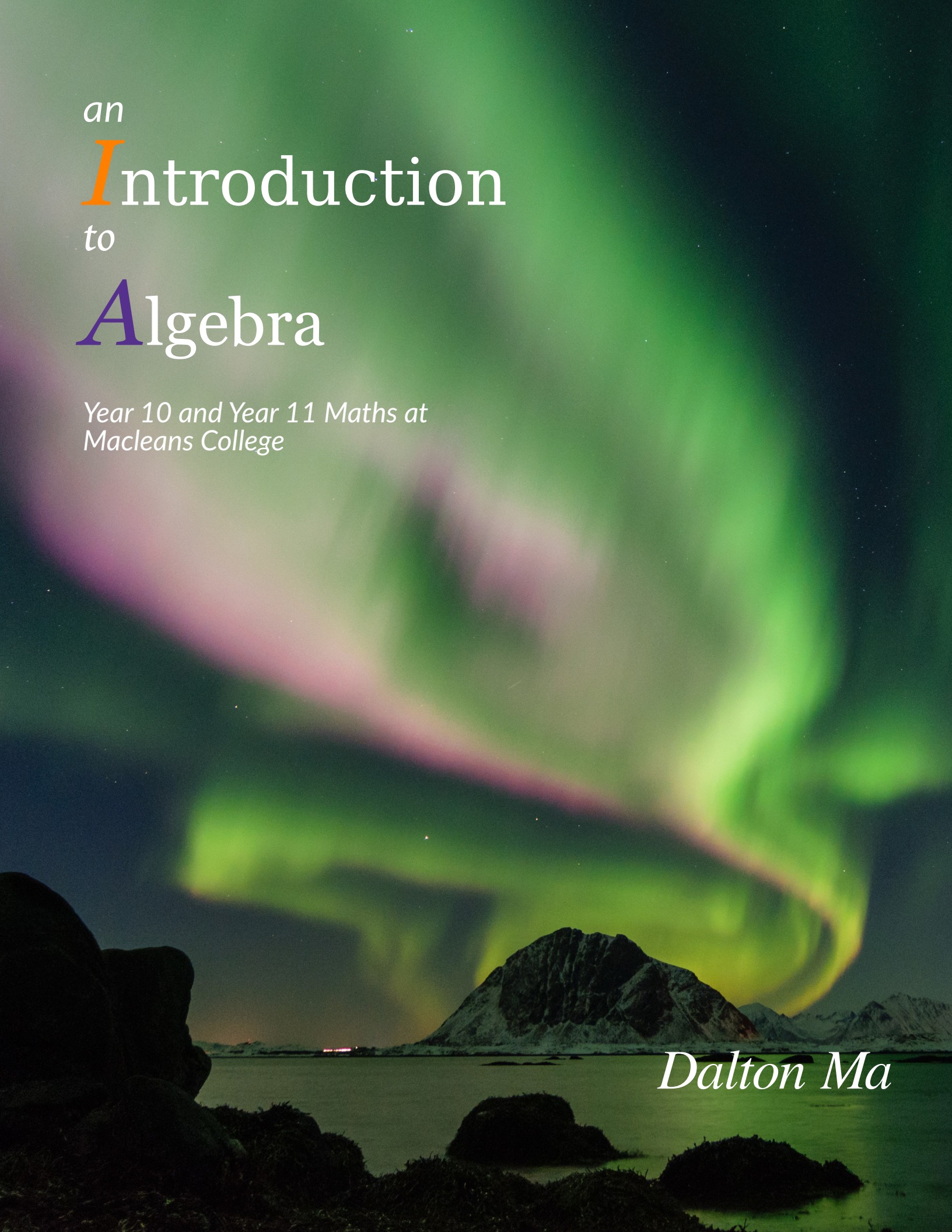

an Introduction *to* Algebra

Year 10 and Year 11 Maths at Macleans College

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Chapter 1

An Introduction to Algebraic Expressions

Welcome! You successfully made it to the first chapter! Depending on where you are reading this, this is more or less of a feat. If you are, say, studying while at war or hiding from aliens, congratulations! It is hard enough to study math normally; studying math while hiding from aliens or war is nearly impossible, I would imagine. If you are at school and are being forced to read this, good job, you are reading. If you are doing this of your own accord, ... why? In this chapter, we will look at "Algebraic Expressions", whatever those are. It would probably be a good idea to keep some scratch paper near you, so that you can work alongside us on the problems. Are you ready? Let's go!

