

## **Homework 8, DATA 556: Due Tuesday, 11/20/2018**

**Alexander Van Roijen**

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Please complete the following:

1. Problem 1 Let  $Z, W$  be bivariate Normal defined as

$$Z = X \quad (1)$$

$$W = \rho X + \sqrt{1 - \rho^2} Y \quad (2)$$

with  $X, Y$  i.i.d  $N(0,1)$  and  $-1 < \rho < 1$ . Find  $E[W|Z]$  &  $\text{Var}(W|Z)$

$$E[W|Z] = E[\rho X + \sqrt{1 - \rho^2} Y | Z = z] = E[\rho X + \sqrt{1 - \rho^2} Y | X = z] \quad (3)$$

$$= \rho z + \sqrt{1 - \rho^2} E[Y] \text{ by linearity of expectation} \quad (4)$$

$$= Z\rho \quad (5)$$

$$\text{Var}(W|Z) = E[W^2|Z] - E[W|Z]^2 = E[(\rho X + \sqrt{1 - \rho^2} Y)^2 | Z = z] - z^2 \rho^2 \quad (6)$$

$$= E[\rho^2 z^2 + (1 - \rho^2) Y^2 + 2\rho z \sqrt{1 - \rho^2} Y] - z^2 \rho^2 \quad (7)$$

$$= \rho^2 z^2 + (1 - \rho^2) E[Y^2] + 2\rho z \sqrt{1 - \rho^2} E[Y] - z^2 \rho^2 \text{ by linearity of expectation} \quad (8)$$

$$\text{We also know } E[Y^2] = \text{Var}(Y) + E[Y]^2 = 1 \quad (9)$$

$$\Rightarrow \text{Var}(W|Z) = \rho^2 z^2 + 1 - \rho^2 + 2\rho z \sqrt{1 - \rho^2} - z^2 \rho^2 = 1 - \rho^2 + 2\rho Z \sqrt{1 - \rho^2} \quad (10)$$

2. Let  $X = (X_1, X_2, X_3, X_4, X_5) \sim \text{Mult}_5(n, p)$  with  $p = (p_1, p_2, p_3, p_4, p_5)$

(a) Find  $E[X_1|X_2]$  and  $\text{Var}(X_1|X_2)$

$$X_1, X_3, X_4, X_5 | X_2 \sim \text{Mult}_4(n - X_2, p') \text{ with} \quad (11)$$

$$p' = (p'_1, p'_3, p'_4, p'_5) \text{ and } p'_i = \frac{p_i}{1 - p_2} \quad (12)$$

$$\text{Now we recognize that } P(X_i = x_i) \text{ in either distribution is } \binom{N'}{x_i} p_1^{x_i} (1 - p_i)^{N' - x_i} \quad (13)$$

$$\text{Where we know that } N' - x_1 = \text{remaining possible successes} \quad (14)$$

$$\text{Thus, we see a binomial distribution for } X_i \quad (15)$$

$$\Rightarrow E[X_1|X_2] = N' * p'_1 = (n - X_2) p'_1 \quad (16)$$

$$\Rightarrow \text{Var}[X_1|X_2] = N' p'_1 (1 - p'_1) = (n - X_2) p'_1 (1 - p'_1) \quad (17)$$

This further makes sense as we expect both values to be functions of our RV  $X_2$

$$(18)$$

(b) Find  $E[X_1|X_2 + X_3]$

Similar to above, we can determine a new multinomial

$$X_1, X_4, X_5|X_2 + X_3 \sim Mult_3(n - (X_2 + X_3), p') \text{ with} \quad (19)$$

$$p' = (p'_1, p'_4, p'_5) \text{ and } p'_i = \frac{p_i}{1 - (p_2 + p_3)} \quad (20)$$

$$\text{By the same logic as above, we determine that } X_1 \sim Binom(N', p'_1) \quad (21)$$

$$\Rightarrow E[X_1|X_2 + X_3] = N' * p'_1 = (n - (X_2 + X_3))p'_1 \quad (22)$$

3. Show that the following version of LOTP follows from Adam's law: for any event A and continuous random variable X with PDF  $f_X$ :

