Integral Representations

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Complex Differentiation of Integrals

Theorem – Complex-Differentiation under the Integral Sign. Let Ω be an open subset of $\mathbb{C}$ and (X, μ) be a measurable space. Let $f : \Omega \times X \to \mathbb{C}$ be a function such that:

1. for every z in Ω, $x \in X \mapsto f(z, x)$ is μ-measurable,
2. for any $z_0 \in \Omega$, there is a neighborhood V of z_0 in Ω and a μ-integrable function $g : X \to \mathbb{R}_+$ such that
$$\forall\, z \in V,\ |f(z, x)| \leq g(x)\ \mu\text{-a.e.}$$
3. for μ-almost every $x \in X$, the function $z \in \Omega \mapsto f(z, x)$ is holomorphic.

Then the function $z \in \Omega \mapsto \int_X f(z, x)\, d\mu(x)$ is holomorphic and its derivative at any order n is

$$\frac{\partial^n}{\partial z^n}\left[\int_X f(z, x)\, d\mu(x)\right] = \int_X \partial_z^n f(z, x)\, d\mu(x).$$

Proof. Let z_0 in Ω and V be as in assumption 2; let $r > 0$ be a radius such that $\overline{D}(z_0, r) \subset V$ and let $\gamma = z_0 + r[\circlearrowleft]$. The Cauchy formula, followed by an integration by parts, yields for μ-almost every $x \in X$ and any $z \in D(z_0, r/2)$

$$\partial_z f(z,x) = \frac{1}{i2\pi} \int_\gamma \frac{\partial_z f(w,x)}{w-z}\, dw = \frac{1}{i2\pi} \int_\gamma \frac{f(w,x)}{(w-z)^2}\, dw,$$

which by the M-L estimation lemma provides the bound

$$|\partial_z f(z,x)| \leq \frac{4|g(x)|}{r}.$$

The difference quotient of $z \mapsto \int_X f(z,x)\, d\mu(x)$ at z_0 is equal to

$$\int_X \frac{f(z_0+h,x) - f(z_0,x)}{h}\, d\mu(x).$$

Let h be a complex number such that $|h| < r/2$. For μ-almost every $x \in X$, the function $\phi : t \in [0,1] \mapsto f(z_0 + th, x)$ is continuous on $[0,1]$, differentiable on $]0,1[$ and satisfies

$$|\phi'(t)| = |\partial_z f(z_0 + th, x)||h| \leq \frac{g(x)}{r}|h|.$$

Hence, the mean value inequality yields

$$\left|\frac{f(z_0+h,x) - f(z_0,x)}{h}\right| = \frac{|\phi(1) - \phi(0)|}{|h|} \leq \frac{4g(x)}{r}.$$

Since

$$\lim_{h \to 0} \frac{f(z_0+h,x) - f(z_0,x)}{h} = \partial_z f(z_0,x) \;\; \mu\text{-a.e.},$$

Lebesgue's dominated convergence theorem provides the result for $n = 1$. Now, the function $\partial_z f$ also satisfies the three assumptions required by the theorem, hence by induction, the theorem statement holds at any order n. ■

Corollary – Complex-Differentiation of Line Integrals. Let $f : \Omega \times \Lambda \to \mathbb{C}$ where Ω and Λ are two subsets of $\mathbb{C}$ and Ω is open. Assume that

1. f is a continuous function.
2. for any $w \in \Lambda$, the function $z \in \Omega \mapsto f(z,w)$ is holomorphic.

Then, for any sequence of rectifiable paths γ of Λ, the function $z \in \Omega \mapsto \int_\gamma f(z,w)\, dw$ is holomorphic and

$$\frac{\partial}{\partial z}\left[\int_\gamma f(z,w)\, dw\right] = \int_\gamma \partial_z f(z,w)\, dw.$$

