LINEAR SYSTEMS

PREAMBLE (CODE)

```
from numpy import *
from numpy.linalg import *
from matplotlib.pyplot import *
from mpl_toolkits.mplot3d import *
from scipy.integrate import solve_ivp
```

PREAMBLE

INPUTS

It’s handy to introduce non-autonomous ODEs.

There are designated as

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, that is

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n.$$

The vector-valued u is the **system input**.

This quantity may depend on the time t

$$u : t \in \mathbb{R} \mapsto u(t) \in \mathbb{R}^m,$$

(actually it may also depend on some state, but we will adress this later).

A solution of

$$\dot{x} = f(x, u) \text{ and } x(t_0) = x_0$$

is merely a solution of

$$\dot{x} = h(t, x) \text{ and } x(t_0) = x_0,$$

where

$$h(t, x) = f(x, u(t)).$$

OUTPUTS

We may complement the system dynamics with an equation

$$y = g(x, u) \in \mathbb{R}^p$$

The vector y refers to the **systems output**, usually the quantities that we can effectively measure in a system (the state x itself may be unknown).

WHAT ARE LINEAR SYSTEMS?

STANDARD FORM

Input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

WHY LINEAR ?

Assume that:

- $\dot{x}_1 = Ax_1 + Bu_1, x_1(0) = x_{10},$
- $\dot{x}_2 = Ax_2 + Bu_2, x_2(0) = x_{20},$

Set

- $x_3 = \lambda x_1 + \mu x_2$,
- $u_3 = \lambda u_1 + \mu u_2$ and
- $x_{30} = \lambda x_{10} + \mu x_{20}$.

for some λ and μ.

Then

$$\dot{x}_3 = Ax_3 + Bu_3, \;\; x_3(0) = x_{30}.$$

INTERNAL + EXTERNAL DYNAMICS

Corollary: Since $(x_0, u) = (x_0, 0) + (0, u)$ the solution of

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$

is the sum of the solutions x_1 and x_2 of:

the **internal dynamics**

$$\dot{x}_1 = Ax_1,\ x_1(0) = x_0$$

(behavior controlled by the initial value only, no input)

and the **external dynamics**:

$$\dot{x}_2 = Ax_2 + Bu,\ x_2(0) = 0$$

(behavior controlled by the input, the systems is initially at rest)

MATRIX SIZE

$$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}.$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

LTI SYSTEMS

They are actually referred to as **linear time-invariant (LTI)** systems:

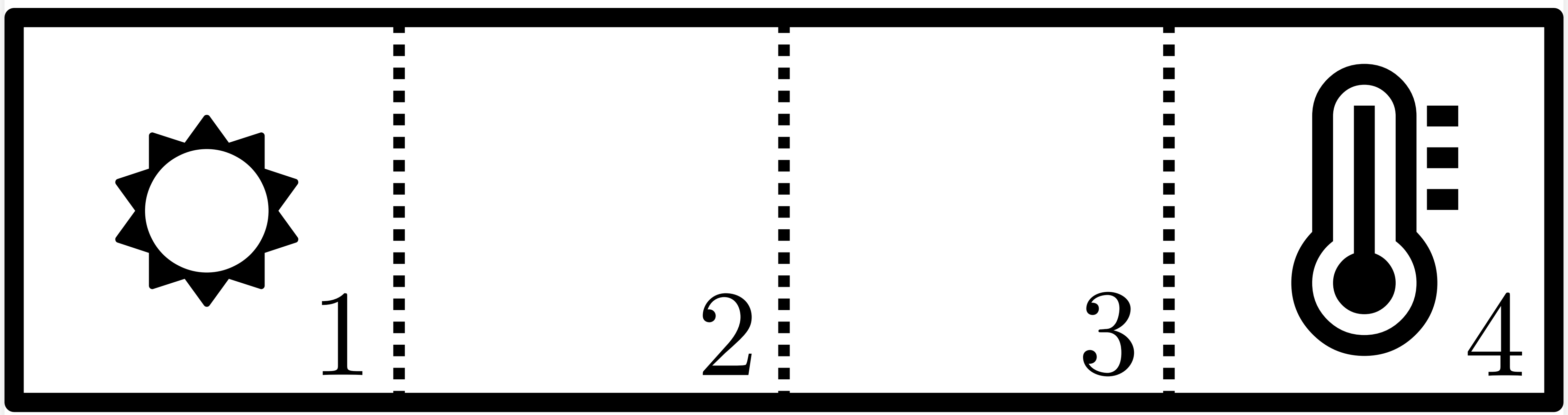
When $x(t)$ is a solution of

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0,$$

then $x(t - t_0)$ is a solution of

$$\dot{x} = Ax + Bu(t - t_0), \;\; x(t_0) = x_0.$$

👁 – LINEAR SYSTEM / HEAT EQUATION



SIMPLIFIED MODEL

- Four cells numbered 1 to 4 are arranged in a row.
- The first cell has a heat source, the last one a temperature sensor.
- The heat sink/source is increasing the temperature of its cell of u degrees by second.
- If the temperature of a cell is T and the one of a neighbor is T_n, T increases of $T_n - T$ by second.

Given the geometric layout:

- $dT_1/dt = u + (T_2 - T_1)$
- $dT_2/dt = (T_1 - T_2) + (T_3 - T_2)$
- $dT_3/dt = (T_2 - T_3) + (T_4 - T_3)$
- $dT_4/dt = (T_3 - T_4)$
- $y = T_4$

Set $x = (T_1, T_2, T_3, T_4)$.

The model is linear and its standard matrices are:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \; D = [0]$$

NONLINEAR TO LINEAR

Consider the nonlinear system

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= g(x, u)\end{aligned}$$

Assume that x_e is an equilibrium when $u = u_e$ (cst):

$$f(x_e, u_e) = 0$$

and let

$$y_e = g(x_e, u_e)$$

Define the error variables

- $\Delta x = x - x_e$,
- $\Delta u = u - u_e$ and
- $\Delta y = y - y_e$.

As long as the error variables stay small

$$f(x,u) \simeq \overbrace{f(x_e,u_e)}^{0} + \frac{\partial f}{\partial x}(x_e,u_e)\Delta x + \frac{\partial f}{\partial u}(x_e,u_e)\Delta u$$

$$g(x,u) \simeq \overbrace{g(x_e,u_e)}^{y_e} + \frac{\partial g}{\partial x}(x_e,u_e)\Delta x + \frac{\partial g}{\partial u}(x_e,u_e)\Delta u$$

Hence, the error variables satisfy *approximately*

$$\begin{aligned} d(\Delta x)/dt &= A\Delta x + B\Delta u \\ \Delta y &= C\Delta x + D\Delta u \end{aligned}$$

with

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} \\ \hline \frac{\partial g}{\partial x} & \frac{\partial g}{\partial u} \end{array}\right](x_e, u_e)$$

ASYMPTOTIC STABILITY

The equilibrium x_e is locally asymptotically stable for

$$\dot{x} = f(x, u_e)$$

$$\iff$$

The equilibrium 0 is locally asymptotically stable for

$$\frac{d\Delta x}{dt} = A\Delta x$$

where $A = \partial f(x_e, u_e)/\partial x$.

👁 – LINEARIZATION

Consider

$$\dot{x} = -x^2 + u, \;\; y = xu$$

If we set $u_e = 1$, the system has an equilibrium at $x_e = 1$ (and also $x_e = -1$ but we focus on the former) and the corresponding y is $y_e = x_e u_e = 1$.

Around this configuration $(x_e, u_e) = (1, 1)$, we have

$$\frac{\partial(-x^2+u)}{\partial x} = -2x_e = -2, \quad \frac{\partial(-x^2+u)}{\partial u} = 1,$$

and

$$\frac{\partial xu}{\partial x} = u_e = 1, \quad \frac{\partial xu}{\partial u} = x_e = 1.$$

Thus, the approximate, linearized dynamics around this equilibrium is

$$
\begin{aligned}
d(x-1)/dt &= -2(x-1) + (u-1) \\
y - 1 &= (x-1) + (u-1)
\end{aligned}
$$

? – LINEARIZED DYNAMICS / PENDULUM

A pendulum submitted to a torque c is governed by

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin \theta = c.$$

We assume that only the angle θ is effectively measured.

- What are the function f and g that determine the nonlinear dynamics of the pendulum when $x = (\theta, \dot{\theta}), u = c$ and $y = \theta$?
- Show that for any angle θ_e we can find a constant value c_e of the torque such that $x_e = (\theta_e, 0)$ is an equilibrium.
- Compute the linearized dynamics of the pendulum around this equilibrium and put it in the standard form (compute A, B, C and D).

INTERNAL DYNAMICS

We study the behavior of the solution

$$\dot{x} = Ax, \;\; x(0) = x_0 \in \mathbb{R}^n$$

We try to get some understanding with the simplest cases first.

SCALAR CASE, REAL-VALUED

$$\dot{x} = ax$$

$$a \in \mathbb{R}, x(0) = x_0 \in \mathbb{R}.$$

Solution:

$$x(t) = e^{at}x_0$$

Proof:

