HWRS 561b Homework Answer Keys #1

- Assigned: Sunday, 25 January 2026
- Due: Thursday, 5 February 2026 (upload answers in PDF or Jupyter Notebook to D2L)
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- Semester: Spring 2026

1. (15 points) Short-answer conceptual questions.

   - (5 points) In the bundle-of-tubes model, we discussed in class that permeability scales linearly with porosity. (a) Show why permeability scales linearly with porosity. (b) Based on the bundle-of-tubes model and this insight, explain why porosity alone is insufficient to characterize permeability. For example, is it possible that two soil media with the same porosity have different permeabilities? Similarly, is it possible that a soil medium with greater porosity has smaller permeability than a soil medium with smaller porosity? Explain why or why not.

   - (5 points) Mechanical dispersion produces spreading of solute plumes at the Darcy scale. Explain why this spreading should not be interpreted as true mixing at the pore scale.

   - (5 points) The advection–diffusion equation is written at the Darcy scale. (a) Which pore-scale processes are being averaged out to obtain this equation? (b) Give one example where this upscaling might fail or produce misleading results.

Answers:

1(a) In the bundle-of-tubes model, take N identical straight tubes of radius r and length L in a REV of total cross-sectional area A. Porosity is $\phi = \frac{N\pi r^2}{A}$. Poiseuille flow in each tube gives flux per tube $q = r^4(\Delta P)/(8\mu L)$. Summing over tubes, the Darcy flux is

$$q_D = \frac{N\pi r^4}{8\mu L}\frac{\Delta P}{A} = \frac{r^2}{8\mu}\phi\frac{\Delta P}{L}$$

Comparing with Darcy's law

$$q_D = -\frac{k}{\mu}\frac{\Delta P}{L}$$

gives $k \propto \phi r^2$, so for fixed tube size (or fixed characteristic pore size) k scales linearly with ϕ.

1(b) Porosity alone does not fix the pore size, connectivity, or tortuosity, all of which strongly affect permeability. Two soils can have the same porosity but different pore sizes or connectivity, producing different k. A soil with larger porosity can even have smaller k if its pores are finer, more tortuous, or poorly connected compared with a lower-porosity soil with larger, well-connected pores. Thus k depends on both ϕ and pore geometry.

2 Mechanical dispersion reflects variability in pore-scale velocities and pathways (fast/slow channels, velocity gradients). It spreads a plume at the Darcy scale by rearranging concentrations across streamlines, but it does not by itself homogenize concentration within pores. True mixing at the pore scale requires molecular diffusion (or possibly local eddying under higher Reynolds number flow).

3(a) The advection–diffusion (or advection–dispersion) equation averages out pore-scale velocity variations, pore geometry, and local diffusion/mixing processes. It replaces the detailed pore-scale transport with averaged (Darcy-scale) advection and an effective diffusion/dispersion term.

3(b) Upscaling can fail when the REV is not representative or when transport is strongly non-Fickian, e.g., in highly heterogeneous media with preferential flow and long tailing, fractured rock where matrix diffusion dominates, or when sharp concentration fronts at small scales persist so that local equilibrium assumptions break down.

2. (20 points) Consider a semi-infinite horizontal soil column undergoing steady-state, saturated horizontal flow. The porosity of the soil is 0.3. The Darcy flux is 0.3 m/day. Now consider a solute transport problem. Initially, the domain is solute-free, so $C(x, t = 0) = 0$. From $t = 0$ to $t = 1$ day, solute with concentration C_0 is injected into the domain from the left boundary at $x = 0$ cm.

   - (10 points) Assuming that the water velocity and solute concentration at the REV scale are always uniform at any cross-section of the soil column, i.e., both flow and solute transport can be considered 1D, derive the governing equation that describes solute transport using the mass conservation and flux laws presented in class. Use the same methodology that we used in class, but do it for this specific 1D problem, i.e., do not derive or use the general 3D equation from class and simplify to 1D. Your control volume is an arbitrary 1D interval of the soil column, and your final derived governing equation should not contain any integrals. Be sure to define all parameters in your equation.
   - (5 points) In your derived equation, explain the physical meaning (i.e., what physical mechanisms they represent) and dimensions of the following parameters: dispersivity (α_L), tortuosity factor (ω), and molecular diffusion coefficient (D_d).
   - (5 points) Sketch the solutions of solute concentration $c(x, t)$ at three times, $t = 2, 5, 10$ days. You do not need to solve the equation to draw the solutions—just sketch what you conceptually think the solution would be. Explain why the shape of the concentration profile changes with time.

