Numerical Computing HW 7

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8.2 Determine the least squares approximation of the form $f(x) = a + b\sin(x)$

x_i	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$
f_i	2	0	-1

The error function in this case is

$$E(a,b) = \sum_{i=1}^{n} (a + b\sin(x_i) - y_i)^2$$

which we need to minimize. Differentiating we get

$$\frac{\partial E}{\partial a} = 2\sum_{i=1}^{n} (a + b\sin(x_i) - y_i)$$

$$\frac{\partial E}{\partial b} = 2\sum_{i=1}^{n} (a + b\sin(x_i) - y_i)\sin(x_i)$$

And so we can write the normal equation

$$\begin{pmatrix} n & \sum_i \sin(x_i) \\ \sum_i \sin(x_i) & \sum_i \sin^2(x_i) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_i y_i \\ \sum_i y_i \sin(x_i) \end{pmatrix}$$

And solve for a and b.

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{n\sum_i \sin^2(x_i) - (\sum_i \sin(x_i))^2} \begin{pmatrix} \sum_i \sin^2(x_i) & -\sum_i \sin(x_i) \\ -\sum_i \sin(x_i) & n \end{pmatrix} \begin{pmatrix} \sum_i y_i \\ \sum_i y_i \sin(x_i) \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

So the approximation is $f(x) = -2\sin(x)$.

8.5(b)-(d) Elliptical path of a comet is described by

$$r = \frac{p}{1 + \epsilon\cos\theta}$$

θ_i	0.00	1.57	3.14	4.71	6.28
r_i	0.62	1.23	16.14	1.35	0.62

(b) *By writing the model function as follows, explain how the nonlinear regression problem can be trandormed into one with model function $R = V_1 + V_2 \cos(\theta)$. Also, what happens to the data values?*

$$\frac{1}{r} = \frac{1 + \epsilon \cos\theta}{p}$$

Let $R = \frac{1}{r}$, $V_1 = \frac{1}{p}$, and $V_2 = \frac{\epsilon}{p}$. This gives the given form of the model function. None of the θ_i change, as they are still part of the model equation, but instead of using r_i, we use $R_i = \frac{1}{r_i}$.

(c) *Writing the model function in part (b) as $R = G(\theta)$, and using the least squares error function, compute V_1 and V_2. Determine p and ϵ.*

We have for the error function

$$E(V_1, V_2) = \sum_{i=1}^{n} [V_1 + V_2 \cos(\theta_i) - R_i]^2$$

which we can differentiate to get

$$\frac{\partial E}{\partial V_1} = 2 \sum_{i=1}^{n} (V_1 + V_2 \cos(\theta_i) - R_i)$$

$$\frac{\partial E}{\partial V_2} = 2 \sum_{i=1}^{n} (V_1 + V_2 \cos(\theta_i) - R_i) \cos(\theta_i)$$

And upon setting these to 0 we get the matrix equation

$$\begin{pmatrix} n & \sum_i \cos(\theta_i) \\ \sum_i \cos(\theta_i) & \sum_i \cos^2(\theta_i) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sum_i R_i \\ \sum_i R_i \cos(R_i) \end{pmatrix}$$

