

WHAT IS...

a Curvature Form?

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Pushing against non-abelianness

Some falling animals land always on their hooves.¹ The manifold E of their postures in physical space fibers over the manifold B of their internal muscle lengths. Amazingly, even cows lacking angular momentum may traverse loops in B that lift to non-loops in E !

Connections as Forms

A **connection** on a bundle $\pi : E \rightarrow B$ is a prescription for how to lift infinitesimal paths through π . We may lift finite paths by integrating — this is **transport**. A connection's **curvature** measures the failure of infinitesimal loops to lift to loops. Thus, in the presence of curvature, the result of traversing a loop γ times depends on γ ; more generally, transport's destination in E depends on one's path in B .

A connection $\tilde{\omega}$ has type $TB \rightarrow E \rightarrow TE$ where we restrict to inputs whose basepoints match and we insist that the output vector projects to the input vector. Thus, any two connections differ at $v \in T_p B$ by a vector field on p 's fiber. We may locally choose a flat connection $\tilde{\omega}_0$ by choosing a trivialization of the bundle; then $\tilde{\omega}$ differs from $\tilde{\omega}_0$ at $v \in T_p B$ by some vector field $\omega(v)$ on the fiber at p . Viewing all those vector fields as living in a Lie algebra \mathfrak{g} , we may package the data of $\tilde{\omega}$ as $\omega : TB \rightarrow \mathfrak{g}$. Note that ω is only defined locally and depends on a local trivialization.

Curvature as a Form

Just as we measure infinitesimal location-dependence using one-forms, we measure infinitesimal path-dependence using two-forms. We therefore expect the curvature $\Omega : \Lambda^2 TB \rightarrow \mathfrak{g}$ to be a two-form valued in \mathfrak{g} .

Intuitively, there are two obstructions to path independence: (**horizontal**) the connection might vary with p and (**vertical**) the vector fields $\omega(\cdot)$ might not commute. To isolate the horizontal phenomenon, suppose \mathfrak{g} is abelian so that transport is just addition in \mathfrak{g} . We then want for disks $h : D^2 \rightarrow B$ that

$$\int_{\partial h} \omega = \int_h \Omega$$

In this case, we must take $\Omega = d\omega$. To isolate the vertical phenomenon, suppose ω is closed, i.e. in local coordinates constant. Then for an infinitesimal rectangle in these coordinates with sides u, v , we want:

$$e^{\omega(-u)} e^{\omega(-v)} e^{\omega(v)} e^{\omega(u)} = e^{\Omega(u,v)}$$

In this case, we must take $\Omega(u, v) = [\omega(u), \omega(v)]$. The general expression for **curvature** is a sum:

$$\Omega(u, v) = d\omega(u, v) + [\omega(u), \omega(v)]$$

For example, a Riemannian manifold B of dimension n has a bundle of orthonormal bases; the relevant algebra is $\mathfrak{o}(n)$. Then $\Omega : \Lambda^2 TB \rightarrow \mathfrak{o}(n)$ has the same data as the Riemann tensor, and it emphasizes the variance and antisymmetries of the indices.

¹Due to dander allergies, we imagine a falling cow.

Chern's Gauss-Bonnet

Chern proved the Gauss-Bonnet Theorem using the curvature form for the bundle UB of unit vectors on a (compact oriented) surface B .

The key insight is to lift this form by

Fix a vector field on B with one non-degenerate zero at p . By Poincaré-Hopf, the zero has degree $\chi(B)$. Regard the graph of the normalized vector field as a smooth submanifold M of $U(B - \{p\})$; the boundary of M winds around the S^1 -shaped fiber $\chi(B)$ times.

Then

$$\int_B \Omega = \int_M \Omega = \int_{\partial M} \omega = \chi(B) \cdot \int_{S^1} \omega$$

We finish by observing that $\int_{S^1} \omega = 2\pi$.

Using double covers, we deduce that G-B also applies to non-orientable manifolds.

Light

We will not mention, for instance, how a connection on a circle bundle elegantly describes light, its curvature form containing as components both the electric and magnetic fields.

Forms are Fun

There is much beyond this tiny tour. All three of our main examples had \mathfrak{g} abelian, so their curvatures' vertical terms vanished.

We will not mention, for instance, that the nuclear forces arise from sphere bundles, their vertical terms indicating that “nuclear light” interacts with itself. Nor will we remark that Chern's proof immediately generalizes beyond surfaces, thus relating geometry and topology in all even dimensions.

Sometimes the journey matters as much as the destination. Let us explain this in the differential geometer's world where all that can be smooth is smooth and all that can be linear is linear.

References

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