Chapter 2

The Maximum Principle: Continuous Time

- Main Purpose: Introduce the maximum principle as a necessary condition to be satisfied by any optimal control.
- Necessary conditions for optimization of dynamic systems.
- General derivation by Pontryagin et al. in 1956-60.
- A simple (but not completely rigorous) proof using dynamic programming.
- Examples.
- Statement of sufficiency conditions.
- Computational method.

Optimal control theory deals with the problem of optimization of dynamic systems.

State equation

$$\dot{x}(t) = f(x(t), u(t), t), \;\; x(0) = x_0, \tag{1}$$

where the vector of *state variables*, $x(t) \in E^n$, the vector of *control variables*, $u(t) \in E^m$, and the function

$$f : E^n \times E^m \times E^1 \to E^n.$$

The function f is assumed to be continuously differentiable.

The path $x(t), t \in [0, T]$, is called a *state trajectory* and $u(t)$, $t \in [0, T]$, is called a *control trajectory*. *Admissible control* $u(t)$, $t \in [0, T]$, is piecewise continuous and satisfies, in addition,

$$u(t) \in \Omega(t) \subset E^m,\ t \in [0, T]. \tag{2}$$

Objective function is defined as follows

$$J = \int_0^T F(x(t), u(t), t)dt + S[x(T), T], \tag{3}$$

where the functions $F : E^n \times E^m \times E^1 \to E^1$ and $S : E^n \times E^1 \to E^1$ are assumed to be continuously differentiable.

2.1.4 The Optimal Control Problem

The problem is to find an admissible control u^*, which maximizes the objective function (3) subject to the state equation (1) and the control constraints (2). We now restate the optimal control problem as:

$$\begin{cases} \max\limits_{u(t)\in\Omega(t)} \left\{ J = \int_0^T F(x,u,t)dt + S[x(T),T] \right\} \\ \text{subject to} \\ \dot{x} = f(x,u,t),\ x(0) = x_0. \end{cases} \tag{4}$$

The control u^* is called an *optimal control* and x^* is called the corresponding *optimal trajectory*. The optimal value of the objective function will be denoted as $J(u^*)$ or J^*.

- Case 1. The optimal control problem (4) is said to be in *Bolza form*.
- Case 2. When $S \equiv 0$, it is said to be in *Lagrange form*.
- Case 3. When $F \equiv 0$, it is said to be in *Mayer form*. If $F \equiv 0$ and S is linear, it is in *linear Mayer form*, i.e.,

$$\begin{cases} \max\limits_{u(t)\in\Omega(t)} \{J = cx(T)\} \\ \text{subject to} \\ \dot{x} = f(x,u,t),\ x(0) = x_0, \end{cases} \tag{5}$$

where $c = (c_1, c_2, \cdots, c_n) \in E^n$.

The Bolza form can be reduced to the linear Mayer form: Introduce a new state vector $y = (y_1, y_2, \ldots, y_{n+1})$, having $n+1$ components defined as follows: $y_i = x_i$ for $i = 1, \ldots, n$, and

$$\begin{aligned} \dot{y}_{n+1} = F(x,u,t) + \frac{\partial S(x,t)}{\partial x} f(x,u,t) + \frac{\partial S(x,t)}{\partial t}&, \\ y_{n+1}(0) = S(x_0, 0)&, \end{aligned} \tag{6}$$

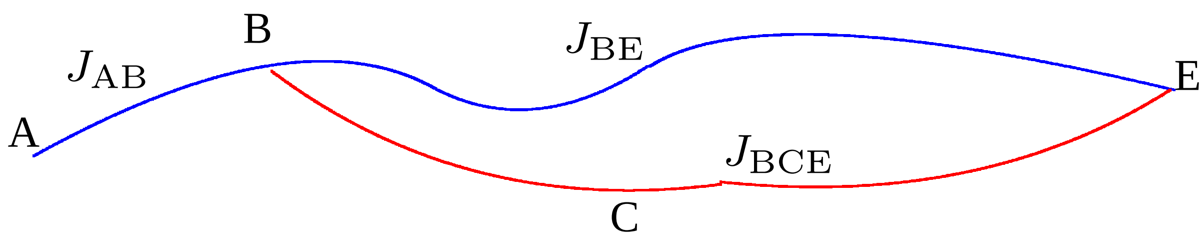
so that the objective function is $J = \hat{c}y(T) = y_{n+1}(T)$, where $\hat{c} = (\underbrace{0, 0, \ldots, 0}_{n}, 1)$. If we now integrate (6) from 0 to T, we have

$$J = \hat{c}y(T) = y_{n+1}(T) = \int_0^T F(x,u,t)dt + S[x(T), T], \tag{7}$$

which is the same as the objective function J in (4).

- Statement of Bellman’s Optimality Principle

  "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."



ASSERTION: *If* ABE *(shown in blue) is an optimal path from* A *to* E, *then* BE *(in blue) is an optimal path from* B *to* E

PROOF. Suppose it is not. Then there is another path (existence is assumed here) BCE (in red), which is optimal from B to E, i.e.,

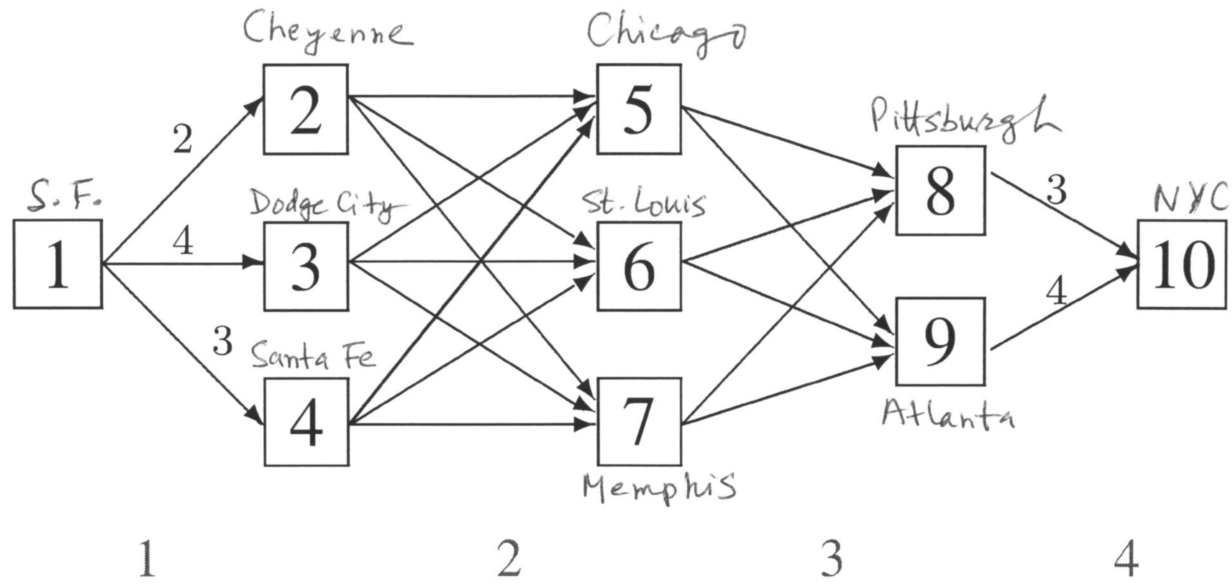
$$J_{\mathrm{BCE}} > J_{\mathrm{BE}}.$$

But then

$$J_{\mathrm{ABE}} = J_{\mathrm{AB}} + J_{\mathrm{BE}} < J_{\mathrm{AB}} + J_{\mathrm{BCE}} = J_{\mathrm{ABCE}}.$$

This contradicts the hypothesis that ABE is an optimal path from A to E.

Stagecoach Problem

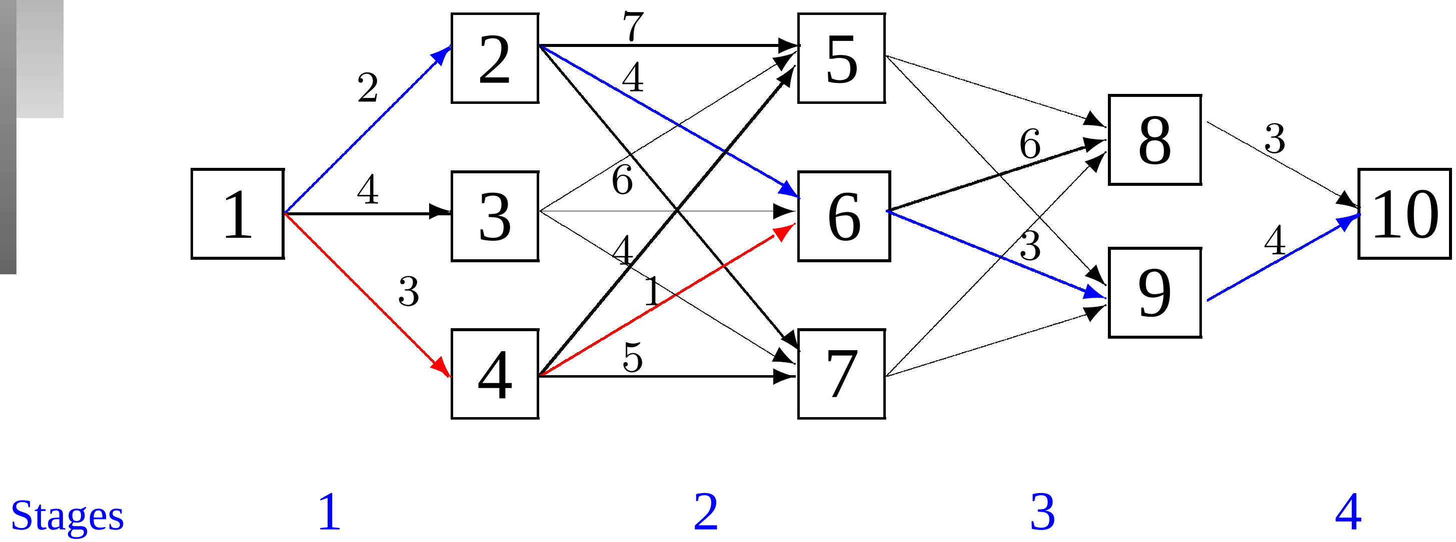


Costs $(c_{i,j})$:

	5	6	7
2	7	4	6
3	3	2	4
4	4	1	5

	8	9
5	1	4
6	6	3
7	3	3

Stagecoach Problem



Note that 1-2-6-9-10 is a greedy (or myopic) path that minimizes cost at each stage. However, this may not be the minimal cost solution. For example, 1-4-6 yields a cheaper cost than 1-2-6.

