Competing Intermediaries*

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Abstract

This paper offers a tractable model of competing intermediaries. We consider two representative modes of intermediation: a middleman mode where an intermediary holds inventories which he stocks from sellers to resell to buyers; a market-making mode where an intermediary offers a platform for buyers and sellers to trade with each other. In a Bertrand competition between two intermediaries where one can combine the two modes and the other is a pure market-maker, we show that a *marketmaking middleman*, who adopts the mixture of these two intermediation modes, can emerge in equilibria.

Keywords: Middlemen, Marketmakers, Platform, Duopoly

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1 Introduction

This paper develops a theory of competing intermediaries who can adopt hybrid business modes. We consider two representative business modes of intermediation. In one mode, an intermediary acts as a *middleman*, who holds inventory and resells to buyers, e.g., supermarkets. In the other mode, an intermediary acts as a *marketmaker*, who offers a platform for fees, where the participating buyers and sellers can search and trade with each other, e.g., auction sites and many real estate agencies.

In most real-life markets intermediaries are not one of those extremes but operate both as a middleman and a market-maker at the same time. This is what we call a *market-making middleman.* One well-known example is the electronic intermediary Amazon, who started as a pure middleman, buying and reselling products in its name. In facing the competition of eBay, Amazon moved toward a market-maker by allowing third-party sellers to join its marketplace. Today, the market-maker sector has accounted for around half of the gross merchandise volume of Amazon. A similar pattern has been observed in financial markets. For example, the New York Stock Exchange (NYSE) adopted an expanded platform "NYSE Arca" after a serious market share drop around 2008. In housing markets, the Trump Organization established a luxury residential real estate brokerage firm, competing with thousands of housing brokers in New York City. Recently, Ikea is exploring some online sales website that combines a middleman mode (Ikea products) and a market-maker mode (rival products), with Alibaba and Amazon in mind.[1]

A common feature of these marketmaking middlemen is the presence of competing intermediaries. Amazon's marketplace is, at least partially, due to the competition with eBay.[2] NYSE's platform sector is clearly driven by the decentralized equity trading — the order-flow in NYSE-listed stocks today is divided among many trading venues, 11 exchanges, more than 40 alternative trading systems and more than 250 broker-dealers in the U.S. (Tuttle, 2014). In short, the hybrid intermediation mode seems to be a way to keep ahead of competing intermediaries.

[1]See a report of Financial Times *https://on.ft.com/2StN4GU* (visited on Feb 25th, 2019).

[2]See details in the book of *The Everything Store: Jeff Bezos and the Age of Amazon* by Stone (2013).

In the precursor to the current paper (Gautier et al., 2018), we have shown that in the presence of outside markets, a marketmaking middlemen mode can be profit-maximizing for a monopolistic intermediary. This is because a middleman with its large inventory holdings can reduce the out-of-stock risk, while a marketmaker with its enrolled third-party sellers can reduce the outside option value of buyers and lower the outside competition pressure. It is not clear, though, whether this intuition holds when the outside market is not passive but is operated by another strategic player. The current paper fills this gap by explaining the emergence of marketmaking middlemen in a duopoly.

In our model, there is a finite mass of buyers and sellers who are unable is meet other than using an intermediary. There are two intermediaries open to agents, call them an incumbent and an entrant. We assume the entrant is restricted to be a pure market-maker, while the incumbent can combine two business modes: as a middleman, he is prepared to serve many buyers at a time by holding inventories; as a marketmaker, he offers a platform and charges transaction fees. Both intermediaries rely on a transaction fee paid ex-post when a transaction takes place between two matched parties. Besides, the incumbent has an additional pricing instrument, the price of its inventory which affects the allocation of the attending buyers among his two business modes.

We formulate the incumbent market as a directed search market to feature the intermediary's technology of spreading price and capacity information efficiently. For example, one can receive instantly all relevant information such as prices, the terms of trade and stocks of individual sellers using the search function in web-based platforms. In this setting, each seller is subject to an inventory capacity of discrete units (normalized to one unit in the model), whereas the middleman has continuous access to production.[3] Naturally, the middleman is more efficient in matching demands with supplies in a directed search equilibrium.

With this setup, we investigate a Bertrand duopoly competition game. A key difference between the two intermediaries is that agents hold an optimistic (pessimistic) belief towards the incumbent (entrant). That is, agents are willing to visit and trade at the incumbent when-

[3] Or equivalently, the middleman is subject to an inventory capacity of a mass K, and assuming a low inventory cost, then the optimal K is chosen to satisfy all demands. Essentially, the out-of-stock risk is zero at the middleman, as is shown in Gautier et al. (2018).

ever possible. This assumption on pessimistic beliefs about the participation decisions of agents on the other side of the market is consistent with Caillaud and Jullien (2003). It is in this sense that the incumbent is more established. Based on the pessimistic beliefs about the entrant, we analyze the existence and sustainability of the market structure in two situations, *single-market search* versus *multiple-market search*.

Under single-market search, agents have to choose which intermediary to visit in advance. Under the pessimistic beliefs against the entrant and the two-sidedness of a platform, it is a classical result that in the equilibrium all agents only visit and trade at the incumbent. That is, market monopolization is the only equilibrium. Given that the middleman mode is more efficient in realizing transactions, the incumbent uses the middleman-mode exclusively when agents search in a single market. This conclusion is the same as Gautier et al. (2018).

In multi-market search, we allow for the possibility that agents search sequentially. As in Gautier et al. (2018), we assume the incumbent intermediary opens prior to the entrant. Accordingly, a pessimistic expectation against the entrant means that agents are willing to *first* visit and trade at the incumbent whenever possible. This changes the nature of competitive strategies — the prices/fees charged by the incumbent must be acceptable relative to the available option at the entrant; otherwise, buyers and sellers can easily switch to the entrant. Thus, under multiple-market search, the entrant represents an outside trading option for agents.

Consistent with Gautier et al. (2018), the incumbent faces a trade-off between trading quantity and marginal trading profit. On the one hand, a larger middleman sector leads to more transactions and consequently to larger profits. On the other hand, sellers are less likely to trade on a smaller-scaled platform, so more sellers are available when a buyer attempts search at the entrant market, a downward restriction on the price/fees that the incumbent can charge. This trade-off determines the middleman scale and eventually the intermediation mode of the incumbent. Beyond the results in Gautier et al. (2018), we further show that a pure middleman incumbent is profitable only when the entrant charges a fee at the monopoly level so that a buyer's expected value at the entrant is zero.

As a critical difference from Gautier et al. (2018), the entrant intermediary in our framework also faces strategic choices.[4] He can either act as a "second source", set a relatively high fee, wait for the leftover agents who are not matched at the incumbent market, or undercut the incumbent fiercely with a transaction fee low enough in order to break the pessimistic beliefs. In the latter case, the entrant, despite the pessimistic beliefs, becomes the sole active source. Therefore, how costly the undercutting determines how profitable a sole source can be. And this, in turn, gives different equilibria.

We first show that a marketmaking middleman incumbent emerges in the equilibria of multi-market search. A pure middleman incumbent can not show up in an equilibrium of pure strategies because the entrant has an incentive to undercut the incumbent, leading to a positive buyer's value at the entrant intermediary. According to the best response of the incumbent, it is profitable for the incumbent to activate its platform trading whenever a buyer's outside value at the entrant is positive. On the other hand, a pure market-making incumbent is also not possible in an equilibrium of pure strategies. This follows the logic of Varian (1980). The entrant has an incentive to undercut the incumbent for a discrete jump in total transactions, as long as transaction fee is positive. And at the fee level of zero, the entrant would rather increase the fee to the highest level, extract full surplus and work as a second source.

Our main result is that there exists a pure strategy equilibrium when undercutting the incumbent is costly. In the equilibrium, the incumbent works as a market-making middleman and is the first source to implement transactions, and the entrant is a second source. Both make positive profits. Moreover, we further extend the insights of Gautier et al. (2018): As the buyers' outside value becomes larger, the incumbent operates a smaller middleman sector and a larger platform sector in the equilibrium. Finally, when undercutting the incumbent is more profitable, we show there exists a mixed strategy equilibrium where the market-maker sector is activated with positive probability.

Literature Review This paper is an extension of Gautier et al. (2018) on competing intermediaries. The incumbent (entrant) here corresponds to the centralized market (decentralized market). So the best response analysis of the incumbent in our model corresponds to the

[4]In the model of Gautier et al. (2018), the decentralized market, a counterpart of the entrant intermediary here, is passive.

profit-maximizing intermediation mode analysis (Proposition 2) in that paper. We show the main conclusion of Gautier et al. (2018) can be extended to the case of duopoly competition despite the strategic behavior of the entrant. In particular, when undercutting the incumbent is not profitable, there exists a pure strategy equilibrium that resembles the market structure as is characterized by Gautier et al. (2018).

This paper is related to the middleman literature and the two-sided market literature, as has been discussed extensively in Gautier et al. (2018). Our model is in particular, closely related to Caillaud and Jullien (2003) who examined a Bertrand competition game between two intermediaries. They considered a rich set of contracting possibilities including the "divide-and-conquer" strategy. Using pessimistic beliefs, they characterized various equilibria assuming both intermediaries are pure market-makers. Compared to them, we abstract from the possibility of using negative participation fees for breaking the pessimistic beliefs and focus on the endogenous intermediation structure out of the market competition. A "divide-and-conquer" strategy would be useful for the entrant to break the disadvantageous beliefs. It is not clear whether the "divide-and-conquer" strategy would reinforce or undermine our results. We leave it for future research.

Our paper is also related to the vast literature of discontinuous games, see, for example, Dasgupta and Maskin (1986b). In our model, an equilibrium of pure strategies exists simply because cutting the transaction fee slightly is not enough to for the entrant to get a discrete jump in demands. As the entrant's fee decreases, a buyer finds it more appealing to visit the incumbent platform, since even he is unmatched, the outside option value — to trade at the entrant — is now higher. Consequently, more sellers are successfully matched at the incumbent platform, leaving fewer sellers available at the entrant. As a result, buyers' expected value at the entrant does not increase with a decreasing transaction fee. This would be the case as long as the incumbent is of a mixed mode. This barrier to an undercutting strategy makes a pure strategy equilibrium possible. When indeed undercutting is more profitable for the entrant, we make use of Proposition 5 in Dasgupta and Maskin (1986a) to show the existence of a mixed strategy equilibrium.

The plan of the article is as follows. Section 2 introduces the model. In Section 3, we consider the case of single-market search, which serves as a benchmark for our subsequent analysis. Section 4 contains our main results about multi-market search. Finally, Section 5 concludes. The more technical proofs are relegated to Appendix A. Appendix B contains our extension to the case that the entrant intermediary is a pure middleman.

2 Setup

Agents We consider a large economy with two populations, a mass B of buyers, a mass S of sellers. Agents of each type are homogeneous. Each buyer has unit demand for a homogeneous good, and each seller is able to sell one unit of that good. The consumption value for the buyers is normalized to 1. Sellers can purchase the good from a wholesale market. We assume the wholesale market is competitive, the demand of all suppliers is always satisfied with a price equal to the marginal cost which is normalized to zero.

