Problem 1

Let $U \sim \mathsf{Unif}(a, b)$.

(a) Use simulations in R (the statistical programming language) to numerically estimate the median and the mode of U for $a = 0$ and $b = 2$.

(b) Find the median and the mode of $U \sim \mathsf{Unif}(a, b)$.

Problem 2

Let $X \sim \mathsf{Expo}(\lambda)$.

(a) Use simulations in R (the statistical programming language) to numerically estimate the median and the mode of $X \sim \mathsf{Expo}(2)$.

(b) Find the median and the mode of $X \sim \mathsf{Expo}(\lambda)$.

Problem 3

Let X be Discrete Uniform on $1, 2, \ldots, n$. Please note that your answers to the questions below can depend on whether n is even or odd.

(a) Use simulations in R (the statistical programming language) to numerically estimate all medians and all modes of X for $n = 1, 2, \ldots, 10$.

(b) Find all medians and all modes of X.

Problem 4

A distribution is called *symmetric unimodal* if it is symmetric (about some point) and has a unique mode. For example, any Normal distribution is symmetric unimodal. Let X have a continuous symmetric unimodal distribution for which the mean exists. Show that the mean, median, and mode of X are all equal.

Problem 5

Let $W = X^2 + Y^2$, with X, Y i.i.d. $\mathsf{N}(0, 1)$. You can assume you know that the MGF of X^2 is $(1 - 2t)^{-1/2}$ for $t < 1/2$. Find the MGF of W.

Problem 6

Let $X \sim \mathsf{Expo}(\lambda)$. You can assume you know that $\lambda X \sim \mathsf{Expo}(1)$, and that the nth moment of an $\mathsf{Expo}(1)$ random variable is $n!$. Find the skewness of X.

Problem 7

Let $X_1, X_2, \ldots, X_n$ be i.i.d. with mean μ, variance σ^2, and MGF M. Let

$$\bar{X}_n = \frac{1}{n}\left(X_1 + X_2 + \ldots + X_n\right).$$

and

$$Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right).$$

(a) Show that Z_n has mean 0 and variance 1.

(b) Find the MGF of Z_n in terms of M, the MGF of each X_j.