Foundations of Computer Science HW 1

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Q1

(c) $A = \{x \mid x^2 = 6 : x \in \mathbb{Z}\}$

$$A = \{-3, 3\}$$

(d) $A = \{x \mid x = x^2 - 1 : x \in \mathbb{R}\}$

$$A = \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$$

Q2

(c) $C = \{1, 2, 4, 7, 11, 16, 22, \dots\}$

$$C = \left\{x \mid x = \frac{n(n-1)}{2} + 1 : n \in \mathbb{N}\right\}$$

(d) $D = \{\dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \dots\}$

$$D = \{x \mid x = 2^n : n \in \mathbb{Z}\}$$

Q3

$B = \{\{a, b\}, a, b, c\}$. What is the power set $\mathscr{P}(B)$

$$\begin{aligned}\mathscr{P} = \Big\{&\{\}, \{a\}, \{b\}, \{c\}, \big\{\{a, b\}\big\}, \{a, b\}, \{a, c\}, \big\{a, \{a, b\}\big\}, \{b, c\}, \big\{b, \{a, b\}\big\}, \\ &\big\{c, \{a, b\}\big\}, \{a, b, c\}, \big\{a, b, \{a, b\}\big\}, \big\{a, c, \{a, b\}\big\}, \big\{b, c, \{a, b\}\big\}, \big\{a, b, c, \{a, b\}\big\}\Big\}\end{aligned}$$

Q4

Problem. List all subsets of $\{a, b, c, d\}$ that contain c but not d.

The subsets are:

$$\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}$$

Q5

$|A| = 7$ and $|B| = 4$.

$0 \le |A \cap B| \le 4$ because it is possible that none of the elements of A and B are the same, and as many as all 4 elements of B could also be in A.

$7 \le |A \cup B| \le 11$ because in the case where all 4 elements of B are in A, the union of the sets is just A. However, as many as 4 new elements from B could be added to A to form the union, if they do not already exist in A.

Q6

Fact. $\sqrt{3}$ is irrational.

Proof.

Let's assume that $\sqrt{3}$ is rational. Then we can write

$$\sqrt{3} = \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots \right\}$$

where $a_i, b_i \in \mathbb{N}$. We use the well ordering principle to assert that there is a minimum b_i, which we will call b. There is of course an a_i corresponding to b, and which we will call a. Since b is the minimum possible value for the denominator, it is true that a and b have no common factor. So let's write our expression for $\sqrt{3}$ and solve for a.

$$\begin{aligned} \sqrt{3} &= \frac{a}{b} \\ a^2 &= 3b^2 \end{aligned}$$

We don't actually have to solve all the way. What we can see here is that either both a and b are odd, or they are both even. Let's say they're both odd. Then

$$\begin{aligned} a &= 2n + 1 \\ b &= 2m + 1 \end{aligned}$$

where $n, m \in \mathbb{N}$. Substituting this into our previous expression, we have

$$\begin{aligned} (2n+1)^2 &= 3(2m+1)^2 \\ 4n^2 + 4n + 1 &= 3(4m^2 + 4m + 1) \\ 2n^2 + 2n &= 6m^2 + 6m + 1 \\ 2(n^2 + n) &= 2(3m^2 + 3m) + 1 \end{aligned}$$

Now we can see that the LHS of the expression is even since it is a multiple of 2, and the RHS is odd. They must not be the same, so we have a contradiction. Therefore, there are no values of a and b which give $\sqrt{3}$, so $\sqrt{3}$ **is irrational**.

Problem. Try to prove that $\sqrt{9}$ is irrational this way.

Assume that $\sqrt{9}$ is rational. As before,

$$\sqrt{9} = \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots \right\}$$

is the set of possible representations, with $a_i, b_i \in \mathbb{N}$. There is a minimum b_i called b and corresponding a. These have no common factor. We can again look at solutions for a.

$$\begin{aligned} \sqrt{9} &= \frac{a}{b} \\ a^2 &= 9b^2 \end{aligned}$$

Again we can use the facts from the previous proof to expand both sides of the expression.

$$
\begin{aligned}
(2n+1)^2 &= 9(2m+1)^2 \\
4n^2+4n+1 &= 9(4m^2+4m+1) \\
2n^2+2n &= 18m^2+18m+4 \\
2(n^2+n) &= 2(9m^2+9m+2)
\end{aligned}
$$

But this time, both the LHS and RHS are even, and nothing about our restrictions for a and b are violated. Since there is no contradiction here, we cannot conclude that $\sqrt{9}$ is irrational.

Q7

p	q	r	$\neg(p \wedge r) \wedge q$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F