Problem 1

(a) Find the mean and the variance of a Discrete Uniform random variable on $1, 2, \ldots, n$.

(b) Let X be a discrete random variable with support $-n, -n+1, \ldots, 0, \ldots, n-1, n$ for some positive integer n. Suppose that the PMF of X satisfies the symmetry property $\mathsf{P}(X = -k) = \mathsf{P}(X = k)$ for all integers k. Find $\mathsf{E}(X)$.

Solution: (a) Let $X \sim \mathsf{DUnif}(1, 2, ..., n)$. We have

$$\mathsf{E}[X] = \sum_{i=1}^{n} i\mathsf{P}(X = i) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Before computing the variance, we calculate the expectation of X^2.

$$\mathsf{E}\left[X^2\right] = \sum_{i=1}^{n} i^2\mathsf{P}(X = i) = \frac{1}{n}\sum_{i=1}^{n} i^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}.$$

Next we calculate the variance:

$$\mathsf{Var}(X) = \mathsf{E}\left[X^2\right] - \mathsf{E}[X]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = (n+1)\left(\frac{2n+1}{6} - \frac{n+1}{4}\right) = \frac{(n+1)(n-1)}{12}.$$

(b) We have

$$\mathsf{E}[X] = \sum_{i=-n}^{n} i\mathsf{P}(X = i) = \sum_{i=-n}^{-1} i\mathsf{P}(X = i) + \sum_{i=1}^{n} i\mathsf{P}(X = i) = -\sum_{k=1}^{n} k\mathsf{P}(X = k) + \sum_{i=1}^{n} i\mathsf{P}(X = i) = 0.$$

Problem 2

Let X have PMF

$$\mathsf{P}(X = k) = -\frac{1}{\log(1-p)}\frac{p^k}{k},$$

for $k = 1, 2, \ldots$. Here p is a parameter with $0 < p < 1$. Find the mean and the variance of X.

Solution: Using the properties of geometric series, we obtain

$$\mathsf{E}[X] = \sum_{i=1}^{\infty} i\mathsf{P}(X = i) = -\frac{1}{\log(1-p)}\sum_{i=1}^{\infty} i\frac{p^i}{i} = -\frac{1}{\log(1-p)}\sum_{i=1}^{\infty} p^i = -\frac{1}{\log(1-p)}\frac{p}{1-p}.$$

Similarly, we have

$$\mathsf{E}\left[X^2\right] = \sum_{i=1}^{\infty} i^2\mathsf{P}(X = i) = -\frac{1}{\log(1-p)}\sum_{i=1}^{\infty} ip^i.$$

We can derive $\sum_{i=1}^{\infty} ip^i$ from the equation $\frac{1}{1-p} = \sum_{i=0}^{\infty} p^i$. Differentiating both sides yields

$$\frac{1}{(1-p)^2} = \sum_{i=0}^{\infty} ip^{i-1} = \sum_{i=1}^{\infty} ip^{i-1}.$$

Multiplying both sides by p yields $\sum_{i=1}^{\infty} ip^i = \frac{p}{(1-p)^2}$. Then

$$\mathsf{E}\left[X^2\right] = -\frac{1}{\log(1-p)}\frac{p}{(1-p)^2}$$

and

$$\mathsf{Var}(X) = \mathsf{E}\left[X^2\right] - \mathsf{E}[X]^2 = -\frac{1}{\log(1-p)}\frac{p}{(1-p)^2}\left(1+\frac{p}{\log(1-p)}\right).$$

Problem 3

(a) Use LOTUS to show that for $X \sim \mathsf{Pois}(\lambda)$ and any function $g(\cdot)$,

$$\mathsf{E}(Xg(X)) = \lambda E(g(X+1)).$$

(b) Find the third moment $E(X^3)$ for $X \sim \mathsf{Pois}(\lambda)$ by using the identity from (a) and a bit of algebra to reduce the calculation to the fact that X has mean λ and variance λ.

