Machine Learning from Data Assignment 5

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Exercise 2.8

(a) *Show that if H is closed under linear combination, then $\overline{g} \in H$.*

The definition of of $\overline{g}(x)$ is

$$\overline{g}(x) = \frac{1}{K}\sum_{k=1}^{K} g_k(x).$$

This means that every value of $\overline{g}$ is a multiple of of the values of g in H. In other words, it is a linear combination of the $g_k \in H$. Since H is closed under linear combination, $\overline{g}$ must also be in H.

(b) *Give a model for which the average function $\overline{g}$ is not in the model's hypothesis set.*

If $H = \emptyset$, then $\overline{g}$ cannot be in H.

(c) *For binary classification, do you expect $\overline{g}$ to be a binary function?*

No, this is unlikely.

Problem 2.14

Let $H_1, H_2, \ldots, H_K$ be K hypothesis sets with finite VC dimension d_{VC}. Let $H = H_1 \cup H_2 \cup \cdots \cup H_K$ be the union of these models.

(a) *Show that $d_{VC}(H) < K(d_{VC} + 1)$.*

The hypothesis sets have VC dimension d_{VC}, meaning they have break point $k^* = d_{VC} + 1$. Recall that the first break point is the sample size for which not all dichotomies can be given by the hypothesis set. So, given K hypothesis sets with this break point, if every hypothesis set can fill out dichotomies that others can't we end up with a largest possible breakpoint of

$$k^*(H) = Kk^*.$$

Since the breakpoint is related to the VC dimension and this is the largest value, we can rewrite this as

$$d_{VC}(H) < K(d_{VC} + 1)$$

since the VC dimension is strictly less than $k*$.

(b) *Suppose that l satisfies $2^l > 2Kl^{d_{VC}}$. Show that $d_{VC}(H) \leq l$.*

This is a straight proof by calculation.

$$
\begin{aligned}
2^l &> 2Kl^{d_{VC}} \\
2^{l-1} &> Kl^{d_{VC}} \\
(l-1)\log_2 2 &> Kd_{VC}\log_2 l \\
d_{VC} &< (l-1)\frac{1}{K\log_2 l} \\
&\leq \frac{l-1}{K}
\end{aligned}
$$

And as we can add this together K times for $d_{VC}(H)$, we get

$$d_{VC}(H) \leq K\frac{l-1}{K} = l-1$$

which implies

$$d_{VC}(H) \leq l.$$

(c) *Hence, show that*

$$d_{VC}(H) \leq \min(K(d_{VC}+1), 7(d_{VC}+K)\log_2(d_{VC}K)).$$

That is, $d_{VC} = O(\max(d_{VC}, K)\log_2\max(d_{VC}, K))$ *is not too bad.*

Obviously, if the bound from (a) is the lower of the two, the inequality above holds.

To show the second term in the min is a bound, we set l to it and substitute into the inequality from (b).

$$
\begin{aligned}
2^{7(d_{VC}+K)\log_2(d_{VC}K)} &> 2K(7(d_{VC}+K)\log_2(d_{VC}K))^{d_{VC}} \\
(d_{VC}K)^{7(d_{VC}+K)} &> 2K(7(d_{VC}+K)d_{VC}\log_2(d_{VC}K)) \\
(d_{VC}K)^{7(d_{VC}+K)-1} &> 2(7(d_{VC}+K)\log_2(d_{VC}K))
\end{aligned}
$$

This inequality, ugly as it is, holds for $K > 1$, so we have shown the previous inequality is true.

Problem 2.15

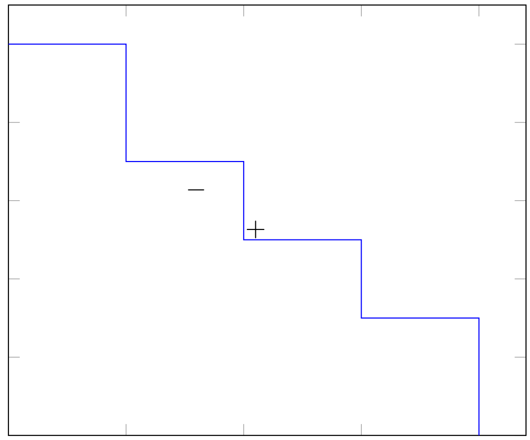
The monotonically increasing hypothesis set is

$$H = \{h \mid x_1 \geq x_2 \implies h(x_1) \geq h(x_2)\},$$

where $x_1 \geq x_2$ *if and only if the inequality is satisfied for every component.*

(a) *Give an example of a monotonic classifier in two dimensions, clearly showing the +1 and -1 regions.*

The monotonic classifier will appear as a sort of decreasing step function, with the upper right side being the +1 region and the -1 region being the bottom left side.



