Propositions

A proposition is either True or False

- may be easy or difficult to assign truth value to proposition
- prop itself should always be precise; unambiguous

connectors

$$\begin{aligned}\text{NOT: } \neg p \equiv & \text{ it is not the case that p} \quad (1)\\ \text{AND: } p \wedge q \equiv & \text{ p and q} \quad (2)\\ \text{OR: } p \vee q \equiv & \text{ p or q} \quad (3)\\ \text{IF THEN: } p \rightarrow q \equiv & \text{ if p then q / p implies q} \quad (4)\end{aligned}$$

implication

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

alright guess we'll just expand the table a bit

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \implies q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

POP QUIZ WOO

$$p \equiv x > 0$$

$$q \equiv y > 1$$

$$r \equiv x < y$$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$
T	T	T	T	T	T
T	T	F	F	T	T
F	T	T	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	F	T	F	F	F
F	T	F	F	F	T
F	F	F	F	F	F

Table 1: Note that row 7 is not actually possible.

Quantifiers

e.g.,

EVERY; A; SOME; ANY; ALL; THERE EXISTS

define *predicate* $P(c)$ where

$$C = \{c \mid c \text{ is a car}\}$$

$$P(c) = \text{"car c has four wheels"}$$

we write the statement "for all c in C, P(c) is true" as

$$\forall c \in C : P(c)$$

e.g., for the function $f(x) = x^2$, we can write

$$\forall x \in \mathbb{R} : f(x) \geq 0$$

More on Proofs.

Direct Proof Template for proving $p \implies q$

Proof.

1. Start by assuming that the statement claimed in p is **T**
2. Restate your assumption in mathematical terms
3. Use mathematical and logical derivations to relate your assumption to q
4. Argue that you have shown that q must be **T**
5. End by concluding that q is **T**

Example.

Thm. If $x, y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$

Proof.

1. Assume that $x, y \in \mathbb{Q}$
2. Then there are integers a, c and natural numbers b, d such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$
3. Then $x + y = (ad + bc)/bd$
4. Since $ad + bc \in \mathbb{Z}$ and $bd \in \mathbb{N}$, $x + y$ is rational.

Another example.

Thm. If $4^x - 1$ is divisible by 3, then 4^{x+1} is divisible by 3 for $x \in \mathbb{R}$.

Proof.

1. Assume that $4^x - 1$ is divisible by 3.
2. So $4^x - 1 = 3k$ for an integer k, i.e. $4^x = 3k + 1$
3. Observe: $4^{x+1} = 4 \cdot 4^x$. So
$$4^{x+1} = 4(3k + 1) = 12k + 4$$
Then $4^{x+1} - 1 = 12k + 3 = 3(4k + 1)$ is a multiple of 3.
4. Since it's a multiple of 3, it must be divisible by 3.
5. ayooo q is **T** ***

*** Note that we don't actually know that $4^x - 1$ is divisible by 3.

Exercise.

Theorem. For all pairs of odd integers m, n, the sume $m + n$ is an even integer.

Proof.

1. Assume m and n are both odd.
2. aighty this means that $m = 2k + 1$ and $n = 2l + 1$.
3. adding these together, we have
$$m + n = 2k + 1 + 2l + 1 = 2 + 2k + 2l = 2(k + l + 1)$$
4. since $m + n$ is a multiple of 2, it is divisible by 2 and thus an even number
5. **QED**

Contraposition Template for $p \implies q$

Proof.

1. Start by assuming that the statement claimed in q is **F**
2. Restate your assumption in mathematical terms
3. Use mathematical and logical derivations to relate your assumption to p
4. Argue that you have shown that p must be **F**
5. End by concluding that p is **F**

Example
Theorem. If x^2 is even, then x is even.
Proof.

1. Assume that x is odd.
2. Then $\exists k \in \mathbb{Z} : x = 2k + 1$
3. Then $x^2 = 2(2k^2 + 2k) + 1$
4. This means x^2 is 1 added to a multiple of 2, so it's odd.
5. x^2 is odd so the proof is over lol

Exercise
Theorem. If r is irrational, then $\sqrt{r}$ is irrational.
Proof.