$$\frac{d}{dt}e^{at}x_0 = ae^{at}x_0 = ax(t)$$

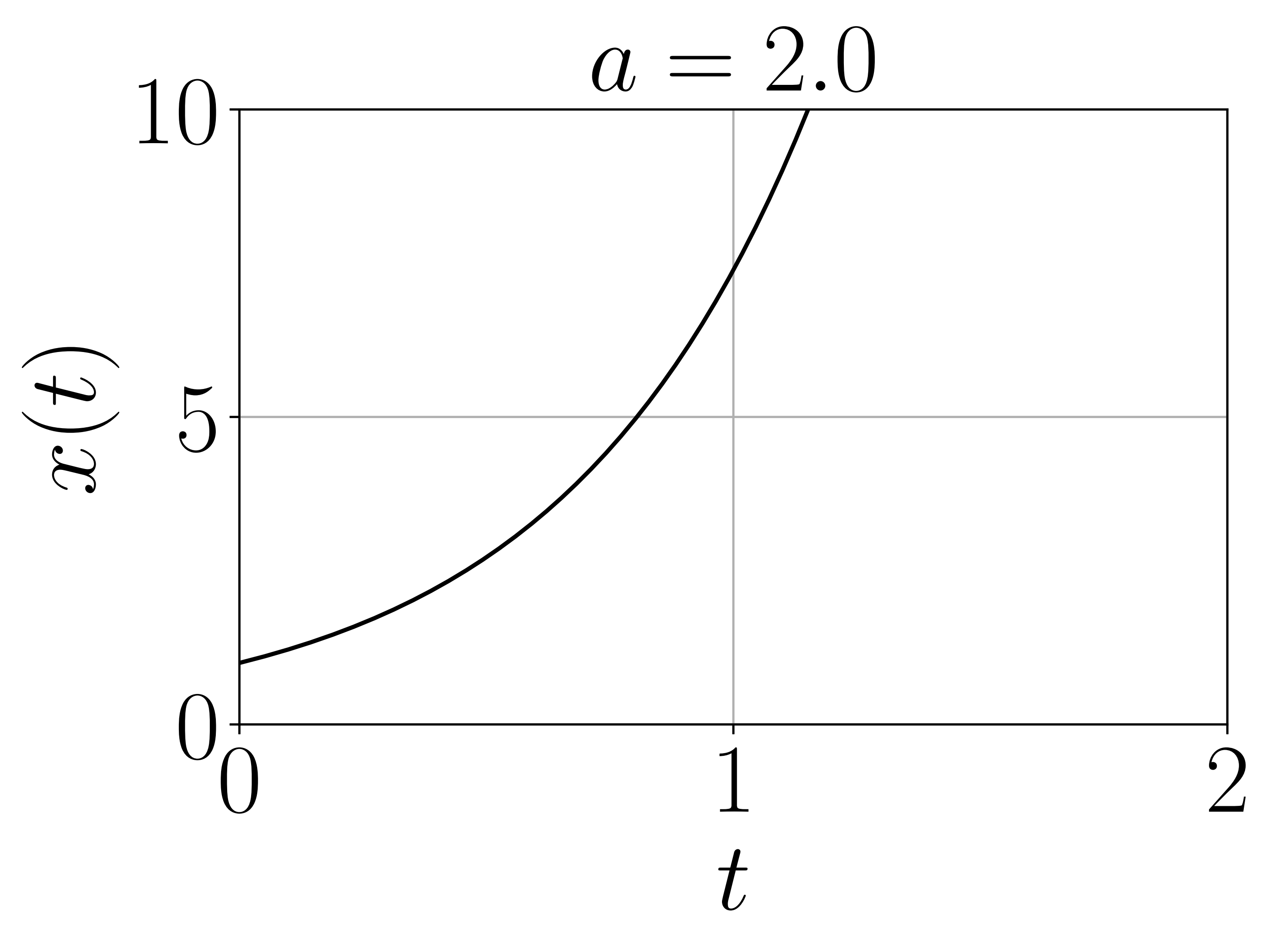
and

$$x(0) = e^{a\times 0}x_0 = x_0.$$

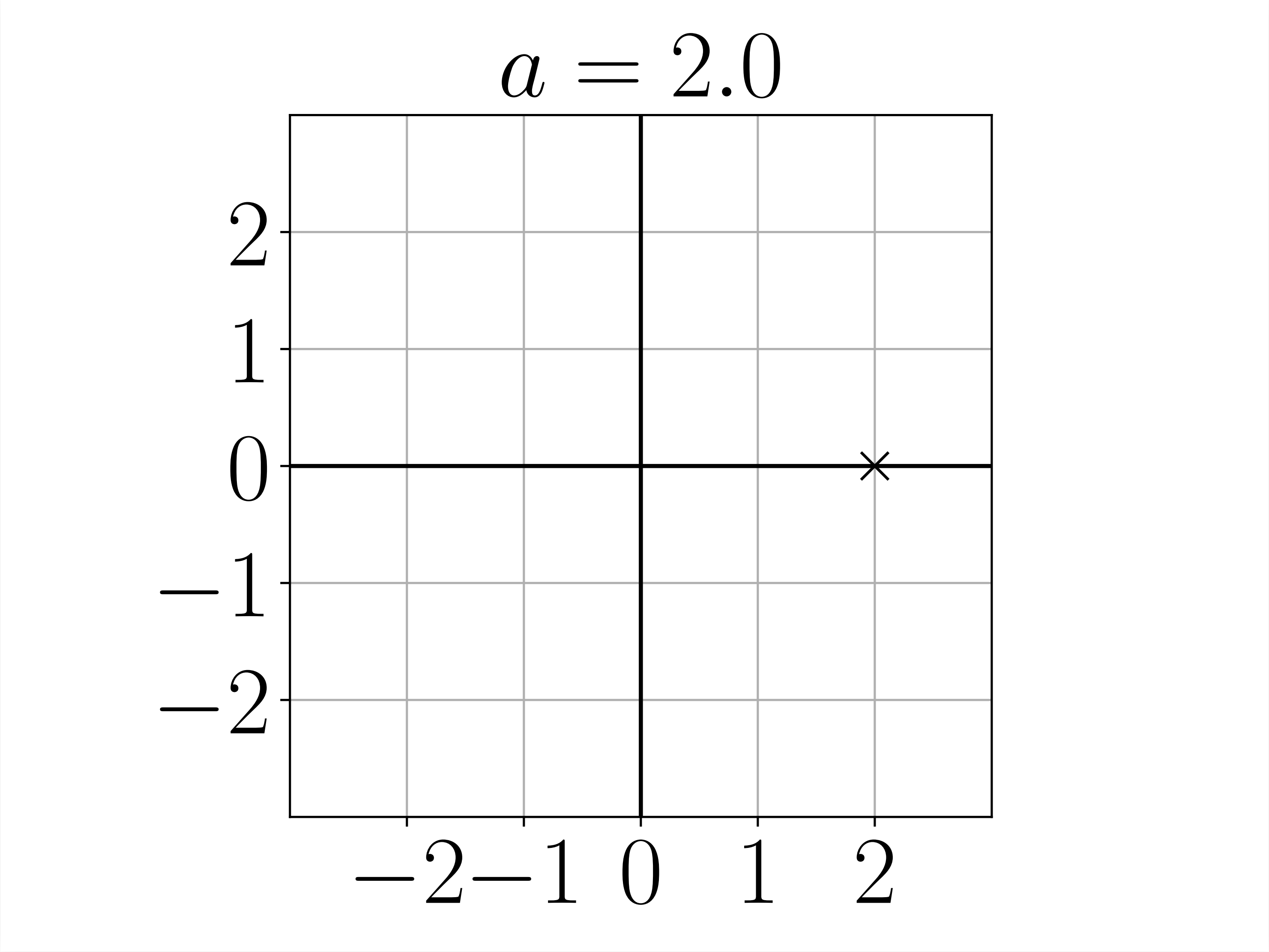
TRAJECTORY

```
a = 2.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

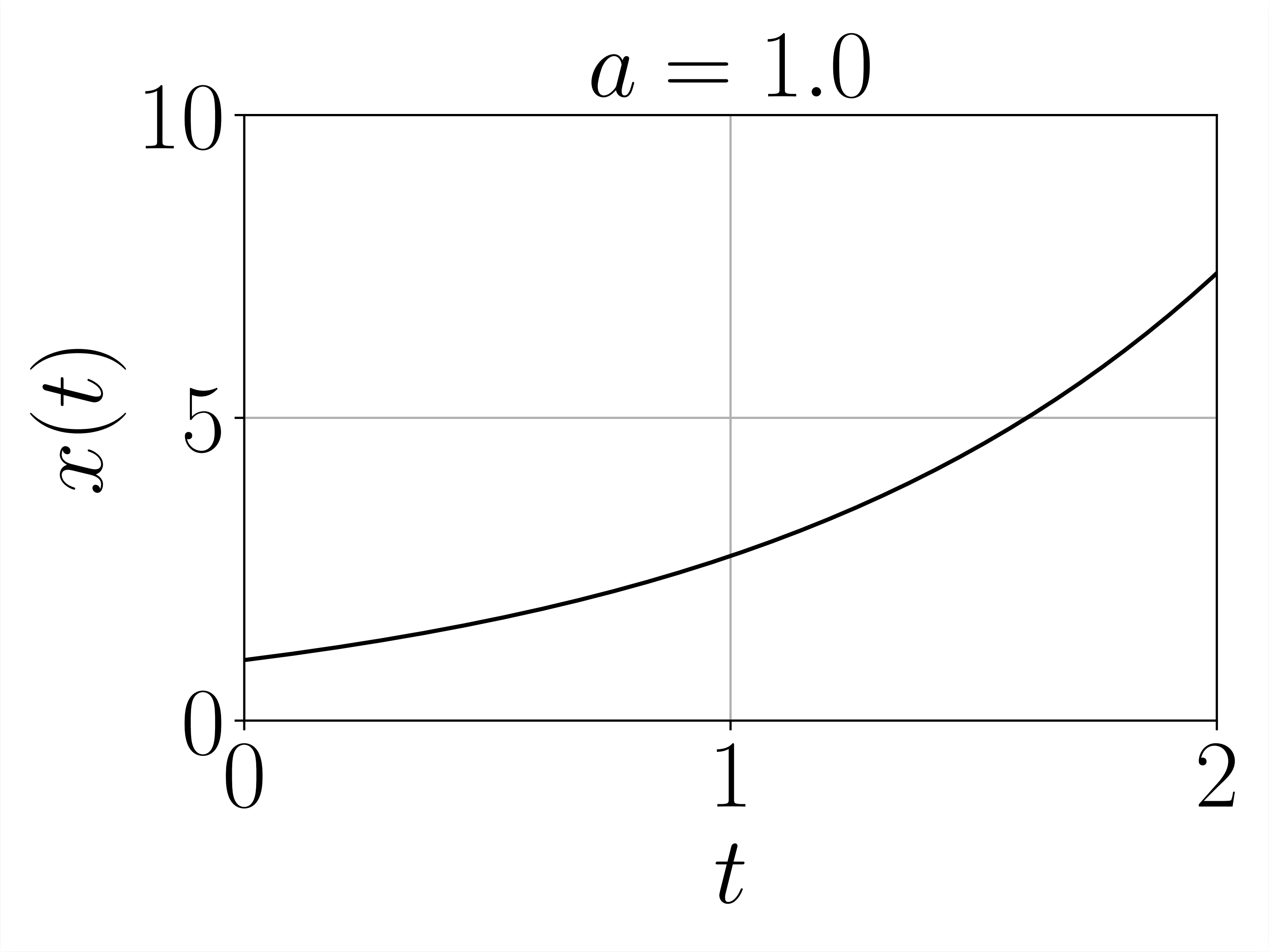
TRAJECTORY



```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```

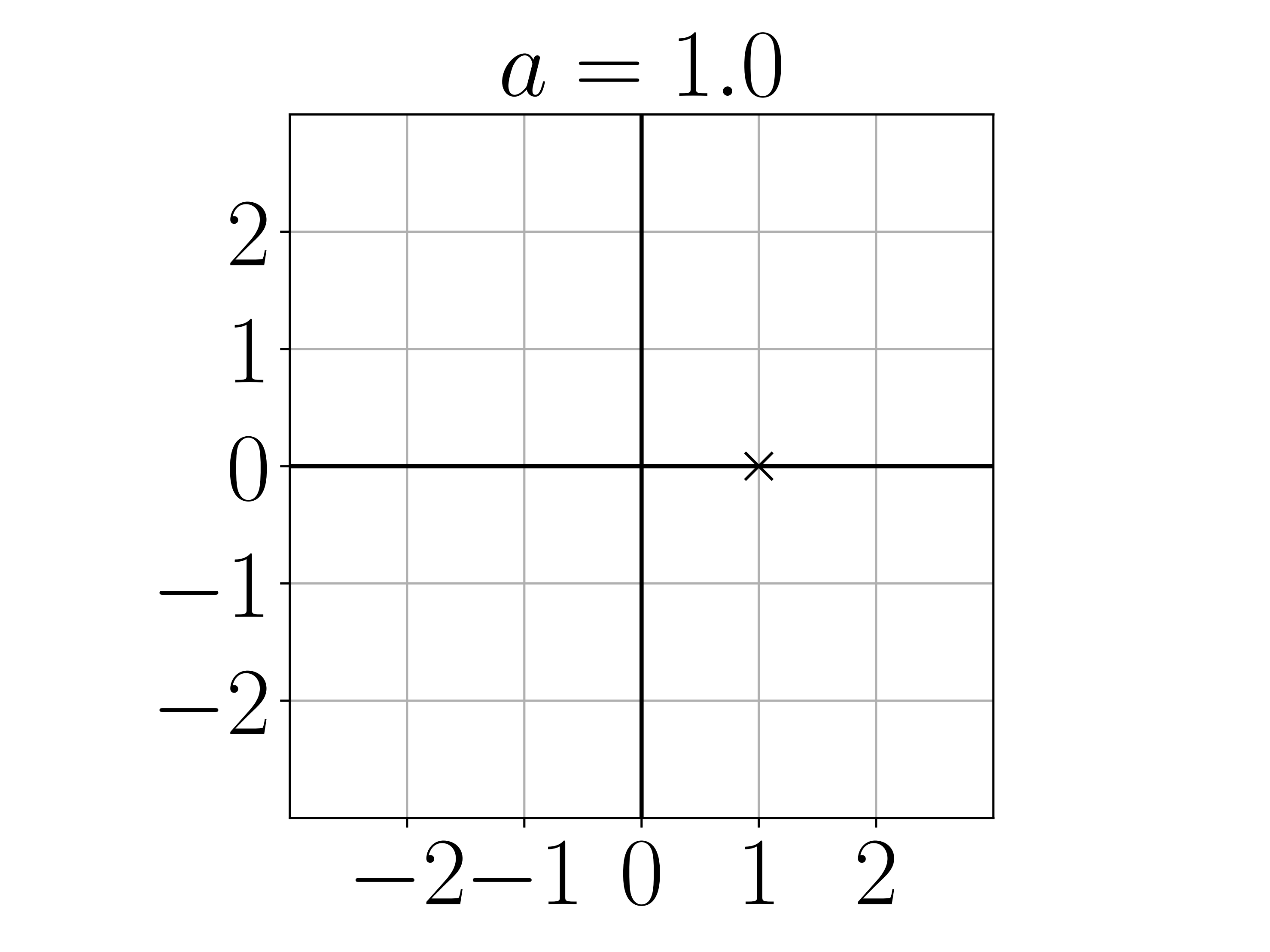

$a = 2.0$
2
1
0
−1
−2
−2
−1
0
1
2

```
a = 1.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

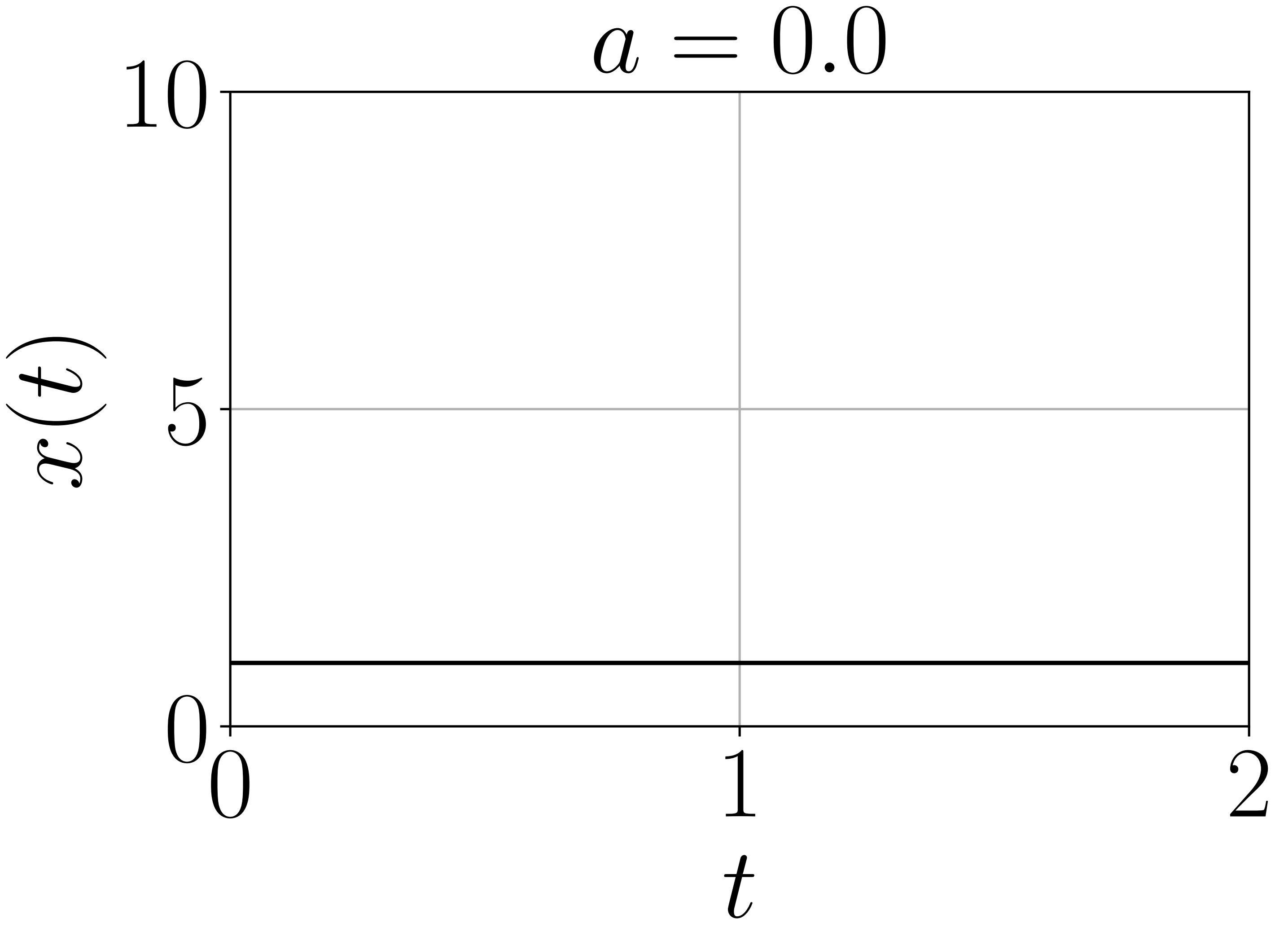


$a = 1.0$
10
5
0
0
1
2
t
$x(t)$

