basic structures of analysis

... her mind is a very thin soil, laid an inch or two upon very barren rock.
— virginia woolf

What's the deal with those function spaces, what with their sometimes non-commuting and absent limits? Those spaces are infinite-dimensional seas with capacity to swallow and disappear whole ships, whether in the infrared (as with $L^\infty - L^1$) or the ultraviolet (as with $L^1 - L^\infty$). What tools can tame this charybdis? Aiming to see familiar ground in new ways, we approach freshman level content at senior level mathematical sophistication. We assume point-set topology and a facility with the language of categories.

Notions of Size

the reals

We take the view that the topological space of reals serves two roles: one covariant and algebraic, as a codomain for metrics and measures; another contravariant and spatial, as a domain for paths. These notes focus on the covariant, algebraic view.

Analysis abstracts physical notions of *size*. For example, the next two definitions abstract *distance* and *density*. A **generalized metric space** over M is a category C and a map $\delta : \mathrm{Mor}_C \to M$ that satisfies the nullary and binary **triangle inequalities**:

$$\delta(\mathrm{id}_x) \leq 0 \qquad \delta(h \circ g) \leq \delta(h) + \delta(g)$$

The latter is the key compositional principle in analysis, just as equality's transitivity is in algebra. Likewise, a **generalized measure space** over M is a lattice P and a non-decreasing map $\mu : P \to M$ that satisfies nullary and binary **additivity**:

$$\mu(\bot) = 0 \qquad \mu(A \vee B) = \mu(A) + \mu(B) \text{ whenever } A \wedge B = \bot$$

In either case, M sets the scale by which we measure size. But what is M? Well, to make sense of the mentioned operations, M ought to be a **costoid**: a total order and an abelian monoid compatible in that:

$$0 \leq a \qquad a + c \leq b + d \text{ whenever } a \leq b \text{ and } c \leq d$$

A map $\phi : M \to M'$ of costoids is a non-decreasing function that is also laxly additive:

$$\phi(0) \leq 0 \qquad \phi(m+n) \leq \phi(m) + \phi(n)$$

As mass has grams and temperature has kelvins, we consider only costoids marked with a distinguished nonzero element and only maps that preserve that element. The punchline? There is a terminal marked costoid: the real interval $[0, \infty]$ marked at 1! This is our universal choice of codomain for metrics and measures.

structures of analysis

A **metric space** is a generalized metric space (X^2, δ) over $\mathbb{R}$ where X^2 is the category with object class X and morphism class $X \times X$ and where δ vanishes only on the diagonal and is symmetric:

$$\delta(x, y) = \delta(y, x)$$

A **measure space** is a generalized measure space (P, μ) over $\mathbb{R}$ where P is a sub-lattice of a powerset $\mathcal{P}(X)$ closed under complement and countable union and where μ enjoys this continuity property:

$$\mu\left(\sup_{N:\mathbb{N}} \bigcup_{n<N} A_n\right) = \sup_{N:\mathbb{N}} \mu\left(\bigcup_{n<N} A_n\right)$$

Perhaps the most celebrated wedding of these two notions is the concept of the **Bourbaki spaces** $L^p(X)$. For $1 \leq p \leq \infty$ and (X, μ) a measure space, $\tilde{L}^p(X)$ is the generalized metric space of measurable real functions on X defined by

$$\delta(f, g)^p = \int d\mu \, |f - g|^p$$

We define $L^p(X)$ as the vector subspace of elements finitely far from 0, mod the vector subspace of elements zero far from 0. This makes $L^p(X)$ a metric space with every distance finite.

Convergence

Write $\gamma_X : (X \sqcup 1) \to X^\star$ for X's one-point compactification. We say that a map $s : X \to Z$ **converges** to a point $z : X \to Z$ when $(s, z) : (X \sqcup 1) \to Z$ factors through γ_X. We're especially interested in the case where $X = \mathbb{N}$ is the countable discrete space; then we say z is a **limit** of s.

Observe that any inclusion $\iota_O : O \hookrightarrow X$ induces inclusions $\iota_{O \sqcup 1}, \iota_{O^\star}$ with the obvious types. When $X - O$ is compact, these inclusions commute with compactification:

$$\gamma_X \circ \iota_{O \sqcup 1} = \iota_{O^\star} \circ \gamma_{O \sqcup 1}$$

So $s : O \to Z$ converges to z if and only if $s \circ \iota_O : X \to Z$ converges to z. Intuitively, convergence of depends only on tail behavior, not on any finite (compact) prefix.

pathologies of measure

metric pathologies

Continuity Theorems: Commuting Limits

fatou-lesbegue

fubini

levy

moore-osgood

Existence Theorems: Convergent Lineages

ascoli

How does the compact-open topology on $(Y \to Z)$ depend on Y's topology? For instance, how does it differ from the product topology $(|Y| \to Z)$? (Here, $|\cdot|$ discretifies.) For one thing, the canonical map $\iota : (Y \to Z) \to (|Y| \to Z)$ (continuous because points (in Y) are compact) has tiny image. Indeed, most elements of the codomain are, as functions from Y to Z, discontinuous. Moreover, ι isn't even typically open: when Y is infinite, a typical compact-open generator isn't much smaller than any cylinder cut out by finitely many pointwise constraints.

Both differences lead us to expect convergence to be rarer in $(Y \to Z)$ than in ι's image.

Still, when Z is a metric space, we have a positive result: any "equicontinuous" subspace $X \subseteq (Y \to Z)$ whose every restriction to a $y : Y$ has compact closure itself has compact closure!

What does **equicontinuous** mean? It means that the metric induced on Y by uniform norm over X is continuous.

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A **countideal** on a poset P is a downward-closed subset closed under countable joins. Each countideal on a powerset $\mathcal{P}(X)$ gives a notion of *negligible subset*. For example, each infinite cardinal κ determines a countideal consisting of subsets of cardinality at most κ. More interestingly, each measure on X determines a countideal consisting of subsets-of-sets-of-zero-measure (**null sets**). Likewise, each topology on X determines a countideal consisting of countable unions of sets-whose-closures-have-empty-interiors (**meager sets**).

banach

riesz

Fourier Theory

To illustrate the above ideas "in the wild", we develop some fourier theory. The classical fourier transform $\mathcal{F} : L^1(S^1) \to L^\infty(\mathbb{Z})$ is celebrated because it diagonalizes both differentiation and translation on the Lie group S^1. The differentiation story generalizes to the harmonic theory of riemannian manifolds; the translation story generalizes to the representation theory of compact groups.