

HWRS 561b: Physical Hydrogeology II

Porous medium models and macroscopic description (Part 2)

Agenda:

1. Soil water characteristics (SWC)
2. Leverett J function; Miller scaling
3. Unsaturated permeability and relative permeability

- ❖ Bundle of cylindrical tubes model
- ❖ Advanced porous medium models
 - Bundle of triangular tubes model
 - Pore-network model
 - Physical porous medium models: micro-channels
 - Direct imaging of soil and rock pore structures
 - Extract pore-network from digital soil/rock images
- ❖ SWC: measurement, physical meaning, mathematical description

Mathematical Description of SWC

Brooks-Corey (1964)

$$p_c = p_d s_e^{-1/\lambda}$$

$$s_e = (s_w - s_{w,r}) / (1 - s_{w,r})$$

p_d is entry pressure

λ is a parameter related to pore size distribution

Notes:

(1) “ λ ” was used in B-C’s original paper. Ty’s notes used “ L ” for “ λ ”.

$$(2) s_e = (p_c/p_d)^{-\lambda}$$

(3) Non-differentiable at $p_c = p_d$

van Genuchten (1980)

$$p_c = \frac{1}{\alpha} (s_e^{-1/m} - 1)^{1/n}$$

α is a parameter related (NOT equal) to $1/p_d$.

n is a parameter related to pore size distribution

$m = 1 - 1/n$ based on the Mualem assumption

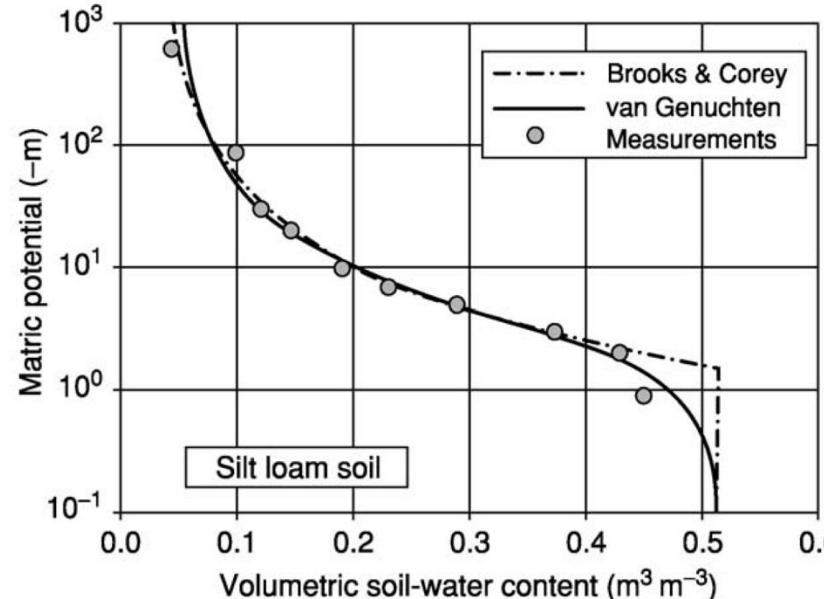


Table 1 Typical van Genuchten model parameters (α , n) including residual (θ_r) and saturated (θ_s) water contents compiled from the UNSODA database