```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```

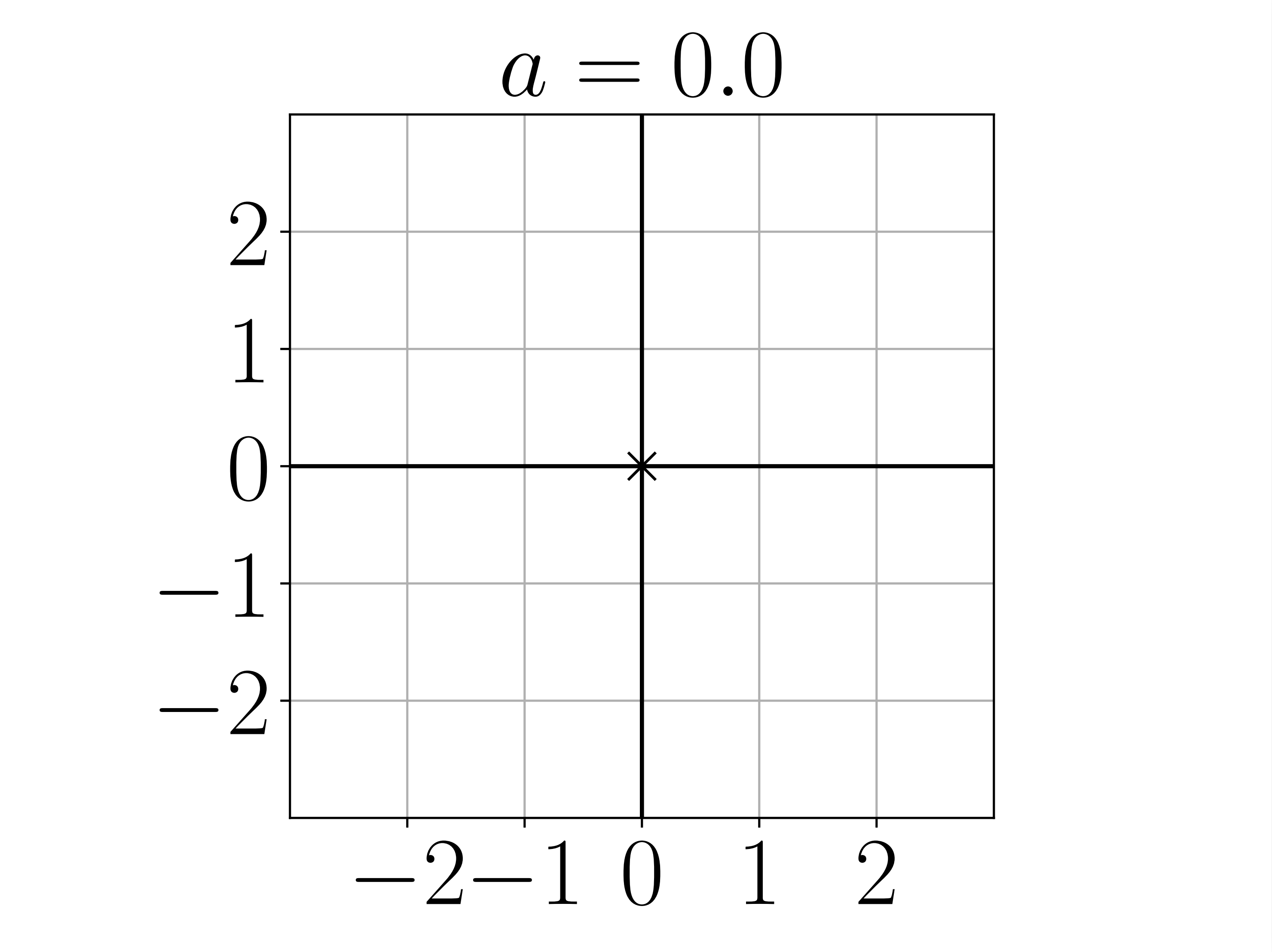

$a = 1.0$
2
1
0
−1
−2
−2
−1
0
1
2

```python
a = 0.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

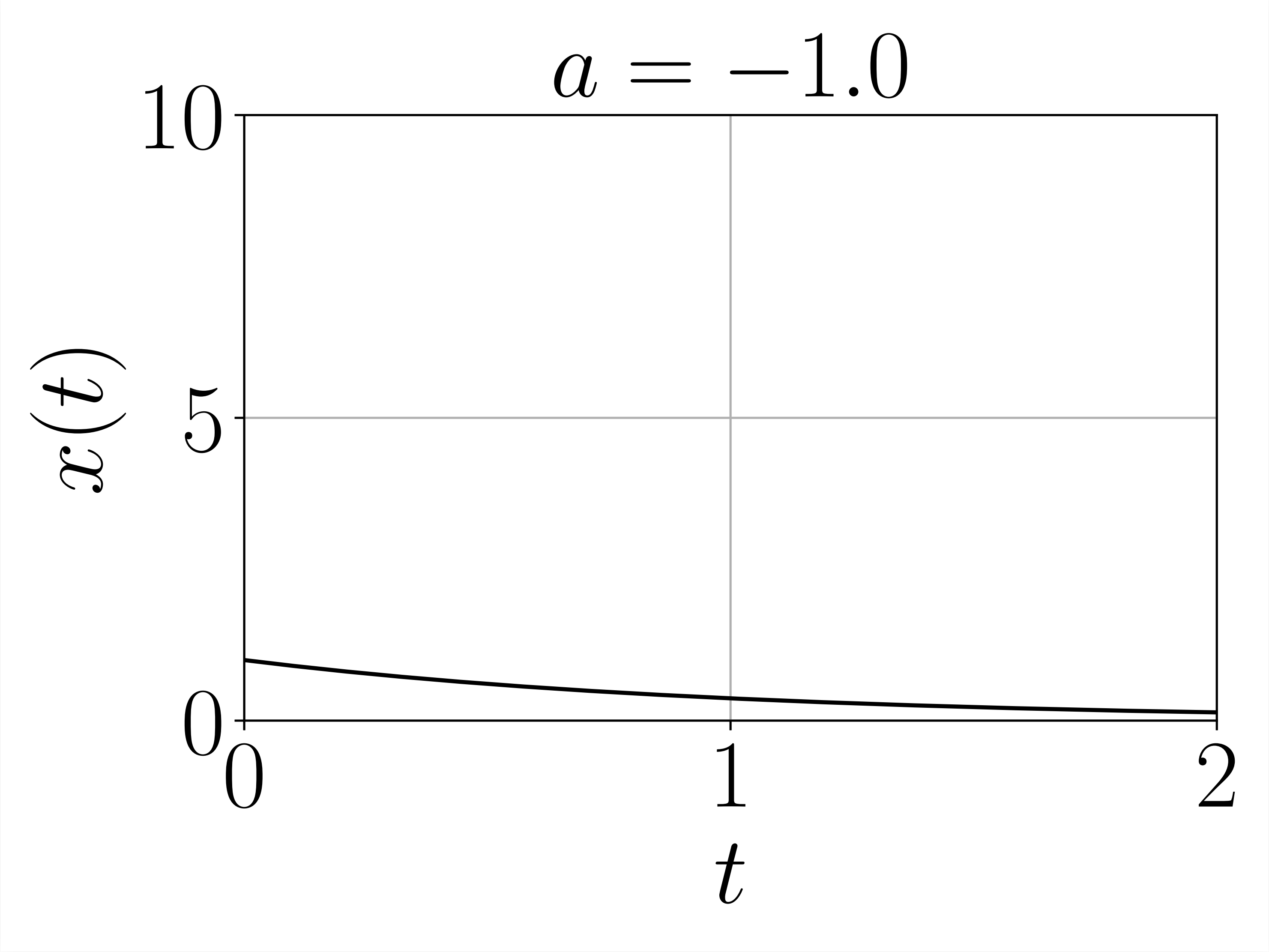


$a = 0.0$
10
5
0
$x(t)$
0
1
2
t

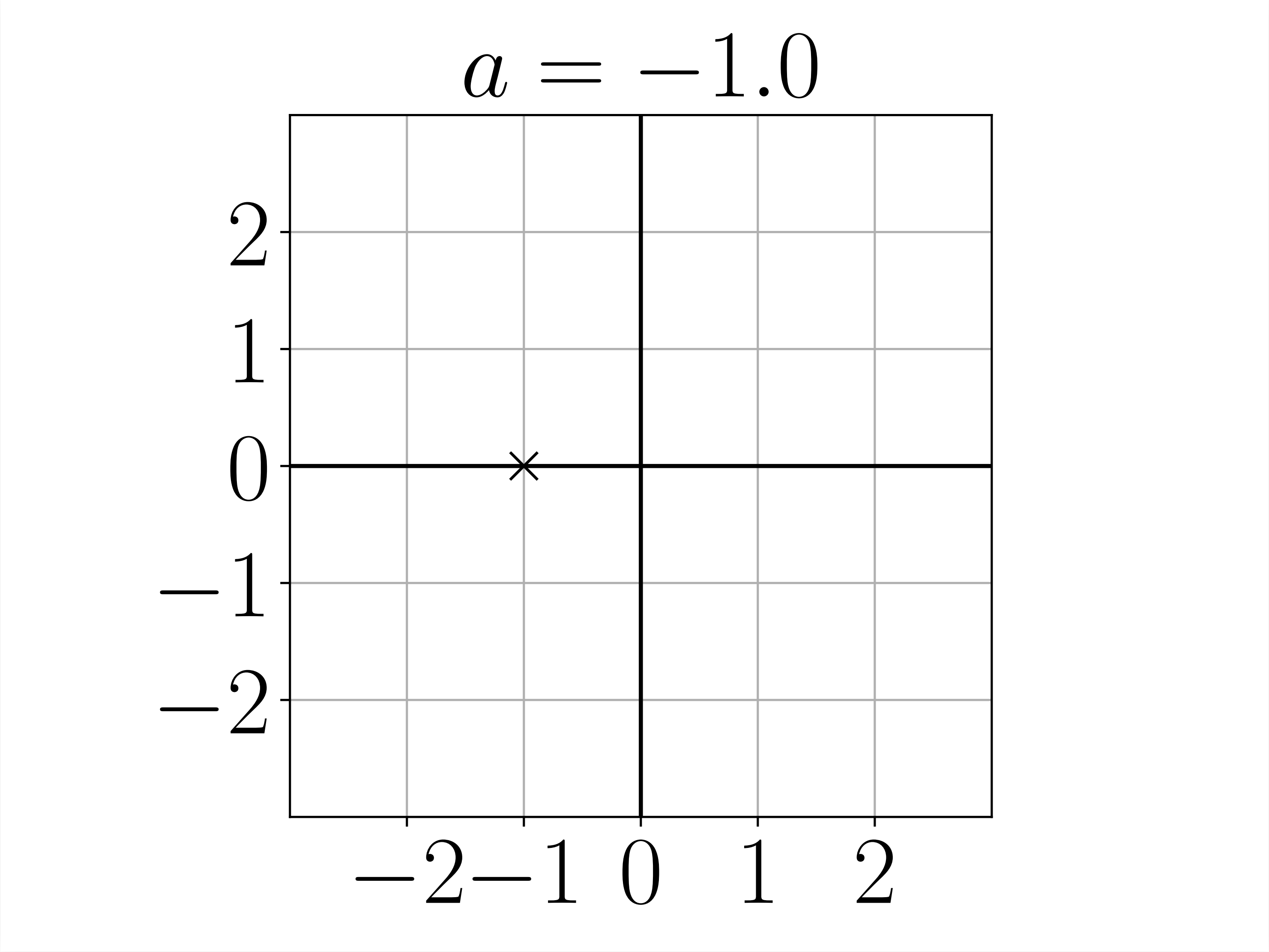
```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```


$a = 0.0$
2
1
0
−1
−2
−2
−1
0
1
2

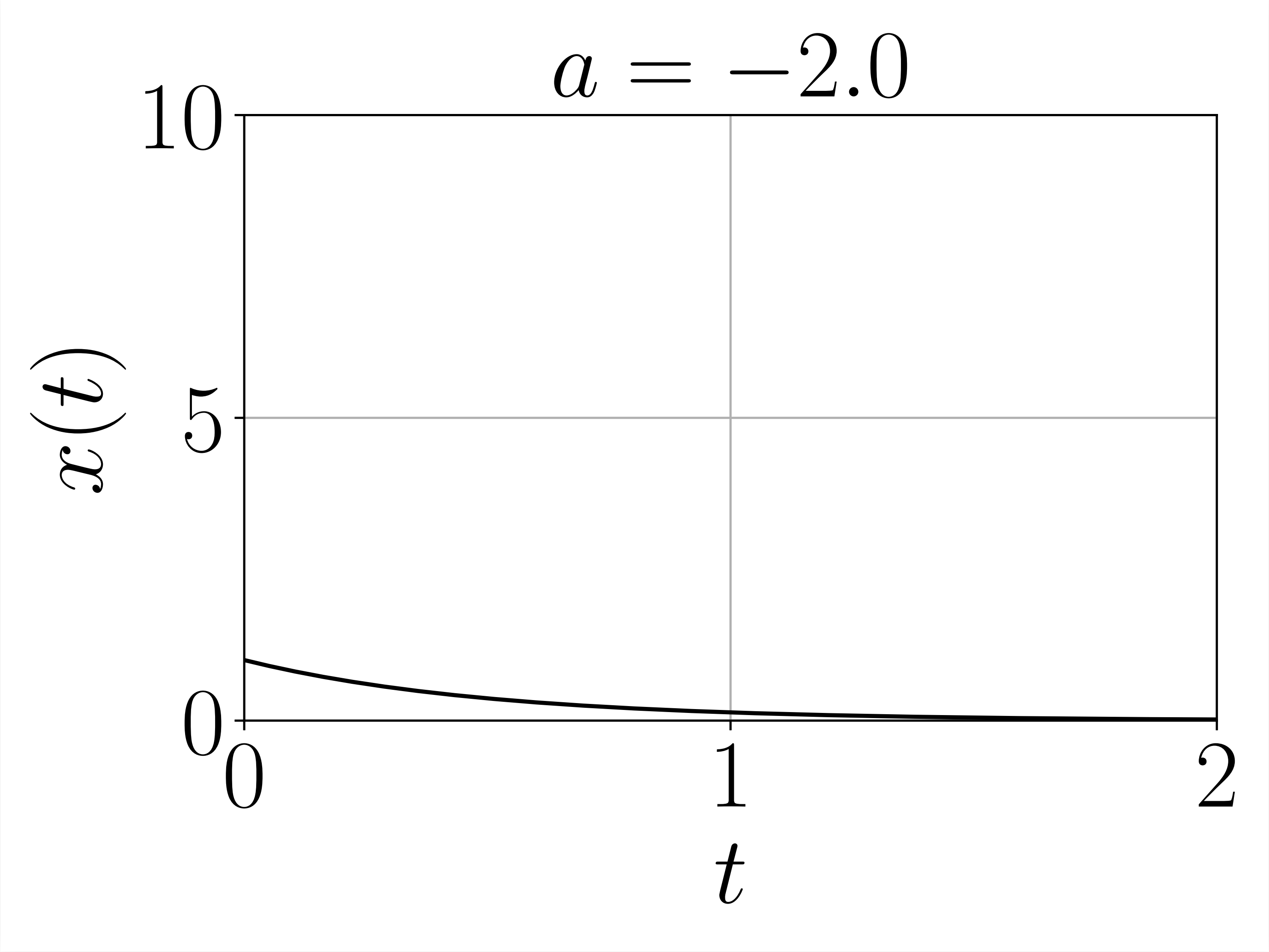
```python
a = -1.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```


$a = -1.0$
10
5
0
$x(t)$
0
1
2
t

