LINEAR MODELS

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CONTROL ENGINEERING WITH PYTHON

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SYMBOLS

🐍	Code	🔍	Worked Example
📈	Graph	🧩	Exercise
🏷️	Definition	💻	Numerical Method
💎	Theorem	🧮	Analytical Method
📝	Remark	🧠	Theory
ℹ️	Information	🗝️	Hint
⚠️	Warning	🔓	Solution

🐍 IMPORTS

```
from numpy import *
from numpy.linalg import *
from scipy.linalg import *
from matplotlib.pyplot import *
from mpl_toolkits.mplot3d import *
from scipy.integrate import solve_ivp
```

🐍 STREAMPLOT HELPER

```
def Q(f, xs, ys):
    X, Y = meshgrid(xs, ys)
    v = vectorize
    fx = v(lambda x, y: f([x, y])[0])
    fy = v(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

PREAMBLE

🏷️ NON-AUTONOMOUS SYSTEMS

Their structure is

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, that is

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n.$$

🏷️ INPUTS

The vector-valued u is the **system input**.

This quantity may depend on the time t

$$u : t \in \mathbb{R} \mapsto u(t) \in \mathbb{R}^m,$$

(actually it may also depend on some state, but we will adress this later).

A solution of

$$\dot{x} = f(x, u), \ x(t_0) = x_0$$

is merely a solution of

$$\dot{x} = h(t, x), \ x(t_0) = x_0,$$

where

$$h(t, x) := f(x, u(t)).$$

🏷️ OUTPUTS

We may complement the system dynamics with an equation

$$y = g(x, u) \in \mathbb{R}^p$$

The vector y refers to the **systems output**, usually the quantities that we can effectively measure in a system (the state x itself may be unknown).

LINEAR SYSTEMS

STANDARD FORM

Input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

MATRIX SHAPE

$$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}.$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

💎 WELL-POSEDNESS

When $u = 0$,

$$\dot{x} = Ax =: f(x) \;\Rightarrow\; \frac{\partial f}{\partial x}(x) = A$$

The vector field f is continuously differentiable

