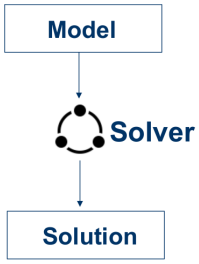
Indice

1 Introduction - what is constraint programming?

The combinatorial decision making is a generic problem where we have to make a decision (obviously) within many cases of a context and with a number of restrictions, usually called constraints. A solution can be any which meets all constraints, but also an optimal solution according to an objective. This problem is quite common in out daily lives, think about hospitals during covid: infected people had to be assigned to hospitals according to some parameters like the severity of illness, the age, the hospital resources. One can object that this can be done with artificial intelligence, but this is very tricky, because for this specific problem we have no data for training and, in general, neural networks are very expensive to train, if we want something which have a useful accuracy. Decision making is typically computationally difficult (NP-hard) and there are many techniques to approach it: integer linear programming; boolean SATisfiability; constraint programming. We will focus on *constraint programming* (CP). But what is constraint programming? It is a declarative programming paradigm for expressing and solving combinatorial optimization problems. These problems have to be expressed as a model which has typically three entities:

- the unknowns, namely decision variables which we have to find a value for;
- the possible values for the unknowns;
- relations between unknowns.

The power of CP stands in the solving, indeed, the user doesn't have to worry about how it can solve a problem, but just how to model it. This is possible thanks to the *solver*, which, given a model for a problem, has the goal of solving the problem, by assigning values to unknowns.



But how does a solver work? It uses a backtracking tree search for guessing the values for variables and it examines model constraints to shrink domains of decision variables in order to avoid incompatible values in the future (this is called *propagation*). But these phases are separated, indeed, it interleaves cycles of variables assigning and propagation. Modelling is critical, since the solver depends on it.

2 Model

Now, we can formalize the concepts around CP. A constraint satisfaction problem (CSP) is a triple $\langle X, D, C \rangle$, where:

- X is a finite set of decision variables $X_1, ..., X_n$ which require a value to be assigned to;
- D is the set of domains of X, so $X_i \in D_i(X_i)$; each D_i is supposed to be finite;
- C is a set of constraints, namely relations over the domains:

$$C_i \subseteq D_j(X_j) \times ... \times D_k(X_k)$$

A constraint optimization problem is 4-tuple $\langle X, D, C, f \rangle$, where f is an objective variable whose value has to be optimized, namely minimizing or maximizing it.

2.1 Constraints

There are two main kinds of constraints representations:

- *extensional* constraints which relies on the fact that any kind of constraint can be expressed as the set of all allowed combinations, for example $C(X_1, X_2) = (0,0), (0,2), (1,3), (2,1)$;
- *intensional* constraints, namely declarative relations on involved entities, for example $X > Y$.

Channeling constraints Channeling constraints makes two different models "comunicate", in the sense that, given two models $m1$ and $m2$ and a channeling constraint c, c brings what has been discovered in $m1$ in $m2$ and viceversa thanks to different propagation algorithms in different models $m1$ and $m2$. This improves propagation because benefits from a model are brought to another model. Think about n-queens problem where we have to place n queens in a $n \times n$ chests board in order to they cannot eat directly any queen. We can have different models for this problem.

First model:

$$X_1, ..., X_n \in 1, ..., n$$
$$\text{alldifferent}([X_1, ..., X_n])$$
$$\text{alldifferent}([X_1 + 1, ..., X_n + n])$$
$$\text{alldifferent}([X_1 - 1, ..., X_n - n])$$

Second model:

$$n \times n \; B_{ij} \in 0, 1$$

$$\sum_{i \in 1,...,n} B_{ij} = 1, \; \forall j \in 1, ..., n$$

$$\sum_{j \in 1,...,n} B_{ij} = 1, \; \forall i \in 1, ..., n$$

$$\sum B_{ij} \leq 1 \text{ on all diagonals}$$

$$\text{lex-lesser-eq}(B, \pi(B)), \; \forall \pi$$

The two models represent the same problem. The first one uses variables to represent queens positions on the board and it exploits the alldifferent constraint, while the second one uses a boolean matrix to represent the positions and it exploits the lexicographic order constraint. They have different benefits on search (not explained which benefits there, take this as granted), so how to take advantage from both? We use a channeling constraint:

$$\forall i, j \; X_i = j \iff B_{ij} = 1$$

Meta-constraints A *meta-constraint* is a constraint occurring in another constraint. For example $\sum_i (X_i > t_i) <= n$; in this case, the inner constraint is $(X_i > t_i)$. This type of constraints is useful when we want to make choices according to the results of other constraints.