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x)f_X(x)dx \quad (23)$$

$$\text{Let's use indicator variables to prove this} \quad (24)$$

$$I_A = \text{Indicator for when event A occurs} \quad (25)$$

$$E[I_A] = P(A) \text{ and } E[I_A|X = x] = P(A|X = x) \text{ By the fundamental bridge} \quad (26)$$

$$E[I_A] = E[E[I_A|X]] \text{ Adam's Law} \quad (27)$$

$$\text{all together we get} \quad (28)$$

$$P(A) = E[E[I_A|X]] = E[P(A|X = x)] = \int_{-\infty}^{\infty} P(A|X = x)f_X(x)dx \text{ by lotus} \square \quad (29)$$

4. Let  $N \sim Pois(\lambda_1)$  be the number of movies that will be released next year. Suppose that for each movie the number of tickets sold is  $Pois(\lambda_2)$ , independently

(a) Find the mean and the variance of the number of movie tickets that will be sold next year.

We can describe the number of tickets sold overall as the conditional probability on tickets sold per movie given the number of movies released that year. We get the

following

$$\text{tickets sold} = E[M_1 + M_2 + \dots + M_n | N = n] \text{ with } M_i \sim \text{Pois}(\lambda_2) \quad (30)$$

$$= E[M_1 | N = n] \dots + E[M_n | N = n] \text{ by linearity of expectation} \quad (31)$$

$$= n * \lambda_2 \Rightarrow S = N * \lambda_2 \text{ where } S \text{ is a random variable expressing tickets sold next year} \quad (32)$$

$$\Rightarrow E[S] = E[N] * \lambda_2 = \lambda_1 * \lambda_2 \quad (33)$$

$$\text{Var}(S) = \text{Var}(\lambda_2 N) = \lambda_2^2 \text{Var}(N) = \lambda_2^2 * \lambda_1 \quad (34)$$

- (b) Use simulations in R (the statistical programming language) to numerically estimate mean and the variance of the number of movie tickets that will be sold next year assuming that the mean number of movies released each year in the US is 700, and that, on average, 800000 tickets were sold for each movie.

5. Show that if  $E(Y|X) = c$  is a constant, then X and Y are uncorrelated.

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} \quad (35)$$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] \quad (36)$$

$$\text{By Adam's law we get } E[E[Y|X]] = E[Y] \Rightarrow E[Y] = E[c] = c \quad (37)$$

$$E[XY] = \sum_{x \in X} E[XY|X = x]P(X = x) = \sum_{x \in X} xE[Y]P(X = x) = c \sum_{x \in X} xP(X = x) = cE[X] \quad (38)$$

$$\Rightarrow \text{Cov}(X, Y) = E[XY] - E[X]E[Y] = cE[X] - cE[X] = 0 \quad (39)$$

$$\text{Thus X and Y are uncorrelated } \square \quad (40)$$

6. Show that for any random variables X and Y,

$$E[Y|E[Y|X]] = E[Y|X] \quad (41)$$

7. Let Y denote the number of heads in n flips of a coin, whose probability of heads is  $\theta$ .

- (a) Suppose  $\theta$  follows a distribution  $P(\theta) = \text{Beta}(a, b)$ , and then you observe y heads out of n flips. Show algebraically that the mean  $E(\theta|Y = y)$  always lies between the

mean  $E(\theta)$  and the observed relative frequency of heads

$$\min \left\{ E[\theta], \frac{y}{n} \right\} \leq E(\theta|Y = y) \leq \max \left\{ E[\theta], \frac{y}{n} \right\} \quad (42)$$

