ASYMPTOTIC BEHAVIOR

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CONTROL ENGINEERING WITH PYTHON

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SYMBOLS

🐍	Code	🔍	Worked Example
📈	Graph	🧩	Exercise
🏷️	Definition	💻	Numerical Method
💎	Theorem	🧮	Analytical Method
📝	Remark	🧠	Theory
ℹ️	Information	🗝️	Hint
⚠️	Warning	🔓	Solution

🐍 IMPORTS

```
from numpy import *
from numpy.linalg import *
from scipy.linalg import *
from matplotlib.pyplot import *
from mpl_toolkits.mplot3d import *
from scipy.integrate import solve_ivp
```

🐍 STREAMPLOT HELPER

```
def Q(f, xs, ys):
    X, Y = meshgrid(xs, ys)
    v = vectorize
    fx = v(lambda x, y: f([x, y])[0])
    fy = v(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

ℹ️ ASSUMPTION

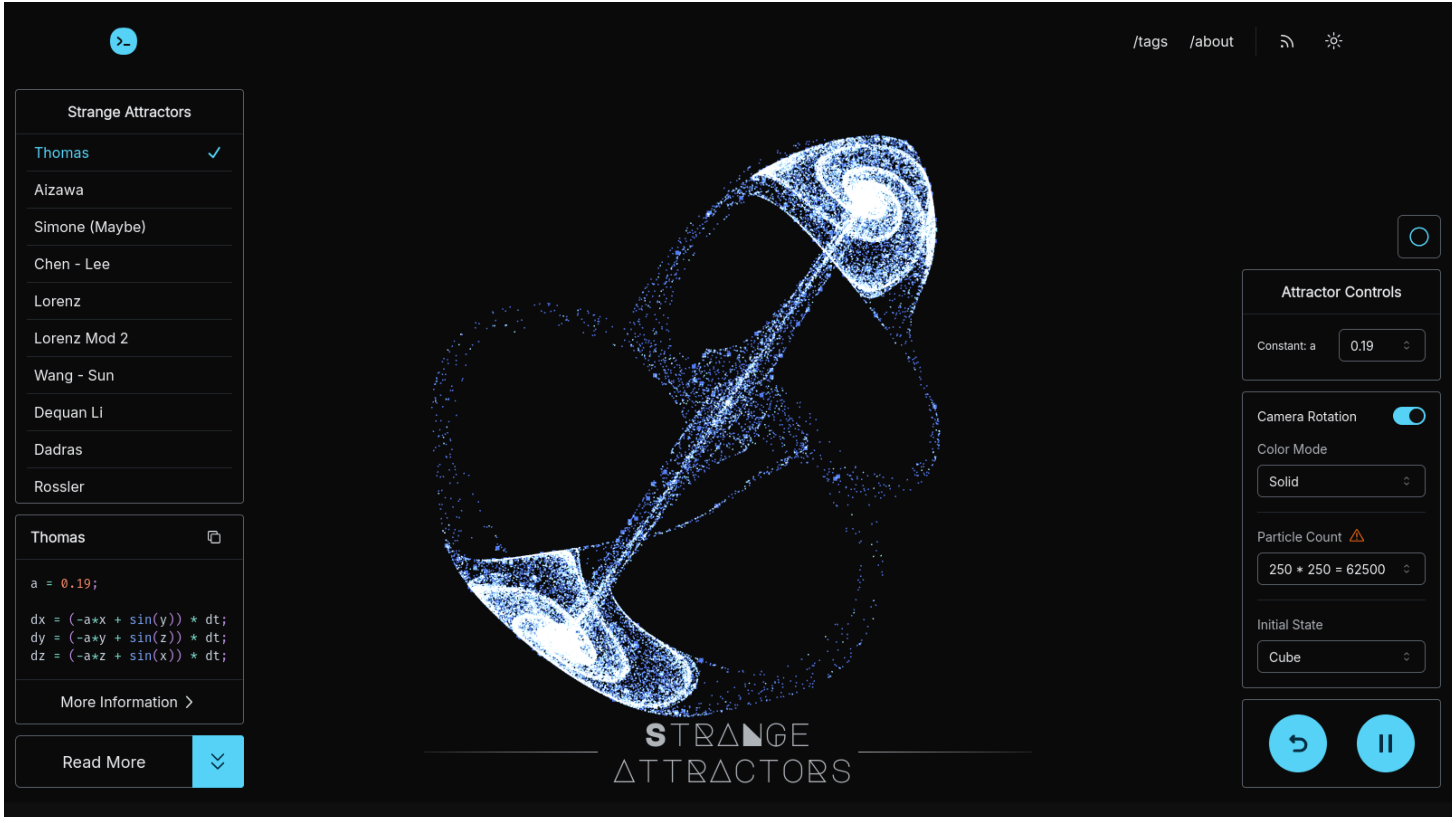
From now on, we only deal with well-posed systems.

🏷️ ASYMPTOTIC

Asymptotic = Long-Term: when $t \to +\infty$

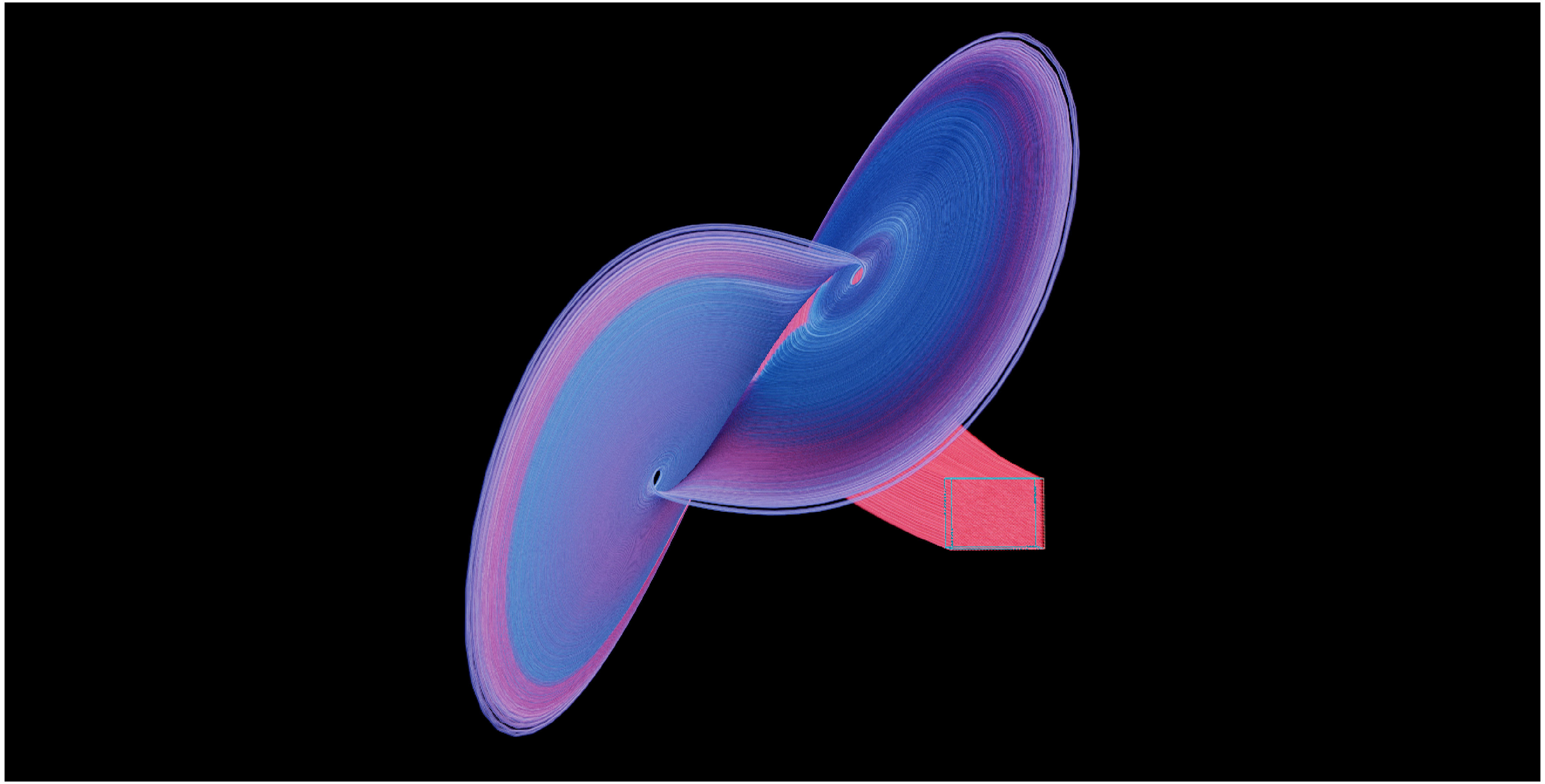
⚠️

Even simple dynamical systems may exhibit **complex asymptotic behaviors.**


/tags
/about
Strange Attractors
Thomas
Aizawa
Simone (Maybe)
Chen - Lee
Lorenz
Lorenz Mod 2
Wang - Sun
Dequan Li
Dadras
Rossler
Thomas
a = 0.19;
dx = (-a*x + sin(y)) * dt;
dy = (-a*y + sin(z)) * dt;
dz = (-a*z + sin(x)) * dt;
More Information
Read More
STRANGE
ATTRACTORS
Attractor Controls
Constant: a
0.19
Camera Rotation
Color Mode
Solid
Particle Count
250 * 250 = 62500
Initial State
Cube

LORENZ SYSTEM

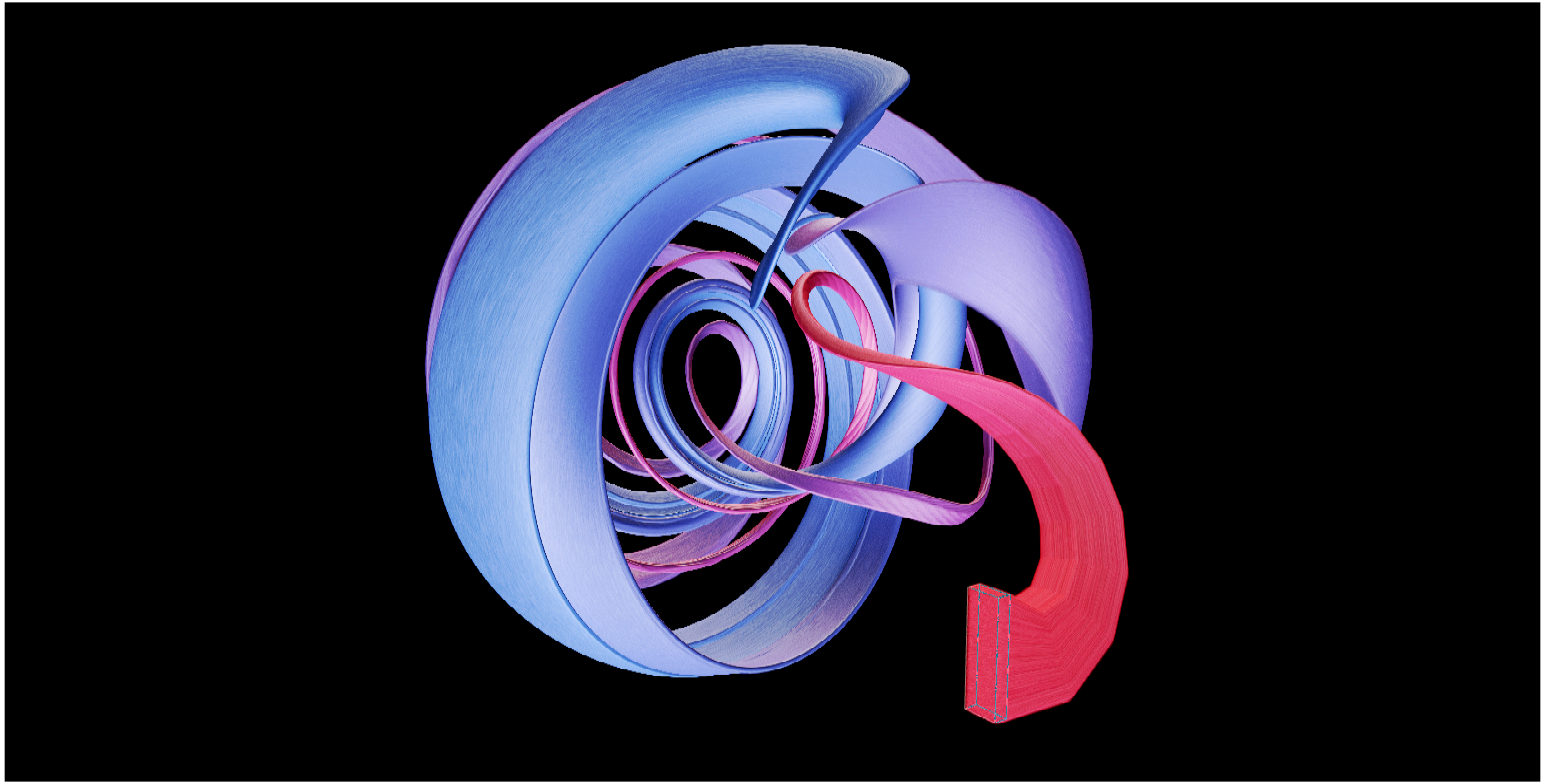