1.1 Terms, Factors, Expressions, oh my!

Learning Intentions

- Identify Terms and Factors in an expression.
- Identify like terms in an expression.

In this section, we will look at terms and factors in expressions. Let's start by introducing some key vocabulary.

Definition 1.1.1

Terms are separated by plusses and minuses.

Before you continue, you might notice all the examples. After most "learning bits", like definitions, strategies, etc., we have an example, which we encourage you to try. After all, you will only learn math by doing problems, not reading about us doing problems.

Example 1.1.2

How many terms are in the expressions below?

1. $a + b + c$
2. $2a$
3. $a + 2(b + c)$

Solution.

1. There are 3 terms: a, b, and c.
2. There is just 1 term: $2a$.
3. There are 2 terms: a and $2(b+c)$. Brackets only count as 1 term.

Definition 1.1.3

Factors are separated by multiplication and division.

Example 1.1.4

What are the factors of $2abc$?

Solution. The factors of $2abc$ are 2, a, b, and c. $2abc$ can be written as $2 \times a \times b \times c$, and there are 3 numbers/variables separated by multiplication and/or division signs.

The next example problem will be a bit harder.

Example 1.1.5

Find the factors of 6.

Solution. The factors of a number are two numbers that multiply to get that number. You might recognise that 6 is divisible by 2 and 3. You might also notice that $2 \times 3 = 6$. Since 2 and 3 are separated by multiplication signs, 2 and 3 are factors of 6.

Next up are a couple of example exercises. You should make sure to try them BEFORE you look at the answers (I know you all too well :)). If you are stuck, you should look at the hint, but PLEASE don't look at the answer. This ruins the whole point of the exercise. It is for you to do it, not us.

Checkpoint 1.1.6

In each part, there is an expression. For each term, state the factor(s).

(a) abc

Hint. Where should the addition signs go? How many are there? Where should the multiplication signs go? How many are there?

(b) $3xy + 2yz + 7ab$

Hint. Pick out all of the terms. See if you can put any multiplication signs anywhere.

Checkpoint 1.1.7

How many terms are in the expression $3a + 4(b + c)$?

Checkpoint 1.1.8

Factor $6axyz$.

1.2 Combining Terms

Learning Intentions

- Identify like terms.
- Combine like terms.

In this section, we will look more closely at terms. What cool stuff can we do with terms? Read carefully; this content will be important later on.

First, we must understand **like terms**. Like terms are terms that have the same variables and the same exponents.

Definition 1.2.1

Like terms are terms that have the same variables and the same exponents.

Example 1.2.2

Identify all the like terms in the expression: $a^2b + a^2c + 2ab + ab + 2a^2c$.

Solution. Let's identify all of the factors in each term.

Term	Factors
a^2b	a^2, b
a^2c	a^2, c
$2ab$	$2, a, b$
ab	a, b
$2a^2c$	2, a^2, c

Next, let's identify the terms that have the same variables (like a), and exponents. a^2b has the variables a and b. Could $2ab$ be a like term? No. The exponents for a^2b are 2 and 1. The exponents for 2ab are 1 and 1. The exponents aren't the same. You cannot add a^2 and a. It turns out there aren't any like terms for a^2b. Let's move on. Are there any like terms for a^2c? Notice the variables are a and c. What other terms have a and c as factors? $2a^2c$ works. Are the exponents the same? Notice there is an a^2 (a with an exponent of 2) in both terms. Both terms also have c (with no exponent), too. So, the variables and exponents are the same. a^2c and $2a^2c$ are like terms.

Your turn. Does 2ab have any like terms? What are its variables? What are its exponents? Are there any other terms that have the same variables and the same exponents?

Checkpoint 1.2.3

Identify the like terms in the expression $x^2y + 3yx + 9x^2 + 3yx^2 + \frac{1}{2}x^2$.

We can add like terms like so:

Example 1.2.4

Simplify the expression $2ab + 3ab$.

Solution. We can write 2ab as "ab + ab" and 3ab as "ab + ab + ab". Thus, 2ab + 3ab = ab + ab + ab + ab + ab. Just as 4 + 4 + 4 = 3 $\times$ 4, ab + ab + ab + ab + ab = 5 $\times ab$ or 5ab. Notice that we also could just add the **coefficients** (like 2 and 3 in 2ab and 3ab) to get 5ab.

Definition 1.2.5

A **coefficient** of a term is a constant factor. For example, the coefficient of 2a would be 2.