Intermediaries Buyers and sellers cannot meet each other other than using intermediaries. There are two intermediaries in the economy, an incumbent I, and an entrant E. Both have some matching technology to facilitate trading. The names "incumbent" and "entrant" are not related to strategic entry deterrence. Instead, they reflect two advantages of I over E: First, the matching technology of I is more advanced in the sense that pricing and capacity information can be spread within the scope of I; Second, I faces favorable beliefs from agents. We will discuss each feature in details, starting with the matching technology.

Matching Technology of E Intermediary E is pure *marketmaker*, or equivalently a *platform*, and it makes profits by extracting a transaction fee denoted by $f^e \in [0,1]$. E owns an inferior matching technology. It is "inferior" because E is not able to spread price and stock information, and a buyer only gets such information after meeting a seller. We characterize this matching technology by random matching and bilateral bargaining. In particular, here we employ a simple linear matching function as follows. Suppose all buyers and sellers participate in E, then a buyers finds a seller with probability λ^b and a seller finds a buyer with probability λ^s, satisfying $B\lambda^b = S\lambda^s$. If a subset of buyers $B^E \leq B$ and sellers $S^E \leq S$

participate, then the meeting probabilities become $\lambda^{b\prime} \equiv \lambda^{b} \frac{S^E}{S}$ and $\lambda^{s\prime} \equiv \lambda^{s} \frac{B^E}{B}$, respectively.[5] Matched partners follow an efficient bargaining process, which yields a linear sharing of the total surplus, with a share of $\beta \in [0,1]$ for the buyer and a share of $1-\beta$ for the seller. In this paper, we focus on the case of $\beta = 1$, i.e. the buyer gets the full trading surplus.

Matching Technology of I Intermediary I can work as a *marketmaking middleman*. First, as a *marketmaker*, I extract transaction fees from platform trade. I's platform has a more advanced matching technology — the prices and capacities of all the individual suppliers are publicly observable — and hence buyers can make use of these information to "direct" their search. Still, given that individual buyers cannot coordinate their search activities, the limited selling capacity of individual sellers creates a possibility that some units remain unsold while at the same time some demands are not satisfied. In this sense, I is subject to coordination frictions. Second, other than the platform, I can also work as a *middleman* — he purchases a good from the wholesale market and resells it to buyers. Therefore, I can be both the manager of the platform (called a marketmaker) and a middleman participant to the platform, so called a "marketmaking middleman".

Compared to a seller, the middleman sector of I has an inventory advantage — he is able to hold more inventory, precisely a continuum of inventory, to lower the out-of-stock risk. For simplicity, we consider an extreme of the flexible inventory technology: A continuous access to production. That is, the middleman can produce as he gets an order from some buyer. Effectively, the out-of-stock probability is zero.[6]

Now let's describe how the directed search works in I. Given a mass of $B^I \in (0, B]$ buyers and a mass of $S^I \in (0, S]$ sellers have decided to participate in I, the matching process in I is specified by a directed search game which consists of the following stages. In the first

[5]To understand $\lambda^{b\prime}$, imagine that all sellers are registered at E, but not all would be available. In our model, a seller is absent from market E if he has sold the unit inventory in other markets. If the seller that a buyer is supposed to meet is not available, then the meeting would fail. Given an mass of $S^E < S$ sells join the platform, the probability a buyer finds an available seller would be $\lambda^{b} \frac{S^E}{S}$. The same logic applies to $\lambda^{s\prime}$. It is easy to verify that $B^E \lambda^{b\prime} = S^E \lambda^{s\prime}$.

[6]An alternative interpretation of the inventory technologies is that individual sellers have to *produce in advance* (up to the inventory constraint of 1 unit) whereas middlemen can *produce to orders*. Suppose the middleman has to "produce in advance" and holds a mass K of inventory. Given a zero inventory cost, it is a weakly dominant strategy to have K larger or equal to the mass of visiting buyers. This gives a zero out-of-stock probability.

stage, all the suppliers, i.e., the participating sellers with the unit selling-capacity and the middleman with a continuous access to production, post a price which they are willing to sell at. We denote the posting price of an individual seller by p^s, and that of the middleman by p^m. Observing the prices and the capacities, all buyers simultaneously decide which supplier to visit in the second stage. As is standard in the literature, we assume that each buyer can visit one supplier, one of the sellers or the middleman. Assuming buyers cannot coordinate their actions over which supplier to visit, we investigate a symmetric equilibrium where all buyers use an identical strategy for any configuration of the announced prices. Each individual seller (if any) has an expected queue x^s of buyers and the middleman has an expected queue x^m of buyers. These quantities should satisfy two requirements. The first requirement is a standard accounting identity,

$$S^I x^s + x^m = B^I, \tag{1}$$

which states that the number of buyers visiting individual sellers $S^I x^s$ and the middleman x^m should sum up to the total population of participating buyers B^I. The second requirement is that buyers search optimally:

$$x^m = \begin{cases} B^I & \text{if } V^m(B^I) \geq V^s(0) \\ (0, B^I) & \text{if } V^m(x^m) = V^s(x^s) \\ 0 & \text{if } V^m(0) \leq V^s(\frac{B^I}{S^I}), \end{cases} \tag{2}$$

where $V^i(x^i)$ is the equilibrium value of buyers in the C market of visiting a seller if $i = s$ and the middleman if $i = m$ (yet to be specified below). Combining (1) and (2) gives the counterpart for $x^s \in [0, \frac{B^I}{S^I}]$. We define the intermediation mode of I as follows.

Definition 1 *Suppose $B^I \in (0, B]$ buyers and $S^I \in (0, S]$ sellers participate in I. Then, given the equilibrium search conditions (1) and (2), we say that the intermediary acts as:*

- *a pure middleman if $x^m = B$;*
- *a market-making middleman if $x^m \in (0, B)$;*

- *a pure market-maker if* $x^m = 0$.

The Game The situations we consider in the following sections all have the same timing structure.

1. Two intermediaries setting fees/prices. Intermediary I decides whether or not to activate the middleman sector and/or the platform, and announces a transaction fee $f^I \in [0,1]$ charged to a seller, a price $p^m \in [0,1]$ charged to a buyer on his own inventory. Intermediary E announces a transaction fee $f^E \in [0,1]$ charged to a seller. Let's denote these prices (or fees, interchangeably) by $P = \{f^I, p^m, f^E\}$.[7]

2. Observing P, buyers and sellers simultaneously decide whether to participate in the two (or one out of two) intermediaries, yielding a distribution of buyers/sellers across intermediaries.

   We consider two different search technologies of buyers/sellers: single-market search, where buyers/sellers can attend only one market, and multi-market search where agents can attend both markets sequentially, first I then E. We make the usual tie-breaking assumption that agents choose to participate in a market if they are indifferent.

3. In the I market, the participating buyers, sellers and middleman are engaged in a directed search game as has been specified. In the E market, agents search randomly and follow the efficient sharing rule for the trade surplus.

Let $\mathcal{N} = \{B^I, B^E, S^I, S^E, x^m\}$ denote a distribution of buyers/sellers across intermediaries with B^I (B^E) the mass of buyers visiting I (E), S^I (S^E) the mass of sellers visiting I (E), and x^m the mass of buyers visiting the middlemen if $B^I > 0$. A market allocation is a mapping $\mathcal{N}(\cdot)$ that associates to each price/fee P an equilibrium distribution of buyers/sellers $\mathcal{N}(P)$. Hence, $\mathcal{N}(\cdot)$ generates a reduced-form price-setting game among intermediaries where a Nash equilibrium can be defined.

[7] Here we assume that transaction fees are charged to sellers for simplicity. But as is shown in Gautier et al. (2018), the results hold if the fee is charged on buyers or split between a matched pair.

Definition 2 *An equilibrium of the game with a market allocation $\mathcal{N}(\cdot)$ is a price vector $P^* = \{f^{I*}, p^{m*}, f^{E*}\}$ and an associated distribution of buyers and sellers $\mathcal{N}(P^*)$ where neither intermediary I nor E has an incentive to deviate under $\mathcal{N}(\cdot)$.*

Some regularities on $\mathcal{N}(\cdot)$ is necessary to proceed. Let's impose a tight structure on $\mathcal{N}(\cdot)$ according to pessimistic beliefs about E which is yet another advantage of I relative to E. A pessimistic market allocation differs slightly under single/multi-market search. In the single-market search, we impose $\mathcal{N}(\cdot)$ such that buyers and sellers coordinate on the distribution with zero market share for E as long as it is in their interests to do so.

Definition 3 (Pessimistic Expectations in Single-market Search) *Under single-market search, a pessimistic market allocation against E, $\mathcal{N}(\cdot)$, is that $B^I = B, S^I = S, B^E = S^E = 0$ as long as*

$$\max\{V^m(\mathcal{N}, P), V^s(\mathcal{N}, P)\} \geq 0,$$

for some $x^m \in [0, B]$; and $B^E = B, S^E = S, B^I = S^I = 0$ otherwise.

In single market search, agents decide which intermediary to join under the expectation that all others visit I. A buyer or a seller only chooses to join E if the fees of I are so high that he would rather take visiting E. Since he expects no counterparts at E, the expected value at E is 0.

In the multi-market search, intermediary I opens first, and participating in I does not rule out the possibility of visiting E later. A pessimistic market allocation against E is that agents first visit and trade at I and then E as long as it is in their interests to do so. We defer a formal definition to Section 4.

3 Single-market Search

In this section, we show that, under single-market search, I chooses to be a pure middleman in the equilibrium. We start by analyzing the best response of I. For any offers from E, buyers have an expected value $V^E = 0$ under favorable beliefs towards I.[8] Given $f^I, p^m \leq 1$, the expected value of an agent at I is non-negative. Thus, for all price P, we have the agent distribution $\mathcal{N}$ that $B^I = B, S^I = S, B^E = S^E = 0$.

Suppose I is a pure middleman, the best response involves setting $p^m = 1 - V^E = 1$, and it gives a profit of B. Since this is the whole trading surplus, it is easy to speculate that the pure middleman mode is the optimal response. This is indeed true: Even I can also get the full surplus for each transaction through the platform, as long as there is some matching frictions that deliver less transactions, using a platform is dominated.

We now derive the matching function of I's platform. Suppose I has an active platform with S sellers, $B - x^m$ buyers, and an intermediation fee $f^I \leq 1$. Then, the platform generates a non-negative trade surplus $1 - f^I \geq 0$. The number of buyers visiting an individual seller is a random variable, denoted by N, which follows a Poisson distribution, $\text{Prob}[N = n] = \frac{e^{-x}x^n}{n!}$, with an expected queue of buyers $x \geq 0$ (due to coordination frictions – see the seminal work by Peters, 1991 and 2001). With limited selling capacity, each seller is able to serve only one buyer. A seller with an expected queue $x^s \geq 0$ has a probability $1 - e^{-x^s}$ ($= \text{Prob}[N \geq 1]$) of successfully selling, while each buyer has a probability $\eta^s(x^s) = \frac{1-e^{-x^s}}{x^s}$ of successfully buying. Hence, the expected value of a seller in the platform with a price p^s and an expected queue x^s is given by

$$W(x^s) = x^s \eta^s(x^s)(p^s - f^I),$$

while the expected value of a buyer who visits the seller is

$$V^s(x^s) = \eta^s(x^s)(1 - p^s).$$

And the platform as a whole generates an expected trading volume of $S(1 - e^{-x^s})$. Hence,

[8] E faces a chicken and egg problem. To get a positive market share despite pessimistic beliefs, E must adopt a divide and conquer strategy. This is analyzed in Caillaud and Jullien (2003).

the resulting profit of I satisfies

$$S(1-e^{-x^s})f^I + x^m p^m < (Sx^s + x^m)\max\{f^I, p^m\} \leq B.$$

The second inequality follows that $f^I, p^m \leq 1$. And the first inequality follows directly from the matching function of the platform. In a nutshell, the pure middleman mode dominates any other modes with an active platform, and in the equilibrium, I is able to gain a monopoly profit B.