```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```

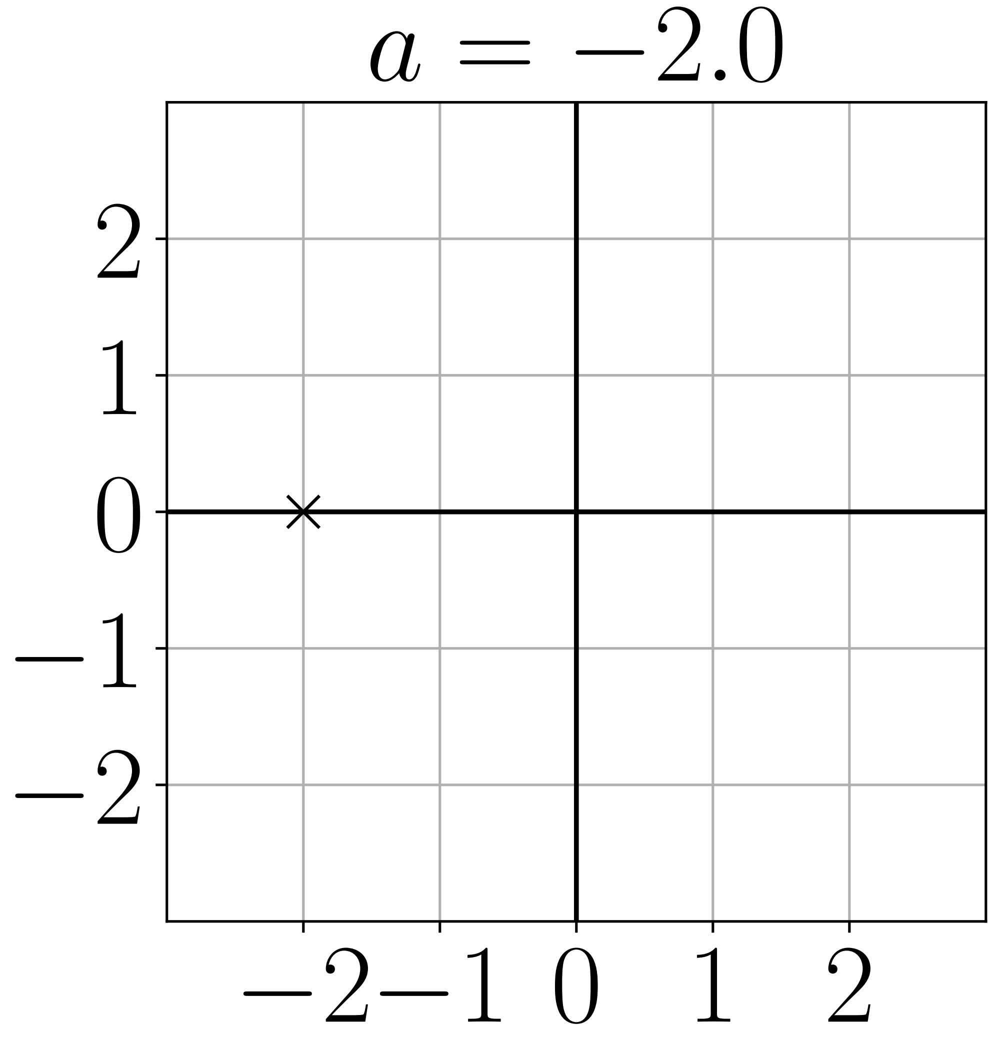

$a = -1.0$
2
1
0
−1
−2
−2
−1
0
1
2

```python
a = -2.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```



$a = -2.0$
10
5
0
$x(t)$
0
1
2
t

```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```


$a = -2.0$
2
1
0
−1
−2
−2
−1
0
1
2

ANALYSIS

- The origin is globally asymptotically stable when $a < 0.0$:
  a is in the open left-hand plane,
- In this case, define the time constant $\tau = -1/a$:

$$x(t) = e^{at} x_0 = e^{-t/\tau} x_0$$

τ controls the time it take for the solution to (almost) reach to the origin:

- when $t = \tau$, $|x(t)|$ is $\simeq 33\%$ of $|x_0|$;
- when $t = 3\tau$, $|x(t)|$ is $\simeq 5\%$ of $|x_0|$.

VECTOR CASE, DIAGONAL, REAL-VALUED

$$\dot{x}_1 = a_1 x_1, \ x_1(0) = x_{10}$$
$$\dot{x}_2 = a_2 x_2, \ x_2(0) = x_{20}$$

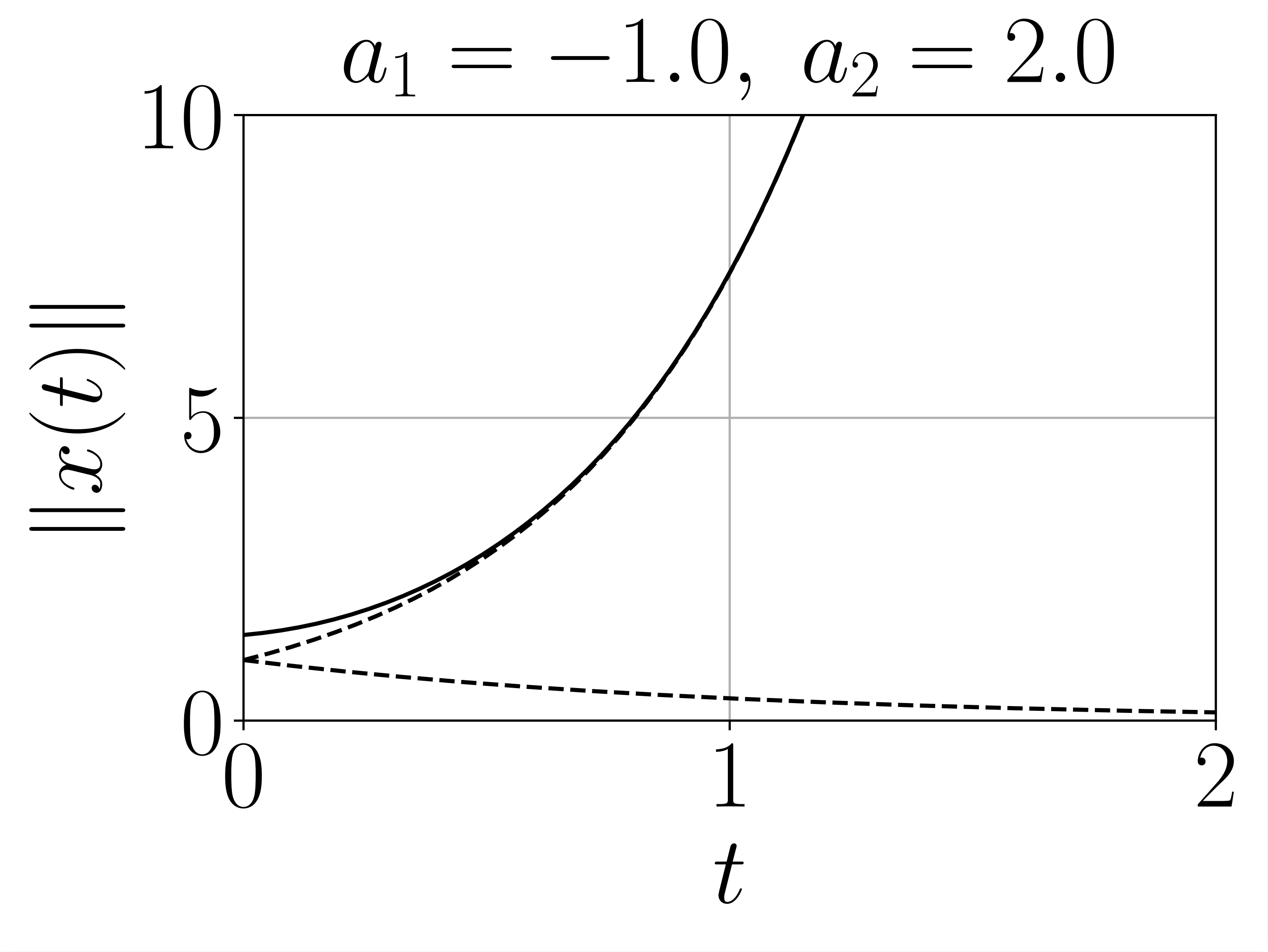
i.e.

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

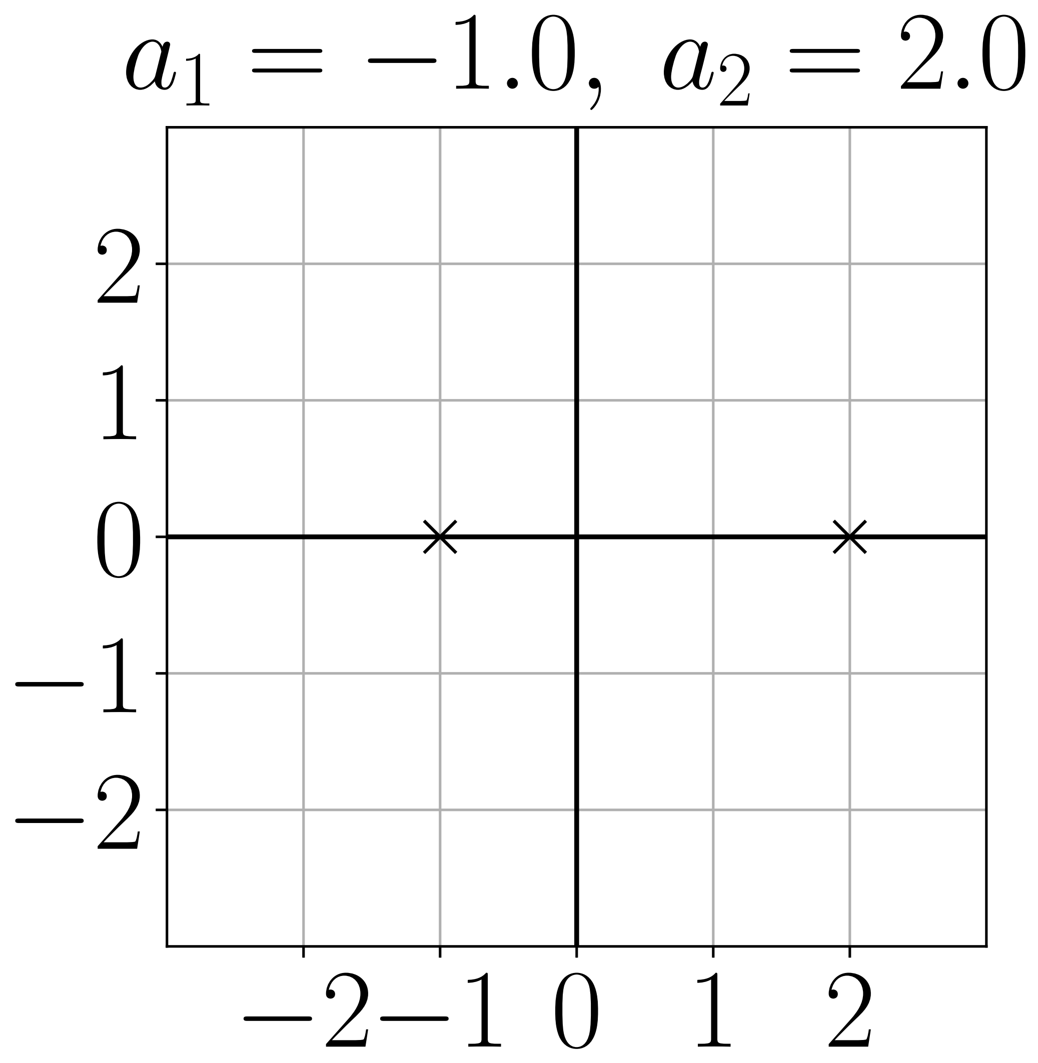
Solution: by linearity

$$x(t) = e^{a_1 t} \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} + e^{a_2 t} \begin{bmatrix} 0 \\ x_{20} \end{bmatrix}$$