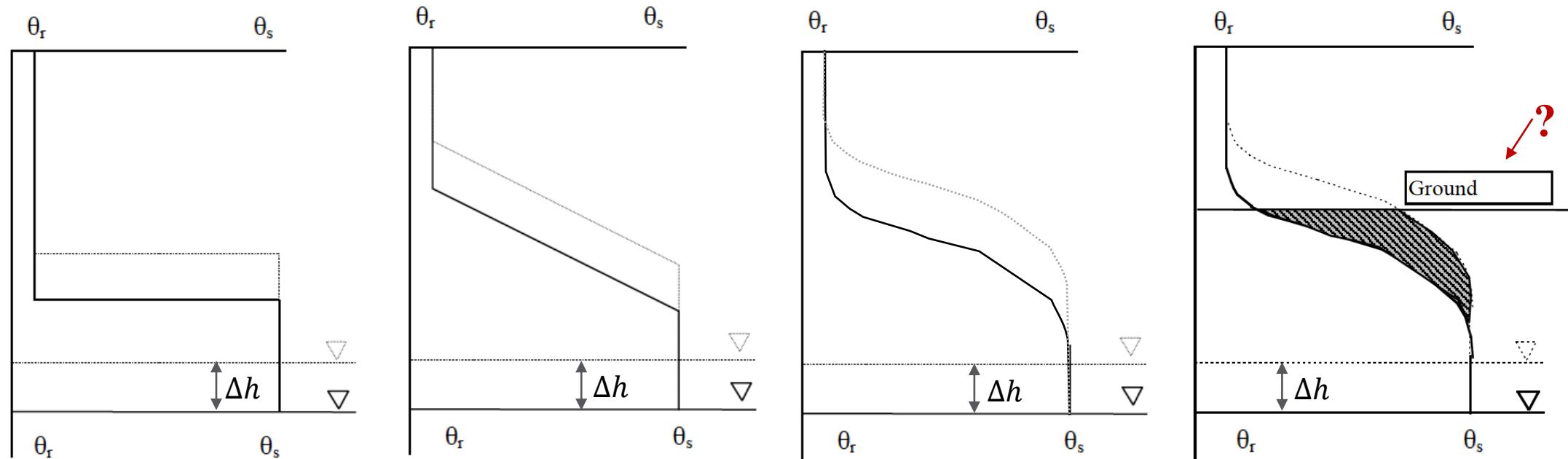
Textural class	N	θ_r ($cm^3 cm^{-3}$)	θ_s ($cm^3 cm^{-3}$)	α (cm^{-1})	n
Sand	126	0.058	0.37	0.035	3.19
Loamy sand	51	0.074	0.39	0.035	2.39
Sandy loam	78	0.067	0.37	0.021	1.61
Loam	61	0.083	0.46	0.025	1.31
Silt	3	0.123	0.48	0.006	1.53
Silt loam	101	0.061	0.43	0.012	1.39
Sandy clay loam	37	0.086	0.40	0.033	1.49
Clay loam	23	0.129	0.47	0.030	1.37
Silty clay loam	20	0.098	0.55	0.027	1.41
Silty clay	12	0.163	0.47	0.023	1.39
Clay	25	0.102	0.51	0.021	1.20

N, the number of soils or samples of a given textural class from which the mean values are compiled.

Reproduced from Leij FJ, Alves WJ, van Genuchten MT, and Williams JR (1996) *The UNSODA Unsaturated Hydraulic Database*. EPA/600/R-96/095. Cincinnati, OH: US Environmental Protection Agency.

Tuller & Or (2004)

Specific Yield and Drainable Porosity



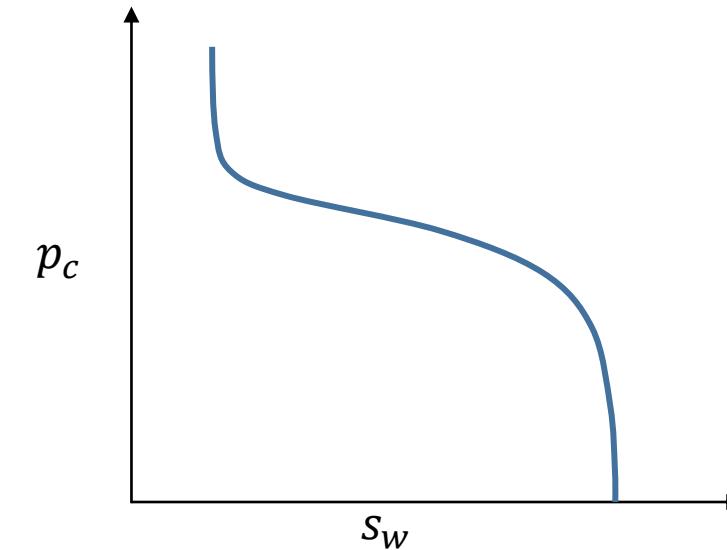
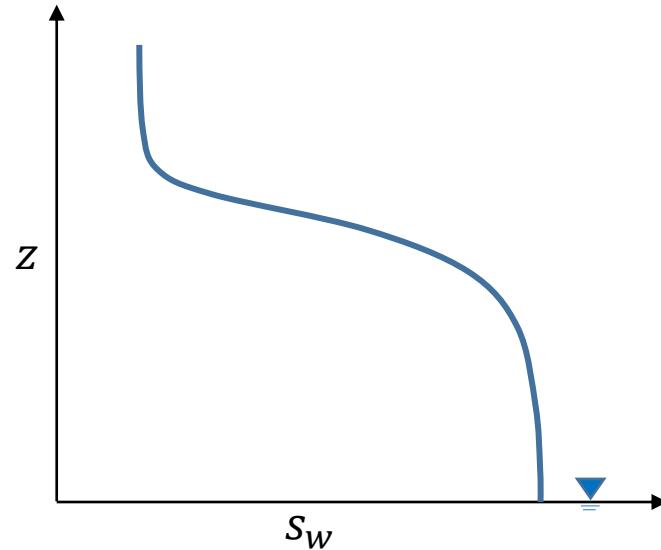
Volume of water release:

$(\theta_s - \theta_r)\Delta h$	$(\theta_s - \theta_r)\Delta h$	$(\theta_s - \theta_r)\Delta h$	$< (\theta_s - \theta_r)\Delta h$
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- Volume of water release is independent of water content distribution if the capillary transition zone is below the ground surface, if θ_s and θ_r remain unchanged.
- How about the time scale for the water release?