Proposition 1 (Pure Middleman) *Given single-market search technologies, in the equilibrium with a pessimistic market allocation against E, I acts as a pure middleman, setting $f^I = p^m = 1$, all buyers and sellers join intermediary I, and E is inactive with $f^E \in [0,1]$.*

4 Multi-market Search

This section is devoted to the analysis of competition between two intermediaries who offer non-exclusive services, so that users may engage in multi-market search.We say an intermediary is the *first source* if buyers/sellers would like to first visit it and trade there, and the *second source* if it is visited only when buyers/sellers did not trade at the first source.

In this section, we assume that intermediary I opens prior to intermediary E and I is the first source by default as in Gautier et al. (2018). This is the pessimistic belief against E under multi-market search. A formal definition will be given in Definition 4. It can be understood as the common belief in the economy and reflects the fact that the incumbent has been well-established in the market.

Under such a belief, when fees/prices of I are comparable to that of E, buyers and sellers will indeed firstly visit I and trade there. It is only when E's fees are *much* lower than that of I that buyers and sellers will forgo trading opportunities of I and trade at E instead. Then, I becomes inactive, and E is the only active intermediary, called the *sole source*.

We start by examining the directed search equilibrium at intermediary I and the best response of I assuming that I is the first source. This analysis is enough to show that equilibrium with pure mode I does not exist. We then turn to the best response analysis of E and the existence of an equilibrium with I of a mixed mode. We show that under certain conditions, there exists an equilibrium of pure strategies that features (1) both intermediaries are active; (2) agents meet and trade first at I and then join E, i.e. I is the first source and E is the second source; (3) I is a marketmaking middleman. Moreover, we show there exists an equilibrium of mixed strategies where the market-maker sector of I is activated with positive probability.

4.1 Directed search equilibrium at intermediary I

We work backward and first describe the directed search equilibrium at intermediary I, given such an equilibrium exists with some $P = \{f^I, p^m, f^E\}$.[9] As in single-market search, any directed search equilibrium has to satisfy (1) and (2). Given the multiple-market search technology, what is new here is that, when deciding whether or not to accept an offer in the I market, buyers expect a non-negative value at E, defined by

$$V^E(x^m, f^E) = \lambda^b e^{-x^s}(1 - f^E).$$

V^E can be understand as follows: The outside payoff is $1 - f^E$ if the buyer is matched with a seller who has failed to trade in the I market. This happens with probability $\lambda^b e^{-x^s}$. Hence, the larger the platform size x^s, the higher the chance that a seller trades in the I market, and the lower the chance that a buyer can trade successfully in the E market and the lower his expected payoff at E.

Whenever I's platform is active $x^s > 0$, it must satisfy the incentive constraints:

$$1 - p^s \geq V^E(x^m, f^E), \tag{3}$$

$$p^s - f^I \geq 0. \tag{4}$$

The constraint of buyers (3) states that the offered price/fee in the platform is acceptable only if the offered payoff, $1 - p^s$, is no less than the expected value in the E market $V^E(x^m, f^E)$.

[9]This section is a simplified version of the directed search equilibrium assuming seller's bargaining power at E is zero. For a full analysis, see section 4.1 of Gautier et al. (2018)

The constraint of sellers (4) states that the payoff in the I market $p^s - f^I$ should be no less than the expected payoff in the E market which in principle depends on a seller's chance of engaging in a trade in the I market. However, since the bargaining power of sellers is zero, so a seller's expected payoff at E is reduced to zero.

We have a similar condition of buyers for the middleman sector:

$$1 - p^m \geq V^E(x^m, f^E), \tag{5}$$

where the middleman's price must be acceptable for buyers relative to the expected payoff in the E market.

Given the outside option of the E market, the equilibrium value of buyers in the I market is $V = \max\{V(x^s), V(x^m)\}$, where

$$V^s(x^s) = \eta^s(x^s)(1 - p^s) + (1 - \eta^s(x^s)) V^E(x^m, f^E) \tag{6}$$

for an active platform $x^s > 0$ and

$$V^m(x^m) = 1 - p^m \tag{7}$$

for an active middleman sector $x^m > 0$. Here, if a buyer visits a seller (or a middleman), then he gets served with probability $\eta^s(x^s)$ (or probability 1) and his payoff is $1 - p^s$ (or $1 - p^m$). If not served at intermediary I, then he enters the E market and he can find an available seller with probability $\lambda^b e^{-x^s}$, and gets a payoff $1 - f^E$. Similarly, the equilibrium value of active sellers in the platform is given by

$$W(x^s) = x^s \eta^s(x^s)(p^s - f^s - c) + (1 - x^s \eta^s(x^s)) \times 0. \tag{8}$$

A seller trades successfully in I's platform with probability $x^s \eta^s(x^s)$ and receives $p^s - f^I$. If not successful, he enters E's platform where he meets a buyer with some probability and get an expected value 0 as the seller's bargaining parameter is assumed to be zero.

We now proceed to determined the equilibrium price p^s take as given the first-stage price/fees $P = \{f^I, p^m, f^E\}$. Suppose a seller deviates to a price $p \neq p^s$ that attracts an expected queue $x \neq x^s$ of buyers. Note that given the limited selling-capacity, this deviation has measure zero and does not affect the expected utility in the I market, V. Since buyers must be indifferent between visiting any seller (including the deviating seller), the equilibrium market-utility should satisfy

$$V = \eta^s(x)(1-p) + (1-\eta^s(x))V^E, \tag{9}$$

where $\eta^s(x) \equiv \frac{1-e^{-x}}{x}$ is the probability that a buyer is served by this deviating seller. If not served, which occurs with probability $1-\eta^s(x)$, his expected utility at intermediary E is $V^E = \lambda^b e^{-x^s}(1-f^E)$. Given market utility V, (9) determines the relationship between x and p, which we denote by $x = x(p|V)$. This yields a downward sloping demand faced by the seller: when the seller raises his price p, the queue length of buyers x becomes smaller, and vice versa.

Given the search behavior of buyers described above and the market utility V, the seller's optimal price should satisfy

$$p^s(V) = \arg\max_p \left(1 - e^{-x(p|V)}\right)\left(p - f^I\right)$$

Substituting out p using (9), the sellers' objective function can be written as

$$W(x) = (1-e^{-x})(1-V^E-f^I) - x(V-V^E),$$

where $x = x(p|V)$ satisfies (9) and $1-V^E$ is the intermediated trade surplus, i.e., the total trading surplus at intermediary I net of the outside options at E. Since choosing a price is isomorphic to choosing a queue, the first order condition is

$$\frac{\partial W(x)}{\partial x} = e^{-x}(1-V^E-f^I) - (V-V^E) = 0$$

The second order condition can be easily verified. Arranging the first order condition using (9) and evaluating it at $x^s = x\left(p^s|V\right)$, we obtain the equilibrium price $p^s = p^s\left(V\right)$ which can be written as

$$p^s - f^I = \left(1 - \frac{x^s e^{-x^s}}{1 - e^{-x^s}}\right)\left(1 - V^E - f^I\right). \tag{10}$$

For the platform to be active, the price and fees must satisfy the incentive constraints (3) and (4). Substituting in (10) yields

$$f^I \leq 1 - V^E(x^m, f^E), \tag{11}$$

which states that for the platform to be active $x^s > 0$, the transaction fee f^I should not be greater than the intermediated trade surplus, $1 - V^E(x^m, f^E)$. Whenever (3) and (4) are satisfied, (11) must hold, and whenever (11) is satisfied, (3) and (4) must hold. Hence, we can say that the market maker faces the constraint (11) for an active platform.

4.2 The Best Response of Intermediary I

For a given fee $f^E \in [0, 1]$, we analyze the most profitable choice of intermediation mode and price/fee $\{f^I, p^m\}$ by I. We impose a "bad expectation" market allocation against E, which we formally define as follows.

Definition 4 (Pessimistic Expectations in Multi-market Search) *Under multi-market search, a pessimistic market allocation against E, $\mathcal{N}(\cdot)$, is that $B^I = B, S^I = S$ as long as*

$$\max\{V^m(\mathcal{N}, P), V^s(\mathcal{N}, P)\} \geq V^E(\mathcal{N}, P), \tag{12}$$

where $B^E = B - x^m - S\left(1 - e^{-x^s}\right)$ and $B^E = S - S\left(1 - e^{-x^s}\right)$ for some $x^m \in [0, B]$. Otherwise, the distribution follows that all users forgo the trading opportunity at I and only visit E, $B^E = B, S^E = S, B^I = S^I = 0$.

Therefore, under the pessimistic expectation allocation, buyers/sellers first visit and trade at I and then E whenever possible. Notice in the definitions above, we do not impose restrictions on x^m. The equilibrium x^m is determined by conditions (1) and (2). Also note that incentive constraints (5) and (11) imply the pessimistic expectation condition (12). So whenever I optimizes its profits obeying the incentive constraints, buyers/sellers follow the pessimistic expectation distribution.

Our next step is to determine the profit of each intermediation mode, denoted by $\tilde{\Pi}(x^m)$.

Pure Middleman: If I does not open the platform, then $x^m = B$ and any encountered seller at E is always available for trade. Hence, the middleman serves all buyers at a highest possible price $p^m = 1 - \lambda^b(1 - f^E)$ with a binding incentive constraint (5) and makes profits

$$\tilde{\Pi}^I(B) = B(1 - \lambda^b(1 - f^E)). \tag{13}$$

Pure Market-maker: When the middleman sector is not open, $x^s = \frac{B}{S}$. Given the equilibrium price p^s in the platform in (10), the intermediary charges a fee f^I in order to maximize

$$S\left(1 - e^{-\frac{B}{S}}\right) f^I,$$

subject to the constraint (11). The constraint is binding and it yields:

$$f^I = V^E(0, f^E) = 1 - \lambda^b e^{-\frac{B}{S}}(1 - f^E).$$

The profit for the market-maker mode is

$$\tilde{\Pi}^I(0) = S(1 - e^{-\frac{B}{S}})(1 - \lambda^b e^{-\frac{B}{S}}(1 - f^E)). \tag{14}$$

Market-making middleman: If the intermediary is a market-making middleman, then $x^m \in (0, B)$ and $x^s \in (0, \frac{B}{S})$, satisfying $V^m(x^m) = V^s(x^s)$. Applying (6), (7), and (10), this indifference condition generates an expression for the price $p^m = p^m(x^m)$:

$$p^m(x^m) = 1 - V^E(x^m, f^E) - e^{-x^s}(1 - V^E(x^m, f^E) - f^I), \tag{15}$$

Together with (1), this equation defines the relationship between p^m and x^m. Applying this expression, we can see that the condition (5) is eventually reduced to (11). The profit for the marketmaking middleman mode is

$$\tilde{\Pi}(x^m) = \max_{x^m, f^I} \Pi\left(x^m, f, K\right) = \max_{x^m, f^I} \left\{ S(1 - e^{-x^s}) f^I + x^m p^m \right\}$$

subject to (11) and $x^m \in (0, B)$.

