of this equation. Even if the equation had more than one solution, only one of them would be the solution to our problem since velocity is a continuous function of the distance. Junction between different values of the velocity is not possible without violating either the equation $\omega_*(e, v, \lambda) = 0$ or the assumption of the continuity of the velocity.

The result presented does not depend on actual forms of the functions describing the efficiency $\mu(v)$, the resistive force $r(v)$ and the energy recovery rate $\sigma(v)$. We do not know these functions accurately very often. [In the available bibliography there is a considerable discrepancy concerning the values of these functions, cf. Kaneko (1990) and Morton (1985).]

Miele's method applies for $a(v)$ not equal to zero. The velocity, for which $a(v) = 0$, is called, in this paper, the critical velocity v_{cr}. For $v = v_{cr}$, the differentials de and dv equate to zero in equations (10) and (12). In such a case, the runner moves with a constant velocity and the rate of energy recovery is equal to the rate of energy consumption at the same time. The curves in the (e, v)-plane degenerate into points. For standard data, critical velocities are much lower than cruising velocities. In the problems under consideration we assume that $a(v) < 0$.

In standard formulation of Miele's method, the final point B is given, whereas the runner's final velocity is not specified in our problem. This difficulty may be overcome by solving the sequence of the minimum-time problems with specified, but different v_B, and minimizing the time of the run with respect to v_B. In such a way, the final velocity v_B may be chosen optimally. The algorithm of the problem's solution may be simplified further. The Lagrange's multiplier λ has only an auxiliary meaning and does not have to be computed. The problem may be reduced to the parametric optimization problem with respect to one variable, v_c.

It may be proved that the admissible domain is bordered by the curves obtained from integration of equations (5) for $\eta = 1$ or $\eta = 0$ on the right, the inequality (7) on the left and the inequality $a(v) < 0$ from the bottom. Details of reasoning are similar to those presented in Miele (1962) and Maroński (1990, 1994).

In conclusion, the distance of the race may be broken into three phases [not four like in Behncke (1987) and Cooper (1990)]:

(1) The early phase of the race:
— the acceleration, for running, where the competitor moves with his/her maximal propulsive force, $\eta = 1$,
— the gliding phase, for swimming, where the propulsive force is equal to zero, $\eta = 0$.
In the (e, v)-plane, this phase is represented by the arc on the border of the admissible domain which joints the initial point A and the singular arc.

(2) The middle phase of the race (the cruise), where the competitor moves with his/her partial propulsive force, $0 < \eta < 1$, and the velocity v is constant. In the (e, v)-plane it is represented by the singular arc $\omega_*(e, v, \lambda) = 0$.

(3) The negative kick at the end, where the competitor after depleting of all his/her energy stores, $e = 0$, supports his motion with the force derived from equation (3), $f_{max}(v)\eta = \sigma(v)\mu(v)/v$. The force is smaller than the one necessary for motion with cruising velocity, therefore his/her velocity decreases. In the (e, v)-plane this phase corresponds to the segment of the straight line on the border of the admissible domain, $e = 0$, which joins the singular arc with the final point B.

The last two phases are the same for running and swimming. The difference appears only in the first stage. For the distances short enough only the first phase may occur.

EXAMPLES

Consider the following problem as an example. The competitor should cover the distance $D = 400$ m in the minimal time. The track is inclined to the horizontal. The inclination angle is $\alpha = 2°$. We will employ the Keller's model of competitive running in which the resistive force linearly depends on the velocity (Keller, 1973, 1974). [The same model is in Townend (1984), a similar model is in Morton (1985).] Then we have

$$r(v) = \frac{v}{\tau} + g\sin\alpha,$$

where τ is the constant damping coefficient. Remaining parameters are constant and they are (cf. Keller, 1973):

$$e_0 = 2409.25\ \mathrm{m^2\,s^{-2}},\ \mu = 1,\ f_{max} = 12.2\ \mathrm{m\,s^{-2}},$$

$$g = 9.81\ \mathrm{m\,s^{-2}},$$

$$v_0 = 0\ \mathrm{m\,s^{-1}},\ \tau = 0.892\ \mathrm{s},\ \sigma = 41.61\ \mathrm{m^2\,s^{-3}}.$$

The admissible domain for this example is given in Fig. 2. The curve OAGHB represents the optimal solution. The optimal velocity graph in the (x, v)-plane has been shown in Fig. 3.