$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) \\
\dot{z} &= xy - \beta z
\end{aligned}
$$



Visualized with Fibre

HADLEY SYSTEM

$$\begin{aligned}\dot{x} &= -y^2 - z^2 - ax + af \\ \dot{y} &= xy - bxz - y + g \\ \dot{z} &= bxy + xz - z\end{aligned}$$



Visualized with Fibre

🏷️ EQUILIBRIUM

An **equilibrium** of system $\dot{x} = f(x)$ is a state x_e such that the maximal solution $x(t)$ such that $x(0) = x_e$

- is global and,
- is $x(t) = x_e$ for any $t > 0$.

💎 EQUILIBRIUM

The state x_e is an equilibrium of $\dot{x} = f(x)$

$$\iff$$

$$f(x_e) = 0.$$

STABILITY

About the long-term behavior of solutions.

- “Stability” subtle concept,
- “Asymptotic Stability” simpler (and stronger),
- “Attractivity” simpler yet, (but often too weak).

ATTRACTIVITY

Context: system $\dot{x} = f(x)$ with equilibrium x_e.

🏷️ GLOBAL ATTRACTIVITY

The equilibrium x_e is **globally attractive** if for every x_0, the maximal solution $x(t)$ such that $x(0) = x_0$

- is global and,
- $x(t) \to x_e$ when $t \to +\infty$.

🏷️ LOCAL ATTRACTIVITY

The equilibrium x_e is **locally attractive** if for every x_0 **close enough to** x_e, the maximal solution $x(t)$ such that $x(0) = x_0$

- is global and,
- $x(t) \to x_e$ when $t \to +\infty$.

🔍 GLOBAL ATTRACTIVITY

The system

$$
\begin{aligned}
\dot{x} &= -2x + y \\
\dot{y} &= -2y + x
\end{aligned}
$$

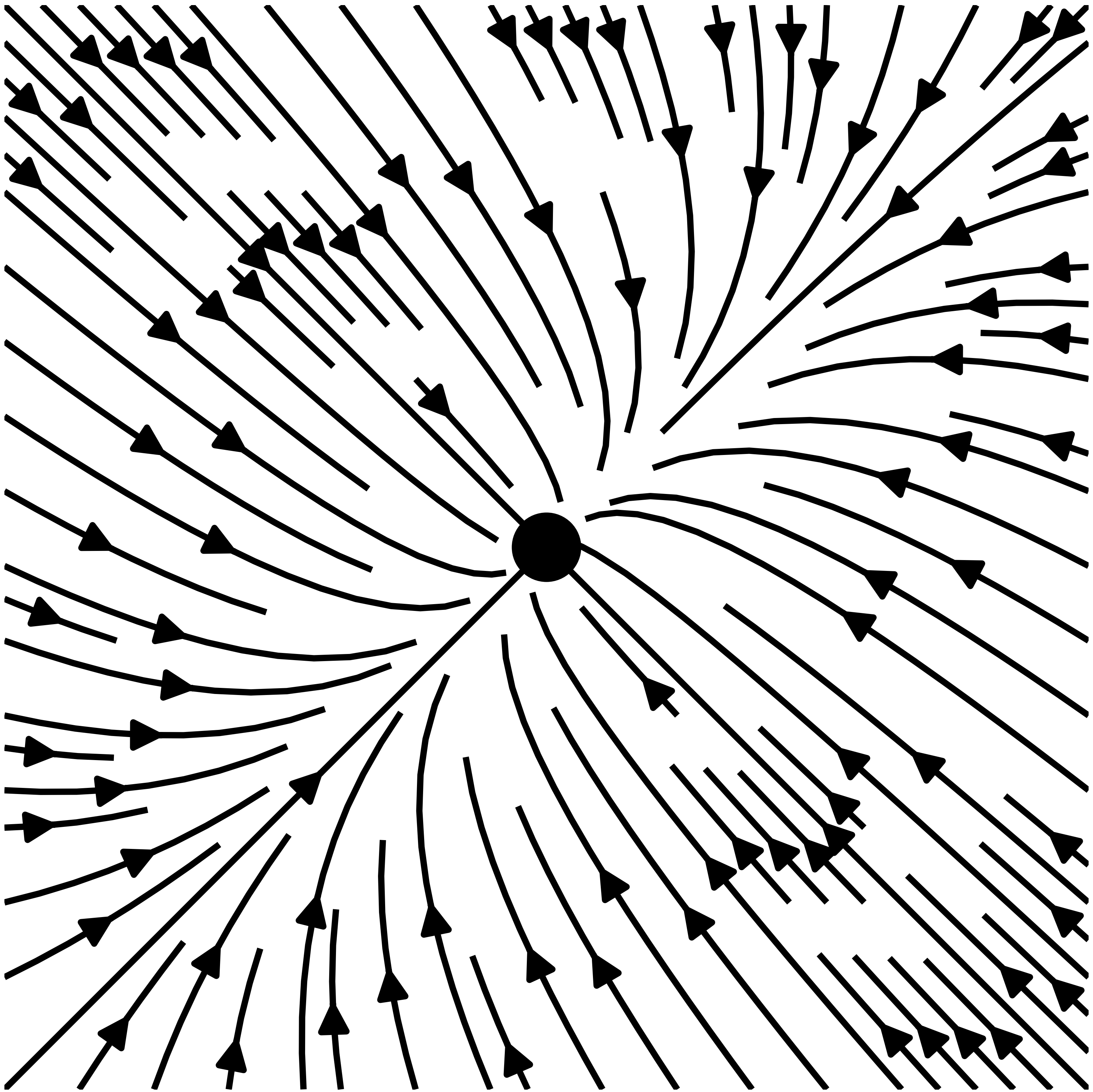
- is well-posed,
- has an equilibrium at $(0, 0)$.

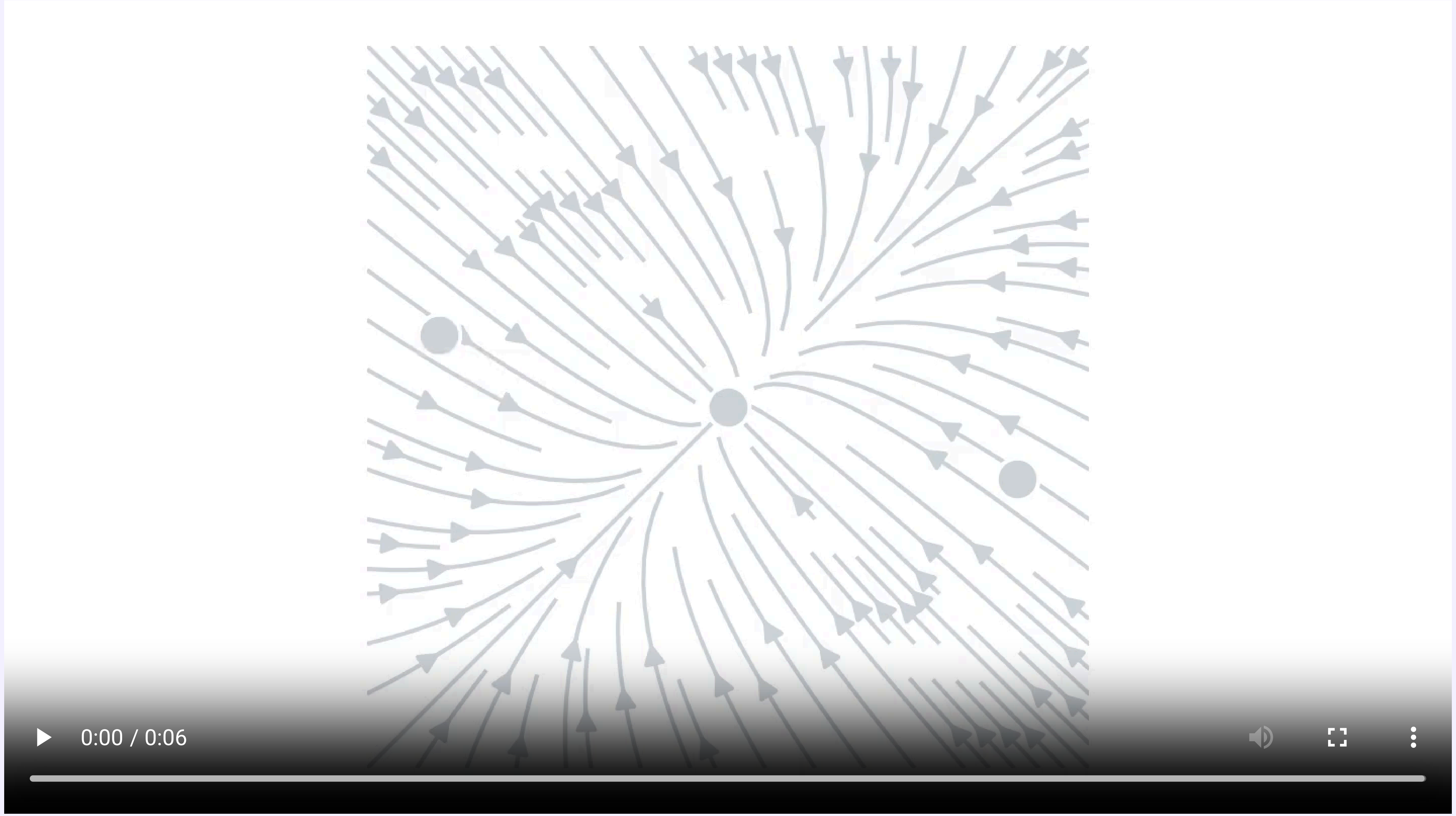
🐍 VECTOR FIELD