Implied and redundant constraints An *implied constraint* is a semantically redundant and they speed up the solver. On the other hand, we refer to redundant constraint as a constraint which is semantically redundant, but it doesn't affect the solver performance. It happens to have a redudant constraint r when we have another constraint s which makes the propagation shrink the domains of variables in the same way r would do. For example, look at this model:

```
include "alldifferent.mzn";
int: n;
array [0..n-1] of var 0..n-1: x;

constraint forall(i in 0..n-1)
    (x[i] = sum (j in 0..n-1)(x[j] == i));

constraint sum(i in 0..n-1)(x[i]) = n;

constraint sum(i in 0..n-1)(x[i]*i) = n;

solve satisfy;
```

In this model, we are expressing the sequence puzzle problem, namely we want to make a sequence x with length n, where for each $i \in 1, ..., n$, i appears exactly x_i times in the sequence x. Note that we don't used global constraints. The last two constraints are implied, so they affect solver performance. Now consider the following model:

```
include "globals.mzn";

int: n;
array [1..n] of var 0..n-1: x;

constraint let {
  array[1..n] of 0..n-1: cover = 0..n-1
} in global_cardinality(x, cover, x);

constraint sum(i in 1..n) (x[i]) = n;

constraint sum(i in 1..n) (x[i] * (i-1)) = n;

solve satisfy;
```

This model is semantically equivalent to the previous one, but the first implied constraint (so the second constraint) became redundant. This happened because the propagation algorithm for `global_cardinality_constraint` is more effective than decomposition and so it "discovers" all the first (ex-)implied constraint would do. This concept will be developed better nextly.

2.2 Symmetry

Often, when we try to solve a problem, tehre are many solutions which are "symmetric". Two solutions are symmetric if one of them is a permutation of the other one. The solver cannot know if two solutions are symmetric and it will look for all of them, but this is a waste of time because two symmetric solutions are actually the same solution, they didn't bring something new. Thus, to avoid symmetry, we can add some constraints which somehow impose an order in order to accept just one solution and exclude all the symmetric ones. Usually, useful constraints are the ones of `lex*` family, namely constraints which pretend lexicographic order on array of variables. TODO

3 Constraints propagation

Propagation is the action of restricting the domains of variables.

3.1 Local consistency

This is a form of inference which detects inconsistent partial assignments. What is a partial assignment? An assignment is literally an assignment of a value to a decision variable:

$$X_i = j$$

If that assignment is inconsistent, then j can be removed from $D(X_i)$ and, as a consequence, it helps propagation.

3.1.1 Generalized Arc Consistency

A *support* $(d_1, ..., d_k) \in (D(X_1), ..., D(X_k))$ for a constraint C is an assignment of decision variables which satisfies C. A constraint C is GAC iff $\forall X_i \in$

$X_i, ..., X_n$, $\forall v \in D(X_i)$, $v \in d_i$, where d_i is a support of C. This is called *Arc consistency* (AC) when $k = 2$.

In other words, a constraint C restricts the domains $D(X_i)$. A question can arise: how does a constraint C grant the property of GAC for domains? The answer is that global constraints have specialized propagation algorithms. These algorithms tries to keep the domains of variables as restricted as possible in order to have always the lowest number assignments.

3.1.2 Bounds consistency

It relaxes the domain of a decision variable X_i to be in a range such that

$$D(X_i) = [min(X_i)...max(X_i)]$$

A *bound support* is a tuple $(d_1, ..., d_k) \in C$ where $d_i \in [min(X_i)...max(X_i)]$.

$C(X_i, ..., X_k)$ is BC iff for all $X_i \in X_1, ..., X_k$, $min(X_i)$ and $max(X_i)$ belong to a bound support. GAC is stronger than BC, however, it's more expensive to achieve sometimes.

