HWRS 561b
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Spring 2026

# HWRS 561b: Physical Hydrogeology II

## Solute transport (Part 3)

Agenda:

1. Analytical solutions for solute transport

# Simplified 1D advection-dispersion equation



3D governing equation for solute transport in saturated porous media

$$\frac{\partial}{\partial t}(\phi C) + \nabla \cdot (\mathbf{q}C - \phi \mathbf{D} \nabla C) = 0$$

1D simplification (assuming constant porosity in space)

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}\left(VC - D\frac{\partial C}{\partial x}\right) = 0$$

1) $\div \phi$
2) $V = V_x$
3) $D = D_{xx}$

Assuming constant porewater velocity and dispersion coefficient, we obtain

$$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0$$

# Simplified 1D advection-dispersion equation



$$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0$$

**Note**:
1) This is a second-order partial differential equation.
2) It is linear, e.g., coefficients does not depend on *C*.
3) It can be solved analytically, i.e., we can obtain a close-form solution for *C* for any *x* and *t*.

To solve the 1D equation, we need to specify one initial condition and two boundary conditions.

First-order in time.

Second-order in space.

# Non-dimensional form and dimensionless number



Define $\tilde{x} = \frac{x}{L}$, $\tilde{t} = \frac{t}{L/V}$, $\tilde{C} = \frac{C}{C_0}$

$$\frac{\partial C}{\partial t} = \frac{\partial(\tilde{C}C_0)}{\partial(\tilde{t}\,L/V)} = \frac{VC_0}{L}\frac{\partial \tilde{C}}{\partial \tilde{t}}$$

$$V\frac{\partial C}{\partial x} = V\frac{\partial(\tilde{C}C_0)}{\partial(\tilde{x}L)} = \frac{VC_0}{L}\frac{\partial \tilde{C}}{\partial \tilde{x}}$$

$$D\frac{\partial^2 C}{\partial x^2} = D\frac{\partial^2(\tilde{C}C_0)}{\partial(\tilde{x}L)^2} = \frac{DC_0}{L^2}\frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2}$$

$$\frac{VC_0}{L}\frac{\partial \tilde{C}}{\partial \tilde{t}} + \frac{VC_0}{L}\frac{\partial \tilde{C}}{\partial \tilde{x}} - \frac{DC_0}{L^2}\frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} = 0$$

$$\Rightarrow \frac{\partial \tilde{C}}{\partial \tilde{t}} + \frac{\partial \tilde{C}}{\partial \tilde{x}} - \frac{D}{VL}\frac{\partial^2 \tilde{C}}{\partial x^2} = 0$$

Define $Pe \equiv \frac{VL}{D}$

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} + \frac{\partial \tilde{C}}{\partial \tilde{x}} - \frac{1}{Pe}\frac{\partial^2 \tilde{C}}{\partial x^2} = 0$$

physical meaning:

$$Pe = \frac{\text{Advective transport rate}}{\text{Dispersive transport rate}} = \frac{\text{Dispersive time scale}}{\text{Advective time scale}}.$$

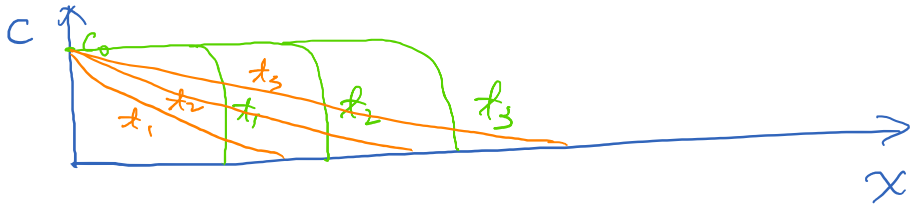
$$\left.\begin{aligned} t_{Adv} &= L/V \\ t_{Disp} &= L^2/D \end{aligned}\right\} \Rightarrow Pe = \frac{L^2/D}{L/V} = \frac{VL}{D}$$

① $Pe \gg 1$ : Advection-dominated
② $Pe \ll 1$ : Dispersion-dominated
③ $Pe \sim O(1)$ : Balanced.

# 1D step change in concentration



Let us first consider a semi-infinite domain.