Let $1 - u_1 - u_2 - u_3 - 10$ be the optimal path. Let $f_n(s, u_n)$ be the minimal cost at stage n given that the current state is s and the decision taken is u_n. Let $f_n^*(s)$ be the minimal cost at stage n if the current state is s. Then

$$f_n^*(s) = \min_{u_n} f(s, u_n) = \min_{u_n} \left\{ c_{s,u_n} + f_{n+1}^*(u_n) \right\}.$$

This is the Recursion Equation of DP. It can be solved by a backward procedure, which starts at the terminal stage and stops at the initial stage.

Solution to the DP Example cont.

- Stage 4.

s	$f_4^*(s)$	u_4^*
8	3	10
9	4	10

- Stage 3.

s	$f_3(s,u_3) = C_{s,u_3} + f_4^*(u_3)$		$f_3^*(s)$	u_3^*
	8	9		
5	$1+3=4$	$4+4=8$	4	8
6	$6+3=9$	$3+4=7$	7	9
7	$3+3=6$	$3+4=7$	6	8

	8	9
5	1	4
6	6	3
7	3	3

- Stage 2.

s	$f_2(s,u_2)=C_{s,u_2}+f_3^*(u_2)$			$f_2^*(s)$	u_2^*
	5	6	7		
2	$7+4=11$	$4+7=11$	$6+6=12$	11	5, 6
3	$3+4=7$	$2+7=9$	$4+6=10$	7	5
4	$4+4=8$	$1+7=8$	$5+6=11$	8	5, 6

	5	6	7
2	7	4	6
3	3	2	4
4	4	1	5

- Stage 1.

s	$f_1(s,u_1)=C_{s,u_1}+f_2^*(u_1)$			$f_1^*(s)$	u_1^*
	2	3	4		
1	$2+11=13$	$4+7=11$	$3+8=11$	11	3, 4

The optimal paths:

$$\left.\begin{array}{l} 1-3-5-8-10 \\ 1-4-5-8-10 \\ 1-4-6-9-10 \end{array}\right\} \text{Total Cost} = 11.$$

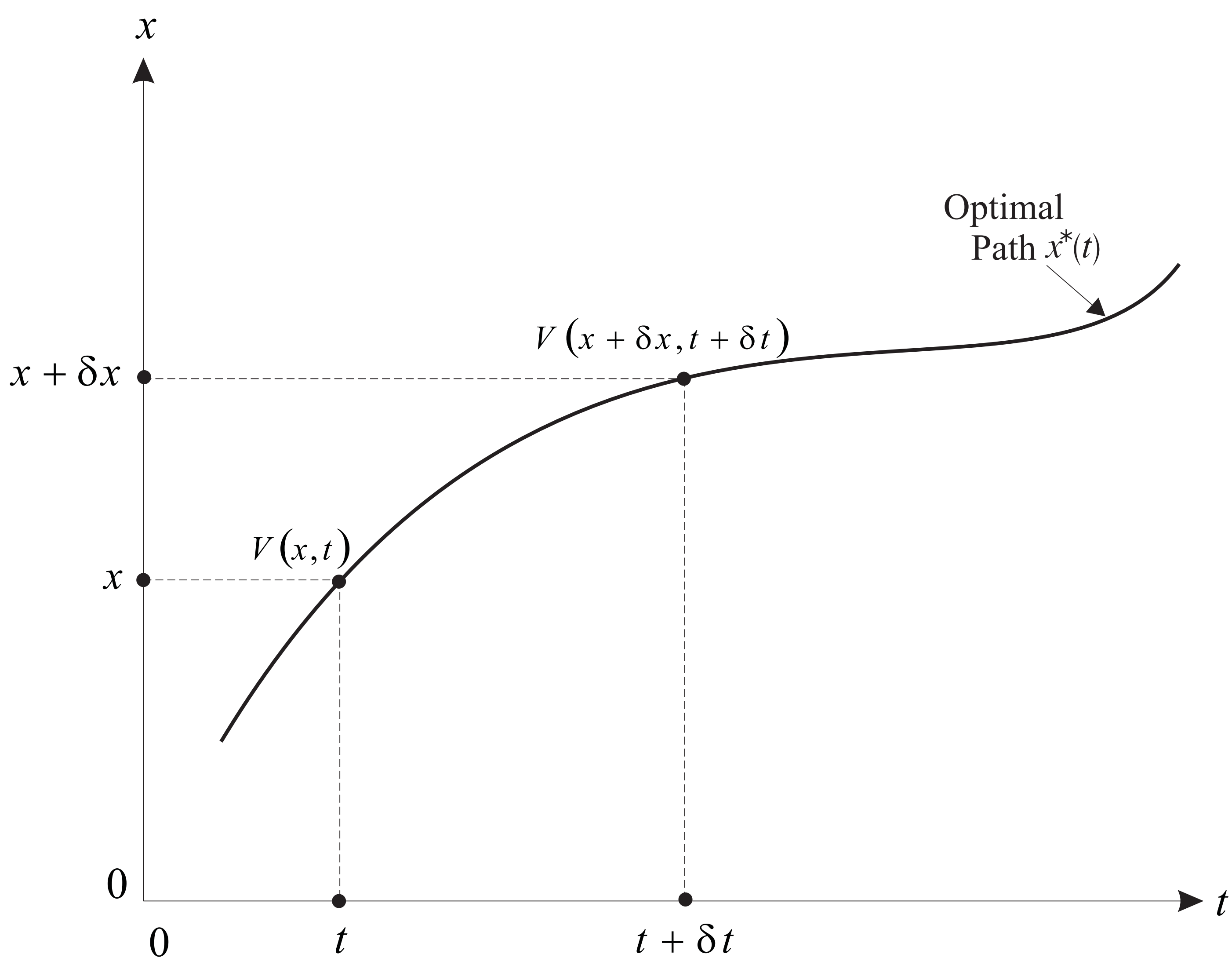
2.2.1 The Hamilton-Jacobi-Bellman Equation

$$V(x,t) = \max_{u(s)\in\Omega(s)} \left\{ \int_t^T F(x(s),u(s),s)ds \; + \; S(x(T),T) \right\}, \tag{8}$$

where for $s \geq t$,

$$\frac{dx}{ds} = f(x(s),u(s),s), \;\; x(t) = x.$$

Figure 2.1 An Optimal Path in the State-Time Space



(i) The incremental change in J from t to $t + \delta t$ is given by the integral of $F(x, u, t)$ from t to $t + \delta t$.

(ii) The value function $V(x + \delta x, t + \delta t)$ at time $t + \delta t$.

In equation form this is

$$V(x,t) = \max_{\substack{u(\tau)\in\Omega(\tau) \\ \tau\in[t,t+\delta t]}} \left\{ \int_t^{t+\delta t} F[x(\tau), u(\tau), \tau] d\tau + V[x(t+\delta t), t+\delta t] \right\}, \quad (9)$$

Since F is a continuous function, the integral in (9) is approximately $F(x, u, t)\delta t$ so that we can rewrite (9) as

$$V(x,t) = \max_{u\in\Omega(t)} \{F(x,u,t)\delta t + V[x(t+\delta t), t+\delta t]\} + o(\delta t). \quad (10)$$

Assume that the value function V is continuously differentiable. Use the Taylor series expansion of V with respect to δt and obtain

$$V[x(t+\delta t), t+\delta t] = V(x,t) + [V_x(x,t)\dot{x} + V_t(x,t)]\delta t + o(\delta t), \tag{11}$$

Substituting for $\dot{x}$ from (1), we obtain

$$\begin{aligned} V(x,t) &= \max_{u \in \Omega(t)} \Big\{ F(x,u,t)\delta t + V(x,t) \\ &\quad + V_x(x,t) f(x,u,t)\delta t + V_t(x,t)\delta t \Big\} + o(\delta t). \end{aligned} \tag{12}$$

Cancelling $V(x,t)$ on both sides and then dividing by δt we get

$$0 = \max_{u \in \Omega(t)} \Big\{ F(x,u,t) + V_x(x,t) f(x,u,t) + V_t(x,t) \Big\} + \frac{o(\delta t)}{\delta t}. \tag{13}$$

Letting $\delta t \to 0$ in (10) gives the equation

$$0 = \max_{u \in \Omega(t)} \left\{ F(x,u,t) + V_x(x,t) f(x,u,t) + V_t(x,t) \right\}, \qquad (14)$$

for which the boundary condition is

$$V(x,T) = S(x,T). \qquad (15)$$

$V_x(x,t)$ can be interpreted as the marginal contribution vector of the state variable x to the maximized objective function. Denote it by the *adjoint* (row) vector $\lambda(t) \in E^n$, i.e.,

$$\lambda(t) = V_x(x^*(t),t) := V_x(x,t) \mid_{x=x^*(t)} . \qquad (16)$$

We introduce the so-called *Hamiltonian*

$$H[x, u, V_x, t] = F(x, u, t) + V_x(x, t)f(x, u, t) \tag{17}$$

or, simply,

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t). \tag{18}$$

We can rewrite equation (14) as the following equation,

$$0 = \max_{u \in \Omega(t)} [H(x, u, V_x, t) + V_t], \tag{19}$$

called the *Hamilton-Jacobi-Bellman equation* or, simply, the HJB equation.

From (19), we can get the Hamiltonian maximizing condition of the maximum principle,

$$\begin{aligned} H[x^*(t), u^*(t), \lambda(t), t] + V_t(x^*(t), t) \quad \geq \quad & H[x^*(t), u, \lambda(t), t] \\ & + V_t(x^*(t), t). \end{aligned} \tag{20}$$

Cancelling the term V_t on both sides, we obtain

$$H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \tag{21}$$

for all $u \in \Omega(t)$.

Remark: H decouples the problem over time by means of $\lambda(t)$, which is analogous to dual variables or shadow prices in Linear Programming.

Let

$$x(t) = x^*(t) + \delta x(t), \tag{22}$$

where $\| \delta x(t) \| < \varepsilon$ for a small positive ε.

