HWRS 561b: Physical Hydrogeology II

Two-phase flow and Richards equation (Part 1)

Today:

1. Mathematical Representation of Two-Phase Flow
2. Unsaturated Flow and the Richards' Equation

Reading: Chapter 11 (Pinder & Celia, 2006).

Review

- ❖ Physical meaning of the SWC models (Brooks-Corey vs. van Genuchten)
- ❖ Specific yield and drainable porosity in the vadose zone
- ❖ Leverett J function/scaling and Miller-Miller scaling
- ❖ Unsaturated permeability (physical meaning) and mathematical description
- ❖ Extended Darcy's Law for unsaturated flow (Buckingham, 1907)

$$\boldsymbol{q}_w = -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w}\nabla(p_w + \rho_w g z)$$

- ❖ How to derive unsaturated permeability from SWC?

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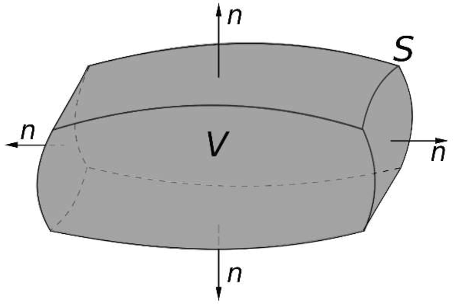
Steady-state saturated flow

Derive the flow equation:

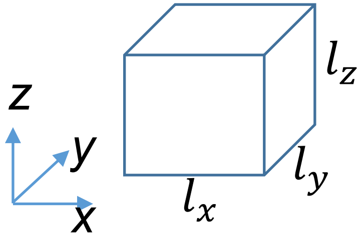
Recall: Divergence theorem

$$\int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{q}\,\mathrm{d}V$$



=> Divergence theorem converts a surface integral to a volume integral.

Mass conservation: Change of mass storage = mass in – mass out.



Rate of mass change: $\frac{d}{dt}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z} \rho(x,y,z,t)\,\phi(x,y,z,t)\,dx\,dy\,dz = \frac{d}{dt}\int_\Omega \rho\phi\,dV$

Net fluxes: $\int_0^{l_y}\int_0^{l_z} q_x\big|_{x=0}\,dy\,dz - \int_0^{l_y}\int_0^{l_z} q_x\big|_{x=l_x}\,dy\,dz$

$+ \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=0}\,dx\,dz - \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=l_y}\,dx\,dz$

$+ \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=0}\,dx\,dy - \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=l_z}\,dx\,dy$

Steady-state saturated flow

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$$\Rightarrow -\int_{\partial\Omega} \underset{\sim}{q}\cdot\underset{\sim}{n}\, ds = -\int_{\Omega} \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\, dv$$ divergence theorem

Thus, $$\frac{d}{dt}\int_{\Omega} \rho\phi\, dv = -\int_{\Omega} \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\, dv$$

$\Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{d}{dt}(\rho\phi)\, dv = -\int_{\Omega} \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\, dv$$

$$\Rightarrow \int_{\Omega} \left[\frac{d}{dt}(\rho\phi) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\right] dv = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\right] dv = 0$$

Integral = 0, Ω is arbitrary $\Rightarrow$ Integrand = 0

$$\left.\begin{aligned} &\Rightarrow \frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q} = 0 \\ &\text{Darcy's law: } \underset{\sim}{q} = -\rho\underset{\approx}{k}\underset{\sim}{\nabla}H \end{aligned}\right\} \Rightarrow \boxed{\frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla}\cdot\left(-\rho\underset{\approx}{k}\underset{\sim}{\nabla}H\right) = 0}$$

General 3D flow equation for a water-saturated porous medium.

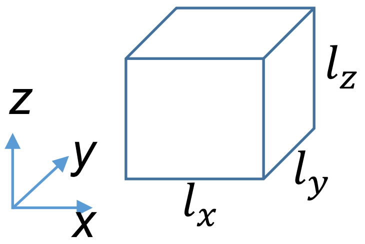
Two-Phase Flow

Flux Law for two-phase flow (Muskat and Meres, 1936)