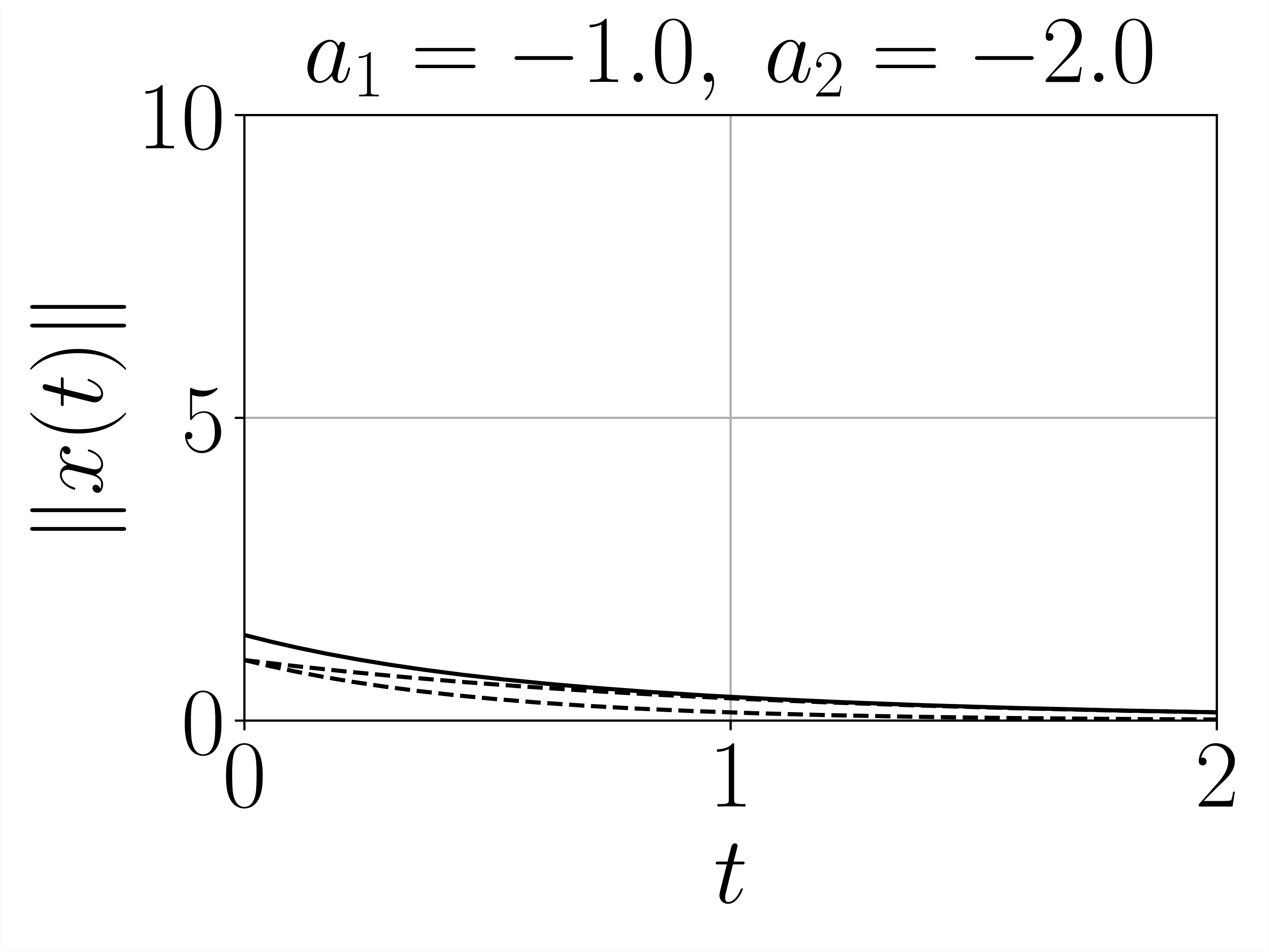
```
a1 = -1.0; a2 = 2.0; x10 = x20 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
x1 = exp(a1*t)*x10; x2 = exp(a2*t)*x20
xn = sqrt(x1**2 + x2**2)
plot(t, xn , "k")
plot(t, x1, "k--")
plot(t, x2 , "k--")
```


$a_1 = -1.0,\ a_2 = 2.0$
10
5
0
0
1
2
t
$\|x(t)\|$

```
figure()
plot(real(a1), imag(a1), "x", color="k", ms=10.0)
plot(real(a2), imag(a2), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
```

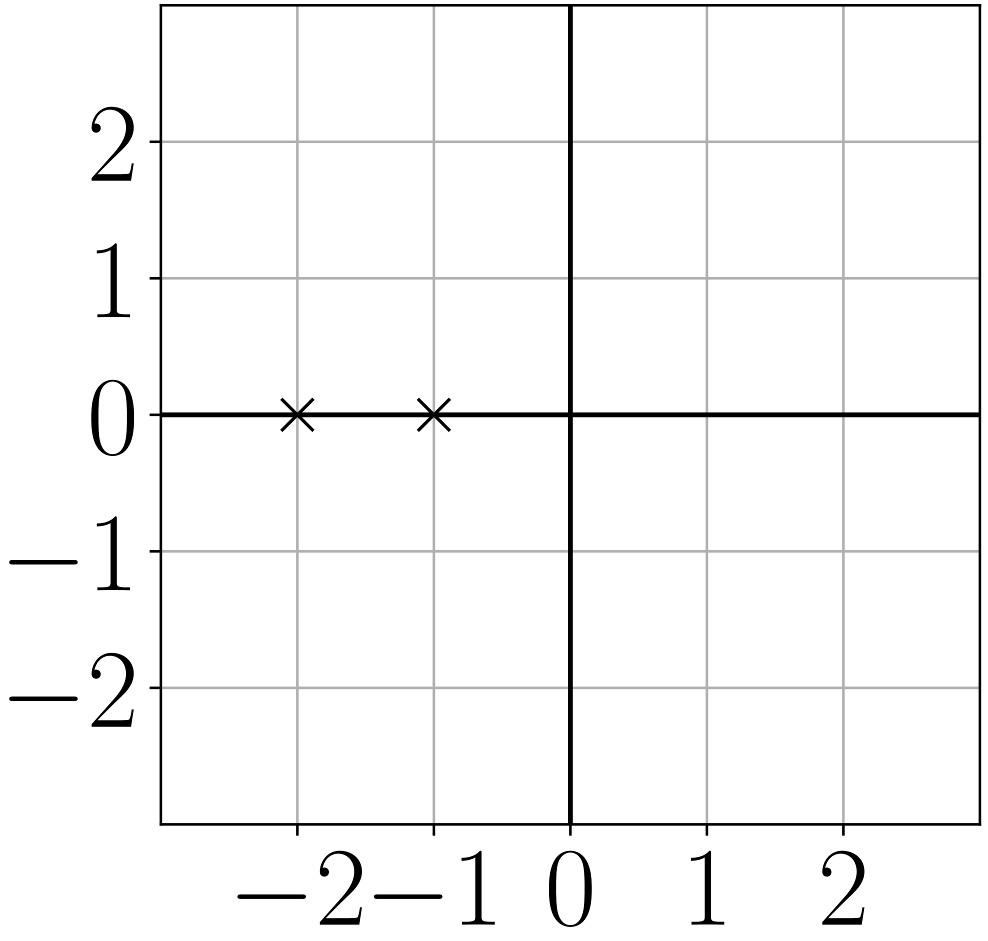

$a_1 = -1.0,\ a_2 = 2.0$
2
1
0
−1
−2
−2
−1
0
1
2

```
a1 = -1.0; a2 = -2.0; x10 = x20 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
x1 = exp(a1*t)*x10; x2 = exp(a2*t)*x20
xn = sqrt(x1**2 + x2**2)
plot(t, xn , "k")
plot(t, x1, "k--")
plot(t, x2 , "k--")
```


$a_1 = -1.0,\ a_2 = -2.0$
10
5
0
$\|x(t)\|$
0
1
2
t

```
figure()
plot(real(a1), imag(a1), "x", color="k", ms=10.0)
plot(real(a2), imag(a2), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
```

$a_1 = -1.0,\ a_2 = -2.0$



ANALYSIS

- The rightmost a_i determines the asymptotic behavior,
- The origin is globally asymptotically stable only when every a_i is in the open left-hand plane.

SCALAR CASE, COMPLEX-VALUED

$$\dot{x} = ax$$

$$a \in \mathbb{C}, x(0) = x_0 \in \mathbb{C}.$$

Solution: formally, the same old solution

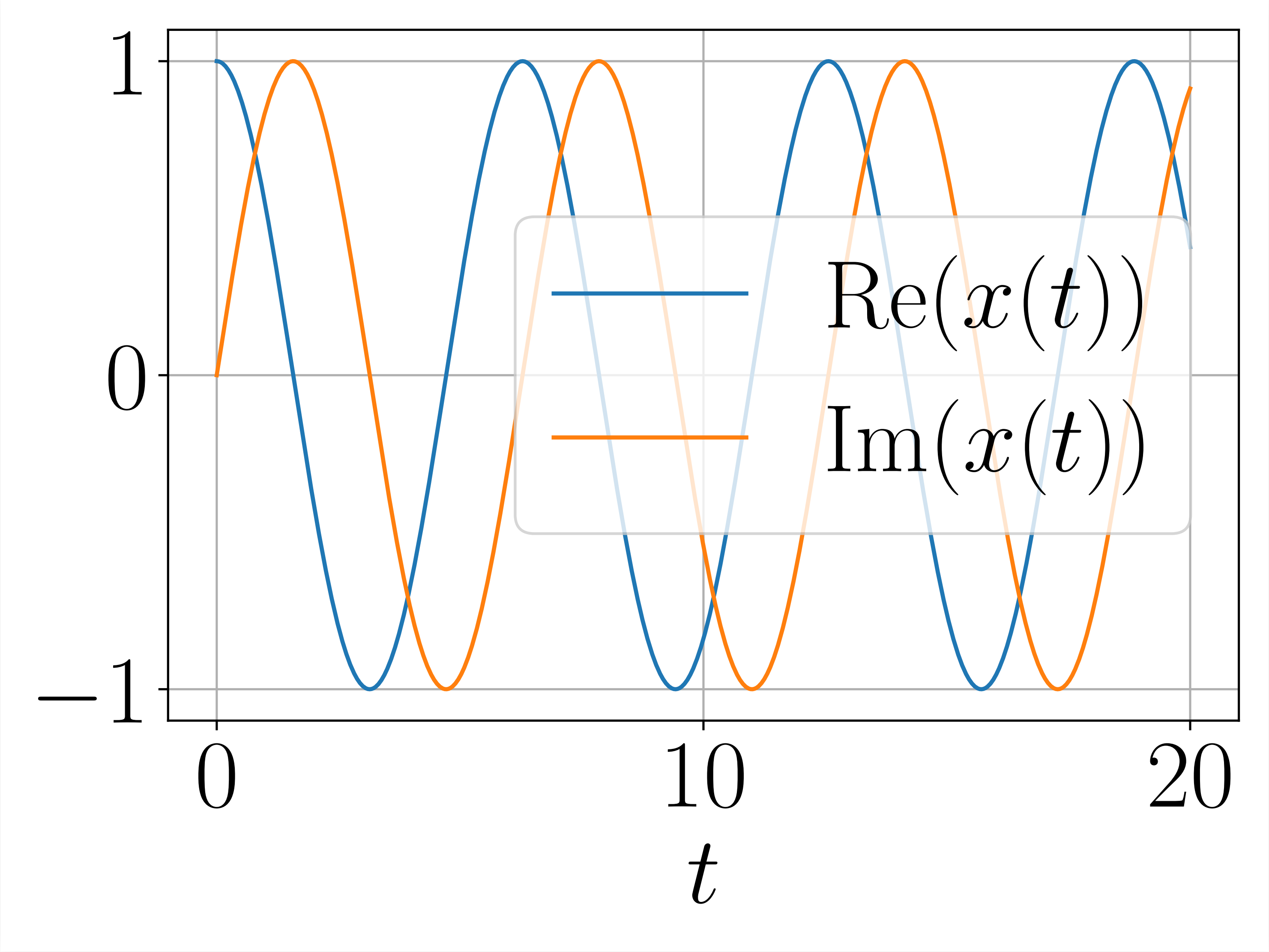
$$x(t) = e^{at} x_0$$

But now, $x(t) \in \mathbb{C}$:

if $a = \sigma + i\omega$ and $x_0 = |x_0| e^{i \angle x_0}$

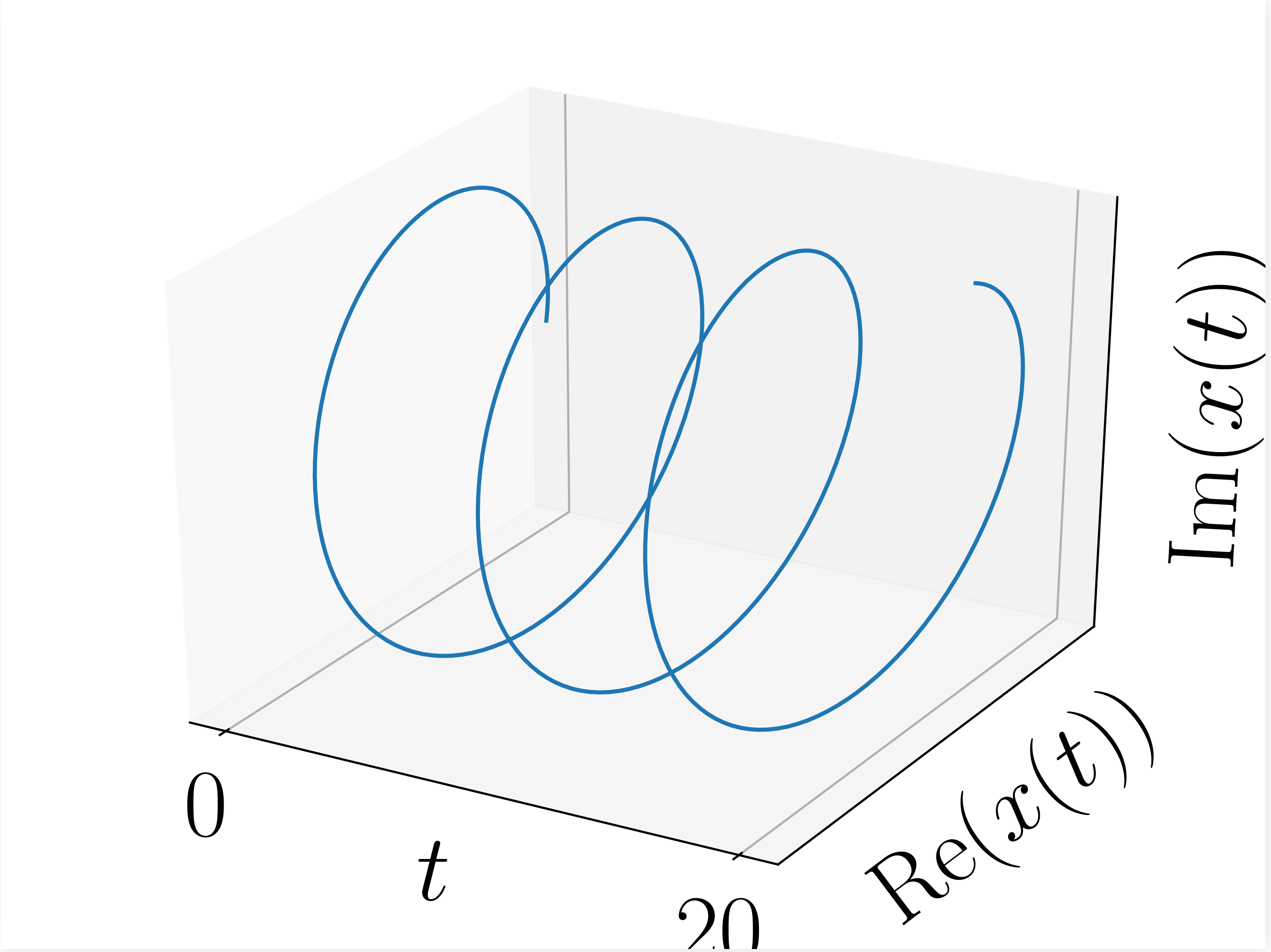
$$|x(t)| = |x_0| e^{\sigma t} \; \text{ and } \; \angle x(t) = \angle x_0 + \omega t.$$

```
a = 1.0j; x0=1.0
figure()
t = linspace(0.0, 20.0, 1000)
plot(t, real(exp(a*t)*x0), label="$\mathrm{Re}
(x(t))$")
plot(t, imag(exp(a*t)*x0), label="$\mathrm{Im}
(x(t))$")
xlabel("$t$")
```



