The Classification of Surfaces

Bird's eye view

The classification of manifolds presents a grand challenge. To begin addressing it, we must decide on a notion of *sameness*. For smooth manifolds, such notions range in strength from homotopy equivalence to diffeomorphism. Do these notions coincide? For instance, we like to distinguish compact (connected, smooth) surfaces by their Euler characteristics. Does this tool miss anything? That is, are there non-diffeomorphic compact surfaces with the same χ? The answer is no: *the Euler characteristic determines compact surfaces up to diffeomorphism*.

We may establish this result in several ways. For example, we use smoothness to construct a triangulation, whereupon we reason combinatorially to bring a surface to standard form. Or, we could use smoothness to construct a metric, whereupon we reason geometrically to bring the surface to a standard form. Just as I regard finite differences to be messier than derivatives, I find the geometric approach more symmetrical than the combinatorial one. In what follows, we'll describe this approach.

Any smooth manifold $\mathcal{M}$ embeds smoothly into some standard sphere and therefore enjoys a smooth Riemannian metric g.[1] We aim to rescale the metric so that it has constant curvature. The point is that we may then compare our surface $\mathcal{M}$ to a standard surface. Indeed, constant curvature implies that $\mathcal{M}$ is locally isometric to a standard (simply connected) model space $\mathcal{S}$. By patching together these local isometries, we isometrically identify $\mathcal{S}$ with the universal cover of $\mathcal{M}$. Since both that isometry and the the deck group act by diffeomorphisms, $\mathcal{M}$ itself has the smooth structure of a standard quotient of $\mathcal{S}$. We finish by recalling that such standard surfaces are determined by their Euler characteristic.

[1] Intuitively, the metric corresponds to an infinitely fine triangulation; it induces a natural volume form measuring the density of highest-dimensional cells (and likewise on each subspace of each tangent space). Refining triangulations thus becomes as easy as scaling the metric.

Ricci flow

So let's focus on the problem of finding a metric of constant curvature. We'll do this by evolving our initial metric by a sort of diffusion equation. Let's see how to do this in the special case where the initial metric g has (potentially varying) negative curvature everywhere. We'd like the regions with extreme negative curvature to expand, thus diluting their curvature. Conversely, we'd like regions that are too flat to contract, thus concentrating their curvature. So, writing R for the scalar curvature (scaled by $1/8\pi^2$ so that it averages to χ) let us scale the metric according to how much R overestimates its average:[2]

$$\dot{g} = (R - \chi) g$$

[2] Intuitively, we split octogonal singularities into pairs of heptagonal singularities so that the graph becomes increasingly regular.

Note that this equation is non-linear. Intuitively, the non-linearity indicates the creation of new (positive and negative) curvature due to the non-uniformity of

g's scaling. This "pair production" complicates our goal of diffusion, but we will see that in 2 dimensions, diffusion prevails.

It's enough to find a solution $g(t, x)$, defined for all time, that converges to a fixed point. To show long-term existence, it's enough to bound $R(t, x)$ over all time and space.

The curvature is never too small

The curvature is never too big