# WELL-POSEDNESS

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### **CONTROL ENGINEERING WITH PYTHON**

- Course Materials
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## **SYMBOLS**

2	Code		Worked Example
	Graph	**	Exercise
	Definition		Numerical Method
	Theorem	D0000 00 000 D000 000000 D000 000000 D00000000	Analytical Method
	Remark		Theory
	Information	Qu.	Hint
1	Warning	1	Solution

# **LIMPORTS**

```
from numpy import *
from numpy.linalg import *
from scipy.integrate import solve_ivp
from matplotlib.pyplot import *
```

## **STREAM PLOT HELPER**

```
def Q(f, xs, ys):
   X, Y = meshgrid(xs, ys)
    fx = vectorize(lambda x, y: f([x, y])[0])
    fy = vectorize(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

# **WELL-POSEDNESS**

Make sure that a system is "sane" (not "pathological"):

### Well-Posedness:

- Existence +
- Uniqueness +
- Continuity.

We will define and study each one in the sequel.

### LOCAL VS GLOBAL

So far, we have mostly dealt with **global** solutions x(t) of IVPs, defined for any  $t \geq t_0$ .

This concept is sometimes too stringent.

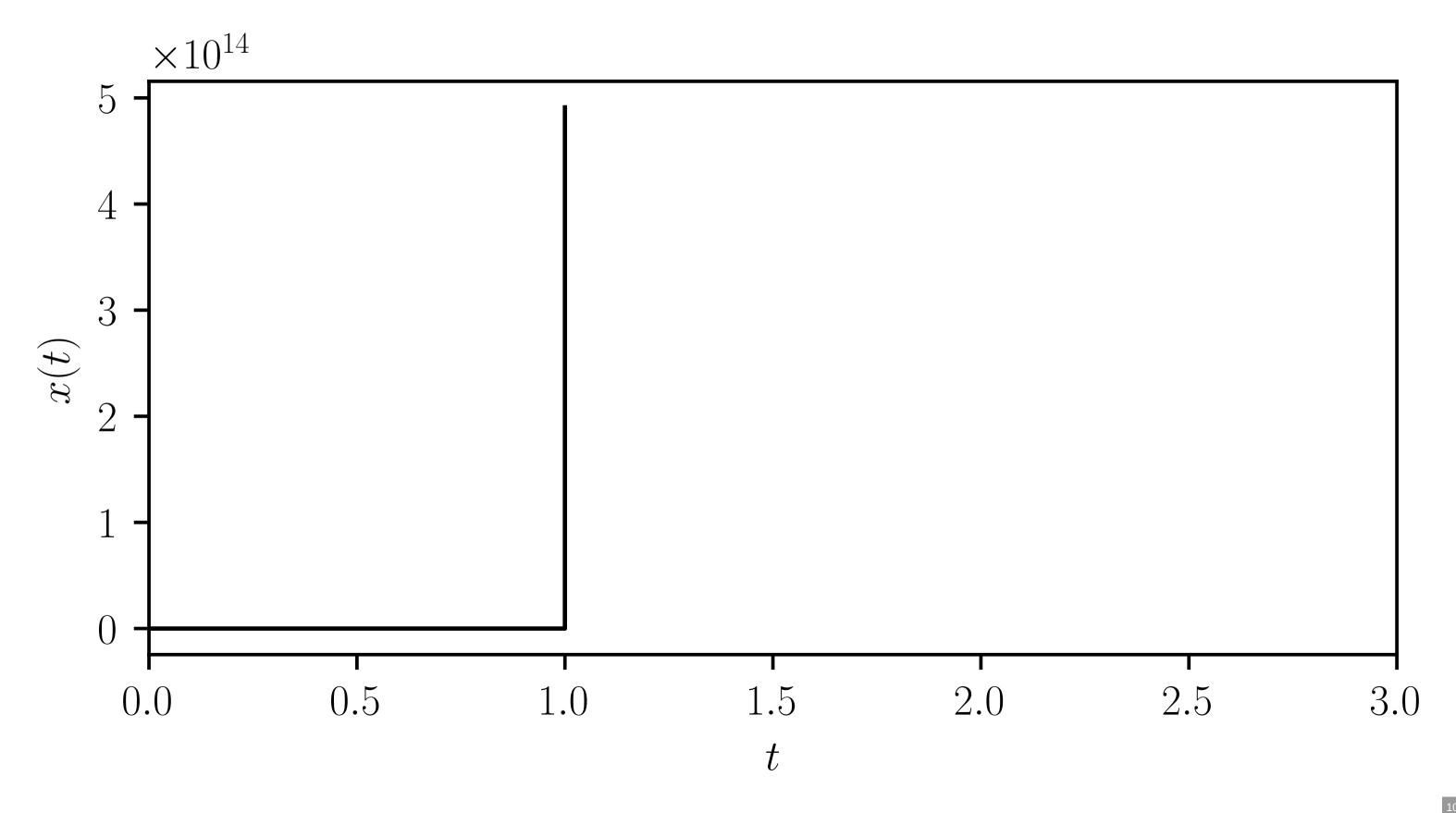
# **FINITE-TIME BLOW-UP**

### Consider the IVP

$$\dot{x} = x^2, \ x(0) = 1.$$