Answers:

(a) Let $c(x, t)$ be aqueous concentration (mass solute per volume water). For a 1D control volume $[x, x + \Delta x]$, mass conservation gives

$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} \phi c \, \mathrm{dx} = -(\mathrm{J}_{x+x} - \mathrm{J}_x),$$

where the 1D total solute flux (mass per area per time) is $J = qc - \phi D_L \frac{\partial c}{\partial x}$. Here q is Darcy flux and the longitudinal hydrodynamic dispersion coefficient is $D_L = \alpha_L v + \omega D_0$, $v = q/\phi$.

Based on the fundamental theorem of calculus (i.e., 1D equivalence of the divergence theorem)

$$J_{x+\Delta x} - J_x = \int_x^{x+\Delta x} \frac{\partial J}{\partial x}.$$

Combining the above two equations gives

$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} \phi c \, \mathrm{dx} = -\int_x^{x+x} \frac{J}{x}.$$

Because the 1D control volume is not time-dependent and it is arbitrary, the integrands on the two sides have to be equal. We obtain the 1D advection–dispersion equation

$$\frac{\partial(\phi c)}{\partial t} + \frac{\partial}{\partial x}\left(qc - \phi D_L \frac{\partial c}{\partial x}\right) = 0.$$

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$$\phi \frac{\partial c}{\partial t} + q \frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(\phi D_L \frac{\partial c}{\partial x}\right).$$

If ϕ and D_L are constant, this simplifies to

$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = D_L \frac{\partial^2 c}{\partial x^2}.$$

Parameters: ϕ is porosity; q is Darcy flux; v is porewater velocity; α_L is longitudinal dispersivity; ω is tortuosity factor; D_0 is molecular diffusion coefficient in free water; D_L is longitudinal dispersion coefficient.

(b)

- **Dispersivity** α_L: measures spreading due to mechanical dispersion from velocity variations; dimension L (e.g., m).

- **Tortuosity** ω: accounts for longer, more tortuous diffusion paths in porous media; dimensionless.

- **Molecular diffusion coefficient** $D_d = \omega D_0$: diffusion in the porous medium; dimension L^2/T.

(c) At $t = 2, 5, 10$ days, the concentration profile is a pulse that moves rightward with the average porewater velocity and spreads over time. It has a leading edge and trailing tail; the peak decreases and the plume widens as t increases because dispersion (mechanical + molecular) spreads solute while advection transports it downstream. After injection stops at $t = 1$ day, the pulse detaches from the boundary and continues to advect and disperse, producing a longer tail at later times.

3. (15 points) Mechanical dispersion and Taylor–Aris dispersion.

   - (5 points) Explain, in your own words, the key insights from the analysis of Taylor–Aris dispersion.
   - (5 points) Use the insights from Taylor–Aris dispersion to conceptually defend the advection–dispersion model that has been widely used to describe solute transport processes in groundwater (while acknowledging the assumptions).
   - (5 points) If molecular diffusion were zero, would Taylor–Aris dispersion occur? Why or why not? Explain.

Answers:

(1) Taylor–Aris dispersion shows that in laminar flow through a tube, velocity varies across the cross-section. Solute in fast and slow streamlines departs from each other, which over time produces an effective longitudinal spreading much larger than molecular diffusion alone. The result is an enhanced, Fickian-like dispersion at long times with an effective coefficient that depends on the velocity profile and diffusion.

(2) These insights justify using a Darcy-scale advection–dispersion model: small-scale velocity variations plus molecular diffusion lead to an emergent, approximately Fickian spreading at larger scales. Thus, it is reasonable to represent unresolved pore-scale heterogeneity with an effective dispersive term, while recognizing assumptions (steady flow, sufficient transverse mixing, scale separation, etc.).

(3) Without molecular diffusion were zero, solute in fast and slow streamlines still departs from each other and Taylor–Aris dispersion would still occur.

4. (25 points) A clay liner of thickness $L = 2\,\text{m}$ separates a landfill from the underlying aquifer. The liner is fully saturated and has an effective diffusion coefficient given by

$$D_d = \omega D_0,$$

where $D_0 = 2.03 \times 10^{-9}\,\text{m}^2/\text{s}$ is the molecular diffusion coefficient in free water and $\omega = 0.5$ is a tortuosity factor.

At time $t = 0$, the landfill side of the liner is suddenly exposed to landfill leachate with a constant chloride concentration of

$$C_0 = 2315\,\text{mg/L}.$$

Initially, the chloride concentration everywhere in the liner is zero. Ignore advection and reactions.