Hydrostatic Water Saturation Distribution

Water saturation distribution under hydrostatic conditions

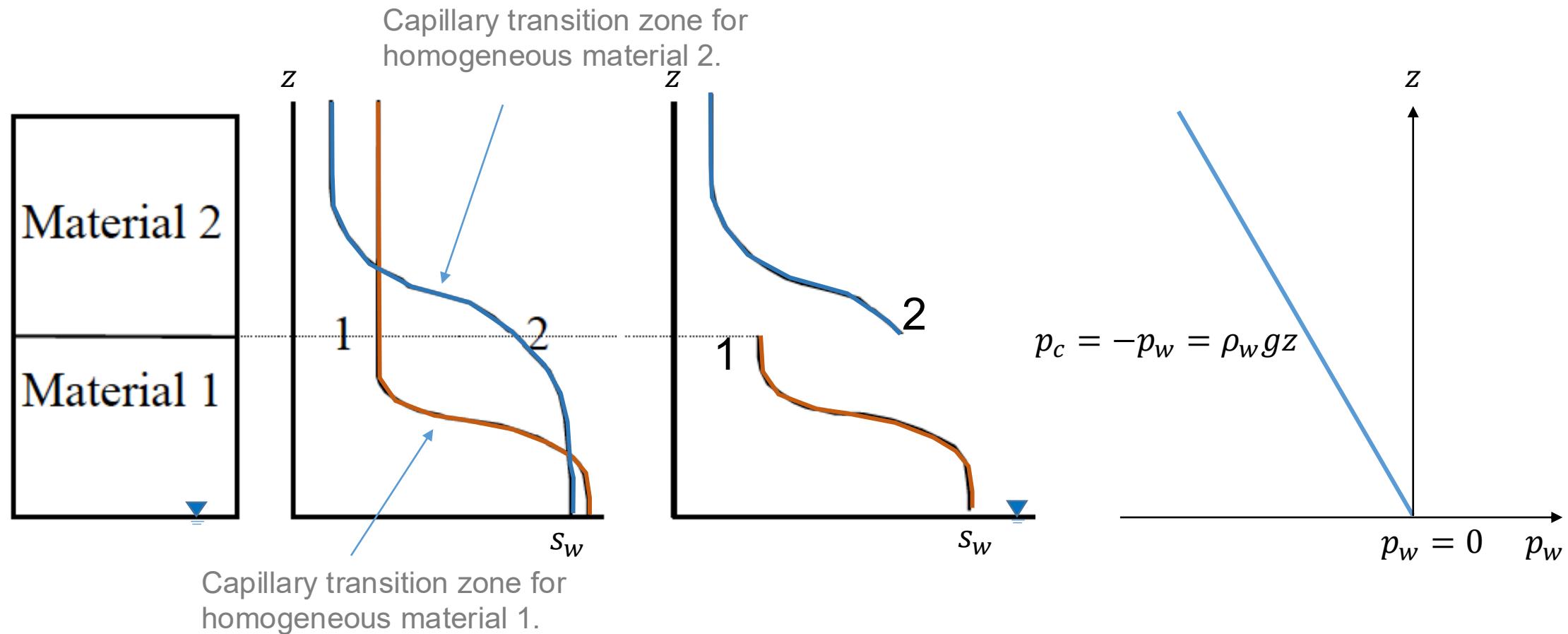


$$\left. \begin{aligned} \text{Hydrostatic} \Rightarrow p_w(z) &= 0 - \rho_w g z \\ p_w &= 0 - p_c = -p_c(S_w) \end{aligned} \right\} \Rightarrow p_c(S_w) = \rho_w g z \Rightarrow p_c(S_w) \sim z$$

- $z(S_w)$ and $p_c(S_w)$ have the same shape. They can be rescaled (“stretched” or “compressed”) along the vertical axis to overlap with each other.

Hydrostatic Water Saturation Distribution

Hydrostatic unsaturated layered systems



- S_w is discontinuous at the material interface, but the p_c and p_w or ψ_w are continuous.

Leverett J function/scaling

Capillary Behavior in Porous Solids

By M. C. LEVERETT,* MEMBER A.I.M.E.