(b) *Compute $m_H(N)$ and hence the VC dimension.*

Consider the case of N monotonically decreasing points in the Cartesian plane. With the monotonically increasing hypothesis set, **all dichotomies are possible**. Thus,

$$m_H(N) = 2^n \qquad \text{and} \qquad d_{VC} = \infty.$$

Problem 2.24

Assume input dimension 1. Assume input variable x is uniformly distributed in the interval [-1,1]. Data set consists of 2 points, $\{x_1, x_2\}$. Assume target function is $f(x) = x^2$. Full data set is $D = \{(x_1, x_1^2), (x_2, x_2^2)\}$. Learning algorithm returns line fitting these two points as g. We are interested in the test performance E_{out} of the learning system w.r.t. the squared error measure, the bias and the variance.

(a) *Give the analytic expression for the average function $\overline{g}(x)$.*

$$\overline{g}(x) \approx \frac{1}{K} \sum_{k=1}^{K} g_k(x) = \mathbb{E}[ax + b]$$

a is the slope of the line obtained from the points, and b the intercept, so we get

$$\begin{aligned}\overline{g}(x) &= \mathbb{E}[(x_1 + x_2)x - x_1 x_2] \\ &= \mathbb{E}[x_1] + \mathbb{E}[x_2])x - \mathbb{E}[x_1]\mathbb{E}[x_2] \\ &= 0\end{aligned}$$

(b) *Describe an experiment that you could run to determine numerically $\overline{g}(x), E_{out}$, bias, and var.*

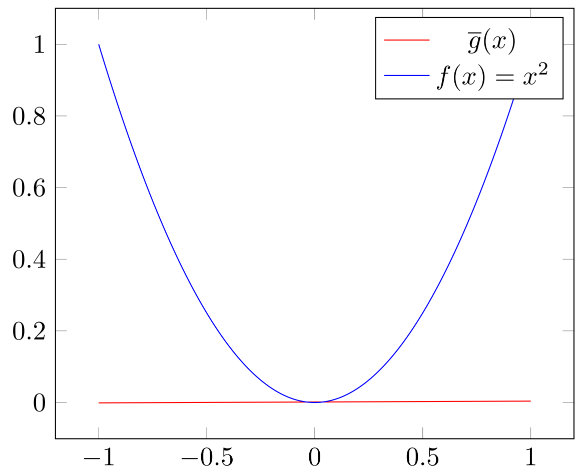
Generate a large number (e.g. 10,000) of 2-point datasets $D = \{(x_1, x_1^2), (x_2, x_2^2)\}$ and generate the set of final hypotheses for all these data sets. Since each g_i is just a line, we can simply store the coefficients a, b of $g = ax + b$, then average the sets of each coefficient to obtain $\overline{g}(x)$. The typical formula for E_{out} can be used for each iteration, then averaged. Bias and variance can both be calculated using $\overline{g}$.

(c) *Run your experiment and report the results. Compare E_{out} with bias+var. Provide a plot of your $\overline{g}(x)$ and $f(x)$.*

Running the experiment for $N = 10000$ gives

$$
\begin{aligned}
\overline{g}(x) &= 0.00244x + 0.00143 \\
E_{out} &= 0.537 \\
\mathbb{E}[bias(x)] &= 0.204 \\
\mathbb{E}[var(x)] &= 0.331
\end{aligned}
$$

So for $bias + var$ we get 0.535, only two thousands from the value first obtained for E_{out}. These values are extremely close, exactly as expected.



(d) *Compute analytically what E_{out}, bias, and var should be.*

For bias we have

$$
\begin{aligned}
bias &= \mathbb{E}_x[\overline{g}(x) - f(x))^2] \\
&= \frac{1}{2}\int_{-1}^{1} x^4 dx \\
&= \frac{1}{5}
\end{aligned}
$$

and for variance we get

$$
\begin{aligned}
var &= \mathbb{E}_x[\mathbb{E}_D[(g_D(X) - \overline{g}(x))^2]] \\
&= \frac{1}{4}\int_{-1}^{1}\int_{-1,1}\int_{-1}^{1} ((y+z)x - yz)^2 dydzdx \\
&= \frac{1}{3}
\end{aligned}
$$

so for E_{out} we get

$$
\begin{aligned}
\mathbb{E}[E_{out}] &= bias + var \\
&= \frac{1}{5} + \frac{1}{3} \\
&= \frac{8}{15} \approx .533
\end{aligned}
$$

All of these values agree well with the numerically obtained values.