Problem 1

Let (Z, W) be Bivariate Normal, defined as

$$\begin{aligned} Z &= X, \\ W &= \rho X + \sqrt{1-\rho^2}\, Y, \end{aligned}$$

with X, Y i.i.d. $\mathsf{N}(0, 1)$, and $-1 < \rho < 1$. Find $\mathsf{E}(W \mid Z)$ and $\mathsf{Var}(W \mid Z)$.

Problem 2

Let $\mathbf{X} = (X_1, X_2, X_3, X_4, X_5) \sim \mathsf{Mult}_5(n, \mathbf{p})$ with $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$.

(a) Find $\mathsf{E}(X_1 \mid X_2)$ and $\mathsf{Var}(X_1 \mid X_2)$.

(b) Find $\mathsf{E}(X_1 \mid X_2 + X_3)$.

Problem 3

Show that the following version of LOTP follows from Adam's law: for any event A and continuous random variable X with PDF f_X:

$$\mathsf{P}(A) = \int_{-\infty}^{\infty} \mathsf{P}(A \mid X = x) f_X(x)\, dx.$$

Problem 4

Let $N \sim \mathsf{Pois}(\lambda_1)$ be the number of movies that will be released next year. Suppose that for each movie the number of tickets sold is $\mathsf{Pois}(\lambda_2)$, independently.

(a) Find the mean and the variance of the number of movie tickets that will be sold next year.

(b) Use simulations in R (the statistical programming language) to numerically estimate mean and the variance of the number of movie tickets that will be sold next year assuming that the mean number of movies released each year in the US is 700, and that, on average, 800000 tickets were sold for each movie.

Problem 5

Show that if $\mathsf{E}(Y \mid X) = c$ is a constant, then X and Y are uncorrelated. Hint: Use Adam's law to find $\mathsf{E}(Y)$ and $\mathsf{E}(XY)$.

Problem 6

Show that for any random variables X and Y,

$$\mathsf{E}(Y \mid \mathsf{E}(Y \mid X)) = \mathsf{E}(Y \mid X).$$

Hint: use Adam's law with extra conditioning.

Problem 6

Let Y denote the number of heads in n flips of a coin, whose probability of heads is θ.

(a) Suppose θ follows a distribution $\mathsf{P}(\theta) = \mathsf{Beta}(a, b)$, and then you observe y heads out of n flips. Show algebraically that the mean $\mathsf{E}(\theta \mid Y = y)$ always lies between the mean $\mathsf{E}(\theta)$ and the observed relative frequency of heads:

$$\min\left\{\mathsf{E}(\theta), \frac{y}{n}\right\} \leq \mathsf{E}(\theta \mid Y = y) \leq \max\left\{\mathsf{E}(\theta), \frac{y}{n}\right\}.$$

Here $\mathsf{E}(\theta \mid Y = y)$ is the mean of the distribution $\mathsf{P}(\theta \mid Y = y)$, and $\mathsf{E}(\theta)$ is the mean of the distribution $\mathsf{P}(\theta) = \mathsf{Beta}(a, b)$.

(b) Show that, if θ follows a uniform distribution,

$$\mathsf{P}(\theta) = \mathsf{Unif}(0, 1),$$

we have

$$\mathsf{Var}(\theta \mid Y = y) \leq \mathsf{Var}(\theta).$$

Here $\mathsf{Var}(\theta \mid Y = y)$ is the variance of the distribution $\mathsf{P}(\theta \mid Y = y)$, and $\mathsf{Var}(\theta)$ is the variance of the distribution $\mathsf{P}(\theta) = \mathsf{Unif}(0, 1)$.

Problem 7

Let A, B and C be independent random variables with the following distributions:

$$\mathsf{P}(A = 1) = 0.4,\ \mathsf{P}(A = 2) = 0.6$$
$$\mathsf{P}(B = -3) = 0.25,\ \mathsf{P}(B = -2) = 0.25,\ \mathsf{P}(B = -1) = 0.25,\ \mathsf{P}(B = 1) = 0.25$$
$$\mathsf{P}(C = 1) = 0.5,\ \mathsf{P}(C = 2) = 0.4,\ \mathsf{P}(C = 3) = 0.1$$

(a) What is the probability that the quadratic equation

$$Ax^2 + Bx + C = 0$$

has two real roots that are different?

(b) What is the probability that the quadratic equation

$$Ax^2 + Bx + C = 0$$

has two real roots that are both strictly positive?