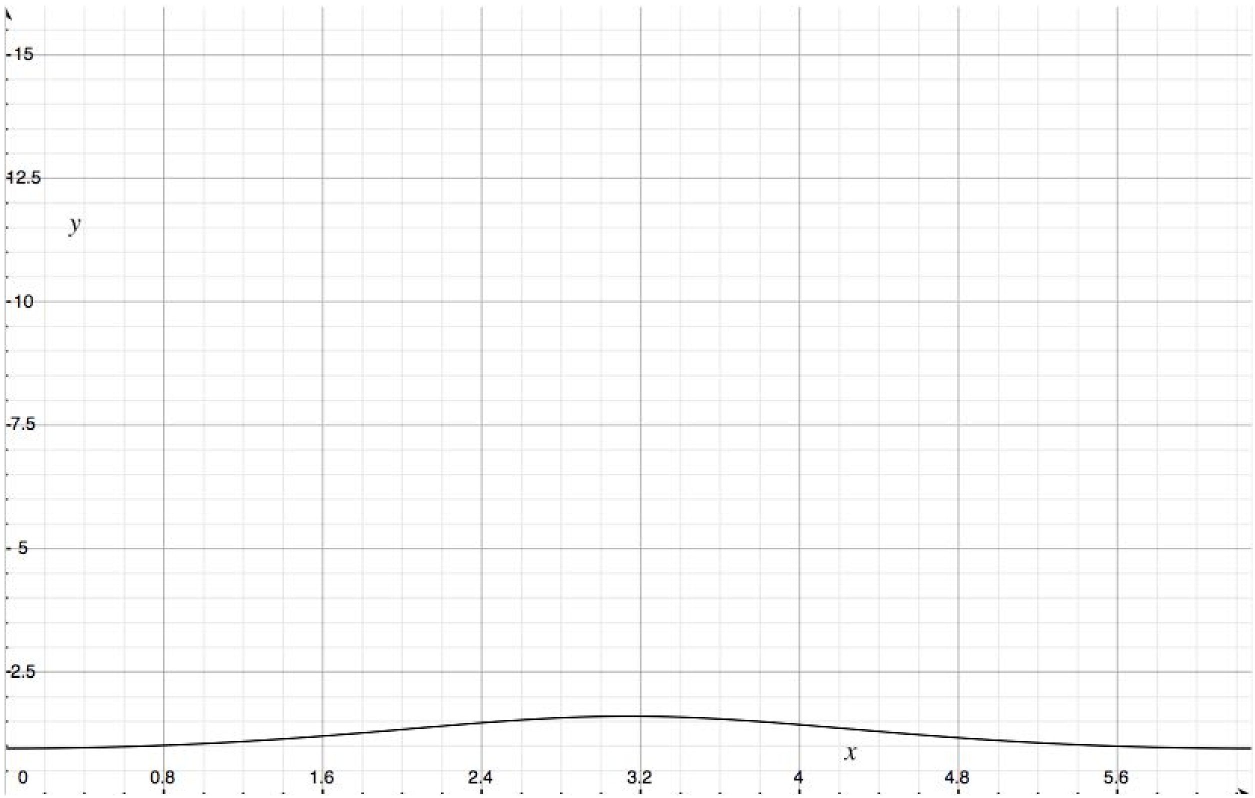
Which can be easily solved for V_1 and V_2

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{n \sum_i \cos^2(\theta_i) - (\sum_i \cos(\theta_i))^2} \begin{pmatrix} \sum_i \cos^2(\theta_i) & -\sum_i \cos(\theta_i) \\ -\sum_i \cos(\theta_i) & n \end{pmatrix} \begin{pmatrix} \sum_i R_i \\ \sum_i R_i \cos(\theta_i) \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0.83885 \\ 0.21575 \end{pmatrix}$$

And now we can solve for p and ϵ:

$$p = \frac{1}{V_1} = 1.192$$
$$\epsilon = pV_2 = 0.257$$



(d) *Redo (c) but use the relative least squares error function:*

$$E_R(V_1, V_2) = \sum_{i=1}^{n} \left(\frac{G(\theta_i) - R_i}{R_i} \right)^2$$

Again, we need to differentiate and set to 0, which gets us

$$\frac{\partial E}{\partial V_1} = 0 = V_1 \sum \frac{1}{R_i^2} + V_2 \sum \frac{\cos\theta_i}{R_i^2} - \sum \frac{1}{R_i}$$
$$\frac{\partial E}{\partial V_2} = 0 = V_1 \sum \frac{\cos\theta_i}{R_i^2} + V_2 \sum \frac{\cos^2\theta_i}{R_i^2} - \sum \frac{\cos\theta_i}{R_i}$$

And produces the matrix equation

$$\begin{pmatrix} \sum \frac{1}{R_i^2} & \sum \frac{\cos\theta_i}{R_i^2} \\ \sum \frac{\cos\theta_i}{R_i^2} & \sum \frac{\cos^2\theta_i}{R_i^2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sum \frac{1}{R_i} \\ \sum \frac{\cos\theta_i}{R_i} \end{pmatrix}$$

Which allows us to solve for V_1 and V_2.

