



# Cambridge International AS & A Level

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## FURTHER MATHEMATICS

9231/23

Paper 2 Further Pure Mathematics 2

October/November 2025

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

### INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

### INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

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**1** Find the values of  $k$  for which the system of equations

$$\begin{aligned}x - y + z &= 0, \\x + ky + 3z &= 0, \\x + 2y + kz &= 0,\end{aligned}$$

does not have a unique solution.

[3]

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**2** The curve  $C$  has parametric equations

$$x = \sinh t \quad \text{and} \quad y = t + \cosh t.$$

- (a) Find  $\frac{dy}{dx}$  in terms of  $t$ .

[2]

- (b) Show that  $\frac{d^2y}{dx^2} = \frac{1 - \sinh t}{\cosh^3 t}$ .

[4]





- (c) Find the Maclaurin's series for  $y$  in terms of  $x$  up to and including the term in  $x^2$ . [2]



3 The integral  $I_n$  is defined by  $I_n = \int_0^1 (1+x^2)^n dx$ .

- (a) By considering  $\frac{d}{dx}(x(1+x^2)^n)$ , or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}.$$

[5]



(b) Find the exact value of  $I_{-2}$ .

[4]



- 4** Find the particular solution of the differential equation

$$5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 5x + 3,$$

given that, when  $x = 0$ ,  $y = \frac{dy}{dx} = 0$ .

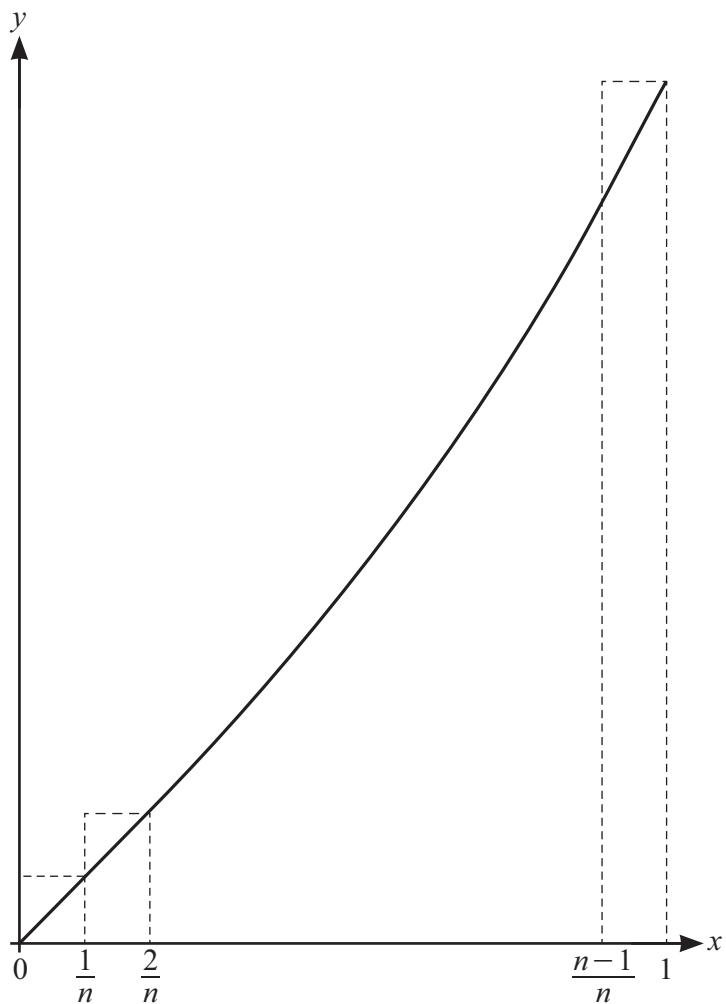
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The diagram shows the curve with equation  $y = \frac{1}{3}x^3 + x$  for  $0 \leq x \leq 1$ , together with a set of  $n$  rectangles of width  $\frac{1}{n}$ .

- (a) By considering the sum of the areas of these rectangles, show that  $\int_0^1 \left( \frac{1}{3}x^3 + x \right) dx < U_n$ , where

$$U_n = \frac{1}{12} \left( 1 + \frac{1}{n} \right) \left( 7 + \frac{1}{n} \right). \quad [5]$$

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- (b) Use a similar method to find, in terms of  $n$ , a lower bound  $L_n$  for  $\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx$ . [4]

- (c) Show that  $\lim_{n \rightarrow \infty} (U_n - L_n) = 0$ . [2]



- 6** The matrix  $\mathbf{P}$  has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

- (a) State, in terms of  $a$ , the eigenvalues of  $\mathbf{P}$ .

