Numerical Computing HW 3

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February 15, 2018

3.2(b) Use Doolittle factorization to solve the following. Calculate $\kappa_\infty(A)$.

$$\begin{aligned} x - 2y &= 0 \\ -x + 4y &= 1 \end{aligned}$$

From this system of equations, we have

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And we want to solve the equation $Az = b$. We factor the matrix A into L and U like so

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

As this is Doolittle factorization, we choose the diagonals of L to be 1, so

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

And at once we see that $u_{11} = 1$ and $u_{12} = -2$. We are left with

$$l_{21}u_{11} = -1 \quad \text{and} \quad l_{21}u_{12} + u_{22} = 4$$

Using the results already obtained, this becomes

$$l_{21} = -1 \quad \text{and} \quad -2l_{21} + u_{22} = 4$$

So $u_{22} = 2$. Now we first solve $Lv = b$ for v:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So clearly we have $v_1 = 0$ and $v_2 = 1$. Now we solve $Uz = v$:

$$\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Which gives $x = 1$ and $y = \frac{1}{2}$.

To calculate $\kappa_\infty(A)$, we first need A^{-1}, which is

$$\frac{1}{2}\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

$||A||_\infty = \max\{1+2, 1+4\} = 5$ and $||A^{-1}||_\infty = \max 2+1, 1/2+1/2 = 3$ so we get

$$\kappa_\infty(A) = ||A|| \cdot ||A^{-1}|| = 15$$

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3.6 Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Find A^{-1}

Use Gaussian elimination to find the inverse, comparing to the identity matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

So the inverse is

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b) Find $\kappa_\infty(A)$.

$$||A|| = \max\{1+1, 1+1, 1+1\} = 2 \quad \text{and} \quad ||A^{-1}|| = \max\{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}\} = \frac{3}{2}$$

So for the infinity norm value of κ,

$$\kappa_\infty = 2 \cdot \frac{3}{2} = 3$$

(c) Find the Doolittle factorization of A.

In the Doolittle factorization, the diagonal elements of L are all 1.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

So immediately we have $u_{11} = 1$, $u_{12} = 1$, and $u_{13} = 0$.

$$l_{21}u_{11} = 0, \quad l_{21}u_{12} + u_{22} = 1, \quad l_{21}u_{13} + u_{23} = 1$$

Which gives $l_{21} = 0$, $u_{22} = 1$, and $u_{23} = 1$. Finally we solve

$$l_{31}u_{11} = 1, \quad l_{31}u_{12} + l_{32}u_{22} = 0, \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$$

So $l_{31} = 1$, $l_{32} = -1$, and $u_{22} = 2$.
Explicitly written out, we have

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

3.7(b) **For the matrix, explain why it's positive definite and find the Cholesky factorization.**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

The matrix is clearly symmetric as it's transpose is the same as the matrix. It is strictly diagonal dominant as the diagonal elements are each greater than the sum of all the other elements in each row, and all the diagonal entries are positive, so by a theorem discussed in class, **this matrix is positive definite.**

The Cholesky Factorization factors a matrix A as $U^T U$, where U is upper triangular, which we write as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

So immediately

$$u_{11}^2 = 1 \qquad u_{11}u_{12} = 0 \qquad u_{11}u_{13} = 0$$

$$u_{12}u_{11} = 0 \qquad u_{12}^2 + u_{22}^2 = 2 \qquad u_{12}u_{13} + u_{22}u_{23} = 1$$

$$u_{13}u_{11} = 0 \qquad u_{13}u_{12} + u_{23}u_{22} = 1 \qquad u_{13}^2 + u_{23}^2 + u_{33}^2 = 5$$

gives $u_{11} = 1$, $u_{12} = 0$, and $u_{13} = 0$. This leads to $u_{22} = \sqrt{2}$ and $u_{23} = \frac{1}{\sqrt{2}}$. These results finally give $u_{33} = \frac{3}{\sqrt{2}}$.

Putting this altogether, the Cholesky factorization is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{pmatrix}$$

3.15(a) **A is 2×2 matrix; A_f is its floating point approximation. Give an example of an invertible matrix A where A_f is the zero matrix. Should also have $\kappa(A) = 1$.**

Consider

$$A = \begin{pmatrix} 10^{-324} & 0 \\ 0 & 10^{-324} \end{pmatrix}$$

The inverse of this matrix is

$$A^{-1} = \frac{1}{10^{-648}} \begin{pmatrix} 10^{-324} & 0 \\ 0 & 10^{-324} \end{pmatrix}$$

Which means for $\kappa(A)$ we get

$$\kappa(A) = 10^{-324} \cdot 10^{648} \cdot 10^{-324} = 1$$

However, in a floating point approximation, all of the elements of A are 0, since 10^{-324} goes to 0. Thus A_f is the zero matrix.

(b) Give an example of a symmetric and positive definite matrix A where A_f is symmetric but not positive definite.

An example of this is

$$A = \begin{pmatrix} 10^{-324} & 0 \\ 0 & 1 \end{pmatrix}$$

which gives

$$A_f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

And clearly A_f is not positive definite. Another example of this is

$$A = \begin{pmatrix} 1+10^{-16} & 1-10^{-16} \\ 1-10^{-16} & 1+10^{-16} \end{pmatrix} \qquad \text{so} \qquad A_f = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

3.17 Use MATLAB to calculate the values in the table.

Consider two ways to solve $Ax = b$, where $A = 3P$ and the exact solution x is the 1-vector. Calculate x_M with the backslash operator, and x_I with the inverse formula, and fill in the table.

n	$\frac{\|\|x-x_M\|\|}{\|\|x\|\|}$	$\frac{\|\|x-x_I\|\|}{\|\|x\|\|}$	$\|\|r\|\|$	$\kappa(A)$	$\epsilon\kappa(A)$
4	1.0214×10^{-14}	1.4211×10^{-14}	0	1.19×10^{3}	2.38×10^{-13}
8	2.7544×10^{-10}	1.1552×10^{-9}	3.6380×10^{-12}	3.9588×10^{7}	7.9176×10^{-9}
12	1.3346×10^{-6}	6.0711×10^{-5}	4.6566×10^{-10}	1.7390×10^{12}	3.4780×10^{-4}
16	3.8994×10^{-4}	1.7193	3.7253×10^{-9}	8.5267×10^{16}	17.0534

(a) Substantial differences?

Up until larger n-values, the two methods are fairly similar in computed values, as described by the relative error. They are generally within an order of magnitude of each other.

(b) Small residual indicate accurate solution? Dependence on n?

A small residule does not seem to be a good indicator of an accurate solution, as the residual remains relatively small for all values of n in the table, while the error drastically increases with greater n-values, along with large increases in the condition number, which is always on the order of 10^n. The residual does increase with n, though not by the same amount as the condition number.

(c) How well does last column predict relative error?

The last column predicts the relative error fairly well, being roughly within and order of magnitude for of the relative error for every n, except in the case of x_M when $n = 16$.