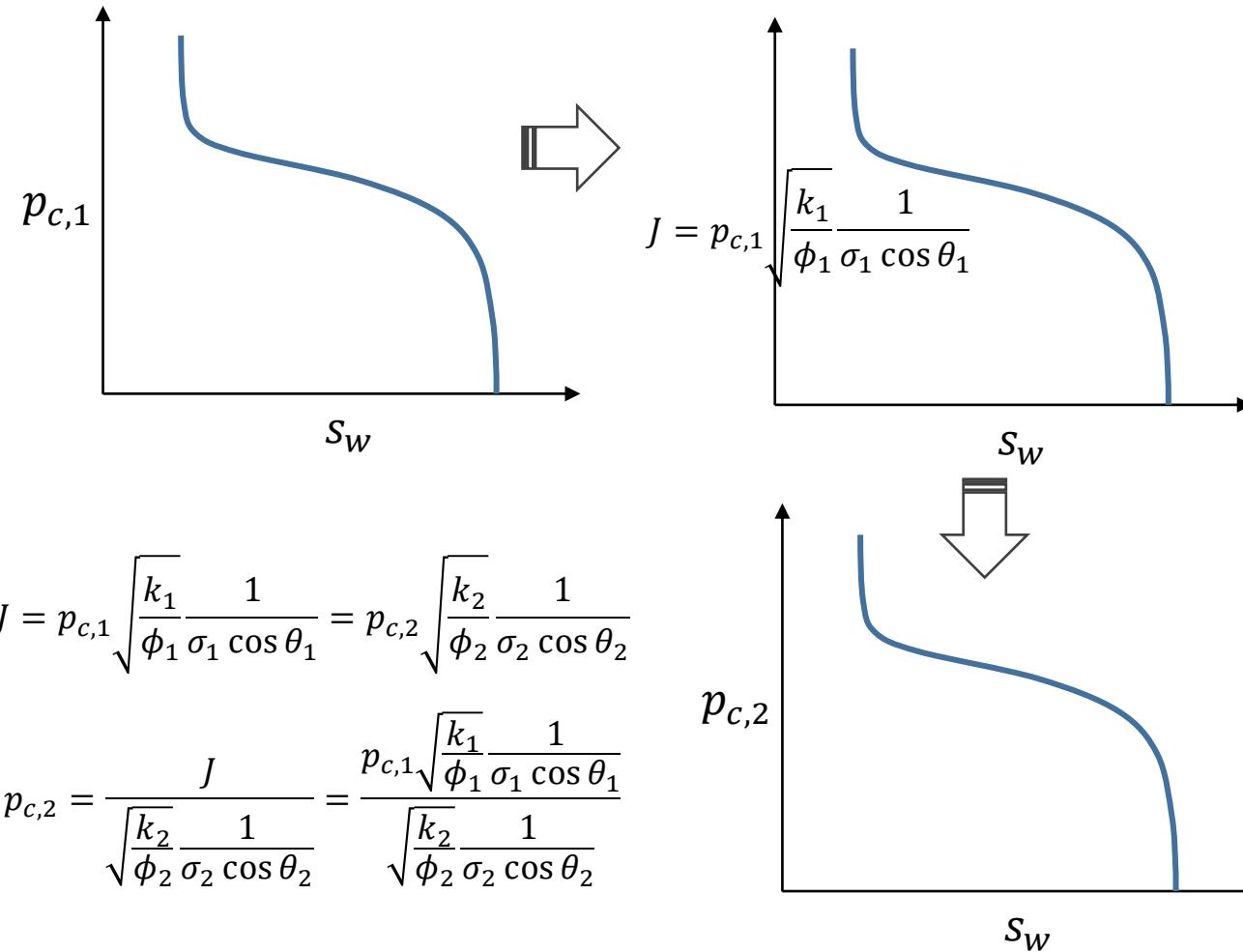
(Tulsa Meeting, October 1940)

Manuscript received at the office of the Institute June 14, 1940. Issued as T.P. 1223 in PETROLEUM TECHNOLOGY, August 1940.

