



# Cambridge International AS & A Level

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## FURTHER MATHEMATICS

9231/23

Paper 2 Further Pure Mathematics 2

October/November 2025

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

### INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

### INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

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**1** Find the values of  $k$  for which the system of equations

$$x - y + z = 0,$$

$$x + ky + 3z = 0,$$

$$x + 2y + kz = 0,$$

does not have a unique solution.

[3]

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**2** The curve  $C$  has parametric equations

$$x = \sinh t \quad \text{and} \quad y = t + \cosh t.$$

**(a)** Find  $\frac{dy}{dx}$  in terms of  $t$ . [2]

[illegible]

**(b)** Show that  $\frac{d^2y}{dx^2} = \frac{1 - \sinh t}{\cosh^3 t}$ . [4]

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(c) Find the Maclaurin’s series for  $y$  in terms of  $x$  up to and including the term in  $x^2$ . [2]

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**3** The integral  $I_n$  is defined by  $I_n = \int_0^1 (1+x^2)^n dx$ .

(a) By considering  $\frac{d}{dx}(x(1+x^2)^n)$ , or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}. \quad [5]$$

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(b) Find the exact value of  $I_{-2}$ .

[4]

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**4** Find the particular solution of the differential equation

$$5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 5x + 3,$$

given that, when  $x = 0$ ,  $y = \frac{dy}{dx} = 0$ . [10]

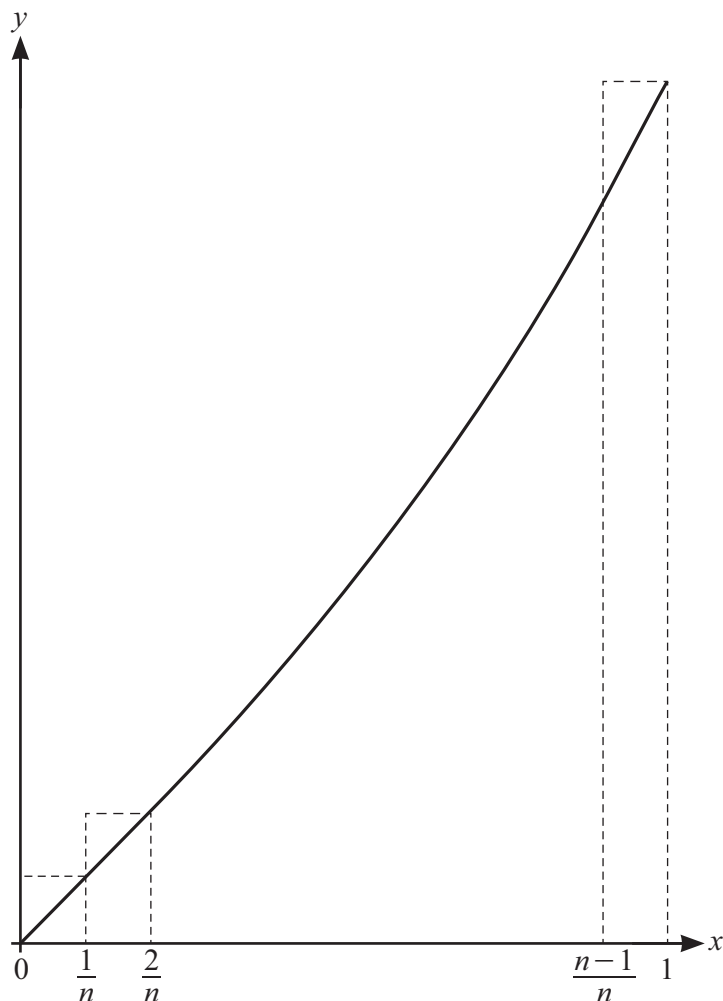
[illegible]





9

[illegible]



The diagram shows the curve with equation  $y = \frac{1}{3}x^3 + x$  for  $0 \leq x \leq 1$ , together with a set of  $n$  rectangles of width  $\frac{1}{n}$ .

- (a) By considering the sum of the areas of these rectangles, show that  $\int_0^1 \left( \frac{1}{3}x^3 + x \right) dx < U_n$ , where

$$U_n = \frac{1}{12} \left( 1 + \frac{1}{n} \right) \left( 7 + \frac{1}{n} \right). \quad [5]$$

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- (b) Use a similar method to find, in terms of  $n$ , a lower bound  $L_n$  for  $\int_0^1 \left( \frac{1}{3}x^3 + x \right) dx$ . [4]

- (c) Show that  $\lim_{n \rightarrow \infty} (U_n - L_n) = 0$ . [2]





**6** The matrix  $\mathbf{P}$  has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

(a) State, in terms of  $a$ , the eigenvalues of  $\mathbf{P}$ . [1]

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**(b) (i)** Find  $\mathbf{P}^2$  in terms of  $a$ . [1]

[illegible]

(ii) Use the characteristic equation of  $\mathbf{P}$  to find  $\mathbf{P}^{-1}$  in terms of  $a$ . [3]

[illegible]



The  $3 \times 3$  matrix  $\mathbf{A}$  has eigenvalues 1, 2, 3 with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

(c) Find  $\mathbf{A}$  in terms of  $a$ .

[5]

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7 (a) Show that  $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$ .

[3]

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(b) Find the solution of the differential equation

$$x \frac{dy}{dx} - y = x^2 \tanh^{-1}x,$$

for  $0 < x < 1$ , given that  $y = 0$  when  $x = \frac{1}{2}$ . Give your answer in an exact form.

[9]

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- 8 (a) State the sum of the series  $z + z^2 + \dots + z^n$ , for  $z \neq 1$ . [1]

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- (b)** By letting  $z = \frac{1}{2}(\cos \theta + i \sin \theta)$  use de Moivre's theorem to deduce that

$$\sum_{m=1}^n \left(\frac{1}{2}\right)^m \sin m\theta = \frac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin \theta}{\frac{5}{4} - \cos \theta}. \quad [6]$$

[illegible]





- (c) Use the result in (b) to find  $\sum_{m=1}^n \left(\frac{1}{2}\right)^m m \cos m\theta$  in terms of  $n$  and  $\theta$ . [You do not need to simplify your answer.] [3]

- (d) Hence find  $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m m \cos m\theta$  in terms of  $\cos \theta$ . [You may assume that  $\frac{n}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$ .] [2]





Additional page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

Lined area for writing answers, consisting of multiple horizontal dotted lines.







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