Initial condition

$$C(x,0) = 0,\ x \geq 0$$

Boundary conditions (assuming semi-infinite domain)

$$C(0,t) = C_0,\ t \geq 0$$

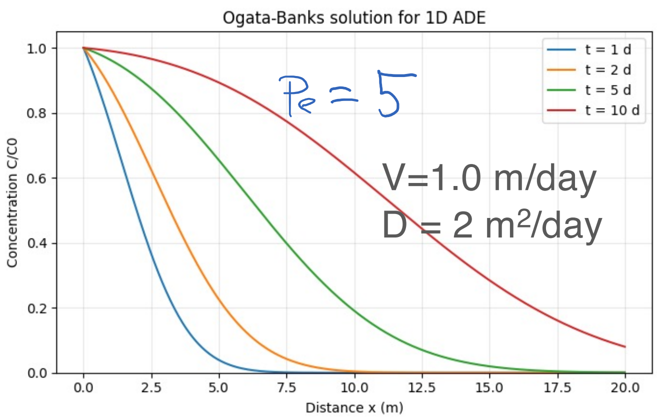
$$C(\infty,t) = 0,\ t \rightarrow \infty$$

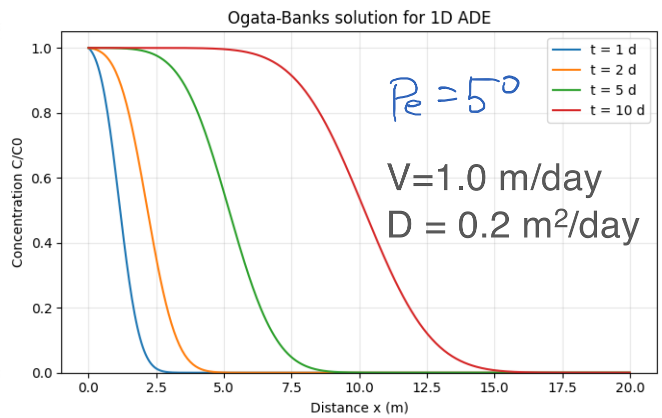
$$C = \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x - Vt}{2\sqrt{Dt}}\right) + \exp\left(\frac{VL}{D}\right)\operatorname{erfc}\left(\frac{x + Vt}{2\sqrt{Dt}}\right)\right]$$

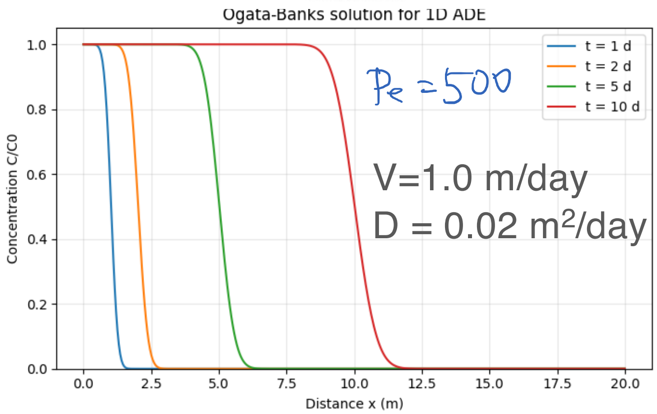
[Ogata and Banks, 1961]

# 1D step change in concentration

$$C = \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x - Vt}{2\sqrt{Dt}}\right) + \exp\left(\frac{VL}{D}\right)\operatorname{erfc}\left(\frac{x + Vt}{2\sqrt{Dt}}\right)\right]$$




$$Pe = \frac{VL}{D}$$

$L = 10$

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# 1D step change in concentration

Special case: *V=0* (only dispersion)

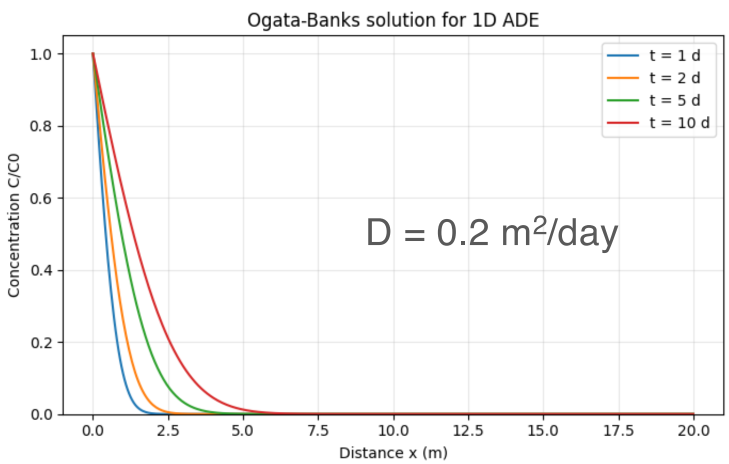
$$C = C_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)$$

