TWO KINDS OF HOLE

Contents

Dimensions 0 *and* 1

Homotopy and Cohomology

When is a nice space — say, a manifold X — connected? On one hand, X is connected when any two points in X may be joined by a path; this has to do with maps *to* X. On the other, X is connected when each locally constant function is globally constant; this has to do with maps *from* X. We formalize these two notions as follows. We work with based spaces. The kth *rational homotopy* of X is the set of formal $\mathbb{Q}$-linear combinations of maps $f : S^k \to X$, mod homotopies between $f, g : S^k \to X$, and mod addition defined below. The kth *rational cohomology* of X is the set of formal $\mathbb{Q}$-linear combinations of maps $f : X \to S^k$, mod homotopies between $f, g : X \to SP(S^k)$, and mod addition defined below. TODO: are $\mathbb{Q}$-linear combinations of maps $f : X \to S^k$ the same as maps from $f : X \to SP(S^k)$ up to homotopy? (Perhaps assume compactness of X or think only about compactly supported cohomology?)

Addition

Multiplication

Graphs and Covers

We might ask for spaces with vanishing 2nd and higher homotopy and cohomology (not just rational). Examples are graphs-with-higher-cells Or we might ask for spaces with vanishing 1st and lower homotopy and cohomology (not just rational). These are the simply connected spaces. For any X we have good maps $C \to X \to G$ from a simply connected space to X to a graph-with-higher-cells.

Dimensions 2 and higher

Differential forms

Poincare duality

Sullivan models

Sample computations

Applications