```
def fun(t, y):
    return y * y
t0, tf, y0 = 0.0, 3.0, array([1.0])
result = solve_ivp(fun, t_span=[t0, tf], y0=y0)
figure()
plot(result["t"], result["y"][0], "k")
xlim(t0, tf); xlabel("$t$"); ylabel("$x(t)$")
```



### LOCAL VS GLOBAL



Ouch.

There is actually no global solution.

However there is a **local** solution x(t),

- ullet defined for  $t\in [t_0, au[$
- for some  $au > t_0$ .

Indeed, the function  $x(t) := rac{1}{1-t}$  satisfies

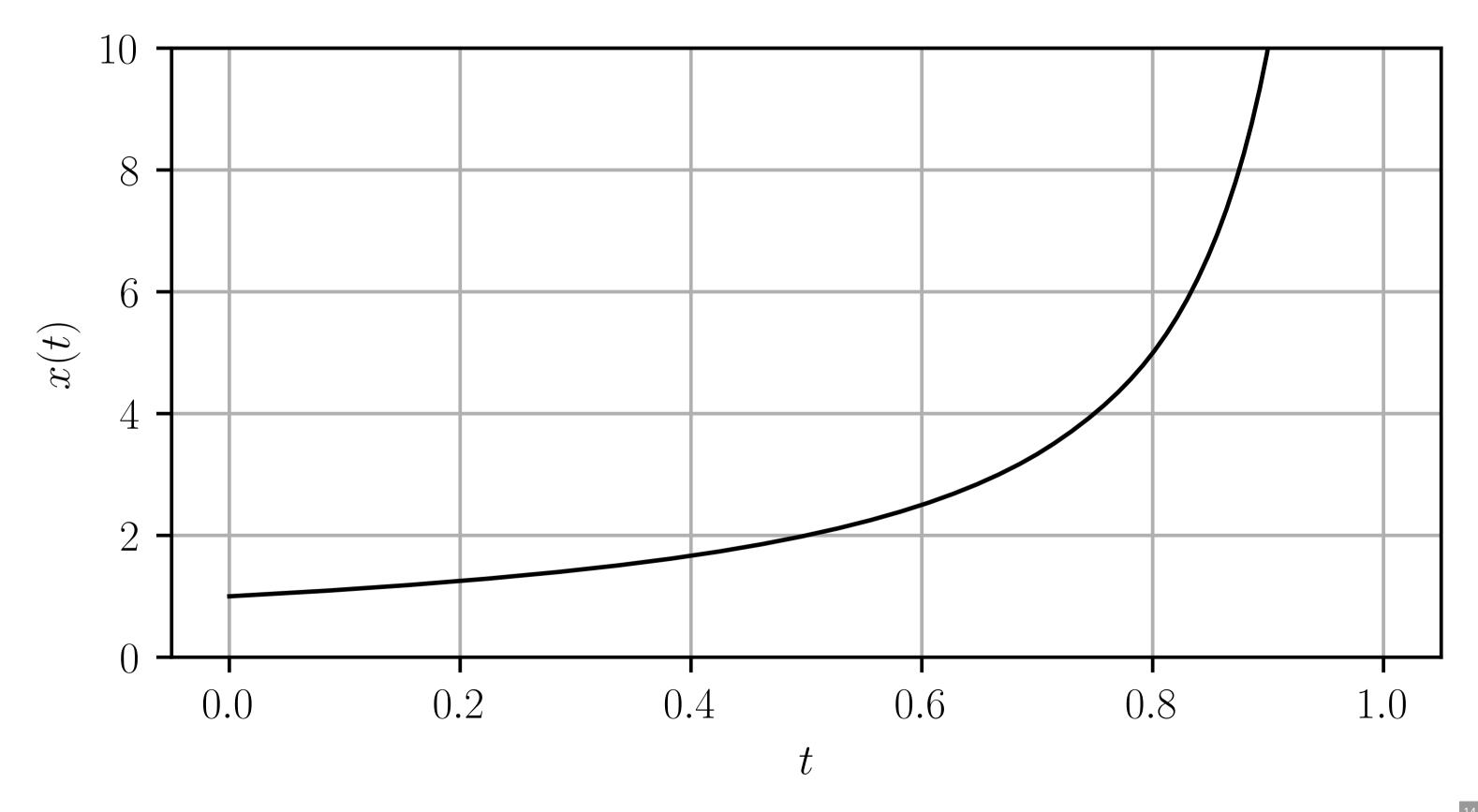
$$\dot{x}(t) = \frac{d}{dt}x(t) = -\frac{-1}{(1-t)^2} = (x(t))^2$$

and x(0) = 1.

 $lap{1}{2}$  But it's defined (continuously) only for t < 1.



