Serie 8, Aufgabe 1

a) vektorielle Form dito Aufgabe 4, Serie 7

$$z' = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ \sin x + 5 - 1.1 z_4 + 0.1 z_3 + 0.3 z_1 \end{pmatrix} \quad z^{(0)} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad h = 0.1$$

$$x_0 = 0$$

Euler:

$$z^{(1)} = z^{(0)} + h \cdot f(x_0, z^{(0)})$$

$$z^{(1)} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + 0.1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ \underbrace{\underbrace{\sin 0}_{0} + 5 - 0 + 0 + 0}_{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0 \\ 0 \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 \\ 2 \\ 0 \\ 0.5 \end{pmatrix}$$

Runge-Kutta:

$$k_1 = f(x_0, z^{(0)}) = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

$$k_2 = f\left(x_0 + \frac{h}{2}, z^{(0)} + \frac{h}{2} \cdot k_1\right) \Rightarrow \quad h/2 \cdot k_1 = 0.05 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ 5 \end{pmatrix} = \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \begin{pmatrix} 0.1 \\ 0 \\ 0 \\ 0.25 \end{pmatrix}$$

$$h/2 = 0.05$$

$$k_2 = \begin{pmatrix} 2 + 0 \\ 0 + 0 \\ 0 + 0.25 \\ \sin(0.05) + 5 - 1.1 \cdot 0.25 + 0.1 \cdot 0 + 0.3 \cdot 0.1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0.25 \\ 0.04998 + 5 - 0.275 + 0.03 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0.25 \\ 4.80498 \end{pmatrix}$$

$$k_3 = f\left(x_0 + \frac{h}{2},\ \cancel{y_0} z_0 + \frac{h}{2}\cdot k_2\right)$$

$$h/_2 \cdot k_2 = 0.05 \cdot \begin{pmatrix} 2 \\ 0 \\ 0.25 \\ 4.80498 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0 \\ 0.0125 \\ 0.240249 \end{pmatrix}$$

$$k_3 = \begin{pmatrix} 2+0 \\ 0+0.0125 \\ 0+0.240249 \\ \sin(0.05)+5-1.1\cdot 0.240249+0.1\cdot 0.0125+0.3\cdot 0.1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ 0.0125 \\ 0.240249 \\ 4.816955 \end{pmatrix}$$

$$k_4 = f\left(x_0 + \frac{h}{2},\ z_0 + h\cdot k_3\right)$$

$$h \cdot k_3 = \overset{0.1}{\cancel{0.05}} \cdot \begin{pmatrix} 2 \\ 0.0125 \\ 0.240249 \\ 4.816955 \end{pmatrix} \cancel{= \begin{pmatrix} 0.1 \\ 0.000625 \\ 0.012012 \\ 0.2408478 \end{pmatrix}} = \begin{pmatrix} 0.2 \\ 0.00125 \\ 0.024025 \\ 0.4816 \end{pmatrix}$$

$$k_4 = \begin{pmatrix} 2+0.00125 \\ 0+0.024025 \\ 0+0.481695 \\ \sin(0.05)+5-1.1\cdot 0.481695+0.1\cdot 0.24025+0.3\cdot 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 2.00125 \\ 0.024025 \\ 0.481695 \\ 4.60414 \end{pmatrix}$$

$$z_1 = \overset{(z_0)}{y_0} + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \frac{0.1}{6} \cdot$$

$$\left(\begin{pmatrix} 2 \\ 0 \\ 0 \\ 5 \end{pmatrix} + 2\cdot \begin{pmatrix} 2 \\ 0 \\ 0.25 \\ 4.80498 \end{pmatrix} + 2\cdot \begin{pmatrix} 2 \\ 0.0125 \\ 0.240249 \\ 4.816955 \end{pmatrix} + \begin{pmatrix} 2.00125 \\ 0.024025 \\ 0.481695 \\ 4.60414 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \frac{0.1}{6} \cdot \begin{pmatrix} 12.00125 \\ 0.049025 \\ 1.462193 \\ 28.84801 \end{pmatrix} = \begin{pmatrix} 0.20002 \\ 2.000817 \\ 0.02437 \\ 0.4808 \end{pmatrix}$$

