A Quantitative Understanding of Exceptions

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Programming Languages

λ-calculus (Pure)

Real (Impure)

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- Simple structure

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- Interact with the real world

Programming Languages

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- Useless for programmers(?)

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How can we add *effects* to pure languages?

[W]e distinguish the object A of values (of type A) from the object TA of computations (of type A).

Eugenio Moggi

in "Notions of Computation and Monads"

Exceptions




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Moggi's CBV Encoding

Effect Operations

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Let E be the type of exceptions.

Then $TA = (\mathcal{E} \oplus A)$:

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& & \qquad\qquad t
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Effect Operations

Let e be an exception name:

- Raising an exception:

$$\mathtt{raise}_e()$$

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Effect Operations

Let e be an exception name:

- Raising an exception:

$$\mathtt{raise}_e()$$

- Handling an exception:

$$\mathtt{handle}_e(t, u)$$

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evaluation length + size of result

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$$\begin{array}{lcl} t_0 & := & y \\ t_n & := & (\lambda x.xx)t_{n-1} \end{array} \quad \rightsquigarrow \quad t_n \to_\beta^n y^{2^n}$$

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Split and Exact Measures

- To obtain split measures

- To obtain exact measures

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counters in judgments + tight constants + persistent typing rules

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$$\underbrace{\text{counters in judgments} + \text{tight constants} + \text{persistent typing rules}}$$

(evaluation length, size of result)

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tight derivations = minimal derivations

- Obtain models capturing exact quantitative computational properties

"t is terminating in exactly X steps with normal form of size Y
iff t is typable with counter (X, Y)"

A Quantitative Understanding of Exceptions

Goal

To build a quantitative model (expressed as a tight type system) that captures exact quantitative properties of a λ-calculus with operations that raise and handle exceptions.

Syntax

$$|\mathtt{raise}_e()| = 1 \quad |x| = 0 \quad |\lambda x.t| = 0 \quad |vt| = 1 + |t| \quad |\mathtt{handle}_e(t,u)| = 1 + |t|$$

Syntax

- We distinguish between and

Values	$::=$
Terms	$::=$

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Syntax

- We distinguish between values v and

Values	v, w	$::=$	$x \mid \lambda x.t$
Terms		$::=$	

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Syntax

- We distinguish between values v and computations t (terms)

$$\begin{array}{llll} \text{Values} & v, w & ::= & x \mid \lambda x.t \\ \text{Terms} & t, u & ::= & v \mid \ldots \end{array}$$

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$$\begin{array}{llll} \text{Values} & v, w & ::= & x \mid \lambda x.t \\ \text{Terms} & t, u & ::= & v \mid vt \mid \dots \end{array}$$

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Syntax

- We distinguish between values v and computations t (terms)
- Applications are restricted to the form vt
- Effect operations are used to raise and handle exceptions

$$\begin{array}{llll} \text{Values} & v, w & ::= & x \mid \lambda x.t \\ \text{Terms} & t, u & ::= & v \mid vt \mid \mathtt{raise}_e() \mid \mathtt{handle}_e(t, u) \end{array}$$

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Weak reduction: we do not reduce inside abstractions

We allow *open* normal forms

Operational Semantics ⚜ Example

$$(\lambda x.\mathtt{handle}_e(xy, x))\ (\lambda z.\mathtt{raise}_e())$$

(0 # b-steps, 0 # exceptions handled, 0# exceptions propagated)

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Encoding Arrow Types

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- $T(!A)$ is a computation wrapping an intersection of value types

$$T[A, \dots, A] = \mathcal{E} \oplus [A, \dots, A]$$

Types

Types

- Values and Neutral Forms

- Computations

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- Values and Neutral Forms

Tight Constants	::=
Value Types	::=
Multi-types	::=
Liftable Types	::=
Types	::=

- Computations

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Tight Constants	$\mathtt{tt}$	$::=$	$\mathtt{v} \mid \mathtt{a} \mid \mathtt{n}$
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Monadic Types	$::=$

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- Computations

Monadic Types	δ	$::=$	$\gamma \oplus \tau$ where $\gamma \in \mathcal{E} \cup \{\star\}$

Type System

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- Judgments are decorated with counters

$$(b \quad , \quad h \quad , \quad p \quad , \quad s)$$

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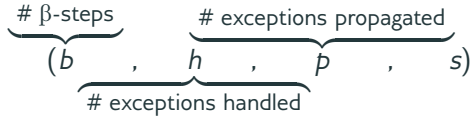

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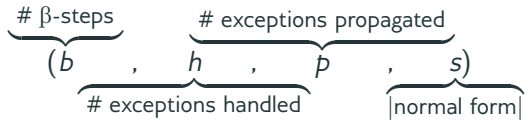

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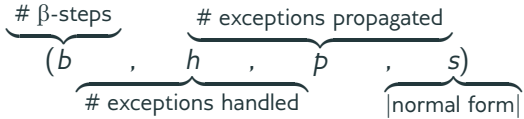

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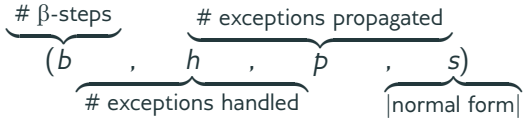

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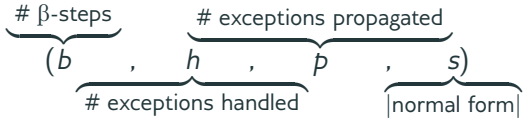
  - *Consuming*: increase only *b*, *h*, and *p* counters

Type System

- Judgments are decorated with counters



- Some typing rules have two (or more) different versions
  - *Consuming*: increase only *b*, *h*, and *p* counters
  - *Persistent*: increase the *s* counter

(Some) Typing Rules

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$$\frac{(\Gamma_i \vdash^{(b_i,h_i,p_i,s_i)} v : \sigma_i)_{i\in I}}{+_{i\in I}\,\Gamma_i \vdash^{(+_{i\in I}b_i,+_{i\in I}h_i,+_{i\in I}p_i,+_{i\in I}s_i)} v : [\sigma_i]_{i\in I}}\ (\texttt{many})$$

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(Some) Typing Rules ⚜ Persistent

$$\frac{}{\vdash^{(0,0,0,0)} \lambda x.t : \mathtt{a}}\ (\mathrm{abs}_p)$$

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Completeness

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If $t \twoheadrightarrow^{(b,h,p)} u$,
then $\exists \Phi \triangleright \Gamma \vdash^{(b,h,p,|u|)} t : \delta$ tight.

Typing Example

Let us consider the term exemplifying the operational semantics:

$$(\lambda x.\mathtt{handle}_e(xy, x))\ (\lambda z.\mathtt{raise}_e())\ \twoheadrightarrow^{(2,1,0)}\ \underbrace{\mid \lambda z.\mathtt{raise}_e() \mid = 0}_{\lambda z.\mathtt{raise}_e()}$$

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- We can build the following tight derivation:

$$\frac{\Phi \qquad \Psi}{y : [] \vdash^{(2,1,0,0)} (\lambda x.\mathtt{handle}_e(xy, x))(\lambda z.\mathtt{raise}_{e'}()) : \star \oplus \mathtt{a}}\ (\mathrm{app})$$

Conclusions

Summary

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- Following a weak (open) CBV strategy
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Future Work

- Different effects: global memory, I/O, non-determinism, ...
- Different Strategies: CBV (unrestricted), CBN, CBNeed, ...
- Unifying frameworks: CBPV, λ!-calculus, EE-calculus, ...


The End