```
tf = 1.0
r = solve_ivp(fun, [t0, tf], y0,
              dense_output=True)
figure()
t = linspace(t0, tf, 1000)
plot(t, r["sol"](t)[0], "k")
ylim(0.0, 10.0); grid();
xlabel("$t$"); ylabel("$x(t)$")
```



This local solution is also maximal:

You cannot extend this solution beyond au=1.0.

## **LOCAL SOLUTION**

A solution  $x:I o \mathbb{R}^n$  of the IVP

$$\dot{x}=f(x),\;x(t_0)=x_0$$

is (forward and) **local** if  $I=[t_0, au[$  for some au[ such that  $t_0 < \tau \leq +\infty$ .



# GLOBAL SOLUTION

A solution  $x:I 
ightarrow \mathbb{R}^n$  of the IVP

$$\dot{x}=f(x),\;x(t_0)=x_0$$

is (forward and) global if  $I = [t_0, +\infty[$ .

## MAXIMAL SOLUTION

A (local) solution  $x:[0,\tau[$  to an IVP is  $\max$  if there is no other solution

- defined on  $[0,\tau'[$  with  $\tau'> au,$
- whose restriction to  $[0, \tau]$  is x.

# MAXIMAL SOLUTIONS

### Consider the IVP

$$\dot{x} = x^2, \; x(0) = x_0 \neq 0.$$

1.

Find a closed-formed local solution x(t) of the IVP.

 $\nearrow$  Hint: assume that x(t) 
eq 0 then compute

$$rac{d}{dt} rac{1}{x(t)}.$$

### 2.

Make sure that your solutions are maximal.

# MAXIMAL SOLUTIONS

1. 🔓

As long as  $x(t) \neq 0$ ,

$$rac{d}{dt}rac{1}{x(t)}=-rac{\dot{x}(t)}{x(t)^2}=1.$$

By integration, this leads to

$$\frac{1}{x(t)} - \frac{1}{x_0} = -t$$

and thus provides

$$x(t) = rac{1}{rac{1}{x_0} - t} = rac{x_0}{1 - x_0 t}.$$

which is indeed a solution as long as the denominator is not zero.

## 2.

- If  $x_0 < 0$ , this solution is valid for all  $t \geq 0$  and thus maximal.
- If  $x_0 > 0$ , the solution is defined until t = 1/x(0) where it blows up. Thus, this solution is also maximal.

# **BAD NEWS (1/3)**

Sometimes things get worse than simply having no global solution.

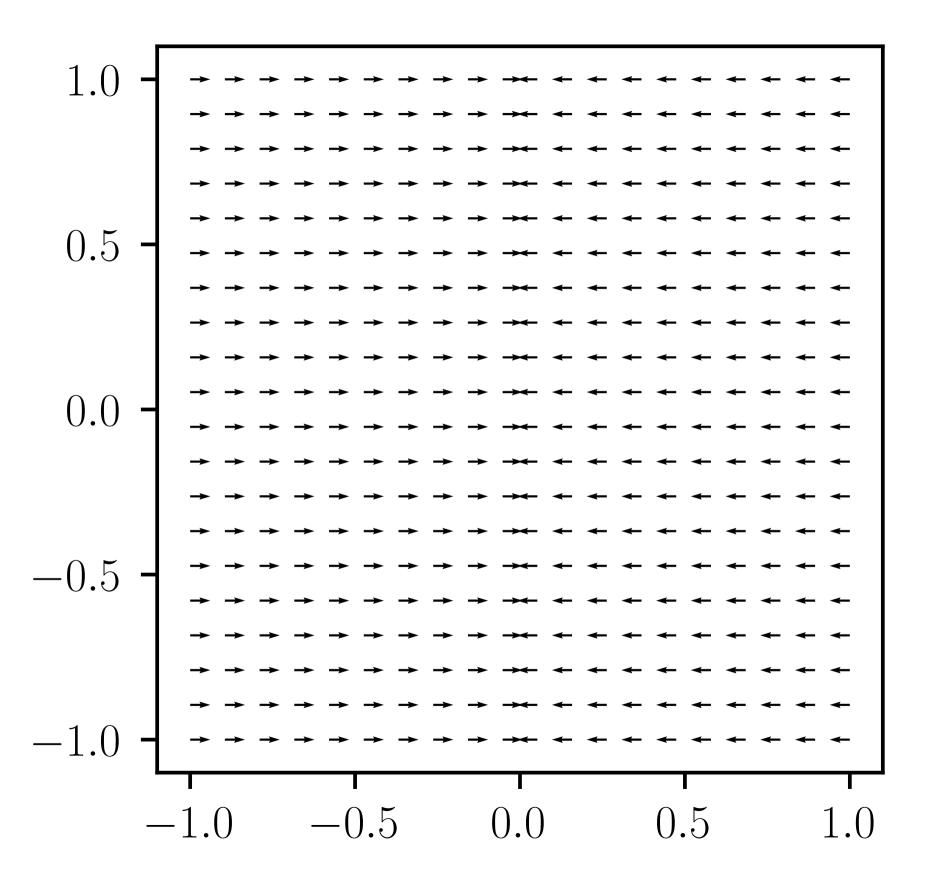
## **NO LOCAL SOLUTION**

Consider the scalar IVP with initial value x(0) = (0,0) and right-hand side

$$f(x_1,x_2) = egin{array}{ccc} (+1,0) & ext{if} & x_1 < 0 \ (-1,0) & ext{if} & x_1 \geq 0. \end{array}$$

### **NO LOCAL SOLUTION**