Fix t and use the HJB equation in (19) as

$$\begin{aligned} H[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_t(x^*(t), t) \\ \geq H[x(t), u^*(t), V_x(x(t), t), t] + V_t(x(t), t). \end{aligned} \tag{23}$$

LHS$= 0$ from (19) since $u^*(t)$ maximizes $H + V_t$. RHS will be zero if $u^*(t)$ also maximizes $H + V_t$ with $x(t)$ as the state. In general $x(t) \neq x^*(t)$, and thus RHS ≤ 0. But then RHS$|_{x(t)=x^*(t)} = 0 \Rightarrow$ RHS is maximized at $x(t) = x^*(t)$.

Since, $x(t)$ is unconstrained, we have $\frac{\partial \text{RHS}}{\partial x}|_{x(t)=x^*(t)} = 0$, or

$$H_x[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_{tx}(x^*(t), t) = 0. \tag{24}$$

By definition of H,

$$F_x + V_x f_x + f^T V_{xx} + V_{tx} = F_x + V_x f_x + (V_{xx} f)^T + V_{tx} = 0. \tag{25}$$

Note: (25) assumes V to be twice continuously differentiable. See (1.16) or Exercise 1.9 for details.

By definition (1.16) of $\lambda(t)$, the row vector $\dot{\lambda}(t)$ is

$$\begin{aligned}\frac{dV_x}{dt} &= \left(\frac{dV_{x_1}}{dt}, \frac{dV_{x_2}}{dt}, \cdots, \frac{dV_{x_n}}{dt}\right)\\ &= (V_{x_1x}\dot{x} + V_{x_1t}, V_{x_2x}\dot{x} + V_{x_2t}, \cdots, V_{x_nx}\dot{x} + V_{x_nt})\\ &= \left(\sum_{i=1}^n V_{x_1x_i}\dot{x}_i, \sum_{i=1}^n V_{x_2x_i}\dot{x}_i, \cdots, \sum_{i=1}^n V_{x_nx_i}\dot{x}_i\right) + (V_x)_t\\ &= (V_{xx}\dot{x})^T + V_{xt}\\ &= (V_{xx}f)^T + V_{tx}.\end{aligned} \tag{26}$$

Note
$V_{x_1x} = (V_{x_1x_1}, V_{x_1x_2}, ..., V_{x_1x_n}),$ and

$$V_{xx}\dot{x} = \begin{pmatrix} V_{x_1x_1} & V_{x_1x_2} & \cdots & V_{x_1x_n}\\ V_{x_2x_1} & V_{x_2x_2} & \cdots & V_{x_2x_n}\\ \vdots & & & \vdots\\ V_{x_nx_1} & V_{x_nx_2} & \cdots & V_{x_nx_n}\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{pmatrix},$$ a column vector.

Using (25) and (26) we have

$$\frac{dV_x}{dt} = -F_x - V_x f_x. \tag{27}$$

Using (16) we have

$$\dot{\lambda} = -F_x - \lambda f_x.$$

From (18) we have

$$\dot{\lambda} = -H_x. \tag{28}$$

$$\lambda(T) = \frac{\partial S(x,T)}{\partial x} \Big|_{x=x(T)} = S_x[x(T),T]. \tag{29}$$

(28) and (29) can determine the adjoint variables.

From (18), we can rewrite the state equation as

$$\dot{x} = f = H_\lambda. \tag{30}$$

From (28), (29), (30) and (1), we get

$$\begin{cases} \dot{x} = H_\lambda, & x(0) = x_0, \\ \dot{\lambda} = -H_x, & \lambda(T) = S_x[x(T),T], \end{cases} \tag{31}$$

called a *canonical system of equations* or *canonical adjoints*.

- In our problem $x(T)$ is free. In this case, $S \equiv 0$ (no salvage value) $\Rightarrow \lambda(T) = 0$
- However, if $x(T)$ is not free, that is, $x(T)$ fixed (given) $\Rightarrow \lambda(T) =$ a constant to be determined. In other words, $\lambda(T)$ is free.

The necessary conditions for u^* to be an optimal control are:

$$\begin{cases} \dot{x}^* = f(x^*, u^*, t), x^*(0) = x_0, \\ \dot{\lambda} = -H_x[x^*, u^*, \lambda, t], \quad \lambda(T) = S_x[x^*(T), T], \\ H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t], \end{cases} \tag{32}$$

for all $u \in \Omega(t), \ t \in [0, T]$.

$$J = \int_0^T F(x, u, t)dt + S[x(T), T],$$

where F is the instantaneous profit rate per unit of time, and $S[x, T]$ is the salvage value. Multiplying (18) formally by dt and using the state equation (1) gives

$$H dt = F dt + \lambda f dt = F dt + \lambda \dot{x} dt = F dt + \lambda dx.$$

$F(x, u, t)dt$: direct contribution to J from t to $t + dt$

λdx: indirect contribution to J due to added capital dx

$H dt$: *total contribution* to J from time t to $t + dt$, when $x(t) = x$ and $u(t) = u$ in the interval $[t, t + dt]$.

By (28) and (29), we have

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x} - \lambda \frac{\partial f}{\partial x}, \;\; \lambda(T) = S_x[x(T), T].$$

Rewrite the first equation as

$$-d\lambda = H_x dt = F_x dt + \lambda f_x dt.$$

$-d\lambda$: marginal cost of holding capital x from t to $t + dt$

$H_x dt$: marginal revenue of investing the capital

$F_x dt$: direct marginal contribution

$\lambda f_x dt$: indirect marginal contribution

Thus, the adjoint equation implies $\Rightarrow$ MC = MR.

Consider the problem:

$$\max \left\{ J = \int_0^1 -x dt \right\} \tag{33}$$

subject to the state equation

$$\dot{x} = u, \ x(0) = 1 \tag{34}$$

and the control constraint

$$u \in \Omega = [-1, 1]. \tag{35}$$

Note that $T = 1$, $F = -x$, $S = 0$, and $f = u$. Because $F = -x$, we can interpret the problem as one of minimizing the (signed) area under the curve $x(t)$ for $0 \leq t \leq 1$.

We form the Hamiltonian

$$H = -x + \lambda u. \tag{36}$$

Because the Hamiltonian is linear in u, the form of the optimal control, i.e., the one that would maximize the Hamiltonian, is

$$u^*(t) = \begin{cases} 1 & \text{if} \quad \lambda(t) > 0, \\ \text{undefined} & \text{if} \quad \lambda(t) = 0, \\ -1 & \text{if} \quad \lambda(t) < 0. \end{cases} \tag{37}$$

It is called Bang-Bang Control. In the notation of Section 1.4,

$$u^*(t) = \text{bang}[-1, 1; \lambda(t)]. \tag{38}$$

Solution of Example 2.1 cont.

To find λ, we write the adjoint equation

$$\dot{\lambda} = -H_x = 1, \;\; \lambda(1) = S_x[x(T), T] = 0. \tag{39}$$

Because this equation does not involve x and u, we can easily solve it as

$$\lambda(t) = t - 1. \tag{40}$$

It follows that $\lambda(t) = t - 1 \leq 0$ for all $t \in [0, 1]$. So $u^*(t) = -1$, $t \in [0, 1)$. Since $\lambda(1) = 0$, we can also set $u^*(1) = -1$, which defines u at the single point $t = 1$. We thus have the optimal control

$$u^*(t) = -1 \quad \text{for} \;\; t \in [0, 1].$$

Substituting this into the state equation (34) we have

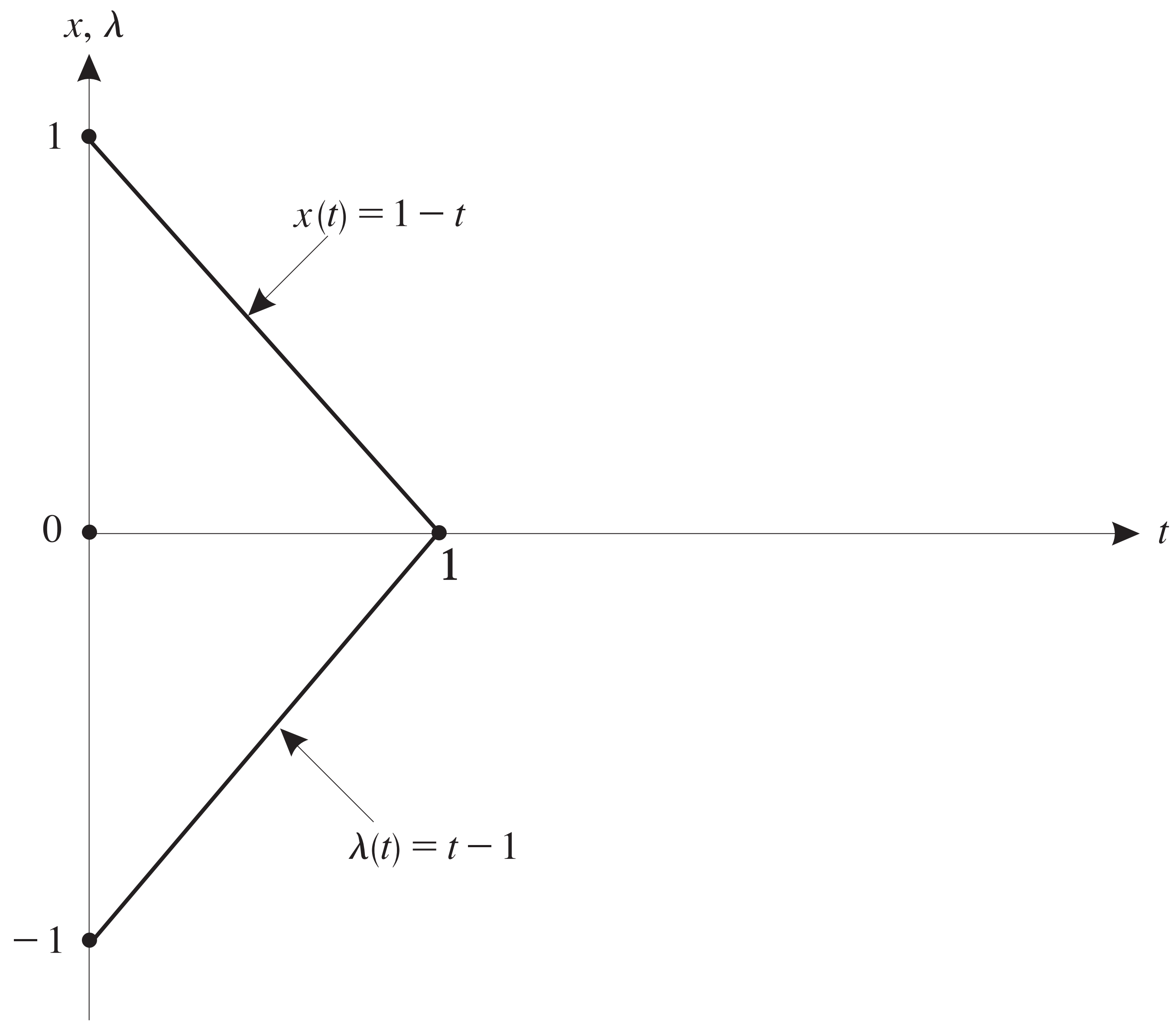
$$\dot{x} = -1, \; x(0) = 1, \tag{41}$$

whose solution is

$$x(t) = 1 - t \text{ for } t \in [0, 1]. \tag{42}$$

The graphs of the optimal state and adjoint trajectories appear in Figure 2.2. Note that the optimal value of the objective function is $J^* = -1/2$.