Profit-maximizing intermediation mode To derive a profit-maximizing intermediation mode, it is important to observe that relative to the pure middleman mode, an active platform at I with multiple-market search can undermine intermediary E by lowering the available supply since $S^E = Se^{-x^s}$, which relaxes the constraint on f^I imposed by the incentive constraint (11).

This influences the middleman's price similarly. With $f^I = 1 - V^E(x^m, f^E)$, the incentive constraint (5) is binding, and the middleman's equilibrium price is given by

$$p^m = 1 - V^E(x^m, f^E)$$

for any $x^s \geq 0$ according to (15). This shows that p^m decreases with x^m: the outside value of buyers depends positively on the size of the middleman sector, since a larger scale of the middleman crowds out the platform and increases the chance that a buyer can find an active seller in the E market (who was unsuccessful in I's platform). Hence, in order to extend the size of the middleman sector, I must lower the price p^m. In other words, a larger platform of I allows for a price increase by reducing agents' outside trade opportunities at E. This key insight from Gautier et al. (2018) holds for any $f^E < 1$.

Proposition 2 (Market-making middleman/Pure Market-maker) *Given multi-market search technologies, there exists a unique directed search equilibrium with active intermediation of I. In particular, I will act as*

- *a pure middleman $x^{m*} = B$ if $f^E = 1$;*
- *a pure market-maker $x^{m*} = 0$ if $\lambda^b e^{-B/S} \geq \frac{1}{2}$ and $f^E \leq 1 - \frac{1}{2\lambda^b e^{-B/S}}$;*

- *a market-making middleman* $x^{m*} \in (0, B)$ *if* $\lambda^b e^{-B/S} < \frac{1}{2}$, *or* $\lambda^b e^{-B/S} \geq \frac{1}{2}$ *and* $f^E > 1 - \frac{1}{2\lambda^b e^{-B/S}}$, *and* x^{m*} *is characterized by* $f^E = b^I(x^{m*})$, *where* $b^I(x^m)$ *is defined by*

$$b^I(x^m) \equiv 1 - \frac{S(1 - e^{-x^s})}{(2S(1 - e^{-x^s}) + x^m)\lambda^b e^{-x^s}}. \tag{16}$$

And in all cases, the optimal prices are set at $p^{m*} = f^{I*} = 1 - \lambda^b e^{-x^{s*}}(1 - f^E)$.

Proof. See the Appendix. ■

When $f^E = 1$, we are essentially back to the single-market search scenario where agents do not have outside options. So the pure middleman with $p^{m*} = 1$ is optimal. Whenever $f^E < 1$, with multiple-market search technologies, there is a cross-market feedback from E to I, which makes using the platform as part or all of I's intermediation activities profitable. Additionally, I must decide whether it wants to operate as a pure market maker. Our result shows that it depends on parameter values. If $\lambda^b e^{-B/S} < \frac{1}{2}$, then the buyers' outside option value is low. In this case, the middleman sector generates high enough profits for the market-making middleman mode to be adopted for any value of f^E. If instead $\lambda^b e^{-B/S} \geq \frac{1}{2}$ then the buyers' outside option value is high, and attracting buyers to the middleman sector is costly. In this case, the intermediary will act as a market-making middleman if $f^E > 1 - \frac{1}{2\lambda^b e^{-B/S}}$, where buyers expect a low value from E market, and as a pure market maker if $f^E \leq 1 - \frac{1}{2\lambda^b e^{-B/S}}$, where buyers expect a high value from the E market.

When the mixed mode is activated, $b^I(\cdot)$ in equation (16) characterizes the optimal intermediation structure that I is willing to pursue ("b" for best response function). It is essentially the best response of I in facing a transaction fee f^E. It is easy to check that $b^I(x^m)$ is monotone increasing in x^m, implying as f^E decreases, I moves towards the pure platform mode. Eventually, as f^E approaches $1 - \frac{1}{2\lambda^b e^{-B/S}}$ (if it is positive), I becomes a pure market-maker.[10] Being different from the optimal intermediation structure shown by Gautier et al. (2018), the pure middleman can be the optimal mode. Moreover, it is not clear which intermediation mode will show up in an equilibrium which we now turn to.

[10] This is so because $\lim_{x^m \to 0} b^I(x^m) = 1 - \frac{1}{2\lambda^b e^{-B/S}}$, $\lim_{x^m \to B} b^I(x^m) = 1$. And if $1 - \frac{1}{2\lambda^b e^{-B/S}} < 0$, then $\lim_{f^E \to 0} x^m > 0$.

4.3 Equilibrium Candidate with I of a Pure Mode

Armed with the characterization in Proposition 2, we can now perform the equilibrium analysis. Doing so, we shall also derive the best responses of E along the analysis. We start by ruling out any pure strategy equilibrium with I a pure middleman or pure market-maker.

A pure middleman intermediary I does not arise in the equilibrium because, given Proposition 2, I only adopts a pure middleman mode with $p^m = 1$ when $f^E = 1$. But facing $p^m = 1$, E would rather set $f^E = 1 - \varepsilon$, for some $\varepsilon > 0$ to become the only active intermediary and make a profit of $B\lambda^b f^E > 0$.[11]

Turn to the case when I is a pure market-maker. Under pessimistic expectation against E, when

$$V^E = \lambda^b e^{-B/S}(1 - f^E) \leq 1 - f^I,$$

E is the second source. Otherwise, I becomes inactive and E is the sole active source.

Consider an equilibrium candidate where E sets a fee $f^{Ec} \in [0, 1]$ ("c" for candidate) and a pure market-maker I sets a fee $f^{Ic} \equiv 1 - \lambda^b e^{-B/S}(1 - f^{Ec})$.[12] In this proposed equilibrium, E wants to undercut I whenever possible, or deviate to $f^{Ed} = 1$ to get the whole trading surplus from its transactions ("d" for deviation). In the case that $f^{Ec} > 0$, it is profitable to undercut f^{Ic} by setting $f^{Ed} = f^{Ec} - \varepsilon$. As such, E becomes the sole source and makes a profit of $B\lambda^b f^{Ed}$. On the other hand, if $f^{Ec} = 0$, then E would rather take the full surplus of each transaction by $f^{Ed} = 1$ and makes a profit of $B\lambda^b e^{-B/S} > 0$.

We summarize these observations in the following lemma.

Lemma 1 *There does no exist a pure strategy equilibrium where I is a pure middleman or a pure market-maker.*

In Gautier et al. (2018), pure middleman mode is suboptimal with a passive outside platform. Lemma 1 shows the same conclusion holds in an equilibrium when the outside

[11] The best response analysis of E when I is a pure middleman is as follows. When I is a pure middleman, E can only make transactions by setting a fee f^E low enough so that buyers' incentive constraint to trade at I, condition (5), is violated, i.e., $1 - p^m < \lambda^b(1 - f^E)$. That is, to undercut I by setting $f^E = 1 - \frac{1-p^m}{\lambda^b} - \varepsilon$ for some $\varepsilon > 0$, as long as this leads to a non-negative f^E. In this way, I becomes inactive and E makes a profit of $B\lambda^b f^E$.

[12] According to Proposition 2, f^{Ic} is required to satisfy the best response of I. Any $f^I \neq f^{Ic}$ would lead to a deviation of I.

market is operated by another intermediary E and responses actively. E has an incentive to undercut I, and this gives a positive outside value for buyers. To lower buyers' outside value, I activates the market-maker mode. Lemma 1 asserts even stronger: The pure market-maker mode can no be in an equilibrium. This follows the intuition of the classical Bertrand-Edgeworth game. Since the matching probability is less than one at the market-maker sector, it is a profitable deviation for E to set $f^E = 1$ to abstract full trading surplus from agents that are not matched at I.

4.4 The Best Response of Intermediary E

Now we turn to the existence of an equilibrium with I of a mixed mode. According to Proposition (2), as a market-making middleman, I's optimal strategy features $p^m = f^I$. Let's denote the price/fee level by ψ. To construct the equilibrium, we first analyze the most profitable fee choice of E taking ψ as given.

Under pessimistic beliefs against E, for any $f^E \in [0,1]$, if there exists an $x^m \in [0, B]$ such that buyers and sellers are willing to first search in I before turning to transaction opportunities in E, as stated in (12), then E remains to be the second source. This means E can have two roles in the multi-market search environment: (1) E can work as a second source following pessimistic beliefs; or (2) E can break the beliefs and act as a sole source by undercutting I.

E works as a sole source Let's first explore the possibility of E being a sole source. Insert $V^m = 1 - \psi$ and $V^m = e^{-x^s}(1 - V^E - \psi) + V^E$, the condition to maintain a bad market allocation (12) becomes $f^E \geq 1 - \frac{1}{\lambda^b e^{-x^s}}(1 - \psi)$, for all $x^s \in [0, B/S]$. The right hand side take the minimum at $x^s = B/S$. Therefore, as long as

$$f^E \geq 1 - \frac{1}{\lambda^b e^{-B/S}}(1 - \psi), \tag{17}$$

E is a second source.

To become the sole source, E can set f^E slightly lower than the right hand side of (17) and make profits of

$$\Pi^E_{sole}(\psi) = B\lambda^b\Big(1 - \frac{1}{\lambda^b e^{-B/S}}(1-\psi) - \varepsilon\Big). \tag{18}$$

Obviously, if $\psi < 1 - \lambda^b e^{B/S}$, then $\Pi^E_{sole} < 0$.

E works as a second source To be a second source, E has to choose an $f^E \geq 1 - \frac{1}{\lambda^b e^{-B/S}}(1 - \psi)$ to satisfy (17). That is, E's profit maximization problem is

$$\Pi^E_{2nd}(\psi) = \max_{f^E \in [1-\frac{1}{\lambda^b e^{-B/S}}(1-\psi),1]} \Big(B - x^m - S(1-e^{-x^s})\Big)\lambda^b e^{-x^s} f^E, \tag{19}$$

subject to the equilibrium constraint (1) and (2). As I, it is in E's interest to have $V^m = V^s$. On the one hand, $V^m > V^s$ leads to a pure middleman incumbent, leaving zero market share to E. On the other hand, $V^m < V^s$ means E needs to set f^E unnecessarily low — E could increase f^E without changing the distribution of agents and make higher profits. This is formally stated in the following lemma.