b) vektorielle Form dito Aufgabe 4, Serie 7

$$z' = \begin{pmatrix} z_2 \\ -\frac{1}{x} \cdot z_2 - \left(\frac{x^2 - n^2}{x^2}\right) \cdot z_1 \end{pmatrix} \quad z^{(0)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad x_0 = 1 \quad h = 0.1$$

$$n^2 = 1$$

Euler:

$$z^{(1)} = z^{(0)} + h \cdot f(x_0, z_0)$$

$$z^{(1)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0.1 \begin{pmatrix} 2 \\ \underbrace{-\frac{1}{1} \cdot 2}_{-2} - \underbrace{\frac{\cancel{4}-1}{\cancel{4}} \cdot 2}_{-0} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot 0.1 \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + 0.2 \\ 2 - 0.2 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 1.8 \end{pmatrix}$$

Runge-Kutta

$$k_1 = h \cdot f(x_0, z^{(0)}) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$k_2 = f\left(x_0 + \frac{h}{2}, z^{(0)} + \frac{h}{2} \cdot k_1\right) \qquad \frac{h}{2} = 0.05$$

$$\frac{h}{2} \cdot k_1 = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix} \quad \Rightarrow \quad z^{(0)} + \frac{h}{2} \cdot k_1 = \begin{pmatrix} 2.1 \\ 1.9 \end{pmatrix}$$

$$k_2 = \begin{pmatrix} 2 - 0.1 \\ -\frac{1}{1.05} \cdot 1.9 - \left(\frac{1.05^2 - 1}{1.05^2}\right) \cdot 2.1 \end{pmatrix} = \begin{pmatrix} 1.9 \\ -2.00476 \end{pmatrix}$$

$$k_3 = f\left(x_0 + \frac{h}{2}, z^{(0)} + \frac{h}{2} \cdot k_2\right)$$

$$\frac{h}{2} \cdot k_2 = 0.05 \begin{pmatrix} 1.9 \\ -2.00476 \end{pmatrix} = \begin{pmatrix} 0.095 \\ -0.100238 \end{pmatrix}$$

$$\Rightarrow \quad z^{(0)} + \frac{h}{2} \cdot k_2 = \begin{array}{c} z_1 \\ z_1 \end{array} \begin{pmatrix} 2.095 \\ +1.89976 \end{pmatrix}$$

$$k_3 = \begin{pmatrix} 2 - 0.100238 \\ -\frac{1}{1.05} \cdot 1.89976 - \overbrace{\left(\frac{1.05^2 - 1}{1.05^2}\right)}^{0.092970522} \cdot 2.095 \end{pmatrix} =$$

$$= \begin{pmatrix} 1.89976 \\ -2.004068 \end{pmatrix}$$

$$k_4 = f(x_0 + h, z^{(0)} + h \cdot k_3)$$

$$h \cdot k_3 = 0.1 \cdot \begin{pmatrix} 1.89976 \\ -2.004068 \end{pmatrix} = \begin{pmatrix} 0.189976 \\ -0.200407 \end{pmatrix}$$

$$z^{(0)} + h \cdot k_3 = \begin{pmatrix} 2.189976 \\ 1.799593 \end{pmatrix}$$

$$k_4 = \begin{pmatrix} 1.799593 \\ -\frac{1}{1.1} \cdot 1.799593 - \left(\frac{1.1^2 - 1}{1.1^2}\right) \cdot 2.189976 \end{pmatrix}$$

$$= \begin{pmatrix} 1.799593 \\ -2.016072 \end{pmatrix}$$

$$z_1 = z_0 + h \cdot \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) =$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{0.1}{6} \cdot \left(\begin{pmatrix} 2 \\ -2 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1.9 \\ -2.00476 \end{pmatrix} + 2 \begin{pmatrix} 1.89976 \\ -2.004068 \end{pmatrix} + \begin{pmatrix} 1.799593 \\ -2.016072 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{0.1}{6} \begin{pmatrix} 11.399113 \\ -12.033728 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.18998 \\ -0.20056 \end{pmatrix}$$

$$= \begin{pmatrix} 2.18998 \\ 1.79944 \end{pmatrix}$$