```
def f(x1x2):
    x1, x2 = x1x2
    dx1 = 1.0 \text{ if } x1 < 0.0 \text{ else } -1.0
    return array([dx1, 0.0])
figure()
x1 = x2 = linspace(-1.0, 1.0, 20)
gca().set_aspect(1.0)
quiver(*Q(f, x1, x2), color="k")
```



### **NO LOCAL SOLUTION**

This system has no solution, not even a local one, when x(0)=(0,0).

### **PROOF**

- Assume that  $x:[0, au[
  ightarrow \mathbb{R}]$  is a local solution.
- Since  $\dot{x}(0)=-1<0$ , for some small enough  $0<\epsilon< au$  and any  $t\in ]0,\epsilon]$ , we have x(t)<0.
- ullet Consequently,  $\dot{x}(t)=+1$  and thus by integration

$$x(\epsilon) = x(0) + \int_0^\epsilon \dot{x}(t) dt = \epsilon > 0,$$

which is a contradiction.

# **GOOD NEWS (1/3)**

However, a local solution exists under very mild assumptions.



### If f is continuous,

- There is a (at least one) local solution to the IVP  $\dot{x}=f(x)$  and  $x(t_0)=x_0$ .
- Any local solution on some  $[t_0, au]$  can be extended to a (at least one) **maximal** one on some  $[t_0, t_{\infty}]$ .



**Note:** a maximal solution is global iff  $t_{\infty} = +\infty$ .

# **MAXIMAL SOLUTIONS**

A solution on  $[t_0, au[$  is maximal if and only if either

- $au = +\infty$  : the solution is global, or
- ullet  $au<+\infty$  and  $\lim_{t o au}\|x(t)\|=+\infty$  .

In plain words: a non-global solution cannot be extended further in time if and only if it "blows up".



Let's assume that a local maximal solution exists.

You wonder if this solution is defined in  $[t_0,t_f[$  or blows up before  $t_f$ .

For example, you wonder if a solution is global (if  $t_f=+\infty$  or  $t_f<+\infty$ .)

### **PROVE EXISTENCE**

**Task.** Show that any solution which defined on some sub-interval  $[t_0, \tau]$  with  $\tau < t_f$  would is bounded.

Then, no solution can be maximal on any such  $[0, \tau[$  (since it doesn't blow up !). Since a maximal solution does exist, its domain is  $[0, t_{\infty}[$  with  $t_{\infty} \geq t_f$ .

 $\Rightarrow$  a solution is defined on  $[t_0,t_f[.$ 

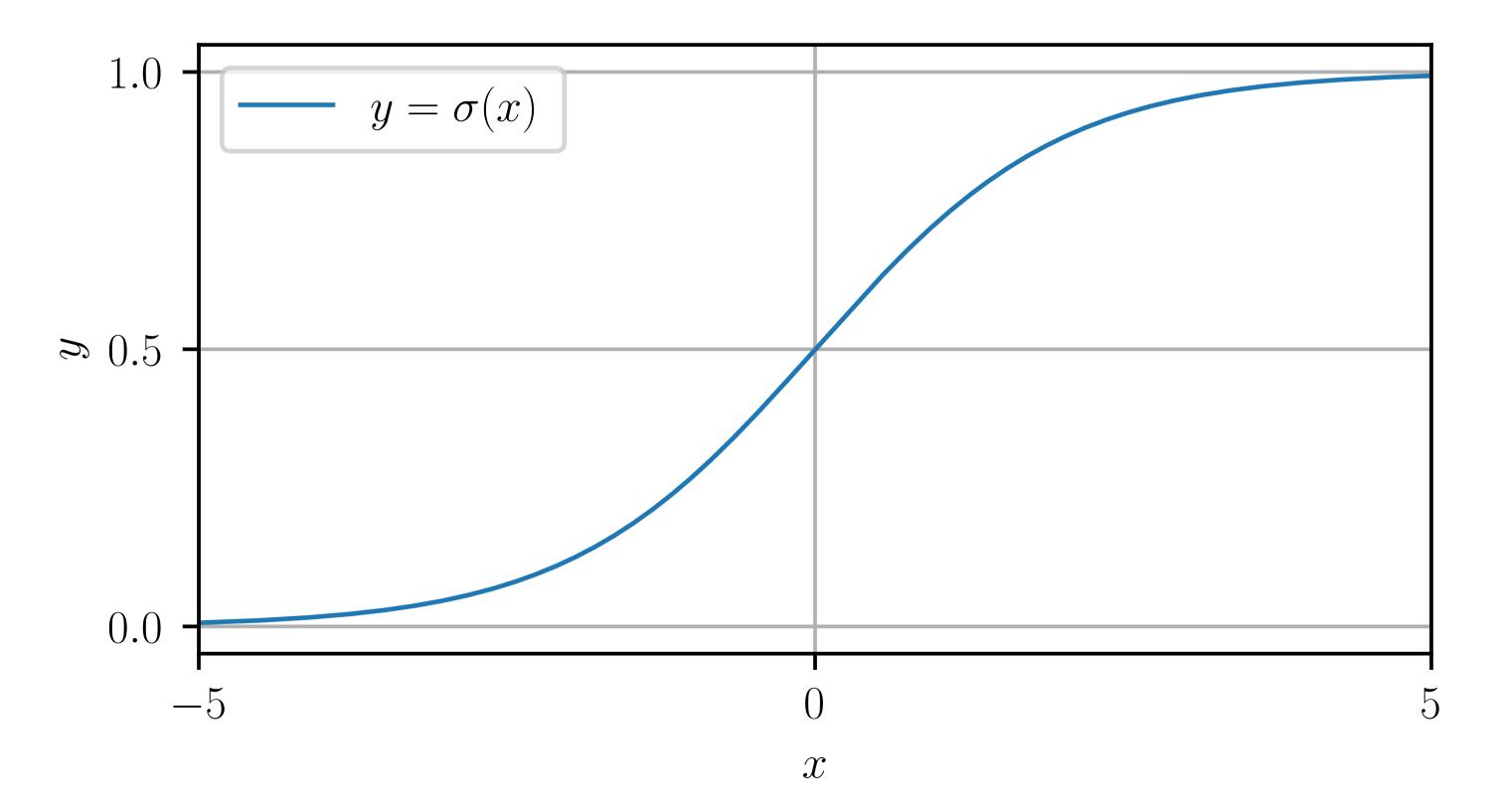


#### Consider the dynamical system

$$\dot{x}=\sigma(x):=rac{1}{1+e^{-x}}.$$