$\Rightarrow$ **The system is well-posed.**

💎 EQUILIBRIUM

When $u = 0$, since

$$\dot{x} = Ax =: f(x)$$

$$f(0) = A0 = 0$$

$\Rightarrow$ **the origin $x = 0$ is always an equilibrium.**

(the only one in the state space if A is invertible).

🤔 WHY “LINEAR” ?

Assume that:

- $\dot{x}_1 = Ax_1 + Bu_1, x_1(0) = x_{10},$
- $\dot{x}_2 = Ax_2 + Bu_2, x_2(0) = x_{20},$

Set

- $u_3 = \lambda u_1 + \mu u_2$ and
- $x_{30} = \lambda x_{10} + \mu x_{20}$.

for some λ and μ.

Then, if

$$x_3 = \lambda x_1 + \mu x_2,$$

we have

$$\dot{x}_3 = Ax_3 + Bu_3,\ x_3(0) = x_{30}.$$

💎 DYNAMICS DECOMPOSITION

The solution of

$$\dot{x} = Ax + Bu, \; x(0) = x_0$$

is the sum $x(t) = x_1(t) + x_2(t)$ where

- $x_1(t)$ is the solution to the **internal dynamics** and
- $x_2(t)$ is the solution to the **external dynamics**.

🏷️ INTERNAL/EXTERNAL

- The **internal dynamics** is controlled by the initial value x_0 only (there is no input, $u=0$).

$$\dot{x}_1 = Ax_1,\ x_1(0) = x_0,$$

- The **external dynamics** is controlled by the input $u(t)$ only (the system is initially at rest, $x_0 = 0$).

$$\dot{x}_2 = Ax_2 + Bu,\ x_2(0) = 0.$$

🏷️ LTI SYSTEMS

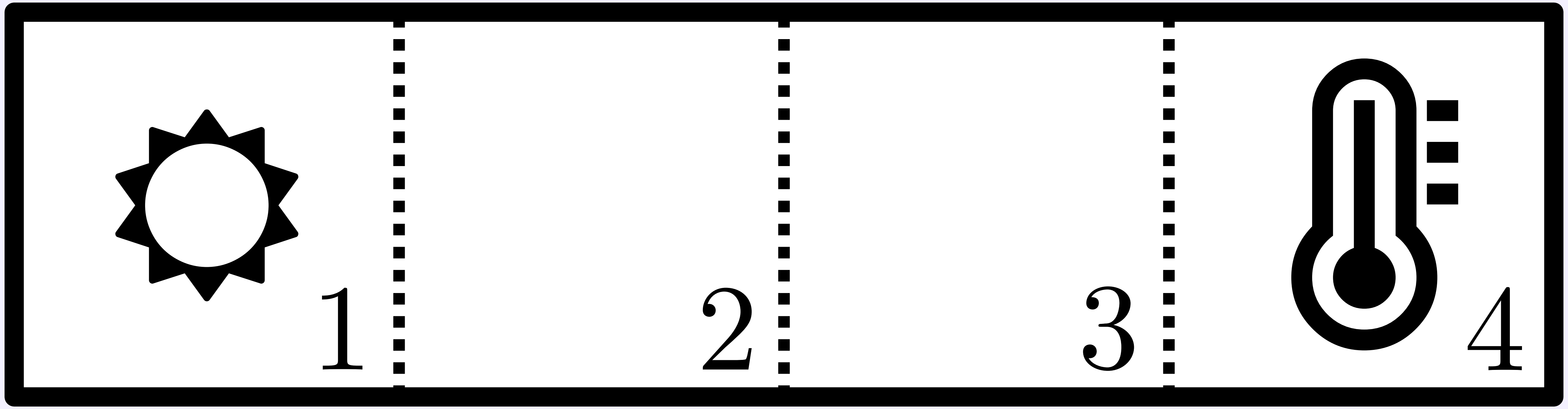
These systems are actually linear and **time-invariant** (hence **LTI**) systems. Time-invariant means that when $x(t)$ is a solution of

$$\dot{x} = Ax + Bu, \; x(0) = x_0,$$

then $x(t-t_0)$ is a solution of

$$\dot{x} = Ax + Bu(t-t_0), \; x(t_0) = x_0.$$

🔍 HEAT EQUATION



SIMPLIFIED MODEL

- Four cells numbered 1 to 4 are arranged in a row.
- The first cell has a heat source, the last one a temperature sensor.
- The heat sink/source is increasing the temperature of its cell of u degrees by second.
- If the temperature of a cell is T and the one of a neighbor is T_n, T increases of $T_n - T$ by second.

Given the geometric layout:

- $dT_1/dt = u + (T_2 - T_1)$