* Humble Oil and Refining Co., Houston, Texas.

$$\left. \begin{array}{l} p_c \sim \frac{\sigma \cos \theta}{r} \\ k \sim \phi r^2 \end{array} \right\}$$

$$\Rightarrow p_c \sim \frac{\sigma \cos \theta}{\sqrt{k}} \Rightarrow p_c = \sqrt{\frac{\phi}{k}} \sigma \cos \theta \quad J(s_w)$$



- Leverett J function allows one to derive the SWC for a porous medium using the SWC from a similar porous medium + some basic hydraulic information (permeability and porosity)

Miller Scaling

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Physical Theory for Capillary Flow Phenomena

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(Received August 8, 1955; revised version received December 1, 1955)

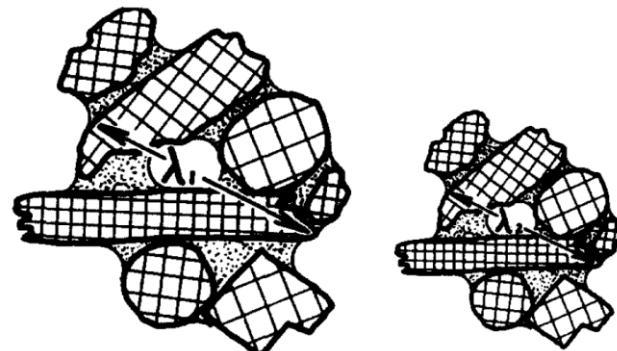


FIG. 2. Illustration of two "similar media" in "similar states." Note that the two characteristic lengths, λ_1 and λ_2 , connect corresponding points in the two media.

Dimensionless capillary pressure
(exact same idea as Leverett J scaling)

$$p_{c,0} = \frac{\lambda p_c}{\sigma} \Rightarrow \frac{\lambda_1 p_{c,1}}{\sigma_1} = \frac{\lambda_2 p_{c,2}}{\sigma_2} \Rightarrow p_{c,2} = \frac{\lambda_1 \sigma_2}{\lambda_2 \sigma_1} p_{c,1}$$

$$K_0 = \frac{\mu K}{\rho g \lambda^2} \Rightarrow \frac{\mu_1 K_1}{\rho_1 g \lambda_1^2} = \frac{\mu_2 K_2}{\rho_2 g \lambda_2^2} \Rightarrow K_2 = \frac{\rho_2 \lambda_2^2 \mu_1}{\rho_1 \lambda_1^2 \mu_2} K_1$$

Dimensionless conductivity

- Miller scaling allows one to derive SWC and conductivity for a porous medium using SWC and conductivity from a similar porous medium + some basic pore size information

Unsaturated Permeability

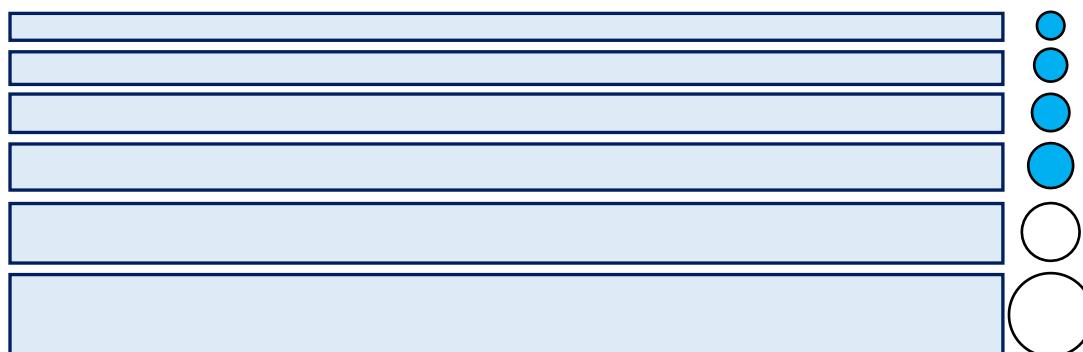
Single-phase Darcy's Law

$$\begin{aligned} q_w &= -\frac{\mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z) \\ &= -\mathbf{K}_{sat} \nabla H \end{aligned}$$

$$\begin{aligned} H &= \frac{p_w}{\rho_w g} + z \\ \mathbf{K}_{sat} &= \frac{\mathbf{k} \rho_w g}{\mu_w} \end{aligned}$$

How about water flow in the presence of air in the porous medium?

