Small Oscillations in Finite Dimension

Modes

Energy minima

Winter — For cosmological reasons, the universe is very cold; for statistical reasons, we expect everyday objects to imperfectly insulated. We thus encounter many interesting systems near their local energy minima. Think of salt crystals or playground swingsets, for instance. These notes study the classical physics of such systems. We restrict ourselves to systems with finitely many degrees of freedom.

Swingset — For example, let's model a swingset as two pendula hanging from a shared bar and hence weakly coupled. The pendula have angles θ_i and angular velocities ω_i for $i \in \{a, b\}$. The hamiltonian might look something like

$$\mathcal{H} = \sum_i m\ell^2\omega_i^2/2 - mg\ell\cos\theta_i \, + \, m^2g^2(\sin\theta_b - \sin\theta_a)^2/2k$$

Crystal — As another example, let's model a (one-dimensional) salt crystal as a line of N ions of equal mass, spaced ℓ apart, and with only nearest-neighbor interactions. The hamiltonian might look something like

$$\mathcal{H} = \sum_{0 \leq i < N} p_i^2/2m + \sum_{0 \leq i, i+1 < N} k \cdot f(|q_j - q_i|/\ell)$$

where f is smooth and bounded on the positive reals and minimized near 1. Perhaps f is Coulomb attraction force plus a repulsive term that models Pauli exclusion classically.

Canonical form — Now we abstract. We posit a manifold M equipped with a symplectic form $\omega : TM \to T^\star M$ and a smooth hamiltonian $\mathcal{H} : M \to \mathbb{R}$. The vector field $v : M \to TM$ such that $\omega_p(v_p) = (d\mathcal{H})_p$ determines evolution through time.

Spectral asymptotics

Dispersion

Waves —

Group velocity —
Waves —
Waves —

Beats

Interactions

Harmonics

Catastrophes

Diagrammatic expansion

Poincare perturbation

Resonance

Forcing resonance

Parametric resonance

Invariant tori

Small divisors

Heat

Thermal conduction

Thermo-elasticity

Significance of phase space measure: Boltzmann's law

Equipartition of energy