Solution: (a) We have

$$\begin{aligned}
\mathsf{E}[Xg(X)] &= \sum_{i=0}^{\infty} ig(i)\,\mathsf{P}(X=i), \\
&= \sum_{i=1}^{\infty} ig(i)\frac{\lambda^i e^{-\lambda}}{i!}, \\
&= \sum_{i=1}^{\infty} g(i)\frac{\lambda^i e^{-\lambda}}{(i-1)!}, \\
&= \lambda\sum_{i=1}^{\infty} g(i)\frac{\lambda^{i-1} e^{-\lambda}}{(i-1)!}, \\
&= \lambda\sum_{i=0}^{\infty} g(i+1)\frac{\lambda^i e^{-\lambda}}{i!}, \\
&= \lambda\sum_{i=0}^{\infty} g(i+1)\,\mathsf{P}(X=i), \\
&= \lambda\mathsf{E}[g(X+1)].
\end{aligned}$$

(b) Let us take $g(x) = x^2$. Then

$$\begin{aligned}
\mathsf{E}\left[X^3\right] &= \lambda\mathsf{E}\left[(X+1)^2\right], \\
&= \lambda\mathsf{E}\left[X^2\right] + 2\lambda\mathsf{E}[X] + \lambda, \\
&= \lambda\left(\mathsf{Var}(X) + \mathsf{E}[X]^2\right) + 2\lambda\mathsf{E}[X] + \lambda, \\
&= \lambda\left(\lambda+\lambda^2\right) + 2\lambda^2 + \lambda, \\
&= \lambda^3 + 3\lambda^2 + \lambda.
\end{aligned}$$

Problem 4

Show that for any events $A_1, \ldots, A_n$,

$$\mathsf{P}(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \sum_{j=1}^{n} \mathsf{P}(A_j) - n + 1. \tag{1}$$

Hint: You might want to prove a similar looking statement about indicator random variables, by interpreting what the events $\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n) = 0$ and $\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n) = 1$ mean.

Solution: **First version**. The inequality (1) is equivalent with the following inequality of indicator functions:

$$\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \sum_{j=1}^{n} \mathsf{I}(A_j) - n + 1. \tag{2}$$

To see why (2) is equivalent with (1), we use the following relations that hold based on the fundamental bridge:

$$\mathsf{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathsf{E}\left[\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n)\right], \quad \mathsf{P}(A_j) = \mathsf{E}[\mathsf{I}(A_j)],\ j = 1, \ldots, n.$$

With these relations, if we take the expectation on both sides of (2), we obtain (1). Therefore, to prove (1), it is sufficient to prove (2). We distinguish to cases (see the hint given in the text of the problem):

(a) Take $\omega \in \Omega$ such that $\omega \in A_1 \cap A_2 \cap \ldots \cap A_n$. Here Ω is the sample space: $A_j \subseteq \Omega$, $j = 1, \ldots, n$. The $\omega \in A_j$ for all $j = 1, \ldots, n$. We have $\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n)(\omega) = 1$, and also $\mathsf{I}(A_j)(\omega) = 1$ for all $j = 1, \ldots, n$. Thus inequality (2) holds in this case because

$$\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n)(\omega) = 1 \geq 1 = \sum_{j=1}^{n} \mathsf{I}(A_j)(\omega) - n + 1.$$

(b) Take $\omega \in \Omega$ such that $\omega \notin A_1 \cap A_2 \cap \ldots \cap A_n$. Then there is at least one $j_0 \in \{1, \ldots, n\}$ such that $\omega \notin A_{j_0}$. This implies $\mathsf{I}(A_{j_0})(\omega) = 0$. Since $\mathsf{I}(A_j)(\omega) \leq 1$ for all $j = 1, \ldots, n$, we have

$$\sum_{j=1}^{n} \mathsf{I}(A_j)(\omega) \leq n - 1 \implies 0 \geq \sum_{j=1}^{n} \mathsf{I}(A_j)(\omega) - n + 1$$

Thus inequality (2) holds in this case because

$$\mathsf{I}(A_1 \cap A_2 \cap \ldots \cap A_n)(\omega) = 0 \geq \sum_{j=1}^{n} \mathsf{I}(A_j)(\omega) - n + 1.$$

This proves (2) for any $\omega \in \Omega$.