Lemma 2 *Given $p^m = f^I = \psi$, the optimal solution of problem (19) features*

$$V^m(p^m) = V^s(x^m, f^I, f^E). \tag{20}$$

Proof. Let $\{\hat{f}^E, \hat{x}^m\}$ denote the optimum. Suppose at the optimum $V^m(p^m) > V^s(\hat{x}^m, f^I, \hat{f}^E)$, then $\hat{x}^m = B$. It follows

$$V^s(\hat{x}^m, f^I, \hat{f}^E) = 1 - f^I = 1 - p^m = V^m(p^m).$$

A contradiction.

Suppose at the optimum, $V^m(p^m) < V^s(\hat{x}^m, f^I, \hat{f}^E)$, or

$$V^s = e^{-B/S}(1-f^I) + (1-e^{-B/S})\lambda^b e^{-B/S}(1-f^E) > 1 - p^m.$$

This implies that $\hat{f}^E < 1 - \frac{1-\psi}{\lambda^b e^{-B/S}}$. But then E gains a higher profit by deviating to $\tilde{f}^E =$

$1-\frac{1-\psi}{\lambda^b e^{-B/S}}$. At $\tilde{f}^E$, E maintains $\hat{x}^m$ while extract higher fees from each transaction. ■

Insert the expression of $V^i, i = m, s, E$, into constraint (20) we have

$$\lambda^b e^{-x^s}(1-f^E) = 1-\psi. \tag{21}$$

Constraint (21) states the trade-off that E faces: increasing f^E leads to less favorable outside option for buyers on I's platform, hence more buyers visit I's middleman sector (x^m increases), and there are more unmatched sellers left for E (e^{-x^s} increases). Substitute for f^E from (21) and insert into (19), we have

$$\Pi^E_{2nd}(\psi) = \max_{x^m\in[0,B]} \Big(B - x^m - S(1-e^{-x^s})\Big)\Big(\lambda^b e^{-x^s} - (1-\psi)\Big). \tag{22}$$

Facing a ψ, by choosing an f^E, E essentially chooses an x^m to balance its demand and supply. A higher x^m implies less buyers join Eafter trading at I, i.e., $B - x^m - S(1-e^{-x^s})$ decreases in x^m. At the same time, more sellers join E since the matching probability is now lower for sellers at I's platform, i.e., e^{-x^s} increases in x^m. So the intermediation structure at I affects the supply, the available sellers, and the demand, the participating buyers at E.

This trade-off crucially depends on the benefit E has by marginally increasing x^m, which is determined by the equilibrium price level ψ. When ψ is high, it is profitable to have more buyers at I's platform by decreasing x^m and ultimately increase participating buyers to E. E can achieve this by decreasing f^E. When ψ is low, E finds it less profitable to have more participating buyers, and the optimal f^E should be higher. So ψ and f^E are strategic substitutes via market structure x^m. From this point of view, this is a quantity competition.

On the other hand, ψ and f^E are complements. Holding x^s constant, (21) implies a Bertrand type price competition. In a nutshell, the game we characterize has both a price competition element (when holding x^s constant) and a quantity competition element (when tracing the change due to x^s).

Finally, to have these trade-offs, there much be some space for E to manipulate f^E. Notice when $\psi < 1-\lambda^b$, there does not exist an $x^m \in [0, B]$ that gives positive profits for E either as a second or a sole source. Hence, such a ψ will not exist in a pure strategy equilibrium. For our equilibrium analysis, we focus on $\psi > 1 - \lambda^b$.

Lemma 3 *There does not exist an (pure strategy) equilibrium where $\psi \leq 1 - \lambda^b$.*

Proof. Suppose in the equilibrium, $\psi \leq 1 - \lambda^b$, then

$$V^m = 1 - \psi \geq e^{-x^s}(1-\psi) + (1 - e^{-x^s})\lambda^b(1 - f^E) = V^s,$$

for all $x^m \in [0, B]$ and $f^E \in [0,1]$, where equality only holds when $x^m = B$, and $f^E = 0$. This means $x^{m*} = B$. According to Proposition 2, this is consistent with I's optimal choice only when $f^{E*} = 1$. But given $f^{E*} = 1.$, I would rather set $\psi = 1$, contradicts with $\psi \leq 1 - \lambda^b$. ■

The following proposition then characterizes the best response of E for $\psi \in (1 - \lambda^b, 1]$.

Proposition 3 *Given multi-market search technologies, and $f^I = p^m = \psi \in (1 - \lambda^b, 1]$, intermediary E's optimal strategy follows that:*

- *For $\psi \in (1 - \lambda^b, 1 - \lambda^b e^{-B/S}]$, E works as the second source, $\Pi^E_{2nd} > 0 \geq \Pi^E_{sole}$, and the optimal $x^{m*} \in (0, B)$ satisfies*

$$1 - \psi = \lambda^b e^{-x^{s*}} \left(1 - \frac{B - x^m - S(1 - e^{-x^s})}{S(1 - e^{-x^s})}\right); \tag{23}$$

Define $\phi(B, S, \lambda^b) \equiv \lambda^b e^{-B/S}\left(1 - \frac{B - S(1 - e^{-B/S})}{S(1 - e^{-B/S})}\right)$,

- *if $\phi(B, S, \lambda^b) \geq 0$, we have*
  - *for $\psi \in (1 - \lambda^b e^{-B/S}, 1 - \phi(B, S, \lambda^b))$, E may work as a second or a sole source since both deliver positive profits, $\Pi^E_{2nd}, \Pi^E_{sole} > 0$, and if E works as the second source, $x^{m*} \in (0, B)$ satisfies (23),*
  - *for $\psi \in [1 - \phi(B, S, \lambda^b), 1]$, E undercuts I to become the sole source $\Pi^E_{sole} > \Pi^E_{2nd} > 0$;*
- *if $\phi(B, S, \lambda^b) < 0$, then for $\psi \in (1 - \lambda^b e^{-B/S}, 1]$, E may work as a second or a sole source since both deliver positive profits, $\Pi^E_{2nd}, \Pi^E_{sole} > 0$, and if E works as a second source, $x^{m*} \in (0, B)$ satisfies (23).*

Equation (23) is the best response function of E. It indicates for a given ψ, the optimal market structure, represented by x^m, that E would like to choose. Consider a range of $[\underline{x}^m, B]$ where the right hand side of (23) is non-negative. It then follows that

$$\frac{\partial \psi}{\partial x^m} = -\frac{1}{S}\left(1 - \psi + \frac{\lambda^b e^{-x^s}(1 - e^{-x^s} - x^s e^{-x^s})}{S^2(1 - e^{-x^s})^2}\right) < 0,$$

for $x^m \in [\underline{x}^m, B]$. This corresponds exactly to the intuition above: as ψ increases, E finds it's more profitable to compete, he lowers f^E to make the middleman sector of I less favorable and x^m decreases.

4.5 Equilibrium Analysis

The equilibrium should jointly solves the optimal responses of two intermediaries, (16) and (23), together with the equilibrium conditions (1) and (21). Insert the equilibrium condition (21) into (23) gives an alternative form of the best response function of E that facilitates our analysis:

$$f^E = b^E(x^m) \equiv \frac{B - x^m - S(1 - e^{-x^s})}{S(1 - e^{-x^s})}. \tag{24}$$

The equilibrium $x^{m*} \in (0, B)$ if exists should solve $b^I(x^m) = b^E(x^m)$. Proposition 4 gives a sufficient condition for the existence and uniqueness of the equilibrium.

Proposition 4 *There exists a unique equilibrium that features a market-making middleman I as a first source, and a second source market-maker E, if*

$$1 - \lambda^b e^{-B/S} \geq \psi > 1 - \lambda^b, \tag{25}$$

where $\psi^* = 1 - \lambda^b e^{-x^{s*}}\left(1 - \frac{B - x^{m*} - S(1-e^{-x^{s*}})}{S(1-e^{-x^{s*}})}\right)$, $x^{s*} = \frac{B - x^{m*}}{S}$, *and* $x^{m*} \in (0, B)$ *solves*

$$\frac{B - x^m}{S(1 - e^{-x^s})} = \frac{S(1 - e^{-x^s})}{\lambda^b e^{-x^s}(2S(1 - e^{-x^s} + x^m)}. \tag{26}$$

The equilibrium is characterized by a distribution of buyers and sellers

$$\mathcal{N}^* = \{B^I = B, B^E = B - x^{m*} - S(1 - e^{-x^{s*}}), S^I = S, S^E = Se^{-x^{s*}}, x^{m*}\},$$

and a price vector P^* that

$$f^{I*} = p^{m*} = \psi^*, f^{E*} = 1 - \frac{1 - \psi^*}{\lambda^b e^{-x^{s*}}}.$$

And both intermediaries make positive profits.

The equilibrium if exists features a first source intermediary I of a mixed mode serving all agents, and a second source intermediary E that serves the rest of agents who are not matched at I. At the equilibrium, the intermediary structure of I, x^{m*}, is derived by $b^I(x^m) = b^E(x^m)$ indicated by (26), which together with $\psi^* \leq 1 - \lambda^b e^{-B/S}$ guarantees x^{m*} is the mutually best response.

Figure 1 demonstrates an equilibrium by two variables, the price/fee level represented by f^E, and the market structure represented by x^m. It plots the two best response functions $b^I(x^m)$ in (16) and $b^E(x^m)$ in (24), and we have marked values as x^m approaches 0 and B.[13] The interaction of the two best responses gives the equilibrium $\{f^{E*}, x^{m*}\}$. The equilibrium distribution and price variables can be accordingly derived, as stated in the proposition.

The figure makes it clear of the comparative statics. Let's consider exogenous changes of buyer's meeting rate λ^b and the buyer population B. As λ^b increases, $b^I(x^m)$ moves upward while $b^E(x^m)$ remains the same, leading to a smaller x^{m*}. This is illustrated in Figure 2. Now

[13]We make use of the following observations:

$$\lim_{x^m \to 0} b^I(x^m) = 1 - \frac{1}{2\lambda^b e^{-B/S}}, \quad \lim_{x^m \to B} b^I(x^m) = 1,$$
$$\lim_{x^m \to 0} b^E(x^m) = \frac{B - S(1 - e^{-B/S})}{S(1 - e^{-B/S})}, \quad \lim_{x^m \to B} b^E(x^m) = 0.$$

We have plotted one particular scenario that $1 - \frac{1}{2\lambda^b e^{-B/S}} > 0$ and $\frac{B - S(1-e^{-B/S})}{S(1-e^{-B/S})} < 1$. But these restrictions are not required for the existence of an equilibrium.