```
def f(xy):
    x, y = xy
    dx = -2*x + y
    dy = -2*y + x
    return array([dx, dy])
```

📈 STREAM PLOT

```
figure()
x = y = linspace(-5.0, 5.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([0], [0], "k.", ms=20.0)
axis("square")
axis("off")
```

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🔍 LOCAL ATTRACTIVITY

The system

$$\begin{array}{rcl}\dot{x} & = & -2x + y^3 \\ \dot{y} & = & -2y + x^3\end{array}$$

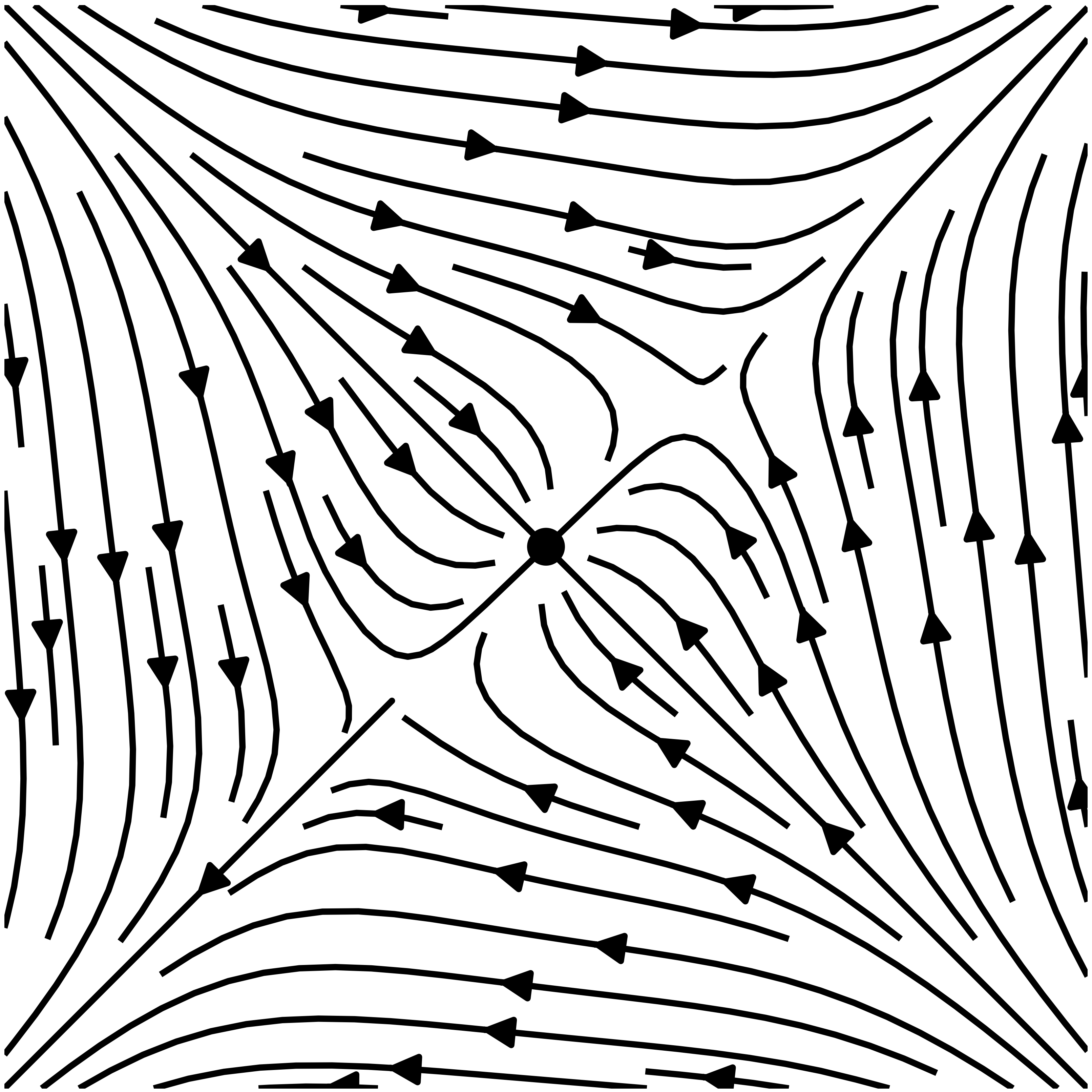
- is well-posed,
- has an equilibrium at $(0, 0)$.

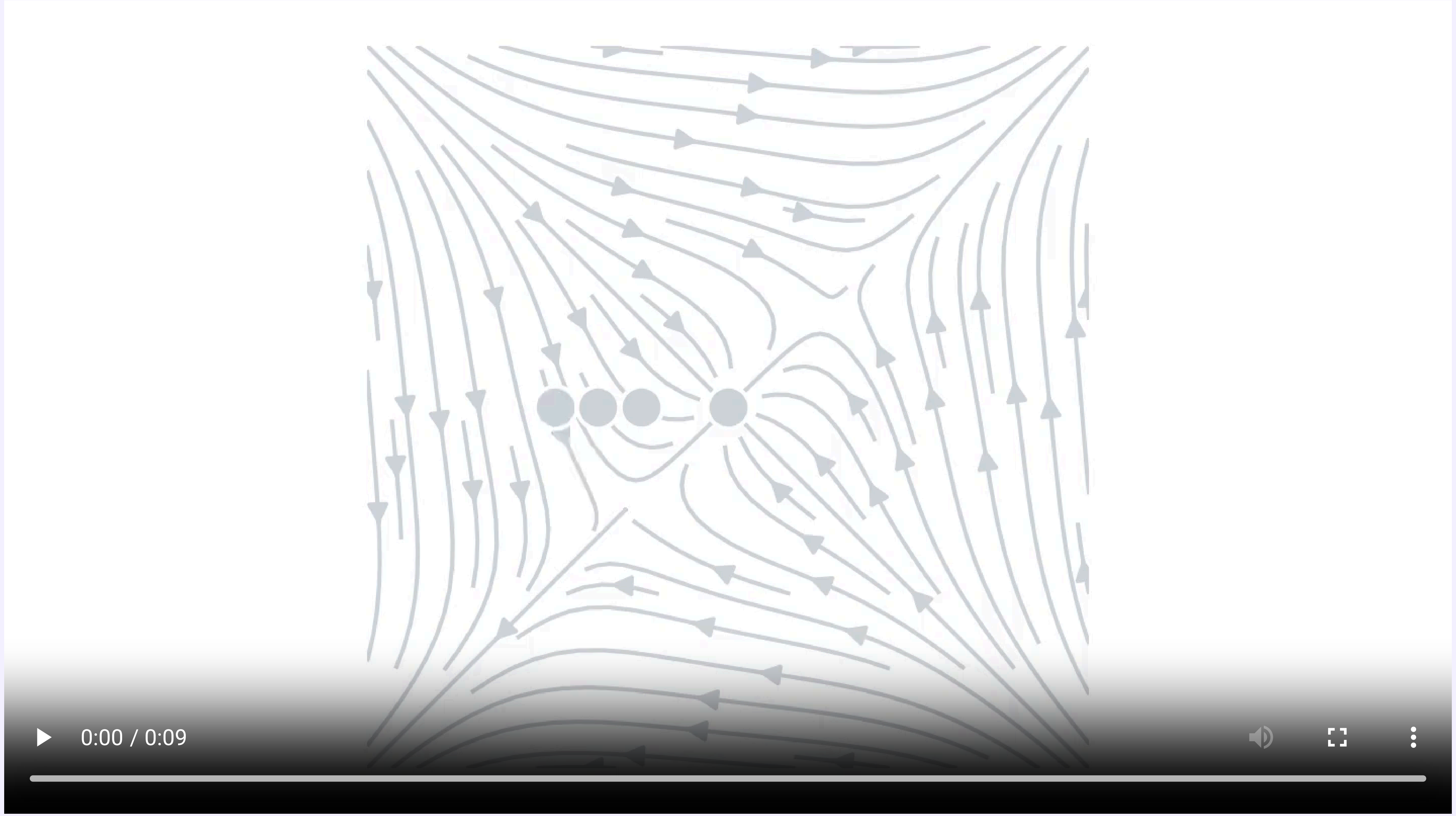
🐍 VECTOR FIELD

```
def f(xy):
    x, y = xy
    dx = -2*x + y**3
    dy = -2*y + x**3
    return array([dx, dy])
```

📈 STREAM PLOT