Look at the following minizinc programs to solve sudoku:

```
include "globals.mzn";

int: n;

array[1..(n*n), 1..(n*n)] of var 1..(n*n): x;

% Rows must be all different
constraint forall(i in 1..(n*n)) (alldifferent(x[i, ..])
    ::domain_propagation
    );

% Columns must be all different
constraint forall(j in 1..(n*n)) (alldifferent(x[.., j])
    ::domain_propagation
    );

% Squares must be all different
constraint forall (i in 1..n) (forall (j in 1..n)
        (alldifferent(x[(1+((i-1)*n))..(n+((i-1)*n)),
                    (1+((j-1)*n))..(n+((j-1)*n))]))
                    ::domain_propagation
        );

solve satisfy;
```

and:

```
include "globals.mzn";

int: n;

array[1..(n*n), 1..(n*n)] of var 1..(n*n): x;

% Rows must be all different
constraint forall(i in 1..(n*n)) (alldifferent(x[i, ..])
```

```
    ::bounds_propagation
    );

% Columns must be all different
constraint forall(j in 1..(n*n)) (alldifferent(x[.., j])
    ::bounds_propagation
    );

% Squares must be all different
constraint forall (i in 1..n) (forall (j in 1..n)
        (alldifferent(x[(1+((i-1)*n))..(n+((i-1)*n)),
                    (1+((j-1)*n))..(n+((j-1)*n))]))
                    ::bounds_propagation
        );

solve satisfy;
```

and:

```
include "globals.mzn";

int: n;

array[1..(n*n), 1..(n*n)] of var 1..(n*n): x;

% Rows must be all different
constraint forall(i in 1..(n*n)) (alldifferent(x[i, ..]));

% Columns must be all different
constraint forall(j in 1..(n*n)) (alldifferent(x[.., j]));

% Squares must be all different
constraint forall (i in 1..n) (forall (j in 1..n)
        (alldifferent(x[(1+((i-1)*n))..(n+((i-1)*n)),
                    (1+((j-1)*n))..(n+((j-1)*n))]))
        );

solve satisfy;
```

We try to solve it with $n = 7$, so we are solving a sudoku 49X49. In the first program, we specify (or better, we suggest) to use domain propagation, namely we keep GAC. Results are these:

```
% time elapsed: 3s 166msec
----------
%%%mzn-stat: failures=1
%%%mzn-stat: initTime=0.0108307
%%%mzn-stat: nodes=1985
%%%mzn-stat: peakDepth=1983
%%%mzn-stat: propagations=101106
%%%mzn-stat: propagators=147
%%%mzn-stat: restarts=0
%%%mzn-stat: solutions=1
%%%mzn-stat: solveTime=2.90563
%%%mzn-stat: variables=2401
%%%mzn-stat-end
%%%mzn-stat: nSolutions=1
%%%mzn-stat-end
Finished in 3s 189msec.
```

Just 3 seconds, not bad! But when we try to use bounds propagation (namely BC) or even not to suggest the propagation way (the third case), the results is

that the solver goes over five minutes with millions of failures. Clearly, this is just a case where domain propagation is better than bounds propagation.

3.2 Propagation

It is the action of achieving a certain level of consistency which is a property (not an action). Indeed, we talk about propagation algorithms. During search, multiple propagation algorithms can interact, usually in way that at a given momentum, only an algorithm is running and it does it until it reaches a level of consistency. Then, there are no other actions to do, so another algorithm runs. An algorithm can run multiple times, since, because of propagation, the decision variables domains change. What is important is that algorithms run until GAC property is granted, if they can, however, GAC is not always possible to be kept, sometimes only BC (or other weaker consistency properties) can be kept.

Complexity What about complexity of algorithms? Assume $|D(X_i)| = d$, from definitions we have that one run of $C(X_1, X_2)$ takes $O(d^2)$. However, we can improve, for example we can think cases where the running time is better, sometimes even $O(1)$.

3.3 Global constraints

They are specific constraints which helps propagation and can state expressions impossible to state with primitive constraints. Often, the propagation algorithms for these constraints keep GAC in polynomial time, namely they simply run in polynomial time and at the end of the process, the domains are exactly the supports for constraints.

Counting constraints They restrict the number of variables satisfying a condition.

3.3.1 Sequencing constraints

They ensures a sequence of variables have certain patterns.

3.3.2 Scheduling constraints

They are all about assigning resources to activities.

3.3.3 Ordering constraints

They obviously care of order of variables values. They include also lexicographic ordering constraint.

The propagation for global constraints is developed by decomposition into smaller constraints for which the propagation algorithm is known. Generally, global constraints integrates a specialized propagation algorithm.

