Numerical Computing HW 5

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6.1 (d), (e)

(e) *Using the composite trapezoirdal rule, how small does the step size h have to be to guarantee that the numerical error is less than 10^{-6}?*

To find h, we set the error formula to 10^{-6} and solve.

$$\begin{aligned} -\frac{1}{12}h^3 f''(\eta) &= 10^{-6} \\ h^3 &= \frac{12 \times 10^{-6}}{f''(\eta)} \\ h &= \Big[\frac{12 \times 10^{-6}}{f''(\eta)}\Big]^{1/3} \end{aligned}$$

Taking $f(x) = x^2$, this means we have h

$$h = 0.01817...$$

(f) *Using the composite Simpson's rule, how small does the step size h have to be to guarantee that the numerical eroor is less than 10^{-6}?*

We solve this the same way.

$$\begin{aligned} -\frac{1}{90}h^5 f''''(\eta) &= 10^{-6} \\ h^5 &= \frac{90 \times 10^{-6}}{f''''(\eta)} \\ h &= \Big[\frac{90 \times 10^{-6}}{f''''(\eta)}\Big]^{1/5} \end{aligned}$$

Takign $f(x) = x^4$, we get for h

$$h = 0.08219...$$

6.4 (a)—(d)

For a linearly elastic material, the stress is given by

$$T = E\frac{du}{dx}$$

where $u(x)$ is the displacement of the material and E is a positive constant known as the Young's modulus. The question considered here is how to determine u from measurements of T:

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
T	1	-1	2	3	4

(a) *Show that*

$$u(x) = u(0) + \frac{1}{E}\int_0^x T(s)ds$$

Let's start with some good old fashioned separation of variables:

$$du = \frac{T}{E}dx$$

We can integrate both of these from 0 to x with respect to dummy variables.

$$\begin{aligned}\int_0^x du &= \frac{1}{E}\int_0^x T(s)ds \\ u(x) - u(0) &= \frac{1}{E}\int_0^x T(s)ds \\ u(x) &= u(0) + \frac{1}{E}\int_0^x T(s)ds\end{aligned}$$

And this final equation is what we wanted.

Note that for the following parts $u(0) = 0$ and $E = 4$.

(b) *Use the trapezoidal rule to find the value of* $u(x)$ *at each nonzero* x *value.*

The trapezoidal rule is

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{h}{2}(f_i + f_{i+1})$$

We take x_i on the LHS to be 0 in this case, and evaluate the RHS of the expression from (a) using this rule.

x	$u(x)$
$\frac{1}{4}$	$\frac{1}{4}(\frac{1}{8}(1 + (-1))) = 0$
$\frac{1}{2}$	$u(1/4) + \frac{1}{4}(\frac{1}{8}(-1 + 2)) = \frac{1}{32}$
$\frac{3}{4}$	$u(1/2) + \frac{1}{4}(\frac{1}{8}(2 + 3)) = \frac{3}{16}$
1	$u(3/4) + \frac{1}{4}\frac{1}{8}(3 + 4) = \frac{13}{32}$

(c) *Use the composite midpoint rule to calculate* $u(1)$.

Since we have no continuous function to evaluate, we can only use two midpoints from the data to evaluate the integral. So we have

$$\begin{aligned}u(1) &= \frac{1}{4}(\frac{1}{2}\cdot(-1) + \frac{1}{2}\cdot 3) \\ u(1) &= \frac{1}{4}\end{aligned}$$

(d) *Use the composite Simpson rule to do the same thing.*

We can be a bit more precise here, and take $h = \frac{1}{4}$.

$$u(1) = \frac{1}{4}\Big[\frac{1/4}{3}(1 + 4(-1) + 2) + \frac{1/4}{3}(-1 + 4(2) + 3) + \frac{1/4}{3}(2 + 12 + 4)\Big]$$
$$u(1) = \frac{1}{4}\Big[\frac{1}{12}(-1) + \frac{1}{12}(10) + \frac{1}{12}(18)\Big]$$
$$u(1) = \frac{1}{4}\Big[\frac{9}{4}\Big]$$
$$u(1) = \frac{9}{16}$$

6.15 (a), (c)

(a) *Given subinterval $t_i \leq t \leq t_{i+1}$, then a_i and a_{i+1} are known. Assuming v_i and y_i habe already been computed, use the trapezoidal rule to obtain the following expressions:*

$$v_{i+1} = v_i + \frac{1}{2}h(a_i + a_{i+1}) \tag{1}$$
$$y_{i+1} = y_i + \frac{1}{2}h(v_i + v_{i+1}) \tag{2}$$

The trapezoidal rule is given by

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{h}{2}(f_{i+1} + f_i)$$

So replacing $f(x)$ with $a(t)$, we get

$$\int_{t_i}^{t_{i+1}} a(t)dt \approx \frac{h}{2}(a_{i+1} + a_i)$$

Of course, integrating $a(t)$ gives us the change in velocity over this interval:

$$\Delta v = v_{i+1} - v_i = \frac{h}{2}(a_{i+1} + a_i)$$

Which can just be rewritten as

$$v_{i+1} = v_i + \frac{h}{2}(a_{i+1} + a_i)$$

And this matches what we wanted for (1). (2) is obtained in exactly the same way, except that $v(t)$ is integrated.

$$\int_{t_i}^{t_{i+1}} v(t)dt \approx \frac{h}{2}(v_{i+1} + v_i)$$

And we use the fact that position is given by the integral of velocity to write

$$y_{i+1} = y_i + \frac{h}{2}(v_{i+1} + v_i)$$

Which is what we wanted for (2).

