Numerology

I've long wanted to take a math class where on the nth day we learn about the number n.[1] These notes attempt to give the reader such an experience. Alas, they are by no means complete.

[1] Looking back, I think I was influenced by John Baez' delightful, fact-dense talks on the numbers five, eight, and twenty-four.

In particular, we present one perspective per n while ignoring n's other important aspects. Any text on numbers must omit something: Guy observes that, since we burden the several small numbers[2] with too many jobs, most small numbers have multiple personalities. In fact, one of the main lessons I learned while an undergraduate was to spend less time compressing my understanding toward One True viewpoint and more time absorbing Many Clarifying viewpoints. The Fourier transform, for instance, is significant both because it diagonalizes translation and because it diagonalizes differentiation. The former viewpoint leads to the representation theory of groups; the latter, the harmonic theory of metric manifolds. Some sheaves simply lack global sections.

[2] n so small, say, that we would recognize n apples by sight alone

With this in mind, the viewpoints I choose to discuss are these: **Zero** names what we'd like to ignore and hence is an accomplice to abstraction. **One** demarcates existence and uniqueness and so highlights the role of coordinates. Because **Two** counts the set $\{\text{algebra}, \text{geometry}\}$, quadratic forms pervade mathematics. **Three**[3] is the beginning of complexity, for degree-three graphs are non-trivial.

[3] also known as Arnold's constant; we will ignore this thread of thought

Most of these notes I can trace to conversations with friends and friendly experts, especially Dean Young and Andrew Snowden. I hope that these notes provide entertainment despite their un-originality.

Zero: Analysis through Equality

Approximation and Abstraction

What do the quotient operations V/W of vector spaces and $S/(\sim)$ of sets have in common? They are both coequalizers; the former, of the two arrows $W \hookrightarrow V$ and $W \to 0 \to V$, and the latter, of $(\sim) \hookrightarrow S \times S \to S$ by left or right projection. For sets, we must specify both sides of the equations $a = b$ to make true; for vector spaces, by contrast, we can get away with specifying only one side of $a - b = 0$ due to the presence of an initial-and-terminal object, 0. Categories with such a **zero object** include those of groups, based spaces, and Noetherian lattices.

Naming our ignorance gives us valuable flexibility as we analyze structures into simpler parts. Indeed, the two grand methods of simplification are **approximation**, where we posit false-but-useful properties, and **abstraction**, where we discard true-but-useless distinctions. The concept of 0 gives us the language for both activities, as illustrated in the concept of **exact sequences** $F \to E \to B$. These are just solid commuting squares as below with respect to which F is terminal and B is initial.[4]

[4] To be explicit, for every choice of X, Y, and dashed arrows that commute with the solid arrows, there should exist unique dotted arrows to make all commute.

$$
\begin{tikzcd}
& & 0 \arrow[dr] \arrow[drr, dashed] & & \\
X \arrow[urr, dashed] \arrow[drr, dashed] \arrow[r, dotted] & F \arrow[ur] \arrow[dr] & & B \arrow[r, dotted] & Y \\
& & E \arrow[ur] \arrow[urr, dashed] & &
\end{tikzcd}
$$

Intuitively, each such square analyzes E into conceptually orthogonal parts F, B. The universal properties ask F, B to hug E as tightly as possible despite the orthogonality imposed by 0. This was Descartes' great idea: to study Euclid axis-by-axis.

Studying $F \to E \to B$ in a concrete category, we sense that F approximates E and B abstracts E.[5] In particular, $F \to E$ represents F as a subset of E, and to factor $X \to E$ through $X \to F$ is to "round" decimal outputs in E to integer outputs in F. Dually, $E \to B$ presents B as a quotient of E, and to factor $E \to Y$ through $E \to B$ is to "collapse" the decimal inputs in E to their fractional parts in B.

[5] For example, consider the groups

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

As another example, consider the category of parameterized distributions $p(Y; X)$ on two real numbers, with maps induced by linear shears $y \mapsto ax + by + c$. Any $p(Y; X)$ induced by a centered normal joint distribution enjoys an exact decomposition

$$p_I \to p \to p_E$$

where $p_I(Y; X = x) = p(Y; X = 0)$ is an independent approximation to p and $p_E(Y = y; X) = \delta(y - \mathbb{E}_p[Y; X])$ is an abstraction of p in terms of a summary statistic.

Distributivity

Multiplication distributes over addition, but not vice versa. As a consequence we typically have $0 \cdot n = 0$ but $1 + n \neq 1$. Thus an ideal (i.e. the pre-image of 0 for some ring map) is closed under both addition and scalar multiplication, but a multiplicative system (i.e. the pre-image of 1 for some ring map) is typically closed only under multiplication and not scalar addition.

Homology

One: Invariance

Two: the Ghost of Pythagoras

Three: Encapsulation through Action

Homology

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