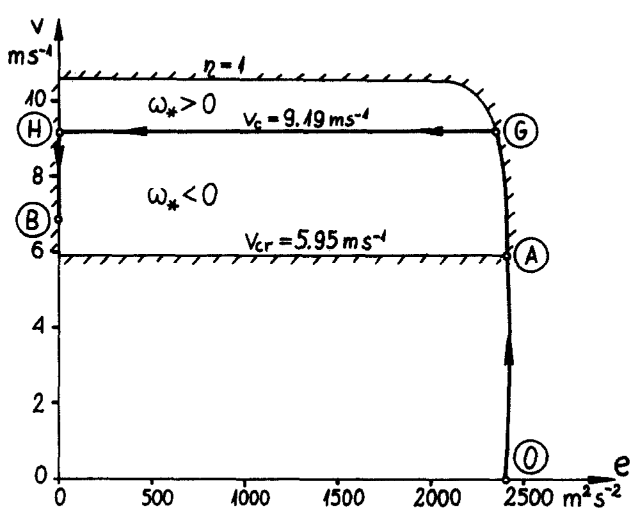

Fig. 2. The admissible domain (undashed area) for the 400 m run and for the slope angle of the track $\alpha = 2°$. The path OAGHB minimizes the time of event. The competitor moves with the maximal propulsive force, $\eta = 1$, on the subarc OAG (the acceleration). The propulsive force is lower than the maximal one, $\eta < 1$, and the velocity is constant on the subarc GH (the cruise). The runner slows down and the condition $e(x) = 0$ is valid on the subarc HB (the negative kick).

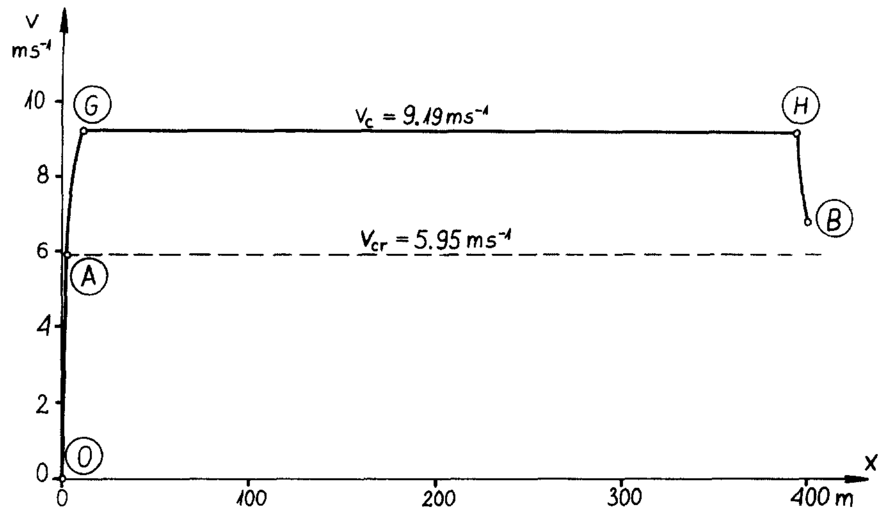

Fig. 3. The optimal velocity of the runner over the distance 400 m for the slope angle of the track $\alpha = 2°$. The characteristic points are the same as in Fig. 2.