Second version. We proceed by induction. For $n = 1$, the statement is just $\mathsf{P}(A_1) \geq \mathsf{P}(A_1)$, which is clearly true. Now suppose it holds for all n up to some k, and let us prove it holds for $n = k + 1$ as well. Define $B = \cap_{j=1}^{k} A_j$, so by assumption we have

$$\mathsf{P}(B) \geq \sum_{j=1}^{k} \mathsf{P}\left(A_j\right) - k + 1.$$

Then

$$\begin{aligned}
\mathsf{P}\left(\bigcap_{j=1}^{k+1} A_j\right) &= \mathsf{P}\left(B \cap A_{k+1}\right), \\
&= \mathsf{P}(B) + \mathsf{P}\left(A_{k+1}\right) - \mathsf{P}\left(B \cup A_{k+1}\right), \\
&\geq \mathsf{P}(B) + P\left(A_{k+1}\right) - 1, \\
&\geq \sum_{j=1}^{k} \mathsf{P}\left(A_j\right) - k + 1 + \mathsf{P}\left(A_k\right) - 1, \\
&= \sum_{j=1}^{k+1} \mathsf{P}\left(A_j\right) - (k+1) + 1.
\end{aligned}$$

Hence the property holds for $n = k + 1$ as well. By induction, it holds for all $n = 1, 2, 3, \ldots$.

Problem 5

(a) For $X \sim \mathsf{Pois}(\lambda)$, find $\mathsf{E}\left(2^X\right)$, if it is finite.

(b) For $X \sim \mathsf{Geom}(p)$, find $\mathsf{E}\left(2^X\right)$ (if it is finite), and $\mathsf{E}\left(2^{-X}\right)$ (if it is finite).

Solution: (a) Using LOTUS, we obtain

$$\begin{aligned}
\mathsf{E}\left[2^X\right] &= \sum_{i=0}^{\infty} 2^i \mathsf{P}(X = i), \\
&= \sum_{i=0}^{\infty} 2^i \frac{\lambda^i e^{-\lambda}}{i!}, \\
&= e^{\lambda} \sum_{i=0}^{\infty} \frac{(2\lambda)^i e^{-2\lambda}}{i!}, \\
&= e^{\lambda} \cdot 1 = e^{\lambda}.
\end{aligned}$$

We used the fact that $\frac{(2\lambda)^i e^{-2\lambda}}{i!}$ is the PMF of $\mathsf{Pois}(2\lambda)$, and thus sums to 1.

(b) By LOTUS we have:

$$\begin{aligned}
\mathsf{E}\left[2^X\right] &= \sum_{i=0}^{\infty} 2^i \mathsf{P}(X = i), \\
&= \sum_{i=0}^{\infty} 2^i (1-p)^i p, \\
&= p \sum_{i=0}^{\infty} (2-2p)^i.
\end{aligned}$$

This summation is finite when $0 \leq 2 - 2p < 1$, i.e. when $\frac{1}{2} < p \leq 1$. In that case, using properties of geometric series, we have

$$\mathsf{E}\left[2^X\right] = \frac{p}{2p-1}.$$

Similarly, we have

$$\mathsf{E}\left[2^{-X}\right] = \sum_{i=0}^{\infty} \frac{1}{2^i} \mathsf{P}(X = i) = p \sum_{i=0}^{\infty} \left(\frac{1}{2} - \frac{p}{2}\right)^i.$$

For any $0 < p \leq 1$, $0 \leq \frac{1}{2} - \frac{p}{2} < 1$, and so this summation is finite. Then

$$\mathsf{E}\left[2^{-X}\right] = p \frac{1}{1 - \left(\frac{1}{2} - \frac{p}{2}\right)} = \frac{2p}{p+1}.$$