```
figure()
x = y = linspace(-5.0, 5.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([0], [0], "k.", ms=10.0)
axis("square")
axis("off")
```

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🔍 NO ATTRACTIVITY

The system

$$\begin{array}{rcl}\dot{x} & = & -2x + y \\ \dot{y} & = & 2y - x\end{array}$$

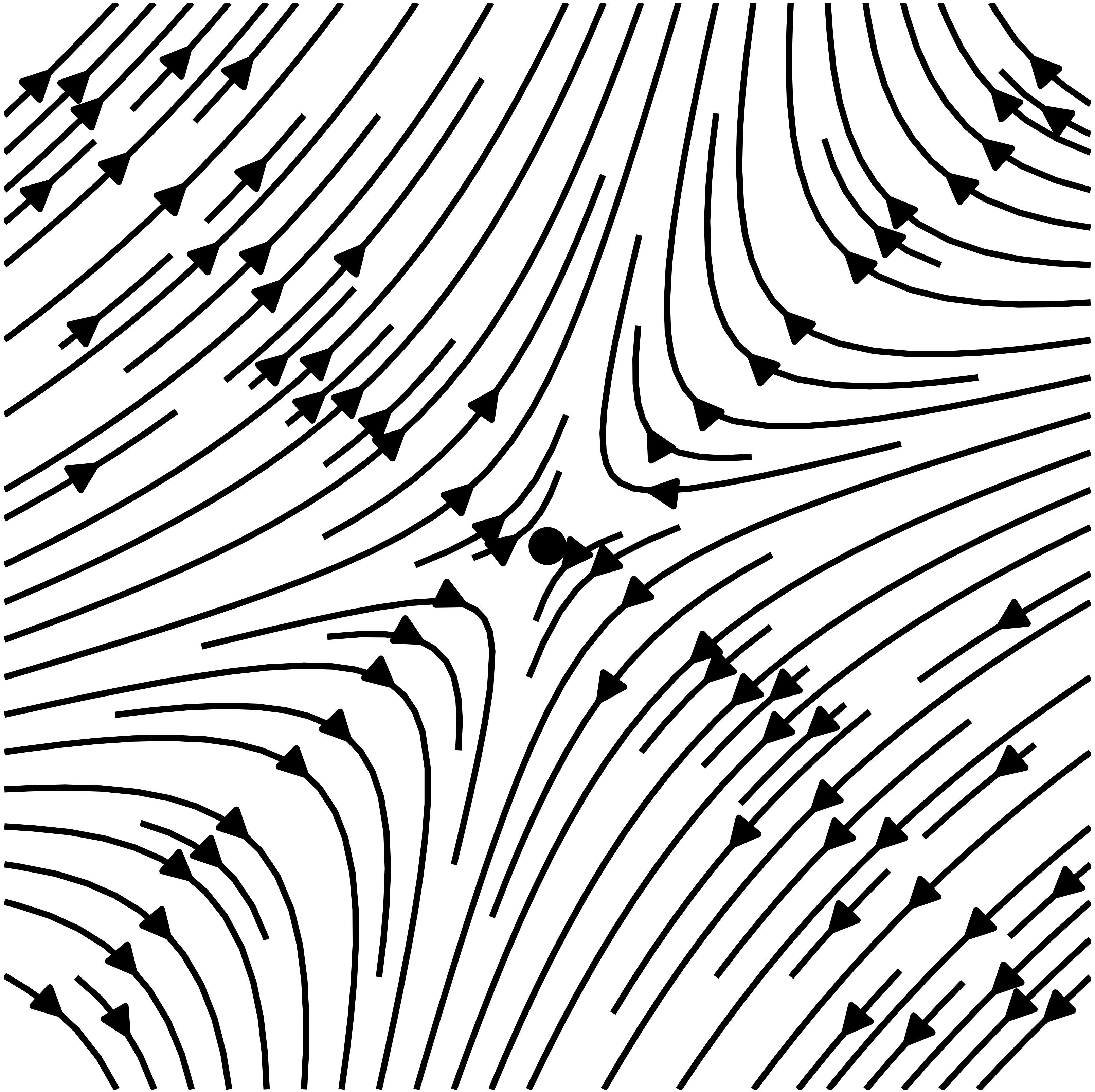
- is well-posed,
- has a (unique) equilibrium at $(0, 0)$.

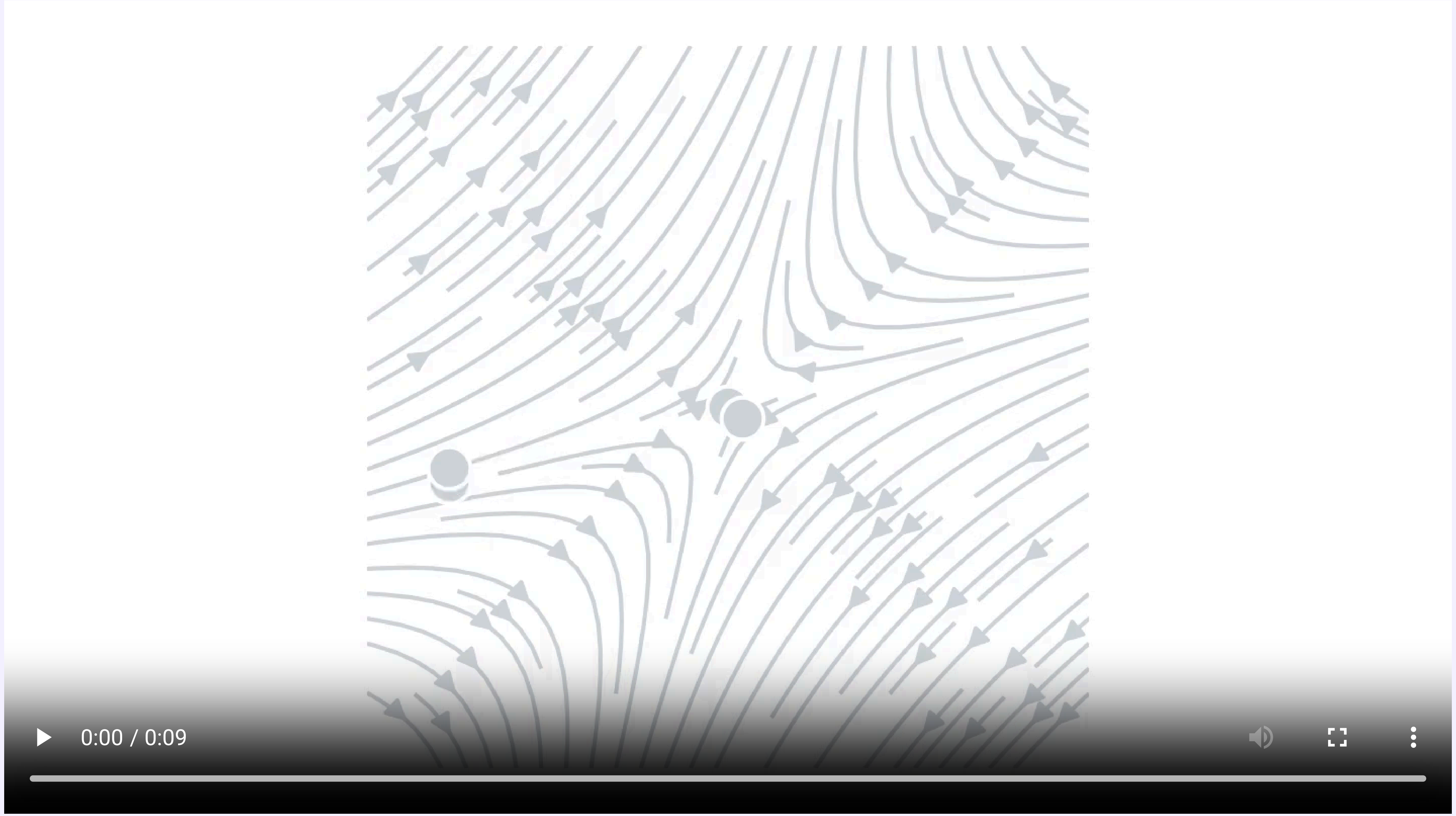
🐍 VECTOR FIELD

```python
def f(xy):
    x, y = xy
    dx = -2*x + y
    dy =  2*y - x
    return array([dx, dy])
```

📈 STREAM PLOT

```
figure()
x = y = linspace(-5.0, 5.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([0], [0], "k.", ms=10.0)
axis("square")
axis("off")
```

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🧩 PENDULUM

The pendulum is governed by the equation

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = 0$$

where $m > 0, \ell > 0, g > 0$ and $b \geq 0$.

1. 🧮

Compute the equilibria of this system.

2. 🧠

Can any of these equilibria be **globally** attractive?

3. 📈

Assume that $m = 1, \ell = 1, g = 9.81$ and $b = 1$.

Make a stream plot of the system.

4. 🔬

Determine which equilibria are locally attractive.

5. 📈 FRICTIONLESS PENDULUM

Assume now that $b = 0$.

Make a stream plot of the system.

6. 🧮 🧠

Prove that the equilibrium at $(0, 0)$ is not locally attractive.

🗝️ **Hint.** Study how the total mechanical energy E

$$E(\theta, \dot{\theta}) := m\ell^2\dot{\theta}^2/2 - mg\ell\cos\theta$$

evolves in time.

🔓 PENDULUM

1. 🔓

The 2nd-order differential equations of the pendulum are equivalent to the first order system

$$\left| \begin{array}{rcl} \dot{\theta} & = & \omega \\ \dot{\omega} & = & (-b/m\ell^2)\omega - (g/\ell)\sin\theta \end{array} \right.$$

Thus, the system state is $x = (\theta, \omega)$ and is governed by $\dot{x} = f(x)$ with

$$f(\theta, \omega) = (\omega, (-b/m\ell^2)\omega - (g/\ell)\sin\theta).$$

Hence, the state (θ, ω) is a solution to $f(\theta, \omega) = 0$ if and only if $\omega = 0$ and $\sin\theta = 0$. In other words, the equilibria of the system are characterized by $\theta = k\pi$ for some $k \in \mathbb{Z}$ and $\omega (= \dot{\theta}) = 0$.

2. 🔓

Since there are several equilibria, none of them can be globally attractive.

Indeed let x_1 be a globally attractive equilibrium and assume that x_2 is any other equilibrium. By definition, the maximal solution $x(t)$ such that $x(0) = x_2$ is $x(t) = x_2$ for every $t \geq 0$. On the other hand, since x_1 is globally attractive, it also satisfies $x(t) \to x_1$ when $t \to +\infty$, hence there is a contradiction.

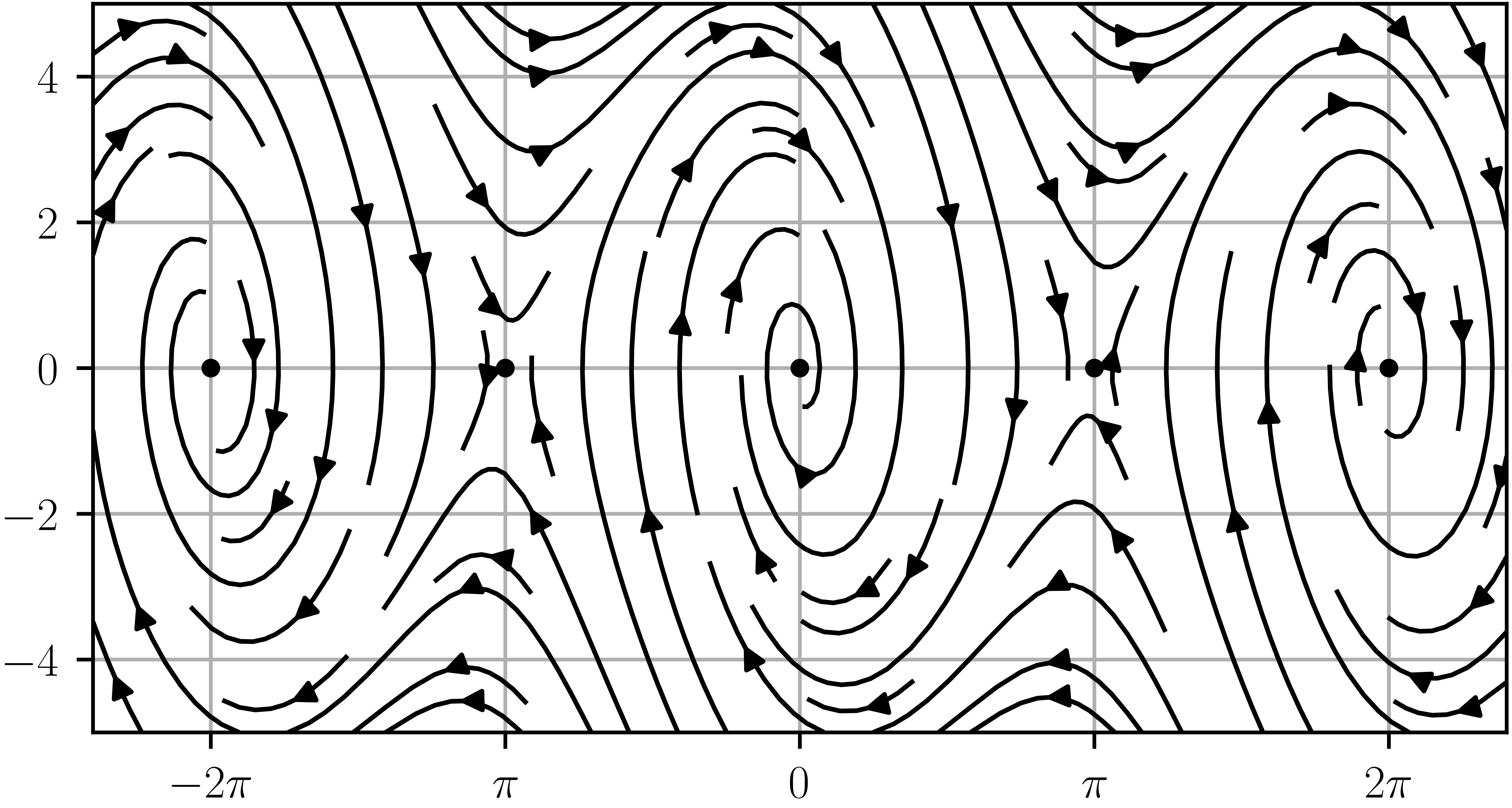
Thus, x_1 is the only possible equilibrium.

3. 🔓