```
def sigma(x):
    return 1 / (1 + exp(-x))
figure()
x = linspace(-7.0, 7.0, 1000)
plot(x, sigma(x), label="$y=\sigma(x)$")
grid(True)
```



### 1. EXISTENCE

Show that there is a (at least one) maximal solution to each initial condition.

## 2. GLOBAL

Show that any such solution is global.



### 1. EXISTENCE

The sigmoid function  $\sigma$  is continuous.

Consequently, **Existence** proves the existence of a (at least one) maximal solution.

### 2. GLOBAL

Let  $x:[0, au[ 
ightarrow \mathbb{R}$  be a maximal solution to the IVP. We have

$$0 \le \dot{x}(t) = \sigma(x(t)) \le 1, \ 0 \le t < \tau$$

and by integration,

$$|x(t)| \le |x(0)| + t$$

Thus, it cannot blow-up in finite time; by Maximal Solutions, it is global.



Consider the pendulum, subject to a torque c

$$ml^2\ddot{ heta} + b\dot{ heta} + mg\ell\sin heta = c( heta,\dot{ heta})$$

We assume that the torque provides a bounded power:

$$P:=c( heta,\dot{ heta})\dot{ heta}\leq P_M<+\infty.$$

Show that for any initial state, there is a global solution  $(\theta, \dot{\theta})$ .

 $\nearrow$  Hint. Compute the derivative with respect to t of

$$E=rac{1}{2}m\ell^2\dot{ heta}^2-mg\ell\cos heta.$$



Since the system vector field

$$( heta,\dot{ heta}) 
ightarrow \left(\dot{ heta},(-b/m\ell^2)\dot{ heta}-(g/\ell)\sin heta+c( heta,\dot{ heta})/m\ell^2
ight)$$

is continuous, **Existence** yields the existence of a (at least one) maximal solution.

#### Additionally,

$$egin{align} \dot{E} &= rac{d}{dt} igg(rac{1}{2} m \ell^2 \dot{ heta}^2 - m g \ell \cos heta igg) \ &= -b \dot{ heta}^2 + c( heta, \dot{ heta}) \dot{ heta} \ &\leq P_M < +\infty. \end{split}$$

#### By integration

$$E(t) = rac{1}{2} m \ell^2 \dot{ heta}^2(t) - mg\ell \cos heta(t) \leq E(0) + P_M t$$

Hence, since  $|\cos \theta(t)| \leq 1$ ,

$$|\dot{ heta}(t)| \leq \sqrt{rac{2E(0)}{m\ell^2} + rac{2g}{\ell} + rac{2P_M}{m\ell^2}}t$$

Thus,  $\dot{ heta}(t)$  cannot blow-up in finite time. Since

$$| heta(t)| \leq | heta(0)| + \int_0^t |\dot{ heta}(s)| ds,$$

heta(t) cannot blow-up in finite time either.

By Maximal Solutions, any maximal solution is global.



Let  $A \in \mathbb{R}^{n \times n}$ .

Consider the dynamical system

$$\dot{x}=Ax,\;x\in\mathbb{R}^n.$$

### 0000-000000

Show that

$$y(t) := \|x(t)\|^2$$

is differentiable and satisfies

$$\dot{y}(t) \leq 2\alpha y(t)$$

for some  $\alpha \geq 0$ .

Let

$$z(t) := y(t)e^{-2\alpha t}.$$

Compute  $\dot{z}(t)$  and deduce that

$$0 \le y(t) \le y(0)e^{2\alpha t}.$$

Prove that for any initial state  $x(0) \in \mathbb{R}^n$  there is a corresponding global solution x(t).

# **LINEAR SYSTEMS**

By definition of y(t) and since  $\dot{x}(t) = Ax(t)$ ,

$$egin{aligned} \dot{y}(t) &= rac{d}{dt} \|x(t)\|^2 \ &= rac{d}{dt} x(t)^t x(t) \ &= \dot{x}(t)^t x(t) + x(t)^t \dot{x}(t) \ &= x(t)^t A^t x(t) + x(t)^t A x(t). \end{aligned}$$

Let  $\alpha$  denote the largest singular value of A (i.e. the operator norm  $\|A\|$ ).

$$\alpha := \sigma_{\max}(A) = ||A||.$$

For any vector  $u \in \mathbb{R}^n$ , we have

$$||Au|| \leq ||A|| ||u||.$$

By the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$egin{aligned} \dot{y}(t) &= \|x(t)^t A^t x(t) + x(t)^t A x(t)\| \ &\leq \|(Ax(t))^t x(t)\| + \|x(t)^t (Ax(t))\| \ &\leq \|Ax(t)\| \|x(t)\| + \|x(t)\| \|Ax(t)\| \ &\leq \|A\| \|x(t)\| \|x(t)\| + \|x(t)\| \|A\| \|x(t)\| \ &\equiv 2\|A\| y(t) \end{aligned}$$

and thus  $\dot{y}(t) \leq 2 \alpha y(t)$  with  $\alpha := \|A\|$  .

Since  $y(t) = \|x(t)\|^2$ , the inequality  $0 \le y(t)$  is clear.

Since 
$$z(t) = y(t)e^{-2\alpha t}$$
,

$$egin{aligned} \dot{z}(t) &= rac{d}{dt} y(t) e^{-2lpha t} \ &= \dot{y}(t) e^{-2lpha t} + y(t) (-2lpha e^{-lpha t}) \ &= (\dot{y}(t) - 2lpha y(t)) e^{-2lpha t} \ &< 0. \end{aligned}$$

#### By integration

$$y(t)e^{-2lpha t} = z(t) = z(0) + \int_0^t \dot{z}(s)\,ds \ \leq z(0) = y(0),$$

hence

$$y(t) \leq y(0)e^{2\alpha t}$$
.

The vector field

$$x \in \mathbb{R}^n o Ax$$

is continuous, thus by  $\bigvee$  Existence there is a maximal solution  $x:[0,t_{\infty}[$  for any initial state x(0).

Moreover,

$$||x(t)|| = \sqrt{||y(t)||} \le \sqrt{y(0)e^{2\alpha t}} = ||x(0)||e^{\alpha t}.$$

Hence there is no finite-time blow-up and the maximal solution is global.

# UNIQUENESS

In the current context, **uniqueness** means uniqueness of the maximal solution to an IVP.

## **BAD NEWS (2/3)**

Uniqueness of solutions, even the maximal ones, is not granted either.



The IVP

$$\dot{x} = \sqrt{x}, \ x(0) = 0$$

has several maximal (global) solutions.

### **PROOF**

For any  $\tau \geq 0$ ,  $x_{\tau}$  is a solution:

$$x_{ au}(t) = egin{array}{cccc} 0 & ext{if } t \leq au, \ 1/4 imes (t- au)^2 & ext{if } t > au. \end{array}$$

## **COOD NEWS (2/3)**

However, uniqueness of maximal solution holds under mild assumptions.



## JACOBIAN MATRIX

$$x = (x_1, \ldots, x_n), f(x) = (f_1(x), \ldots, f_n(x)).$$

### Jacobian matrix of f:

$$egin{array}{c} rac{\partial f}{\partial x} := egin{bmatrix} rac{\partial f_1}{\partial x_1} & \cdots & rac{\partial f_1}{\partial x_n} \ dots & dots & dots \ rac{\partial f_n}{\partial x_1} & \cdots & rac{\partial f_n}{\partial x_n} \end{bmatrix}$$

## UNIQUENESS

If  $\partial f/\partial x$  exists and is continuous, the maximal solution is unique.

## **BAD NEWS (3/3)**

An infinitely small error in the initial value could result in a finite error in the solution, even in finite time.

That would severely undermine the utility of any approximation method.

## CONTINUITY

Instead of denoting x(t) the solution, use  $x(t,x_0)$  to emphasize the dependency w.r.t. the initial state.

Continuity w.r.t. the initial state means that if  $x(t,x_0)$  is defined on  $[t_0, au]$  and  $t\in[t_0, au]$ :

$$x(t,y) 
ightarrow x(t,x_0) ext{ when } y 
ightarrow x_0$$

and that this convergence is uniform w.r.t. t.

## **COOD NEWS (3/3)**

However, continuity w.r.t. the initial value holds under mild assumptions.



Assume that  $\partial f/\partial x$  exists and is continuous.

Then the dynamical system is continous w.r.t. the initial state.

## **PREY-PREDATOR**

Let

$$\dot{x} = \alpha x - \beta xy$$
 $\dot{y} = \delta xy - \gamma y$ 

with 
$$\alpha=2/3, \beta=4/3, \delta=\gamma=1.0$$
.



