HWRS 561b Homework Answer Keys #3

- Assigned: Sunday, 22 February 2026
- Due: Sunday, 8 March 2026 (upload answers in PDF or Jupyter Notebook to D2L)
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- Semester: Spring 2026

1. (10 points) Propose two questions that you think can be used as midterm oral exam questions. The questions should focus on some of the most critical concepts that we have covered so far in this course. Keep in mind that these are oral exam questions, so they should not involve detailed calculations; rather, they should focus on testing whether one can draw or develop conceptual pictures and formulate physical arguments. Make sure to describe your questions concisely and accurately. Ask one of your classmates to answer your questions. Then, summarize her/his answers below. Please also write your own comments on the answers. You should also put down her/his name.

Note: No answer is provided for this problem.

2. (20 points) Consider a confined aquifer into which an LNAPL has been spilled. The LNAPL accumulates below the upper aquitard. The aquifer and aquitard materials are water-wet with contact angle $\theta = 0°$. A representative pore radius for the aquitard is 0.05 mm. A representative pore diameter for the confined aquifer is 1 mm. The hydraulic head of water in the aquitard (above the LNAPL) is identical to the hydraulic head of water in the aquifer (below the LNAPL), and water is hydrostatic everywhere. The densities of water and LNAPL are 1.0 and 0.975 g cm^{-3}, respectively. The interfacial tension between LNAPL and water is 0.02 N m^{-1}.

   - (a) (10 points) Draw a capillary-tube representation of this system. (Hint: approximate both aquifer and aquitard as bundles of tubes; for simplification, one tube for the aquifer and one tube for the aquitard is sufficient.)
   - (b) (10 points) Write a general expression (in terms of r_{aquitard}, r_{aquifer}, ρ_{water}, ρ_{LNAPL}, $\sigma_{\text{water-LNAPL}}$, and h_{LNAPL}) for the maximum LNAPL thickness that can exist below the aquitard before LNAPL first enters the aquitard. Then use the values above to compute that thickness.

Answers:
2(a) Capillary-tube representation (conceptual)
Treat the system as two vertical water-wet tubes:

- upper smaller tube for the aquitard: radius r_{aquitard},
- lower bigger tube for the aquifer: radius r_{aquifer}.

LNAPL forms a column of thickness h_{LNAPL} below the aquitard. Because both media are water-wet ($\theta = 0°$), LNAPL is the nonwetting phase and must overcome capillary entry pressure to invade the aquitard.

2(b) Maximum LNAPL thickness before entry into aquitard
At upper LNAPL-water interface at the entry point into the aquitard,

$$p_{n,T} - p_{w,T} = \frac{2\sigma_{\text{water-LNAPL}} \cos\theta}{r_{\text{aquitard}}},$$

while at the lower water-LNAPL interface in the aquifer,

$$p_{n,B} - p_{w,B} = \frac{2\sigma_{\text{water-LNAPL}} \cos\theta}{r_{\text{aquifer}}}.$$

Hydrostatic differences over the LNAPL column give

$$p_{n,B} - p_{n,T} = \rho_{\text{LNAPL}} g h_{\text{LNAPL}}, \qquad p_{w,B} - p_{w,T} = \rho_{\text{water}} g h_{\text{LNAPL}}.$$

Subtracting yields

$$(\rho_{\text{water}} - \rho_{\text{LNAPL}}) g h_{\text{LNAPL,max}} = 2\sigma_{\text{water-LNAPL}} \cos\theta \left(\frac{1}{r_{\text{aquitard}}} - \frac{1}{r_{\text{aquifer}}} \right).$$

Therefore,

$$h_{\text{LNAPL,max}} = \frac{2\sigma_{\text{water-LNAPL}} \cos\theta}{(\rho_{\text{water}} - \rho_{\text{LNAPL}}) g} \left(\frac{1}{r_{\text{aquitard}}} - \frac{1}{r_{\text{aquifer}}} \right).$$

Use

$$r_{\text{aquitard}} = 0.05\,\text{mm} = 5 \times 10^{-5}\,\text{m}, \quad r_{\text{aquifer}} = 0.5\,\text{mm} = 5 \times 10^{-4}\,\text{m},$$

$$\rho_{\text{water}} = 1000\,\text{kg/m}^3, \quad \rho_{\text{LNAPL}} = 975\,\text{kg/m}^3, \quad \sigma_{\text{water-LNAPL}} = 0.02\,\text{N/m}, \quad \cos\theta = 1.$$

Then

$$h_{\text{LNAPL,max}} \approx 2.94\,\text{m}.$$

3. (20 points) Use the bundle of cylindrical capillary tubes to complete the following two problems.