- $dT_2/dt = (T_1 - T_2) + (T_3 - T_2)$
- $dT_3/dt = (T_2 - T_3) + (T_4 - T_3)$
- $dT_4/dt = (T_3 - T_4)$
- $y = T_4$

Set $x = (T_1, T_2, T_3, T_4)$.

The model is linear and its standard matrices are:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 0 \end{bmatrix}$$

🧭 LINEARIZATION

NONLINEAR TO LINEAR

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x, u) \end{aligned}$$

Assume that x_e is an equilibrium when $u = u_e$ (cst):

$$f(x_e, u_e) = 0$$

and let

$$y_e := g(x_e, u_e).$$

Define the error variables

- $\Delta x := x - x_e$,
- $\Delta u := u - u_e$ and
- $\Delta y := y - y_e$.

As long as the error variables stay small

$$f(x,u) \simeq \overbrace{f(x_e,u_e)}^{0} + \frac{\partial f}{\partial x}(x_e,u_e)\Delta x + \frac{\partial f}{\partial u}(x_e,u_e)\Delta u$$

$$g(x,u) \simeq \overbrace{g(x_e,u_e)}^{y_e} + \frac{\partial g}{\partial x}(x_e,u_e)\Delta x + \frac{\partial g}{\partial u}(x_e,u_e)\Delta u$$

Hence, the error variables satisfy *approximately*

$$\begin{array}{rcl} d(\Delta x)/dt & = & A\Delta x + B\Delta u \\ \Delta y & = & C\Delta x + D\Delta u \end{array}$$

with

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u} \\ \hline \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial u} \end{array}\right](x_e, u_e)$$

🔍 EXAMPLE

The system

$$\begin{aligned} \dot{x} &= -2x + y^3 \\ \dot{y} &= -2y + x^3 \end{aligned}$$

has an equilibrium at $(0, 0)$.

The corresponding error variables satisfy $\Delta x = x$ and $\Delta y = y$, thus

$$\frac{d\Delta x}{dt} = \dot{x} = -2x + y^3 = -2\Delta x + (\Delta y)^3 \approx -2\Delta x$$

$$\frac{d\Delta y}{dt} = \dot{y} = -2y + x^3 = -2\Delta y + (\Delta x)^3 \approx -2\Delta y$$

$$
\begin{aligned}
\dot{x} &= -2x + y^3 \\
\dot{y} &= -2y + x^3
\end{aligned}
$$

$\rightarrow$

$$
\begin{aligned}
\dot{x} &\approx -2x \\
\dot{y} &\approx -2y
\end{aligned}
$$

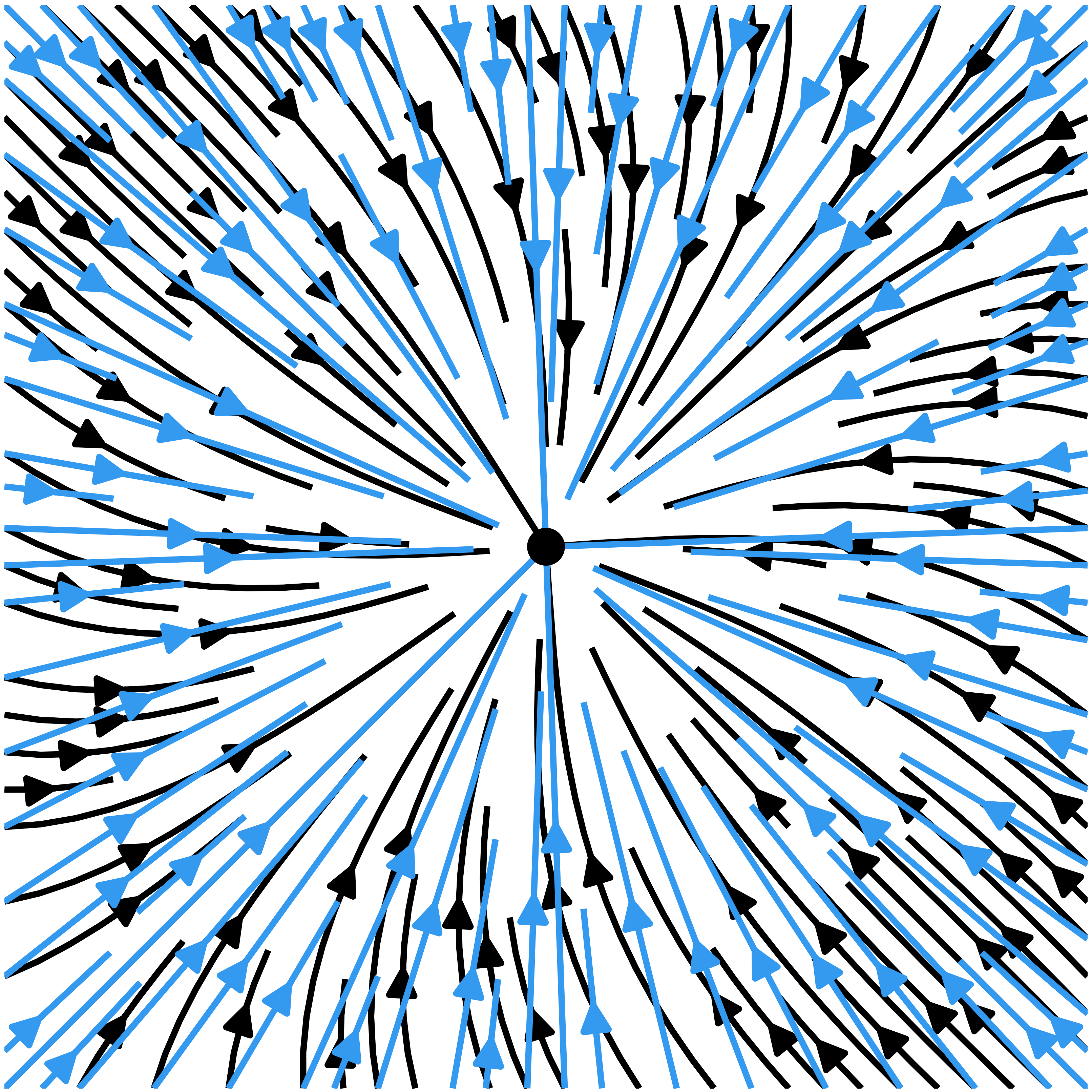
🐍 VECTOR FIELDS