Figure 2.2 Optimal State and Adjoint Trajectories for Example 2.1



Let us solve the same problem as in Example 2.1 over the interval $[0, 2]$ so that the objective is to

$$\text{maximize } \left\{ J = \int_0^2 -x dt \right\}. \tag{43}$$

As before, the dynamics and constraints are (33) and (34), respectively. Here we want to minimize the *signed* area between the horizontal axis and the trajectory of $x(t)$ for $0 \le t \le 2$.

As before the Hamiltonian is defined by (36) and the optimal control is as in (38). The adjoint equation

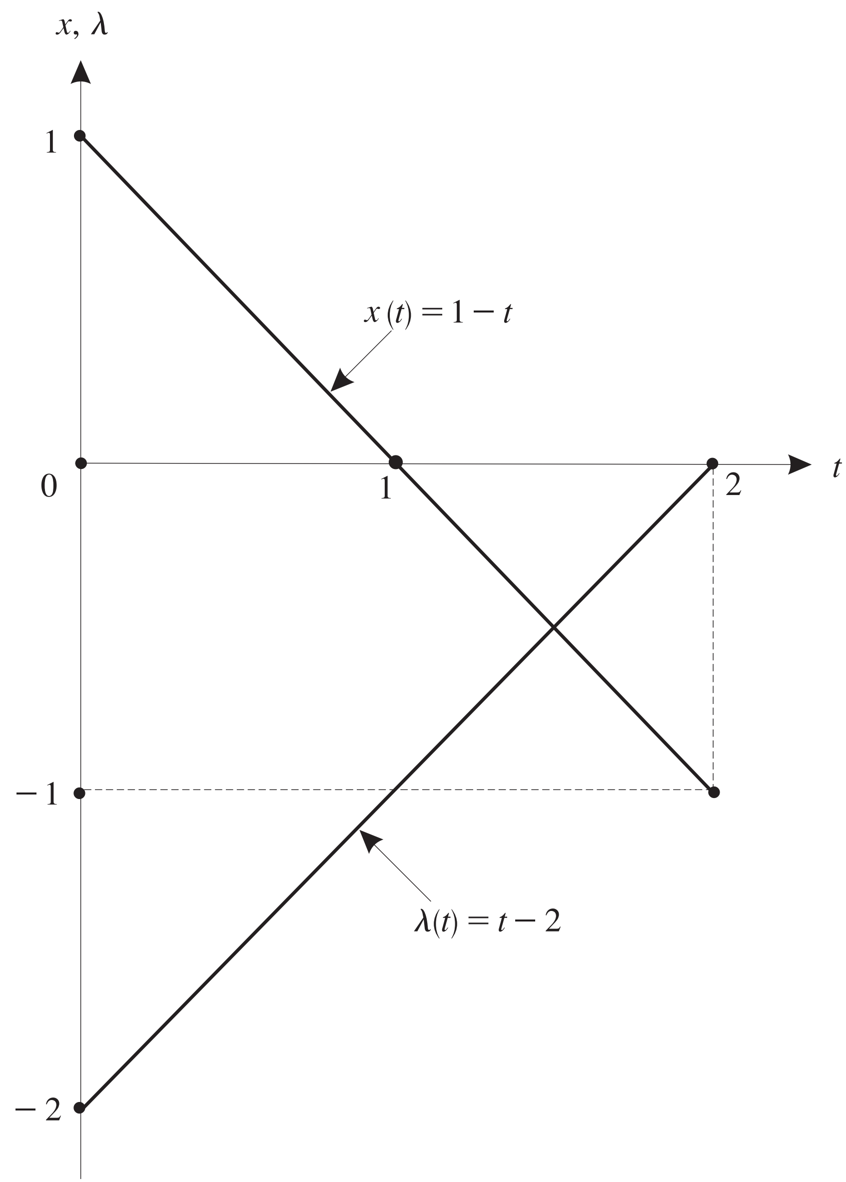
$$\dot{\lambda} = 1,\ \lambda(2) = 0 \tag{44}$$

is the same as (39), except that now $T = 2$ instead of $T = 1$. The solution of (44) is easily found to be

$$\lambda(t) = t - 2,\ t \in [0, 2]. \tag{45}$$

Hence the state equation (41) and its solution (42) are exactly the same. The graph of $\lambda(t)$ is shown in Figure 2.3. The optimal value of the objective function is $J^* = 0$.

FIGURE 2.3 OPTIMAL STATE AND ADJOINT TRAJECTORIES FOR EXAMPLE 2.2



The next example is:

$$\max \left\{ J = \int_0^1 -\frac{1}{2}x^2 dt \right\} \tag{46}$$

subject to the same constraints as in Example 2.1, namely,

$$\dot{x} = u,\ x(0) = 1,\ u \in \Omega = [-1, 1]. \tag{47}$$

Here $F = -(1/2)x^2$ so that the interpretation of the objective function (46) is that we are trying to find the trajectory $x(t)$ in order that the area under the curve $(1/2)x^2$ is minimized.

The Hamiltonian is

$$H = -\frac{1}{2}x^2 + \lambda u, \tag{48}$$

which is linear in u so that the optimal policy is

$$u^*(t) = \text{ bang } [-1, 1; \lambda]. \tag{49}$$

The adjoint equation is

$$\dot{\lambda} = -H_x = x, \ \lambda(1) = 0. \tag{50}$$

Here the adjoint equation involves x so that we cannot solve it directly. Because the state equation (47) involves u, which depends on λ, we also cannot integrate it independently without knowing λ.

A way out of this dilemma is to use some intuition. Since we want to minimize the area under $(1/2)x^2$ and since $x(0) = 1$, it is clear that we want x to decrease as quickly as possible. Let us therefore temporarily *assume* that λ is nonpositive in the interval $[0, 1]$ so that from (49) we have $u = -1$ throughout the interval. (In Exercise 2.5, you will be asked to show that this assumption is correct.) With this assumption, we can solve (47) as

$$x(t) = 1 - t. \tag{51}$$

Substituting this into (50) gives

$$\dot{\lambda} = 1 - t.$$

Integrating both sides of this equation from t to 1 gives

$$\int_t^1 \dot{\lambda}(\tau)d\tau = \int_t^1 (1 - \tau)d\tau,$$

or

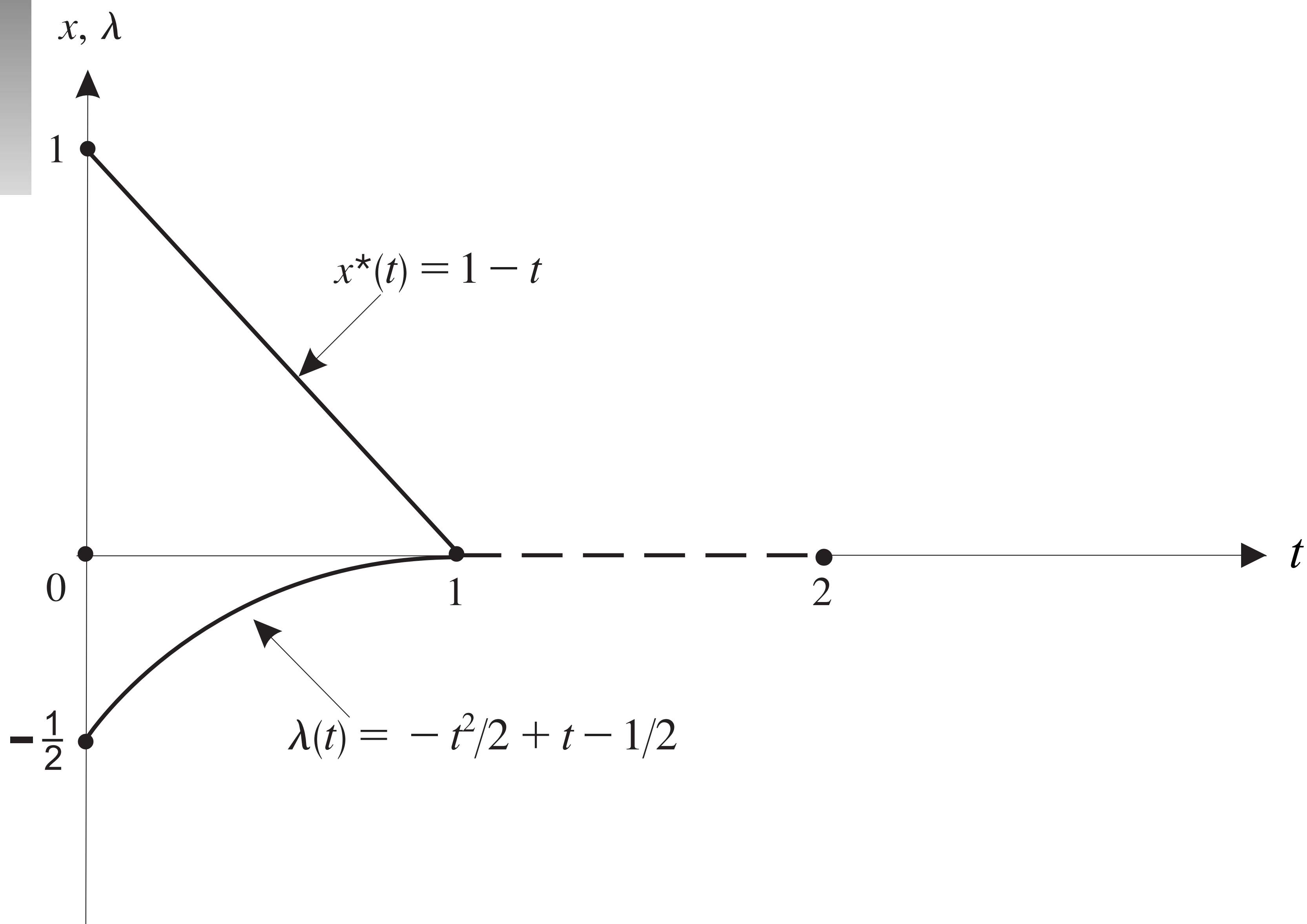
$$\lambda(1) - \lambda(t) = (\tau - \frac{1}{2}\tau^2) \mid_t^1,$$

which, using $\lambda(1) = 0$, yields

$$\lambda(t) = -\frac{1}{2}t^2 + t - \frac{1}{2}. \tag{52}$$

The reader may now verify that $\lambda(t)$ is nonpositive in the interval $[0, 1]$, justifying our original assumption. Hence, (51) and (52) satisfy the necessary conditions. In Exercise 2.6, you will be asked to show that they satisfy sufficient conditions derived in Section 2.4 as well, so that they are indeed optimal. Figure 2.4 shows the graphs of the optimal trajectories.