1. Let's assume $\sqrt{r}$ is rational.
2. So $\exists a, b \in Z : \sqrt{r} = \frac{a}{b}$
3. What happens when we square it?
$$\sqrt{r}^2 = (\frac{a}{b})^2$$
$$r = \frac{a^2}{b^2}$$
a and b are both integers, so r must be rational
4. So it's clear that r is not irrational in this case (it is rational).
5. unnecessary restatement of concluding p is **F**

Equivalence: sort of a sidenote

IF AND ONLY IF

$$p \iff q$$

This just means you have to prove the implication **both ways**.

Contradictions

e.g.,

$$1 = 2;\ n^2 < n \text{ for } n \in \mathbb{N};\ |x| < x;\ p \wedge \neg p$$

Wowie these look **FISHY** don't they?

Proof Template

1. To derive a contradiction, assume that p is **F**
2. Restate your assumption in mathematical terms
3. Derive a **FISHY** statement?a contradiction that must be false
4. Thus, the assumption in step 1 is false, and p is **T**

Exercise

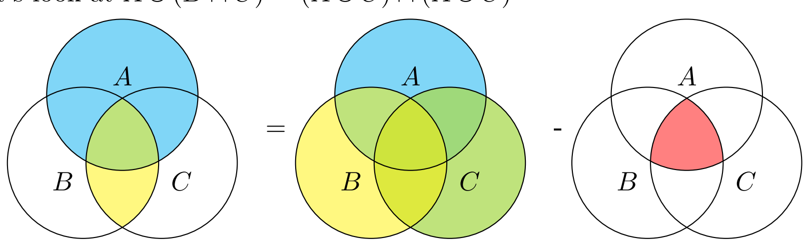
Theorem. Let a, b be integers. Then $a^2 - 4b \neq 2$

Proof.

1. Say $a^2 - 4b = 2$
2. Then
$$a^2 = 2 + 4b = 2(1 + 2b)$$
$$a = \sqrt{2}\sqrt{1 + 2b}$$
3. $\sqrt{2}$ is irrational, so a must be irrational, so it's not an integer. but a is an integer. **FISHY.**
4. alright so we must have
$$a^2 - 4b \neq 2$$

Proofs about Sets

Let's look at $A \cup (B \cap C) = (A \cup C) \cap (A \cup C)$



Induction

Template

1. Show $P(1)$
2. Assume $P(n)$
3. Show $P(n) \implies P(n+1)$

More Proof-y Things

Well-Ordering Principle

Any non-empty set of natural numbers has a minimum element.

This is important because induction follows form well ordering. e.g.

Take some predicate $P(n)$. If $P(1)$, and $P(n) \implies P(n+1)$, then $P(n)$ for $n \geq 1$.

Proof. Suppose $P(1)$ and $P(n) \implies P(n+1)$ for $n \geq 1$.

Assume $P(n)$ false for some values of n, with $n*$ representing the smallest counterexample for $P(n)$. Here, $n* > 1$ because $P(1)$ is true.

Given this assumption, $n*-1$ is not a counterexample because $n*$ is the smallest counterexample, so $P(n*-1)$ is true.

But since $P(n*-1)$ is true, we must have $P(n*-1) \implies P(n*)$. So we have a contradiction. Therefore $P(n)$ is true for all $n \geq 1$.

An example

$$n < 2^n \text{ for } n \geq 1$$

Proof.

Induction.

$P(1)$ is true because $1 < 2$. Assume $P(n)$ true. Then

$$n+1 \leq n+n = 2n \leq 2 \cdot 2^n = 2^{n+1}$$

So $P(n+1)$ is true and therefore $P(n)$ is true.

Well-ordering

Assume that there is an $n \geq 1$ such that $n \geq 2^n$. Let $n*$ be the minimum example of this, so $n* \geq 2^n$.

We know $1 < 2^1$, so $n* \geq 2$, which gives $\frac{1}{2}n* \geq 1$. So

$$n*-1 \geq n*-\frac{1}{2}n* = \frac{1}{2}n* \geq \frac{1}{2} \cdot 2^{n*} = 2^{n*-1}$$

which means that $n*-1$ is a smaller counterexample! ooOOoOOOO.

Harder

Prove $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$.

Proof.

$P(1)$: $1 \leq 2 \cdot \sqrt{1}$ is true.