   - (a) (10 points) The relative permeability function is almost always nonlinear. Examples are the Brooks-Corey and van Genuchten models. Explain why the relative permeability function is nonlinear. Additionally, explain how the relative permeability functions for wetting and nonwetting fluid phases are different and why.
   - (b) (10 points) Explain the algorithmic steps to derive a relative permeability curve if an SWC is measured for a soil. You are not allowed to use any existing functional forms for the relative permeability.

Answers:

3(a) Why k_r is nonlinear and phase-dependent

Relative permeability is nonlinear because effective permeability depends on which pores are occupied and connected, not just on bulk saturation. In a pore bundle, intrinsic permeability contribution from a tube scales with r^2. Therefore, draining/filling a small fraction of large pores can change phase permeability disproportionately.

Wetting vs nonwetting curves differ because the wetting phase tend to remain in small pores, while the nonwetting phase mainly occupies larger pores and often needs an entry threshold before forming connected pathways. Thus, at the same saturation, wetting-phase k_{rw} and nonwetting-phase k_{rn} are generally different.

3(b) Algorithm to derive k_r from measured SWC

1. Given the measure SWC, convert each p_c to an equivalent pore-throat radius via Young-Laplace:

$$r_c = \frac{2\sigma\cos\theta}{p_c}.$$

2. Infer cumulative pore-size occupancy from $S_w(p_c)$, i.e., the fraction of pores still filled by wetting phase at each capillary pressure.
3. Discretize pore radii into bins and assign occupancy for each phase at each saturation state.
4. Compute phase permeability from occupied pore classes (via Poiseuille weighting):

$$k_w^*(S_w) \propto \sum_{i \in \text{wetting-filled}} r_i^4, \qquad k_n^*(S_w) \propto \sum_{i \in \text{nonwetting-filled}} r_i^4.$$

5. Normalize by each phase's saturated reference to obtain relative permeabilities:

$$k_{rw} = \frac{k_w^*(S_w)}{k_{w,\text{sat}}^*}, \qquad k_{rn} = \frac{k_n^*(S_w)}{k_{n,\text{sat}}^*}.$$

6. Repeat across the saturation range to generate full $k_{rw}(S_w)$ and $k_{rn}(S_w)$ curves.

4. (20 points) Unsaturated flow and relative permeability.

   In 1D vertical flow (positive z upward) for a homogeneous medium, use extended Darcy's law for the wetting phase:

$$q_w = -K_{sat} k_{rw}(S_e) \left(\frac{dh}{dz} + 1 \right).$$

   Use the Brooks-Corey-Mualem type relation

$$k_{rw} = S_e^{(2+3\lambda)/\lambda}.$$

   Parameters: $\theta_r = 0.08$, $\theta_s = 0.38$, $\lambda = 0.50$, $K_{sat} = 1.2 \times 10^{-5}$ m/s, and $dh/dz = 0$.

   - (a) (4 points) In this 1D vertical-flow setting, identify the driving force for water flow under the condition above.
   - (b) (8 points) Plot k_{rw} and q_w as functions of water saturation, $S_w = \theta/\theta_s$, over $S_w \in [\theta_r/\theta_s, 1]$. Clearly label axes and units.
   - (c) (8 points) Based on your plots from (b), explain why direct measurement of relative permeability curves is difficult.

Answers:
4(a) With $dh/dz = 0$,

$$q_w = -K_{sat} k_{rw}(S_e)(0 + 1) = -K_{sat} k_{rw}(S_e).$$

So the only driving force is gravity (unit gravitational gradient). Because z is positive upward, negative q_w indicates downward water flow.

4(b) First convert from S_w to effective saturation:

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \frac{\theta_s S_w - \theta_r}{\theta_s - \theta_r}, \qquad S_w \in \left[\frac{\theta_r}{\theta_s}, 1 \right].$$