Figure 2.4 Optimal Trajectories for Example 2.3 and Example 2.4



Let us rework Example 2.3 with $T = 2$, i.e., solve the problem:

$$\max \left\{ J = \int_0^2 -\frac{1}{2}x^2 dt \right\} \tag{53}$$

subject to the constraints

$$\dot{x} = u,\ x(0) = 1,\ u \in \Omega = [-1, 1].$$

It would be clearly optimal if we could keep $x^*(t) = 0$, $t \in [1, 2]$. This is possible by setting $u^*(t) = 0,\ t \in [1, 2]$. Note $u^*(t) = 0$, $t \in [1, 2]$, is a singular control, since it gives $\lambda(t) = 0$, $t \in [1, 2]$. See Fig. 2.4.

The problem is:

$$\max \left\{ J = \int_0^2 (2x - 3u - u^2)dt \right\} \tag{54}$$

subject to

$$\dot{x} = x + u, \ x(0) = 5 \tag{55}$$

and the control constraint

$$u \in \Omega = [0, 2]. \tag{56}$$

The Hamiltonian is

$$\begin{aligned} H &= (2x - 3u - u^2) + \lambda(x + u) \\ &= (2 + \lambda)x - (u^2 + 3u - \lambda u). \end{aligned} \tag{57}$$

The optimal control is

$$u(t) = \frac{\lambda(t) - 3}{2}. \tag{58}$$

The adjoint equation is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2 - \lambda, \ \lambda(2) = 0. \tag{59}$$

Using the integrating factor as shown in Appendix A.6, its solution is

$$\lambda(t) = 2(e^{2-t} - 1).$$

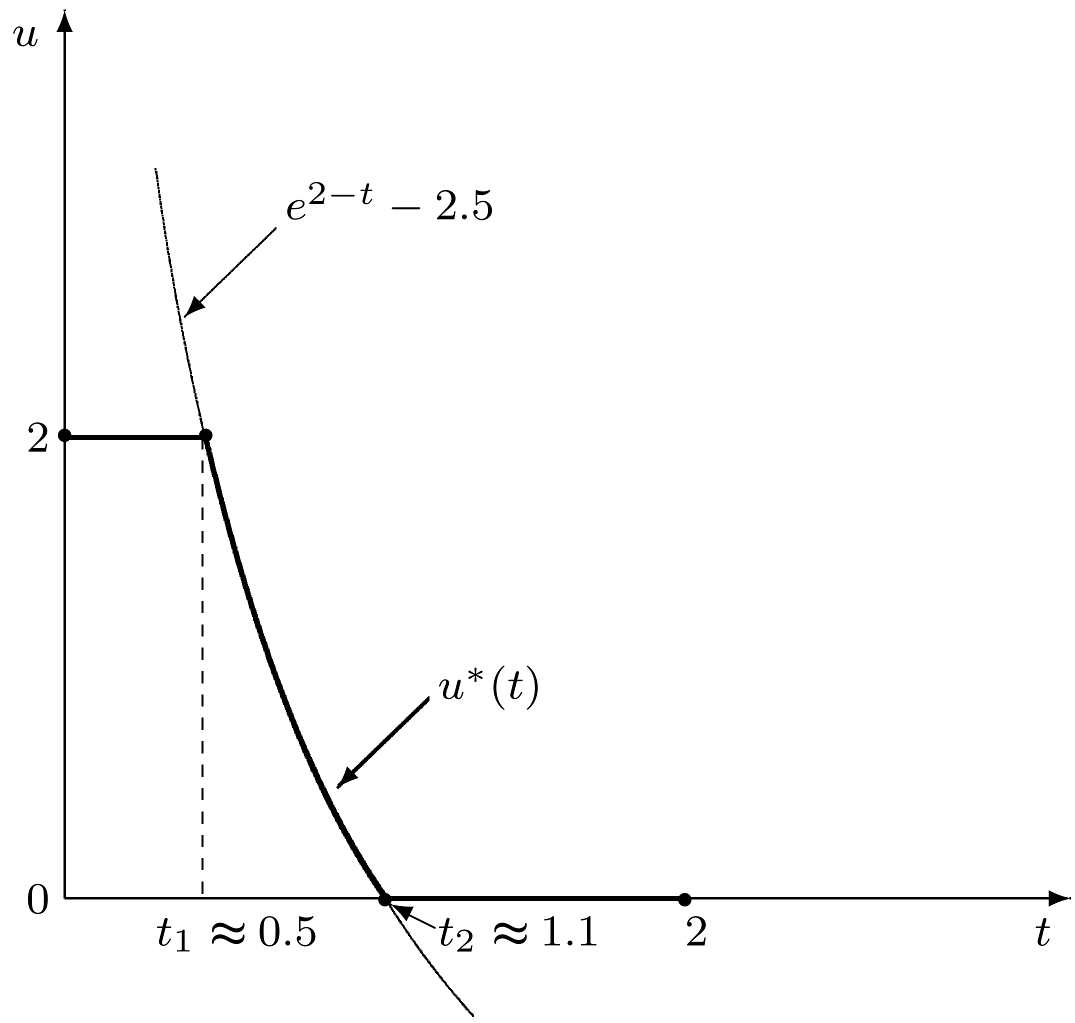
The optimal control is

$$u^*(t) = \begin{cases} 2 & \text{if} \quad e^{2-t} - 2.5 > 2, \\ e^{2-t} - 2.5 & \text{if} \quad 0 \leq e^{2-t} - 2.5 \leq 2, \\ 0 & \text{if} \quad e^{2-t} - 2.5 < 0. \end{cases} \tag{60}$$

It can be written as

$$u^*(t) = \text{ sat } [0, 2; e^{2-t} - 2.5].$$

Figure 2.5 Optimal control for Example 2.5



$$H^0(x,\lambda,t) = \max_{u\in\Omega(t)} H(x,u,\lambda,t). \tag{61}$$

$$H^0(x,\lambda,t) = H(x,u^0,\lambda,t). \tag{62}$$

The **Envelope Theorem** gives

$$H_x^0(x,\lambda,t) = H_x(x,u^0,\lambda,t). \tag{63}$$

Here is why?

$$H_x^0(x,\lambda,t) = H_x(x,u^0,\lambda,t) + H_u(x,u^0,\lambda,t)\frac{\partial u^0}{\partial x}. \tag{64}$$

But

$$H_u(x,u^0,\lambda,t)\frac{\partial u^0}{\partial x} = 0, \tag{65}$$

since either $H_{u_i} = 0$ or $\partial u_i^0/\partial x_j = 0$ or both.

Theorem 2.1

$($*Sufficiency Conditions*$)$ *Let* $u^*(t)$, *and the corresponding* $x^*(t)$ *and* $\lambda(t)$ *satisfy the maximum principle necessary condition* (32) *for all* $t \in [0, T]$. *Then,* u^* *is an optimal control if* $H^0(x, \lambda(t), t)$ *is concave in* x *for each* t *and* $S(x, T)$ *is concave in* x.

By definition of H^0,

$$H[x(t), u(t), \lambda(t), t] \leq H^0[x(t), \lambda(t), t]. \tag{66}$$

Since H^0 is differentiable and concave,

$$\begin{aligned} H^0[x(t), \lambda(t), t] \ \leq \ & H^0[x^*(t), \lambda(t), t] \\ & + H^0_x[x^*(t), \lambda(t), t][x(t) - x^*(t)]. \end{aligned} \tag{67}$$

Using the Envelope Theorem, we can write

$$\begin{aligned} H[x(t), u(t), \lambda(t), t] \ \leq \ & H[x^*(t), u^*(t), \lambda(t), t] \\ & + H_x[x^*(t), u^*(t), \lambda(t), t][x(t) - x^*(t)]. \end{aligned} \tag{68}$$

By the definition of H and the adjoint equation,

$$\begin{aligned} & F[x(t), u(t), t] + \lambda(t) f[x(t), u(t), t] \\ & \quad \leq F[x^*(t), u^*(t), t] + \lambda(t) f[x^*(t), u^*(t), t] \\ & \quad -\dot{\lambda}(t)[x(t) - x^*(t)]. \end{aligned} \tag{69}$$

Using the state equation,

$$\begin{aligned} & F[x^*(t), u^*(t), t] - F[x(t), u(t), t] \\ & \quad \geq \dot{\lambda}(t)[x(t) - x^*(t)] + \lambda(t)[\dot{x}(t) - \dot{x}^*(t)]. \end{aligned} \tag{70}$$

Since S is differentiable and concave

$$S[x(T),T] \le S[x^*(T),T] + S_x[x^*(T),T][x(T) - x^*(T)] \tag{71}$$

or,

$$S[x^*(T),T] - S[x(T),T] + S_x[x^*(T),T][x(T) - x^*(T)] \ge 0. \tag{72}$$

Integrating (70) and adding (72),

$$\begin{aligned} J(u^*) - J(u) &+ S_x[x^*(T),T][x(T) - x^*(T)] \\ &\ge \lambda(T)[x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)], \end{aligned} \tag{73}$$

$$J(u^*) \ge J(u). \tag{74}$$

- Let us show that the problems in Examples 2.1 and 2.2 satisfy the sufficient conditions. We have from (36) and (61),

$$H^0 = -x + \lambda u^0,$$

where u^0 is given by (37). Since u^0 is a function of λ only, $H^0(x, \lambda, t)$ is certainly concave in x for any t and λ. Since $S(x, T) = 0$, the sufficient conditions hold.