```
alpha = 2 / 3; beta = 4 / 3; delta = gamma = 1.0

def fun(t, y):
    x, y = y
    u = alpha * x - beta * x * y
    v = delta * x * y - gamma * y
    return array([u, v])
```



```
tf = 3.0
result = solve_ivp(
  fun,
  t_{span} = (0.0, tf),
  y0=[1.5, 1.5],
  max_step=0.01)
x, y = result["y"][0], result["y"][1]
```



```
def display_streamplot():
    ax = gca()
    xr = yr = linspace(0.0, 2.0, 1000)
    def f(y):
        return fun(0, y)
    streamplot(*Q(f, xr, yr), color="grey")
```

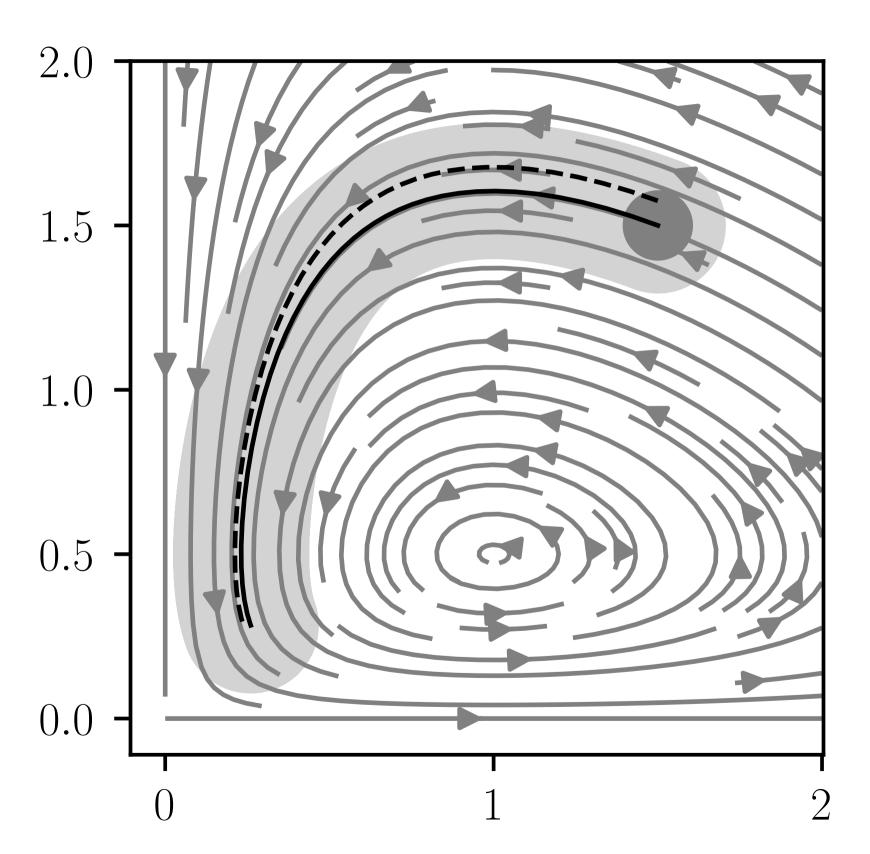


