Problem 1

Let X and Y be i.i.d. $\textsf{Expo}(\lambda)$, and $T = \log(X/Y)$. Find the CDF and PDF of T.

Solution: Because X and Y are independent, their joint distribution has PDF:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \lambda^2 e^{-\lambda(x+y)}, \text{ for } x>0, y>0.$$

We find the distribution of T by transforming the random vector (X, Y) into the vector (Z, W) where

$$\begin{aligned} Z &= X, \\ T &= \log(X/Y) = \log(X) - \log(Y). \end{aligned}$$

The inverse transformation that maps (Z, W) into (X, Y) is

$$\begin{aligned} X &= Z, \\ Y &= \frac{Z}{e^T}. \end{aligned}$$

The Jacobian matrix is

$$\frac{\partial(z,t)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{x} & -\frac{1}{y}. \end{pmatrix}$$

The absolute value of the determinant of this Jacobian matrix is $\frac{1}{y}$. The change of variables formula gives the joint PDF of Z and T:

$$\begin{aligned} f_{Z,T}(z,t) &= f_{X,Y}(x,y) \cdot \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix}^{-1}, \\ &= \lambda^2 e^{-\lambda\left(z+ze^{-t}\right)} z e^{-t}, \\ &= \lambda^2 z e^{-\left(\lambda z + \lambda z e^{-t} + t\right)}. \end{aligned}$$

We integrate out Z to obtain the marginal PDF of T:

$$f_T(t) = \int_0^\infty f_{Z,T}(z,t)\,\mathrm{d}z = \lambda^2 e^{-t} \int_0^\infty z e^{-\lambda\left(1+e^{-t}\right)z}\,\mathrm{d}z.$$

We recognize the integrand $ze^{-\lambda\left(1+e^{-t}\right)z}$ as the kernel of the Gamma distribution $\textsf{Gamma}\left(2, \lambda\left(1+e^{-t}\right)\right)$. Thus:

$$\int_0^\infty z e^{-\lambda\left(1+e^{-t}\right)z}\,\mathrm{d}z = \frac{\Gamma(2)}{\left[\lambda\left(1+e^{-t}\right)\right]^2}.$$

Since $\Gamma(2) = 1$, we obtain that the PDF of T is

$$f_T(t) = \frac{e^{-t}}{\left(1+e^{-t}\right)^2}, \text{ for } t \in \mathbb{R}.$$

The CDF of T is

$$F_T(t) = \int_{-\infty}^t \frac{e^{-q}}{\left(1+e^{-q}\right)^2}\,\mathrm{d}q = \left.\left(\frac{1}{1+e^{-q}}\right)\right|_{-\infty}^t = \frac{1}{1+e^{-t}}, \text{ for } t \in \mathbb{R}.$$

Problem 2

Let X and Y be i.i.d. $\textsf{Expo}(\lambda)$, and transform them to $T = X + Y$, $W = X/Y$.

(a) Find the joint PDF of T and W. Are they independent?

(b) Find the marginal PDFs of T and W.

Solution: (a) Because X and Y are independent, their joint distribution has PDF:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \lambda^2 e^{-\lambda(x+y)},\ \text{for } x > 0, y > 0.$$

The inverse of the transformation that maps (X, Y) into (T, W)

$$\begin{aligned} T &= X + Y, \\ W &= \frac{X}{Y}, \end{aligned}$$

is

$$\begin{aligned} X &= \frac{TW}{W+1}, \\ Y &= \frac{T}{W+1}. \end{aligned}$$

The Jacobian matrix is

$$\frac{\partial(x,y)}{\partial(t,w)} = \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{pmatrix} = \begin{pmatrix} \frac{w}{w+1} & \frac{t}{(w+1)^2} \\ \frac{1}{w+1} & -\frac{t}{(w+1)^2} \end{pmatrix}.$$

The absolute value of the determinant of this Jacobian matrix is $\frac{t}{(w+1)^2}$. The joint PDF of T and W is given by the change of variables formula:

$$f_{T,W}(t,w) \quad = \quad f_{X,Y}(x,y) \cdot \left| \begin{matrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{matrix} \right| = \lambda^2 e^{-\lambda\left(\frac{t}{w+1} + \frac{tw}{w+1}\right)} \frac{t}{(w+1)^2} = \lambda^2 \frac{1}{(w+1)^2} te^{-\lambda t},\ \text{for } t > 0, w > 0.$$

The random variables T and W are independent because their joint PDF factorizes as the product of two functions that depend only on T and only on W, respectively:

$$f_{T,W}(t,w) = g(w)h(t),$$

where

$$g(w) = \lambda^2 \frac{1}{(w+1)^2}, \qquad \text{and } h(t) = te^{-\lambda t}.$$

(b) The marginal PDF of T is obtained by marginalizing (integrating out) W in the joint PDF of T and W:

$$f_T(t) = \int_0^\infty f_{T,W}(t,w)\, \mathrm{d}w = \lambda^2 te^{-\lambda t} \int_0^\infty \frac{1}{(w+1)^2}\, \mathrm{d}w = \lambda^2 te^{-\lambda t} \left(-\frac{1}{w+1} \right) \Big|_0^\infty = \lambda^2 te^{-\lambda t},\ \text{for } t > 0.$$

We recognize this PDF to be a Gamma random variable: $T \sim \textsf{Gamma}(2, \lambda)$.

