Cauchy's Integral Theorem – Global Version

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Contents

Exercises

Cauchy's Converse Integral Theorem

Question

Let Ω be an open subset of $\mathbb{C}$.

Suppose that for every holomorphic function $f : \Omega \to \mathbb{C}$,

$$\int_\gamma f(z)\, dz = 0$$

for some sequence γ of closed rectifiable paths of Ω. What conclusion can we draw? What if the property holds for every sequence γ of closed rectifiable paths of Ω?

Answer

For any $w \in \mathbb{C} \setminus \Omega$, the function $f : z \in \Omega \mapsto 1/(z-w)$ is defined and holomorphic, thus

$$\mathrm{ind}(\gamma, w) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z-w} = 0$$

and therefore $\operatorname{Int}\gamma \subset \Omega$. Now, suppose that this conclusion holds for any sequence γ of closed rectifiable paths of Ω. Since the winding number is locally constant and since for any closed path μ of Ω and any $\epsilon > 0$, there is a closed rectifiable path γ of Ω such that

$$\forall\, t \in [0,1],\ |\gamma(t) - \mu(t)| < \epsilon,$$

we also have $\operatorname{ind}(\mu, w) = \operatorname{ind}(\gamma, w) = 0$. Therefore $\operatorname{Int}\mu \subset \Omega$: the set Ω is simply connected.

Cauchy Transform of Power Functions

Question

Compute for any $n \in \mathbb{Z}$ and any $z \in \mathbb{C}$ such that $|z| \neq 1$ the line integral

$$\phi(z) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{w^n}{w - z} dw.$$

Answer

For any $z \in \mathbb{C}$ such that $|z| \neq 1$, the function

$$\psi_z : w \mapsto \frac{w^n}{w - z}$$

is defined and holomorphic on $\Omega = \mathbb{C} \setminus \{z\}$ if $n \geq 0$; it is defined and holomorphic on $\Omega = \mathbb{C} \setminus \{0, z\}$ if $n < 0$. The interior of $[\circlearrowleft]$ is the open unit disk.

We now study separately four configurations.

1. Assume that $n \geq 0$ and $|z| > 1$. The interior of $[\circlearrowleft]$ is included in Ω, hence by Cauchy's integral theorem, $\phi(z) = 0$.

   Alternatively, Cauchy's formula was also applicable.

2. Assume that $n \geq 0$ and $|z| < 1$. The unique singularity of ψ_z is $w = z$; it satisfies $\operatorname{ind}([\circlearrowleft], z) = 1$. Let $\gamma(r) = z + r[\circlearrowleft]$; we have

$$\operatorname{res}(\psi_z, z) = \lim_{r \to 0} \frac{1}{i2\pi} \int_{\gamma(r)} \frac{w^n}{w - z} dw = \lim_{r \to 0} \int_0^1 (z + re^{i2\pi t})^n dt = z^n,$$

   hence by the residues theorem, $\phi(z) = z^n$.

   Alternatively, Cauchy's formula was also applicable.

3. Assume that $n < 0$ and $|z| > 1$. We have

$$\phi(z) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{w^{|n|}(w - z)} dw.$$

If $n = -1$, Cauchy's formula provides the answer:

$$\phi(z) = \frac{1}{0 - z} = -z^{-1}.$$

Otherwise $n < -1$, we may use integration by parts (several times):

$$\begin{aligned}
\phi(z) &= \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{w^{|n|}(w-z)}\, dw \\
&= -\frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{-1}{|n|-1} \frac{1}{w^{|n|-1}} \frac{-1}{(w-z)^2}\, dw \\
&= \dots \\
&= (-1)^{|n|-1} \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{(|n|-1)!} \frac{1}{w} \frac{(|n|-1)!}{(w-z)^{|n|}}\, dw \\
&= -\frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{w} \frac{1}{(z-w)^{|n|}}\, dw.
\end{aligned}$$

At this point, Cauchy's formula may be used again and we obtain

$$\phi(z) = -z^n.$$

Alternatively, we may perform the change of variable $w = 1/\xi$:

$$\begin{aligned}
\phi(z) &= \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{w^n}{w-z} dw \\
&= -\frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{\xi^{-n}}{\xi^{-1}-z} \left(-\frac{d\xi}{\xi^2}\right) \\
&= -\frac{1}{z} \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{\xi^{-n-1}}{\xi - z^{-1}} d\xi.
\end{aligned}$$

As $-n-1 \geq 0$ and $|z^{-1}| < 1$, we may invoke the result obtained for the first configuration: it provides $\phi(z) = -z^n$.