```
def display_reference_solution():
    for xy in zip(x, y):
        x_{-}, y_{-} = xy
        gca().add_artist(Circle((x_, y_),
                          0.2, color="#d3d3d3"))
    gca().add_artist(Circle((x[0], y[0]), 0.1,
                      color="#808080"))
    plot(x, y, "k")
```





```
figure()
display_streamplot()
display_reference_solution()
display_alternate_solution()
axis([0,2,0,2]); axis("square")
```



## **CONTINUITY**

Let  $h \geq 0$  and  $x^h(t)$  be the solution of the IVP

$$\dot{x} = x, \; x^h(0) = 1 + h.$$

Let  $\epsilon > 0$  and  $\tau > 0$ .

Find the largest  $\delta>0$  such that  $|h|<\delta$  ensures that

$$\text{for any } t \in [t_0,\tau], |x^h(t)-x^0(t)| \leq \epsilon$$

What is the behavior of  $\delta$  when au goes to infinity?



The solution  $x^h(t)$  to the IVP is

$$x^h(t) = (1+h)e^t.$$

Hence,

$$|x^h(t) - x^0(t)| = |(1+h)e^t - e^t| = |h|e^t$$
  $\max_{t \in [0, au]} |x^h(t) - x^0(t)| = |h|e^ au.$ 

Thus, the smallest  $\delta$  such that  $|h| \leq \delta$  yields

$$\max_{t \in [0, au]} |x^h(t) - x^0(t)| \leq \epsilon.$$

is 
$$\delta = \varepsilon e^{-\tau}$$
 .

For any  $\varepsilon > 0$ ,

$$\lim_{ au o +\infty} \delta = 0.$$



#### Consider the IVP

$$\dot{x}=\sqrt{|x|},\ x(0)=x_0\in\mathbb{R}.$$

Solve numerically this IVP for  $t \in [0,1]$  and  $x_0 = 0$  and plot the result.

Then, solve it again for  $x_0=0.1, x_0=0.01,$  etc. and plot the results.

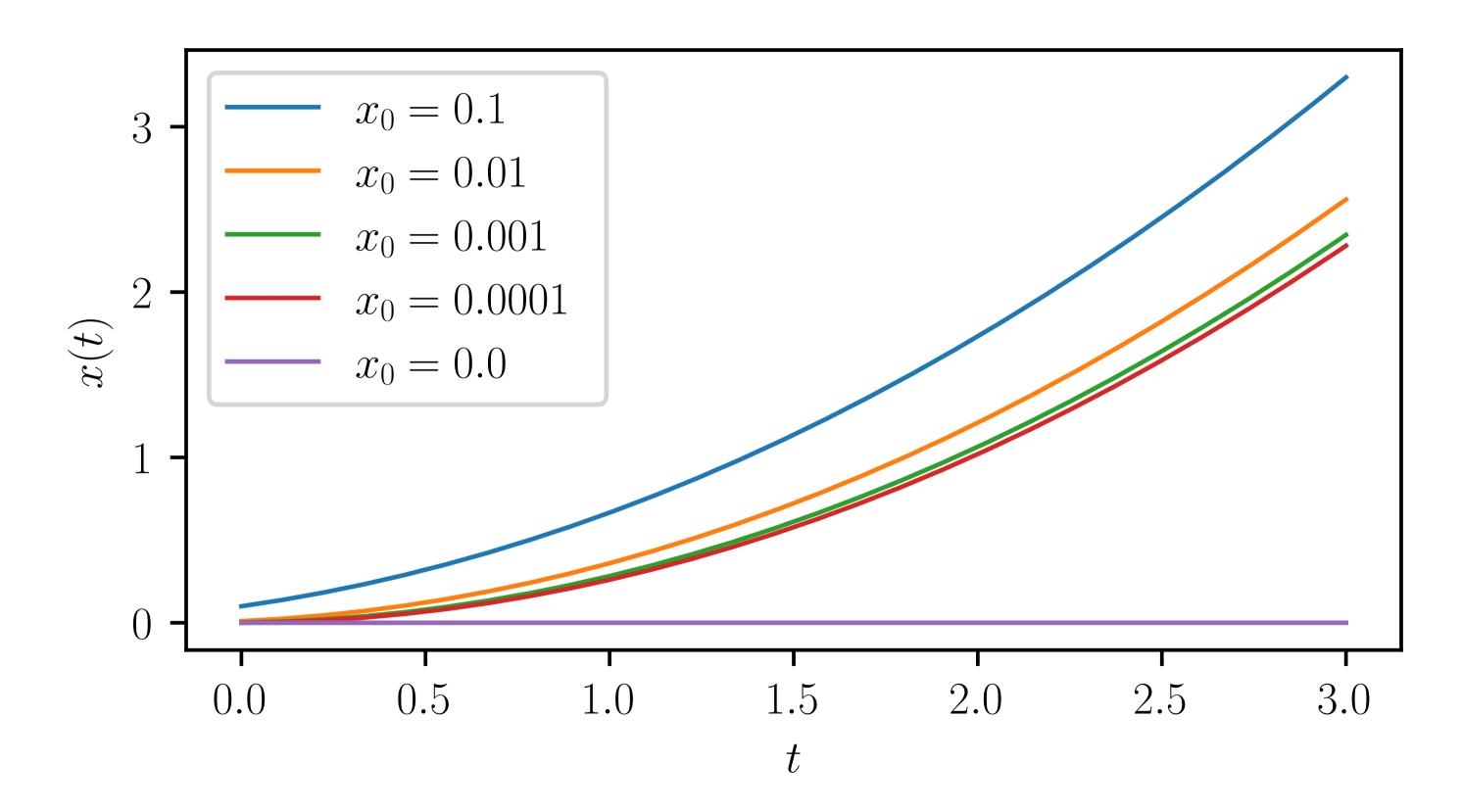
Does the solution seem to be continuous with respect to the initial value?

Explain this experimental result.

# **CONTINUITY ISSUES**