The marginal PDF of W is obtained by marginalizing (integrating out) T in the joint PDF of T and W:

$$f_W(w) = \int_0^\infty f_{T,W}(t,w)\, \mathrm{d}t = \frac{1}{(w+1)^2},\ \text{for } w > 0.$$

Problem 3

Let $U \sim \mathsf{Unif}(0,1)$ and $X \sim \mathsf{Expo}(\lambda)$, independently. Find the PDF of $U + X$.

Solution: We denote $W = U+X$. The PDF of U is $f_U(u) = 1$ for $u \in (0,1)$, and the PDF of X is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$. The PDF of W is obtained from the convolution formula:

$$f_W(w) = \int_0^1 f_X(w-u) \mathsf{I}_{\{w \geq u\}} f_U(u) \, \mathrm{d}u.$$

If $w < 1$, we have

$$\begin{aligned} f_W(w) &= \int_0^w f_X(w-u) f_U(u) \, \mathrm{d}u, \\ &= \lambda \int_0^w e^{-\lambda(w-u)} \, \mathrm{d}u, \\ &= \lambda e^{-\lambda w} \left(\frac{e^{\lambda u}}{\lambda} \right) \Big|_0^w, \\ &= e^{-\lambda w} \left(e^{\lambda w} - 1 \right), \\ &= 1 - e^{-\lambda w}. \end{aligned}$$

For $w > 1$, we have:

$$\begin{aligned} f_W(w) &= \lambda \int_0^1 e^{-\lambda(w-u)} \, \mathrm{d}u, \\ &= \lambda e^{-\lambda w} \int_0^1 e^{\lambda u} \, \mathrm{d}u, \\ &= \lambda e^{-\lambda w} \left(\frac{e^{\lambda u}}{\lambda} \right) \Big|_0^1, \\ &= \left(e^{\lambda} - 1 \right) e^{-\lambda w}. \end{aligned}$$

Problem 4

Let X and Y be i.i.d. $\mathsf{Expo}(\lambda)$. Use a convolution integral to show that the PDF of $L = X - Y$ is

$$f_L(l) = \frac{\lambda}{2} e^{-\lambda |l|},$$

for all real l.

Solution: The PDF of $X \sim \mathsf{Expo}(\lambda)$ is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$. Similarly, the PDF of $Y \sim \mathsf{Expo}(\lambda)$ is $f_Y(x) = \lambda e^{-\lambda y}$ for $y > 0$.

Consider the random variable $Z = -Y$. The inverse of this transformation is $Y = -Z$. The PDF of Z is given by the change of variables formula:

$$f_Z(z) = f_Y(y) \left| \frac{\mathrm{d}y}{\mathrm{d}z} \right| = \lambda e^{\lambda z}, \text{ for } z < 0.$$

We have $L = X - Y = X + (-Y) = X + Z$. The PDF of L is the convolution of the PDFs of X and Z:

$$\begin{aligned} f_L(l) &= \int_0^\infty f_X(x) f_Z(l-x) \mathrm{I}_{\{l \le x\}} \,\mathrm{d}x, \\ &= \lambda^2 e^{\lambda l} \int_{\max(l,0)}^\infty e^{-2\lambda x} \,\mathrm{d}x. \end{aligned}$$

Therefore, for $l < 0$, we have

$$f_L(l) = \lambda^2 e^{\lambda l} \left(\frac{e^{-2\lambda x}}{-2\lambda} \right) \Big|_0^\infty = \frac{\lambda}{2} e^{\lambda l} = \frac{\lambda}{2} e^{-\lambda |l|}.$$

For $l \ge 0$, we have

$$f_L(l) = \lambda^2 e^{\lambda l} \left(\frac{e^{-2\lambda x}}{-2\lambda} \right) \Big|_l^\infty = \frac{\lambda}{2} e^{-\lambda l} = \frac{\lambda}{2} e^{-\lambda |l|}.$$

Problem 5

Use a convolution integral to show that if $X \sim \mathsf{N}(\mu_1, \sigma^2)$ and $Y \sim \mathsf{N}(\mu_2, \sigma^2)$ are independent, then

$$T = X + Y \sim \mathsf{N}(\mu_1 + \mu_2, 2\sigma^2).$$

You can use a standardization (location-scale) idea to reduce to the standard Normal case before setting up the integral. Hint: complete the square.

