WHAT IS...

a Curvature Form?

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Pushing against non-abelianness

Some falling animals land always on their hooves.[1] The manifold E of their postures in physical space fibers over the manifold B of their internal joint angles. Amazingly, even cows *in vacuo* and lacking angular momentum may effect net rotations by adjusting their muscles appropriately! Such cows traverse loops in B that lift to non-loops in E.

B is contractible, so path-lifting is uninteresting from a topological point of view. To account for this amazing bovine behavior, we must model the physics and hence geometry of E and B.

Connections as Forms

A **connection** on a bundle $\pi : E \to B$ is a prescription for how to lift infinitesimal paths (that is, vectors) through π. We may lift finite paths by integrating — this is **transport**. A connection's **curvature** measures the failure of infinitesimal loops to lift to loops. Thus, in the presence of curvature, transport's destination in E depends on one's path in B.

A connection $\tilde{\omega}$ has type $TB \times_B E \to TE$ where we insist that the output vector projects to the input vector. Thus, any two connections differ at $v \in T_pB$ by a vector field on p's fiber. We may choose a flat connection $\tilde{\omega}_0$ locally by choosing a local trivialization; then $\tilde{\omega}$ differs from $\tilde{\omega}_0$ at $v \in T_pB$ by some vector field $\omega(v)$ on the fiber at p. Viewing those vector fields as living in a Lie algebra $\mathfrak{g}$, we may package the data of $\tilde{\omega}$ as $\omega : TB \to \mathfrak{g}$. Note that ω is defined only locally and in reference to a local trivialization.

Curvature as a Form

Just as we measure infinitesimal location-dependence using one-forms, we measure infinitesimal path-dependence using two-forms. We therefore expect the curvature $\Omega : \Lambda^2 TB \to \mathfrak{g}$ to be a two-form valued in $\mathfrak{g}$.

Intuitively, there are two obstructions to path independence: (**horizontal**) the connection might vary with p and (**vertical**) the vector fields $\omega(\cdot)$ might not commute. To isolate the horizontal phenomenon, suppose $\mathfrak{g}$ is abelian so that transport is just addition in $\mathfrak{g}$. We then want for disks $h : D^2 \to B$ that

$$\int_{\partial h} \omega = \int_h \Omega$$

In this case, we must take $\Omega = d\omega$. To isolate the vertical phenomenon, suppose ω is closed, i.e. in local coordinates constant. Then for an infinitesimal rectangle in these coordinates with sides u, v, we want:

$$e^{\omega(-u)} e^{\omega(-v)} e^{\omega(v)} e^{\omega(u)} = e^{\Omega(u,v)}$$

In this case, we must take $\Omega(u, v) = [\omega(u), \omega(v)]$. The general expression for **curvature** is a sum:

$$\Omega(u, v) = d\omega(u, v) + [\omega(u), \omega(v)]$$

Unlike ω, this sum is canonically defined everywhere.

For example, a Riemannian manifold B of dimension n has a bundle of orthonormal bases; the relevant algebra is $\mathfrak{o}(n)$. Then $\Omega : \Lambda^2 TB \to \mathfrak{o}(n)$ has the same data as the Riemann tensor, and its notation emphasizes the variance and antisymmetries of the indices.

[1] Due to dander allergies, we imagine a falling cow.

Chern's Gauss-Bonnet

Chern proved the Gauss-Bonnet Theorem using the curvature form for the bundle UB of unit vectors on a (compact oriented) surface B. The fibers are oriented circles transported isometrically, so the relevant algebra is $\mathfrak{o}(2)$ and is abelian. So $\Omega = d\omega$.

Chern's insight is that Ω, pulled back to UB, is exact. On UB, we have $\Omega = d\eta$ with η globally defined.

Fix a vector field on B with one non-degenerate zero at p. By Poincaré-Hopf, the zero has degree $\chi(B)$. Regard the graph of the normalized vector field as a smooth submanifold M of $U(B - \mathring{p})$, where $\mathring{p}$ is a tiny open disk around p. Then ∂M winds around p's fiber $\chi(B)$ many times and

$$\int_B \Omega = \int_M \Omega = \int_{\partial M} \eta = \chi(B) \cdot \int_{S^1} \eta$$

We finish by observing that $\int_{S^1} \eta = 2\pi$.

Using double covers, we deduce that G-B also applies to non-orientable manifolds.

Light

We will not mention, for instance, how a connection on a circle bundle elegantly describes light, its curvature form containing as components both the electric and magnetic fields.

Forms are Fun

There is much beyond this tiny tour. Indeed, our main examples all had $\mathfrak{g}$ abelian, so their curvatures' vertical terms vanished. We will thus avoid mentioning that the nuclear forces arise from sphere bundles, their vertical terms indicating that "nuclear light" interacts with itself. Nor will we remark that Chern's proof immediately generalizes beyond surfaces, thus relating geometry and topology in all even dimensions. We won't even mention how curvature forms help to tame the holonomies that complicate robotics.

In other words, we Sometimes the journey matters as much as the destination.