```
def fun(t, y):
    x = y[0]
    dx = sqrt(abs(y))
    return [dx]
tspan = [0.0, 3.0]
t = linspace(tspan[0], tspan[1], 1000)
```

```
figure()
for x0 in [0.1, 0.01, 0.001, 0.0001, 0.0]:
    r = solve_ivp(fun, tspan, [x0],
        dense_output=True)
    plot(t, r["sol"](t)[0],
         label=f''$x_0 = {x0}$")
xlabel("$t$"); ylabel("$x(t)$")
legend()
```



The solution does not seem to be continuous with respect to the initial value since the graph of the solution seems to have a limit when  $x_0 \to 0^+$ , but this limit is different from x(t) = 0 which is the numerical solution when  $x_0 = 0$ .

3. 🔓

The jacobian matrix of the vector field is not defined when x=0, thus the continuity was not guaranted to begin with. Actually, uniqueness of the solution does not even hold here, see  $\bigcirc$  Non-Uniqueness. The function x(t)=0 is valid when  $x_0=0$ , but so is

$$x(t)=rac{1}{4}t^2$$

and the numerical solution seems to converge to the second one when  $x_0 o 0^+.$ 



#### Consider the system

$$\dot{x} = \alpha x - \beta xy$$
 $\dot{y} = \delta xy - \gamma y$ 

where  $\alpha$ ,  $\beta$ ,  $\delta$  and  $\gamma$  are positive.

Prove that the system is well-posed.

Prove that all maximal solutions such that x(0)>0 and y(0)>0 are global and satify x(t)>0 and y(t)>0 for every  $t\geq 0$ .

**Hint**  $\nearrow$  . Compute the ODE satisfied by  $u=\ln x$  and  $v=\ln y$  and then the derivative w.r.t. time of

$$V := \delta e^u - \gamma u + \beta e^v - \alpha v.$$

# PREY-PREDATOR



The jacobian matrix of the system vector field

$$f(x,y) = (\alpha x - \beta xy, \delta xy - \gamma y)$$

is defined and continuous:

$$rac{\partial f}{\partial (x,y)} = egin{bmatrix} lpha - eta y & -eta x \ \delta y & \delta x - \gamma \end{bmatrix}$$

thus the sytem is well-posed.



The (continuously differentiable) change of variable

$$F:(x,y)\mapsto (u,v):=(\ln x,\ln y)$$

is a bijection between  $]0,+\infty[^2$  and  $\mathbb{R}^2$ .

#### Since

$$rac{d}{dt} \ln x = rac{\dot{x}}{x}, \; rac{d}{dt} \ln y = rac{\dot{y}}{y}$$

the prey-predator ODE is equivalent to

$$egin{array}{lll} \dot{u} &=& lpha - eta e^v \ \dot{v} &=& \delta e^u - \gamma \end{array}$$

#### Accordingly,

$$egin{aligned} rac{d}{dt}V &= \delta e^u \dot{u} - \gamma \dot{u} + \beta e^v \dot{v} - lpha \dot{v} \ &= (\delta e^u - \gamma) \dot{u} + (\beta e^v - lpha \dot{v}) \ &= (\delta e^u - \gamma) (lpha - eta e^v) + (eta e^v - lpha) (\delta e^u - \gamma) \ &= 0 \end{aligned}$$

Therefore V(u(t), v(t)) is constant.

#### Now, the function

$$\phi(u) := \delta e^u - \gamma u, \ \psi(v) := \beta e^v - \alpha v$$

are continuous and

$$\lim_{|u| o +\infty} \phi(u) = +\infty, \ \lim_{|v| o +\infty} \phi(v) = +\infty.$$

As 
$$V(u,v)=\phi(u)+\psi(v)$$
,

$$\lim_{\|(u,v)\| o +\infty}V(u,v)=+\infty.$$

Consequently, since V(x(t),y(t)) is constant, the solution (u(t),v(t)) cannot blow up (either in finite or infinite time).

Therefore the solution (u(t),v(t)) is global as is the solution in the original variables (x(t),y(t)).

Since  $(x,y)=F^{-1}(u,v)$  and the domain of F is  $]0,+\infty[^2,x(t)>0$  and y(t)>0 for any  $t\geq 0$ .