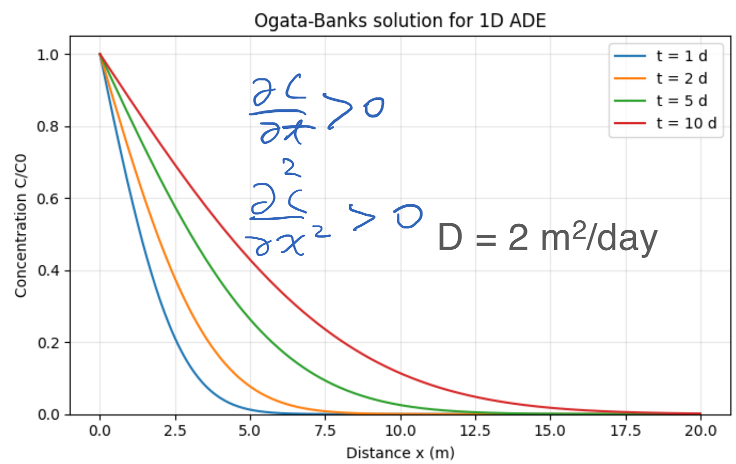
$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

Gaussian distribution

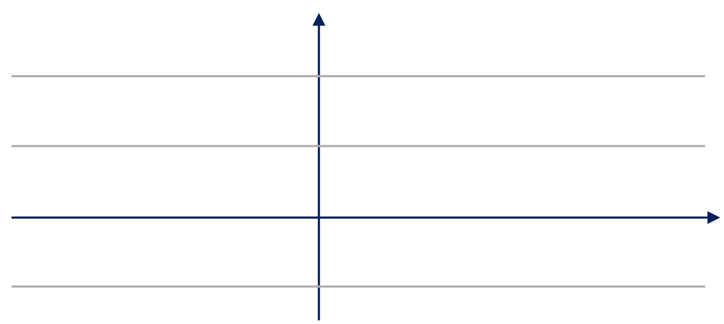
PDF $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

CDF $\Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$




$$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = 1 - \operatorname{erf}(x)$$



# 1D step change in concentration

Characteristic time scales for advection and dispersion

# 1D pulse change in concentration



Pulse change in concentration

$C(0,t) = C_0,\ 0 \leq t \leq t_0$

$C(0,t) = 0,\ t \geq t_0$

Let the solution for the step change be $(t \geq t_0)$

$$F(x,t) = \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x - Vt}{2\sqrt{Dt}}\right) + \exp\left(\frac{VL}{D}\right)\operatorname{erfc}\left(\frac{x + Vt}{2\sqrt{Dt}}\right)\right]$$

Then, the solution for pulse change can be written as

$0 \leq t < t_0$

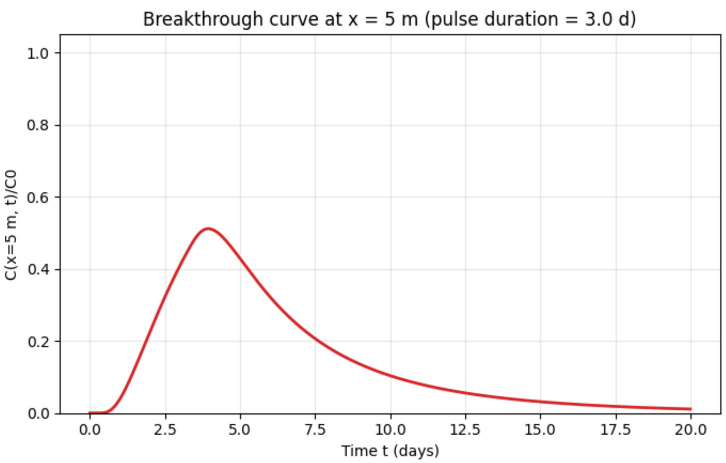
$$C = F(x,t)$$

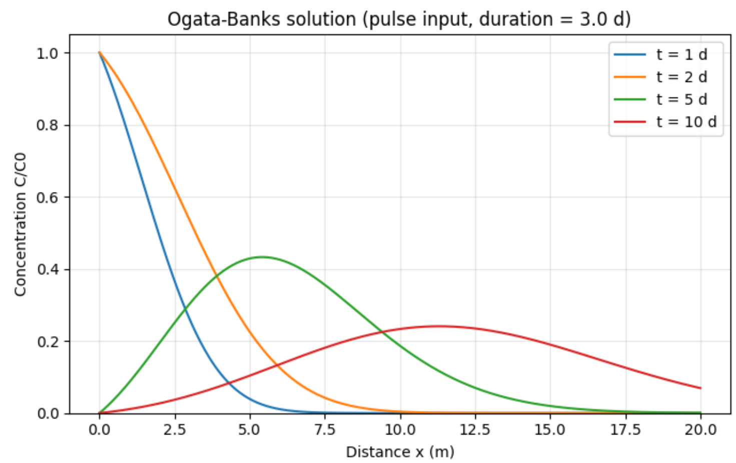
$t \geq t_0$

$$C = F(x,t) - F(x, t - t_0)$$

# Moment analysis






Pulse injection

$$C(0,t) = C_0,\ 0 \le t \le t_0$$
$$C(0,t) = 0,\ t \ge t_0$$
$$t_0 = 3\,\mathrm{d}$$

$$M_t^0 = \int_0^\infty C\,dt$$

Total mass passing the observation point.