Here  $E(\theta|Y = y)$  is the mean of the distribution  $P(\theta|Y = y)$ , and  $E(\theta)$  is the mean of the distribution  $P(\theta) = \text{Beta}(a, b)$ .

$$E[\theta|Y = y] = \int_{-\infty}^{\infty} \theta f(\theta|Y = y) d\theta \quad (43)$$

$$f(\theta|Y = y) = \frac{P(Y|\theta) * P(\theta)}{P(Y)} \text{ By bayes rule} \quad (44)$$

$$\text{We recognize that the numerator is a new beta distribution with new } \alpha\beta \quad (45)$$

Resultingly, depending on which one is large, y and n or a and b, the expected alue will lie between

(46)

(b) Show that if  $P(\theta) = \text{Unif}(0, 1)$

We have  $\text{Var}(\theta|Y = y) \leq \text{Var}(\theta)$

Here  $\text{Var}(\theta|Y = y)$  is the variance of the distribution of  $P(\theta|Y = y)$  and  $\text{Var}(\theta)$  is the variance of the distribution  $P(\theta) = \text{Unif}(0, 1)$

8. Let A, B and C be independent random variables with the following distributions:

$$P(A = 1) = 0.4; P(A = 2) = 0.6 \quad (47)$$

$$P(B = -3) = 0.25; P(B = -2) = 0.25; P(B = -1) = 0.25; P(B = 1) = 0.25 \quad (48)$$

$$P(C = 1) = 0.5; P(C = 2) = 0.4; P(C = 3) = 0.1 \quad (49)$$

(a) What is the probability of the quadratic equation  $Ax^2 + Bx + C = 0$  has two real

roots that are different?

$$\text{we are solving } \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (50)$$

$$\text{For our roots to be real and different, we require } B^2 - 4AC > 0 \text{ \&} B^2 - 4AC \neq 0 \quad (51)$$

$$\text{With some thinking, we can see that this is only satisfied in the following cases} \quad (52)$$

$$A = 1, C = 1, B = -3 \quad (53)$$

$$A = 1, C = 2, B = -3 \quad (54)$$

$$A = 2, C = 1, B = -3 \quad (55)$$

$$(56)$$

If A or C were both 2 or either was great than two, we would only have negative roots as  $B^2$  can only take the maximum value of 9 at any time.

Thus we simply add the probabilities of these three joint events

$$P(A = 1, C = 1, B = -3) + P(A = 2, C = 1, B = -3) + P(A = 1, C = 2, B = -3) \quad (57)$$

$$= .4 * .5 * .25 + .6 * .5 * .25 + .4 * .4 * .25 = 0.165 \quad (58)$$

(b) What is the probability that the quadratic equation  $Ax^2 + Bx + C = 0$  has two real

roots and are both strictly positive.

$$\text{we are solving } \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (59)$$

$$\text{For our roots to be real positive, we require } B^2 - 4AC \geq 0 \text{ \& } -B - \sqrt{B^2 - 4AC} > 0 \quad (60)$$

With some thinking, we can see that this is only satisfied in the following cases

$$(61)$$

$$A = 1, C = 1, B = -2 \quad (62)$$

$$A = 1, C = 1, B = -3 \quad (63)$$

$$A = 1, C = 2, B = -3 \quad (64)$$

$$A = 2, C = 1, B = -3 \quad (65)$$

$$(66)$$

We can see that unlike the last problem, we can allow our roots to be one in the same, so we add another case where  $-B - \sqrt{B^2 - 4AC} = 0$

we get the following

$$P(A = 1, C = 1, B = -3) + P(A = 2, C = 1, B = -3) + P(A = 1, C = 2, B = -3) + P(A = 1, C = 1, B = -2) \quad (67)$$

$$= .4 * .5 * .25 + .6 * .5 * .25 + .4 * .4 * .25 + .4 * .5 * .25 = 0.215 \quad (68)$$

Have a great thanksgiving!