- (5 points) Clearly state the governing equation and boundary conditions describing chloride transport in the liner.
- (5 points) Explain physically why diffusion, rather than advection, is the dominant transport mechanism in this system.
- (5 points) Estimate the characteristic diffusion time scale across the liner and use it to assess whether 50 years is a "short" or "long" time relative to that.
- (5 points) Using an appropriate analytical solution, estimate the chloride concentration at the downgradient side of the liner after 50 years.
- (5 points) If the clay liner were twice as thick but had the same material properties, how would the breakthrough time scale change? Explain without recalculating.

Answers:

(a) Governing equation (1D diffusion in a slab, no advection/reaction): $\frac{\partial C}{\partial t} = D_d \frac{\partial^2 C}{\partial x^2}$, $0 < x < L$. Initial condition: $C(x,0) = 0$. Boundary condition at the landfill side: $C(0,t) = C_0$ for $t > 0$. At the aquifer side, assume the aquifer is large and initially clean so the liner can be treated as diffusion into a semi-infinite medium; then the far-field condition is $C(x \to \infty, t) = 0$, and we evaluate the solution at $x = L$.

(b) The liner is clay with very low hydraulic conductivity, so advective flux is negligible. Solute transport is therefore dominated by molecular diffusion through the water-filled pores, represented by $D_d = \omega D_0$.

(c) $D_d = 0.5 \times 2.03 \times 10^{-9} = 1.015 \times 10^{-9}\,\text{m}^2/\text{s}$. Diffusion time scale $t_c \sim L^2/D_d = (2^2)/(1.015 \times 10^{-9}) \approx 3.94 \times 10^9\,\text{s} \approx 125\,\text{years}$. Thus 50 years is shorter than the characteristic diffusion time (early to intermediate time).

(d) For constant concentration at $x = 0$ and semi-infinite medium, the solution is

$$\frac{C(x,t)}{C_0} = \text{erfc}\left(\frac{x}{2\sqrt{D_d t}}\right).$$

We use the semi-infinite domain solution as an approximation. At $x = L = 2\,\text{m}$ and $t = 50\,\text{yr} = 1.58 \times 10^9\,\text{s}$, $\frac{L}{2\sqrt{D_d t}} = \frac{2}{2\sqrt{(1.015 \times 10^{-9})(1.58 \times 10^9)}} \approx 0.79$. erfc$(0.79) \approx 0.26$, so $C(L, 50\,\text{yr}) \approx 0.26\,C_0 \approx 6.0 \times 10^2\,\text{mg/L}$.

(e) Diffusion time scale grows with L^2. Doubling thickness to $2L$ increases breakthrough time by a factor of 4.

5. (25 points) A canal leaks water into an underlying saturated aquifer. At time $t = 0$, the canal water becomes contaminated with a conservative, nonreactive organic compound (1,4-dioxane) at a constant concentration of

$$C_0 = 1.245\,\text{mg/L}.$$

Groundwater flows horizontally away from the canal with a constant average porewater velocity of

$$v = 0.544\,\text{m/day}.$$

Assume one-dimensional transport away from the canal, neglect transverse spreading, and ignore dispersion and diffusion.

- (5 points) Write the governing transport equation for this problem and clearly state all assumptions.
- (5 points) Describe in words how the concentration signal propagates through the aquifer under these assumptions.
- (5 points) Calculate the travel time for groundwater to move 25 m away from the canal.
- (5 points) Based on your result from above, determine the concentration of 1,4-dioxane at a location 25 m downgradient from the canal after 45 days.
- (5 points) If longitudinal dispersion were included, how would the concentration at 25 m after 45 days change qualitatively? Discuss arrival time, peak concentration, and tailing.

Answers:

(a) Governing equation (1D advection only, no dispersion/diffusion/reaction): $\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial x} = 0$, $x > 0, t > 0$, with boundary condition $C(0, t) = C_0$ for $t > 0$ and initial condition $C(x, 0) = 0$.

Assumptions: 1D flow, constant v, conservative solute, no dispersion/diffusion, no sorption or reactions, constant boundary concentration.

(b) The concentration signal propagates as a sharp front moving at speed v with no spreading. Upstream of the front, $C = C_0$; downstream, $C = 0$.

(c) Travel time to $x = 25\,\text{m}$: $t = x/v = 25/0.544 \approx 45.96\,\text{days}$.

(d) After 45 days, the front has not quite reached 25 m (needs ~45.96 days). So $C(25\,\text{m}, 45\,\text{days}) \approx 0$ (just before breakthrough). Shortly after 46 days it jumps to C_0.

(e) With longitudinal dispersion, the front would smear: earlier arrival of a small concentration (breakthrough before 46 days), the peak concentration will still be C_0, but it takes time to gradually approach toward C_0.