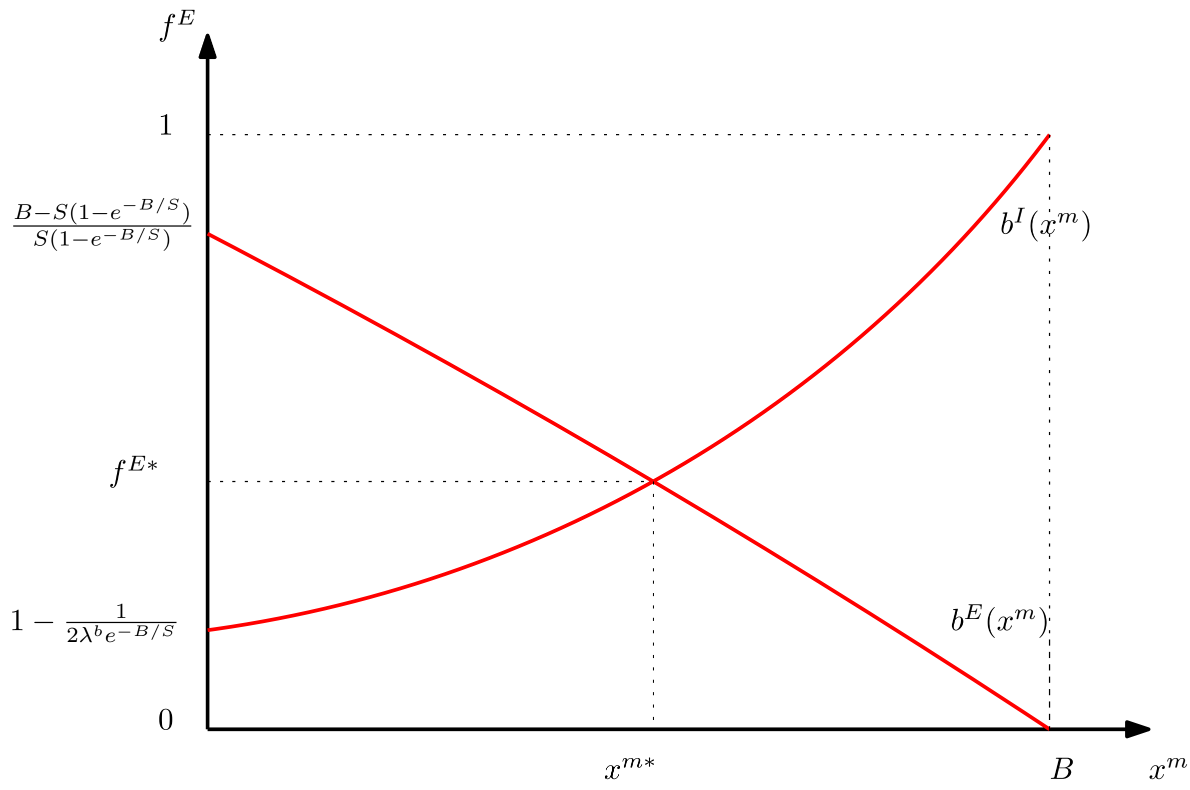

Figure 1: Equilibrium under multi-market search

consider an exogenous change of B, it follows that

$$\frac{\partial b^E(x^m)}{\partial B} = \frac{1-e^{-x^s}-x^s e^{-x^s}}{S(1-e^{-x^s})} > 0,$$
$$\frac{\partial b^I(x^m)}{\partial B} = -\frac{\lambda^b x^m e^{-2x^s}}{S^2(1-e^{-x^s})^2} < 0.$$

That is, as the population of buyers B increases, $b^I(x^m)$ moves downward while $b^E(x^m)$ moves upward, leading to a higher x^{m*}. This is illustrated in Figure 3. Similar comparative statics can be done on the seller population S. We summarize these observations in Corollary 1.

Corollary 1 (Comparative statics) *Consider a parameter space in which a pure strategy equilibrium exits. Then, an increase buyer's meeting rate λ^b in the D market, or a decrease in the buyer-seller population ratio B/S, leads to a smaller middleman sector x^m and a larger platform x^s of I.*

Numerically, it is easy to verify that the sufficient condition (25) is satisfied in some parameter space. For example, taking $B = S = 1$, and set a grid of λ^b with two decimals from 0.01 to 0.99, then (25) holds for all λ^b grids between smaller than 0.95. While for λ^b grids

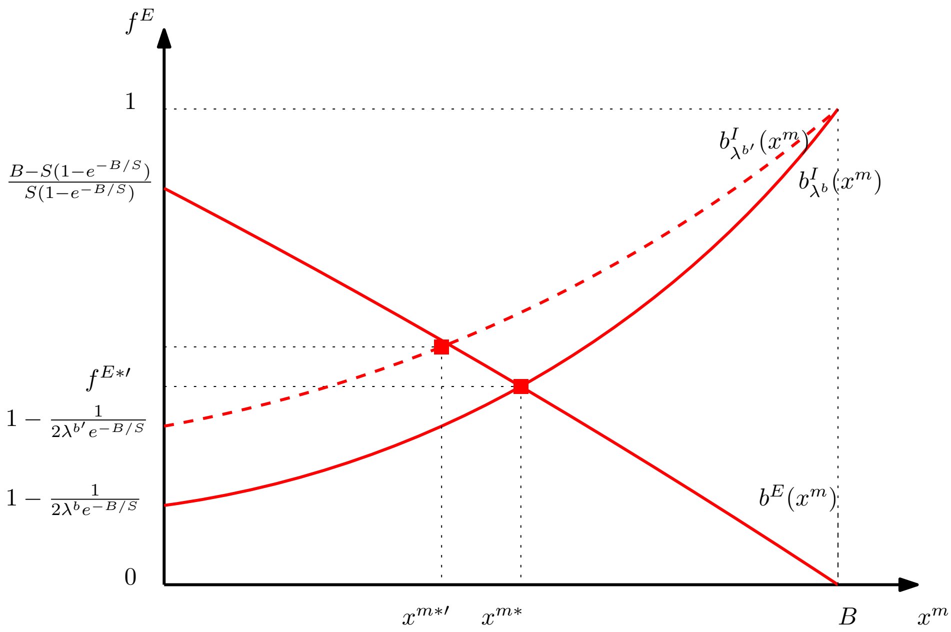

Figure 2: Comparative Statics w.r.t. λ^b

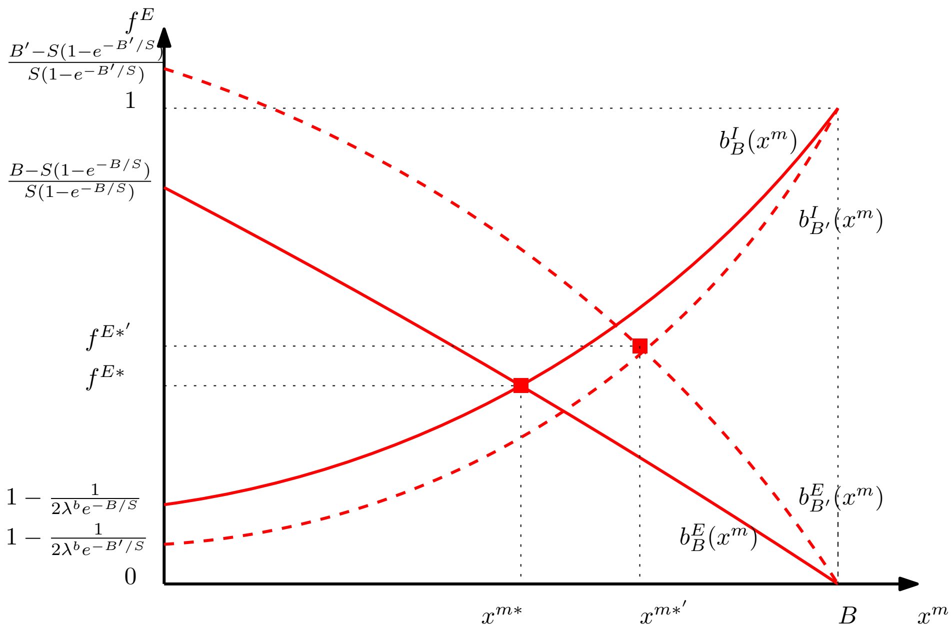

Figure 3: Comparative Statics w.r.t. B

between 0.95 and 0.98 (25) is violated, it can be verified that being the second source is more profitable than being a sole source for E, so a pure strategy equilibrium still exits. For the grid $\lambda^b = 0.99$, E finds it more profitable to undercut I and become the sole source. Then there does not exist a pure strategy equilibrium.

Equilibrium of mixed strategies When the pure strategies equilibrium does not exist, by applying Theorem 5* of Dasgupta and Maskin (1986a), we show that because Π^I (Π^E) is bounded and weakly lower semi-continuous in f and p^m (in f^E), and $\Pi^I + \Pi^E$ is upper semi-continuous, there exists a mixed strategy equilibrium.

Proposition 5 *There exists a mixed strategy equilibrium under multi-market search.*

Given that $f^E \in [0, 1)$ is selected with positive probability, according to Proposition 2, I activates its platform with positive probability in equilibrium.

Corollary 2 *In the mixed strategies equilibrium, I's platform is activated with positive probability.*

5 Conclusion

This paper has proposed a framework to analyze imperfect competition between two intermediaries, with a particular focus on intermediation mode. We considered two representative business modes of intermediation that the incumbent can adopt: a market-making mode and a middleman mode. We show that the incumbent faces a trade-off between more transactions by using the middleman mode and more profits per transaction by using the market-making mode. This trade-off determines its intermediation model in the competition game with entrant. Therefore, the main insight of Gautier et al. (2018) holds with competing intermediaries.

Appendix A Proofs

A.1 Proof for proposition 2

Using (15), the intermediary's problem can be written as

$$\begin{aligned}
\Pi\left(x^m, f^I\right) &= \max_{x^m, f^I} S(1-e^{-x^s})f^I + x^m p^m \\
&= \max_{x^m, f^I} S(1-e^{-\frac{B-x^m}{S}})f^I + x^m(1-V^E(x^m,f^E)) - x^m e^{-x^s}(1-V^E(x^m,f^E)-f^I)
\end{aligned}$$

subject to (11) and $0 < x^m < B$.

Observe that: $\lim_{x^m \to B} \Pi\left(x^m, f^I\right) = \tilde{\Pi}(B)$ and $\lim_{x^m \to 0} \Pi\left(x^m, f, K\right) = \tilde{\Pi}(0)$, where $\tilde{\Pi}(B)$ is the profit for the pure middleman mode (13) and $\tilde{\Pi}(0)$ is the profit for the pure market-maker mode (14). Hence, we can compactify the constraint set and set up a general problem to pin down a profit-maximizing intermediation mode using the following Lagrangian:

$$\mathcal{L} = \Pi\left(x^m, f^I\right) + \mu_0 x^m + \mu_b(B-x^m) + \mu_f\left(1 - V^E(x^m,f^E) - f^I\right),$$

where the μ's ≥ 0 are the lagrange multiplier of each constraint. The following first order conditions are necessary:

$$\frac{\partial \mathcal{L}}{\partial x^m} = \frac{\partial \Pi\left(x^m, f^I\right)}{\partial x^m} + \mu_0 - \mu_b - \mu_f \frac{\partial V^E\left(x^m, f^I\right)}{\partial x^m} = 0, \tag{27}$$

$$\frac{\partial \mathcal{L}}{\partial f^I} = \frac{\partial \Pi\left(x^m, f^I\right)}{\partial f^I} - \mu_f = 0, \tag{28}$$

The solution is characterized by these and the complementary slackness conditions of the three constraints.

