Notes on Evaluation of Game Trees

sam t, 2022-01

1 Alpha Beta Search

1.1 Game Trees

We wish to determine the minimax score of a zero-sum two-player full-information game tree. Say the tree has depth D and branching factor B and that its leaves are valued in the real numbers. We write r for the root node, $m(n)$ for the result of applying move m to node n, and $s(n)$ for the value of a leaf node n. The depth $d(n)$ of a node obeys $d(r) = 0$ and $d(m(n)) = d(n) + 1$.

Without loss, the first player seeks to maximize the score. Then we extend s to non-leaf nodes n via $s(n) = \max_m{}^{d(n)} s(m(n))$. Here, $\max^i$ denotes max or min as i is even or odd. this inductive definition of tree value translates directly to the recursive **minimax** algorithm evaluating game trees.

1.2 Alpha Beta Search

We can often inspect fewer than all B^D leaves. The crucial observation is that, whether or not $(a \wedge b) \geq c$ holds or fails independently of d — and when it does hold, $(a \wedge b) \vee (c \wedge d)$ also doesn't depend on d. So we may verify and exploit the hypothesis while only ever inspecting only 3 of the 4 leaves. What permits these savings is that, though we wish to evaluate $a \wedge b$ exactly, we merely seek to bound $c \wedge d$. The same idea may save more nodes in more elaborate cases. For instance, imagine a 16-leaf binary tree each of whose four 4-leaf sub-trees verifies the above bound and whose sub-trees' values also verify the above bound. Then we may get by while inspecting only 9 of the 16 leaves.

But we may improve still further over recursive application of that 3/4 factor. Say $(a \wedge b) \geq c$ and $(a \wedge b) \leq (e \wedge f)$ and $(A \vee B) \geq C$ and $(a \vee b) \geq (A \vee B)$. Then

$$\begin{aligned} &(((a \wedge b) \vee (c \wedge d)) \ \wedge \ ((e \wedge f) \vee (g \wedge h))) \quad \vee \\ &(((A \wedge B) \vee (C \wedge D)) \ \wedge \ ((E \wedge F) \vee (G \wedge H))) \end{aligned}$$

does not depend on d, g, h, D, E, F, G, H. Naïve recursion with the 3/4s would have inspected the leaf g. But due to an "out-of-phase" or "staggered" application of the key observation, the hypothesis $((a \wedge b) \ \vee \ (c \wedge d)) \ \leq \ (e \wedge f))$ implies we may ignore the entire pair $(g \wedge h)$ In this favorable circumstance, we may get by while inspecting only 8 of the 16 leaves.

WRITE DOWN ALPHA BETA

1.3 Crude Analyses

BEST CASE ANALYSIS: $b^{d/2}1^{d/2}$

VERY ROUGH AVERAGE CASE ANALYSIS: $b^{d/2}(b/2)^{d/2}$

1.4 Move Ordering, Interior Estimators

1.5 Principle Variation Search

A further refinement

2 Analysis with Markov Scores

2.1 Model

We now model the distribution of leaf values for games such as chess. (A chess engine might seek to approximate the value of a tree with $B = D = 30$).

Chess often features nodes "quiet" in the sense that their direct children are pretty similar in quality and score. The closed, positional games of Tigran Petrosian and Anatoly Karpov give a wealth of examples. Due to quietness, nodes with late common ancestors tend to have similar values. We model this by using the game tree's underlying graph as a graphical model of "hidden values" $h(n)$ that agree with the score $s(n)$ at the leaves. Specifically, let $h(r) = 0$ and that $h(m(n)) \sim L(h(n), 1)$ obeys a two-sided variance-1 exponential distribution around $h(n)$. We assume the steps $\epsilon(m, n) = h(m(n)) - h(n)$ are all independent. (If we allow the variance to depend on depth, then the case where the pre-leaf-to-leaf steps account for all the variance is the case where the leaf scores are all independent).

Let's write $v(n) = s(n) - h(n)$ for the centered sub-tree value, which is interesting because the centered values at a given depth are all i.i.d. and because $v(r) = s(r)$. We have $v(n) = 0$ on leaves and

$$v(n) = \max_m{}^{d(n)} v(m(n)) + \epsilon(m, n)$$

on branches. Suppose that $v(m(n))$ has mean μ and variance σ^2. Meanwhile, $\epsilon(m, n)$ has mean 0 and variance 1.

We expect the maximum to be near the top $1/(B + 1) = 1/(2B')$th quantile of the distribution of $v(m(n)) + \epsilon(m, n)$. This distribution has mean μ and variance $\sigma^2 + 1$. In turn, this quantile is roughly

$$\tilde{\mu} \approx \mu + \sqrt{\sigma^2 + 1} \cdot \log(B')$$

The typical scale will also be $\sqrt{\sigma^2 + 1} \cdot 1/B'$. We see that the scale will converge to a fixed point of $\tilde{\sigma}^2 \approx (\tilde{\sigma}^2 + 1)/B'$ or

$$\tilde{\sigma}^2 \approx 1/B'$$

Since we alternate maximizations and minimizations, μ doesn't accumulate much net drift. This, for nodes far above the leaves, including for the root, $v(n)$ tends to lie within some interval of width roughly $(\log(B') + C)/B'$ for some universal constant C.

2.2 Naïve Ordering