The `alldifferent` constraint propagation algorithm is based on bipartite graphs. One part of the graph is the variables, the other part is the possible values of the variables. A *matching* M in a graph G is a set of non-adjacent

edges such that no two edges share common vertices. A *maximal matching* for a graph G is a matching which is not subset of any other matching in G. The propagation algorithm for `alldifferent` constraint considers edges as the assignments of variables to the values; finding a maximal matching means finding a possible assignment for variables. Before analyzing it, let's give some definition:

- an edge is *matching* if it belongs in a matching;
- an edge is *free* if it is not matching;
- a node is *matched* if it is incident to a matching edge;
- a node is *free* if it is not mathed;
- an edge is *vital* if it belongs to every maximal matching;

Let's see the algorithm:

- compute all maximal matchings;
- if no maximal matching exists, then fail;
- if an edge is free in all maximal matchings, then:
  - remove the edge;
  - remove the corresponding value to the domain of the associated variable;
- if an edge is vital, then:
  - keep the edge;
  - assign the corresponding value to the associated variable;
- if an edge is matching (but not vital), then keep the edge.

The problem with this algorithm is that calculating all maximal matching in a naive way is too expensive. Let's see these further definitions:

- An *alternating path* is a simple path with edges alternating free and matching;
- An *alternating cycle* is a path with edges alternating free and matching;
- An *even path* or *cycle* is such if the number of edges is even.

An important can help us to in finding maximal matchings. An edge e belongs to a maximal matching iff for some arbitrary maximal matching M:

- either e belongs to M;
- or e belongs to even alternating path starting at a free node;
- or e belongs to an even alternating cycle.

Next, we have to perform the following actions:

- we choose an arbitrary maximal matching M;
- all free edges in M changes their direction in order to they go from values to variables;
- all matching edges in M changes their direction in order to they go from variables to values;
- starting from a free node, search for all nodes on directed simple path and mark all the edges;
- find the strongly connected components and mark edges which are part of the SCCs;
- all the marked edges and the edges which are parts of M are part of some maximal matching. Other edges are discarded and domains can be updated.

3.4 Table constraint

Sometimes, we know the exact combinations of values that variables can take. In those cases, table constraint can be really useful. An example can be crossword puzzle:

$$table([X_1, X_2, X_3], dictionary)$$

$$table([X_1, X_{13}, X_{16}], dictionary)$$

$$table([X_4, X_5, X_6, X_7], dictionary)$$

...

3.5 Regular constraint

Sometimes, we need that variables follow a certain pattern. In those cases, deterministic finite-state automaton can be very useful, indeed, *regular* constraint is based on dfsa: $regular([X_1, ..., X_k], A)$ holds iff $\langle X_1, ...X_k \rangle$ forms a string accepted by A. This constraint has an efficient propagation algorithm which keeps GAC property, however, decomposition is very efficient for this constraint as well. The advantage over table constraint is that the latter needs all solutions have to be computed firstly. Anyway, even if many constraints are just instances of regular, it is advisable to use specific constraints when possible, firstly for readability of the code; regular constraint are useful for complicated patterns.

4 Search

This phase is carried out by the constraint solver which performs a backtracking tree search, where nodes are variables and branches are decisions on variables. The first point to focus on is that the tree is not built immediately as whole, but subtrees are built gradually (at the need). The search is helped by propagation: without it, the search should build all the possible subtrees to guess a solution.

Remember that propagation and search are two different phases: propagation is concerned with domains shrinking and it's done when "evaluating" a constraint; search phase is done after propagation and it is concerned with the "growing" of the tree. Before doing search, it's necessary to perform propagation.

4.1 Depth-first visit

There are two types of branching:

d-way branching for a variable X (which corresponds to a node), k branches are created, where k is the cardinality of the actual domain of X;

2-way branching a subset S of the domain of a variable X is set, then two branches are created, one if $X \in S$, one if $X \notin S$.

There are different types of heuristics, which are related to the choice of variables that have to come in the next branches and their values.

Static variable ordering heuristics A variable is associated a priori to each level of the search tree, regardless how the search will be carried out.

Dynamic variable ordering heuristics At any node, any variable can be considered. This can be more expensive than static heuristics, but the current state of the tree can be kept, while state heuristics cannot do this. There's no a best heuristics from this point of view, it depends from the problem.

Generic dynamic variable ordering heuristics There are many types of heuristics, two of them are fail-first, namely discarding inconsistent subtrees as soon as possible. The advantage is that making a choice like this allows to propagation to find a larger number of inconsistent values with a greater probability.

Minimum domain (dom) the variable to be chosen is the one with the minimum domain size.

