Alexander Van Roiden. Homework 1, pg 1

1 a) Show
$$P(A)+P(B)-1 \le P(A\cap B) \le P(A\cup B) \le P(A)+P(B)$$

We know $0 \le P(A) \le 1$, $0 \le P(B) \le 1$, $0 \le P(A\cup B) \le 1$, $0 \le P(A\cap B) \le 1$

and for $\cancel{A \le B}$ $X \le Y \Rightarrow Y - X \ge 0$.

so all probabilities are non-negative $\Rightarrow$ (by definition)

$$P(A\cup B) = P(A)+P(B) - P(A\cap B) \ge 0 \Rightarrow$$

$$P(A)+P(B) \ge P(A\cap B)$$

Further, since $P(A\cap B) \le 1 \Rightarrow P(A)+P(B) - P(A\cap B) \le P(A)+P(B)$

Even further, since $0 \le P(A\cup B) \le 1 \Rightarrow P(A\cup B) \ge P(A\cap B)$ as

$$P(A\cup B) = P(A)+P(B) - P(A\cap B) \ge 0$$

so we have $P(A\cap B) \le P(A\cup B) \le P(A)+P(B)$

Now, since we know $0 \le P(A)+P(B) - P(A\cap B) \le 1$ and $0 \le P(A\cap B) \le 1$

$$\Rightarrow -1 \le P(A)+P(B) - P(A\cap B) - 1 \le 0 \Rightarrow P(A)+P(B) - 1 \le P(A\cap B)$$

$$\Rightarrow P(A)+P(B) - 1 \le P(A\cap B) \le P(A\cup B) \le P(A)+P(B) \blacksquare$$

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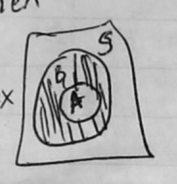
b) Show if $A \subseteq B$, then $P(B\setminus A) = P(B) - P(A)$

$P(B\setminus A) = P(B\cap A^c)$, since $\forall x \in A$ are also $x \in B$ $(A \subseteq B)$ then

$B \cap A^c = \forall x \in B$ s.t. $x \notin A$, $\Rightarrow P(B\setminus A) = P(B) - P(A)$ $\square$

ex 

① $A \cap B^c \cap B \cap A^c = \underbrace{A \cap A^c}_{\emptyset} \cap \underbrace{B \cap B^c}_{\emptyset} = \emptyset$

c) show $P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$\Rightarrow P(A \Delta B) = P((A \cap B^c) \cup (B \cap A^c)) = \cancel{P(A \cap B^c) + P(B \cap A^c) - P(}$

$= P(A \cap B^c) + P(B \cap A^c) + P(A \cap B^c \cap B \cap A^c)^{①} = P(A \cap B^c) + P(B \cap A^c)$

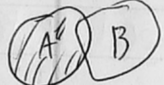
2 cases

Case 1: $P(A \cap B) = P(\emptyset) = 0 \Rightarrow P(A \cap B^c) = P(A)$ & $P(B \cap A^c) = P(B)$

$\Rightarrow P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$

(0)

Case 2: $P(A \cap B) > 0 \Rightarrow P(A \cap B^c) = P(A) - P(A \cap B)$



$P(B \cap A^c) = P(B) - P(B \cap A) \Rightarrow$

$P(A \setminus B) = P(A) - P(A \cap B)$

$P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$ ☒

I realize now that the cases were irrelevant! Sorry!

2) if A & B are independent, show $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

$= 1 - P(A^c)P(B^c)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ by definition $\Rightarrow P(A \cup B) = P(A) + P(B) - P(A)P(B)$

by def of independent

$P(A) + P(B) - P(A)P(B) = P(A)(1 - P(B)) + P(B) = P(A)(P(B^c)) + P(B)$

$= P(A)P(B^c) + (1 - P(B^c)) = P(B^c)(P(A) - 1) + 1 = P(B^c) - P(A^c) + 1$

$= 1 - P(B^c)P(A^c)$ ☒

Alexander Van Roijen HW 1 pg 2

1

3) show $P(A|B) \leq P(A) \Rightarrow P(A|B^c) \geq P(A)$

$$P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)} \geq P(A) = P(A \cap B^c) \geq P(A) \cdot P(B^c) = P(A)(1-P(B))$$

$$P(A) - P(A \cap B) \geq P(A) - P(A)P(B)$$

Need to get to here from 1

1 $P(A|B) = \frac{P(A \cap B)}{P(B)} \leq P(A) \Rightarrow P(A \cap B) \leq P(A) \cdot P(B)$

$= P(A \cap B) \leq P(A) - P(A)P(B^c)$

$= P(A)P(B^c) \leq P(A) - P(A \cap B)$

$P(A) \cdot P(B^c) = P(A)(1 - P(B))$

$= P(A) - P(A) \cdot P(B) \leq P(A) - P(A \cap B)$ ▨

4) if $P(A) = 1$ then $P(A|B) = 1$ for any B w/ $P(B) > 0$

By definition $P(A|B) = \frac{P(A \cap B)}{P(B)}$, further we have $P(B) > 0$ and $P(A) = 1$,

lets assume $B \not\subseteq A \Rightarrow \exists\, b \in B$ s.t. $b \notin A \Rightarrow P(A^c \cap B) > 0$

~~$P(A \cap B) = P(A) + P(B) - P(A \cap B) = P(A$~~ However, $P(A) = 1 \Rightarrow P(A^c) = P(\emptyset) = 0$

$P(\emptyset \cap B) = P(\emptyset) > 0$? contradiction $\Rightarrow B \subseteq A$. Thus

$$P(A \cap B) = P(B) \Rightarrow A(A|B) = \frac{P(B)}{P(B)} = 1$$ ▨

5) ~~Problem~~ if A & B indep, and $C = A \cup B$ then A & B are <u>not</u> conditionally independent given C

further ($P(A \cap B) > 0$ and $P(A \cup B) < 1$)

NTS $P(A \mid B \cap C) < P(A \mid C)$

$$= \frac{P(A \cap B \cap (A \cup B))}{P(B \cap (A \cup B))} < \frac{P(A \cap (A \cup B))}{P(A \cup B)} \Rightarrow \frac{P(A \cap B)}{P(B)} < \frac{P(A)}{P(A \cup B)}$$ by absorption laws

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Absorption laws:

$A \cap A \cup B \Leftrightarrow A$ (if and only if)

$\Rightarrow$ $x \in A \cap (A \cup B) \Rightarrow x \in A$ and $(x \in A$ or $x \in B) \Rightarrow$ $x \in A$ □

$\Leftarrow$ $x \in A \Rightarrow x \in A$ or $B \Rightarrow x \in A$ and $x \in A$ or B

□

$$= P(A) < \frac{P(A)}{P(A \cup B)}$$

since A indep B

and since $P(A \cup B) < 1$

~~$P(A$~~ $< P($

$P(A \mid B, C) < P(A \mid C)$

as any $x \in \mathbb{R}$ divided by $0 < y < 1$ is $> x$.

□