Homework 1

- Assigned: Thursday, 28 August 2025
- Due: Thursday, 4 September 2025
- Instructor: Bo Guo, University of Arizona
- Semester: Fall 2025

1. For the linear vector space $P_n[\alpha, \beta]$, derive a criterion by which the set of vectors

$\{p_i(x)\}_{i=1}^{n+1}$, with

$$p_i(x) = \alpha_{i0} + \alpha_{i1}x + \alpha_{i2}x^2 + \cdots + \alpha_{in}x^n,$$

can be tested for linear independence.

2. Lagrange polynomials are often used to interpolate data. Consider $n+1$ data points $\{U_i\}_{i=1}^{n+1}$, measured at discrete locations $\{x_i\}_{i=1}^{n+1}$. An n-th degree interpolation polynomial can then be defined by

$$U(x) = \sum_{i=1}^{n+1} U_i \, \ell_{ni}(x)$$

Show that $U(x)$ exactly matches the measurements at each node point. An alternative interpolation procedure is to use piecewise Lagrange polynomials. These are standard Lagrange polynomials that are of degree less than n and are defined over only "pieces" of the domain. For example, to "connect the data points by straight lines," piecewise linear Lagrange polynomials are used as follows:

$$U(x) = \sum_{i=1}^{n+1} U_i \, \phi_i(x)$$

where $\phi_i(x)$ is a piecewise linear Lagrange polynomial defined as

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \le x \le x_{i+1} \\ \\ 0, & \text{all other } x \end{cases}$$

This interpolation retains the property that $U(x_k) = U_k$, but differs from the previous one in that $U(x)$ is only $C^0[x_1, x_{n+1}]$ rather than $C^\infty[x_1, x_{n+1}]$. Furthermore, the value of $U(x)$ at any $x_i \le x \le x_{i+1}$ is dependent only on the values U_i and U_{i+1} in the piecewise case, while in the previous case the dependence is on all of the $U_j,\ j = 1, 2, \ldots, n+1$. For the data points below, compute $U(x)$ using both of the interpolation methods above, and sketch the results.

i	x_i	U_i
1	0	1
2	2	1
3	3	2
4	4	0

3. Prove that the expression

$\langle f(x), g(x)\rangle = \int_\alpha^\beta w(x) f(x) g(x)\, dx, \quad w(x) > 0$

where $w(x), f(x), g(x) \in C^0[\alpha, \beta]$ is a valid inner product for the space $C^0[\alpha, \beta]$.

4. Determine whether or not the differential operator

$$\mathcal{L} = \frac{d^2}{dx^2} + (x^2)\frac{d}{dx}$$

is linear. State an appropriate domain and range for this operator.

5. Show that the expression

$$\|u(x)\| = \int_\alpha^\beta \left\{ [u(x)]^2 + \left[\frac{du}{dx}\right]^2 \right\}^{1/2} dx$$

is a valid norm for $u(x) \in C^1[\alpha, \beta]$. Is this a valid norm for $u(x) \in C^0[\alpha, \beta]$?

6. (5 points) Watch this video on YouTube (https://youtu.be/ly4S0oi3Yz8), which provides a very nice introduction to partial differential equations using the Heat Equation as an example. The purpose of this video and the problem is to help you understand (or review in case you took partial

differential equation classes before) what our primary target in this course (i.e., partial differential equations) is.

- Summarize the key points and insights. Use bullet points instead of paragraphs.
- If you take the 1-D advection-diffusion equation presented on slide 12 of Lecture 1 and assume the fluid velocity V=0 and the source/sink term R=0, you obtain a so-called diffusion equation that describes the transport of a contaminant driven by only molecular diffusion. How is this diffusion equation related to the heat equation introduced in the video? Compare the variables and coefficients of the two equations directly.