First, (28) implies that we must have

$$\mu_f = S(1-e^{-x^s}) + x^m e^{-x^s} > 0,$$

which implies the binding constraint (11),

$$f^I = V^E\left(x^m, f^E\right). \tag{29}$$

Second, we show that the pure middleman mode is optimal if $f^E = 1$. Given $f^E = 1$, then $V^E(x^m, f^E = 1) = 0$. Using (29) and (15), the intermediary's problem can be written as $\Pi\left(x^m\right) = \max_{x^m} S(1-e^{-x^s}) + x^m$, where $\Pi\left(x^m, f^I\right)$ is concave in x^m. The first order condition with respect to x^m is $1 - e^{-x^s} = 0$. Therefore, at the optimal $x^{m*} = B$.

Next, given $f^E < 1$, we show that I's platform is active, i.e., $x^m < B$ at the optimum. Substituting μ_f into (27),

$$\begin{aligned}
\mu_b - \mu_0 &= (1-e^{-x^s})(1-\lambda^b e^{-x^s}(1-f^E)) - (x^m + S(1-e^{-x^s}))\frac{\lambda^b}{S}e^{-x^S}(1-f^E) \\
&\equiv \phi(x^m \mid B, S, \lambda^b, f^E).
\end{aligned} \tag{30}$$

Suppose that the solution is $x^{m*} = B$. Then, (30) yields $\phi(B \mid \cdot) = \mu_b = -\frac{B}{S}\lambda^b(1-f^E) < 0$, which contradicts $\mu_b \geq 0$. Hence, the solution must satisfy $x^{m*} < B$ (which implies $\mu_b = 0$). Suppose that the solution is $x^{m*} = 0$. Then, (30) yields $\phi(0 \mid \cdot) = -\mu_0 = (1-e^{-B/S})(1-$

$2\lambda^b e^{-B/S}(1-f^E))$, which requires

$$f^E \le 1 - \frac{1}{2\lambda^b e^{-B/S}}.$$

This leads to the conditions in the proposition. Set $\mu_b = \mu_0 = 0$, we have $\phi(x^m \mid B, S, \lambda^b, f^E) = 0$ according to (30). Since $x^m < B$, we have $1 - e^{-x^s} > 0$. This gives condition (16).

Finally, it is straightforward to verify the second order condition using the Hessian of $\mathcal{L}$ with respect to $[f^I, x^m]$.■

A.2 Proof for proposition 3

First, from (18) and (22), it is straightforward to see that if $\psi \le 1 - \lambda^b$, then $\Pi^E_{2nd} \le 0$; and if $\psi \le 1 - \lambda^b e^{-B/S}$, then $\Pi^E_{sole} \le 0$. These observations give the signs of the profits in all three cases of the proposition.

Second, the first order condition of Π^E_{2nd} with respect to x^m is

$$\frac{\partial \Pi^E_{2nd}(x^m, \psi)}{\partial x^m}|_{x^m} = -S(1-e^{-x^s})(\lambda^b e^{-x^s} - (1-\psi)) + (B - x^m - S(1-x^{-x^S}))\lambda^b e^{-x^S} = 0. \tag{31}$$

Since $\Pi^E_{2nd}(x^m)$ is continuously differentiable on $[0, B]$, the maximum point is among $x^m = 0$, $x^m = B$ or an $\hat{x}^m$ such that $\frac{\partial \Pi^E_{2nd}(x^m,\psi)}{\partial x^m}|_{\hat{x}^m} = 0$. Under $\psi > 1 - \lambda^b$, $x^m = B$ is not a maximum point since $\Pi^E_{2nd}(x^m = B) = 0$. When $\frac{\partial \Pi^E_{2nd}}{\partial x^m}|_{x^m=0} > 0$, that is $1 - \lambda^b e^{-B/S}\left(1 - \frac{B - S(1-e^{-B/S})}{S(1-e^{-B/S})}\right) > \psi$, $x^m = 0$ is not a maximum point. The following discussion depends on the sign of $\phi(B, S, \lambda^b) \equiv \lambda^b e^{-B/S}\left(1 - \frac{B-S(1-e^{-B/S})}{S(1-e^{-B/S})}\right)$.

If $\phi(B, S, \lambda^b) \ge 0$, then for $\psi \in [1 - \phi(B, S, \lambda^b), 1]$, as a second source E chooses $x^{m*} = 0$, which is strictly dominated by being the first source (lowering f^E slightly by some $\varepsilon > 0$)

$$\Pi^E_{2nd}(\psi) = (B - S(1-e^{-B/S})(\lambda^b e^{-B/S} - (1-\psi)) < B(\lambda^b e^{-B/S} - (1-\psi) - \varepsilon) = \Pi^E_{sole}(\psi).$$

For $\psi \in (1 - \lambda^b e^{-B/S}, 1 - \phi(B, S, \lambda^b))$, E compares Π^E_{sole} and Π^E_{2nd} to decide which is more profitable.

If $\phi(B, S, \lambda^b) < 0$, then for $\psi \in [1 - \lambda^b e^{-B/S}, 1]$, one needs to compare the profit of being second source and sole source. These observations give that $0 < x^{m*} < B$ in the first two bullet points of the proposition.

Finally, rearranging (31) gives (23). ■

A.3 Proof for proposition 4

Define

$$g(x^m) \equiv b^I(x^m) - b^E(x^m).$$

First, notice $g(x^m)$ is continuous differentiable with respect to x^m. Taking the limits of x^m to 0 and B, we have

$$
\begin{aligned}
\lim_{x^m\to 0} g(x^m) &= \lim_{x^m\to 0} b^I(x^m) - \lim_{x^m\to 0} b^E(x^m) \\
&= 1 - \frac{1}{2\lambda^b e^{-x^s}} - \frac{B - S(1-e^{-B/S})}{S(1-e^{-B/S})} \\
&= -\frac{1}{2\lambda^b e^{-x^s}} - \frac{B}{S(1-e^{-B/S})} < 0; \\
\lim_{x^m\to B} g(x^m) &= \lim_{x^m\to B} b^I(x^m) - \lim_{x^m\to B} b^E(x^m) \\
&= 1 - 0 > 0.
\end{aligned}
$$

According to the Intermediate Value Theorem, there exits an $x^{m*} \in (0, B)$ such that $b^I(x^{m*}) = b^E(x^{m*})$.

Second, x^{m*} is unique. This is because g is monotone increasing in x^m on $(0, B)$. Taking the first order derivate with respect to x^m, we have

$$g'(x^m) = b^{I'}(x^m) - b^{E'}(x^m) > 0,$$

where

$$
\begin{aligned}
b^{I'}(x^m) &= \frac{(1+2(1-e^{-x^s}))S(1-e^{-x^s}) + x^m}{(2S(1-e^{-x^s}) + x^m)^2 \lambda^b e^{-x^s}} > 0, \\
b^{E'}(x^m) &= -\frac{1}{S^3(1-e^{-x^s})^2}(1 - e^{x^s} - x^s e^{-x^s}) < 0.
\end{aligned}
$$

Thirdly, if $\psi = 1 - \lambda^b e^{-x^{s*}}\left(1 - \frac{B - x^{m*} - S(1-e^{-x^{s*}})}{S(1-e^{-x^{s*}})}\right) \in (1-\lambda^b, 1-\lambda^b e^{-B/S})$, then according to Proposition 3, E has no incentive to deviate to the sole source since $\Pi^E_{sole} < 0$.

Finally, according to Proposition 2 and Proposition 3, both intermediaries make positive profits. ■

A.4 Proof for corollary 1

Let's consider a marginal increase in λ^b, B and S in turns.

Consider an increase in λ^b: $\lambda^{b\prime} = \lambda^b + \varepsilon$ with $\varepsilon > 0$. We show the equilibrium market structure under $\lambda^{b\prime}$ follows $x^{m*\prime} < x^{m*}$, where x^{m*} is the in the equilibrium under λ^b and $x^{m*\prime}$ is the in the equilibrium under $\lambda^{b\prime}$. We denote the best response function under λ^b by $b^i_{\lambda^b}(x^m)$ and that under $\lambda^{b\prime}(x^m)$ by $b^i_{\lambda^{b\prime}}(x^m)$, $i = I, E$. Since $\frac{\partial b^E(x^m)}{\partial \lambda^b} = 0$, $\frac{\partial b^I(x^m)}{\partial \lambda^b} > 0$, for $x^m \in (0, B)$, we have $b^E_{\lambda^{b\prime}}(x^m) = b^E_{\lambda^b}(x^m)$, and $b^I_{\lambda^{b\prime}}(x^m) > b^I_{\lambda^b}(x^m)$.

Suppose $x^{m*\prime} = x^{m*}$, then

$$b^I_{\lambda^{b\prime}}(x^{m*\prime}) > b^I_{\lambda^b}(x^{m*\prime}) = b^I_{\lambda^b}(x^{m*}) = b^E_{\lambda^b}(x^{m*}) = b^E_{\lambda^{b\prime}}(x^{m*}) = b^E_{\lambda^{b\prime}}(x^{m*\prime}).$$

But $b^I_{\lambda^{b\prime}}(x^{m*\prime}) > b^E_{\lambda^{b\prime}}(x^{m*\prime})$ implies that $x^{m*\prime} = x^{m*}$ is not in an equilibrium.

Suppose $x^{m*\prime} > x^{m*}$, then

$$b^I_{\lambda^{b\prime}}(x^{m*\prime}) > b^I_{\lambda^b}(x^{m*\prime}) > b^I_{\lambda^b}(x^{m*}) = b^E_{\lambda^b}(x^{m*}) = b^E_{\lambda^{b\prime}}(x^{m*}) > b^E_{\lambda^{b\prime}}(x^{m*\prime}).$$

Again, $b^I_{\lambda^{b\prime}}(x^{m*\prime}) > b^E_{\lambda^{b\prime}}(x^{m*\prime})$ implies that $x^{m*\prime} > x^{m*}$ can not be in an equilibrium.

Consider an increase in B: $B' = B + \varepsilon$ with $\varepsilon > 0$. We show the equilibrium market structure under B' follows $x^{m*\prime} > x^{m*}$. Since

$$\frac{\partial b^E(x^m)}{\partial B} = \frac{1 - e^{-x^s} - x^s e^{-x^s}}{S(1 - e^{-x^s})} > 0,$$

$$\frac{\partial b^I(x^m)}{\partial B} = -\frac{2S(1 - e^{-x^s})^2 + x^m}{(2S(1 - e^{-x^s}) + x^m)^2 \lambda^b e^{-x^s}} < 0,$$

for $x^m \in (0, B)$, we have $b^E_{B'}(x^m) > b^E_B(x^m)$, and $b^I_{B'}(x^m) < b^I_B(x^m)$.

Suppose $x^{m*\prime} = x^{m*}$, then

$$b^I_{B'}(x^{m*\prime}) < b^I_B(x^{m*\prime}) = b^I_B(x^{m*}) = b^E_B(x^{m*}) < b^E_{B'}(x^{m*}) = b^E_{B'}(x^{m*\prime}).$$

But $b^I_{B'}(x^{m*\prime}) < b^E_{B'}(x^{m*\prime})$ implies that $x^{m*\prime} = x^{m*}$ can not be in an equilibrium.