$$\left.\begin{aligned}\boldsymbol{q}_w &= -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w}\boldsymbol{\nabla}(p_w + \rho_w g z)\\ \boldsymbol{q}_{nw} &= -\frac{k_{r,nw}(S_w)\mathbf{k}}{\mu_{nw}}\boldsymbol{\nabla}(p_{nw} + \rho_{nw} g z)\end{aligned}\right\}\quad \begin{aligned}\boldsymbol{q}_\alpha &= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha}\boldsymbol{\nabla}(p_\alpha + \rho_\alpha g z), \qquad \alpha = nw \text{ or } w\\ &= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha}(\boldsymbol{\nabla}p_\alpha - \rho_\alpha \boldsymbol{g})\end{aligned}$$

$$\boldsymbol{\nabla}(\rho_\alpha g z) = -\rho_\alpha \boldsymbol{g}$$

The derivation of the governing equations for two-phase flow shares the same procedure as that used for sing-phase flow. The only difference is that a governing equation is needed for each fluid phase.



Rate of mass change:

$$\frac{\partial}{\partial t}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z}\rho_\alpha(x,y,z,t)\,\phi(x,y,z,t)\,S_\alpha(x,y,z,t)\,dx\,dy\,dz = \frac{\partial}{\partial t}\int_\Omega \rho_\alpha \phi S_\alpha\, dv$$

net fluxes:

$$\left.\begin{aligned}&\int_0^{l_y}\int_0^{l_z}\rho_\alpha q_x|_{x=0}\, l_y l_z - \int_0^{l_y}\int_0^{l_z}\rho_\alpha q_x|_{x=l_x}\, l_y l_z\\ &\int_0^{l_x}\int_0^{l_z}\rho_\alpha q_y|_{y=0}\, l_x l_z - \int_0^{l_x}\int_0^{l_z}\rho_\alpha q_y|_{y=l_y}\, l_x l_z\\ &\int_0^{l_x}\int_0^{l_y}\rho_\alpha q_z|_{z=0}\, l_x l_y - \int_0^{l_x}\int_0^{l_y}\rho_\alpha q_z|_{z=l_z}\, l_x l_y\end{aligned}\right\} \Rightarrow -\int_{\partial\Omega}\rho_\alpha \underset{\sim}{q_\alpha}\cdot\underset{\sim}{n}\,ds$$

$$= -\int_\Omega \underset{\sim}{\nabla}\cdot(\rho_\alpha \underset{\sim}{q_\alpha})\,dv$$

Divergence theorem

Two-Phase Flow

$$\Rightarrow \frac{\partial}{\partial t}\int_{\Omega} \rho_\alpha \phi S_\alpha \, dV = -\int_{\Omega} \underset{\sim}{\nabla} \cdot (\rho_\alpha \underset{\sim}{q_\alpha}) \, dV$$

$\swarrow \Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{\partial}{\partial t}(\rho_\alpha \phi S_\alpha) \, dV = -\int_{\Omega} \underset{\sim}{\nabla} \cdot (\rho_\alpha \underset{\sim}{q_\alpha}) \, dV$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t}(\rho_\alpha \phi S_\alpha) + \underset{\sim}{\nabla} \cdot (\rho_\alpha \underset{\sim}{q_\alpha})\right] dV = 0$$

$\swarrow \Omega$ is arbitrary

$$\Rightarrow \frac{\partial}{\partial t}(\rho_\alpha \phi S_\alpha) + \underset{\sim}{\nabla} \cdot (\rho_\alpha \underset{\sim}{q_\alpha}) = 0$$

Flux law: $\underset{\sim}{q_\alpha} = -\frac{k_{r,\alpha} \underline{\underline{k}}}{\mu_\alpha} \underset{\sim}{\nabla}(P_\alpha + \rho_\alpha g z)$

$\Rightarrow$

$$\boxed{\frac{\partial}{\partial t}(\rho_\alpha \phi S_\alpha) - \underset{\sim}{\nabla} \cdot \left[\rho_\alpha \frac{k_{r,\alpha} \underline{\underline{k}}}{\mu_\alpha} \underset{\sim}{\nabla}(P_\alpha + \rho_\alpha g z)\right] = 0}$$

General governing equation for two-phase flow

Two-Phase Flow

The governing equations for the two fluid phases:

Wetting phase
$$\frac{\partial}{\partial t}(\rho_w \phi S_w) - \boldsymbol{\nabla} \cdot \left[\rho_w \frac{k_{r,w}\mathbf{k}}{\mu_w} \boldsymbol{\nabla}(p_w + \rho_w g z)\right] = 0 \quad (1)$$

Nonwetting phase
$$\frac{\partial}{\partial t}(\rho_{nw} \phi S_{nw}) - \boldsymbol{\nabla} \cdot \left[\rho_{nw} \frac{k_{r,nw}\mathbf{k}}{\mu_{nw}} \boldsymbol{\nabla}(p_{nw} + \rho_{nw} g z)\right] = 0 \quad (2)$$

Unknowns: $p_w, p_{nw}, S_w, S_{nw}, k_{r,w}, k_{r,nw}$ (Assuming $\rho_\alpha, \mu_\alpha, \phi, \mathbf{k}$ are known.)