Solution: Consider the random variables $X_1 = \frac{X - \mu_1}{\sigma}$ and $Y_1 = \frac{Y - \mu_2}{\sigma}$. We have X_1 and Y_1 i.i.d. $\mathsf{N}(0,1)$. We also have:

$$T_1 = \frac{T - (\mu_1 + \mu_2)}{\sigma} = \frac{X - \mu_1}{\sigma} + \frac{Y - \mu_2}{\sigma} = X_1 + Y_1.$$

Proving that $T = X + Y \sim \mathsf{N}(\mu_1 + \mu_2, 2\sigma^2)$ is equivalent to proving that

$$T_1 = X_1 + Y_1 \sim \mathsf{N}(0, 2).$$

We obtain the PDF of T_1 by applying the convolution formula for X_1 and Y_1:

$$\begin{aligned} f_{T_1}(t) &= \int_{-\infty}^\infty f_{X_1}(x) f_{Y_1}(t-x) \,\mathrm{d}x, \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-\frac{1}{2}\left(x^2 + (t-x)^2\right)} \,\mathrm{d}x, \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-\left[\left(x - \frac{t}{2}\right)^2 + \frac{t^2}{4}\right]} \,\mathrm{d}x, \\ &= \frac{1}{2\pi} e^{-\frac{t^2}{4}} \int_{-\infty}^\infty e^{-\frac{\left(x - \frac{t}{2}\right)^2}{2\frac{1}{2}}} \,\mathrm{d}x. \end{aligned}$$

But the integrand in the expression above is the kernel of the $\mathsf{N}\left(\frac{t}{2}, \frac{1}{2}\right)$, thus:

$$\int_{-\infty}^\infty e^{-\frac{\left(x - \frac{t}{2}\right)^2}{2\frac{1}{2}}} \,\mathrm{d}x = \sqrt{2\pi} \frac{1}{\sqrt{2}}.$$

Therefore:

$$f_{T_1}(t) = \frac{1}{\sqrt{2} \cdot \sqrt{2\pi}} e^{-\frac{t^2}{2 \cdot 2}}.$$

This proves that $T_1 \sim \mathsf{N}(0, 2)$.

Problem 6

Let W_1 and W_2 be two random variables with the joint distribution:

$$\mathsf{P}(W_1 \leq w_1, W_2 \leq w_2) = \int_{-\infty}^{w_1} \int_{-\infty}^{w_2} \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x^2+y^2)\right] \mathrm{d}x\,\mathrm{d}y.$$

Consider two other random variables $Z_1 = |W_1|$ and $Z_2 = |W_2|$. In words, Z_1 is the absolute value of W_1, Z_2 is the absolute value of W_2.

(a) Show that Z_1 is independent of Z_2.

(b) Show that Z_1 and Z_2 have the same distribution, and find that distribution.

Solution: (a) We write:

$$\mathsf{P}(W_1 \leq w_1, W_2 \leq w_2) \;=\; \left[\int_{-\infty}^{w_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x\right] \cdot \left[\int_{-\infty}^{w_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \mathrm{d}y\right]$$

Take $g_1(w_1) = \int_{-\infty}^{w_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x$ and $g_2(w_2) = \int_{-\infty}^{w_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \mathrm{d}y$. We have

$$\mathsf{P}(W_1 \leq w_1, W_2 \leq w_2) = g_1(w_1) g_2(w_2).$$

This implies that the random variables W_1 and W_2 are independent. Denote $g(x) = |x|$. Then $Z_1 = g(W_1)$ and $Z_2 = g(W_2)$. Therefore Z_1 and Z_2 are independent because they are functions of two independent random variables. Moreover, we have:

$$\mathsf{P}(W_1 \leq w_1) = \int_{-\infty}^{w_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x, \quad \mathsf{P}(W_2 \leq w_2) = \int_{-\infty}^{w_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \mathrm{d}y.$$

(b) We can write for any $t > 0$:

$$\begin{aligned}
\mathsf{P}(Z_1 \leq t) &= \mathsf{P}(-t \leq W_1 \leq t) = \mathsf{P}(W_1 \leq t) - \mathsf{P}(W_1 \leq -t), \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x - \int_{-\infty}^{-t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x, \\
&= \int_{-t}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x.
\end{aligned}$$

Similarly, we have

$$\mathsf{P}(Z_2 \leq t) = \int_{-t}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x.$$