Proof. We prove the result for any continuously differentiable path γ of Λ (the case of a sequence of rectifiable paths is a simple corollary). By definition of the line integral,

$$\int_\gamma f(z,w)\, dw = \int_{[0,1]} f(z,\gamma(t))\gamma'(t)\, dt.$$

Now,

1. For any $z \in \Omega$, the function $t \in [0,1] \mapsto f(z, \gamma(t))\gamma'(t)$ is continuous and therefore Lebesgue measurable.
2. Let $z_0 \in \Omega$ and let $r > 0$ be such that $K = \overline{D}(z_0, r) \subset \Omega$. The restriction of f to the compact set $K \times \gamma([0,1])$ is bounded by some constant κ. Therefore, for any $z \in D(z_0, r)$, the function $t \in [0,1] \mapsto f(z, \gamma(t))\gamma'(t)$ is dominated by $t \in [0,1] \mapsto \kappa|\gamma'(t)|$ which is Lebesgue integrable.
3. For any $t \in [0,1]$, the function $z \in \Omega \mapsto f(z, \gamma(t))\gamma'(t)$ is holomorphic; its derivative is $\partial_z f(z, \gamma(t))\gamma'(t)$.

Consequently, the differentiation of Lebesgue integrals theorem provides the existence of $\partial_z \left[\int_\gamma f(z,w)\,dw\right]$ and its value:

$$\frac{\partial}{\partial z}\left[\int_\gamma f(z,w)\,dw\right] = \int_{[0,1]} \partial_z f(z, \gamma(t))\gamma'(t)\,dt.$$

The right-hand side is equal to $\int_\gamma \partial_z f(z,w)\,dw$. ■

The Laplace Transform

Definition – The Laplace Transform. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a Lebesgue measurable function. We denote by σ the extended real number defined by

$$\sigma \in [-\infty, +\infty] = \inf \left\{ \sigma^+ \in \mathbb{R} \;\middle|\; \int_{\mathbb{R}_+} |f(t)| e^{-\sigma^+ t}\,dt < +\infty \right\}.$$

If $s \in \mathbb{C}$ and $\mathrm{Re}(s) > \sigma$, the function $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is Lebesgue integrable. The *Laplace transform* of f is the function

$$\mathcal{L}[f] : \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\} \to \mathbb{C}$$

defined by

$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} f(t) e^{-st}\,dt.$$

Proof – Definition of the Laplace Transform. For any $s \in \mathbb{C}$, the function $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is Lebesgue measurable . If additionally $\mathrm{Re}(s) > \sigma$, then there is some σ^+ such that $\sigma < \sigma^+ < \mathrm{Re}(s)$ and $t \mapsto |f(t)|e^{-\sigma^+ t}$ is Lebesgue integrable. Thus,

$$\int_{\mathbb{R}_+} |f(t)e^{-st}|\,dt = \int_{\mathbb{R}_+} |f(t)| e^{-\mathrm{Re}(s)t}\,dt \leq \int_{\mathbb{R}_+} |f(t)| e^{-\sigma^+ t}\,dt < +\infty.$$

and therefore $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is Lebesgue integrable. ■

Example – Laplace Transform of Exponential Functions. For any $\lambda \in \mathbb{C}$, the function $t \in \mathbb{R}_+ \mapsto e^{\lambda t}$ is Lebesgue measurable. Additionally,

$$\forall\, t \geq 0,\ |f(t)|e^{-\sigma^+ t} = e^{-(\sigma^+ - \mathrm{Re}(\lambda))t},$$

hence the function $t \in \mathbb{R}_+ \mapsto |f(t)|e^{-\sigma^+ t}$ is Lebesgue integrable if and only if $\sigma^+ > \mathrm{Re}(\lambda)$. The infimum σ of all such σ^+ is therefore $\mathrm{Re}(\lambda)$. Now, if $\mathrm{Re}(s) > \mathrm{Re}(\lambda)$,

$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} e^{(\lambda - s)t}\, dt = \left[\frac{e^{(\lambda-s)t}}{\lambda - s}\right]_0^{+\infty} = \frac{1}{s - \lambda}.$$

Theorem – Derivative of the Laplace Transform. The Laplace transform of a Lebesgue measurable function $f : \mathbb{R}_+ \to \mathbb{C}$ is holomorphic on its domain of definition and

$$(\mathcal{L}[f])'(s) = \mathcal{L}[t \mapsto -tf(t)](s).$$

Proof. Let $\Omega = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\}$.

