Problem 1

Let A and B be two events.

1. Show that
$$\mathsf{P}(A) + \mathsf{P}(B) - 1 \leq \mathsf{P}(A \cap B) \leq \mathsf{P}(A \cup B) \leq \mathsf{P}(A) + \mathsf{P}(B).$$

2. The difference $B \setminus A$ is defined to be the set of all elements of B that are not in A. Show that, if $A \subseteq B$, then
$$\mathsf{P}(B \setminus A) = \mathsf{P}(B) - \mathsf{P}(A).$$

3. The symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is defined to be the set of all elements that are in A or B but not both. Show that
$$\mathsf{P}(A \triangle B) = \mathsf{P}(A) + \mathsf{P}(B) - 2\mathsf{P}(A \cap B).$$

Solution: 1. From Theorem 1.6.2 part 3,

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cup B) \geq \mathsf{P}(A) + \mathsf{P}(B) - 1$$

since $\mathsf{P}(A \cup B) \leq 1$. Part 2 of Theorem 1.6.2 gives us

$$A \cap B \subseteq A \cup B \implies \mathsf{P}(A \cap B) \leq \mathsf{P}(A \cup B).$$

Lastly, noting that probabilities are always non-negative,

$$\mathsf{P}(A \cup B) \leq \mathsf{P}(A \cap B) + \mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B),$$

where the last equality arises from rearranging part 3 of Theorem 1.6.2.

2. First, note that $B \backslash A \equiv B \cap A^c$. Then $\mathsf{P}(B) = \mathsf{P}(A) + \mathsf{P}(B \cap A^c)$ and

$$\mathsf{P}(B \backslash A) = \mathsf{P}(B \cup A^c) = \mathsf{P}(B) - \mathsf{P}(A).$$

3. By definition, we have

$$\mathsf{P}(A \triangle B) = \mathsf{P}(A \backslash B) + \mathsf{P}(B \backslash A) + \mathsf{P}((A \backslash B) \cap (B \backslash A)) = \mathsf{P}(A \backslash B) + \mathsf{P}(B \backslash A),$$

since any element in $A \backslash B$ cannot be in B (and hence cannot be in $B \backslash A \subseteq B$), implying $\mathsf{P}((A \backslash B) \cap (B \backslash A)) = 0$. Then, noting again $A \backslash B \equiv A \cap B^c$ and $B \backslash A \equiv B \cap A^c$,

$$\begin{aligned} \mathsf{P}(A \triangle B) &= \mathsf{P}(A \cap B^c) + P(A^c \cap B) \\ &= \mathsf{P}(A) - \mathsf{P}(A \cap B) + \mathsf{P}(B) - \mathsf{P}(A \cap B) \\ &= \mathsf{P}(A) + \mathsf{P}(B) - 2\mathsf{P}(A \cap B). \end{aligned}$$

Problem 2

Show that, if A and B are independent events, then

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A)\mathsf{P}(B) = 1 - \mathsf{P}(A^c)\mathsf{P}(B^c).$$

Solution: Because A and B are independent events, we have $\mathsf{P}(A \cap B) = \mathsf{P}(A)\,\mathsf{P}(B)$, hence

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A)\,\mathsf{P}(B)\,.$$

Substituting $\mathsf{P}(A) = 1 - \mathsf{P}(A^c)$ and $\mathsf{P}(B) = 1 - \mathsf{P}(B^c)$ yields

$$\begin{aligned}
\mathsf{P}(A \cup B) &= 1 - \mathsf{P}(A^c) + 1 - \mathsf{P}(B^c) - (1 - \mathsf{P}(A^c))\,(1 - \mathsf{P}(B^c))\,, \\
&= 1 - \mathsf{P}(A^c) + 1 - \mathsf{P}(B^c) - 1 + \mathsf{P}(A^c) + \mathsf{P}(B^c) - \mathsf{P}(A^c)\,\mathsf{P}(B^c)\,, \\
&= 1 - \mathsf{P}(A^c)\,\mathsf{P}(B^c)\,.
\end{aligned}$$

Problem 3

Show that $\mathsf{P}(A \mid B) \leq \mathsf{P}(A)$ implies $\mathsf{P}(A \mid B^c) \geq \mathsf{P}(A)$.