```
def f(xy):
    x, y = xy
    dx = -2*x + y**3
    dy = -2*y + x**3
    return array([dx, dy])
```

```python
def f1(xy):
    x, y = xy
    dx = -2*x
    dy = -2*y
    return array([dx, dy])
```

📈 STREAM PLOT

```
figure()
x = y = linspace(-1.0, 1.0, 1000)
streamplot(*Q(f, x, y), color="k")
blue_5 = "#339af0"
streamplot(*Q(fl, x, y), color=blue_5)
plot([0], [0], "k.", ms=10.0)
axis("square")
axis("off")
```



🔍 LINEARIZATION

Consider

$$\dot{x} = -x^2 + u,\ y = xu$$

If we set $u_e = 1$, the system has an equilibrium at $x_e = 1$ (and also $x_e = -1$ but we focus on the former) and the corresponding y is $y_e = x_e u_e = 1$.

Around this configuration $(x_e, u_e) = (1, 1)$, we have

$$\frac{\partial(-x^2 + u)}{\partial x} = -2x_e = -2,\ \frac{\partial(-x^2 + u)}{\partial u} = 1,$$

and

$$\frac{\partial xu}{\partial x} = u_e = 1,\ \frac{\partial xu}{\partial u} = x_e = 1.$$

Thus, the approximate, linearized dynamics around this equilibrium is

$$\begin{aligned} d(x-1)/dt &= -2(x-1) + (u-1) \\ y-1 &= (x-1) + (u-1) \end{aligned}$$

💎 ASYMPTOTIC STABILITY

The equilibrium 0 is locally asymptotically stable for

$$\frac{d\Delta x}{dt} = A \Delta x$$

where $A = \partial f(x_e, u_e) / \partial x$.

$\Rightarrow$

The equilibrium x_e is locally asymptotically stable for

$$\dot{x} = f(x, u_e).$$

⚠️ CONVERSE RESULT

- The converse is not true : the nonlinear system may be asymptotically stable but not its linearized approximation (e.g. consider $\dot{x} = -x^3$).
- If we replace local **asymptotic stability** with local **exponential stability**, the requirement that locally

$$\|x(t) - x_e\| \leq Ae^{-\sigma t}\|x(0) - x_e\|$$

for some $A > 0$ and $\sigma > 0$, then it works.

🧩 PENDULUM

A pendulum submitted to a torque c is governed by

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin \theta = c.$$

We assume that only the angle θ is measured.

1. 🧮

Let $x = (\theta, \dot{\theta})$, $u = c$ and $y = \theta$.

What are the function f and g that determine the nonlinear dynamics of the pendulum?

2. 🧮

Show that for any angle θ_e there is a constant value c_e of the torque such that $x_e = (\theta_e, 0)$ is an equilibrium.

3. 🧮

Compute the linearized dynamics of the pendulum around this equilibrium and put it in the standard form (compute A, B, C and D).

🔓 PENDULUM

1. 🔓

The 2nd-order differential equation

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = c.$$

is equivalent to the first-order differential equation

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -(b/m\ell^2)\omega - (g/\ell)\sin\theta + c/m\ell^2\end{bmatrix}$$

Hence, with $x = (\theta, \dot{\theta})$, $u = c$ and $y = \theta$, we have

$$
\begin{array}{rcl}
\dot{x} & = & f(x, u) \\
y & = & g(x, u)
\end{array}
$$

with

$$
\begin{array}{rcl}
f((\theta, \omega), c) & = & \left(\omega, -(b/m\ell^2)\omega - (g/\ell)\sin\theta + c/m\ell^2\right) \\
g((\theta, \omega), c) & = & \theta.
\end{array}
$$

2. 🔓

Let θ_e in $\mathbb{R}$. If $c = c_e$, the state $x_e := (\theta_e, 0)$ is an equilibrium if and only if $f((\theta_e, 0), c_e) = 0$, that is

$$\begin{bmatrix} 0 \\ 0 - (g/\ell)\sin\theta_e + c_e/m\ell^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which holds if and only if

$$c_e = mg\ell \sin\theta_e.$$

3. 🔓

We have

$$A = \frac{\partial f}{\partial x}(x_e, c_e) = \begin{bmatrix} 0 & 1 \\ -(g/\ell)\cos\theta_e & -(b/m\ell^2) \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(x_e, u_e) = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix}$$

$$C = \frac{\partial g}{\partial x_e}(x_e, u_e) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; D = \frac{\partial g}{\partial u_e}(x_e, u_e) = 0$$

Thus,

$$\begin{aligned}
\frac{d}{dt}\Delta\theta &\approx \Delta\omega \\
\frac{d}{dt}\Delta\omega &\approx -(g/\ell)\cos(\theta_e)\Delta\theta - (b/m\ell^2)\Delta\omega + \Delta c/m\ell^2
\end{aligned}$$

and obviously, as far as the output goes,

$$\Delta\theta \approx \Delta\theta.$$