Homework 7, DATA 556: Due Tuesday, 11/13/2018

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Please complete the following:

1. Problem 1 Let X and Y be i.i.d. Expo(λ) and T = $\log(X/Y)$. Find CDF and PDF of T

$$F_t(t) = P(T \leq t) = P(\frac{X}{Y} \leq e^t) = \int_0^\infty P(\frac{X}{Y} \leq e^t | Y = y)P(Y = y)dy \tag{1}$$

$$= \int_0^\infty P(X \leq ye^t | Y = y)f(y)dy = \int_0^\infty \int_0^{ye^t} f_X(x)f_Y(y)dxdy \tag{2}$$

$$= \int_0^\infty \int_0^{ye^t} \lambda e^{-\lambda x} e^{-\lambda y} dxdy = \int_0^\infty (1 - e^{-\lambda y e^t})\lambda e^{-\lambda y} dy \tag{3}$$

$$= \int_0^\infty \lambda e^{-\lambda y} dy - \int_0^\infty e^{-\lambda y e^t} \lambda e^{-\lambda y} dy = 1 - \int_0^\infty \lambda e^{-\lambda y(e^t+1)} dy = 1 - \frac{1}{e^t + 1} e^{-\lambda y(e^t+1)} |_0^\infty \tag{4}$$

$$= 1 - (0 + \frac{1}{e^t + 1}) = 1 - \frac{1}{e^t + 1} \tag{5}$$

$$=> f_T(t) = \frac{d}{dt}(1 - \frac{1}{e^t + 1}) = \frac{e^t}{(e^t + 1)^2} \tag{6}$$

The above used the LOTP as well as properties of independent exponential distributions, what they integrate to, and similar principles. This result is easily verified to hold all the properties necessary of a PDF or CDF.

2. Let X and Y be i.i.d. Expo(λ), and transform them to T = X+Y and W=X/Y

   (a) Find the joint PDF of T and W. Are they independent?

$$f_{t,w} = f_{x,y} * \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{bmatrix} = f_x(x) * f_y(y) \begin{bmatrix} \frac{1}{1+t} & \frac{t}{1+w} - \frac{tw}{(1+w)^2} \\ \frac{1}{1+w} & \frac{-t}{(1+w)^2} \end{bmatrix} \tag{7}$$

$$= \lambda^2 e^{-\lambda(x+y)} | \frac{-tw}{(1+w)^3} - \frac{t(1+w)}{(1+w)^3} + \frac{tw}{(1+w)^3} | = \lambda^2 e^{-\lambda t} \frac{t(1+w)}{(1+w)^3} \tag{8}$$

$$= \lambda^2 t e^{-\lambda t} \frac{1}{(1+w)^2} \tag{9}$$

Thus they are independent as we are able to separate their joint distribution into two functions of their respective variables in the form of $g(t) * h(w)$

$$g(t) = \lambda^2 t e^{-\lambda t} \ \& h(w) = \frac{1}{(1+w)^2} \tag{10}$$

(b) Find the marginal PDFs of T and W.

$$f(t) = \int_{-\infty}^{\infty} f(t,w)dw = \int_{-\infty}^{\infty} \lambda^2 te^{-\lambda t}\frac{1}{(1+w)^2}dw \text{ w/}0 \leq w \tag{11}$$

$$\text{This is obvious as } W = \frac{X}{Y} \text{ and } X \geq 0 Y \geq 0 \tag{12}$$

$$=> \int_{0}^{\infty} \lambda^2 te^{-\lambda t}\frac{1}{(1+w)^2}dw = \lambda^2 te^{-\lambda t}\frac{-1}{(1+w)}|_0^\infty = \lambda^2 te^{-\lambda t} \tag{13}$$

$$f(w) = \int_{-\infty}^{\infty} f(t,w)dt = \int_{0}^{\infty} \lambda^2 te^{-\lambda t}\frac{1}{(1+w)^2}dw \text{ w/}0 \leq t \tag{14}$$

$$\text{This is similarly obvious as } T = X + Y \text{ and } X \geq 0 Y \geq 0 \tag{15}$$

$$=> \frac{1}{(1+w)^2} - e^{-\lambda t}(\lambda t + 1)|_0^\infty = \frac{1}{(1+w)^2}(0 + 1(0+1)) = \frac{1}{(1+w)^2} \tag{16}$$

$$\text{The first term is evaluated to zero as } e^{\lambda t} \text{ grows faster than } \lambda t \tag{17}$$

This confirms our previous result that they are independent as $f_t(t) * f_w(w) = f_{w,t}(w,t)$