Important!

When combining like terms, we can just add the coefficients.

Let's do an example regarding coefficients.

Example 1.2.7

For each term below, find the coefficient.

1. ac
2. 4mx
3. 9x
4. 17p
5. $-4y$

Solution.

1. The coefficient is 1. If no coefficient is given, assume the coefficient is 1.

   Warning 1.2.8

   The coefficient is NOT a. a is a variable, not a constant.

2. The coefficient is 4.
3. The coefficient is 9.
4. The coefficient is 17.
5. The coefficient is -4. Were you about to say 4 (positive 4)? Remember your signs!

Important!

Remember your signs! -4 is not the same as (positive) 4!

Example 1.2.10

Simplify the expression $3x^2y + xy^2 + 2xy^2 + xy$.

Solution. This problem is slightly more complex. First, we must identify like terms. We can see that xy^2 and $2xy^2$ are like terms. There are no other sets of like terms in this expression. Next, we need to add our like terms. We can write xy^2 as $1xy^2$. Then, we can add the coefficients. $1xy^2 + 2xy^2 = 3xy^2$. Let's put that back into the expression. $3x^2y + 3xy^2 + xy$ is our final answer.

Example 1.2.11

Simplify the expression 4x - x.

Solution. Let's write this out. x means 1x, so -1x would just be -x. 4x would be x + x + x + x. Writing this out, we get, x + x + x + x - x. x - x is 0, so we get x + x + x or 3x. Notice we could also have noticed that the coefficient of -x is -1, and you might have just added 4 and -1 to get the coefficient 3.

Example 1.2.12

Simplify the expression $3xy \times 4xy$.

Solution. What's wrong with this solution?

Bogus Solution.

Since the variables and exponents are the same, we can combine 3xy and 4xy to get 7xy.

Were you about to make that error? It turns out that 3xy and 4xy are not like terms. In fact, they are not terms at all. There is only one term in the expression given. Do you see why?

Warning 1.2.13

Multiplication is not addition. You cannot add coefficients of factors. Multiplying two factors is NOT combining like terms.

Checkpoint 1.2.14

Simplify the following expressions.

(a) $9am - 3am$

(b) $4ac^2 + 4ab^3$

(c) 8abc - 7ac + 3abc

Checkpoint 1.2.15

Identify the coefficient in each term in the expression below.

$$6m - 5u^2 + 3p^3$$

1.3 Combining Factors I

Learning Intentions

- Combine factors

In the last lesson, we learned how to combine like terms. We introduced the idea of coefficients. In this lesson, we will introduce combining factors. This lesson, we will jump right into the examples.

Example 1.3.1

Simplify the expression 4×3.

Solution. This is relatively straightforward, 4×3 is $\boxed{12}$.

Example 1.3.2

Simplify the expression a $\times$ b.

Solution. There isn't anything to simplify here. We can just rewrite this as ab to save space.

Example 1.3.3

Simplify the expression 4a $\times$ 3b.

Solution. We can rearrange this as $4 \times 3 \times a \times b$. As before, we can simplify 4×3 to get 12. We can rewrite this as $12 \times ab$ or 12ab.

Example 1.3.4

Simplify the expression $\frac{6m}{2}$.

Solution. We can rewrite this as $\frac{6}{2}m$. $\frac{6}{2}$ is 3, so our answer is 3m.

This is a rather short lesson. Before we can cover the rest of this topic, we need to talk about exponent rules, which is coming up in the next chapter. We'll see you there! But first, do these quick checkpoint exercises and the exercises for this chapter.

Checkpoint 1.3.5

Simplify the expression $3m \times 2b$.

1.4 Exercises

1. How many terms are in the expression $8x^yz + 3xz^2$?
2. List the terms in the expression $15x^2 + 6x + 2$.
3. List the terms in the expression $3 \times 5x^2$
4. List the terms in the expression $3 + 5x$.
5. List all of the like terms in the expression $4x + 4y$.
6. List all of the like terms in the expression $8ab + 9ab + 3ac$.
7. Determine if $8a^2c$ and $9ac^2$ are like terms.
8. Determine if 9mb and 11mb are like terms.
9. Simplify 11mb + 9mb.
10. Simplify 18m + 12mb + 13m $\times$ 2b.

Chapter 2

Exponent Rules

Woah! You survived Chapter 1! Good job. It was hard. Unfortunately, it gets harder from here. This chapter, we will look more at multiplying expressions. We will look at multiplying expressions with the same variables.