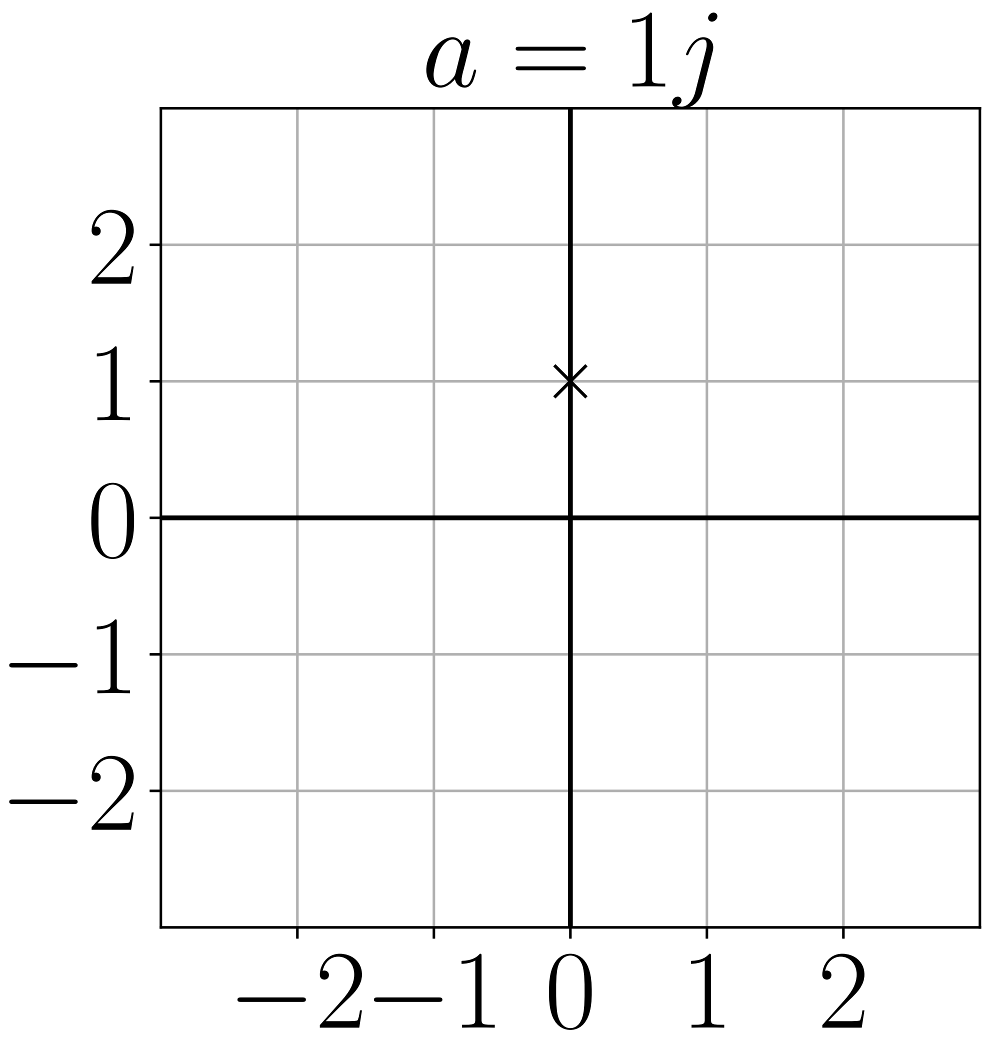
1
0
−1
0
10
20
t
Re(x(t))
Im(x(t))

```
fig = figure()
ax = fig.add_subplot(111, projection="3d")
zticks = ax.set_zticks
ax.plot(t, real(exp(a*t)*x0), imag(exp(a*t)*x0))
xticks([0.0, 20.0]); yticks([]); zticks([])
ax.set_xlabel("$t$")
ax.set_ylabel("$\mathrm{Re}(x(t))$")
ax.set_zlabel("$\mathrm{Im}(x(t))$")
```



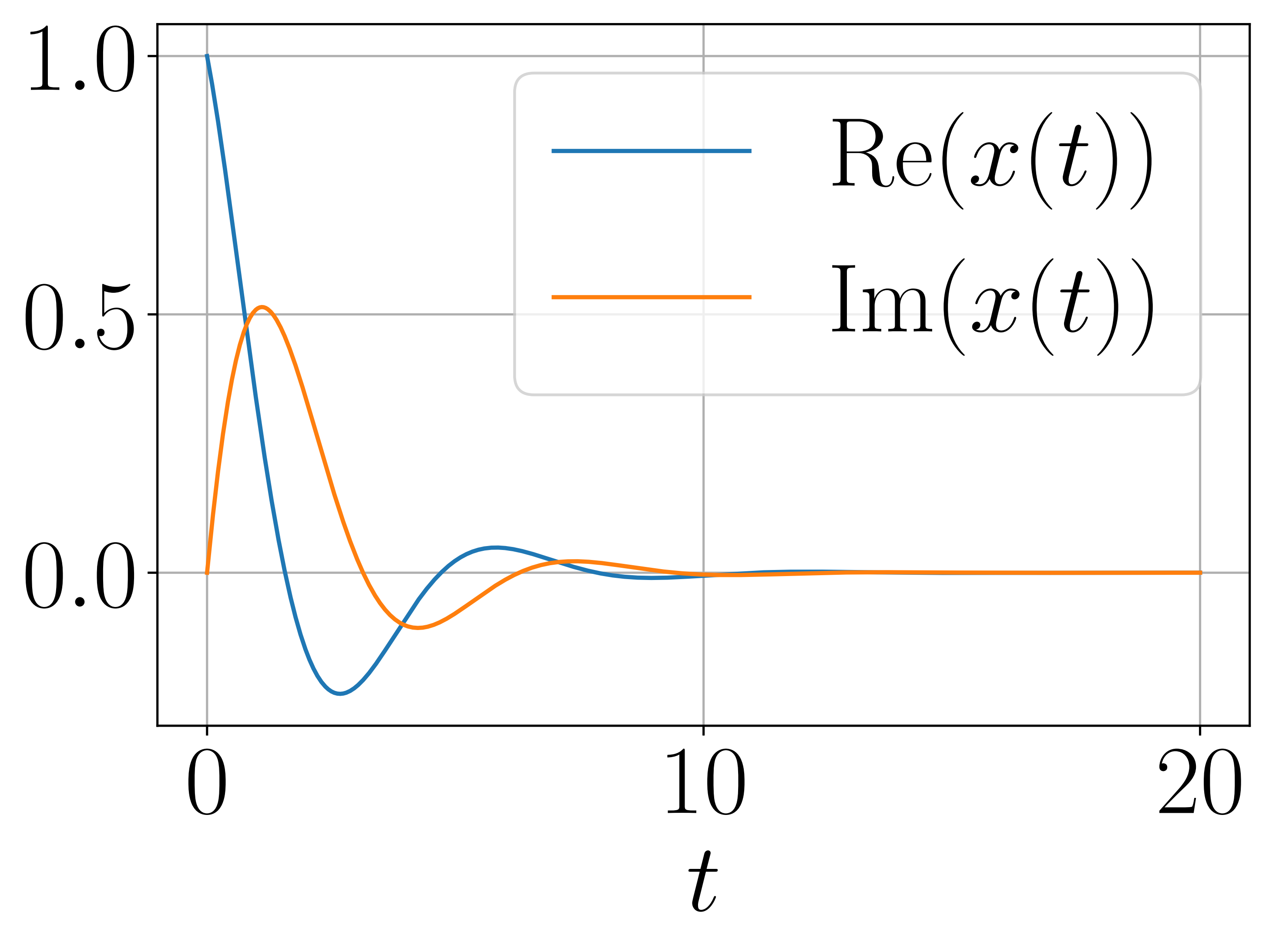
0
t
20
$\mathrm{Re}(x(t))$
$\mathrm{Im}(x(t))$

```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```

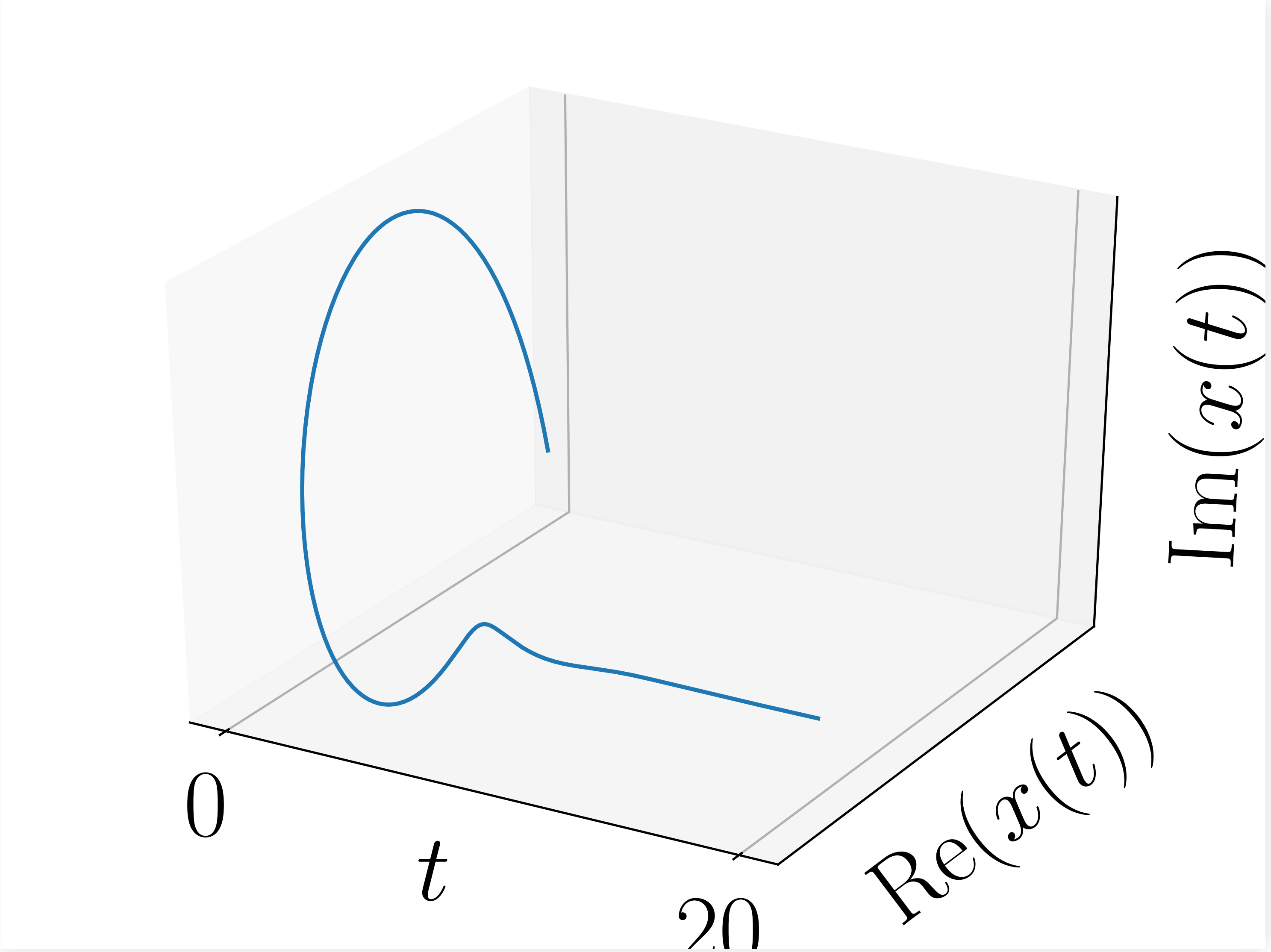


$a = 1j$
2
1
0
−1
−2
−2
−1
0
1
2

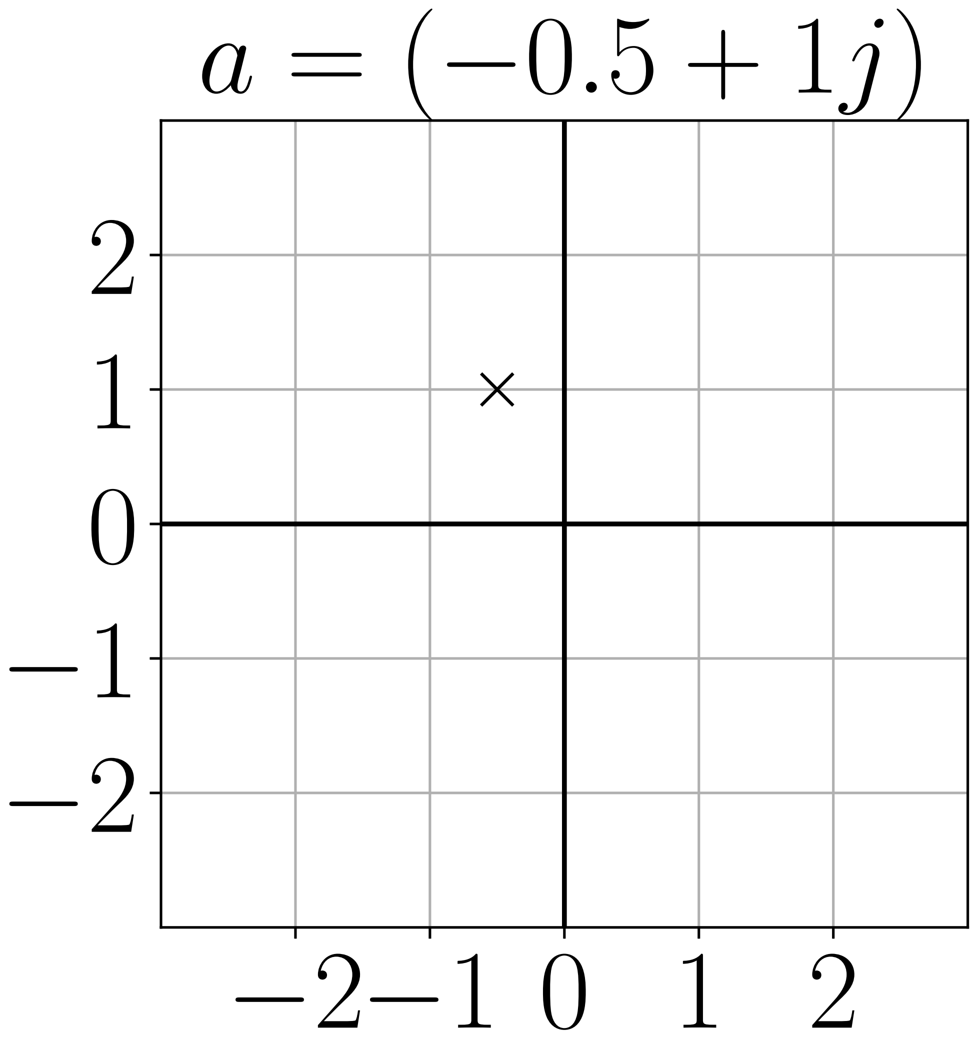
```
a = -0.5 + 1.0j; x0=1.0
figure()
t = linspace(0.0, 20.0, 1000)
plot(t, real(exp(a*t)*x0), label="$\mathrm{Re}(x(t))$")
plot(t, imag(exp(a*t)*x0), label="$\mathrm{Im}(x(t))$")
xlabel("$t$")
```


1.0
0.5
0.0
0
10
20
t
$\mathrm{Re}(x(t))$
$\mathrm{Im}(x(t))$

