Problem 1

Let X be a continuous random variable with PDF $f(x)$ and CDF $F(x)$. For a fixed number x_0 such that $F(x_0) < 1$, define the function:

$$g(x) = \begin{cases} \frac{f(x)}{1-F(x_0)}, & \text{for } x \geq x_0, \\ 0, & \text{for } x < x_0. \end{cases}$$

Prove that $g(x)$ is a PDF.

Problem 2

Let X be a random variable with PDF

$$f(x) = \frac{1}{2}(1+x), \text{ for } -1 < x < 1.$$

(a) Find the PDF of the random variable $Y = X^2$.

(b) Find the mean $\mathsf{E}(Y)$ and the variance $\mathsf{Var}(Y)$ of the random variable Y.

Problem 3

Let X be a random variable with moment generating function $M_X(t)$, for $-h < t < h$. Prove that

(a) $\mathsf{P}(X \geq a) \leq e^{-at} M_X(t)$, for $0 < t < h$.

(b) $\mathsf{P}(X \leq a) \leq e^{-at} M_X(t)$, for $-h < t < 0$.

Problem 4

Let $X \sim \mathsf{N}(\mu, \sigma^2)$. Find the values of μ and σ^2 such that

$$\mathsf{P}(|X| < 2) = \frac{1}{2}.$$

Prove or disprove that these values of μ and σ^2 are unique.

Problem 5

Find $\mathsf{P}\left(X > \sqrt{Y}\right)$ if X and Y are jointly distributed with PDF

$$f(x, y) = x + y, \text{ for } 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

Problem 6

Suppose X and Y are independent $\mathsf{N}(0, 1)$ random variables.

(a) Find $\mathsf{P}\left(X^2 < 1\right)$.

(b) Find $\mathsf{P}\left(X^2 + Y^2 < 1\right)$.

Problem 7

Let $X \sim \mathsf{N}\left(\mu, \sigma^2\right)$, and let $Y \sim \mathsf{N}\left(\gamma, \sigma^2\right)$. Suppose X and Y are independent. Define two random variables: $U = X + Y$ and $V = X - Y$.

(a) Show that U and V are independent Normal random variables.

(b) Find the distribution of U.

(c) Find the distribution of V.

Problem 8

A random variable X is defined by the transformation $Z = \log(X)$, where the mean of the random variable Z is $\mathsf{E}(Z) = 0$. Is $\mathsf{E}(X)$ greater than, less than, or equal to 1?