Most constrained (deg) the variable to be chosen is the one with the greatest number of constraints.

dom and *deg* are often combined to take advantages from both. A variation of *deg* is the weighted degree heuristic, where weights are assigned to constraints, initially set to 1, then during propagation of a constraint, its weight is increased by 1 if the constraint fails. Minizinc has an annotation (`dom_w_deg`) which consists in choosing the variable with the following smallest value: domain size divided by the weighted degree, namely the number of times a variable caused a failure earlier in the search.

4.1.1 Heavy tail behaviour

Sometimes, particular instances of a problem can be difficult to solve and they take much more time than other instances. But this is not due to instances directly! Sometimes, the combination of instances and heuristics make solution

of those instances difficult to find, but this can be solved by changing heuristics. Sometimes, randomization can make problems easier to solve! Randomization can regard to parameters of heuristics in search but also the choice of variables and even values. Restarting the search can be sometimes a good idea as well, performing it under certain conditions, for instance when a certain amount of resources have been consumed. Any kind of depth first search for solving optimization problems suffers from the problem that wrong decisions made at the top of the search tree can take an exponential amount of search to undo. One common way to ameliorate this problem is to restart the search from the top thus having a chance to make different decisions. The usefulness comes when, at the restart, the search is performed differently. But how do we restart?

constant restart , restart after using L resources;

geometric restart , restart after using L resources, but setting a new limit at the next restart, so we restart after $a^n * L$, where n is the number of restarts.

Luby restart , restart after using $s[i] * L$, where $s[i]$ is the i-th number of the Luby sequence which is a sequence of powers of 2, but before adding a new power in the sequence, the previous subsequence is repeated in the sequence, so:

$$[1]$$

$$[1, 1, 2]$$

$$[1, 1, 2, 1, 1, 2, 4]$$

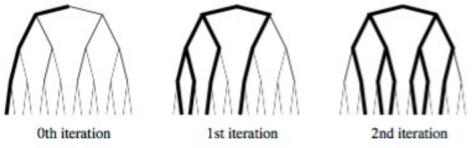
$$[1, 1, 2, 1, 1, 2, 4, 1, 1, 2, 1, 1, 2, 4, 8]$$

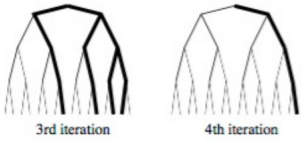
$$...$$

This is the most used.

4.1.2 Limited discrepancy search

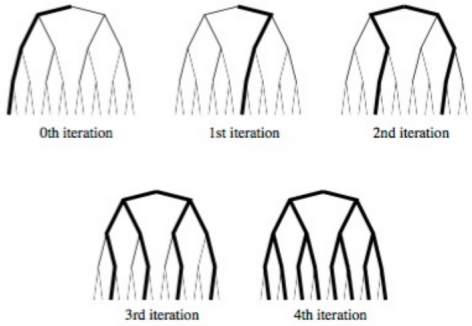
A *discrepancy* is any decision in a search tree that does not follow the heuristic. The point of LDS is that it starts with left depth-first search at the first loop, then, if the solution is not found, the search is repeated by going at the first right branch of each variable encountered at the previous loop.





4.1.3 Depth-bounded discrepancy search

Since it is more convenient to perform discrepancy at the highest levels of the tree instead of the deepest ones, the discrepancy performing is done a bit different from LDS: at the i-th iteration, discrepancy branches are visited only if they are at the i-th level, otherwise the original heuristic is followed.



The problem with LDS is that it acts on the deepest branches, but often it is useful to make different decisions at the highest levels because the search is less informative and the bad decisions are likely to be taken tehre. DDS is better than LDS from this point of view.

4.2 Optimization problems

An optimization problem is a 4-tuple $\langle X, D, C, f \rangle$ where X, D and C are the decision variables, domains of variables and constraints respectively while f is an optimization criterion in the form of an objective function. For example we can set the goal of a problem as the $minf$, so minimizing the value of f (remember that f is a variable, not a function). Another way to set the goal can be:

- defining $f = max(X)$;
- defining the goal as the $minf$.

In this way, we are searching the minimum number of f value such that it is useful to solve a certain problem.

Destructive lower bound A possible technique to solve minimization (or maximization or stuff like those) can be searching "through the domain of f", namely try assigning values to f at each iteration starting from the minimum of $D(f)$ until we reach a solution. At each iteration, the value is increased. Is is named "destructive" because intermediate results are discarded. The solution is proved to be optimal. Disadvantages for this algorithm are: if resources of search are limited, then it's not granted that it finds a solution; it performs small steps (just one). An advantage is that it makes constraints tighter at each iteration, so propagation is helped. Another advantage is that it provides lower bounds.