Suppose $x^{m*\prime} < x^{m*}$, then

$$b^I_{B'}(x^{m*\prime}) < b^I_B(x^{m*\prime}) < b^I_B(x^{m*}) = b^E_B(x^{m*}) < b^E_{B'}(x^{m*}) < b^E_{B'}(x^{m*\prime}).$$

Again, $b^I_{B'}(x^{m*\prime}) < b^E_{B'}(x^{m*\prime})$ implies that $x^{m*\prime} < x^{m*}$ can not be in an equilibrium.

Consider an increase in S: $S' = S + \varepsilon$ with $\varepsilon > 0$. We show the equilibrium market structure under S' follows $x^{m*\prime} < x^{m*}$. Since

$$\frac{\partial b^E(x^m)}{\partial S} = -\frac{(1 - e^{-x^s} - x^s e^{-x^s})x^s}{S(1 - e^{-x^s})^2} < 0,$$

$$\frac{\partial b^I(x^m)}{\partial S} = \frac{x^s\left[2(1 - e^{-x^s})^2 + \frac{x^m}{S}(1 - \frac{1-e^{-x^s}}{x^s})\right]}{S(2(1 - e^{-x^s}) + x^m/S)^2 \lambda^b e^{-x^s}} > 0,$$

for $x^m \in (0, B)$, we have $b^E_{S'}(x^m) < b^E_S(x^m)$, and $b^I_{S'}(x^m) > b^I_S(x^m)$.

Suppose $x^{m*\prime} = x^{m*}$, then

$$b^I_{S'}(x^{m*\prime}) > b^I_S(x^{m*\prime}) = b^I_S(x^{m*}) = b^E_S(x^{m*}) > b^E_{S'}(x^{m*}) = b^E_{S'}(x^{m*\prime}).$$

Sut $b^I_{S'}(x^{m*\prime}) > b^E_{S'}(x^{m*\prime})$ implies that $x^{m*\prime} = x^{m*}$ can not be in an equilibrium.

Suppose $x^{m*\prime} > x^{m*}$, then

$$b^I_{S'}(x^{m*\prime}) > b^I_S(x^{m*\prime}) > b^I_S(x^{m*}) = b^E_S(x^{m*}) > b^E_{S'}(x^{m*}) > b^E_{S'}(x^{m*\prime}).$$

Again, $b^I_{S'}(x^{m*\prime}) > b^E_{S'}(x^{m*\prime})$ implies that $x^{m*\prime} > x^{m*}$ can not be in an equilibrium. This completes the proof of corollary 1.■

A.5 Proof for proposition 5

Consider a game between I, who selects $(f, p^m) \in [0, \bar{f}] \times [0, \bar{p}]$ with a payoff $\Pi^I = \Pi^I(f, p^m \mid f^E)$, and E, who selects $f^E \in [0, 1]$ with a payoff $\Pi^E = \Pi^E(f^E \mid f, p^m)$. Here, we set $\bar{f}, \bar{p} > 1$ and $f > 1$ ($\bar{p} > 1$) leads to an inactive platform (middleman sector). We apply Theorem 5 of Dasgupta and Maskin (1986) to show there exists a mixed strategy equilibrium.

Given Theorem 5 of Dasgupta and Maskin (1986a), it is sufficient to show that Π^I (Π^E) is bounded and weakly lower semi-continuous in f and p^m (in f^E), and $\Pi^I + \Pi^E$ is upper semi-continuous. Clearly, Π^I (Π^E) is bounded in $(f, p^m) \in [0, \bar{f}] \times [0, \bar{p}]$ (in $f^E \in [0, 1]$).

Both of the profit functions are continuous except at

$$\min\{f, p^m\} = 1 - V^E(f^E), \tag{32}$$

where $V^E(f^E)$ is evaluated at $x^m = 0$. So we shall pay attention to this discontinuity point.

First, we show that $\Pi^I(f, p^m \mid f^E)$ is weakly lower semi-continuous in (f, p^m). Give the discontinuous point in (32), we have

$$\Pi^I(f, p^m \mid f^E) = \begin{cases} S(1-e^{-x^s})f + x^m p^m, & \text{if } \min\{f, p^m\} \le 1 - V^E(f^E) \\ 0 & \text{otherwise,} \end{cases}$$

where in the second situation, the price/fee of I is not competitive to the fee of E, hence agents will trade via E, rather than I, and so I will become inactive. Consider some $f_\varepsilon \in [0,1]$, and some $f, p^m > 0$ such that $\min\{f, p^m\} = 1 - V^E(f^E)$. For any sequence $\{(f^{(j)}, p^{m(j)})\}$ converging to (f, p^m) such that no two $f^{(j)}$'s, and no two $p^{m(j)}$'s are the same, and $f^{(j)} \le f, p^{m(j)} \le p^m$, we must have $\min\{f^{(j)}, p^{m(j)}\} \le 1 - V^E(f^E)$. Hence,

$$\lim_{j\to\infty} \Pi^I(f^{(j)}, p^{m(j)} \mid f^E) = \Pi^I(f, p^m \mid f^E),$$

satisfying the definition of weakly lower semi-continuity (see Definition 6 in page 13 of Dasgupta and Maskin, 1986, or condition (9) in page 384 of Maskin, 1986).

Second, we shall show that $\Pi^E(f^E \mid f, p^m)$ is lower semi-continuous in f^E. Consider a potential discontinuity point $f_0 \in (0,1)$ satisfying (32) such that

$$\Pi^E(f^E \mid f, p^m) = \begin{cases} B\lambda^b f^E, & \text{if } f^E < f_0 \\ (B - x^m - S(1-e^{-x^s}))\lambda^b f^E & \text{if } f^E \ge f_0. \end{cases}$$

Clearly, this function is lower semi-continuous, since for every $\epsilon > 0$ there exists a neighborhood U of f_0 such that $\Pi^E(f^E \mid \cdot) \ge \Pi^E(f_0 \mid \cdot) - \epsilon$ for all $f^E \in U$.

Finally, we prove the upper semi-continuity of $\Pi^I + \Pi^E$. For this purpose, consider all sequences of $\{f^{(j)}, p^{m(j)}, f^{E(j)}\}$ that converges to $\{\hat{f}, \hat{p}^m, \hat{f}^E\}$ that satisfies $\min\{\hat{f}, \hat{p}^m\} = 1 - V^E(\hat{f}^E)$.

Consider first an extreme in which case $\min\{f^{(j)}, p^{m(j)}\} \le 1 - V^E(f^{E(j)})$ for all j. As the equilibrium is that I is visited prior to E, we must have

$$\lim_{j\to\infty} \Pi^I(f^{(j)}, p^{m(j)} \mid f^{E(j)}) + \Pi^E(f^{E(j)} \mid f^{(j)}, p^{m(j)}) = \Pi^I(\hat{f}, \hat{p}^m \mid \hat{f}^E) + \Pi^E(\hat{f}^E \mid \hat{f}, \hat{p}^m).$$

Consider next the other extreme in which $\min\{f^{(j)}, p^{m(j)}\} > 1 - V^E(f^{E(j)})$ for all j. Then, in the equilibrium only E is active and we must have

$$\lim_{j\to\infty} \Pi^I(f^{(j)}, p^{m(j)} \mid f^{E(j)}) + \Pi^E(f^{E(j)} \mid f^{(j)}, p^{m(j)}) = B\lambda^b \hat{f}^E.$$

If $\hat{f} \ge \hat{p^m}$, then

$$\Pi^I(\hat{f}, \hat{p}^m \mid \hat{f}^E) + \Pi^E(\hat{f}^E \mid \hat{f}, \hat{p}^m) = B\hat{p}^m = B(1 - \lambda^b(1 - \hat{f}^E)) > B\lambda^b \hat{f}^E.$$

If $\hat{f} < \hat{p^m}$, then

$$\Pi^I(\hat{f}, \hat{p}^m \mid \hat{f}^E) + \Pi^E(\hat{f}^E \mid \hat{f}, \hat{p}^m) = B(1 - e^{-\frac{B}{S}})\hat{f} + B\lambda^b e^{-\frac{B}{S}} \hat{f}^E > B[(1 - e^{-\frac{B}{S}}) + \lambda^b e^{-\frac{B}{S}}]\hat{f}^E > B\lambda^b \hat{f}^E.$$

Thus,

$$\lim_{j\to\infty} \Pi^I(f^{(j)}, p^{m(j)} \mid f^{E(j)}) + \Pi^E(f^{E(j)} \mid f^{(j)}, p^{m(j)}) < \Pi^I(\hat{f}, \hat{p}^m \mid \hat{f}^E) + \Pi^E(\hat{f}^E \mid \hat{f}, \hat{p}^m).$$

As these two extreme cases give the upper and lower bounds respectively, all the other sequences give some limits in between. Therefore,

$$\lim_{j\to\infty} \Pi^I(f^{(j)}, p^{m(j)} \mid f^{E(j)}) + \Pi^E(f^{E(j)} \mid f^{(j)}, p^{m(j)}) \leq \Pi^I(\hat{f}, \hat{p}^m \mid \hat{f}^E) + \Pi^E(\hat{f}^E \mid \hat{f}, \hat{p}^m),$$

for any of the sequences converging to $\{\hat{f}, \hat{p}^m, \hat{f}^E\}$, and so $\Pi^I + \Pi^E$ is upper semi-continuous. This completes the proof of Proposition 5. ■

Appendix B The Game With A Pure Middlamn E

⊙ Single-market search. First of all, suppose E is a middleman with a price p^E, then $V^E = 1 - p^E$. If I is a pure middleman, then it makes a profit of Bp^m with price $p^m = 1 - V^E$. If I activates a platform, it must satisfy the participation constraints,

$$\eta^s(x^s)(1 - p^s) \geq V^E,$$
$$p^s - f^I \geq 0.$$

Under these conditions, it holds that

$$f^I \leq 1 - V^E.$$

Hence, the resulting profit of I satisfies $S(1 - e^{-x^s})f^I + x^m p^m < (Sx^s + x^m)\max\{f^I, p^m\} \leq B(1 - V^E)$. That is, the pure middleman mode dominates any other modes with an active platform. Therefore, under single-market search, I must be a pure middleman in all possible equilibria.

⊙ Multi-market search: E is a pure middleman. With multi-market search, when E is a pure middleman, an active platform of I has to satisfy the incentive constraints,

$$\begin{aligned} 1 - p^s &\geq 1 - p^E \\ p^s - f^I &\geq 0. \end{aligned}$$

These constraints imply: $f^I \leq p^E$. Similarly, an active middleman sector has to satisfy $p^m \leq p^E$. Then, if $\max\{p^m, f^I\} \leq p^E$, then I can be a market-making middleman, and if

$$\min\{p^m, f\} \leq p^E,$$

then trade can occur in either one of the sectors, and so I can be an active intermediary. The profit of I is

$$S(1 - e^{-x^s})f + x^m p^m.$$

Noting $x^s = \frac{B - x^m}{S}$, we see from this expression that the profit maximization requires that $x^m = B$ with $p^m = f = p^E$. Hence an active platform is not profitable. Then, since the two intermediaries compete with price, any equilibrium must be subject to the Bertrand undercutting, leading to $p^m = p^E = 0$ and zero profits.

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