$$M_s^0 = \int_0^\infty C\,dx$$

Total mass in the system at a given time.

$$M_t^1 = \frac{\int_0^\infty C\,t\,dt}{\int_0^\infty C\,dt} = \frac{\int_0^\infty C\,t\,dt}{M_t^0}$$

Arrival time

$$M_x^1 = \frac{\int_0^\infty C\,x\,dx}{\int_0^\infty C\,dx} = \frac{\int_0^\infty C\,t\,dx}{M_x^0}$$

Center of mass

$$M_{t,adj}^1 = \frac{\int_0^\infty C\,t\,dt}{M_t^0} - \frac{1}{2}T_0$$

Travel time

($T_0$ is the pulse duration)

# Moment analysis



**TABLE 2.2** One-dimensional moments.

| **Moment** | **Temporal Moments** | | **Spatial Moments** | |
|---|---|---|---|---|
| Zeroth Absolute Moment | $M_t^0 = \int_0^\infty C\,dt$ | Total mass passing the observation point. | $M_s^0 = \int_0^\infty C\,dx$ | Total mass in the system at a given time. |
| First Normalized Moment | $M_t^1 = \dfrac{\int_0^\infty Ct\,dt}{\int_0^\infty C\,dt} = \dfrac{\int_0^\infty Ct\,dt}{M_t^0}$ | Arrival time | $M_s^1 = \dfrac{\int_0^\infty Cx\,dx}{\int_0^\infty C\,dx} = \dfrac{\int_0^\infty Cx\,dx}{M_x^0}$ | Center of mass |
| Adjusted First Temporal Moment | $M_{adj}^1 = \dfrac{\int_0^x Ct\,dt}{M_t^0} - \dfrac{1}{2}T_0$ | Travel time<br>($T_0$ is the pulse duration) | Not defined | |
| Second Central Moment | $M_t^2 = \dfrac{\int_0^\infty \left(t - M_t^1\right)^2 C\,dt}{M_t^0}$ | Temporal variance | $M_s^2 = \dfrac{\int_0^\infty \left(x - M_s^1\right)^2 C\,dx}{M_s^0}$ | Spatial variance |
| Third Central Moment | $M_t^3 = \dfrac{\int_0^\infty \left(t - M_t^1\right)^3 C\,dt}{M_t^0}$ | Skewness of arrival times | $M_s^3 = \dfrac{\int_0^\infty \left(x - M_s^1\right)^3 C\,dx}{M_s^0}$ | Plume asymmetry in space |
| Fourth Central Moment | $M_t^4 = \dfrac{\int_0^\infty \left(t - M_t^1\right)^4 C\,dt}{M_t^0}$ | Tailedness (kurtosis) of the breakthrough curve | $M_s^4 = \dfrac{\int_0^\infty \left(x - M_s^1\right)^4 C\,dx}{M_s^0}$ | Plume intermittency and extreme spreading |

# Moment analysis



- First normalized moment can be used to determine velocity (assuming no retardation)

$$M_t^1 = x/V$$

Note: The first normalized moment is not related to dispersion.

- The second temporal moment and second spatial moment are the temporal and spatial variances, which can both be used to determine the dispersion coefficient.

$$M_t^2 = \frac{\int_0^\infty \left(t - M_t^1\right)^2 C\,dt}{M_t^0} = 2Dt \Rightarrow D = \frac{M_t^2}{2t}$$

$$M_s^2 = \frac{\int_0^\infty \left(x - M_x^1\right)^2 C\,dx}{M_s^0} = 2Dx \Rightarrow D = \frac{M_s^2}{2x}$$

# Continuous injection (3rd-type boundary condition)



Initial condition

$$C(x,0) = 0,\ x \geq 0$$

Boundary conditions (assuming semi-infinite domain)

$$\left(-D\frac{\partial C}{\partial x} + VC\right)\bigg|_{x=0} = VC_0$$

$$\frac{\partial C}{\partial x}\bigg|_{x\to\infty} = (\text{finite})$$

[van Genuchten, 1981]