[1]

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- (b) (i) Find  $\mathbf{P}^2$  in terms of  $a$ .

[1]

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- (ii) Use the characteristic equation of  $\mathbf{P}$  to find  $\mathbf{P}^{-1}$  in terms of  $a$ .

[3]





The  $3 \times 3$  matrix  $\mathbf{A}$  has eigenvalues 1, 2, 3 with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

- (c) Find A in terms of  $a$ .

[5]



7 (a) Show that  $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ .

[3]

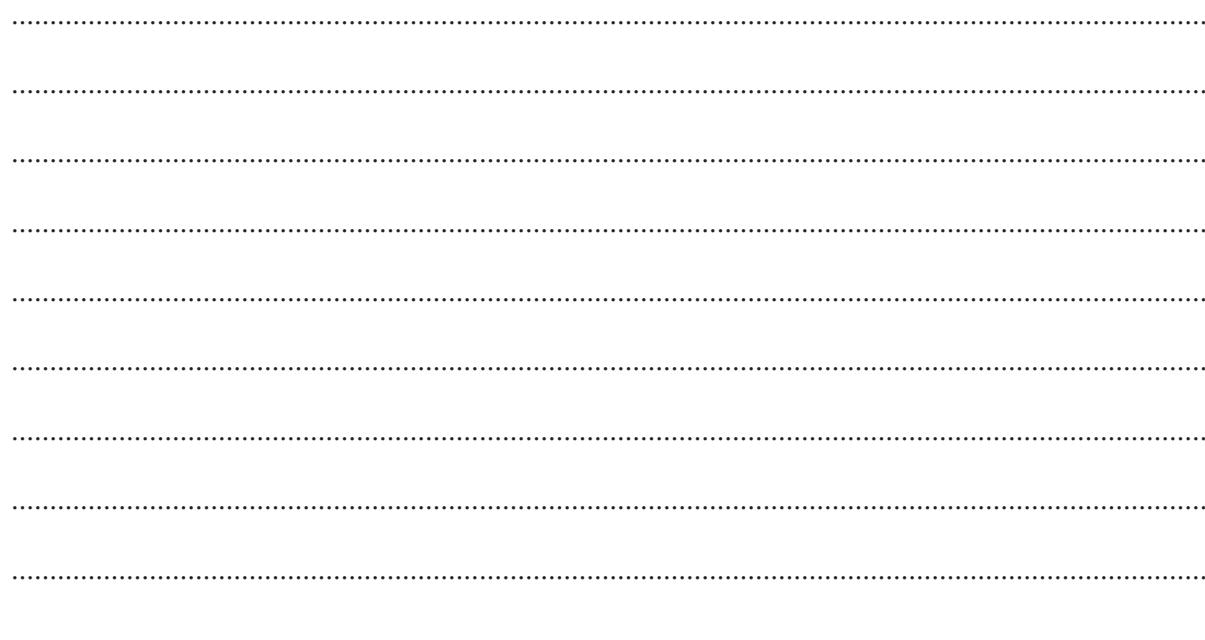





(b) Find the solution of the differential equation

$$x \frac{dy}{dx} - y = x^2 \tanh^{-1} x,$$

for  $0 < x < 1$ , given that  $y = 0$  when  $x = \frac{1}{2}$ . Give your answer in an exact form. [9]



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[Turn over

- 8** (a) State the sum of the series  $z + z^2 + \dots + z^n$ , for  $z \neq 1$ .

[1]



- (b) By letting  $z = \frac{1}{2}(\cos \theta + i \sin \theta)$  use de Moivre's theorem to deduce that

$$\sum_{m=1}^n \left(\frac{1}{2}\right)^m \sin m\theta = \frac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin \theta}{\frac{5}{4} - \cos \theta}.$$

[6]





- (c) Use the result in (b) to find  $\sum_{m=1}^n \left(\frac{1}{2}\right)^m m \cos m\theta$  in terms of  $n$  and  $\theta$ . [You do not need to simplify your answer.] [3]

- (d) Hence find  $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m m \cos m\theta$  in terms of  $\cos \theta$ . [You may assume that  $\frac{n}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$ .] [2]



## **Additional page**

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