

Homework 3, DATA 556: Due Tuesday, 10/16/2018

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Please complete the following:

1. Problem 1

- (a) Find the mean and the variance of a Discrete Uniform random variable on $1, 2, \dots, n$.

Let $X \sim \text{DUnif}$

$$P(X = x) = \frac{1}{n} \text{ by the def of uniform distrib} \quad (1)$$

$$E[X] = \sum_{j=0}^n P(X = j) * j = \sum_{j=0}^n \frac{1}{n} * j = \frac{1}{n} * \sum_{j=0}^n j = \frac{1}{n} * \frac{n(n+1)}{2} = \frac{n+1}{2} \quad (2)$$

It is a similar proof with a different substitution for the expected value of X squared (3)

$$E[X^2] = \sum_{j=0}^n P(X = j) * (j^2) \text{ LOTUS} \quad (4)$$

$$= \sum_{j=0}^n \frac{1}{n} * (j^2) = \frac{1}{n} * \sum_{j=0}^n (j^2) = \frac{1}{n} * \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6} \quad (5)$$

$$E[X^2] - (E[X])^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \quad (6)$$

$$\frac{4 * (n+1)(2n+1)}{24} - \left(\frac{6 * (n+1)^2}{24}\right) = \frac{n^2 - 1}{12} \quad (7)$$

- (b) Let X be a discrete random variable that satisfies the properties listed in the homework. Find $E[X]$

$$\text{We have a symmetry such that } P(X = x) = P(X = -x) \quad (8)$$

$$E[X] = \sum_{x=-n}^n P(X = x) * x \quad (9)$$

$$= P(X = -n) * -n + P(X = -(n-1)) * (-(n-1)) + \dots + P(X = n) * n \quad (10)$$

$$= -1 * (P(X = -n) * n - P(X = n) * n) + \dots P(X = 0) * 0 \quad (11)$$

$$= -1 * (P(X = n) * n - P(X = n) * n) + \dots P(X = 0) * 0 = 0 + 0 + \dots + 0 \quad (12)$$

$$\text{due to the symmetry of the probs} \quad (13)$$

$$\Rightarrow E[X] = 0 \quad (14)$$

2. We have X with PMF $P(X = k) = \frac{-1 * p^k}{\log(1-p) * k}$ for $k = 1, 2, \dots$. Here p is a parameter with $0 < p < 1$. Find the mean and the variance of X

First, note that the value $\frac{-1}{\log(1-p)}$ is a positive constant

$$E[X] = \sum_{k=1}^{\infty} P(X = k) * k = \sum_{k=1}^{\infty} \frac{-1 * p^k}{\log(1-p) * k} * k = \sum_{x=1}^{\infty} \frac{-1 * p^k}{\log(1-p)} \quad (15)$$

$$= \frac{-1}{\log(1-p)} \sum_{x=1}^{\infty} p^k \quad (16)$$

$$\Rightarrow \sum_{x=1}^{\infty} p^k = S \Rightarrow S * p = \sum_{x=2}^{\infty} p^k \Rightarrow S - Sp = p \Rightarrow S = \frac{p}{(1-p)} \quad (17)$$

$$\Rightarrow E[X] = \frac{-1}{\log(1-p)} * \frac{p}{(1-p)} \text{ similarly, the } E[X^2] \text{ is proved in a similar manner except} \quad (18)$$

$$S - Sp = \sum_{x=1}^{\infty} p^k \quad (19)$$

which we already proved so by substituting we get (20)

$$S - Sp = \frac{p}{(1-p)} \Rightarrow S = \frac{p}{(1-p)^2} \quad (21)$$

$$\Rightarrow E[X^2] = \frac{-1}{\log(1-p)} * \frac{p}{(1-p)^2} \quad (22)$$

$$\Rightarrow Var(X) = E[X^2] - E[X]^2 = \frac{p(p - \log(1-p))}{\log(1-p)^2 * (1-p)^2} \quad (23)$$

3. (a) Use LOTUS to show that for $X \sim \text{Pois}(\lambda)$ and any function $g(\cdot)$, $E[Xg(X)] = \lambda E[g(X+1)]$

$$E[Xg(X)] = \sum_{x=0}^{\infty} x * g(x) \frac{e^{-\lambda} * \lambda^x}{x!} = \sum_{x=1}^{\infty} g(x) * \lambda \frac{e^{-\lambda} * \lambda^{x-1}}{(x-1)!} \quad (24)$$

now let there be a substitution such that $j = x-1$ (25)

$$= \sum_{j=0}^{\infty} g(j+1) * \lambda \frac{e^{-\lambda} * \lambda^j}{j!} = \lambda * \sum_{j=0}^{\infty} g(j+1) \frac{e^{-\lambda} * \lambda^j}{j!} \quad (26)$$

considering $g(x)$ is any valid function, we conclude (27)

$$E[Xg(X)] = \lambda * E[g(X+1)] \quad (28)$$

We bring this back to this form using LOTUS and our lack of knowledge on $g(x)$ (29)

