Random Numbers

Adrian Dobra
adobra@uw.edu

Uniform Random Numbers

Task: producing $X_1, \ldots, X_n \sim U[0,1]$. Start with some initial number X_0 and use a transformation D such that $X_{i+1} = D(X_i)$.
Example: linear congruential generators have a period M

$$x_{i+1} = a \cdot x_i + b(\text{modulo} M).$$

Usually M is very large (2^{59}).
Remember: any sequence of random numbers will eventually repeat itself. You need to set the seed of your random number generator to obtain a different sequence of numbers. Getting the same sequence of numbers is actually good for replicating results and debugging.

The generalized inverse of a non-decreasing function F is:

$$F^{-}(u) = \inf\{x : F(x) \geq u\}.$$

Lemma

If $U \sim U[0,1]$, then the random variable $F^{-}(U)$ has distribution F.

Proof.

$$P(F^{-}(U) \leq x) = P(U \leq F(x)) = F(x).$$

$\square$

Example

If $X \sim Exp(1)$ we have $F(x) = 1 - \exp(-x)$. Thus $F^{-}(u) = -log(1-u)$. Therefore, if you simulate $U \sim U[0,1]$, the r.v. $X = -\log(U) \sim Exp(1)$.

Exponential(1) Leads to Other Distributions

Let $X_1, X_2, \ldots$ be i.i.d. $Exp(1)$. Then:

$$
\begin{aligned}
2\sum_{j=1}^{k} X_j &\sim \chi^2_{2k}, \\
\beta\sum_{j=1}^{a} X_j &\sim Gamma(a, \beta), \\
\frac{\sum_{j=1}^{a} X_j}{\sum_{j=1}^{a+b} X_j} &\sim Beta(a, b),
\end{aligned}
$$

and so on.

Generating Random Normals N(0,1)

The Box-Muller Algorithm

- Generate U_1, U_2 i.i.d. $U[0,1]$.
- Define

$$
\begin{aligned}
x_1 &= \sqrt{-2\log(u_1)}\cos(2\pi u_2), \\
x_2 &= \sqrt{-2\log(u_2)}\sin(2\pi u_2).
\end{aligned}
$$

- Take x_1 and x_2 as two independent draws from $N(0,1)$.

If $N \sim Pois(\lambda)$ and $X_1, X_2, \dots$ i.i.d. $Exp(\lambda)$, then

$$P(N = k) = P(X_1 + \ldots + X_k \leq 1 < X_1 + \ldots X_k + X_{k+1}).$$

You simulate i.i.d. $Exp(\lambda)$ random variables until their sum exceeds 1. The number of variables you simulated is a random draw from $Pois(\lambda)$.

Generating Discrete Random Variables

Let X be a r.v. that takes values $\{1, 2, \ldots, K\}$. Calculate the cumulative probabilities:

$$
\begin{aligned}
p_0 &= 0, \\
p_1 &= P(X \leq 1) = P(X = 1), \\
p_2 &= P(X \leq 2) = p_1 + P(X = 2), \\
&\vdots \\
p_K &= P(X \leq K) = p_{K-1} + P(X = k) = 1.
\end{aligned}
$$

Generate $U \sim U[0, 1]$. Take $X = k$ if $p_{k-1} < U \leq p_k$.

Generating Multivariate Normal Random Variables

Need to generate from the p-dimensional normal $N_p(\mu, \Sigma)$. Obtain the Cholesky factorization of Σ

$$\Sigma = (\Sigma^{1/2})^T \Sigma^{1/2},$$

where $\Sigma^{1/2}$ is an upper triangular matrix with positive elements (use BLAS!). Simulate p independent $N(0,1)$ r.v. $X_1, \ldots, X_p$. Then

$$\mu + \Sigma^{1/2} \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} \sim N_p(\mu, \Sigma).$$

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