Alternatively, we may perform a partial fraction decomposition of $w \mapsto 1/(w^{|n|}(w-z))$. Since

$$1 - \left(\frac{w}{z}\right)^{|n|} = \left(1 - \frac{w}{z}\right) \left(1 + \frac{w}{z} + \cdots + \left(\frac{w}{z}\right)^{|n|-1}\right),$$

we have

$$\frac{1}{w-z} = -\frac{1}{z} \left(1 + \frac{w}{z} + \cdots + \left(\frac{w}{z}\right)^{|n|-1}\right) + \frac{w^{|n|}/z^{|n|}}{w-z}$$

and therefore

$$\frac{1}{w^{|n|}(w-z)} = -\left(\frac{z}{w^{|n|}} + \frac{1/z^2}{w^{|n|-1}} + \cdots + \frac{1/z^{|n|}}{w^{-1}}\right) + \frac{1/z^{|n|}}{w-z}.$$

The integral along γ of $w \in \mathbb{C} \mapsto 1/w^p$ is zero for $p > 1$ since this function has a primitive. The integral of $w \mapsto 1/(w-z)$ is also zero since $|z| > 1$. Finally,

$$\phi(z) = \frac{1}{i2\pi} \int_\gamma -\frac{1/z^{|n|}}{w^{-1}} dw = -z^n.$$

4. Assume that $n < 0$ and $|z| < 1$. There are two singularities of ψ_z in the interior of $[\circlearrowleft]$, $w = 0$ and $w = z$, unless of course if $z = 0$.

   If $z = 0$, we have

   $$\phi(z) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} w^{n-1} dw = 0$$

   because $w \in \mathbb{C}^* \mapsto w^n/n$ is a primitive of $w \in \mathbb{C}^* \mapsto w^{n-1}$.

   We now assume that $z \neq 0$. The residue associated to $w = z$ can be computed directly with Cauchy's formula; with $\gamma(r) = z + r[\circlearrowleft]$, we have

   $$\mathrm{res}(\psi_z, z) = \lim_{r \to 0} \frac{1}{i2\pi} \int_{\gamma(r)} \frac{w^n}{(w-z)} dw = z^n.$$

   On the other hand, using computations similar to those of the previous question, we can derive

   $$\mathrm{res}(\psi_z, 0) = \lim_{r \to 0} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{w^n}{(w-z)} dw = -z^n.$$

   Consequently, $\phi(z) = 0$.

   In the case $z \neq 0$, we may perform again the change of variable $w = 1/\xi$ that provides

   $$\phi(z) = -\frac{1}{z} \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{\xi^{-n-1}}{\xi - z^{-1}} d\xi.$$

   As $-n - 1 \geq 0$ and $|z^{-1}| > 1$, we may invoke the result obtained for the first configuration: it yields $\phi(z) = 0$.

   There is yet another method: we can notice that for $r > 1$, the interior of the path sequence $(r[\circlearrowleft], [\circlearrowleft]^{\leftarrow})$, which is the annulus $\{z \in \mathbb{C} \mid 1 < |z| < r\}$, is included in Ω. Cauchy's integral theorem provides

   $$\forall\, r > 1,\ \phi(z) = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{w^n}{w - z} dw.$$

   and the M-L estimation lemma

   $$\forall\, r > 1,\ |\phi(z)| \leq \frac{1}{r^{|n|-1}(r - |z|)}.$$

   The limit of the right-hand side when $r \to +\infty$ yields $\phi(z) = 0$.

Finally, we may use again the partial fraction decomposition of $w \mapsto 1/(w^{|n|}(w-z))$:

$$\frac{1}{w^{|n|}(w-z)} = -\left(\frac{z}{w^{|n|}} + \frac{1/z^2}{w^{|n|-1}} + \cdots + \frac{1/z^{|n|}}{w^{-1}}\right) + \frac{1/z^{|n|}}{w-z}.$$

The integral along γ of $w \in \mathbb{C} \mapsto 1/w^p$ is zero for $p > 1$ since this function has a primitive. Therefore

$$\phi(z) = \frac{1}{i2\pi} \int_\gamma -\frac{1/z^{|n|}}{w^{-1}} dw + \frac{1}{i2\pi} \int_\gamma \frac{1/z^{|n|}}{w-z} dw = 0.$$