Foundations of Computer Science HW 2

Greg Stewart

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Q1. 3.25 (c) - (d)

Given: $P(x, h) =$ "Person x has hair h" and $M(x) =$ "Hair h is grey"

(c) Nobody is bald.

$$\forall x, h : P(x, h)$$

or

$$\neg\exists x, h : \neg P(x, h)$$

(d) Kilam does not have all grey hair

$$\exists x, h : \neg M(x) \wedge x = "Kilam"$$

Q2. 3.29 (c) - (d)

(c) $\forall x : (M(x) \rightarrow \neg F(x))$

For any person, if they are a math major, then they are not a freshman.

(d) $\neg\exists x : (M(x) \wedge \neg F(x))$

There aren't any people that are a math major *and* a freshman.

Q3. 3.36 (c) - (d)

(c)

$\forall x : (\exists y : x^2 = y)$
This claim is **T** in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

(d)

$\forall y : (\exists x : x^2 = y)$
This claim is **T** in $\mathbb{R}, \mathbb{Q}$

Q4. 4.8 (c) -(d)

(c)

Prove by contraposition that $2^n - 1$ is prime $\implies$ n is prime.

Begin by assuming that n is not prime. Then there is some way to factor n such that $n = xy$. So

$$\begin{aligned} 2^{xy} - 1 &= (2^x)^y - 1 \\ &= (2^y - 1)(2^{y(x-1)} + 2^{y(x-2)} + \cdots + 2^y + 1) \end{aligned}$$

We know that $y > 1$ so for the first factor we know

$$2^y - 1 > 1$$

But this means that $2^n - 1$ is a composite number, and thus not prime.

Thus the implication must be true.

(d)

Prove by contraposition that n^3 is odd $\rightarrow n$ is odd.

Assume that n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Then

$$n^3 = (2k)^3 = 8k^3$$

But this is an even number! So the original implication must be true.

Q5. 4.10 (f) & (h): Prove by contradiction

(f)

$(x, y) \in \mathbb{Z}^2 \rightarrow x^2 - 4y - 3 \neq 0$

Assume that the implication is false. That is, $(x, y) \in \mathbb{Z}^2 \rightarrow x^2 - 4y - 3 = 0$. Then

$$\begin{aligned} 4y &= x^2 - 3 \\ y &= \frac{x^2 - 3}{4} = \frac{x^2}{4} - \frac{3}{4} \end{aligned}$$

It is obvious now, since the right(most) hand side of the above equation subtracts a fraction, that y is not an integer, so we have a contradiction. Therefore it must be true that

$$(x, y) \in \mathbb{Z}^2 \rightarrow x^2 - 4y - 3 \neq 0$$

(h)

$\forall (a, b, c) \in \mathbb{Z}^3 : (a^2 + b^2 = c^2) \rightarrow (a \text{ or } b \text{ is even})$.

Let's assume this is false. So

$$\exists (a, b, c) \in \mathbb{Z}^3 : (a^2 + b^2 = c^2) \rightarrow (a \wedge b \text{ odd})$$

Then we can say that $a = 2k + 1$ and $b = 2l + 1$ for some integers k and l. So

$$(2k + 1)^2 + (2l + 1)^2 = c^2 = 4k^2 + 4k + 1 + 4l^2 + 4l + 1 = 4(k^2 + l^2 + k + l) + 2$$

There is no constraint on c^2, so either $c = 2m$ or $c = 2m + 1$ for some integer m. This leads to either $c^2 = 4m^2$ or $c^2 = 4m^2 + 4m + 1$. In either case, we cannot have

$$a^2 + b^2 = c^2$$

because in both cases, the two sides have different remainders when divided by 4. Thus we have a contradiction and the original statement must be true.

Q6. 4.14 (o)

Thm. There exists $x, y \in \mathbb{Z}$ for which $2x^2 + 5y^2 = 14$.

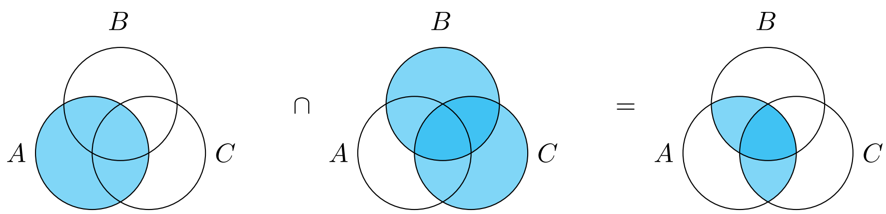
Suppose that there is indeed such a pair x, y which satisfies the equation. Then since 14 and $2x^2$ are both even, $5y^2$ must also be even. Since 5 is odd, this implies that y^2 is even, so y is even and $y = 2k$ for some $k \in \mathbb{Z}$. So we have

$$2x^2 + 20k^2 = 14$$
$$x^2 + 10k^2 = 7$$

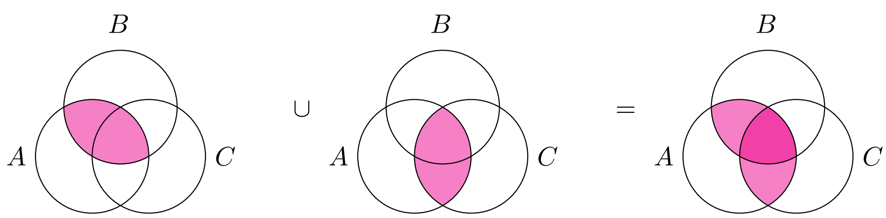
However, $10k^2 > 7$ for $k \neq 0$, so we must have 0 for that term. Then $x^2 = 7$, which means $x \notin \mathbb{Z}$, so we have a contradiction. Therefore the theorem is false.

Q7. 4.26 (c)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



A $(B \cup C)$



$(A \cap B)$ $(A \cup C)$

Q8. 4.27 (c)

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let there be $x \in A \cap (B \cup C)$ so that $x \in A$ and $x \in B \vee C$. So $x \in A \wedge B$ or $x \in A \wedge C$. So $x \in (A \cap B) \cup (A \cap C)$. Thus we have

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

Now the other way. Let there be $y \in (A \cap B) \cup (A \cap C)$. So $y \in A \cap B$ or $y \in A \cap C$. In either case, $y \in A$, and either $y \in B$ or $y \in C$, so $y \in B \cup C$. Then $y \in A \cap (B \cup C)$. Therefore

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Thus the equality is true.