WHAT IS...

the Curvature Form?

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Connections as Forms

If insanity means redoing actions while expecting new results, then curvature is a local measure of insanity.

Let us explain this in the differential geometer's ink-saving world where all that can be smooth is smooth and all that can be linear is linear. A **connection** on a bundle $\pi : E \to B$ is a prescription for how to lift infinitesimal paths through π. We may lift finite paths by integrating — this is **transport**. A connection's **curvature** measures the failure of infinitesimal loops to lift to loops. Thus, in the presence of curvature, the result of traversing a loop n times depends on n; more generally, transport's destination is path-dependent.

A connection is a $\tilde{\omega} : TB \to E \to TE$ where we restrict to inputs whose basepoints match and we insist that the output vector projects to the input vector. Thus, any two connections differ at $v \in T_pB$ by a vector field on p's fiber. We may locally choose a flat connection $\tilde{\omega}_0$ by choosing a trivialization of the bundle; relative to $\tilde{\omega}_0$, $\tilde{\omega}$ has the data of a bunch of vector fields on fibers, comparable between different fibers by means of $\tilde{\omega}_0$. Viewing all those vector fields as living in a Lie algebra $\mathfrak{g}$, we may package the data of $\tilde{\omega}$ as $\omega : TB \to \mathfrak{g}$. Note that ω is only defined locally and depends on a local trivialization.

Curvature as a Form

Just as we measure infinitesimal location-dependence using one-forms, we measure infinitesimal path-dependence using two-forms. We therefore expect the curvature $\Omega : \Lambda^2 TB \to \mathfrak{g}$ to be a two-form valued in $\mathfrak{g}$.

Intuitively, there are two obstructions to path-independence: (**horizontal**) the connection might vary with p and (**vertical**) the vector fields $\omega(\cdot)$ might not commute. To isolate the horizontal phenomenon, suppose $\mathfrak{g}$ is abelian so that transport is just addition in $\mathfrak{g}$. We then want for any disk $h : D^2 \to B$ that

$$\int_{\partial h} \omega = \int_h \Omega$$

In this case, we must take $\Omega = d\omega$. To isolate the vertical phenomenon, suppose ω is closed, i.e. in local coordinates constant. Then for an infinitesimal rectangle in these coordinates with sides u, v, we want:

$$e^{\omega(-u)} e^{\omega(-v)} e^{\omega(v)} e^{\omega(u)} = e^{\Omega(u,v)}$$

In this case, we must take $\Omega(u, v) = [\omega(u), \omega(v)]$. The general expression for **curvature** is a sum:

$$\Omega(u, v) = d\omega(u, v) + [\omega(u), \omega(v)]$$

For example, a Riemannian manifold B of dimension d has a bundle of orthonormal bases; the relevant algebra is $\mathfrak{o}(d)$. Then $\Omega : \Lambda^2 TB \to \mathfrak{o}(d)$ has the same data as the Riemann tensor, and it emphasizes the variance and antisymmetry of the indices.

Chern's Gauss-Bonnet

Chern proved the Gauss-Bonnet Theorem using the curvature form for the bundle UB of unit vectors on a (compact oriented) surface B.

The key insight is to lift this form by

Fix a vector field on B with one non-degenerate zero at p. By Poincaré-Hopf, the zero has degree $\chi(B)$. Regard the graph of the normalized vector field as a smooth submanifold M of $U(B-\{p\})$; the boundary of M winds around the S^1-shaped fiber $\chi(B)$ times.

Then

$$\int_B \Omega = \int_M \Omega = \int_{\partial M} \omega = \chi(B) \cdot \int_{S^1} \omega$$

We finish by observing that $\int_{S^1} \omega = 2\pi$.

Using double covers, we deduce that G-B also applies to non-orientable manifolds.

A Useful Holonomy

It is observed with awe that certain pets, upon falling, land always on their feet. Due to dander allergies, we will imagine a falling cow.

Forms are Fun

There is much beyond this tiny tour. We will not mention, for instance, how a connection on a circle bundle elegantly describes light, its curvature form containing as components both the electric and magnetic fields. Nor will we remark that Chern's proof immediately generalizes beyond surfaces, thus relating geometry and topology in all even dimensions.

References

We thank Professor Ralf Spatzier for teaching us to love pictures.

When not thinking about machines that learn, Sam can often be found pretending to be a cow. Sam enjoys memory over experience, pet snails over pet spinach, left adjoints over right adjoints, and analogies over lists.

The author, courtesy of Karl Winsor