Solution: By the law of total probability (LOTP), we have

$$\begin{aligned}
\mathsf{P}(A) &= \mathsf{P}(A \mid B)\,\mathsf{P}(B) + \mathsf{P}(A \mid B^c)\,\mathsf{P}(B^c)\,, \\
&= \mathsf{P}(A \mid B)\,(1 - \mathsf{P}(B^c)) + \mathsf{P}(A \mid B^c)\,\mathsf{P}(B^c)\,, \\
&\leq \mathsf{P}(A)\,(1 - \mathsf{P}(B^c)) + \mathsf{P}(A \mid B^c)\,\mathsf{P}(B^c)\,.
\end{aligned}$$

Here we used $\mathsf{P}(B) = 1 - \mathsf{P}(B^c)$, and $\mathsf{P}(A \mid B) \leq \mathsf{P}(A)$. Rearranging terms yields $\mathsf{P}(A)\,\mathsf{P}(B^c) \leq \mathsf{P}(A \mid B^c)\,\mathsf{P}(B^c)$. Thus, if $\mathsf{P}(B^c) > 0$, we have the desired inequality $\mathsf{P}(A) \leq P(A \mid B^c)$.

Problem 4

Show that if $\mathsf{P}(A) = 1$, then $\mathsf{P}(A \mid B) = 1$ for any B with $\mathsf{P}(B) > 0$.

Solution: We write

$$\begin{aligned}
\mathsf{P}(A) &= \mathsf{P}(A \mid B)\,\mathsf{P}(B) + \mathsf{P}(A \mid B^c)\,\mathsf{P}(B^c)\,, \\
&= \mathsf{P}(A \mid B)\,\mathsf{P}(B) + \mathsf{P}(A \mid B^c)\,(1 - \mathsf{P}(B))\,, \\
&\leq \mathsf{P}(A \mid B)\,\mathsf{P}(B) + 1 - \mathsf{P}(B)\,.
\end{aligned}$$

since probabilities must be less than 1. Taking $\mathsf{P}(A) = 1$ and rearranging terms, we get $\mathsf{P}(B) \leq \mathsf{P}(A \mid B)\,\mathsf{P}(B)$. We can divide both sides by $\mathsf{P}(B)$ if $\mathsf{P}(B) > 0$ to get

$$\mathsf{P}(A \mid B) \geq 1 \implies \mathsf{P}(A \mid B) = 1,$$

since probabilities must be between 0 and 1.

Problem 5

Show that if A and B are independent and $C = A \cup B$, then A and B are conditionally dependent (that is, A and B are not conditionally independent) given C (as long as $\mathsf{P}(A \cap B) > 0$ and $\mathsf{P}(A \cup B) < 1$), with

$$\mathsf{P}(A \mid B, C) < \mathsf{P}(A \mid C).$$

Solution: We write:

$$\mathsf{P}(A \mid B, C) = \frac{\mathsf{P}(A \cap B \cap C)}{\mathsf{P}(B \cap C)} = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} = \mathsf{P}(A).$$

Then, noting $\mathsf{P}(C) = \mathsf{P}(A \cup B) < 1$, we obtain

$$\mathsf{P}(A \mid B, C) = \mathsf{P}(A) = \mathsf{P}(A \cap C) = \mathsf{P}(A \mid C)\,\mathsf{P}(C) < \mathsf{P}(A \mid C).$$

To show that A and B are conditionally dependent given C, note

$$\mathsf{P}(A \mid C) > \mathsf{P}(A \mid B, C) = \frac{\mathsf{P}(A \cap B \mid C)}{P(B \mid C)}.$$

Thus $\mathsf{P}(A \cap B \mid C) \neq \mathsf{P}(A \mid C)\,\mathsf{P}(B \mid C)$.