=> 6 unknowns, 2 equations. We need 4 more equations to have a mathematically closed system.

$$S_w + S_{nw} = 1 \quad (3)$$

Constitutive equations (e.g., VG or B-C models)
$$\begin{cases} p_{nw} = p_w + p_c(S_w) & (4) \\ k_{r,w} = k_{r,w}(S_w) & (5) \\ k_{r,nw} = k_{r,nw}(S_w) & (6) \end{cases}$$

- 6 equations, 6 unknowns. The system of equations is mathematically closed.
- Substituting Eqs. (3-6) to Eqs. (1-2) => 2 equations and 2 unknowns.
- Several options for the primary variables. Two common options:
  - ✓ One phase pressure + one phase saturation (e.g., p_w, S_{nw})
  - ✓ Two phase pressures (e.g., p_w, p_{nw})

Unsaturated Flow: Richards' Equation

Let's now think about the air-water system in the vadose zone.

At a common condition: 1 atm, 20 °C

$$\left.\begin{array}{l}\rho_w \approx 1000\ \mathrm{kg/m^3} \\ \rho_a \approx 1\ \mathrm{kg/m^3}\end{array}\right\} \rho_a \ll \rho_w \qquad \left.\begin{array}{l}\mu_w \approx 10^{-3}\ \mathrm{Pa\cdot s} \\ \mu_a \approx 1.8\times 10^{-5}\ \mathrm{Pa\cdot s}\end{array}\right\} \mu_a \ll \mu_w$$

Richards' assumptions

Air flux: $\boldsymbol{q}_a = -\dfrac{k_{r,a}\mathbf{k}}{\mu_a}\boldsymbol{\nabla}(p_a + \rho_a g z)$

$\uparrow$ $\mu_a \ll \mu_w$. Air is essentially inviscid. In that sense, almost no gradient of energy potential is required to drive air flow.

$$\Rightarrow \underset{\sim}{\nabla}(P_a + \rho_a g z) \simeq 0 \Rightarrow P_a + \rho_a g z \simeq \text{const}$$

At the ground surface ($z=0$), $P_a = 0$

$P_a(z) = 0 - \rho_a g z$

$\rho_a \simeq 1\ \mathrm{kg/m^2}$

$\Rightarrow P_a \simeq 0$ in the unsaturated zone.

$\rho_a\, g\, z \ll \text{atm}$

$1 \times 10 \times 100$

$\mathrm{kg/m^3}\ \ \mathrm{m/s^2}\ \ \mathrm{m}$

Richards' Equation

Governing equations for air-water flow in porous media:

$p_a = 0$. Thus, no need to solve equation (1)

$$\frac{\partial}{\partial t}(\rho_a \phi s_a) - \nabla \cdot \left[\frac{\rho_a k_{r,a} \mathbf{k}}{\mu_a} \nabla(p_a + \rho_a g z) = 0\right] \quad (1)$$

$p_w(s_w)$

$$\boxed{\frac{\partial}{\partial t}(\rho_w \phi s_w) - \nabla \cdot \left[\frac{\rho_w k_{r,w} \mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z) = 0\right]} \quad (2)$$

now, this is the only eqn we need to solve. It's essentially the Richards' eqn, but in a more general form.

$s_a + s_w = 1$ (3)

$k_{r,a} = k_{r,a}(s_w)$ (4)

$k_{r,w} = k_{r,w}(s_w)$ (5)

$p_a - p_w = p_c(s_w)$ (6)

If we simply by assuming constant density and porosity:

$$\boxed{\phi \frac{\partial s_w}{\partial t} - \nabla \cdot \left[\frac{k_{rw} \mathbf{k}}{\mu_w} \nabla (p_w + \rho_w g z)\right] = 0}$$

Richards' Eqn.