Bijections of Finite Sets

We review a small part of basic combinatorics while emphasizing conceptual unity. For us, combinatorics is the study of bijections between finite sets. This may seem silly seeing as every finite set is isomorphic to some standard ordinal $[n] \triangleq \{k \in \mathbb{N} : k < n\}$. But then again, every real space is isomorphic to some free space $\mathbb{R}^{\oplus I}$, yet linear algebra does not collapse into triviality. In both cases, those isomorphisms are typically *not canonical*, and we find that, while finite sets or vectors spaces by themselves are boring, the maps between them are quite rich.

Natural Structures

Joyal's Species

What is a Species? — How may we structure elements into trees, into partitions, etc? We'll focus on what remains when we ignore the elements' internal structure. The categorical concept of *naturality* formalizes this focus: we deem a concept significant only when it coheres under renaming of elements.

Let Fin be the category of functions between finite sets and let FinBij be its core groupoid. A *combinatorial species* is a functor : $\mathsf{FinBij} \to \mathsf{Fin}$. For example, the inclusion $I : \mathsf{FinBij} \to \mathsf{Fin}$ is a species. We'll study the category $\mathsf{Species} = [\mathsf{FinBij}, \mathsf{Fin}]$ of natural maps between species. It is an orienting exaggeration to say that combinatorics is the study of $\mathsf{Species}$.

Note that post-composition with I gives an eidetic functor $[\mathsf{FinBij}, \mathsf{FinBij}] \to [\mathsf{FinBij}, \mathsf{Fin}]$. Thus, we may compose species by composing their corresponding endofunctors. The inclusion I is this composition's identity.

A functor is *eidetic* when it surjects on objects, arrows between those objects, and equalities between those arrows. The three conditions separately we call *esophageal*, *full*, and *faithful*. Indeed, it is through the esophagus that one becomes full of objects.

Maps between Species —

Using Finiteness — We restrict attention to *finitary* combinations, since their study admits pidgeon-hole principles (e.g. in the presence of a bijection, every injection is a surjection). Transfinite induction fails to prove such principles in general because limits do not preserve those principles.

Beyond Dimension Zero

Permutations vs Orders —

Yoneda Lemma —

Eidetic Species — Perhaps the best species are the *eidetic* ones. It turns out there are exactly four eidetic species up to isomorphism! Two of them swap size-0 and size-1 sets. One of the non-swapping ones is I; the other one, E, acts like I except on sets of size 6.

Say S has 6 elements. Call a size-3 subset of S a 'face'. $\mathbb{Z}/2\mathbb{Z}$ acts on the faces by set complement. A q-*structure* on S is a partition of S's 20 faces into two regular classes interchanged by $\mathbb{Z}/2\mathbb{Z}$. (A size-10 class of faces is *regular* if each singleton $\{s\}$ includes into the same number (5) of faces and each doublet $\{s, t\}$ includes into the same number (2) of faces). We define $E(S)$ to be the set of q-structures on S.

Theorem. *The eidetic species with functor composition form a group canonically isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$.

Composing Species

Algebraic Data Types — Since the data of a species is the data of an endofunctor on FinBij, we may compose combinatorial species (we use $\square$ instead of the standard circle). The identity under composition we have already named (I). Moreover, any k-ary operation on Fin that preserves FinBij induces a k-ary operation on Species. Thus we have 'pointwise' (co)product $(+, \times)$ and their identities $(0, 1)$.

For example, we denote $I \times F$ by $\star F$ — this is called 'pointing'; we denote $F \square (I + 1)$ by F' — this is called 'puncturing'. Fans of algebraic data types will recognize these operations as dear friends.

Partitions — Each species F maps canonically to the species Part of partitions as follows. For $t \in F(S)$, let π_t be the coarsest partition of S such that every π_t-preserving automorphism of S also preserves t.

This induces a species $\mathring{F}$ that pointwise includes into Part.

As an illustration, we present Joyal's calculation:

$$\begin{aligned} \text{Ord} \times \text{Endo} &\cong \text{Ord} \times (\text{Perm} \circ (\star\text{Tree})) \\ &\cong \text{Ord} \times (\text{Ord} \circ (\star\text{Tree})) \\ &\cong \text{Ord} \times (\star\star\,\text{Tree}) \end{aligned}$$

Because this map preserves its first factor, we summarize as follows.

Theorem. *In the presence of a total order, endomorphisms correspond naturally with bipointed trees.*

An interesting corollary is Cayley's: there are n^{n-2} trees on n nodes.

An endofunction's core is the categorical limit of $\cdots \to S \to S$. To give a pointed tree is to give an endofunction with size-one core (pidgeonhole).

To give a permutation is to give an endofunction with full-size core.

Atomic Decomposition — Each S_n-set decomposes canonically as a sum of orbits. We call such orbits *molecules*. With coproduct and cauchy product, Species is a (class-sized) commutative semiring with multiplicative unit $X_0 = [-, 0]$. A molecule p is *atomic* when it is irreducible with respect to the cauchy product.

Theorem. *Moo*

There are $0, 1, 1, 2, 6, 6, 27, 20, \cdots$ many atoms on $0, 1, 2, 3, 4, 5, 6, 7, \cdots$ elements.

Making Choices

Products don't Reflect Isomorphisms —

Finitary Choice Principles — A size-n choice principle on a set S is a simultaneous pointing for each of S's size-n subsets. Let the species Ch_n give a set's size-n choice principles.

$$\mathrm{Ch}_n = E \,\square\, \mathrm{Subs}_n$$

For example, we may visualize C_2 as sending S to the set of complete directed graphs on S.

What data can witness a choice principle? That is, which species map into C_n?

Suppose we have a way to choose an element from each k-element set. Do we then have a canonical way to choose an element from each K element set S?

For example, say $k = 2$ and $K = 4$. The following method works. Take the set P of two-part partitions of S into size-two subsets. For each of P's three elements, apply the choice function to select an element of S. The image is a multi-set M of three elements in S. If the elements are all distinct, let us output the one missing element. Otherwise, there is an element that occurs strictly more times in M than any other element occurs; return that element.

More precisely, we have demonstrated a natural map between species:

$$\mathrm{Ch}_2 \times (\mathrm{Ch}_2 \,\square\, \mathrm{Subs}_2) \to [\mathrm{Part}_{2+2}, I] \to \mathrm{Ch}_4$$

Here, Subs_2 gives a set's size-2 subsets; Part_{2+2} gives a set's partitions of type $2+2$; and $[,]$ is the pointwise exponential.

Abstraction and Symmetry

Polyá Counting

Approximation and Order

Möbius Inversion