Numerical Computing HW 3

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5.2

x	-1	0	1	2
y	0	1	1	0

(a) *Find the piecewise linear interpolation function $g(x)$ that fits these data.*

We need the interpolation function

$$\begin{aligned} g(x) &= y_1G_1(x) + y_2G_2(x) + y_3G_3(x) + y_4G_4(x) \\ &= 0 + G_2(x) + G_3(x) + 0 \end{aligned}$$

To do this we need the "hat" function

$$G_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x-x_{i+1}}{x_i-x_{i+1}} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Obviously, we only need G_2 and G_3:

$$G_2(x) = \begin{cases} \frac{x+1}{0+1} = x+1 & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x-1}{0-1} = 1-x & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$G_3(x) = \begin{cases} \frac{x-0}{1-0} = x & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x-2}{1-2} = 2-x & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

So in the end we have

$$g(x) = G_2(x) + G_3(x)$$

(b) *Find the global interpolation polynomial $p_3(x)$ that fits these data.*

To find this interpolation we just need to solve a relatively simple matrix equation guaranteeing all data are plotted by the interpolation, which is

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solving this we for a_i, we get $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$, and $a_3 = 0$, so we can write the polynomial interpolation function

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{2}x^2$$

5.8(b) For the following functions, determine the step size h that will guarantee that the error is less than 10^{-6} using piecewise linear interpolation.

(c) $f(x) = x^{10}$ for $-1 \leq x \leq 1$.

$f''(x) = 90x^8$, and the maximum value of $f''(x)$ in the given interval is 90. We need

$$\begin{aligned} 10^{-6} &\leq \frac{1}{8}h^2 ||f''||_\infty = \frac{1}{8}h^2 \cdot 90 \\ 8.888 \times 10^{-8} &\leq h^2 \\ 2.981 \times 10^{-4} &\leq h \end{aligned}$$

5.9(b) Do that again but for clamped cubic spline.

Here we need $f''''(x) = 5040x^6$ which has a maximum on the interval of 5040. Now we can solve for h:

$$\begin{aligned} 10^{-6} &\leq \frac{1}{3}h^2(5040) \\ 5.952 \times 10^{-10} &\leq h^2 \\ 2.44 \times 10^{-5} &\leq h \end{aligned}$$

5.11(a)-(c)

$$g(x) = \begin{cases} 2 + 3x^2 + \alpha x^3 & \text{if } -1 \leq x \leq 0 \\ 2 + \beta x^2 - x^3 & \text{if } 0 \leq x \leq 1 \end{cases}$$

(a) *For what values of α and β, if any, is $g(x)$ a cubic spline for the total interval?* In order to have a cubic spline, we must have that the first and second derivates of each piecewise element are equal at their shared data point, $x = 0$:

$$\begin{aligned} 6(0) + 3\alpha(0)^2 &= 2\beta(0) - 3(0)^2 \\ 0 &= 0 \end{aligned} \qquad \begin{aligned} 6 + 6\alpha(0) &= 2\beta - 6(0) \\ \beta &= 3 \end{aligned}$$

So we must have that $\beta = 3$, but there is no such constraint for α.

(b) *What were the data points that gave rise to this cubic spline?*

For this we just need to plug in the end points of each piecewise spline's interval, which gets:

x	y
-1	$2 + 3 - \alpha$
0	2
1	$2 + \beta - 1$

x	y
-1	$5 - \alpha$
0	2
1	4

(c) *For what values of α and β is $g(x)$ a natural cubic spline?*

For natural cubic spline, we must have that $s_1''(-1) = s_2''(1) = 0$, so setting this up we have

$$\begin{aligned} 6 + 6\alpha(-1) &= 0 \\ \alpha &= 1 \end{aligned} \qquad\qquad \begin{aligned} 2\beta - 6(1) &= 0 \\ \beta &= 3 \end{aligned}$$

So for a natural cubic spline we must have $\alpha = 1$ and $\beta = 3$.

