Problem 1

Let A and B be two events.

1. Show that
$$\mathsf{P}(A) + \mathsf{P}(B) - 1 \leq \mathsf{P}(A \cap B) \leq \mathsf{P}(A \cup B) \leq \mathsf{P}(A) + \mathsf{P}(B).$$

2. The difference $B \setminus A$ is defined to be the set of all elements of B that are not in A. Show that, if $A \subseteq B$, then
$$\mathsf{P}(B \setminus A) = \mathsf{P}(B) - \mathsf{P}(A).$$

3. The symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is defined to be the set of all elements that are in A or B but not both. Show that
$$\mathsf{P}(A \triangle B) = \mathsf{P}(A) + \mathsf{P}(B) - 2\mathsf{P}(A \cap B).$$

Problem 2

Show that, if A and B are independent events, then

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A)\mathsf{P}(B) = 1 - \mathsf{P}(A^c)\mathsf{P}(B^c).$$

Problem 3

Show that $\mathsf{P}(A \mid B) \leq \mathsf{P}(A)$ implies $\mathsf{P}(A \mid B^c) \geq \mathsf{P}(A)$.

Problem 4

Show that if $\mathsf{P}(A) = 1$, then $\mathsf{P}(A \mid B) = 1$ for any B with $\mathsf{P}(B) > 0$.

Problem 5

Show that if A and B are independent and $C = A \cup B$, then A and B are conditionally dependent (that is, A and B are not conditionally independent) given C (as long as $\mathsf{P}(A \cap B) > 0$ and $\mathsf{P}(A \cup B) < 1$), with

$$\mathsf{P}(A \mid B, C) < \mathsf{P}(A \mid C).$$