Destructive upper bound It is the same mechanism of DLB, but it starts with $maxD(f)$. However, when we find a solution, it isn't proved that it is the optimal one, so we continue to decrease values and search solutions. When we arrive at the case where it fails to find the solution, then the previous iteration of the current case is proven to be the optimal solution. The pros and cons of this algorithm are exactly complementary to DLB.

Binary search The main idea is to consider upper bound and lower bound in an array (the solutions domain). At each iteration, these bounds are restricted and come closer in this way:

- we set $lb < f < (lb + ub/2)$, where lb is lower bound and ub is upper bound;
- we try to solve the problem for that value of f;
- if the problem is feasible, then we update ub, else we update lb;
- we repeat these procedures until we reach $f = lb + 1$, which is the optimal solution.

It takes advantages from both DLB and DUB. But it's not the best way, because all information is discarded at each iteration, so we are performing duplicated work.

Branch and bound It uses a single search tree, incorporating bounds in the search. At each iteration, when a solution is found, a new bound constraint is set to ensure the future solutions are better, then it backtracks and looks for a new solution. This is repeated until the problem is infeasible, at that point, the last solution is demonstrated to be optimal.

5 Scheduling

It concerns with ordering resources and/or tasks over time and it has a lot of applications. We have:

- a set of resources with fixed capacities;
- a set of tasks with durations and resource requirements;
- a set of temporal constraints, for example we may want task 1 to run before or after task 2;
- a performance metric, namely something to measure the goodness (usually the objective variable).

In this kind of problem, usually, we have to decide when tasks have to start. The decision variables correspond to the operations/activities to be performed. An activity a_i has: a starting time s_i, a duration d_i and an ending time e_i. We will refer to the earliest start time as EST_i, to the latest start time as LST_i, to the earliest end time as EET_i and to latest end time LET_i. Note that EST_i, EET_i, LST_i and LET_i (the latter called deadline) correspond to the possible earliest or latest times (not necessarily the real ones). There are two types of activity:

- preemptive activities, which can be interrupted any time, and it holds $s_i + d_i \leq e_i$
- non-preemtpive activities, which cannot be interrupted by external agents and it holds $s_i + d_i = e_i$.

A resource is something available and limited for tasks to run. We have:

cumulative/parallel resources which allow multiple activities to run at the same time, for example, they can be a multi-core CPU. Formally, a resource r_k is associated to a capacity c_k. Each activity a_i has requirements $rq_{ik} \geq 0$ on resource r_k. Clearly, it must hold $rq_{ik} \leq c_k$. We have cumulative constraint which handles the resource requirements for activities;

unary/disjunctive/sequential resources which allow activities to execute only one at a time regardless the resource capacity. The disjunctive constraint handles this type of resources.

5.0.1 Temporal constraints

Temporal constraints can model the fact some actvities must come before or after other activities. But they can be finer: they can define *time-legs*, namely they bound the difference between the end time and the start time of two activities; they can define also *time-windows*, namely pretending one or more activities runs in a certain time window.

5.0.2 Cost function

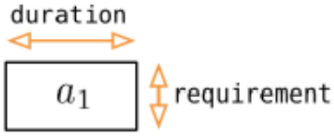
We define *makespan* as the completion time of the last activity. The RCPSP (cumulative resources) and job shop (disjunctive resources) scheduling cost functions are makespan and the objective is usually to minimize the makespan. A makespan can be modeled in different ways, for example introducing a dummy activity with duration 0 and that must run as the last activity or taking the maximum of the starting times of activities plus their durations (obtaining the ending times).

5.1 RCPSP

Take a look at a sample for RCPSP:

- a project graph $\langle A, E \rangle$, where A is the set of activities and E are their ending time;
- a set R of resource r_k with a capacity c_k;
- each activity a_i has a duration d_i and a resource requirement r_{ik}

We can see activities as:



5.1.1 Search heuristics

What are best heuristics for RCPSP? We must pose the following questions: which variable to pick next? Which value to assign? Let's focus on the value selection. Since the makespan has to be minimized, a good choice can be using the minimum value (the EST_i). As far as variables choice is concerned, a good choice can be selecting the one with the minimum LET_i. These are just heuristics, so the optimal solution is not granted to be found immediately, sometimes backtracking is necessary. Generally, since the objetive is to minimize the makespan, increasing an S_i cannot improve the makespan (we say these problems have *regular cost metrics*), so a good choice is the EST_i and this is true for many scheduling problems. But this is just a greedy solution for a fast first solution, indeed, in order to reach optimality we have to do backtracking. How to backtrack? Given a non-optimal solution, we select an activity a_i and instead of choosing the value for that variable, we postpone it and we visit a subtree. Doing this procedure will grant optimality, but it's often very expensive in terms of time. This technique regards both value and variable choices.