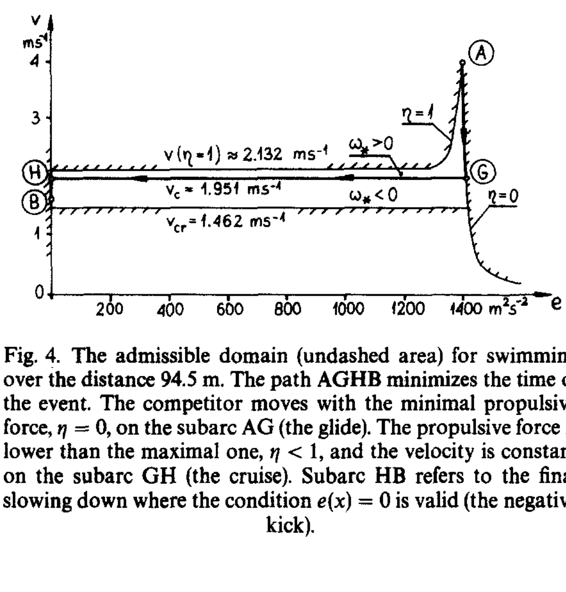

Fig. 4. The admissible domain (undashed area) for swimming over the distance 94.5 m. The path AGHB minimizes the time of the event. The competitor moves with the minimal propulsive force, $\eta = 0$, on the subarc AG (the glide). The propulsive force is lower than the maximal one, $\eta < 1$, and the velocity is constant on the subarc GH (the cruise). Subarc HB refers to the final slowing down where the condition $e(x) = 0$ is valid (the negative kick).

The second example refers to swimming. The athlete should cover the distance $D = 94.5$ m in the minimal time (the distance on which the competitor's centre of gravity moves in the water during the 100 m race) (see Figs 4 and 5). The model of the competitor's motion is similar to that of Behncke (1987). The resistive force is the quadratic function of the velocity

$$r(v) = dv^2,$$

where $d = 0.55\ \mathrm{m}^{-1}$. The remaining data for calculations are (cf. Behncke, 1987):

$$e_A = 1400\ \mathrm{m}^2\ \mathrm{s}^{-2},\ \mu = 0.08, f_{\max} = 2.5\ \mathrm{m\,s}^{-2},$$

$$v_A = 4\ \mathrm{m\,s}^{-1},\ \sigma = 21.5\ \mathrm{m}^2\,\mathrm{s}^{-3}.$$

The admissible domain and the character of the optimal solution for swimming are similar to the previous ones. The differences appear at early phase of the race.

DISCUSSION

Determination of the time to cover a given distance is possible via integration of the equations of the competitor's motion. In middle- and long-distance events, however, such approach gives nonunique results because different competitor's tactics during the race may produce different times for the run. The optimal tactic of the race generates the minimal time of the event. It is represented by the competitor's optimal velocity profile along the distance. Determination of the optimal tactic is possible by applying optimal control methods. Such an approach has been used by Keller (1973, 1974) for competitive running, Behncke (1987) for running and swimming and Cooper (1990) for wheelchair athletics. In those papers instead of the minimum-time problem of covering the given distance, a reformulated problem of maximization of the distance for the given time is considered. The proof given by Keller is elegant, however, it is based on a relatively simple model of competitor's motion in which the resistive force linearly depends on velocity and the maximal propulsive force is constant. Models presented in Behncke's and Cooper's papers are more general. The same method of solution of the problem is used in both papers. It employs variable Lagrange's multipliers which have no clear physical interpretation. The optimal solution depends on the behaviour of these multipliers, therefore the reasoning used in these papers is very complicated.

Application of Miele's method, used in this paper, makes it possible to solve, in relatively simple way, the original problem (minimization of the time for the given distance) for a more general case than Keller's model. This method has a clear graphical interpretation in the (e, v)-plane. It reduces a relatively complex problem of optimal control to the minimization problem of a function of one variable. Application of Miele's method is possible if the time of the race to be minimized can be represented as a line integral of the form shown in equation (10). The isoperimetric constraint is the distance also given as a line integral (12). That is why the model of motion used in this paper cannot be too complicated. The model is represented by two ordinary equations resulting from Newton's second law and the power balance of metabolism, initial conditions and inequality constraints. The method presented does not work for velocities lower than or equal to the critical velocity v_{cr}. For running, the solution below this velocity has been obtained by assuming that the character of the optimal solution does not change. For minimum-time swimming such weakness does not appear.

The results of this paper for the first two phases of the race, the acceleration (the glide) and the cruise with a constant velocity, are approximately consistent with observation. However, the results indicating that the competitor finishes with decreasing velocity may be questionable. From the theory employed in this