3. Let U $\sim$ Unif(0,1) and X $\sim$ Expo(λ), independently. Find the PDF of U+X

   We have $0 \leq U \leq 1$ for U and $0 \leq X \leq \infty$

$$T = U + X => f(t) = \int_{-\infty}^{\infty} f_x(x)f_u(t-x)dx \tag{18}$$

$$\text{We require that } x \geq 0 \& \, 0 \leq t - x \leq 1 => t \geq x \geq 1 - t \text{ By the properties of our distribs} \tag{19}$$

$$\text{Thus we need to match all properties and get the following} \tag{20}$$

$$\int_{max(0,t-1)}^{t} f_x(x)f_u(t-x)dx => \int_0^t f_x(x)f_u(t-x)dx = \int_0^t \lambda e^{-\lambda(x)}dx = 1 - e^{-\lambda(t)} \tag{21}$$

$$=> \int_{t-1}^{t} f_x(x)f_u(t-x)dx => \int_{t-1}^{t} f_x(x)f_u(t-x)dx = \int_{t-1}^{t} \lambda e^{-\lambda(x)}dx = e^{-\lambda(t-1)} - e^{-\lambda t} = e^{-\lambda t}(e^\lambda - 1) \tag{22}$$

$$f(t) = \begin{cases} e^{-\lambda t}(e^\lambda - 1) & t > 1 \\ 1 - e^{-\lambda(t)} & 0 \leq t \leq 1 \\ 0 & t < 0 \end{cases} \tag{23}$$

4. Let X and Y be i.i.d. Expo(λ). Use a convolution integral to show that the PDF of L =

X - Y is

$$f_L(l) = \frac{\lambda}{2}e^{-\lambda|l|} \tag{24}$$

$$\text{Note that } L = X + (-Y) \tag{25}$$

$$\text{Further, we know } x - l \geq 0 => x \geq l \& x \geq 0 \tag{26}$$

$$\text{if } L < 0 \text{ we get the following} \tag{27}$$

$$f_L(l) = \int_0^\infty f_x(x)f_{-y}(l-x)dx = \int_0^\infty f_x(x)f_y(x-l)dx \text{ This is easily seen as} \tag{28}$$

$$f_{-y}(t-x) = f_y(-1(t-x)) \text{ as } g^{-1}(-y) = y \text{ as } g(x) = -x \text{ and } |\frac{d}{dy}(-y)| = 1 \tag{29}$$

$$= \int_0^\infty \lambda^2 e^{-\lambda(2x-l)}dx = e^{\lambda l}\int_0^\infty \lambda e^{-\lambda(2x)}dx = e^{\lambda l} * -\frac{\lambda e^{-\lambda 2x}}{2}|_0^\infty = e^{\lambda l} * -\frac{\lambda e^{-\lambda 2x}}{2}|_0^\infty = e^{\lambda l} * (0 + \frac{\lambda}{2}) = \frac{\lambda e^{\lambda l}}{2} \tag{30}$$

$$\text{Now for } l \geq 0 \tag{31}$$

$$f_L(l) = \int_l^\infty f_x(x)f_{-y}(l-x)dx = \int_l^\infty f_x(x)f_y(x-l)dx \tag{32}$$

$$= \int_l^\infty \lambda^2 e^{-\lambda(2x-l)}dx = e^{\lambda l}\int_l^\infty \lambda e^{-\lambda(2x)}dx = e^{\lambda l} * -\frac{\lambda e^{-\lambda 2x}}{2}|_l^\infty = e^{\lambda l} * -\frac{\lambda e^{-\lambda 2x}}{2}|_l^\infty \tag{33}$$

$$= e^{\lambda l} * (0 + \frac{\lambda}{2}e^{-\lambda 2l}) = \frac{\lambda e^{-\lambda l}}{2} \tag{34}$$

$$\text{we get } f_L(l) = \begin{cases} \frac{\lambda e^{-\lambda l}}{2} & l \geq 0 \\ \frac{\lambda e^{\lambda l}}{2} & l < 0 \end{cases} \text{which is equivalent to saying } f_L(l) = \frac{\lambda}{2}e^{-\lambda|l|}\square \tag{35}$$

5. Use a convolution integral to show that if X $\sim N(\mu_1, \sigma)$ and Y $\sim N(\mu_2, \sigma)$ are independent, then

$$T = X + Y \sim N(\mu_1 + \mu_2, 2\sigma^2) \tag{36}$$

$$f(t) = \int_{-\infty}^{\infty} f_x(x) f_y(t-x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1((t-x)-\mu_2)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1(x-\mu_1)^2}{2\sigma^2}} dx \tag{37}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma^2} e^{-1\frac{(t-x-\mu_2)^2+(x-\mu_1)^2}{2\sigma^2}} dx \tag{38}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma^2} e^{-1\frac{t^2+x^2+\mu_2^2-2tx-2t\mu_2+2x\mu_2+x^2-2\mu_1 x+\mu_1^2}{2\sigma^2}} dx \tag{39}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma^2} e^{-1\frac{2x^2-2x(t-\mu_2+\mu_1)+t^2+\mu_2^2-2t\mu_t+\mu_1^2}{2\sigma^2}} dx \tag{40}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma^2} e^{-1\frac{x^2-x(t-\mu_2+\mu_1)+\frac{1}{2}(t^2+\mu_2^2-2t\mu_t+\mu_1^2)}{\sigma^2}} dx \tag{41}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma^2} e^{-1\frac{(x-\frac{t-\mu_2+\mu_1}{2})^2-\frac{(t-\mu_2+\mu_1)^2}{4}+\frac{(t-\mu_2)^2+\mu_1^2}{2}}{\sigma^2}} dx \tag{42}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2-\mu_2^2-\mu_1^2+2t\mu_2-2t\mu_1+2\mu_2\mu_1+2t^2+2\mu_1^2+2\mu_2^2-4t\mu_2}{4\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-1\frac{(x-\frac{t-\mu_2+\mu_1}{2})^2}{\sigma^2}} dx \tag{43}$$