The terminal values of the state variables are completely specified, i.e., $x(T) = k \in E$, where k is a vector of constants. Recall that the proof of the sufficient conditions requires (73) to hold, i.e.,

$$\begin{aligned} J(u^*) - J(u) + S_x[x^*(T), T][x(T) - x^*(T)] \\ \geq \lambda(T)[x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)]. \end{aligned}$$

Since $x(T) - x^*(T) = 0$, the RHS vanishes regardless of the value of $\lambda(T)$ in this case. This means that the sufficiency result would go through for any value of $\lambda(T)$. Not surprisingly, therefore, the transversality condition is

$$\lambda(T) = \beta. \tag{75}$$

- Example 2.7. Consider the problem:

$$\max \left\{ J = \int_0^1 -\frac{1}{2}(x^2 + u^2)dt \right\}$$

subject to

$$\dot{x} = -x^3 + u, \; x(0) = 5. \tag{76}$$

- Solution:

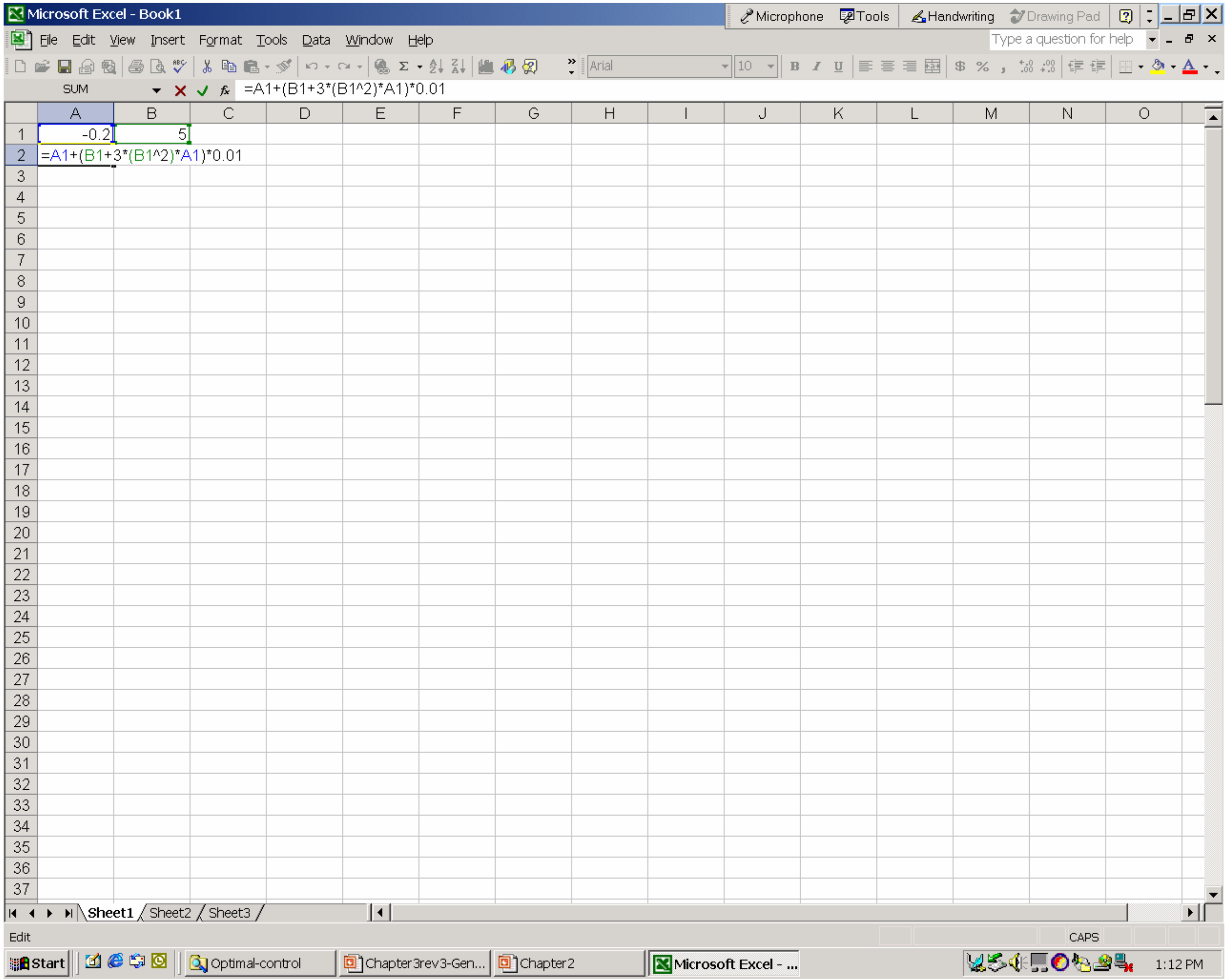
$$\dot{\lambda} = x + 3x^2\lambda, \;\; \lambda(1) = 0. \tag{77}$$

$$\dot{x} = -x^3 + \lambda, \;\; x(0) = 5. \tag{78}$$

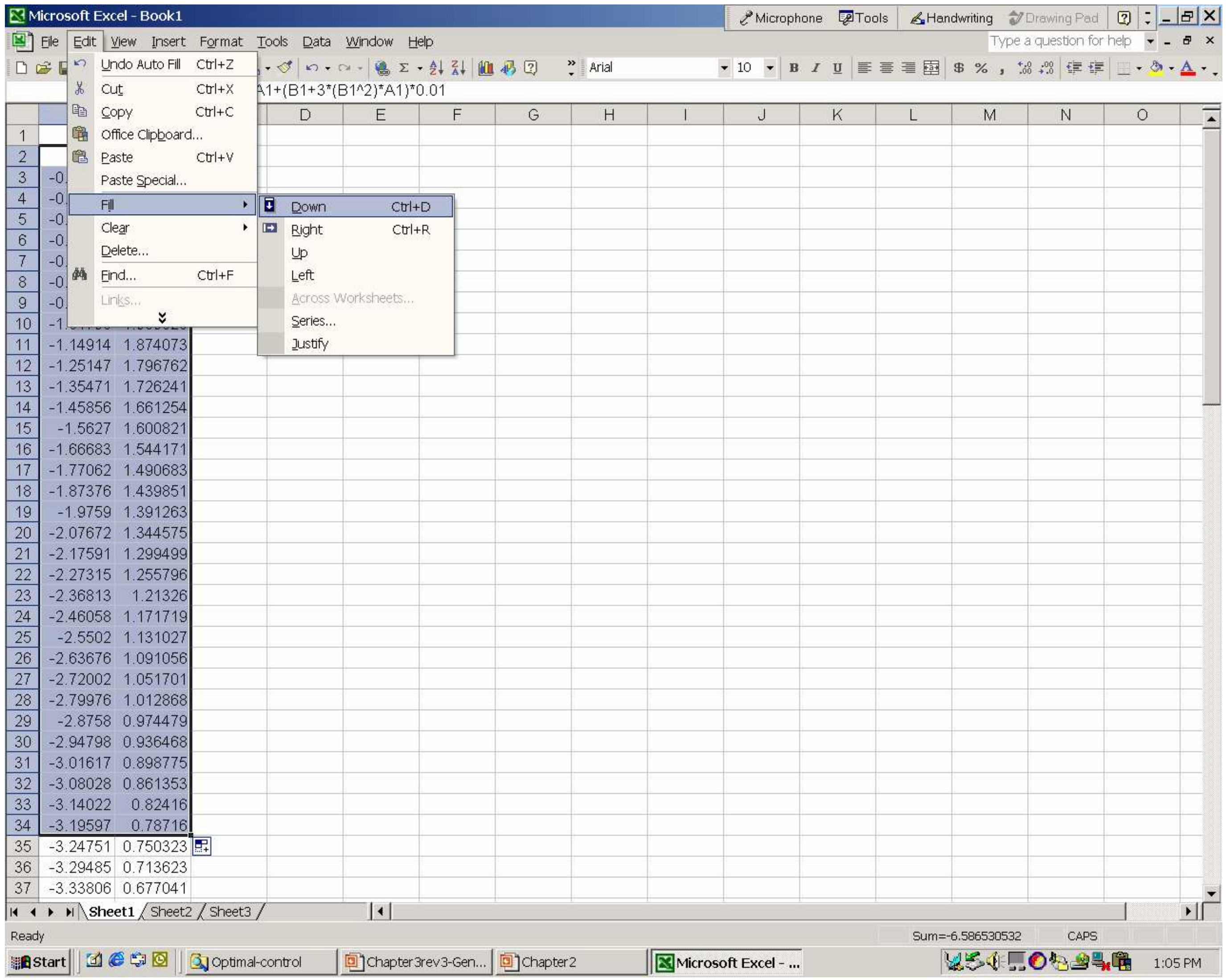
$$x(t + \triangle t) = x(t) + \left[-x(t)^3 + \lambda(t)\right] \triangle t, \;\; x(0) = 5, \tag{79}$$