```python
m = l = b = 1; g=9.81

def f(theta_omega):
    theta, omega = theta_omega
    d_theta = omega
    d_omega = - b / (m * l * l) * omega
    d_omega -=  (g / l) * sin(theta)
    return (d_theta, d_omega)
```

```
figure()
theta = linspace(-2*pi*(1.2), 2*pi*(1.2), 1000)
d_theta = linspace(-5.0, 5.0, 1000)
streamplot(*Q(f, theta, d_theta), color="k")
plot([-2*pi, -pi, 0 ,pi, 2*pi], 5*[0.0], "k.")
xticks([-2*pi, -pi, 0 ,pi, 2*pi],
[r"$-2\pi$", r"$\pi$", r"$0$", r"$\pi$", r"$2\pi$"])
grid(True)
```


4
2
0
−2
−4
−2π
π
0
π
2π

4. 🔓

From the streamplot, we see that the equilibria

$$(\theta, \dot{\theta}) = (2k\pi, 0), \; k \in \mathbb{Z}$$

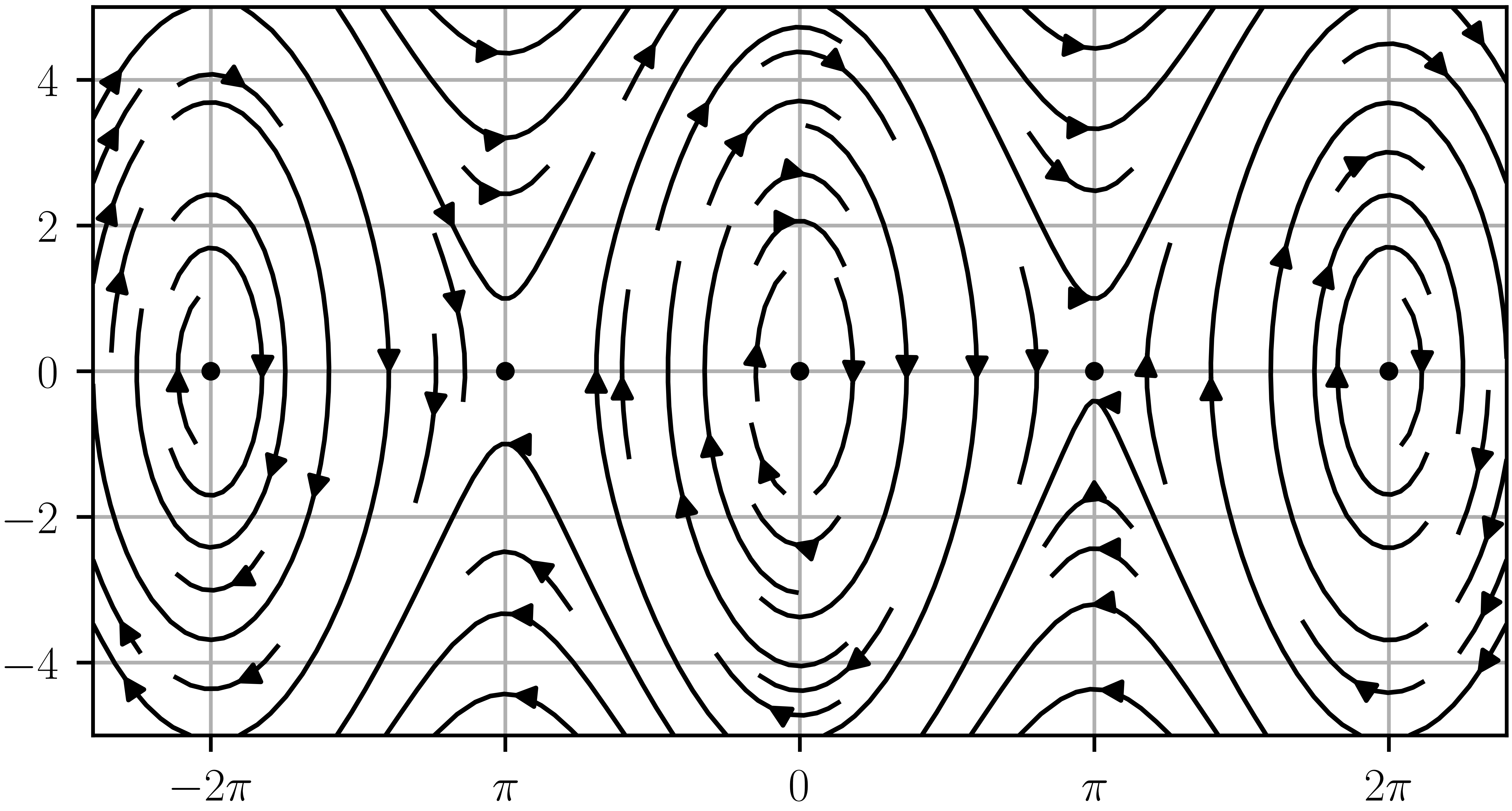
are asymptotically stable, but that the equilibria

$$(\theta, \dot{\theta}) = (2(k+1)\pi, 0), \; k \in \mathbb{Z}$$

are not (they are not locally attractive).

5. 🔓