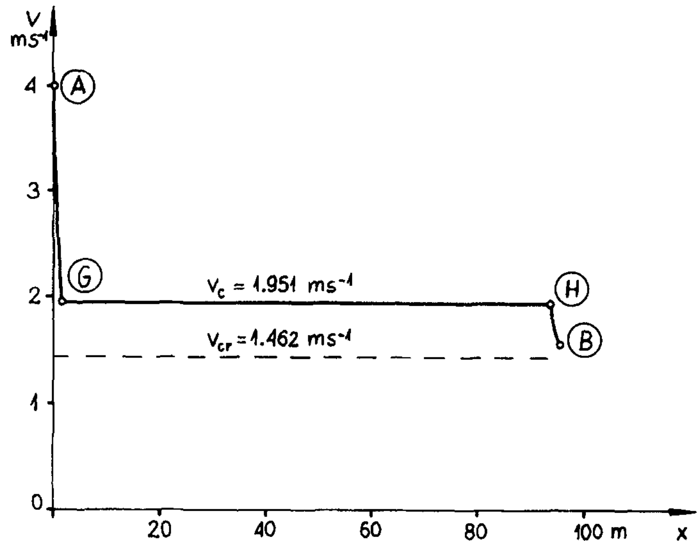

Fig. 5. The optimal velocity of the swimmer over the distance 94.5 m. The characteristic points are the same as in Fig. 4.

paper it follows that increasing the velocity in the final stage of the race is not optimal.

The question formulated at the beginning is this: what is the competitor's optimal velocity profile along the distance minimizing the time of the event. For the presented model of motion it has been shown that the race consists of three phases: the acceleration for running, or the glide for swimming; the cruise with a constant velocity; and the final slowing down. If the distance is short enough only the first phase occurs. This result is essentially consistent with the earlier results of Keller and Behncke. It confirms the often used practice where the time of the event may be obtained by dividing the distance by the constant velocity. The error of such an approximation is small for distances which are sufficiently long where the early and the final phases of the race may be neglected.

In this paper, the problem of time minimization during the race over the given distance is reconsidered. The model of the competitor's motion is more general than Keller's (1973, 1974) and is similar to the model of Behncke (1987). The method of problem solution based on Green's theorem (Miele's method) is simpler than the procedures used in the papers mentioned above. It gives a clear graphical interpretation of the optimal solution in the (e, v)-plane. The reasoning used in this paper enables one to reduce the optimal control problem with the so-called singular control to the parametric optimization problem with respect to one variable.

REFERENCES

Behncke, H. (1987) Optimization models for the force and energy in competitive sports. *Math. Meth. Appl. Sci.* **9,** 298–311.

Cooper, R. A. (1990) A force/energy optimization model for wheelchair athletic. *IEEE Trans. Systems, Man, Cybernet.* **20,** 444–449.

Dapena, J. and Feltner, M. E. (1987) Effects of wind and altitude on the times of 100-meter sprint races. *Int. J. Sport Biomechanics* **3,** 6–39.

Kaneko, M. (1990) Mechanics and energetics in running with special reference to efficiency. *J. Biomechanics* **23,** Suppl. 1, 57–63.

Keller, J. B. (1973) A theory of competitive running. *Phys. Today* **26,** 42–47.

Keller, J. B. (1974) Optimal velocity in a race. *Am. Math. Mon.* **81,** 474–480.

Lloyd, B. B. (1966) The energetics of running: an analysis of world records. *Adv. Sci.* **22,** 515–530.

Maroński, R. (1990) On optimal running downhill on skis. *J. Biomechanics* **23,** 435–439.

Maroński, R. (1994) On optimal velocity during cycling. *J. Biomechanics* **27,** 205–213.

Miele, A. (1962) Extremization of linear integrals by Green's theorem. In *Optimization Techniques with Applications to Aerospace Systems* (Edited by Leitmann, G.), pp. 69–98. Academic Press, New York.

Morton, R. H. (1985) Comment on "A model for the calculation of mechanical power during distance running". *J. Biomechanics* **18,** 161–162.

Peronnet, F. and Thibault, G. (1989) Mathematical analysis of running performance and world running records. *J. Appl. Physiology* **67,** 453–465.

Townend, M. S. (1984) *Mathematics in Sport,* pp. 23–28. Horwood, Chichester.

Ward-Smith, A. J. (1985a) A mathematical theory of running, based on the first law of thermodynamics, and its application to the performance of word class athletes. *J. Biomechanics* **18,** 337–349.

Ward-Smith, A. J. (1985b) A mathematical analysis of the influence of adverse and favourable winds on sprinting. *J. Biomechanics* **18,** 351–357.

Ward-Smith, A. J. (1987) Reply to "Comment on "A mathematical theory of running" and the applications of this theory". *J. Biomechanics* **20,** 94–95.