



Cambridge International AS & A Level

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FURTHER MATHEMATICS

9231/21

Paper 2 Further Pure Mathematics 2

October/November 2025

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [].

This document has **20** pages. Any blank pages are indicated.

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1 Find the values of k for which the system of equations

$$\begin{aligned}x - y + z &= 0, \\x + ky + 3z &= 0, \\x + 2y + kz &= 0,\end{aligned}$$

does not have a unique solution.

[3]



2 The curve C has parametric equations

$$x = \sinh t \quad \text{and} \quad y = t + \cosh t.$$

- (a) Find $\frac{dy}{dx}$ in terms of t .

[2]

- (b) Show that $\frac{d^2y}{dx^2} = \frac{1 - \sinh t}{\cosh^3 t}$.

[4]





- (c) Find the Maclaurin's series for y in terms of x up to and including the term in x^2 . [2]





- 3 The integral I_n is defined by $I_n = \int_0^1 (1+x^2)^n dx$.

(a) By considering $\frac{d}{dx}(x(1+x^2)^n)$, or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}.$$

[5]

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(b) Find the exact value of I_{-2} .

[4]



- 4** Find the particular solution of the differential equation

$$5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 5x + 3,$$

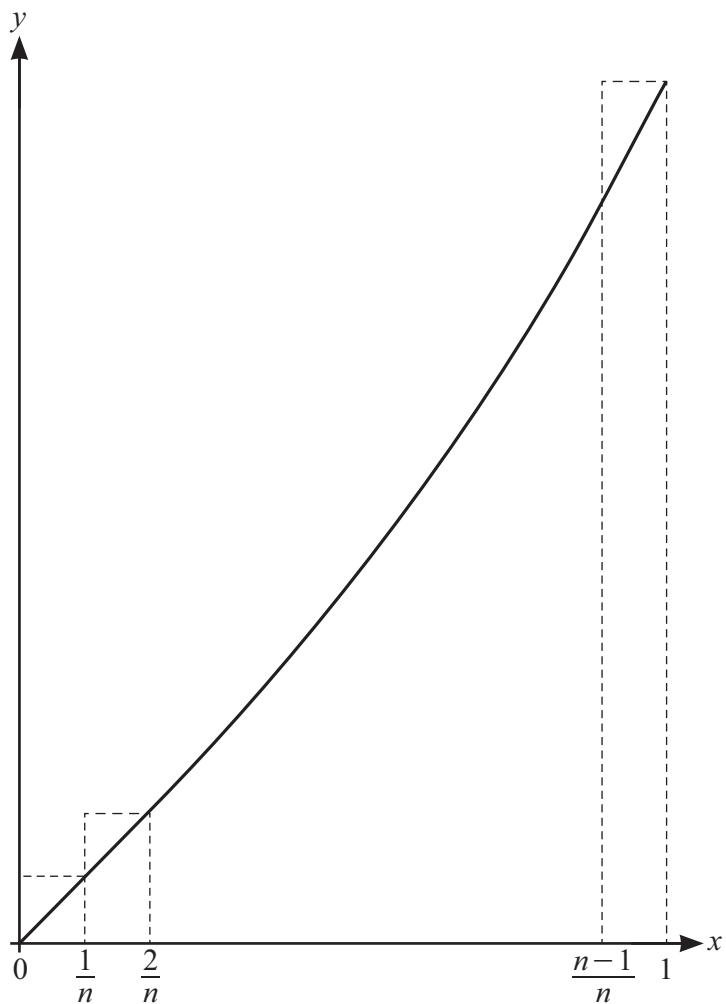
given that, when $x = 0$, $y = \frac{dy}{dx} = 0$.

[10]

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The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \leq x \leq 1$, together with a set of n rectangles of width $\frac{1}{n}$.

- (a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx < U_n$, where

$$U_n = \frac{1}{12} \left(1 + \frac{1}{n}\right) \left(7 + \frac{1}{n}\right). \quad [5]$$



- (b) Use a similar method to find, in terms of n , a lower bound L_n for $\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx$. [4]
- (c) Show that $\lim_{n \rightarrow \infty} (U_n - L_n) = 0$. [2]



- 6** The matrix \mathbf{P} has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

- (a) State, in terms of a , the eigenvalues of \mathbf{P} .

[1]

.....

- (b) (i) Find \mathbf{P}^2 in terms of a .

[1]

- (ii) Use the characteristic equation of \mathbf{P} to find \mathbf{P}^{-1} in terms of a .

[3]



The 3×3 matrix \mathbf{A} has eigenvalues 1, 2, 3 with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

- (c) Find A in terms of a .

[5]



7 (a) Show that $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$.

[3]

(b) Find the solution of the differential equation

$$x \frac{dy}{dx} - y = x^2 \tanh^{-1} x,$$

for $0 < x < 1$, given that $y = 0$ when $x = \frac{1}{2}$. Give your answer in an exact form.

[9]



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[Turn over

- 8** (a) State the sum of the series $z + z^2 + \dots + z^n$, for $z \neq 1$.

[1]



- (b) By letting $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ use de Moivre's theorem to deduce that

$$\sum_{m=1}^n \left(\frac{1}{2}\right)^m \sin m\theta = \frac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin \theta}{\frac{5}{4} - \cos \theta}.$$

[6]





- (c) Use the result in (b) to find $\sum_{m=1}^n \left(\frac{1}{2}\right)^m m \cos m\theta$ in terms of n and θ . [You do not need to simplify your answer.] [3]

- (d) Hence find $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m m \cos m\theta$ in terms of $\cos \theta$. [You may assume that $\frac{n}{2^n} \rightarrow 0$ as $n \rightarrow \infty$.] [2]



Additional page

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