And what about variable selection? Precedence constraints help us because they make variables domains shrink. A good greedy choice can be selecting the task with the minimum LET_i, so the one with the first deadline in order to discard it immediately and to have the lowest probability for it to get a blind spot. To reach optimality, refer to the technique explained before.

6 Heuristic search

We have two types of methods: complete methods, which guarantee the optimal solution to each finite-size instance of a problem in a finite time; approximate methods, which don't guarantee neither optimality, nor termination in case of infeasibility, but they can find better solutions than complete methods in a lower amount of time. We have three types approximate methods: constructive heuristics, local search, metaheuristics.

6.1 Constructive heuristics

This is the fastest approximation and it consists in constructing the solution from scratch by repeatedly extending the current partial assignment until a solution is found or the stopping conditions are satisfied. An example can be the greedy heuristics, but they don't guarantee optimality, but even though this, they are quick and they are used as initialization step for other methods.

6.2 Local search

It gives often better solutions than constructive heuristics in terms of optimality. It starts with an initial solution and it iteratively tries to replace the current solution a with a better "neighbourhood" (a near solution), by applying small changes. Now we have to define what neighbourhood is, let's see combinatorial optimization from a different perspective: given the problem $\langle X, D, C, f \rangle$, S the set of all solutions, we have to find $s^* \in S$ such that $\forall s \in S.\ f(s^*) \leq f(s)$. We define a function $N : S \mapsto \wp(S)$ that assigns to every $s \in S$ a set of neighbours $N(s) \subseteq S$. $N(s)$ is called the *neighbourhood* of s. It is often implicitly defined

by defining the modifications to s to reach $N(s)$. We now define a *locally minimal solution* with respect to a neighbourhood N as a solution s' such that $\forall s \in N(s').\ f(s') \leq f(s)$. A structure of local search algorithm is done like this:

- Generate a solution s;
- while $\exists s' \in N(s).\ f(s') \leq f(s)$, assign to s an improving neighbor.

This algorithm stops when it finds a local minimum. There are different choices of the neighbor: the first one or the best one. The first solution can be generated randomly or heuristically. But how to define neighbourhood structure? It comes useful the notion *K-exchange neighbourhood*, where K is the number of modifications to do on the graph. In the problem of travelling salesman problem, we can have that a starting solution is a hamiltonian tour of the graph, then if we choose 2-exchange, we have to switch two archs, if we choose 3-exchange, we have to switch three archs, etc.. Clearly, the more K grows, the more neighbourhoods grows exponentially. Generally, a key issue is how to define the neighbourhood. Too small neighbourhood is fast to find, but the quality of local minima is low, while too large neighbourhood improves the quality of local minima, but they are expensive to calculate.

6.3 Metaheuristics

They consist in higher level strategies which "guide" heuristics to find a solution. They can be seen, in simply terms, as heuristics on heuristics. We have two types of policies: *intensification* which exploits previous experience; *diversification* which explores the search space in the large. A good balance between them is at the base of effectiveness of metaheuristics.

6.3.1 Local search methods

It is similar to local search, but differently to it, this method tries to escape to local minimum and it does it by allowing worsening solutions, by changing neighbourhood structure during search or by changing objective function during search. Clearly, we have to interrupt the algorithm at a certain point since it cannot stop; some criteria are setting a maximum CPU time, number of iterations, etc.. Here some methods.

Simulated annealing Like local search, but it accepts worsening moves with a certain probability. The probability decreases at each iteration. It favours intensification over diversification.

Variable neighbourhood search It changes neighbourhood structure during search, more precisely this happens whenever a new local optima is reached.