```
b = 0
figure()
streamplot(*Q(f, theta, d_theta), color="k")
plot([-2*pi, -pi, 0 ,pi, 2*pi], 5*[0.0], "k.")
xticks([-2*pi, -pi, 0 ,pi, 2*pi],
[r"$-2\pi$", r"$\pi$", r"$0$", r"$\pi$", r"$2\pi$"])
grid(True)
```


4
2
0
−2
−4
−2π
π
0
π
2π

6. 🔓

$$
\begin{aligned}
\dot{E} &= \frac{d}{dt}\left(m\ell^2\dot{\theta}^2/2 - mg\ell\cos\theta\right) \\
&= m\ell^2\ddot{\theta}\dot{\theta} + mg\ell(\sin\theta)\dot{\theta} \\
&= \left(m\ell^2\ddot{\theta} + mg\ell\sin\theta\right)\dot{\theta} \\
&= \left(-b\dot{\theta}\right)\dot{\theta} \\
&= 0
\end{aligned}
$$

Therefore, $E(t)$ is constant.

On the other hand,

$$\min \{E(\theta, \dot{\theta}) \mid (\theta, \dot{\theta}) \in \mathbb{R}^2\} = E(0,0) = -mgl.$$

Moreover, this minimum is locally strict. Precisely, for any $0 < |\theta| < \pi$,

$$E(0,0) < E(\theta, \dot{\theta}).$$

If the origin was locally attractive, for any $\theta(0)$ and $\dot{\theta}(0)$ small enough, we would have

$$E(\theta(t), \dot{\theta}(t)) \to E(0,0) \;\; \text{when} \;\; t \to +\infty$$

(by continuity). But if $0 < |\theta(0)| < \pi$, we have

$$E(\theta(0), \dot{\theta}(0)) > E(0,0)$$

and that would contradict that $E(t)$ is constant.

Hence the origin is not locally attractive.

💎 ATTRACTIVITY (LOW-LEVEL)

The equilibrium x_e is **globally attractive** iff:

- for any state x_0 and for any $\epsilon > 0$ there is a $\tau \geq 0$,
- such that the maximal solution $x(t)$ such that $x(0) = x_0$ is global and,
- satisfies:

$$\|x(t) - x_e\| \leq \epsilon \; \text{ when } \; t \geq \tau.$$

⚠️ WARNING

- Very close values of x_0 could **theoretically** lead to very different “speed of convergence” of $x(t)$ towards the equilibrium.
- This is not contradictory with the well-posedness assumption: continuity w.r.t. the initial condition only works with finite time spans.

🔍 (PATHOLOGICAL) EXAMPLE

$$\begin{aligned}
\dot{x} &= x + xy - (x+y)\sqrt{x^2+y^2} \\
\dot{y} &= y - x^2 + (x-y)\sqrt{x^2+y^2}
\end{aligned}$$

Equivalently, in polar coordinates:

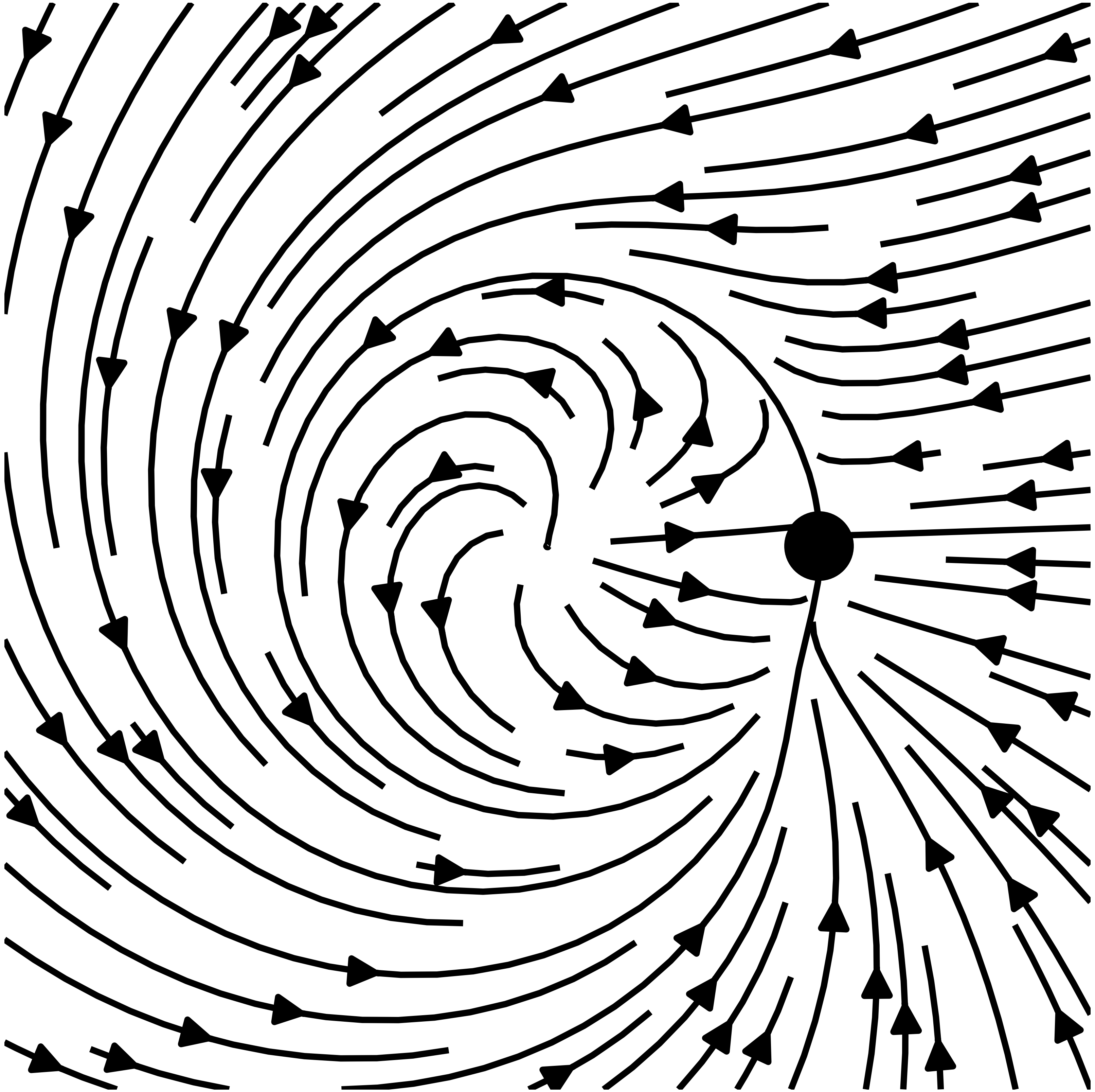
$$
\begin{aligned}
\dot{r} &= r(1-r) \\
\dot{\theta} &= r(1-\cos\theta)
\end{aligned}
$$

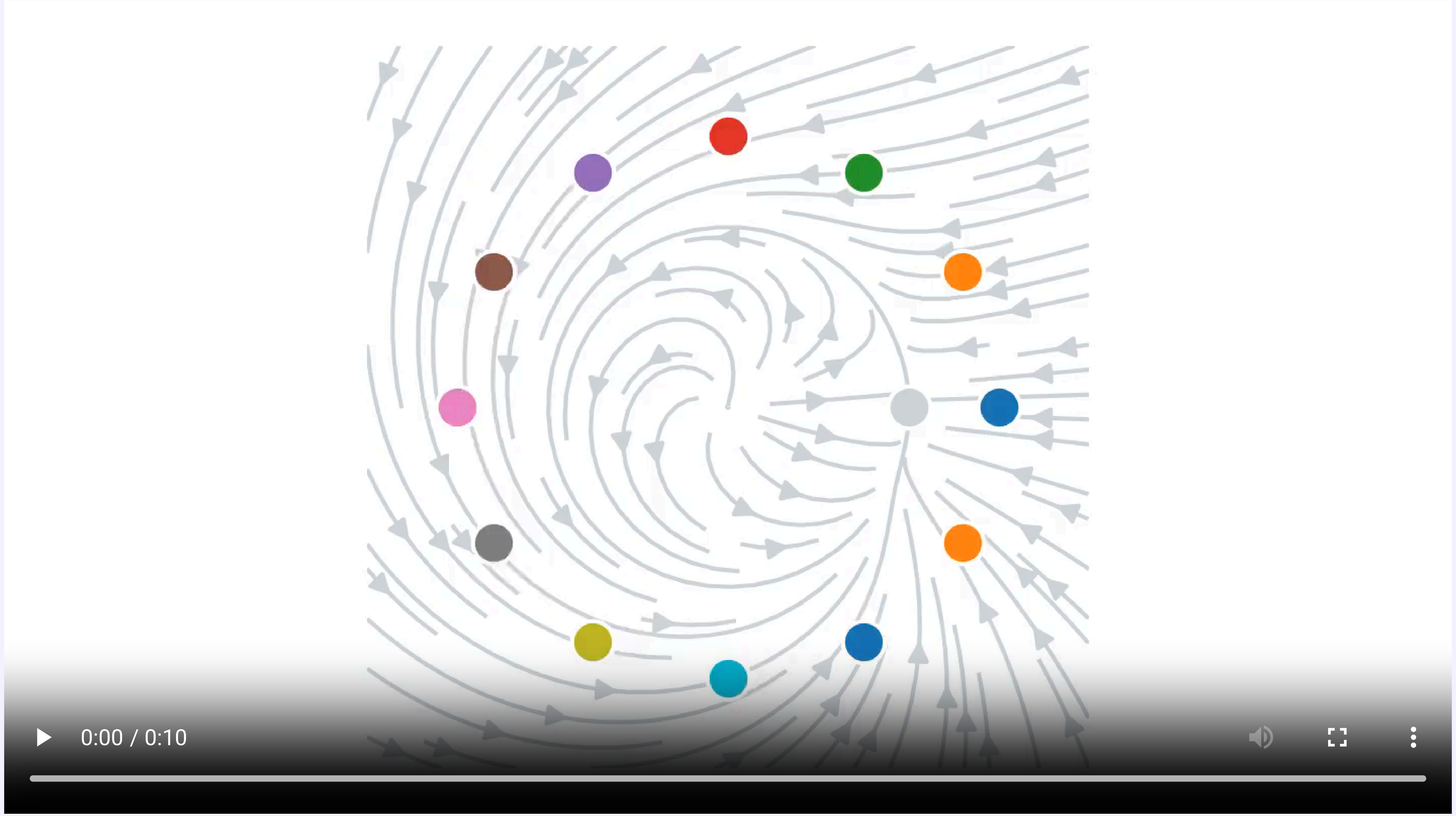
🐍 VECTOR FIELD

```python
def f(xy):
    x, y = xy
    r = sqrt(x*x + y*y)
    dx = x + x * y - (x + y) * r
    dy = y - x * x + (x - y) * r
    return array([dx, dy])
```

📈 STREAM PLOT