2.1 Basic Exponents

Yay! We will now look at basic exponents! Let's dive right in!

Example 2.1.1

Write $3 \times 3 \times 3$ as an exponent.

Solution. We are multiplying 3 by itself 3 times. We can write that as 3^3.

Example 2.1.2

Write $x \times x \times x$ as an exponent.

Solution. Similar to the previous example, we are multiplying x times itself 3 times and we would write it as x^3.

Example 2.1.3

Write x as an exponent.

Solution. We are multiplying x by itself 1 time to get just x. We can write this as x^1.

Let's do a few exercises. You should really try to understand the solutions for these, as you will need these skills next lesson. Have fun!

Checkpoint 2.1.4

Write $y \times y \times y \times y$ as an exponent.

Checkpoint 2.1.5

Write the expression $x \times x \times x \times x \times x$ as an exponent.

Checkpoint 2.1.6

Write $4 \times 4 \times 4 \times x$.

Checkpoint 2.1.7

Write $5 \times x \times x \times x \times y \times y$ as an exponent.

That's all for this lesson! We'll continue with multiplying exponents in the next lesson!

2.2 Multiplying Exponents

Hopefully, you feel relatively comfortable with the ideas presented last lesson. We will continue on this idea by disassembling exponents and exploring properties of these exponents.

Example 2.2.1

Expand the exponent 2^3

Solution. We know that 2^3 is 2 multiplied by itself 3 times, so we write it as $2 \times 2 \times 2$.

Example 2.2.2

Expand the exponent x^3.

Solution. Let's use the logic from the previous example to write this as $x \times x \times x$.

Example 2.2.3

Simplify the expression $x^3 \times x^2$.

Solution. Well, let's start by expanding x^3, which we write as $x \times x \times x$. We will now expand x^2, which we write as $x \times x$. Let's write this out fully: $x \times x \times x \times x \times x$. Now, we can write that as an exponent, similar to what we did in Lesson 1. We are multiplying x by itself 5 times, so we get x^5 as our final answer.

In the last example, notice how we combined the exponents.

Important!

When multiplying exponents, you can add the exponents. Or, in math terms, $x^a \times x^b = x^{a+b}$.

Now let's do some more examples.

Example 2.2.5

Simplify the expression $x^4 \times x^2$.

Solution. Let's do this 2 ways. First, we can do it the long way that writes it out, and then let's do it the short way. First up, let's do the long way! We can write x^4 as $x \times x \times x \times x$ and x^2 as $x \times x$. Multiplying those together, we get $x \times x \times x \times x \times x \times x$ or x^6. Let's do it the 2nd way. We will add the exponents first: $4 + 2$ is 8, so $x^4 \times x^2$ is x^6.

Example 2.2.6

Simplify the expression $x^2 \times x^3$.

Solution. Let's do this the fast way: $2 + 3 = 5$, so $x^2 \times x^3 = x^5$.

Example 2.2.7

Simplify the expression $x^2 + x^3$.

Solution. What's wrong with this solution?

Bogus Solution.

Let's add the exponents. $2 + 3 = 5$, so $x^2 + x^3 = x^5$.

Do you remember the rules for adding terms? You can only add terms if their variables and their exponents are the same. Is x^2 the same as x^3? No. The exponents are different. So, there are no like terms, and you cannot simplify. Did you make this mistake? (The solution should be $x^2 + x^3$.)

Warning 2.2.8

You cannot add terms with different exponents.

Example 2.2.9

Simplify the expression $y^3 \times y$.

Solution. What's wrong with this solution?

Bogus Solution.

There isn't any exponent for y, so the exponent for y stays as 3. Thus, the solution is y^3.

Remember, if no exponent is written, you should write the exponent as 1. Thus, the expression would be $y^3 \times y^1$, and we can simplify that to be y^4.

Important!

If no exponent is specified, write the exponent as 1.

Example 2.2.11

Simplify the expression $y^2 \times x^2$.

Solution. What's wrong with this solution?

Bogus Solution.

We can add the exponents, getting y^4x^4.

We can only combine exponents if the base is the same. $3^2 \times 2^2$ is NOT 3^4. Thus, the answer should be x^2y^2.

Warning 2.2.12

We can only combine exponents whose bases are the same.

Checkpoint 2.2.13

Simplify the expression $p^9 \times p$.

Checkpoint 2.2.14

Simplify the expression $q^3 \times q^2 \times q$.