Tabu search It keeps track of a "tabu list" of solutions or moves and it forbids them. So it doesn't change the neighbourhood, but it restricts the neighbors which it can move to. Anyway, storing solutions is very inefficient, moves are cheaper even if that could eliminate good solutions which haven't been visited yet. The tabu list size is another important issue: the more it's big, the more diversification grows against intensification. In general, the size can be increased

in case of repetitions (so when diversification is needed) or decreased when there are no improvements (so intensification is needed).

Guided local search It changes the objective function. But one could argue it changes the semantics of the claimed solution! The idea is to penalize some solution characteristics which occur frequently, so the result is that some solutions get worse than how they are really.

6.3.2 Population-based methods

We work on a set of solutions and the basic principle is to find common characteristics among solutions. We have a probabilistic model from which we obtain sample solutions to analyse. After this analysis, we update our model and we restart the process. This process is similar to what ants do when they come across an obstacle: the ants will try to find a path to escape from the obstacle, then they will start to take the shortest path because it will be the one which is walked by the majority of ants and the ants drop pheromones when they walk and follow paths where the concentration of pheromones is higher. From this phenomenon, we have the so-called *pheromone model*: we have a set of *pheromone values* which act as the memory with the goal to focus the search on certain paths. A way to make this concept formal can be: a *pheromone value* is a value $t(X_i, v_i)$, $\forall X_i \in X$ and $v_i \in D(X_i)$ which represents the desirability of assigning v_i to X_i. A common usage is to have bounding for pheromone values t_{min} and t_{max} and to set initially values to t_{max}. As a consequence, we have that decreasing values augment diversification because, we will have different values from what we have had before (so it's like ants forget older solutions to get close to newer solutions), while increasing will help intensification.

In order to begin, we use "artificial ants". They simply consist in using constructive heuristics for an initial solution. So the algorithm has the following schema:

- start from an initial solution, using constructive heuristics;
- iteratively choose a variable X_i according to some heuristic (this is parametric) and then choose the value $v_i \in D(X_i)$ according to the probability:

$$p(x_i, v_i) = \frac{[\tau(x_i, v_i)]^\alpha \cdot [\eta(x_i, v_i)]^\beta}{\sum_{v_j \in D(x_i)} [\tau(x_i, v_j)]^\alpha \cdot [\eta(x_i, v_j)]^\beta}$$

Heuristic factor

  where α and β are parameters used to balance the pheromone and heuristic factors.

Pheromone values in the i-th round are the ones which comes from $(i-1)$-th round solutions. Then, at each round, pheromone values are updated. If they are decreased, diversification is boosted, else if they are increased, then intensification is boosted. The update is done in order to improve the quality of future solutions.

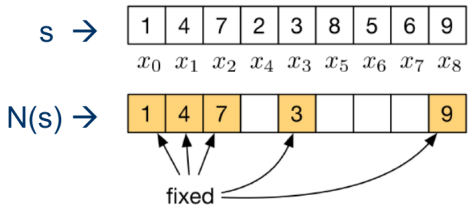
At the end, which one to use? Local search metaheuristics or population-based metaheuristics? It depends from the problem. If in the problem, moving from neighbor to neighbor is easy and cheap in terms of computational cost,

then local search-based algorithms is good. On the other hand, if neighbourhood is difficult to build, the cost of moves is not so cheap and solutions can be built as composition of blocks, then population-based methods is a good idea. Generally, local-based methods boost intensification because we act on subsets of all solutions, while population-based methdos have the ability to boost diversification (always according to how the pheromones are updated). Hybrid methods tries to get the best from both methods types.

Metaheuristics is effective in finding first good-quality solutions, but it struggles with complex constraints. For this reason, mixing metaheuristics and complete methods in different ways can improve the quality of solutions in a time unit, since complete methods, on the other hand, can be bad and inefficient if we have loose bounds of objective function. For example, a complete method can apply a metaheuristic to improve a solution or, on the contrary, metaheuristics use a complete methods to efficiently to explore neighbourhood.

Large neighbourhood search And local search? How can it be combined with complete methods? Remember the issue about small and large neighbourhoods; from this point, we can use large neighbourhood and explore it with a complete method. We can see exploration of neighbourhood as the solution of a sub-problem. So, given a solution s, we have to possible moves to apply to each variable:

- fix part of the variables value;
- relax the remaining variables.



The good fact about this neighbourhood structure is that it works with all problems. Now, two issues are: which percentage of variables to fix? Which variables to fix? There are many ways to follow and it can depend from the problem.

Generally, after wee saw all these different methods, the important thing is to know how to mix these methods; to fix an optimization problem, the general good way is to use a hybrid method.