```
figure()
x = y = linspace(-2.0, 2.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([1], [0], "k.", ms=20.0)
axis("square")
axis("off")
```

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ASYMPTOTIC STABILITY

Asymptotic stability is a stronger version of attractivity which is by definition robust with respect to the choice of the initial state.

🏷️ GLOBAL ASYMPT. STABILITY

The equilibrium x_e is **globally asympt. stable** iff:

- for any state x_0 and for any $\epsilon > 0$ there is a $\tau \geq 0$,
- and there is a $r > 0$ such that if $\|x_0' - x_0\| \leq r$,
- such that the maximal solution $x(t)$ such that $x(0) = x_0'$ is global and,
- satisfies:

$$\|x(t) - x_e\| \leq \epsilon \; \text{ when } \; t \geq \tau.$$

SET OF INITIAL CONDITIONS

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $X_0 \subset \mathbb{R}^n$.

Let $X(t)$ be the image of X_0 by the flow at time t:

$$X(t) := \{x(t) \mid \dot{x} = f(x),\ x(0) = x_0,\ x_0 \in X_0\ \}.$$

💎 GLOBAL ASYMPT. STABILITY

An equilibrium x_e is globally asympt. stable iff

- for every bounded set X_0 and any $x_0 \in X_0$ the associated maximal solution $x(t)$ is global and,
- $X(t) \to \{x_e\}$ when $t \to +\infty$.

🏷️ LIMITS OF SETS

$$X(t) \to \{x_e\}$$

to be interpreted as

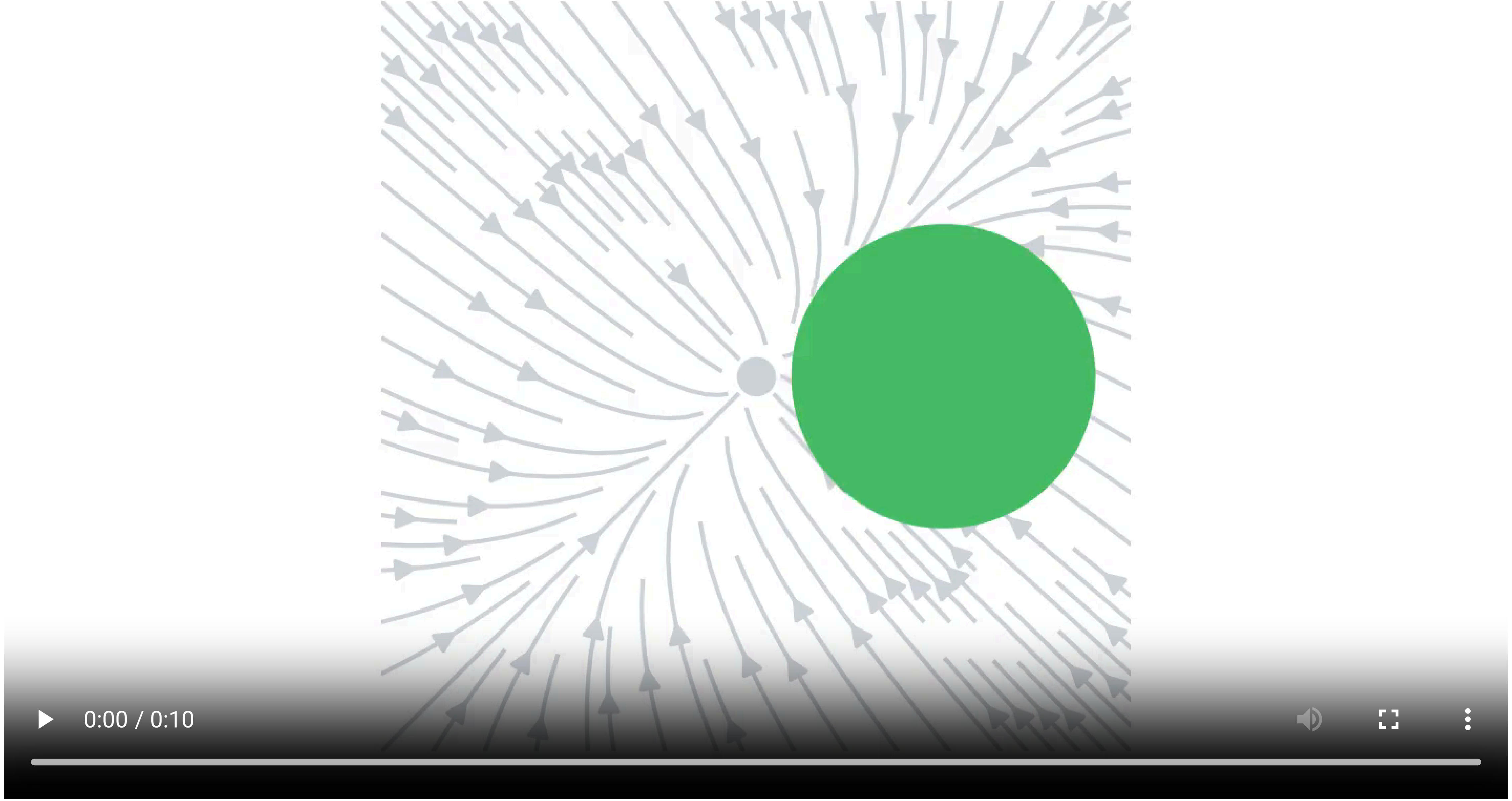
$$\sup_{x(t) \in X(t)} \|x(t) - x_e\| \to 0.$$

🏷️ HAUSDORFF DISTANCE

$$\sup_{x(t)\in X(t)} \|x(t) - x_e\| = d_H(X(t), \{x_e\})$$

where d_H is the **Hausdorff distance** between sets:

$$d_H(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}.$$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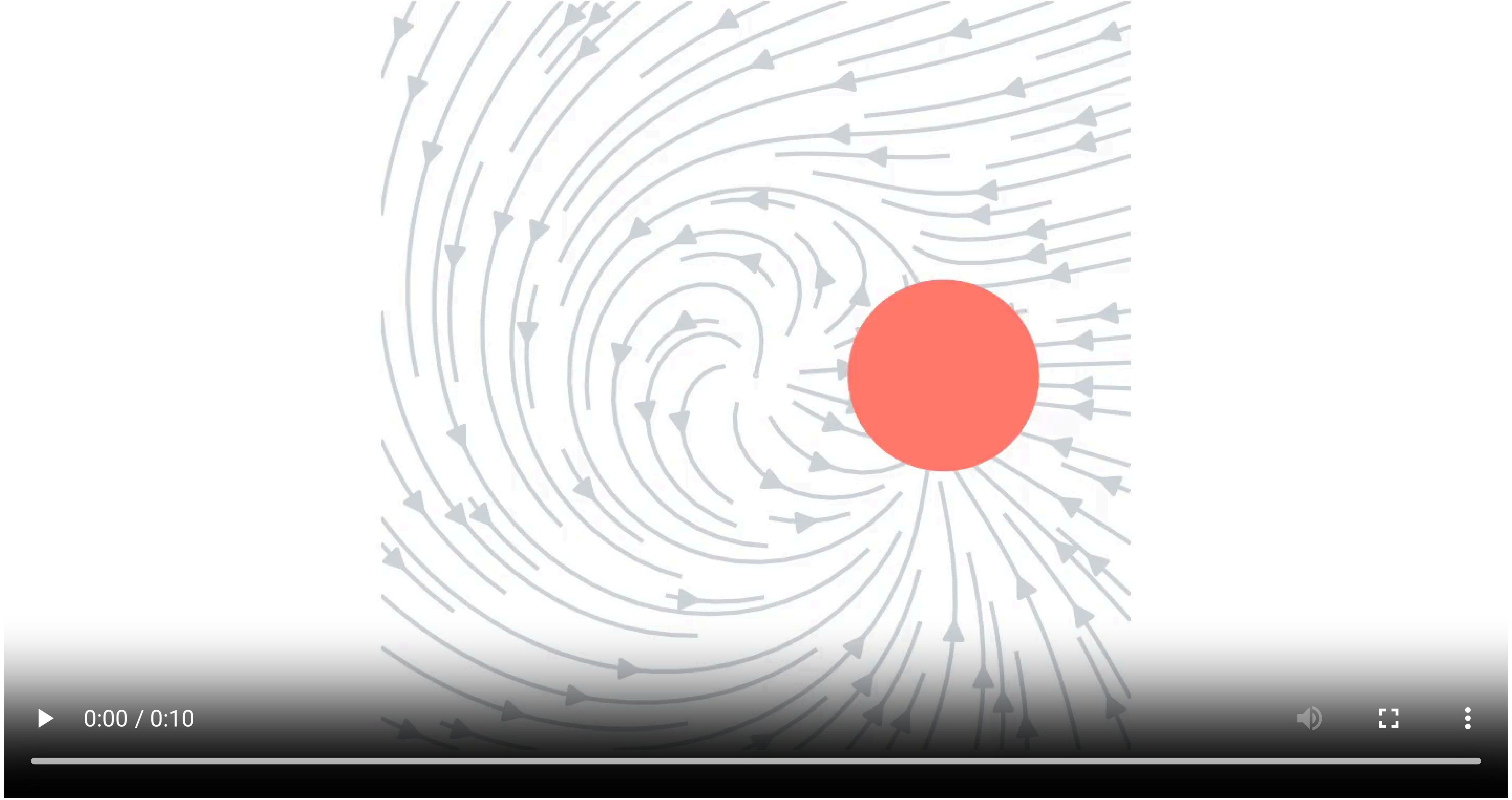
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🏷️ LOCAL ASYMPTOTIC STABILITY

The equilibrium x_e is **locally asympt. stable** iff:

- there is a $r > 0$ such that for any $\epsilon > 0$,
- there is a $\tau \geq 0$ such that,
- if $\|x_0 - x_e\| \leq r$, the maximal solution $x(t)$ such that $x(0) = x_0$ is global and satisfies:

$$\|x(t) - x_e\| \leq \epsilon \;\text{ when }\; t \geq \tau.$$

💎 LOCAL ASYMPT. STABILITY

An equilibrium x_e is locally asympt. stable iff:

There is a $r > 0$ such that for every set X_0 such that

$$X_0 \subset \{x \mid \|x - x_e\| \leq r\},$$

and for any $x_0 \in X_0$, the associated maximal solution $x(t)$ is global and

$$X(t) \to \{x_e\} \text{ when } t \to +\infty.$$

🏷️ STABILITY

An equilibrium x_e is **stable** iff:

- for any $r > 0$,
- there is a $\rho \leq r$ such that if $|x(0) - x_e| \leq \rho$, then
- the solution $x(t)$ is global,
- for any $t \geq 0$, $|x(t) - x_e| \leq r$.

VINOGRAD SYSTEM

Consider the system:

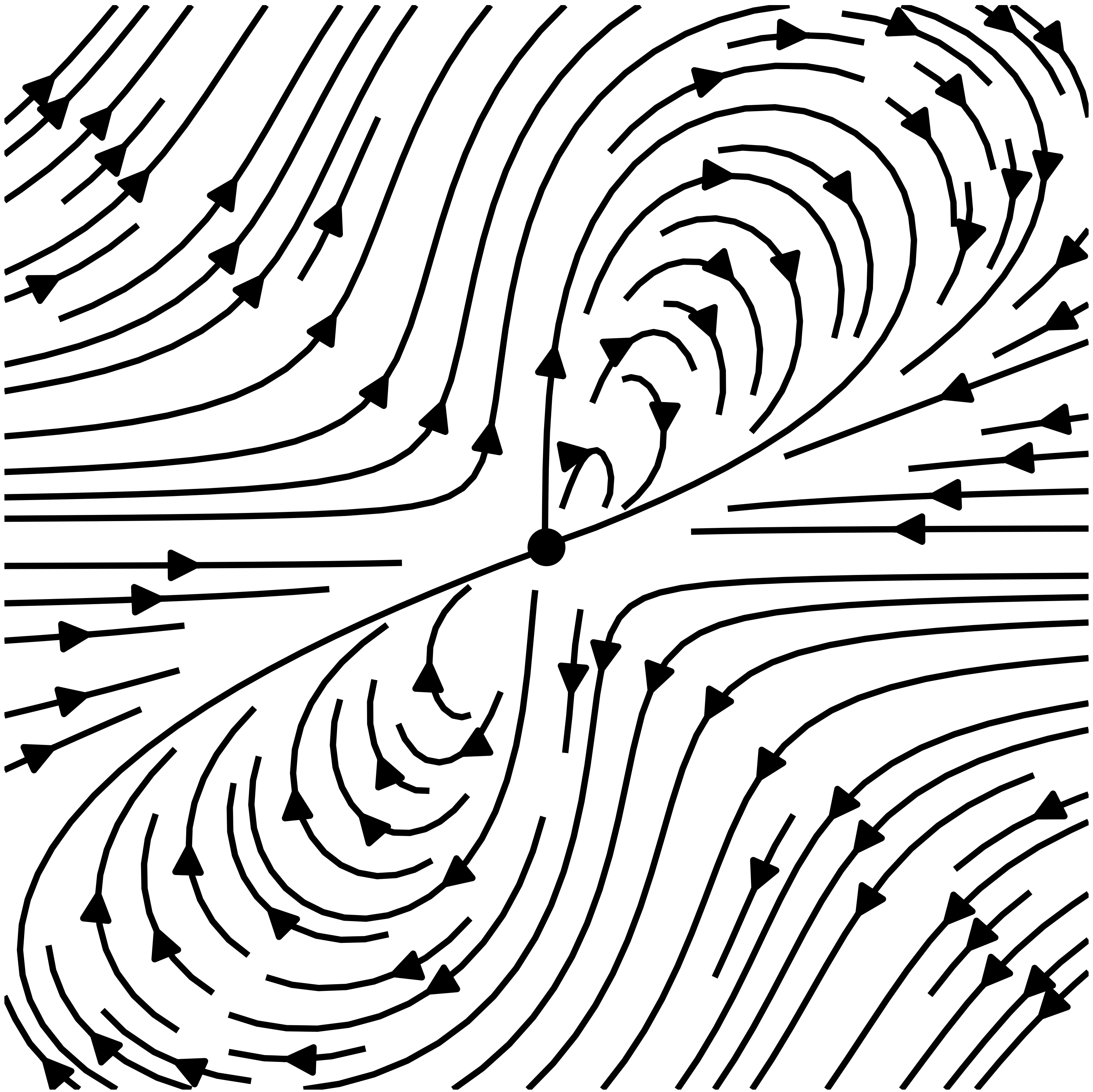
$$\begin{aligned}\dot{x} &= (x^2(y-x)+y^5)/(x^2+y^2(1+(x^2+y^2)^2)) \\ \dot{y} &= y^2(y-2x)/(x^2+y^2(1+(x^2+y^2)^2))\end{aligned}$$

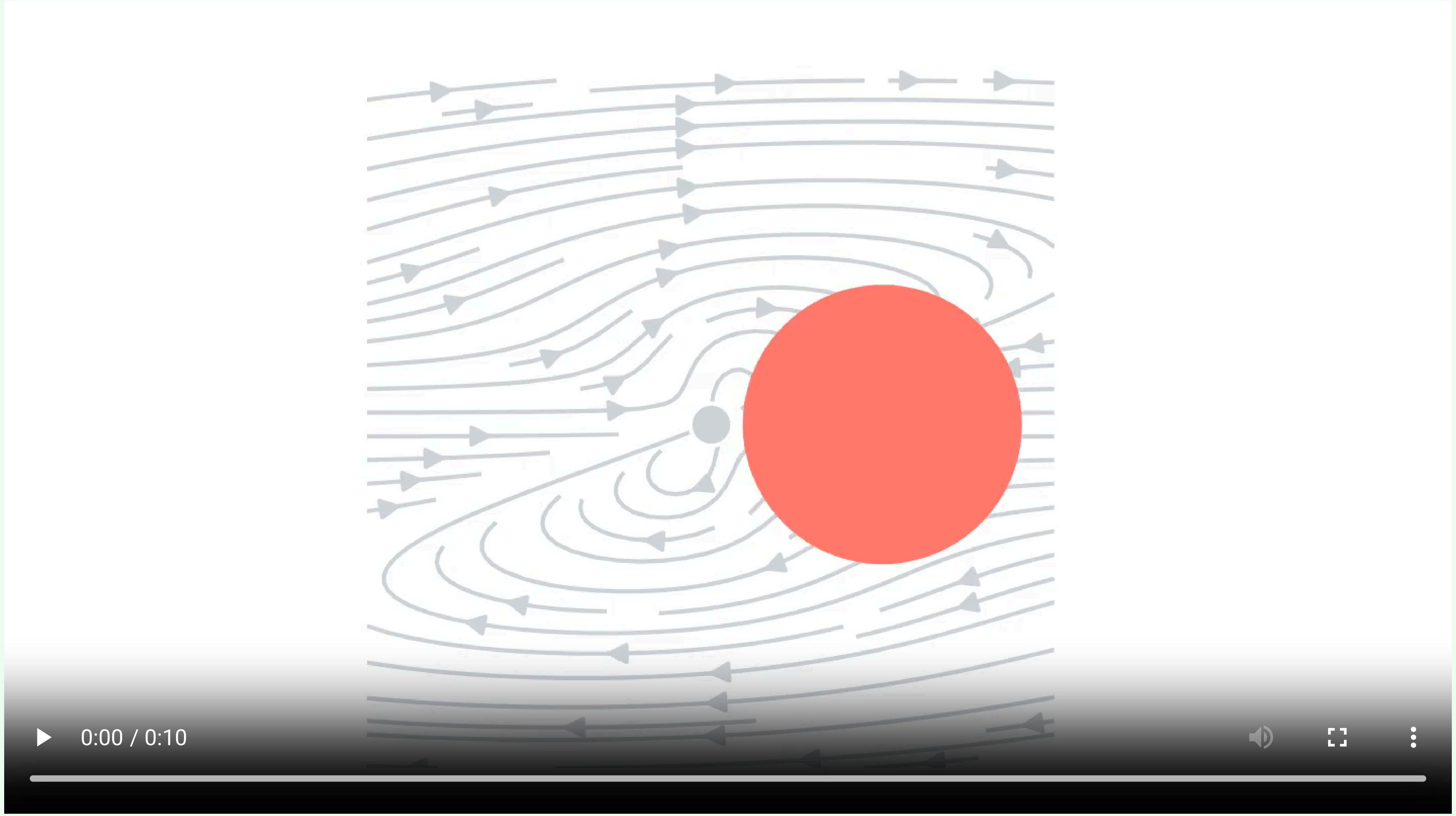
🐍 VECTOR FIELD