1. For any $s \in \Omega$, the function $t \mapsto f(t)e^{-st}$ is Lebesgue measurable.
2. Let $s \in \Omega$ and let $r > 0$ be such that $\epsilon = \mathrm{Re}(s) - \sigma - r > 0$. For any $w \in D(s, r)$, we have $\mathrm{Re}(w) > \mathrm{Re}(s) - r = \sigma + \epsilon$, thus

   $$\int_{\mathbb{R}_+} |f(t)e^{-wt}|\, dt = \int_{\mathbb{R}_+} |f(t)|e^{-\mathrm{Re}(w)t}\, dt \leq \int_{\mathbb{R}_+} |f(t)|e^{-(\sigma+\epsilon)t}\, dt < +\infty.$$

3. For almost any $t \geq 0$, $s \mapsto f(t)e^{-st}$ is holomorphic and

   $$\partial_s[f(t)e^{-st}] = -tf(t)e^{-st}.$$

We can therefore differentiate under the integral sign and obtain

$$\frac{\partial}{\partial s}\int_0^{+\infty} f(t)e^{-st}\, dt = \int_0^{+\infty} -tf(t)e^{-st}\, dt = \mathcal{L}[t \mapsto -tf(t)](s)$$

as expected. ■

Example – Laplace Transform of Polynomials. The constant function defined by $f(t) = 1$ for $t \geq 0$ is an exponential function (as $1 = e^{0\times t}$); its Laplace transform is defined for $\mathrm{Re}(s) > 0$ and equal to $1/s$. Now, this Laplace transform has a derivative at every of order n which is

$$\frac{(-1)^n n!}{s^{n+1}}.$$

It is also the Laplace transform of $t \in \mathbb{R}_+ \mapsto (-t)^n$. Thus, by linearity, the Laplace transform of the polynomial $f(t) = \sum_{p=0}^{n} a_p t^p$ is

$$\mathcal{L}[f](s) = \sum_{p=0}^{n} a_p p! \frac{1}{s^{p+1}}.$$

Cauchy's Integral Theorem – Dixon's Proof

In (Dixon 1971), John D. Dixon provides a short proof of the global version of Cauchy's Formula, using the local Cauchy theory. The proof relies on the following key result:

Lemma – Integral of the Difference Quotient. Let Ω be an open subset of the complex plane, f be a holomorphic function on Ω and γ be a sequence of rectifiable closed paths of Ω. The function

$$z \in \Omega \setminus \gamma([0,1]) \mapsto \int_\gamma \frac{f(z)-f(w)}{z-w} dw$$

has a holomorphic extension on Ω.

Proof. We may define the function $g : \Omega \times \Omega \to \mathbb{C}$ by

$$g(z,w) = \frac{f(z)-f(w)}{z-w} \;\text{ if }\; z \neq w \;\text{ and }\; g(w,w) = f'(w).$$

The continuity and complex-differentiability of g at any point $(z,w) \in \Omega^2$ such that $z \neq w$ is plain. Now, let $c \in \Omega$ and let $r > 0$ be a radius such that the closure of the disk $D = D(c,r)$ is included in Ω. Using the Taylor expansion of f in this disk, we derive for any $z \in D$ and $w \in D$:

$$\begin{aligned}\frac{f(z)-f(w)}{z-w} &= \frac{1}{z-w}\sum_{n=0}^{+\infty} a_n((z-c)^n - (w-c)^n) \\ &= \sum_{n=1}^{+\infty} a_n \left[\sum_{p=0}^{n-1} (z-c)^{n-1-p}(w-c)^p\right]\end{aligned}$$

The right-hand side of this equation is a uniformly convergent sum of continuous functions of $(w,z) \in D^2$. Thus, its limit is a continuous function of (w,z) and we have

$$\lim_{(w,z)\to(c,c), w\neq z} \frac{f(z)-f(w)}{z-w} = \sum_{n=1}^{+\infty} n a_n (w-c)^{n-1} = f'(w) = g(w,w),$$

thus this continuous function is actually g. Additionally, for every $w \in D$, every function of the sum is a holomorphic function with respect to z, hence its uniform limit $z \in D \mapsto g(z,w)$ is also holomorphic.