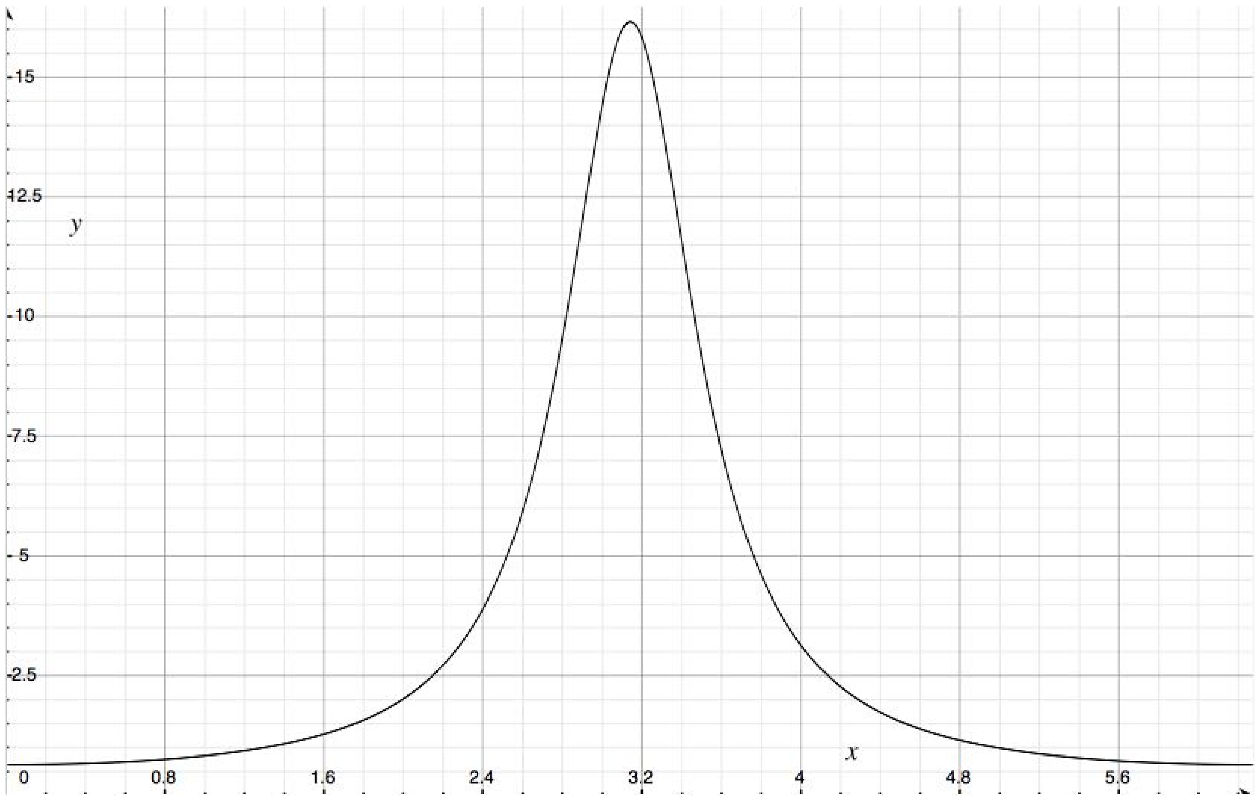
$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{\sum \frac{1}{R_i^2} \sum \frac{\cos^2\theta_i}{R_i^2} - \sum \frac{\cos\theta_i}{R_i^2} \sum \frac{\cos\theta_i}{R_i^2}} \begin{pmatrix} \sum \frac{\cos^2\theta_i}{R_i^2} & -\sum \frac{\cos\theta_i}{R_i^2} \\ -\sum \frac{\cos\theta_i}{R_i^2} & \sum \frac{1}{R_i^2} \end{pmatrix} \begin{pmatrix} \sum \frac{1}{R_i} \\ \sum \frac{\cos\theta_i}{R_i} \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ V_1 \end{pmatrix} = \frac{1}{1672.52299} \begin{pmatrix} 1344.92834 \\ 1241.63015 \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} 0.80413 \\ 0.74237 \end{pmatrix}$$

And we use these values to calculate p and ϵ:

$$p = \frac{1}{V_1} = 1.244$$
$$\epsilon = pV_2 = 0.923$$



8.12 Consider the equation:

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(a) *What is the quadratic form associated with this equation? Write it out as a polynomial.*

$$F(\mathbf{v}) = \frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v} - \mathbf{b} \cdot \mathbf{v}$$

$$\begin{aligned} F(\mathbf{v}) &= \frac{1}{2}\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \frac{1}{2}(3v_1^2 + 2v_1v_2 + 2v_2^2 - v_1 + v_2) \end{aligned}$$

(b) *In this question you are to use the SDM. Taking* $\mathbf{v_1} = (-1, 2)^T$*, calculate* $\mathbf{v_2}$*.*

We need $\mathbf{r_1}$, $\mathbf{q_1}$, and α_1 to get $\mathbf{v_2}$.

$$\mathbf{r_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\mathbf{q_1} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

$$\alpha_1 = \frac{\mathbf{r_1} \cdot \mathbf{r_1}}{\mathbf{r_1} \cdot \mathbf{q_1}} = \frac{20}{28} = \frac{5}{7}$$

With this we can get the next point:

$$\mathbf{v_2} = \mathbf{v_1} + \alpha_1 \mathbf{r_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{5}{7} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \end{pmatrix}$$

(c) *In this question you are to use the CGM. Taking* $\mathbf{v_1} = (-1, 2)^2$, *calculate* $\mathbf{v_2}$ *and* $\mathbf{v_3}$.

