Line Integrals & Primitives

Sébastien Boisgérault, Mines ParisTech, under CC BY-NC-SA 4.0

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Exercises

Primitives of Power Functions

Question

Determine the primitives of the power $z \mapsto z^n$ – defined on $\mathbb{C}$ if n nonnegative and on $\mathbb{C}^*$ otherwise – or prove that no such function exist.

Answer

If $n \neq -1$, the function $z \mapsto z^{n+1}/(n+1)$ is a primitive of $z \mapsto z^n$. As $\mathbb{C}$ and $\mathbb{C}^*$ are path-connected, the other primitives differ from this one by a constant.

If $n = -1$, no primitive exist: the function $\gamma : t \in [0,1] \to e^{i2\pi t}$ is a closed rectifiable path of $\mathbb{C}^*$ and

$$\int_\gamma \frac{dz}{z} = \int_0^1 \frac{e^{i2\pi t} i2\pi}{e^{i2\pi t}} \, dt = i2\pi,$$

which is nonzero.

Primitive of a Rational Function

Question

Let $\Omega = \mathbb{C} \setminus \{0,1\}$ and let $f : \Omega \to \mathbb{C}$ be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

Show that f has no primitive on Ω, but that it has a primitive on $\mathbb{C} \setminus [0,1]$ and determine its expression.

Answer

We have

$$f(z) = -\frac{1}{z} + \frac{1}{z-1}.$$

The function $z \mapsto -1/z$ has no primitive on $D(0,1) \setminus \{0\}$: indeed if $\gamma(t) = 1/2 \times e^{i2\pi t}$, we have

$$\int_\gamma \frac{dz}{z} = i2\pi \neq 0.$$

On the other hand, on the same set, $z \mapsto \log(z-1)$ is a primitive of $z \mapsto 1/(z-1)$. Hence $f(z)$ has no primitive.

The function

$$g(z) = \log \frac{z-1}{z} = \log\left(1 - \frac{1}{z}\right)$$

is defined on $\mathbb{C} \setminus [0,1]$ and is a primitive of f. Indeed $g(z)$ is defined as long as neither of the conditions $z = 0$ and $1 - 1/z \in \mathbb{R}_-$ are met; they are equivalent to the condition $z \in [0,1]$, which is excluded. Moreover, g satisfies

$$g'(z) = \frac{1/z^2}{1 - 1/z} = \frac{1}{z(z-1)}$$

hence it is a primitive of f.

Reparametrization of Paths

Questions

Let $\alpha : [0,1] \to \mathbb{C}$ be a continuously differentiable path. Let $\phi : [0,1] \to [0,1]$ be a continuously differentiable function such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi'(t) > 0$ for any $t \in [0,1]$.

1. Show that $\beta = \alpha \circ \phi$ is a rectifiable path which has the same initial point, terminal point and image as α.
2. Prove that for any continuous function $f : \alpha([0,1]) \to \mathbb{C}$,
$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz.$$
3. Prove that the paths α and β have the same length.

Answers

1. The statement about the initial and terminal points is obvious. The one relative to the image holds because, under the assumptions that were made, the function ϕ is a bijection from $[0,1]$ on itself (and its inverse is also continuously differentiable).
2. We have
$$\int_\beta f(z)\, dz = \int_0^1 (f \circ \beta)(t)\beta'(t)\, dt = \int_0^1 (f \circ \alpha)(\phi(t))\alpha'(\phi(t))\,(\phi'(t)dt).$$
The change of variable $s = \phi(t)$ leads to
$$\int_\beta f(z)\, dz = \int_0^1 (f \circ \alpha)(s)\alpha'(s)\, ds = \int_\alpha f(z)\, dz.$$
3. We have
$$\int_0^1 |\beta'(t)|\, dt = \int_0^1 |\alpha'(\phi(t))\phi'(t)|\, dt = \int_0^1 |\alpha'(\phi(t))|\, \phi'(t)dt$$
The change of variable $s = \phi(t)$ leads to
$$\int_0^1 |\beta'(t)|\, dt = \int_0^1 |\alpha'(s)|\, ds,$$
hence the lengths of α and β are equal.

The Logarithm: Alternate Choices

Question

Show that for any $\alpha \in \mathbb{R}$, the function $z \in \mathbb{C}_\alpha \mapsto 1/z$ defined on

$$\mathbb{C}_\alpha = \mathbb{C} \setminus \{re^{i\alpha} \mid r \geq 0\}.$$

has a primitive; describe the set of all its primitives.

Answer

Let γ be a closed rectifiable path of $\mathbb{C}_\alpha$. The path $\mu : [0,1] \mapsto e^{i(\pi-\alpha)}\gamma(t)$ is closed, rectifiable and its image is included in $\mathbb{C} \setminus \mathbb{R}_-$. Additionally

$$\int_\gamma \frac{dz}{z} = \int_\gamma \frac{d(e^{i(\pi-\alpha)}z)}{e^{i(\pi-\alpha)}z} = \int_\mu \frac{dz}{z}.$$

Since the principal value of the logarithm is a primitive if $z \mapsto 1/z$ on $\mathbb{C} \setminus \mathbb{R}_-$, the integral of $z \mapsto 1/z$ on μ is equal to zero. Therefore, there are primitives of $z \mapsto 1/z$ on $\mathbb{C}_\alpha$; since $\mathbb{C}_\alpha$ is connected, they all differ from an arbitrary constant.

Alternatively, we can build explicitly such a primitive: the function

$$f : z \mapsto \log(ze^{i(\pi-\alpha)});$$

it is defined and holomorphic on $\mathbb{C}_\alpha$ and for any $z \in \mathbb{C}_\alpha$,

$$f'(z) = \frac{1}{ze^{i(\pi-\alpha)}} \times e^{i(\pi-\alpha)} = \frac{1}{z}.$$