Now the function

$$z \in \Omega \mapsto \int_\gamma g(z,w) dw$$

clearly extends the function of the lemma statement. It also satisfies the assumptions of the complex-differentiation of line integrals result, thus it is holomorphic. ■

For completeness, here is Dixon's proof of Cauchy's formula:

Proof – Cauchy's Integral Formula. Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function. Let γ be a sequence of rectifiable closed paths of Ω such that $\mathrm{Int}\,\gamma \subset \Omega$.

Introduce the holomorphic extension h to Ω of

$$z \in \Omega \setminus \gamma([0,1]) \mapsto \frac{1}{i2\pi} \int_\gamma \frac{f(z) - f(w)}{z - w} dw$$

and define the function $\phi : \mathbb{C} \mapsto \mathbb{C}$ by

$$\phi(z) = h(z) \text{ if } z \in \Omega,\; \phi(z) = -\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w - z}\, dw \text{ if } z \in \mathrm{Ext}\,\gamma.$$

This definition is unambiguous: if $z \in \Omega \cap \mathrm{Ext}\,\gamma$, then

$$\begin{aligned} h(z) &= \frac{1}{i2\pi} \int_\gamma \frac{f(z) - f(w)}{z - w} dw \\ &= f(z)\mathrm{ind}(\gamma, z) - \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{z - w} dw. \\ &= -\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{z - w} dw \end{aligned}$$

The function ϕ is holomorphic on Ω and also on $\mathrm{Ext}\,\gamma$ by the complex-differentiation of line integrals theorem. Hence, it is holomorphic on $\mathbb{C}$. Additionally, if $|z| > r = \max\{|w| \mid w \in \gamma([0,1])\}$, then $z \in \mathrm{Ext}\,\gamma$, thus if M is an upper bound of f on the image of γ,

$$|\phi(z)| \leq \frac{1}{2\pi} \frac{M}{|z| - r} \times \ell(\gamma)$$

and $|\phi(z)| \to 0$ when $|z| \to +\infty$. By Liouville's Theorem, ϕ is identically zero; hence, if $z \in \Omega$,

$$\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{z - w} dw = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{z - w} dw = \mathrm{ind}(\gamma, z) f(z),$$

which is Cauchy's integral formula. ■

The Π Function

Definition – Π Function. The Π function is defined for all complex numbers z such that $\mathrm{Re}(z) > -1$ by

$$\Pi(z) = \int_0^{+\infty} t^z e^{-t}\, dt$$

It is a holomorphic function whose n-th order derivative is given by

$$\Pi^{(n)}(z) = \int_0^{+\infty} (\ln t)^n t^z e^{-t} \, dt.$$

Proof – Π **Function.** For any $z \in \mathbb{C}$ and any $t > 0$,

$$t^z e^{-t} = e^{z \ln t - t} \quad \text{and} \quad |t^z e^{-t}| = e^{\mathrm{Re}(z) \ln t - t} = t^{\mathrm{Re}(z)} e^{-t}.$$

Thus, if $\mathrm{Re}(z) > -1$, the function $t \in \mathbb{R}_+^* \mapsto t^z e^{-t}$ is Lebesgue integrable. Let $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > -1$ and let $r = (\mathrm{Re}(z) + 1)/2 > 0$. For any $h \in \mathbb{C}$ such that $|h| < r$ and any $t > 0$,

$$|t^{(z+h)} e^{-t}| = t^{\mathrm{Re}(z+h)} e^{-t} < \max(t^{\mathrm{Re}(z)-r}, t^{\mathrm{Re}(z)+r}) e^{-t}$$

and the right-hand side of this inequality is a Lebesgue integrable function of t. Finally, for any $t > 0$, the function $z \mapsto t^z e^{-t}$ is holomorphic on the domain of the Π function and at any order n,

$$\partial_z^n t^z e^{-t} = \partial_z^n e^{z \ln t - t} = (\ln t)^n t^z e^{-t}.$$

The assumptions of differentiation under the integral sign are met and the application of this theorem provides the desired result. ■

References

Dixon, John D. 1971. "A Brief Proof of Cauchy's Integral Theorem." *Proceedings of the American Mathematical Society* 29. American Mathematical Society (AMS), Providence, RI: 625–26. doi:10.2307/2038614.