# Homework 3, DATA 556: Due Tuesday, 10/16/2018

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Please complete the following:

1. Problem 1

(a) Find the mean and the variance of a Discrete Uniform random variable on 1, 2, ...,n. Note that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ are well known summations that can be proved via mathematical induction.

Refer to http://mathforum.org/library/drmath/view/56920.html for an example.

Let $X \sim$ DUnif

$$P(X = x) = \frac{1}{n} \text{ by the def of uniform distrib} \tag{1}$$

$$E[X] = \sum_{j=1}^{n} P(X = j) * j = \sum_{j=1}^{n} \frac{1}{n} * j = \frac{1}{n} * \sum_{j=1}^{n} j = \frac{1}{n} * \frac{n(n+1)}{2} = \frac{n+1}{2} \tag{2}$$

It is a similar proof with a different subsitution for the expected value of X squared (3)

$$E[X^2] = \sum_{j=1}^{n} P(X = j) * (j^2) \text{ LOTUS} \tag{4}$$

$$= \sum_{j=1}^{n} \frac{1}{n} * (j^2) = \frac{1}{n} * \sum_{j=1}^{n} (j^2) = \frac{1}{n} * \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6} \tag{5}$$

$$E[X^2] - (E[X])^2 = \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 \tag{6}$$

$$\frac{4 * (n+1)(2n+1)}{24} - (\frac{6 * (n+1)^2}{24}) = \frac{n^2 - 1}{12} \tag{7}$$

(b) Let X be a discrete random variable that satisfies the properties listed in the homework. Find E[X]

$$\text{We have a symmetry such that } P(X = x) = P(X = -x) \tag{8}$$

$$E[X] = \sum_{x=-n}^{n} P(X = x) * x \tag{9}$$

$$= P(X = -n) * -n + P(X = -(n-1)) * (-(n-1)) + ... + P(X = n) * n \tag{10}$$

$$= -1 * (P(X = -n) * n - P(X = n) * n) + ...P(X = 0) * 0 \tag{11}$$

$$= -1 * (P(X = n) * n - P(X = n) * n) + ...P(X = 0) * 0 = 0 + 0 + ... + 0 \tag{12}$$

$$\text{due to the symmetry of the probs} \tag{13}$$

$$=> E[X] = 0 \tag{14}$$

2. We have X with PMF $P(X = k) = \frac{-1*p^k}{log(1-p)*k}$ for k = 1, 2, . . . .. Here p is a parameter with 0 ¡ p ¡ 1. Find the mean and the variance of X

First, note that the value $\frac{-1}{log(1-p)}$ is a positive constant

$$E[X] = \sum_{k=1}^{\infty} P(X = k) * k = \sum_{k=1}^{\infty} \frac{-1 * p^k}{log(1-p) * k} * k = \sum_{x=1}^{\infty} \frac{-1 * p^k}{log(1-p)} \tag{15}$$

$$= \frac{-1}{log(1-p)} \sum_{x=1}^{\infty} p^k \tag{16}$$

$$=> \sum_{x=1}^{\infty} p^k = S => S * p = \sum_{x=2}^{\infty} p^k => S - Sp = p => S = \frac{p}{(1-p)} \tag{17}$$

$$=> E[X] = \frac{-1}{log(1-p)} * \frac{p}{(1-p)} \text{similarly, the } E[X^2] \text{is proved in a similar manner except} \tag{18}$$

$$S - Sp = \sum_{x=1}^{\infty} p^k \tag{19}$$

$$\text{which we already proved so by substituting we get} \tag{20}$$

$$S - Sp = \frac{p}{(1-p)} => S = \frac{p}{(1-p)^2} \tag{21}$$

$$=> E[X^2] = \frac{-1}{log(1-p)} * \frac{p}{(1-p)^2} \tag{22}$$

$$=> Var(X) = E[X^2] - E[X]^2 = \frac{p(p - log(1-p))}{log(1-p)^2 * (1-p)^2} \tag{23}$$

3. (a) Use LOTUS to show that for X ∼ Pois($\lambda$) and any function g(.), $E[Xg(X)] = \lambda E[g(X + 1)]$

$$E[Xg(X)] = \sum_{x=0}^{\infty} x * g(x) \frac{e^{-\lambda} * \lambda^x}{x!} = \sum_{x=1}^{\infty} g(x) * \lambda \frac{e^{-\lambda} * \lambda^{x-1}}{(x-1)!} \tag{24}$$

$$\text{now let there be a substitution such that j = x-1} \tag{25}$$

$$= \sum_{j=0}^{\infty} g(j+1) * \lambda \frac{e^{-\lambda} * \lambda^j}{j!} = \lambda * \sum_{j=0}^{\infty} g(j+1) \frac{e^{-\lambda} * \lambda^j}{j!} \tag{26}$$

$$\text{We see how we can convert this back to a form we understand using LOTUS} \tag{27}$$