Checkpoint 2.2.15

Simplify the expression $t^2 \times t^9$

Checkpoint 2.2.16

Simplify the expression $z^{-1} \times z^3$.

Hint. What would you do here? Do the normal rules apply?

The next exercise will be more challenging.

Checkpoint 2.2.17

Simplify the expression $3z^2 \times 2z^3 + z^5$.

Checkpoint 2.2.18

Simplify the expression $y^a \times y^b$.

Did these exercises make sense to you? Next section, we will look at dividing exponents. Are you ready?

2.3 Dividing Exponents

Hopefully, you feel relatively comfortable with the ideas presented last lesson. We will continue on this idea by disassembling exponents and exploring properties of these exponents.

Example 2.3.1

1. Write out x^5.
2. Write out x^3.
3. Hence, write out $\frac{x^5}{x^3}$.

Solution. We can write x^5 as $x \times x \times x \times x \times x$ and x^3 as $x \times x \times x$. We can thus write $\frac{x^5}{x^3}$ as $\frac{x \times x \times x \times x \times x}{x \times x \times x}$.

Example 2.3.2

Simplify $\frac{2}{2}$.
Solution. $\frac{2}{2}$ is 1.

We can use this same logic to divide variables (and exponents).

Example 2.3.3

Simplify $\frac{x}{x}$.
Solution. Just like the previous example, $\frac{x}{x} = 1$.

Example 2.3.4

Simplify $\frac{x^4}{x^2}$.
Solution. We can write out x^4 as $x \cdot x \cdot x \cdot x$ and x^2 as $x \times x$. Now we can write out a fraction:

$$\frac{x \times x \times x \times x}{x \times x}$$
$$\frac{\color{red}{x \times x} \times x \times x}{\color{red}{x \times x}}$$

We know that $\frac{x}{x}$ is 1, and so $\frac{x \times x}{x \times x}$ can also be written as 1. Note that we can turn $\frac{x \times x}{x \times x}$ into $\frac{x}{x} \times \frac{x}{x}$, just as we can write $\frac{1}{2} \times \frac{3}{4}$ as $\frac{1 \times 3}{2 \times 4}$, and vice versa. So, we can do the same thing to $\frac{\color{red}{x \times x} \times x \times x}{\color{red}{x \times x}}$.

$$\frac{x}{x} \times \frac{x}{x} \times x \times x$$

Thus, we get $1 \times 1 \times x \times x$, or just x^2.

Let's try another example. Now that we have a general idea, we can try creating a rule.

Example 2.3.5

Simplify $\frac{x^5}{x}$.

Solution. We can write x^5 as $x \times x \times x \times x \times x$. Let's write out the expanded fraction:

$$\frac{x \times x \times x \times x \times x}{x}$$
$$\frac{x}{x} \times x \times x \times x \times x$$
$$x \times x \times x \times x$$
$$\boxed{x^4}$$

Let's compare the exponents. x can be written as x^1. We notice that 4 is $5 - 1$.

We have a broad rule now. Let's test it by putting it into action!

Example 2.3.6

Simplify $\frac{x^3}{x^2}$

Solution. We need to do this both ways. Let's do this the "new" way first. Our exponents are 3 and 2. We can just subtract 2 from 3: $3 - 2 = 1$. So, our answer should be x^1 or just x.

Now let's do it the long way. x^3 is $x \times x \times x$, and x^2 is $x \times x$.

$$\frac{x \times x \times x}{x \times x}$$
$$\frac{x}{x} \times \frac{x}{x} \times x$$
$$1 \times 1 \times x$$
$$x$$

As expected, we get x. We can now form a generalization statement.

Important!

We can write $\frac{x^m}{x^n}$ as x^{m-n}.

Appendix A

Hints

1 · An Introduction to Algebraic Expressions
1.1 · Terms, Factors, Expressions, oh my!

Checkpoint 1.1.6

(a) Where should the addition signs go? How many are there? Where should the multiplication signs go? How many are there?

(b) Pick out all of the terms. See if you can put any multiplication signs anywhere.

2 · Exponent Rules
2.2 · Multiplying Exponents

Checkpoint 2.2.16 What would you do here? Do the normal rules apply?

Appendix B

Solutions

1 · An Introduction to Algebraic Expressions
1.1 · Terms, Factors, Expressions, oh my!

Checkpoint 1.1.6

(a) Answer. There is one term, whose factors are a, b, c.