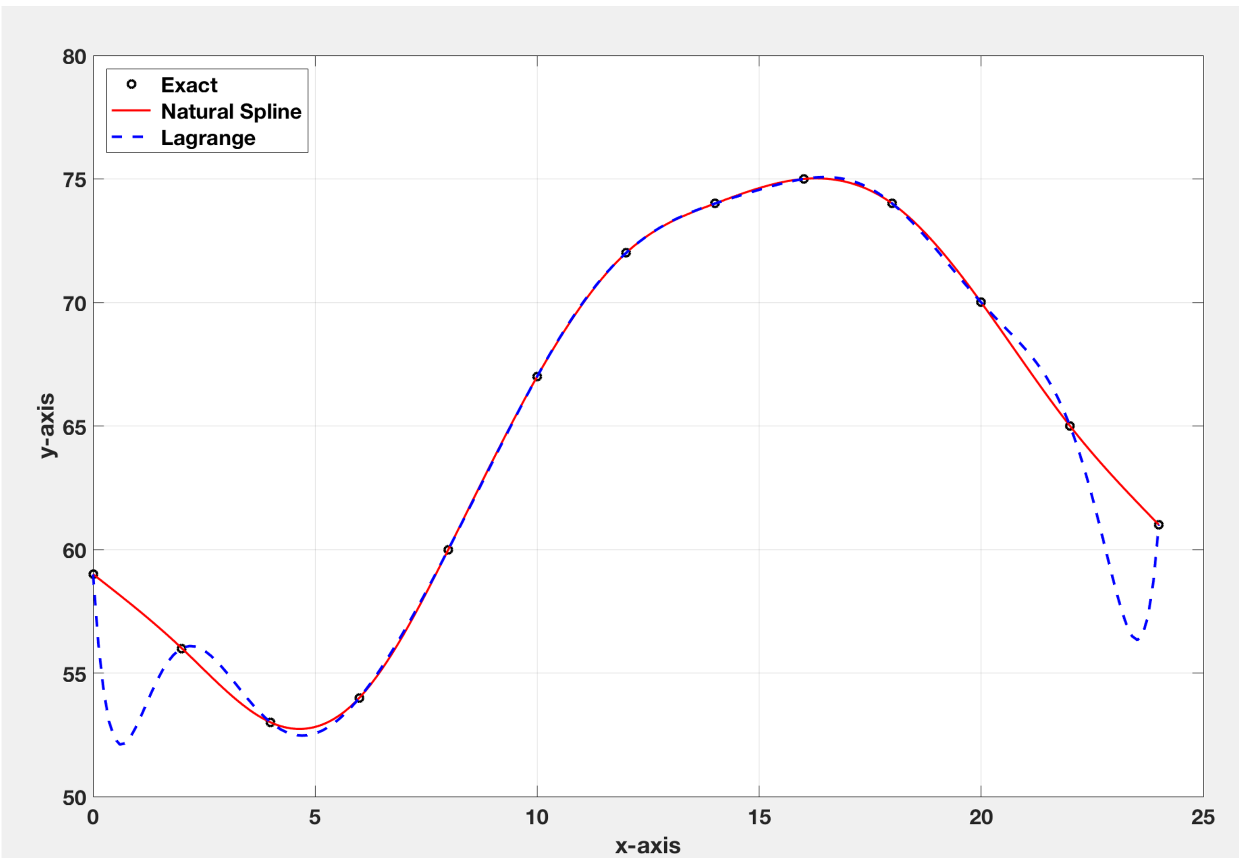
5.15

x	0	2	4	6	8	10	12	14	16	18	20	22	24
y	59	56	53	54	60	67	72	74	75	74	70	65	61

(a) *Fit data and plot on the same axis with:*

(i) Lagrange interpolation

(ii) Natural cubic spline



(b) *What do each of the interpolation functions give us as the temperature at 11AM?*

Lagrange interpolation gives 69.9 degrees.

Natural spline gives 69.9 degrees.

(c) *What do the two inperpolation functions predict the temperatue will be at 1AM the next day?*

Lagrange interpolation gives 152 degrees.

Natural spline gives 58.0 degrees.

(d) *What do the two inperpolation functions predict the temperature will be at 9AM the next day? Explain the result of the spline interpolation.*

Lagrange interpolation gives 452,300 degrees.

Natural spline gives 0 degrees. This occurs due to the end conditions on natural spline, namely that $s_1''(x_1) = 0$ and $s_n''(x_{n+1}) = 0$. What happens past the endpoints of natural spline is restricted by this, and causes the function to tend to 0 as we look to extrapolate well beyond the end of the interpolation.