```
fig = figure()
ax = fig.add_subplot(111, projection="3d")
zticks = ax.set_zticks
ax.plot(t, real(exp(a*t)*x0), imag(exp(a*t)*x0))
xticks([0.0, 20.0]); yticks([]); zticks([])
ax.set_xlabel("$t$")
ax.set_ylabel("$\mathrm{Re}(x(t))$")
ax.set_zlabel("$\mathrm{Im}(x(t))$")
```


0
t
20
Re(x(t))
Im(x(t))

```
figure()
plot(real(a), imag(a), "x", color="k", ms=10.0)
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
```



$a = (-0.5 + 1j)$
2
1
0
−1
−2
−2
−1
0
1
2

ANALYSIS

- the origin is globally asymptotically stable if a is in the open left-hand plane:
$$\mathrm{Re}(a) < 0$$
- if $a = \sigma + i\omega$,
  - $\tau = -1/\sigma$ is the time constant related of the speed of convergence,
  - ω the (rotational) frequency of the (damped) oscillations.

Only one step left before the (almost) general case …

EXPONENTIAL MATRIX

If $M \in \mathbb{C}^{n \times n}$, the **exponential** is defined as:

$$e^M = \sum_{i=0}^{+\infty} \frac{M^n}{n!} \in \mathbb{C}^{n \times n}$$

⚠

The exponential of a matrix M is *not* the matrix with elements $e^{M_{ij}}$ (the elementwise exponential).

- elementwise exponential: **`exp`** (`numpy` module),
- exponential: **`expm`** (`scipy.linalg` module).

❓ EXPONENTIAL MATRIX

Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- [$\mathbf{x^2}$] Compute the exponential of M.

  🔑 **Hint:** $\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}.$

- [⚗] Check the results with `expm`.

Note that

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\sum_{n=0}^{+\infty}\frac{A^n}{n!}t^n \\
&= \sum_{n=1}^{+\infty}\frac{A^n}{(n-1)!}t^{n-1} \\
&= A\sum_{n=1}^{+\infty}\frac{A^{n-1}}{(n-1)!}t^{n-1} = Ae^{At}
\end{aligned}$$

Thus, for any $A \in \mathbb{C}^{n \times n}$ and $x_0 \in \mathbb{C}^n$,

$$\frac{d}{dt}(e^{At}x_0) = A(e^{At}x_0)$$

INTERNAL DYNAMICS

The solution of

$$\dot{x} = Ax \;\text{ and }\; x(0) = x_0$$

is

$$x(t) = e^{At} x_0.$$

STABILITY CRITERIA

Let $A \in \mathbb{C}^{n \times n}$.

The origin of $\dot{x} = Ax$ is globally asymptotically stable

$$\iff$$

all eigenvalues of A have a negative real part.

? G.A.S. $\Leftrightarrow$ L.A.

Show that for a linear systems $\dot{x} = Ax$, it is enough that the origin is locally attractive for the system to be globally asymptotically stable.

WHY DOES THIS CRITERIA WORK?

Assume that A is diagonalizable with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.

(Very likely unless A has some special structure)

Then, there is an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Thus, if $y = P^{-1}x, \dot{x} = Ax$ is equivalent to

$$\left| \begin{array}{lcl} \dot{y}_1 & = & \lambda_1 y_1 \\ \dot{y}_2 & = & \lambda_2 y_2 \\ \vdots & = & \vdots \\ \dot{y}_n & = & \lambda_n y_n \end{array} \right.$$

The system is G.A.S. iff each component of the system is, which holds iff $\mathrm{Re}\lambda_i < 0$ for each i.

❓ STABILITY / 2ND-ORDER SYSTEM

Consider the scalar ODE

$$\ddot{x} + kx = 0, \quad \text{with } k > 0$$

- [$\mathbf{x^2}$] Determine the representation of this system as a first-order ODE with state $(x, \dot{x})$.
- [💡, $\mathbf{x^2}$] Is this system asymptotically stable?

- [💡, **x²**] If its solutions oscillate, determine its (rotational) frequency ω?
- [💡, **x²**] Characterize the asymptotic behavior of $x(t)$ when $\ddot{x} + b\dot{x} + kx = 0$ for some $b > 0$.

❓ STABILITY / INTEGRATORS

Consider the system

$$\dot{x} = Jx \;\text{ with }\; J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- [💡, $\mathbf{x^2}$] Compute the solution when

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

then for any initial condition.

- [💡, **x²**] Same questions when $\dot{x} = (\lambda I + J)x$ for some $\lambda \in \mathbb{C}$.
- [💡] Is the system asymptotically stable ? Why does it matter in general?

I/O BEHAVIOR

CONTEXT

- Assume that the system is “initially at rest”:

$$x(0) = 0$$

- Forget about the state $x(t)$ (may be unknown)
- Study the input/output (I/O) relationship:

$$u \to y$$

In this context, we have:

$$y(t) = \int_0^t Ce^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t)$$

CAUSAL SIGNALS

- extend $u(t)$ and $y(t)$ by 0 when $t < 0$ (as **causal signals**).
- introduce the **Heaviside function** defined by

$$e(t) = \left| \begin{array}{ll} 1 & \text{if } \ t \geq 0, \\ 0 & \text{if } \ t < 0. \end{array} \right.$$

IMPULSE RESPONSE

The system **impulse response** is defined by:

$$H(t) = (Ce^{At}B) \times e(t) + D\delta(t) \in \mathbb{R}^{p \times m}$$

works for general or **MIMO** systems.

MIMO = multiple-input & multiple-output systems.

$\delta(t)$ is the **unit impulse**, we'll get back to it (in the meantime, you may assume that $D = 0$).

SISO SYSTEMS

When

$$p = m = 1$$

(single-input & single-output or **SISO** systems),

the 1×1 matrix $H(t)$ is identified with a scalar $h(t)$:

$$H(t) = [h(t)]$$

Then, we have:

$$y(t) = \int_{-\infty}^{+\infty} H(t - \tau)u(\tau)\, d\tau$$

and denote $*$ this operation between H and u:

$$y(t) = (H * u)(t)$$

It’s called a **convolution**.

👁 IMPULSE RESPONSE

Consider the SISO system

$$\left| \begin{array}{rcl} \dot{x} & = & ax + u \\ y & = & x \end{array} \right.$$

where $a \neq 0$.

We have

$$
\begin{aligned}
H(t) &= (Ce^{At}B) \times e(t) + D\delta(t) \\
&= [1]e^{[a]t}[1]e(t) + [0]\delta(t) \\
&= [e(t)e^{at}]
\end{aligned}
$$

When $u(t) = e(t)$ for example,

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} e(t-\tau)e^{a(t-\tau)}e(\tau)\,d\tau \\
&= \int_0^t e^{a(t-\tau)}\,d\tau \\
&= \int_0^t e^{a\tau}\,d\tau \\
&= \frac{1}{a}\left(e^{at}-1\right)
\end{aligned}
$$

? IMPULSE RESPONSE / INTEGRATOR

- [$\mathbf{x^2}$] Compute the impulse response of the system

$$\left| \begin{array}{rcl} \dot{x} & = & u \\ y & = & x \end{array} \right.$$

where $u \in \mathbb{R}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

? IMPULSE RESPONSE / DOUBLE INTEGRATOR

- [**x²**] Compute the impulse response of the system

$$\left| \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & u \\ y & = & x_1 \end{array} \right.$$

where $u \in \mathbb{R}, x = (x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$.

❓ IMPULSE RESPONSE / GAIN

- [$\mathbf{x}^2$] Compute the impulse response of the system

$$y = Ku$$

where $u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and $K \in \mathbb{R}^{p \times m}$.

? IMPULSE RESPONSE / MIMO SYSTEM

- [**x²**] Find a linear system with matrices A, B, C, D whose impulse response is

$$H(t) = \left[\begin{array}{cc} e^t e(t) & e^{-t} e(t) \end{array} \right]$$

- [**x²**] Is there another set of matrices A, B, C, D with the same impulse response? With a matrix A of a different size?

LAPLACE TRANSFORM

Associate to a scalar signal $x(t) \in \mathbb{R}, t \in \mathbb{R}$, the function of a complex argument $s \in \mathbb{C}$:

$$x(s) = \int_{-\infty}^{+\infty} x(t) e^{-st}\, dt.$$

defined when $\mathrm{Re}\,(s) > \sigma$ if $\|x(t)\| \leq K e^{\sigma t}$.

⚠ NOTATION

We use the same symbol (here “x”) to denote:

- a signal $x(t)$ and
- its Laplace transform $x(s)$

They are two equivalent representations of the same “object”, but different mathematical “functions”.

If you fear some ambiguity, use named variables, e.g.:

$$x(t = 1) \text{ or } x(s = 1) \text{ instead of } x(1).$$

VECTOR/MATRIX-VALUED SIGNALS

The Laplace transform

- of a vector-valued signal $x(t) \in \mathbb{R}^n$ or
- of a matrix-valued signals $X(t) \in \mathbb{R}^{m \times n}$

are computed elementwise.

$$x_i(s) = \int_{-\infty}^{+\infty} x_i(t)e^{-st}\,dt.$$

$$X_{ij}(s) = \int_{-\infty}^{+\infty} X_{ij}(t)e^{-st}\,dt.$$

RATIONAL & CAUSAL SIGNALS

We will only deal with **rational & causal** signals:

$$x(t) = \left(\sum_{\lambda \in \Lambda} p_\lambda(t) e^{\lambda t} \right) e(t)$$

where:

- Λ is a finite subset of $\mathbb{C}$,
- for every $\lambda \in \Lambda, p_\lambda(t)$ is a polynomial in t.

- Such signals are **causal** since

  $x(t) = 0$ when $t < 0$.

  Causality $\Leftrightarrow \deg n(s) \leq \deg d(s)$.

- They are **rational** since

$$x(s) = \frac{n(s)}{d(s)}$$

  where $n(s)$ and $d(s)$ are polynomials.

👁 LAPLACE TRANSFORM / EXPONENTIAL

Set $x(t) = e(t)e^{at}$

$$x(s) = \int_0^{+\infty} e^{at} e^{-st}\, dt = \int_0^{+\infty} e^{(a-s)t}\, dt$$

$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{+\infty} = \frac{1}{s-a}$$

(If $\mathrm{Re}(s) \geq \mathrm{Re}\,(a) + \epsilon$, then $|e^{(a-s)t}| \leq e^{-\epsilon t}$)

SYMBOLIC COMPUTATIONS

```
import sympy
from sympy.abc import t, s, a
from sympy.integrals.transforms import
laplace_transform
def L(f):
    return laplace_transform(f, t, s)[0]
```