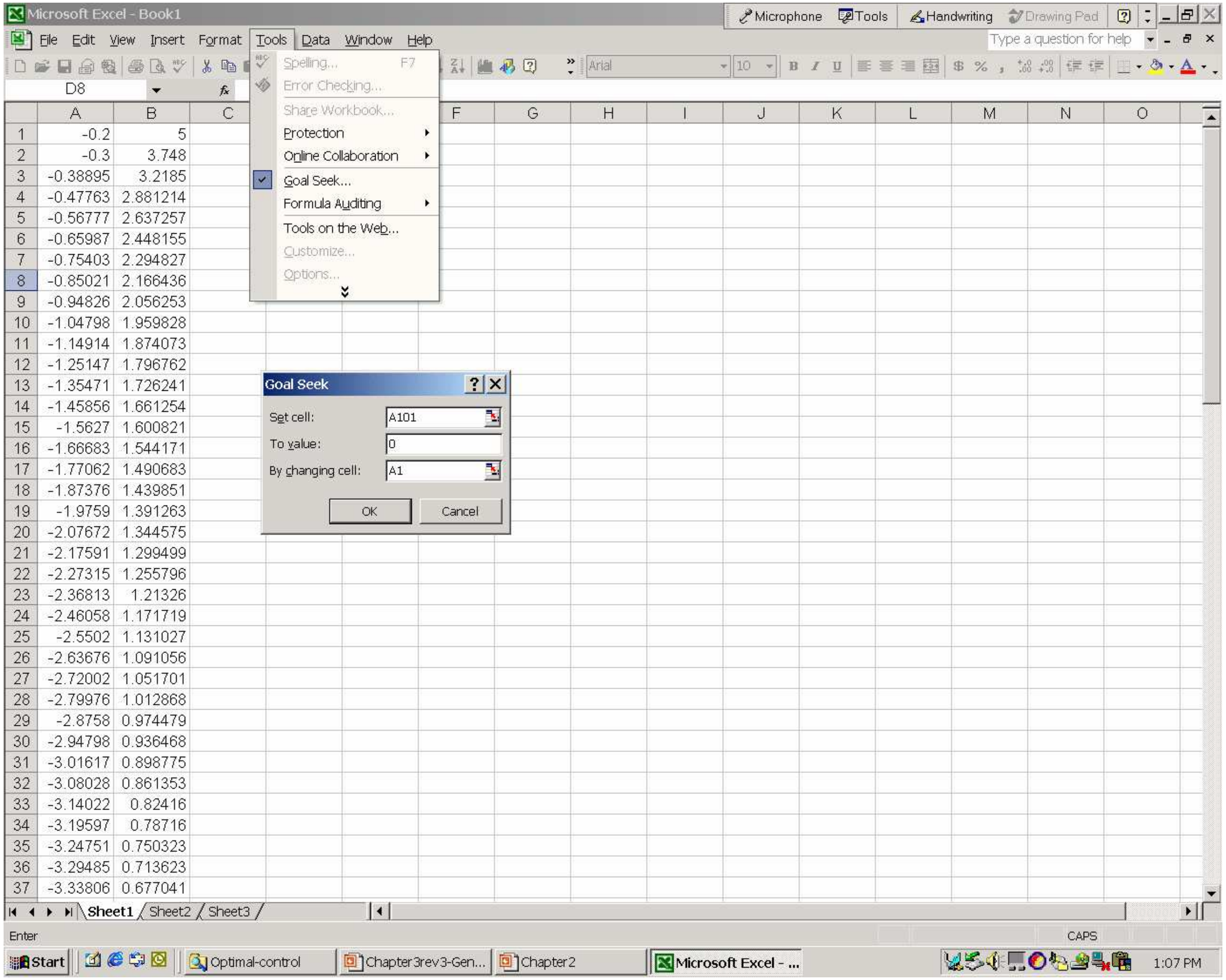
$$\lambda(t + \triangle t) = \lambda(t) + \left[x(t) + 3x(t)^2\lambda(t)\right] \triangle t, \;\; \lambda(1) = 0. \tag{80}$$

Set $\triangle t = 0.01$, $\lambda(0) = -0.2$ and $x(0) = 5$.



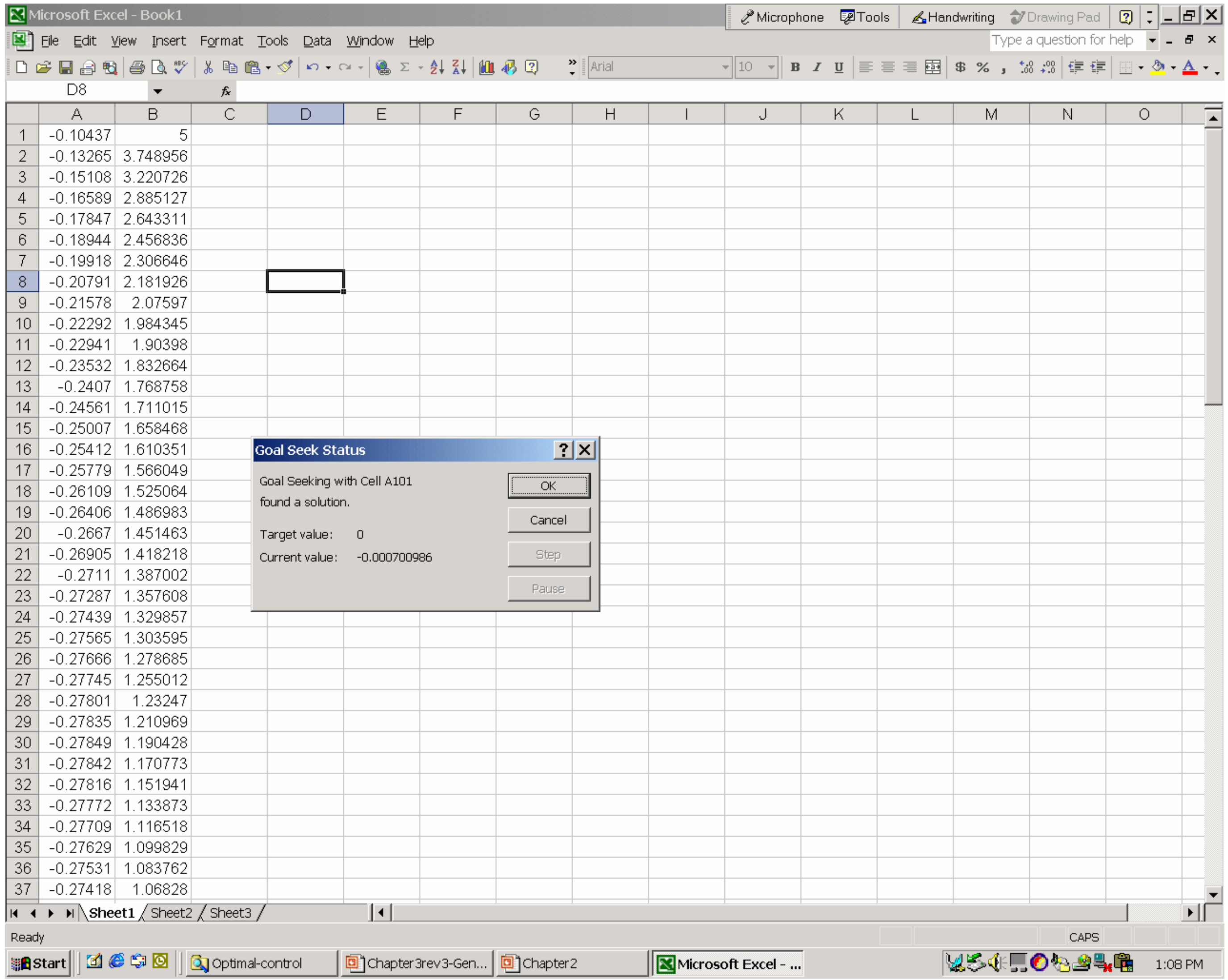
SOLVING A TPBVP BY USING SPREADSHEET SOFTWARE CONT.





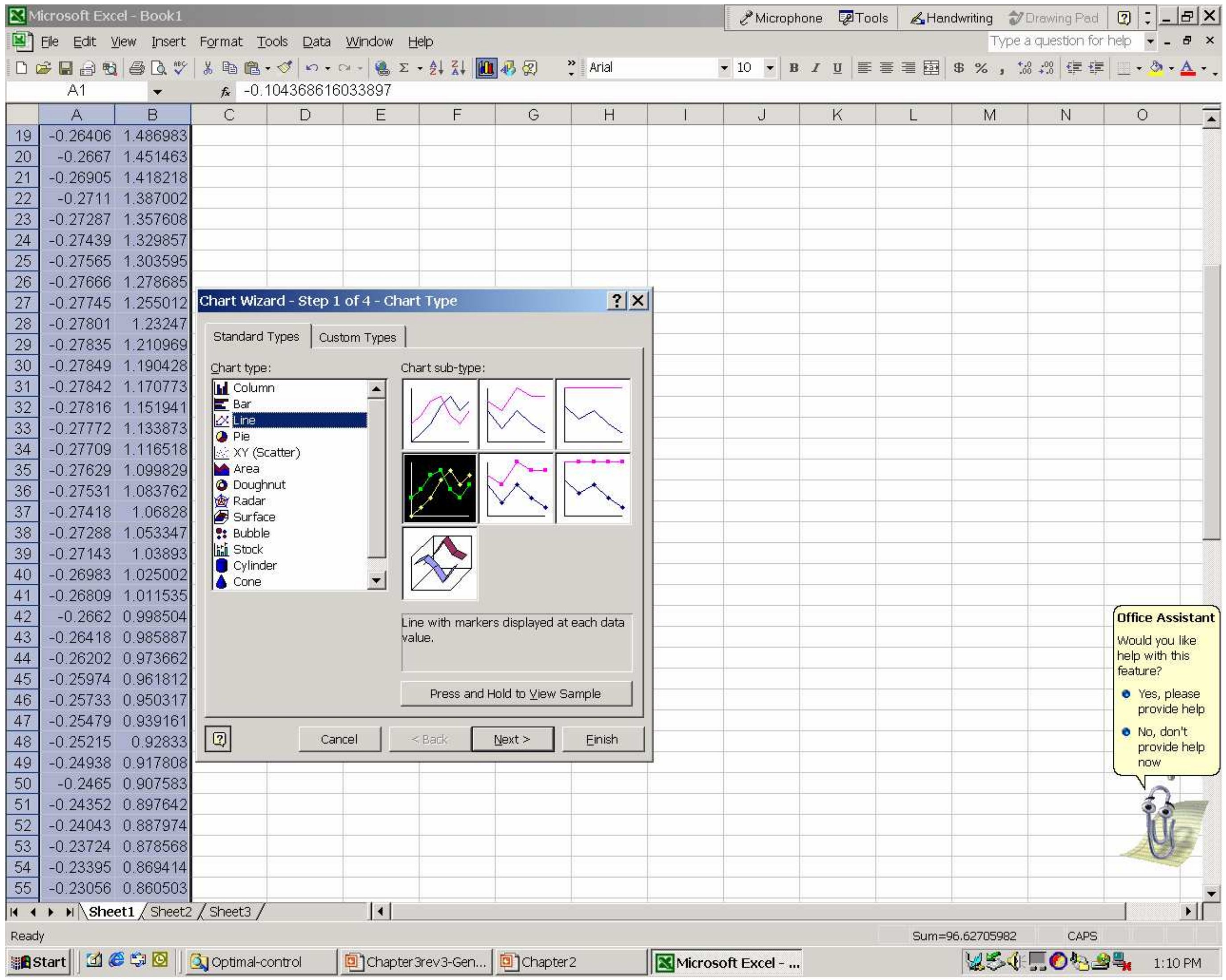
	A	B
1	-0.2	5
2	-0.3	3.748
3	-0.38895	3.2185
4	-0.47763	2.881214
5	-0.56777	2.637257
6	-0.65987	2.448155
7	-0.75403	2.294827
8	-0.85021	2.166436
9	-0.94826	2.056253
10	-1.04798	1.959828
11	-1.14914	1.874073
12	-1.25147	1.796762
13	-1.35471	1.726241
14	-1.45856	1.661254
15	-1.5627	1.600821
16	-1.66683	1.544171
17	-1.77062	1.490683
18	-1.87376	1.439851
19	-1.9759	1.391263
20	-2.07672	1.344575
21	-2.17591	1.299499
22	-2.27315	1.255796
23	-2.36813	1.21326
24	-2.46058	1.171719
25	-2.5502	1.131027
26	-2.63676	1.091056
27	-2.72002	1.051701
28	-2.79976	1.012868
29	-2.8758	0.974479
30	-2.94798	0.936468
31	-3.01617	0.898775
32	-3.08028	0.861353
33	-3.14022	0.82416
34	-3.19597	0.78716
35	-3.24751	0.750323
36	-3.29485	0.713623
37	-3.33806	0.677041