Bundle of cylindrical capillary tubes model

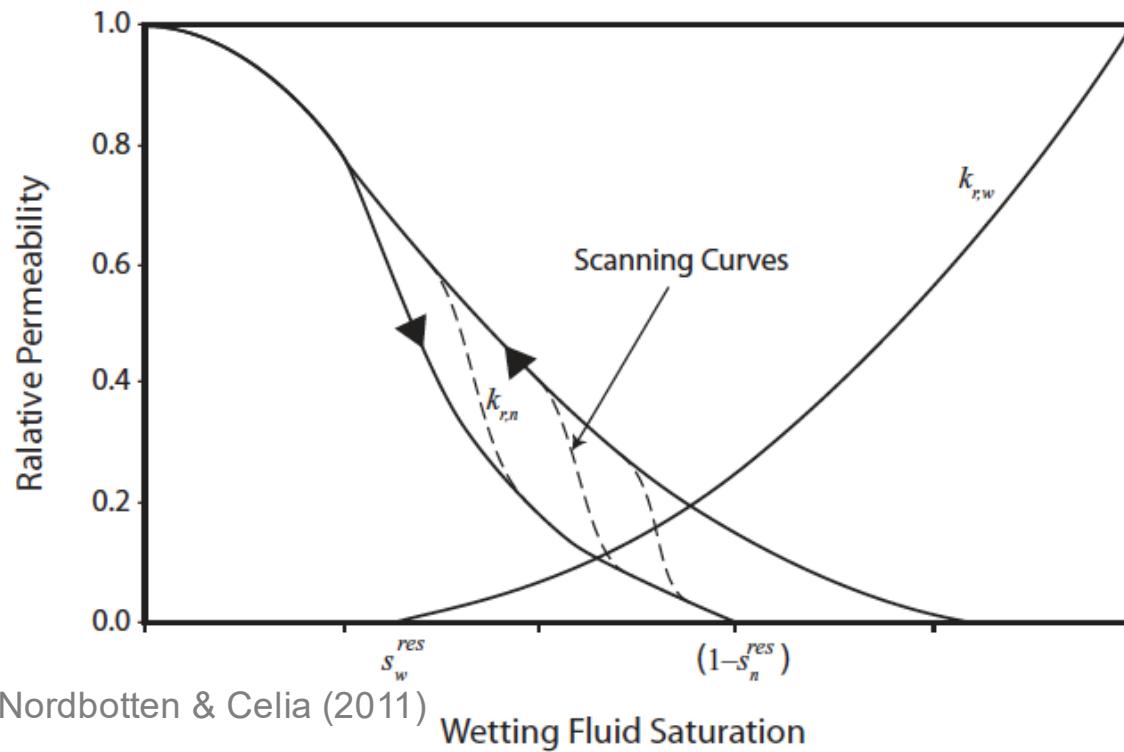


Extended Darcy's Law for unsaturated flow
(Buckingham, 1907)

$$\begin{aligned} q_w &= -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z) \\ &= -k_{r,w}(S_w) \mathbf{K}_{sat} \nabla H \end{aligned}$$

Relative Permeability

Typical forms of relative permeabilities



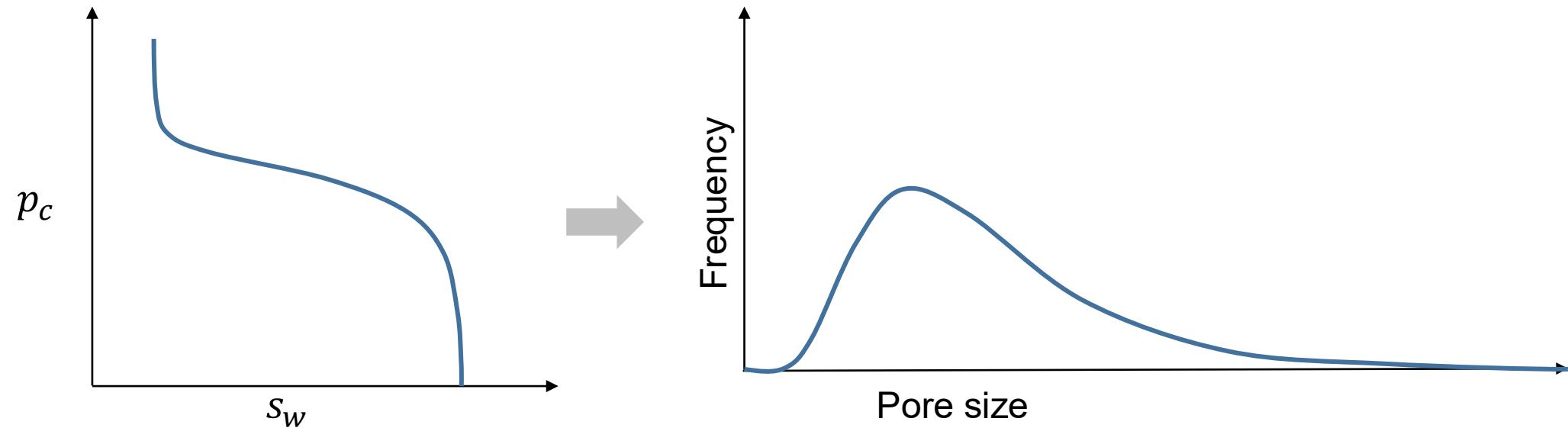
Bundle of cylindrical capillary tubes model



Note: Direct measurement of relative permeability curves is difficult and time-consuming.