(b) Find the third moment $E(X^3)$ for $X \sim \text{Pois}(\lambda)$

$$\text{let } g(x) = x^2 \Rightarrow E(Xg(X)) = E[g(X+1)] = \lambda * E[(X+1)^2] \quad (30)$$

$$= \lambda * E[X^2 + 2X + 1] = \lambda * (E[X^2] + 2E[X] + 1) \quad (31)$$

using the known properties of the variance and mean of the uniform dist we get

$$(32)$$

$$= \lambda * (\lambda^2 + \lambda + 2\lambda + 1) = \lambda^3 + 3\lambda^2 + \lambda \quad (33)$$

4. Show that for any events $A_1 \dots A_n$, $P(A_1 \cap \dots \cap A_n) \geq \sum_{j=1}^n P(A_j) - n + 1$

lets prove this through induction

$$\text{Base case, } n=1: P(A) \geq P(A) - 1 + 1 \Rightarrow P(A) \geq P(A) \quad (34)$$

$$\text{Now we assume this is true for } 1 \dots n, \text{ NTS for } n+1 \quad (35)$$

$$n+1 \text{ case: we have } \sum_{j=1}^{n+1} P(A_j) - (n+1) + 1 = \sum_{j=1}^n P(A_j) + P(A_{n+1}) - (n+1) + 1 \quad (36)$$

$$\text{Now notice we can rewrite } P(A_1 \cap \dots \cap A_{n+1}) = P(B \cap A_{n+1}) \text{ where } B = A_1 \cap \dots \cap A_n \quad (37)$$

$$\Rightarrow P(B \cap A_{n+1}) = P(B) + P(A_{n+1}) - P(A_{n+1} \cup B) \text{ by definition} \quad (38)$$

$$\text{By the induction hypothesis, we know } P(B) \geq \sum_{j=1}^n P(A_j) - n + 1 \quad (39)$$

$$\text{That means all that is left to show is } P(A_{n+1}) - P(A_{n+1} \cup B) \geq P(A_{n+1}) - 1 \quad (40)$$

$$\Rightarrow P(A_{n+1}) - P(A_{n+1} \cup B) \geq P(A_{n+1}) - 1 \Rightarrow -P(A_{n+1} \cup B) \geq -1 \Rightarrow P(A_{n+1} \cup B) \leq 1 \quad (41)$$

$$\text{We know this inequality is true as } 0 \leq P(X) \leq 1 \quad (42)$$

$$\text{Thus, we have proved the induction step. } \square \quad (43)$$

5. For $X \sim \text{Pois}(\lambda)$, find $E[2^X]$, if it is finite.

$$E[2^X] = \sum_{n=0}^{\infty} \frac{2^n e^{-\lambda} \lambda^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda} (2 * \lambda)^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(2 * \lambda)^n}{n!} \quad (44)$$

$$\text{now let } \lambda' = 2 * \lambda \Rightarrow e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda')^n}{n!} \quad (45)$$

we recognize this is the taylor expansion of the exponential function (46)

$$\text{we get } \sum_{n=0}^{\infty} \frac{(\lambda')^n}{n!} = e^{\lambda'} = e^{2\lambda} \Rightarrow E[2^X] = e^{-\lambda} e^{2\lambda} = e^{\lambda} \square \quad (47)$$

6. For $X \sim \text{Geom}(p)$, find $E[2^X]$ and $E[2^{-X}]$ (if it is finite).

$$E[2^X] = \sum_{n=1}^{\infty} 2^n (1-p)^{n-1} * p \text{ let } S = \sum_{n=1}^{\infty} 2^n (1-p)^{n-1} = 2 + 4(1-p) + \dots \quad (48)$$

$$\Rightarrow S * 2(1-p) = 4(1-p) + 8(1-p)^2 + \dots \Rightarrow S - S * 2(1-p) = 2 \quad (49)$$

$$S - 2S + 2Sp = -S + 2Sp = S(-1 + 2p) = 2 \Rightarrow S = \frac{2}{2p-1} \Rightarrow E[2^X] = p * S = \frac{2p}{2p-1} \quad (50)$$

$$\text{Similarly } E[2^{-X}] = \sum_{n=1}^{\infty} 2^{-n} (1-p)^{n-1} * p \text{ let } S = \sum_{n=1}^{\infty} 2^{-n} (1-p)^{n-1} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}}{2^n} \quad (51)$$

$$= \frac{(1-p)^0}{2^1} + \frac{(1-p)^1}{2^2} + \dots \Rightarrow S * \frac{1-p}{2} = \frac{(1-p)^1}{2^2} + \frac{(1-p)^2}{2^3} \quad (52)$$

$$\Rightarrow S - S * \frac{1-p}{2} = S(1 - \frac{1-p}{2}) = \frac{1+p}{2} = \frac{1}{2} \Rightarrow S = \frac{1}{1+p} \quad (53)$$

$$\Rightarrow E[2^{-X}] = \frac{p}{1+p} \square \quad (54)$$