Assume $P(n)$. Then for $P(n+1)$ we have

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$$

We can use the assumption of $P(n)$ to rewrite this

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \\ &\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}\end{aligned}$$

And here we use a *Lemma.* $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$

Which we prove by contradiction:

$$\begin{aligned}2\sqrt{n} + \frac{1}{\sqrt{n+1}} &> 2\sqrt{n+1}\\ 2\sqrt{n(n+1)} + 1 &> 2(n+1)\\ 4n(n+1) &> 4(n+1)^2\\ 4n &> 4n+4\end{aligned}$$

Wow fishy.
Back to the proof:

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} &\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}\\ &\leq 2\sqrt{n+1}\end{aligned}$$

So $P(n)$ is true for all $n \geq 1$.

Prove $n^2 \leq 2^n$ for $n \geq 4$

$$4^2 = 16 \leq 2^4 = 16$$

Assume that $n^2 \leq 2^n$ and that $2n+1 \leq 2^n$. Then

$$(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^n + 1$$

the tile problem

Can you tile a $2^n \times 2^n$ patio missing one of the center squares, using only the corner shaped tile?

let $P(n) :=$ the $2^n \times 2^n$ grid minus a center square can be L-tiled.

Suppose $P(n)$ is **T**. WELL. The $2^{n+1} \times 2^{n+1}$ patio can be separated into four $2^n \times 2^n$ patios.

Think about adding the center L to this first. Then all four of the subtiles were/are missing a corner square. Thus we can revise the original claim to be

$Q(n)$:

(i) the $2^n \times 2^n$ grid missing a center square can be L-tiled.

(ii) the $2^n \times 2^n$ grid missing a corner square can be L-tiled.

So add base cases and complete the proof.

Different Problem

$P(n) : n^3 < 2^n$ for $n \geq 10$

Suppose $P(n)$ is true. Consider $P(n+1) : (n+1)^3 < 2^{n+2}$??

$$\begin{aligned}(n+2)^3 &= n^3 + 6n^2 + 12n + 8\\ &< n^3 + nn^2 + n^2n + n^3\\ &< 4n^3 < 4 \cdot 2^n = 2^{n+2}\end{aligned}$$

so

$$P(n) \implies P(n+2)$$

We can have two base cases to cover all cases—$P(10)$ and $P(11)$ are both true.

THE FUNDAMENTAL THEOREM OF ARITHMETIC

SUPPOSE $n \geq 2$. Then (i) n can be written as a product of prime factors, and (2) the representation of n as a product of primes is unique.

We could use $P(n)$: n is a product of primes. But this is hard. So let's use

$$Q(n) : P(2) \wedge P(3) \wedge P(4) \wedge \cdots P(n)$$

Proof. Q(1) claims 2 is a product of primes, which is true.

Assume that $Q(n)$ is true, so each of $2, 3, \ldots, n$ are prime products. Since we know $Q(n)$, to prove $Q(n+1)$, we just need to show that $n+1$ is a product of primes. There are some possible cases here:

- $n+1$ is prime. Fin.
- $n+1$ not prime, so $n+1 = kl$ where $2 \leq k, l \leq n$

In the second case, we know that $P(k)$ and $P(l)$ are both true, so k and l are both products of primes. Thus kl is a product of primes, so $n+1$ is a product of primes. $Q(n+1)$ is true for all $n \geq 2$.

Strong Induction

To prove $P(n) \forall n \geq 1$ by strong induction, use induction to prove the *stronger* claim that $Q(n)$: each of $P(1), P(2), \ldots, P(n)$ are true.

	Ordinary Induction	Strong Induction
Base Case	Prove $P(1)$	Prove $Q(1) = P(1)$
Induction Step	$P(n) \implies P(n+1)$	$Q(n) = P(1) \wedge \cdots \wedge P(n) \implies P(n+1)$

Induction is Important

Applications of Inudction

- Tournament rankings

- Greedy or recursive algorithms

- Games of strategy

Equal Pile Nim.
$P(n)$: Player 2 can win the ame that starts with n pennies in each row.

Player 2 can always return the game ot smaller equal piles. If player 2 wins the smaller game, Player 2 wins the larger game. Strong Induction!