$$E[Xg(X)] = \lambda * E[g(X + 1)] \tag{28}$$

$$\tag{29}$$

(b) Find the third moment E($X^3$) for X Pois($\lambda$)

$$\text{let } g(x) = x^2 => E(Xg(X)) = \lambda E[g(X+1)] = \lambda * E[(X+1)^2] \tag{30}$$

$$= \lambda * E[X^2 + 2X + 1] = \lambda * (E[X^2] + 2E[X] + 1) \tag{31}$$

using the known properties of the variance and mean of the uniform dist we get

$$\tag{32}$$

$$= \lambda * (\lambda^2 + \lambda + 2\lambda + 1) = \lambda^3 + 3\lambda^2 + \lambda \tag{33}$$

4. Show that for any events $A_1..A_n$, $P(A_1 \cap ... \cap A_n) \geq \sum_{j=1}^{n} P(A_j) - n + 1$

lets prove this through induction

$$\text{Base case, n=1:} P(A) \geq P(A) - 1 + 1 => P(A) \geq P(A) \tag{34}$$

$$\text{Now we assume this is true for 1 ... n, NTS for n+1} \tag{35}$$

$$\text{n+1 case:we have} \sum_{j=1}^{n+1} P(A_j) - (n+1) + 1 = \sum_{j=1}^{n} P(A_j) + P(A_{n+1}) - (n+1) + 1 \tag{36}$$

$$\text{Now notice we can rewrite} P(A_1 \cap .. \cap A_{n+1}) = P(B \cap A_{n+1}) \text{where} B = A_1 \cap ... \cap A_n \tag{37}$$

$$=> P(B \cap A_{n+1}) = P(B) + P(A_{n+1}) - P(A_{n+1} \cup B) \text{by definition} \tag{38}$$

$$\text{By the induction hypothesis, we know} P(B) \geq \sum_{j=1}^{n} P(A_j) - n + 1 \tag{39}$$

$$\text{That means all that is left to show is} P(A_{n+1}) - P(A_{n+1} \cup B) \geq P(A_{n+1}) - 1 \tag{40}$$

$$=> P(A_{n+1}) - P(A_{n+1} \cup B) \geq P(A_{n+1}) - 1 => -P(A_{n+1} \cup B) \geq -1 => P(A_{n+1} \cup B) \leq 1 \tag{41}$$

$$\text{We know this inequality is true as } 0 \leq P(X) \leq 1 \tag{42}$$

$$\text{Thus, we have proved the induction step.} \square \tag{43}$$

5. For X $\sim$ Pois($\lambda$), find $E[2^X]$ , if it is finite.

$$E[2^X] = \sum_{n=0}^{\infty} \frac{2^n e^{-\lambda} \lambda^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda}(2*\lambda)^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(2*\lambda)^n}{n!} \tag{44}$$

$$\text{now let } \lambda' = 2*\lambda => e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda')^n}{n!} \tag{45}$$

$$\text{we recognize this is the taylor expansion of the exponential function} \tag{46}$$

$$\text{we get} \sum_{n=0}^{\infty} \frac{(\lambda')^n}{n!} = e^{\lambda'} = e^{2\lambda} => E[2^X] = e^{-\lambda} e^{2\lambda} = e^{\lambda} \square \tag{47}$$

6. For X $\sim$ Geom(p), find $E[2^X]$ and $E[2^{-X}]$ (if it is finite).

$$E[2^X] = \sum_{n=1}^{\infty} 2^n (1-p)^{n-1} * p \text{ let} S = \sum_{n=1}^{\infty} 2^n (1-p)^{n-1} = 2 + 4(1-p) + ... \tag{48}$$

$$=> S * 2(1-p) = 4(1-p) + 8(1-p)^2 + .. => S - S * 2(1-p) = 2 \tag{49}$$

$$S - 2S + 2Sp = -S + 2Sp = S(-1+2p) = 2 => S = \frac{2}{2p-1} => E[2^X] = p * S = \frac{2p}{2p-1} \tag{50}$$

$$\text{Similarly } E[2^{-X}] = \sum_{n=1}^{\infty} 2^{-n} (1-p)^{n-1} * p \text{ we let} S = \sum_{n=1}^{\infty} 2^{-n} (1-p)^{n-1} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}}{2^n} \tag{51}$$

$$= \frac{(1-p)^0}{2^1} + \frac{(1-p)^1}{2^2} + ... => S * \frac{1-p}{2} = \frac{(1-p)^1}{2^2} + \frac{(1-p)^2}{2^3} \tag{52}$$

$$=> S - S * \frac{1-p}{2} = S(1 - \frac{1-p}{2}) = S * \frac{1+p}{2} = \frac{1}{2} => S = \frac{1}{1+p} \tag{53}$$

$$=> E[2^{-X}] = \frac{p}{1+p} \square \tag{54}$$