Estimate Relative Permeability from SWC

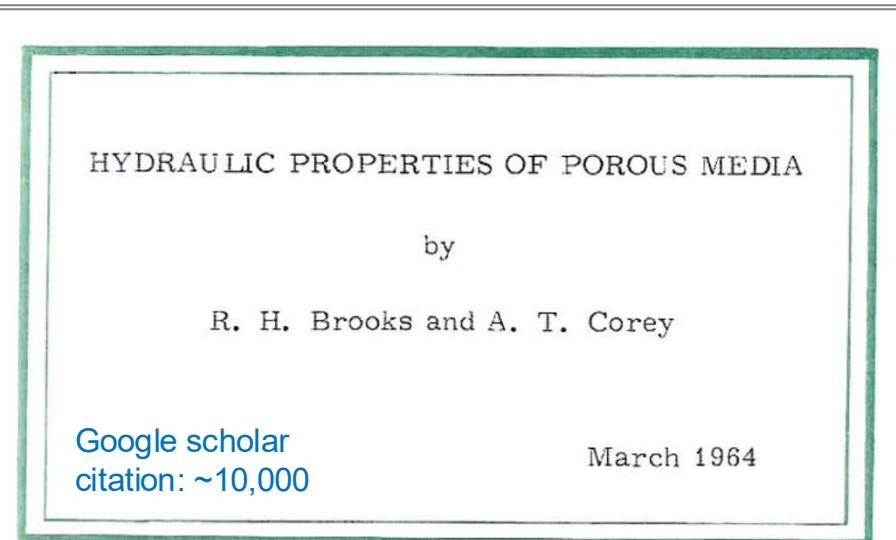
1. Extraction of radii distribution of the Bundle of Cylindrical Capillary tubes (based on BCC)



2. For a given p_c or s_w , compute the fluxes for the cylindrical tubes filled by water ($Q = \frac{\pi r^4}{8\mu_w} \frac{\Delta p}{L}$) and sum them up to obtain the total flux, and then to compute the overall conductivity of the bundle.