For CGM, $\mathbf{v_2}$ is exactly the same as for SDM, so we have

$$\mathbf{v_2} = \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \end{pmatrix}$$

Which we can use to calculate $\mathbf{v_3}$. Using the formulae for $\mathbf{r_2}, \beta_1, \mathbf{d_2}, \mathbf{q_2}$, and α_2 from the text, it's easy to obtain

$$\begin{aligned}
\mathbf{r_2} &= \frac{1}{7} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\
\beta_1 &= \frac{1}{49} \\
\mathbf{d_2} &= \frac{1}{49} \begin{pmatrix} 30 \\ 10 \end{pmatrix} \\
\mathbf{q_2} &= \frac{1}{49} \begin{pmatrix} 100 \\ 50 \end{pmatrix} \\
\alpha_2 &= \frac{20}{1/49(35000)} = \frac{98}{350} \\
\mathbf{v_3} &= \mathbf{v_2} + \alpha_2 \mathbf{d_2} \\
&= \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix}
\end{aligned}$$

8.24 Consider the following function, where A is a symmetric positive definite 2 $\times$ 2 matrix, and $\mathbf{v} = (v_1, v_2)^T$.

$$F(\mathbf{v}) = \frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v} - \mathbf{b} \cdot \mathbf{v}$$

(a) *Shown is the starting point* $\mathbf{v_1}$ *and the point* $\mathbf{v_2}$ *calculated using the CGM. Determine where the next point* $\mathbf{v_3}$ *is located.*

Since we're using CGM here, by theorem 8.2 in the text, we must come to a solution in $m \leq n+1$ steps. n in this case is 2, so it can take a maximum of 3 steps to reach the minimum. Since we're on v_2 and only have one step left, we are left with the fact that $\mathbf{v_3}$ will in fact be at the minimum.

(b) *If one uses the same starting point for SDM, where is* $\mathbf{v_2}$

Using SDM, $\mathbf{v_2}$ will be in the same spot as it is for the case of CGM, since in for $\mathbf{v_2}$ and *only* $\mathbf{v_2}$, the formula for its calculation is the same in both CGM and SDM.

(c) *What starting point* $\mathbf{v_1}$ *should be used so the point* $\mathbf{v_2}$ *computed by the SDM is exactly at the minimum?*

To get it in one go, $\mathbf{v_1}$ must be on a contour which is orthognal to the direction of the minimum. Thus, it could be at approximately (4,2), or around (2.5, -4). Any of these type of points would work.

8.33 Consider $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \mathbf{Ax}$, $\mathbf{x} = (1, 1, \ldots, 1)^T$, and A is the tridiagonal symmetric positive definite matrix as defined.

(a) *Use MATLAB to fill out the table. x_c is the solution computed using CGM. The computing time is how long it takes to execute the CGM algorithm.*

n	$\frac{\lvert\lvert x-x_c \rvert\rvert}{\lvert\lvert x \rvert\rvert_\infty}$	$\kappa_\infty(A)$	Number Iterations	Compute Time
100	1.3323×10^{-15}	5000	52	0.009302s
500	5.3291×10^{-15}	1.25×10^5	254	0.034839s
1000	2.0761×10^{-14}	5×10^5	505	0.221704s
2000	1.5321×10^{-14}	2×10^6	1009	1.114833s

(b) *Based on your results, can you predict the number iterations when $n = 4000$? Computing time?*

The number of iterations should be about $4000/2 = 2000$, and based on the trends perhaps about 2014.

The computing time should be on the order of about 10 seconds.

(c) *Using MATLAB command ones, determine the smallest integer value of k where MATLAB states there is an "Error using ones" and that the "array exceeds maximum array size preference." With this, run the CGM code with $n = (k-1) \cdot 1000$ and report ont he value of n you used, the number of iterations, and the computing time. Then, do this with $sparse(A)$, and report again.*

The smallest k in my case is 32, so I'll run CGM with $n = 32000$.

Unfortunately, neither of the representations of A led to anything close to a quick calculation, and I terminated the processes after a few minutes.