(c) *An accurate computed value at $t = 3$ is $y(3) = .72732289075\ldots$. What is the difference between this value and what you compute for $y(3)$ at $n = 10, 20, 40$? How large need n be so that the error between the two is less than 10^{-8}?*

Based on my MATLAB results and checking, n must be about 4000 to achieve an error of less than 10^-8.

3/26/18 9:36 AM MATLAB Command Window 1 of 2

```
>> sixfifteen

exactish =

    0.7273


n =

    10


position =

    0.5562


error =

    0.1711


n =

    20


position =

    0.9515


error =

    0.2242


n =

    40


position =

    0.7034


error =

    0.0240
```

```
function sixfifteen
ns = [10 20 40 100 4000];
exactish = .72732289075
for i=1:5
    h = 3/ns(i);
    t = [];
    for j=1:(ns(i)+1)
        tmp = (j-1)*h;
        t = [t tmp];
    end
    t;
    a = zeros(ns(i),1);
    [ugh, sze] = size(t);
    for j=1:sze
        t(j);
        a(j) = sin((t(j))^4);
    end
    a;

    v = zeros(ns(i),1);
    for j=2:sze
        v(j) = v(j-1) + .5*h*(a(j-1) + a(j));
    end
    v;

    y = zeros(ns(i),1);
    for j=2:sze
        y(j) = y(j-1) + .5*h*(v(j-1) + v(j));
    end
    n = ns(i)
    position = y(end)
    error = abs(y(end) - .72732289075)

end
```

6.21

Suppose the integration rule has form

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx w_1 f(x_i) + w_2 f(z)$$

This is an example of Radau quadrature, which means that only one of the points used is an endpoint.

(a) *Find the values of w_1, w_2, z that maximize the precision.*

To do this we will take z to have the form $z = x_i + \alpha h$ and solve for alpha in addition to the other two unknowns.

$$
\begin{aligned}
f(x) = 1 &\implies w_1 + w_2 = h \\
f(x) = x &\implies h(x_i + \frac{h}{2}) = w_1 x_i + w_2(x_i + \alpha h) \\
&\quad h(x_i + \frac{h}{2}) = (w_1 + w_2)x_i + w_2 \alpha h \\
&\quad h(x_i + \frac{h}{2}) = h x_i + w_2 \alpha h \\
&\quad \frac{h}{2} = w_2 \alpha h \\
&\quad \alpha = \frac{h}{2w_2} \\
f(x) = x^2 &\implies h(x_i^2 + h x_i + \frac{1}{3}h^2) = w_1 x_i^2 + w_2(x_i^2 + 2x_i \alpha h + \alpha^2 h^2) \\
&\quad h(x_i^2 + h x_i + \frac{1}{3}h^2) = (w_1 + w_2)x_i^2 + w_2 \frac{x_i h^2}{w_2} + w_2 \frac{h^4}{4w_2^2} \\
&\quad \frac{1}{3}h^3 = \frac{h^4}{4w_2} \\
&\quad w_2 = \frac{3h}{4} \\
&\quad \implies w_1 + \frac{3h}{4} = h \\
&\quad \implies w_1 = \frac{h}{4} \\
&\quad \implies \alpha = \frac{4h}{6h} = \frac{2}{3}
\end{aligned}
$$

(b) *The error is know to have form*

$$\int_{x_i}^{x_{i+1}} f(x)dx = w_1 f(x_i) + w_2 f(z) + K h^4 f'''(\eta)$$

where, as usual, η is a point somewhere in the interval. Find K.

We solve for K by taking $f(x) = x^3$, which means that $f'''(\eta) = 6$. Thus we have

$$
\begin{aligned}
h(x_i^3 + \frac{3hx_i^2}{2} + h^2 x_i + \frac{h^3}{4}) &= \frac{h}{4}x_i^3 + \frac{3h}{4}(x_i + \frac{2h}{3})^3 + 6h^4 K \\
&= \frac{hx_i^3}{4} + \frac{3h}{4}(x_i^3 + 3 \cdot \frac{2hx_i^2}{3} + 3 \cdot \frac{4h^2 x_i}{9} + \frac{8h^3}{27} + 6h^4 K \\
&= hx_i^3 + \frac{3h^2 x_i^2}{2} + h^3 x_i + \frac{2h^4}{9} + 6h^4 K \\
\frac{h^4}{4} &= \frac{2h^4}{9} + 6h^4 K \\
\frac{1}{36} &= 6K \\
K &= \frac{1}{216}
\end{aligned}
$$