```
xt = sympy.exp(a*t)
xs = L(xt) # 1/(-a + s)
```

? LAPLACE TRANSFORM / RAMP

Compute the Laplace Transform of

$$x(t) = te(t)$$

CONVOLUTION & LAPLACE

Let $H(t)$ be the impulse response of a system.

Its Laplace transform $H(s)$ is called the system **transfer function**.

For LTI systems in standard form, we have

$$H(s) = C[sI - A]^{-1}B + D$$

OPERATIONAL CALCULUS

The Laplace transform turns convolution into products:

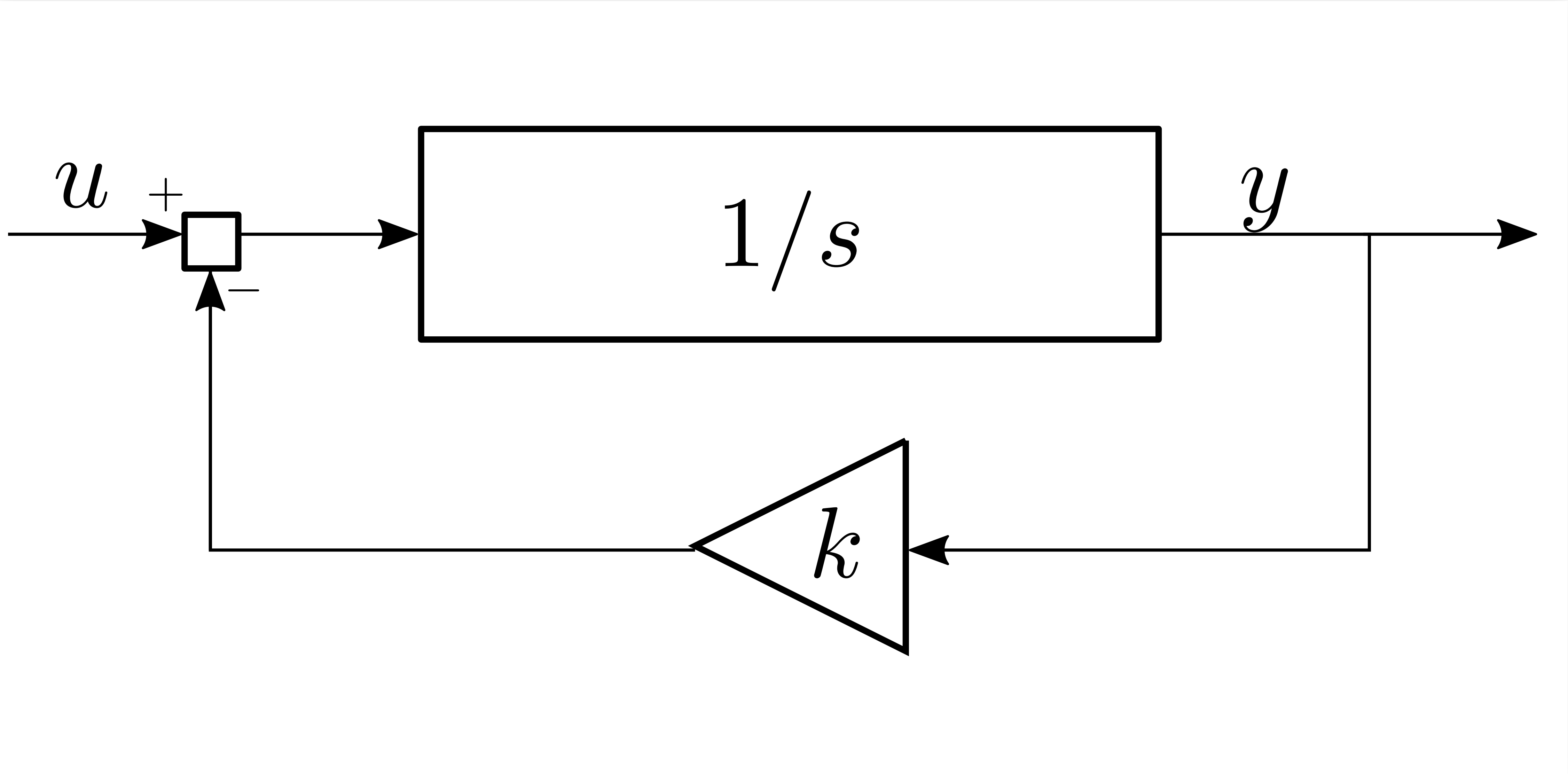
$$y(t) = (H * u)(t) \iff y(s) = H(s) \times u(s)$$

GRAPHICAL LANGUAGE

Control engineers used *block diagrams* to describe (combinations of) dynamical systems, with

- “boxes” to determine the relation between input signals and output signals and
- “wires” to route output signals to inputs signals.

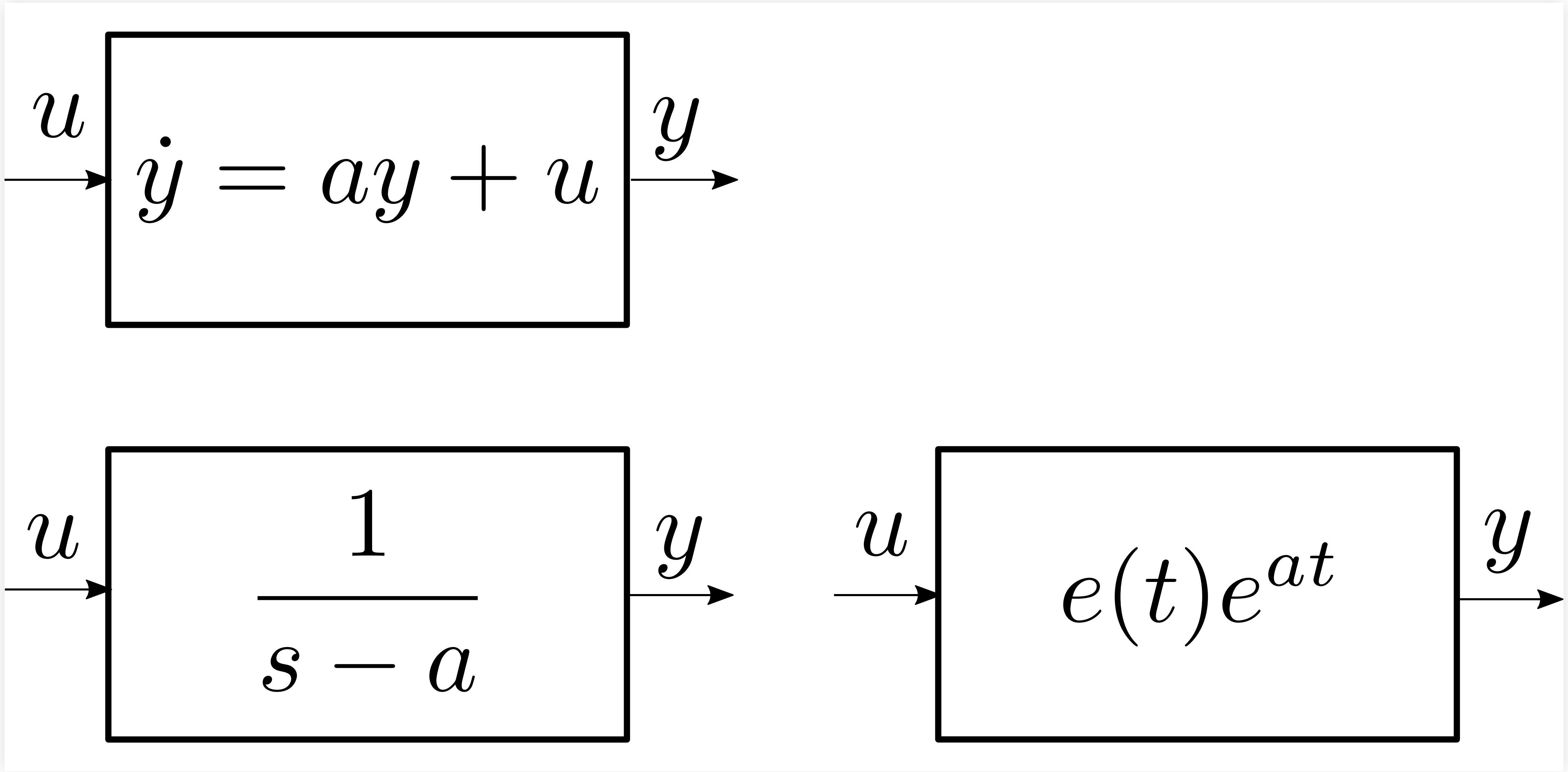
👁 BLOCK-DIAGRAM / FEEDBACK



- **Triangles** denote **gains** (scalar or matrix multipliers),
- **Adders** sum (or substract) signals.

- **LTI systems** can be specified with:
  - (differential) equations,
  - the impulse response,
  - the transfer function,

EQUIVALENT SYSTEMS



? BLOCK-DIAGRAM / FEEDBACK

Compute the transfer function $H(s)$ of the system depicted in the feedback block-diagram example.

IMPULSE RESPONSE

Why refer to $h(t)$ as the system “impulse response”?

By the way, what’s an impulse?

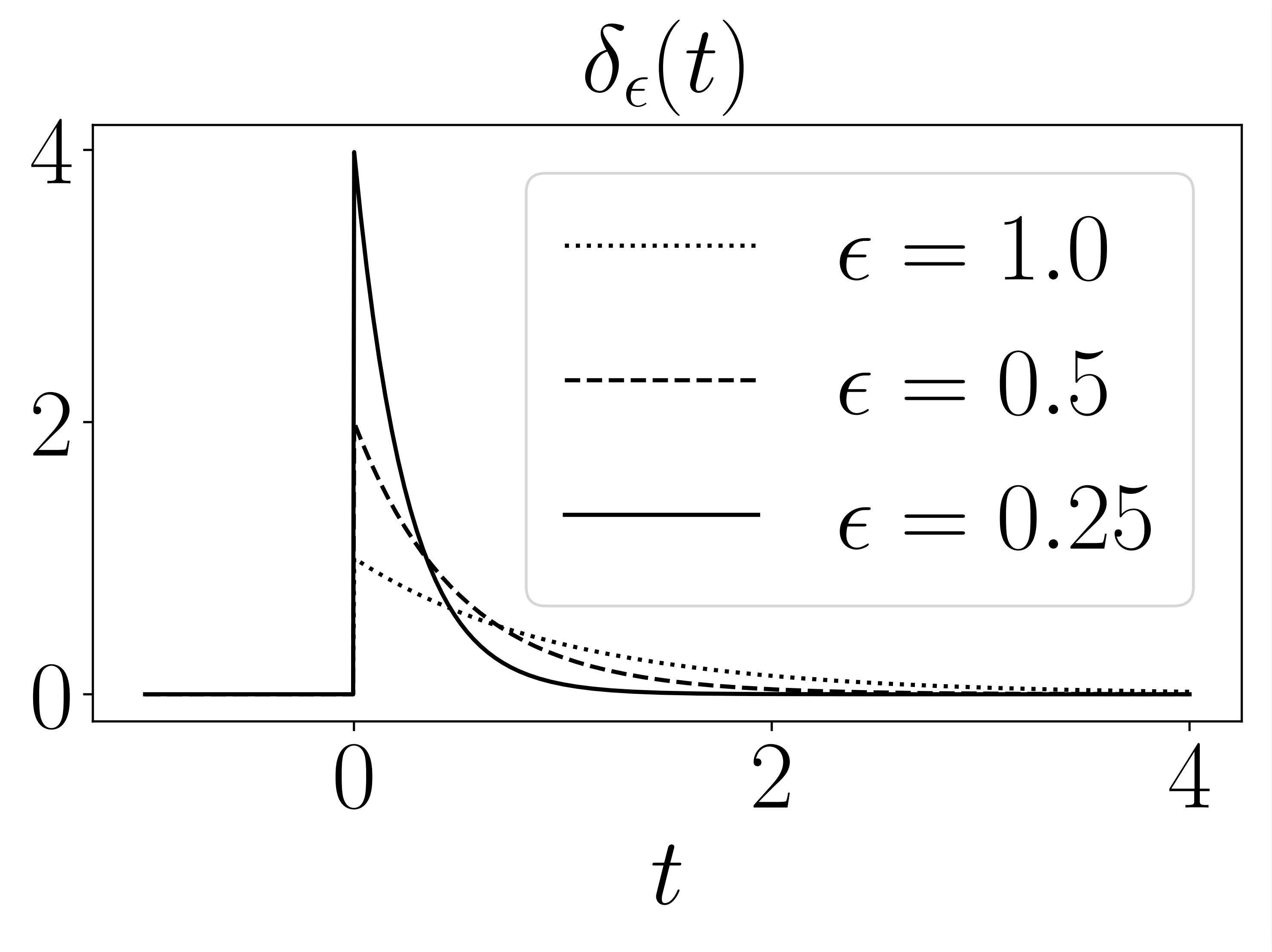
👁 IMPULSES APPROXIMATIONS

Pick a time constant $\epsilon > 0$ and define

$$\delta_\epsilon(t) = \frac{1}{\epsilon} e^{-t/\epsilon} e(t)$$

```python
def delta(t, eps=1.0):
    return exp(-t / eps) / eps * (t >= 0)
```

```
figure()
t = linspace(-1,4,1000)
plot(t, delta(t, eps=1.0), "k:",
label="$\epsilon=1.0$")
plot(t, delta(t, eps=0.5), "k--",
label="$\epsilon=0.5$")
plot(t, delta(t, eps=0.25), "k",
label="$\epsilon=0.25$")
```



$\delta_\epsilon(t)$
4
2
0
$\epsilon = 1.0$
$\epsilon = 0.5$
$\epsilon = 0.25$
0
2
4
t

IMPULSES IN THE LAPLACE DOMAIN

$$
\begin{aligned}
\delta_\epsilon(s) &= \int_{-\infty}^{+\infty} \delta_\epsilon(t) e^{-st}\, dt \\
&= \frac{1}{\epsilon} \int_0^{+\infty} e^{-(s+1/\epsilon)t}\, dt \\
&= \frac{1}{\epsilon} \left[\frac{e^{-(s+1/\epsilon)t}}{-(s+1/\epsilon)} \right]_0^{+\infty} = \frac{1}{1+\epsilon s}
\end{aligned}
$$

- The "limit" of the signal $\delta_\epsilon(t)$ when $\epsilon \to 0$ is not defined *as a function* (issue for $t = 0$) but as a *generalized function* $\delta(t)$, the **unit impulse**.
- This technicality can be avoided in the Laplace domain where

$$\delta(s) = \lim_{\epsilon \to 0} \delta_\epsilon(s) = \lim_{\epsilon \to 0} \frac{1}{1 + \epsilon s} = 1.$$

Thus, if $y(t) = (h * u)(t)$ and

1. $u(t) = \delta(t)$ then
2. $y(s) = h(s) \times \delta(s) = h(s) \times 1 = h(s)$
3. and thus $y(t) = h(t)$.

Conclusion: the impulse response $h(t)$ is the output of the system when the input is the unit impulse $\delta(t)$.

I/O STABILITY

A system is **I/O-stable** if there is a $K \geq 0$ such that

$$\text{for any } t \geq, \|y(t)\| \leq KM$$

whenever

$$\text{for any } t \geq, \|u(t)\| \leq M$$

There is a bound on the amplification of the input signal that the system can provide.

Also called **BIBO-stability** (for “bounded input, bounded output”)

TRANSFER FUNCTION POLES

A **pole** of the transfer function $H(s)$ is a $s \in \mathbb{C}$ such that for at least one element $H_{ij}(s)$,

$$|H_{ij}(s)| = +\infty.$$

I/O-STABILITY CRITERIA

A system is I/O-stable if and only if all its poles are in the open left-plane, i.e. such that

$$\mathrm{Re}(s) < 0.$$

INTERNAL STABILITY VS I/O-STABILITY

If the system $\dot{x} = Ax$ is asymptotically stable, then for any matrices B, C, D of appropriate sizes,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is I/O-stable.

FULLY ACTUATED & MEASURED SYSTEM

If $B = I, C = I$ and $D = 0$, that is

$$\dot{x} = Ax + u, \;\; y = x$$

then $H(s) = [sI - A]^{-1}$.

Therefore, s is a pole of H iff it's an eigenvalue of A.

Thus, in this case, asymptotic stability and I/O-stability are equivalent.

This equivalence holds under much weaker conditions.