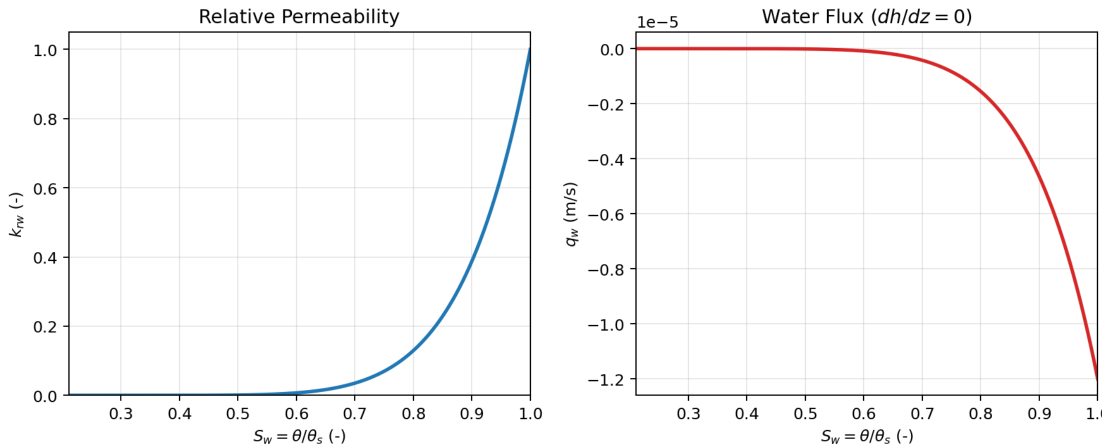
Given $\lambda = 0.50$,

$$\frac{2 + 3\lambda}{\lambda} = \frac{2 + 1.5}{0.5} = 7,$$

so

$$k_{rw} = S_e^7, \qquad q_w = -1.2 \times 10^{-5} S_e^7 \text{ m/s}.$$

Generated curves:



4(c) Direct measurement is difficult because k_{rw} is extremely small over much of the unsaturated range (here $k_{rw} = S_e^7$), so measured fluxes can be near instrument detection limits, and the duration for conducting the experiments will be impractical. There are other reasons, but this represents the primary reason.

5. (30 points) Richards assumptions and Richards equation.

- (a) (6 points) Starting from the two-phase air-water equations, state the key assumptions used to derive the Richards equation for variably saturated water flow in the vadose zone.
- (b) (6 points) If the pressure in the air phase of an unsaturated soil remains essentially at atmospheric pressure, is it then reasonable to conclude that the air does not move? Why or why not? Does this type of air movement violate the Richards assumptions?
- (c) (6 points) When deriving the Richards equation, if air density cannot be considered negligible compared to water density, how would you modify the Richards equation or functions associated with it to account for that?
- (d) (6 points) Suppose you wish to write a mathematical model for water flow in the unsaturated zone that can also include zones of full saturation. Of the three equations (mixed, h-based, and θ-based), which is most appropriate? Which (if any) is not appropriate? Explain your answers.
- (e) (6 points) Give one field scenario where the Richards assumptions may fail (or become weak), and explain the expected modeling consequence.

Answers:

5(a) Key assumptions behind Richards equation

From two-phase air-water flow equation (assuming it is already valid), Richards equation is obtained by assuming:

- air pressure is spatially uniform (typically $p_a \approx p_{atm}$).
- air-phase density is neglibile.

5(b) Does near-atmospheric air pressure imply no air motion?

No. Air can still move even if pressure is close to atmospheric everywhere. Air viscosity is very small; tiny pressure gradient can generate substantial flow.

5(c) Modification when air density is not negligible

Use z positive upward. If the air phase is hydrostatic,

$$\frac{dp_a}{dz} = -\rho_a g, \qquad p_a(z) = p_{a,0} - \rho_a g z.$$

(With gauge reference $p_{a,0} = 0$ at $z = 0$, $p_a(z) = -\rho_a g z$.)

You do not need to change the form of the Richards equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\left(\frac{dh}{dz} + 1\right)\right].$$

The change is in the constitutive relation through matric potential (capillary-pressure head), defined as

$$\psi_m = \frac{p_a - p_w}{\rho_w g} = h_a - h,$$

where $h_a = p_a/(\rho_w g)$ and $h = p_w/(\rho_w g)$. Therefore, the soil water characteristic curve should be expressed as

$$\theta = \theta(\psi_m) = \theta(h_a - h),$$

instead of

$$\theta = \theta(\psi_m) = \theta(-h),$$

5(d) Best form for combined unsaturated + saturated zones

- Most appropriate: mixed form (pressure head + water content), because it is robust across unsaturated/saturated transitions and is mass-conservative.
- Also usable: h-based form, but it can be less robust numerically in very dry ranges and may show mass conservation issues.
- Not appropriate for this purpose: pure θ-based form, because in saturated zones $\theta = \theta_s$ is constant and cannot represent pressure gradients that drive saturated flow.

5(e) Example failure scenario and consequence

Example: intense infiltration into a layered soil where air becomes trapped above a low-permeability lens. Air pressure then becomes non-uniform and dynamically coupled to water flow.

Expected modeling consequence if Richards equation is used anyway: infiltration rate and wetting-front timing are biased (often overpredicted infiltration), and transient water saturation will be inaccurate.