Solution. Write abc as $a \times b \times c$. Since there are 3 items separated by multiplication signs, there are 3 factors.

(b) Answer.

$Term$	Factors
3xy	3, x, y
2yz	2, y, z
7ab	7, a, b

Checkpoint 1.1.7 Answer. 2

Checkpoint 1.1.8 Answer. $6 \times x \times y \times z$

1.2 · Combining Terms

Checkpoint 1.2.3 Answer. x^2y and $3yx^2$, $9x^2$ and $\frac{1}{2}x^2$, 3yx has no like terms

Checkpoint 1.2.14

(a) Answer. 6am

Solution. Add the coefficients. 9 + (-3) is 6. Thus, the coefficient must be 6; we get 6am.

(b) Answer. $4ac^2 + 4ab^3$

Solution. Notice that there are no like terms in $4ac^2 + 4ab^3$ because the variables are not the same. Thus, this expression cannot be simplified.

(c) Answer. 5abc - 7ac

Checkpoint 1.2.15 Answer. The coefficient of 6m is 6. The coefficient of $-5u^2$ is -5. The coefficient of $3p^3$ is 3.

1.3 · Combining Factors I

Checkpoint 1.3.5 Answer. 6bm

Solution. 2×3 is 6. m × b is mb. 6 × mb is 6mb.

1.4 · Exercises

1.4.1. Answer. 2

1.4.2. Answer. $15x^2$, 6x, 2

1.4.3. Answer. $15x^2$ or $3 \times 5x^2$

1.4.4. Answer. 3, 5x

1.4.5. Answer. There are no like terms.

1.4.6. Answer. 8ab and 9ab are like terms.

Solution. Both 8ab and 9ab have the variables a and b. 3ac has no like terms because there are no other terms with the variables a and c.

1.4.7. Answer. $8a^2c$ and $9ac^2$ are not like terms.

1.4.8. Answer. 9mb and 11mb are like terms.

1.4.9. Answer. 20mb

Solution. Add the coefficients: mb(11 + 9) = 20mb.

1.4.10. Answer. 18m + 38mb

Solution. First, do multiplication: 13m $\times$ 2b is 13 × 2 × mb $\rightarrow$ 26mb. Next add: 18m + 12mb + 26mb. 12mb and 26mb are like terms, so we can add the coefficients: 38mb. So, the final answer is $\boxed{38mb}$.

2 · Exponent Rules

2.1 · Basic Exponents

Checkpoint 2.1.4 Answer. y^4.

Solution. We are multiplying y by itself 4 times, to get y^4.

Checkpoint 2.1.5 Answer. x^5

Checkpoint 2.1.6 Answer. 4^3x

Solution. We are multiplying 4 by itself 3 times, so we write 4^3, and we are multplying that by x. Thus, we can write 4^3x.

Checkpoint 2.1.7 Answer. $5x^3y^2$

Solution. We are multiplying x by itself 3 times and y by itself 2 times. We can write those as x^3 and y^2. Since we are multiplying all of those together, we can write it as $5x^3y^2$.

2.2 · Multiplying Exponents

Checkpoint 2.2.13 Answer. p^{10}

Solution. Write p as p^1. Thus, we add the exponents as follows: $p^9 \times p^1 = p^{9+1} = p^{10}$. Thus, the answer is p^{10}.

Checkpoint 2.2.14 Answer. q^6

Solution. Write q as q^1. We add the exponents: 3 + 2 + 1 = 6. Thus, the final answer should be q^6.

Checkpoint 2.2.15 Solution. Add the exponents: 2 + 9 = 11, so $t^2 \times t^9 = t^11$.

Checkpoint 2.2.16 Answer. z^2

Solution. Add the exponents as normal: -1 + 3 = 2, so $z^{-1} \times z^3 = z^2$.

Checkpoint 2.2.17 Solution. First, we need to do multiplication. We can write $3z^2 \times 2z^3$ as $3 \times z^2 \times 2 \times z^3$, which we can rearrange as $3 \times 2 \times z^2 \times z^3$. First do 3×2, which is 6. Add the exponents: $z^2 \times z^3 = z^5$. So, we have $6z^5 + z^5$. Now we must add like terms: $6z^5$ and z^5 are like terms, so adding those gets $7z^5$. Thus, our answer is $\boxed{7z^5}$.

Checkpoint 2.2.18 **Solution**. This isn't any different to previous problems. We just add the exponents to get $y^a \times y^b = y^{a+b}$.

Colophon

By Dalton Ma