Estimate Relative Permeability from SWC

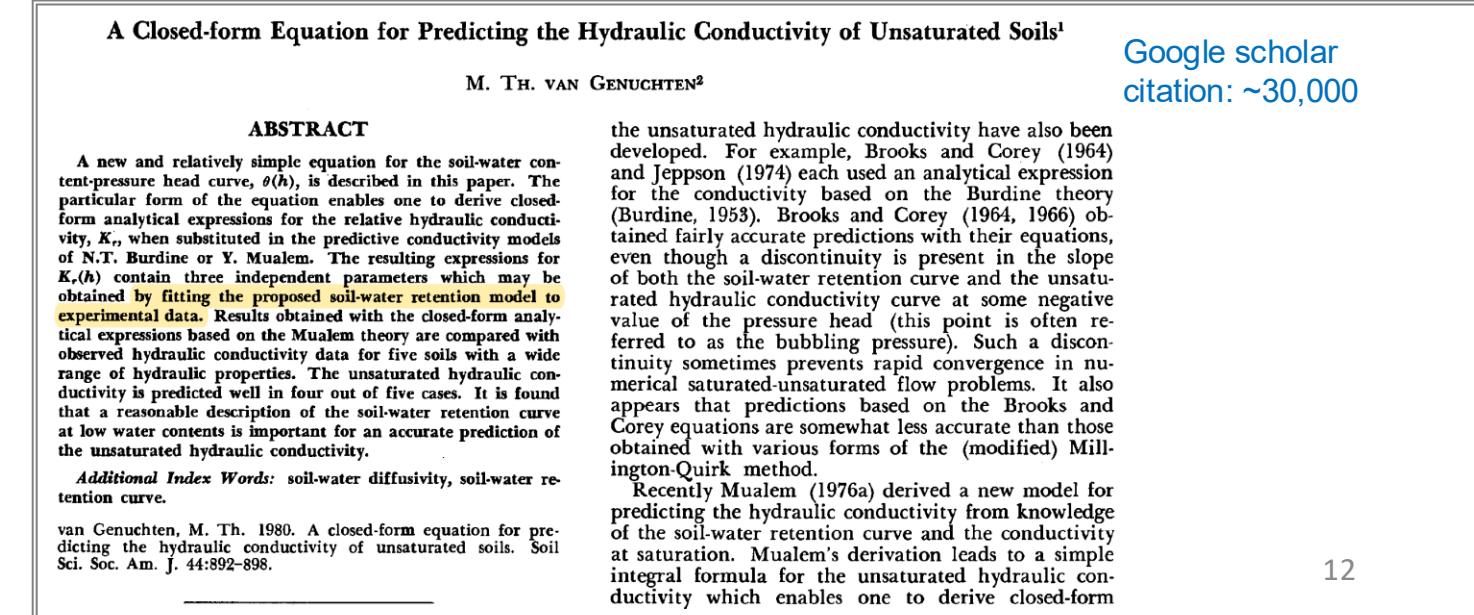
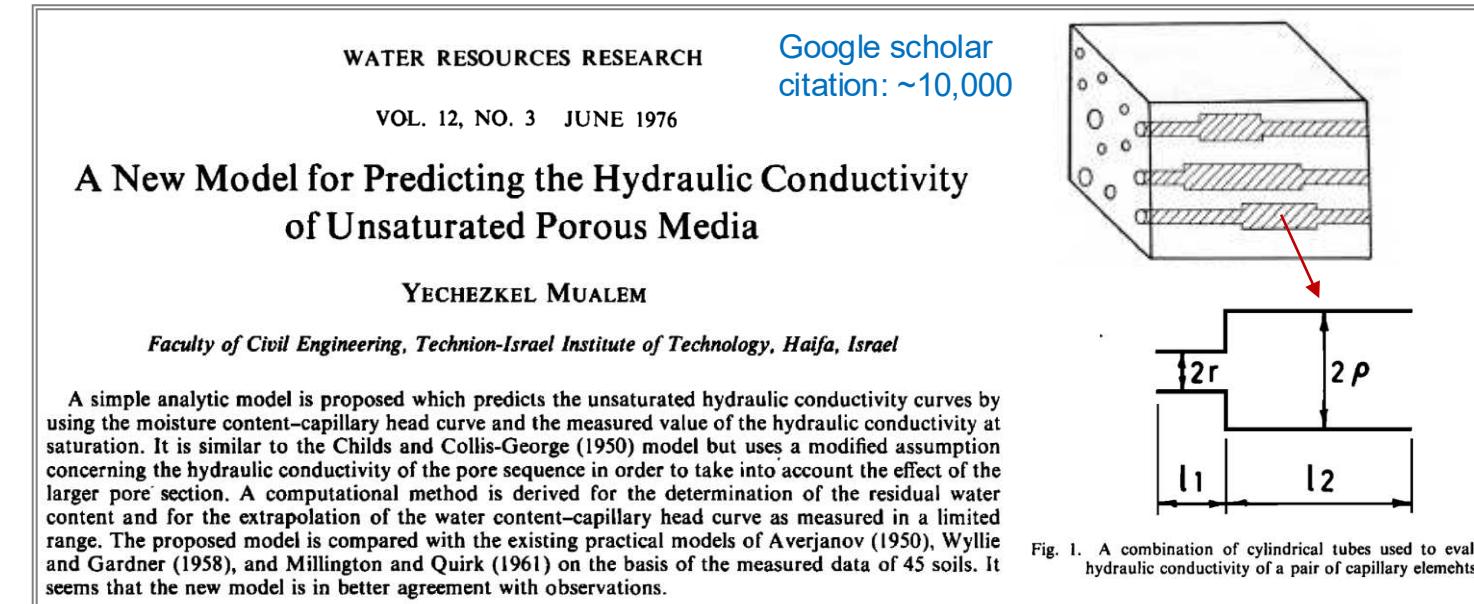
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Following the Burdine approach, a theory is presented that develops the functional relationships among saturation, pressure difference, and the permeabilities of air and liquid in terms of hydraulic properties of partially saturated porous media. Procedures for determining these hydraulic properties from capillary pressure - desaturation curves are described. Air and liquid permeabilities as a function of saturation and capillary pressure are predicted from the experimentally determined hydraulic properties.

tional relationship among effective permeability, saturation and capillary pressure is essential. This paper develops these functional relationships using a model of porous media developed by investigators [3, 4, 11, 19] in the petroleum industry. Specifically, the analysis presented here is an extension of the work of Burdine [3].

Based on a bundle of tubes with irregular cross sections.



Relative Permeability

Brooks-Corey (1964)

$$k_{r,w} = s_e^{(2+3\lambda)/\lambda}$$

$$k_{r,nw} = (1 - s_e)^2 (1 - s_e^{(2+\lambda)/\lambda})$$

λ is a parameter related to pore size distribution

van Genuchten (1980) – Mualem (1976)

$$k_{r,w} = s_e^{1/2} \left[1 - \left(1 - s_e^{1/m} \right)^m \right]^2$$

$k_{r,nw}$ was not proposed by V-G. Often the $k_{r,nw}$ from B-C is used

α is a parameter related to the inverse of entry pressure

n is a parameter related to pore size distribution

$m = 1 - 1/n$ based on the Mualem assumption