```
def f(xy):
    x, y = xy
    q = x**2 + y**2 * (1 + (x**2 + y**2)**2)
    dx = (x**2 * (y - x) + y**5) / q
    dy = y**2 * (y - 2*x) / q
    return array([dx, dy])
```

📈 STREAM PLOT

```
figure()
x = y = linspace(-1.0, 1.0, 1000)
streamplot(*Q(f, x, y), color="k")
xticks([-1, 0, 1])
plot([0], [0], "k.", ms=10.0)
axis("square")
axis("off")
```

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1. 🧮

Show that the origin $(0, 0)$ is the unique equilibrium.

2. 📈🔬

Does this equilibrium seem to be attractive (graphically) ?

3. 🧠

Show that for any equilibrium of a well-posed system:

💎 **(locally) asymptotically stable $\Rightarrow$ stable**

4. 🧪📈

Does the origin seem to be stable (experimentally ?)

Conclude accordingly.

🔓 VINOGRAD SYSTEM

1. 🔓

(x, y) is an equilibrium of the Vinograd system iff

$$\begin{aligned} (x^2(y-x)+y^5)/(x^2+y^2(1+(x^2+y^2)^2)) &= 0 \\ y^2(y-2x)/(x^2+y^2(1+(x^2+y^2)^2)) &= 0 \end{aligned}$$

or equivalently

$$y^2(y-2x)=0 \;\text{ and }\; x^2(y-x)+y^5=0.$$

If we assume that $y = 0$, then:

- $y^2(y - 2x) = 0$ is satisfied and
- $x^2(y - x) + y^5 = 0 \Leftrightarrow -x^3 = 0 \Leftrightarrow x = 0.$

Under this assumption, $(0, 0)$ is the only equilibrium.

Otherwise (if $y \neq 0$),

- $y^2(y - 2x) = 0$ yields $y = 2x$,
- The substitution of y by $2x$ in $x^2(y - x) + y^5 = 0$ yields

  $$x^3(1 + 32x^2) = 0$$

  and therefore $x = 0$.
- $y^2(y - 2x) = 0$ becomes $y^3 = 0$ and thus $y = 0$.

The initial assumption cannot hold.

Conclusion:

The Vinograd system has a single equilibrium: $(0, 0)$.

2. 🔓

Yes, the origin seems to be (globally) attractive.

As far as we can tell, the streamplot displays trajectories that ultimately all converge towards the origin.

3. 🔓

Let's assume that x_e is a (locally) asymptotically stable of a well-posed system.

Let $r > 0$ such that this property is satisfied and let

$$B := \{x \in \mathbb{R}^n \mid \|x - x_e\| \leq r\} \subset \operatorname{dom} f.$$

The set $x(t, B)$ is defined for any $t \geq 0$ and since B is a neighbourhood of x_e, there is $\tau \geq 0$ such that for any $t \geq \tau$, the image of B by $x(t, \cdot)$ is included in B.

$$t \geq \tau \;\Rightarrow\; x(t, B) \subset B.$$

Additionally, the system is well-posed.

Hence there is a $r' > 0$ such that for any x_0 in the closed ball B' of radius r' centered at x_e and any $t \in [0, \tau]$, we have

$$\|x(x_0, t) - x(x_e, t)\| \leq r.$$

Since x_e is an equilibrium, $x(t, x_e) = x_e$, thus $\|x(x_0, t) - x_e\| \leq r$. Equivalently,

$$0 \leq t \leq \tau \;\Rightarrow\; x(t, B') \subset B.$$

Note that since $x(0, B') = B'$, this inclusion yields $B' \subset B$. Thus, for any $t \geq 0$, either $t \in [0, \tau]$ and $x(t, B') \subset B$ or $t \geq \tau$ and since $B' \subset B$,

$$x(t, B') \subset x(t, B) \subset B.$$

Conclusion: we have established that there is a $r > 0$ such that $B \subset \mathrm{dom}\, f$ and a $r' > 0$ such that $r' \leq r$ and

$$t \geq 0 \;\Rightarrow x(t, B') \subset B.$$

In other words, **the system is stable!** 🎉

4. 🔓

No! We can pick initial states $(0, \varepsilon)$, with $\varepsilon > 0$ which are just above the origin and still the distance of their trajectory to the origin will exceed 1.0 at some point:

```python
def fun(t, xy):
    return f(xy)
eps = 1e-10; xy0 = (0, eps)
sol = solve_ivp(
  fun=fun,
  y0=xy0,
  t_span=(0.0, 100.0),
  dense_output=True)["sol"]
```

```
t = linspace(0.0, 100.0, 10000)
xt, yt = sol(t)
figure()
x = y = linspace(-1.0, 1.0, 1000)
streamplot(*Q(f, x, y), color="#ced4da")
xticks([-1, 0, 1])
plot([0], [0], "k.", ms=10.0)
plot(xt, yt, color="C0")
```