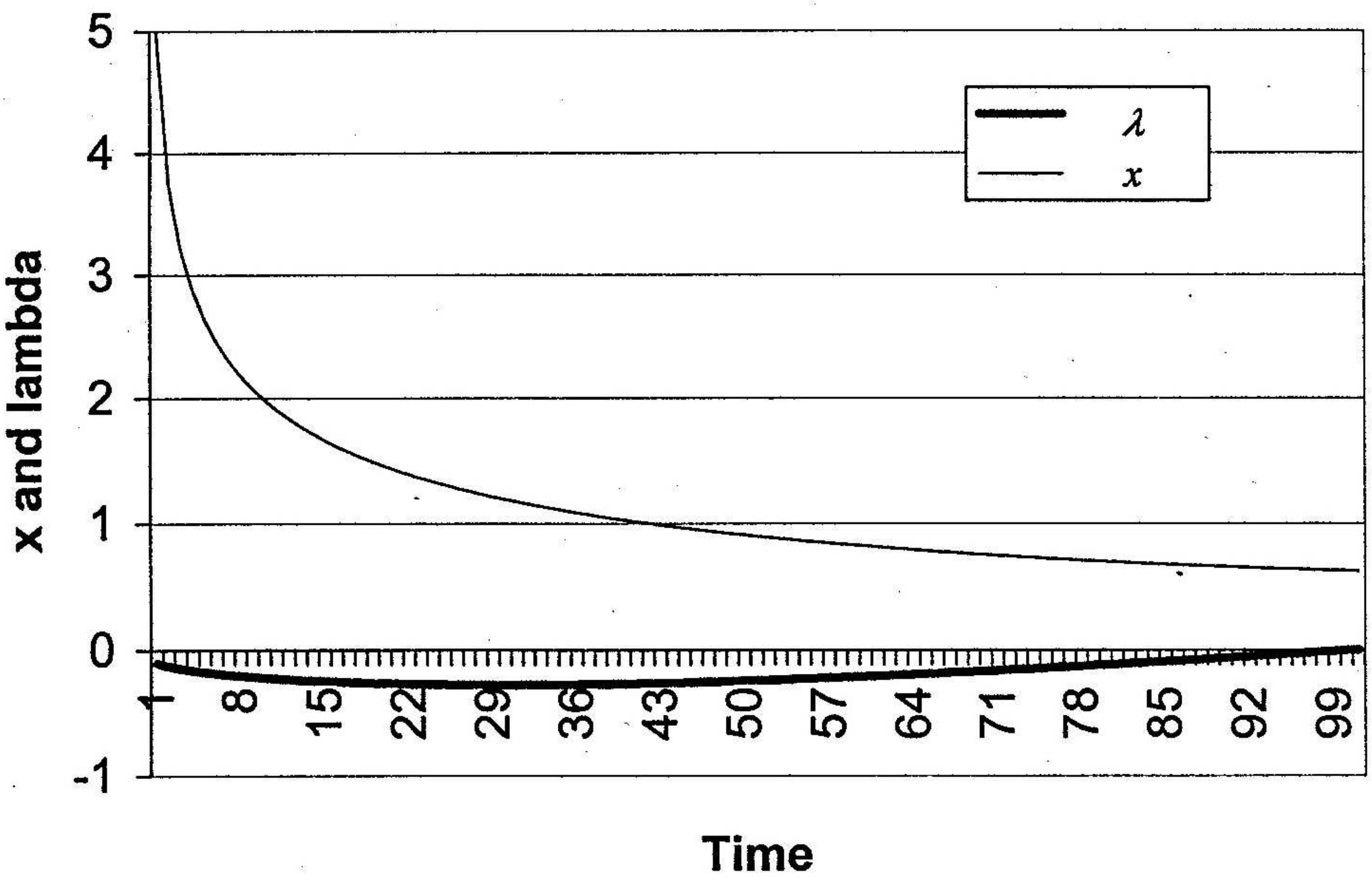
Solving a TPBVP by Using Spreadsheet Software cont.

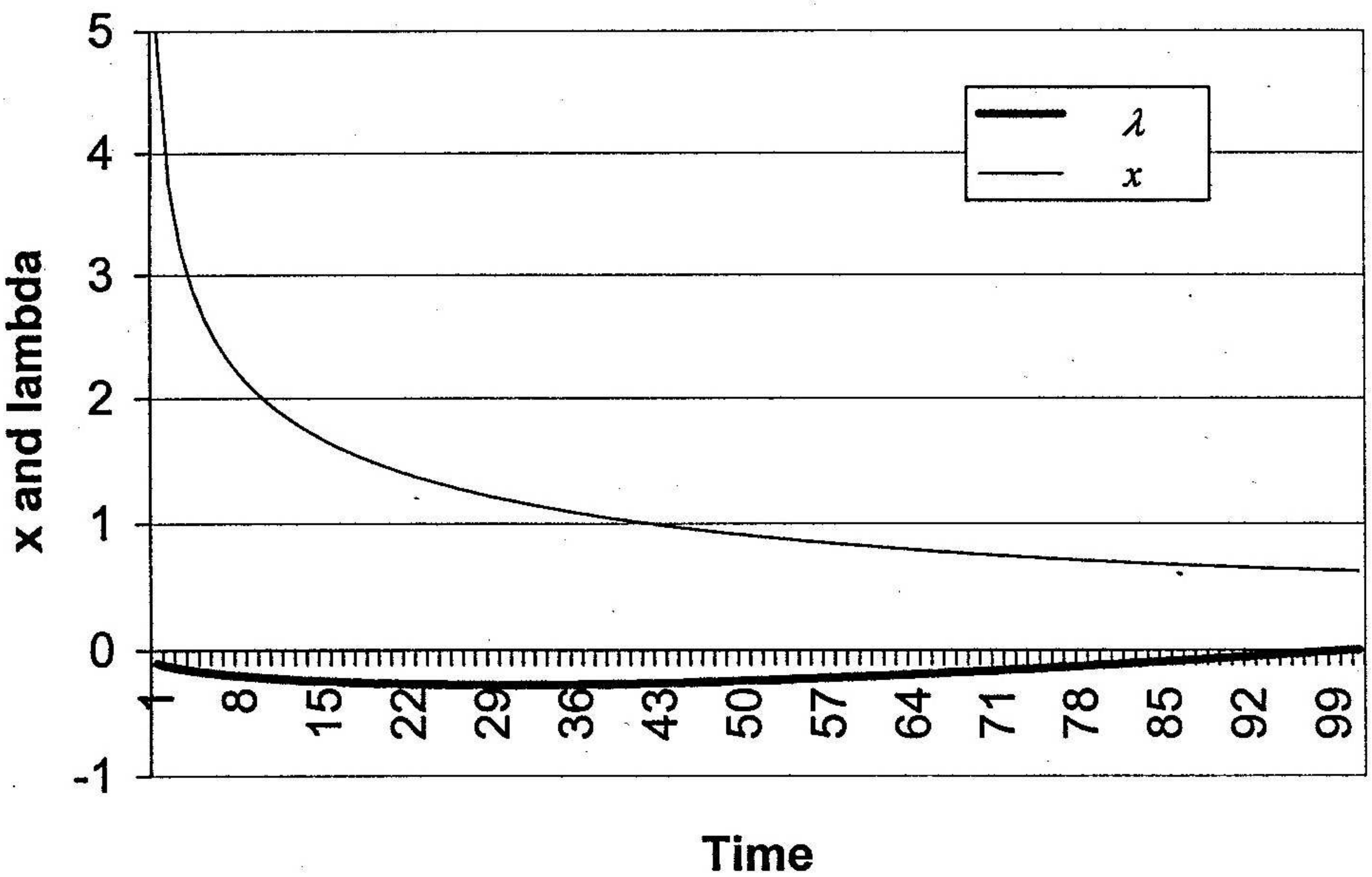


	A	B
1	-0.10437	5
2	-0.13265	3.748956
3	-0.15108	3.220726
4	-0.16589	2.885127
5	-0.17847	2.643311
6	-0.18944	2.456836
7	-0.19918	2.306646
8	-0.20791	2.181926
9	-0.21578	2.07597
10	-0.22292	1.984345
11	-0.22941	1.90398
12	-0.23532	1.832664
13	-0.2407	1.768758
14	-0.24561	1.711015
15	-0.25007	1.658468
16	-0.25412	1.610351
17	-0.25779	1.566049
18	-0.26109	1.525064
19	-0.26406	1.486983
20	-0.2667	1.451463
21	-0.26905	1.418218
22	-0.2711	1.387002
23	-0.27287	1.357608
24	-0.27439	1.329857
25	-0.27565	1.303595
26	-0.27666	1.278685
27	-0.27745	1.255012
28	-0.27801	1.23247
29	-0.27835	1.210969
30	-0.27849	1.190428
31	-0.27842	1.170773
32	-0.27816	1.151941
33	-0.27772	1.133873
34	-0.27709	1.116518
35	-0.27629	1.099829
36	-0.27531	1.083762
37	-0.27418	1.06828

Solving a TPBVP by Using Spreadsheet Software cont.




5
4
3
2
1
0
-1
x and lambda
λ
x
1
8
15
22
29
36
43
50
57
64
71
78
85
92
99
Time


5
4
3
2
1
0
-1
x and lambda
λ
x
1
8
15
22
29
36
43
50
57
64
71
78
85
92
99
Time