$$= \frac{1}{\sqrt{2\pi}2\sigma} e^{\frac{t^2+\mu_2^2+\mu_1^2-2t\mu_2-2t\mu_1+2\mu_2\mu_1}{4\sigma^2}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}\sigma} e^{-1\frac{(x-\frac{t-\mu_2+\mu_1}{2})^2}{\sigma^2}} dx \tag{44}$$

$$= \frac{1}{\sqrt{2\pi}2\sigma} e^{\frac{-1(t-(\mu_1+\mu_2))^2}{4\sigma^2}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}\sigma} e^{-1\frac{(x-\frac{t-\mu_2+\mu_1}{2})^2}{\sigma^2}} dx = \frac{1}{\sqrt{2\pi}2\sigma} e^{\frac{-1(t-(\mu_1+\mu_2))^2}{4\sigma^2}} \tag{45}$$

$$\text{This is true as in the integral we have a Normal Distribution with} \mu = \frac{t-\mu_2+\mu_1}{2} \text{ and Variance } \frac{\sigma}{2} \tag{46}$$

$$\text{Thus when integrated over its whole region is equal to one} \tag{47}$$

$$\text{What we are left with is } f_t(t) = \frac{1}{\sqrt{2\pi}2\sigma} e^{\frac{-1(t-(\mu_1+\mu_2))^2}{4\sigma^2}} \tag{48}$$

$$\text{We recognize this as the pdf of a normal distribution with parameters} \tag{49}$$

$$\mu_t = (\mu_1 + \mu_2) \text{ and } \sigma_t^2 = 2\sigma^2 \square \tag{50}$$

I apologize in advance for any slight typos in the proof above, as there are likely some due to its complexity.

6. Let W_1, W_2 be two R.V. with joint distrib
   $P(W_1 \leq w_1, W_2 \leq w_2) = \int_{-\infty}^{w_1} \int_{-\infty}^{w_2} \frac{1}{2\pi} e^{\frac{-1(x^2+y^2)}{2}} dxdy$
   consider two other RVs $Z_1 = |W_1|$ and $Z_2 = |W_2|$. In words Z_1 is the absolute value of W_1 and similar for Z_2

   (a) Show Z_1 independent of Z_2

Note that it is very easy to see that $W_1 \,\&\, W_2 \sim N(0,1)$ i.i.d.

$$\int_{-\infty}^{w_1}\int_{-\infty}^{w_2} \frac{1}{2\pi} e^{\frac{-1(x^2+y^2)}{2}} dxdy = \int_{-\infty}^{w_1} \frac{1}{\sqrt{2\pi}} e^{\frac{-1(y^2)}{2}} dy \int_{-\infty}^{w_2} \frac{1}{\sqrt{2\pi}} e^{\frac{-1(x^2)}{2}} dx = \phi(w_1)\phi(w_2) \tag{51}$$

$$\tag{52}$$

We further know that functions of independent random variables are independent, and thus $Z_1 \,\&\, Z_2$ are independent

(b) Show that Z_1 and Z_2 have the same distribution, and find it

$$P(Z_1 \le z_1) = P(|W_1| \le z_1) = P(-z_1 \le W_1 \le z_1) = \varphi(z_1) - \varphi(-z_1) = \varphi(z_1) - (1 - \varphi(z_1)) \tag{53}$$

$$= \int_0^{z_1} \frac{1}{\sqrt{2\pi}} (e^{\frac{-1*w_1^2}{2}} + e^{\frac{-1*w_1^2}{2}}) dw_1 \;\; z_1 > 0 \tag{54}$$

The same is trivially done for Z_2. Now lets analyze what kind of distribution this is. It is clear that the distributions are the same, as they both are composed of a sum of two normal distributions over the same range of values $0 \to \infty$.

These both also are strictly increasing and sum to one.

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi}} (e^{\frac{-1*w_1^2}{2}}) dw_1 + \int_0^{\infty} \frac{1}{\sqrt{2\pi}} (e^{\frac{-1*w_1^2}{2}}) dw_1 = \frac{1}{2} + \frac{1}{2} = 1 \tag{55}$$

Thinking about these random variables, since they are the absolute value of a normal distribution, which is symmetric around mean 0, it appears to reflect the negative area back on to its positive area. Effectively "doubling up" the area under our curve.

$$\frac{1}{\sqrt{2\pi}} (e^{\frac{-1*w_1^2}{2}}) + \frac{1}{\sqrt{2\pi}} (e^{\frac{-1*w_1^2}{2}}) = \frac{2}{\sqrt{2\pi}} e^{\frac{-1*w_1^2}{2}} \;\; w_1 > 0 \tag{56}$$

Thus we have two RVs that follow the same "double positive" normal distribution with $N(0,1)$