Power Series

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Convergence of Power Series

Definition & Theorem – Radius of Convergence. Let $c \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for every $n \in \mathbb{N}$. The *radius of convergence* of the power series

$$\sum_{n=0}^{+\infty} a_n (z-c)^n$$

is the unique $r \in [0, +\infty]$ such that the series converges if $|z-c| < r$ and diverges if $|z-c| > r$. The disk $D(c, r)$ – the largest open disk centered on c where the series converges – is the *open disk of convergence* of the series.

The radius of convergence r is the inverse of the *growth ratio* of the sequence a_n, defined as the infimum in $[0, +\infty]$ of the set of values $\sigma \in [0, +\infty)$ such that a_n is eventually dominated by σ^n:

$$\exists\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ (n \geq m) \ \Rightarrow\ |a_n| \leq \sigma^n.$$

(or equivalently, such that $\exists\, \kappa > 0$, $\forall\, n \in \mathbb{N}$, $|a_n| \leq \kappa \sigma^n$.) This growth ratio is equal to $\limsup_{n \to +\infty} |a_n|^{1/n}$, which leads to the Cauchy-Hadamard formula[1]:

$$r = \frac{1}{\displaystyle \limsup_{n \to +\infty} |a_n|^{1/n}}.$$

[1] to compute the *limit superior* of a sequence of (extended) real numbers, consider all subsequences that converge (as extended real numbers: in $[-\infty, +\infty]$) and take the supremum of their limits.

By convention here, $1/0 = +\infty$ and $1/(+\infty) = 0$.

Proof. Let ρ be the growth ratio of the sequence a_n. If a complex number z satisfies $|z-c| < \rho^{-1}$, ρ is finite and there is a $\sigma > \rho$ such that $|z-c| < \sigma^{-1}$. Eventually, we have $|a_n| \leq \sigma^n$ and thus

$$|a_n(z-c)^n| \leq (\sigma|z-c|)^n.$$

As $\sigma|z-c| < 1$, the series $\sum_{n=0}^{+\infty} a_n(z-c)^n$ is convergent. Conversely, if $|z-c| > \rho^{-1}$, $\rho > 0$ and there is a $\sigma < \rho$ such that $|z-c| > \sigma^{-1}$. As $\sigma < \rho$, there is a strictly increasing sequence of $n \in \mathbb{N}$ such that $|a_n| > \sigma^n$ and thus $|a_n(z-c)^n| > (\sigma\sigma^{-1})^n = 1$. Since its terms do not converge to zero, the series $\sum_{n=0}^{+\infty} a_n(z-c)^n$ is divergent.

We now prove that the growth ratio of $|a_n|$ is equal $\limsup_n |a_n|^{1/n}$. Indeed, for any σ greater than the growth ratio ρ, eventually $|a_n| \leq \sigma^n$, hence $|a_n|^{1/n} \leq \sigma$ and $\limsup_n |a_n|^{1/n} \leq \sigma$, therefore $\limsup_n |a_n|^{1/n} \leq \rho$. Conversely, if σ is smaller than the growth ratio, there is a strictly increasing sequence of $n \in \mathbb{N}$ such that $|a_n| > \sigma^n$, hence $|a_n|^{1/n} > \sigma$ and $\limsup_n |a_n|^{1/n} \geq \sigma$, thus $\limsup_n |a_n|^{1/n} \geq \rho$. ■

Example – A Geometric Series. Consider the power series

$$\sum_{n=0}^{+\infty} (-1/2)^n z^n.$$

Since $|(-1/2)^n| = 1/2^n \leq \sigma^n$ eventually if and only if $\sigma \geq 1/2$, the growth bound of the geometric sequence $(-1/2)^n$ is $1/2$. Thus the open disk of convergence of this power series is $D(0,2)$.

Example – A Lacunary Series. Consider the power series:

$$\sum_{n=0}^{+\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots.$$

The "lacunary" adjective refers to the large gaps between nonzero coefficients; These coefficients are defined by

$$a_n = \left| \begin{array}{ll} 1 & \text{if } \exists\, p \in \mathbb{N},\ n = 2^p, \\ 0 & \text{otherwise.} \end{array} \right.$$

It is plain that $|a_n| \leq \sigma^n$ eventually if and only if $\sigma \geq 1$. Hence the growth bound of the sequence if 1 and the open disk of convergence of the power series is $D(0,1)$.

Lemma – Multiplication of Power Series Coefficients. The radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n b_n (z-c)^n$ is at least the product of the radii of convergence of the series $\sum_{n=0}^{+\infty} a_n(z-c)^n$ and $\sum_{n=0}^{+\infty} b_n(z-c)^n$. In particular, for any nonzero polynomial sequence

$$a_n = \alpha_0 + \alpha_1 n + \cdots + \alpha_p n^p,$$

the radii of convergence of $\sum_{n=0}^{+\infty} a_n b_n (z-c)^n$ and $\sum_{n=0}^{+\infty} b_n (z-c)^n$ are identical.

Proof. Denote by ρ_a and ρ_b the respective growth bounds of the sequences a_n and b_n; the growth bound of the product sequence $a_n b_n$ is at most $\rho_a \rho_b$: for any $\sigma > \rho_a \rho_b$, we may find some $\sigma_a > \rho_a$ and $\sigma_b > \rho_b$ such that $\sigma = \sigma_a \sigma_b$. Since $|a_n| \leq (\sigma_a)^n$ and $|b_n| \leq (\sigma_b)^n$ eventually, $|a_n b_n| \leq \sigma^n$ eventually.

The growth bound of any polynomial sequence a_n is at most 1: the inequality

$$|\alpha_0 + \alpha_1 n + \cdots + \alpha_p n^p| \leq \rho^n$$

holds for any $\rho > 1$ eventually. Now, for any nonzero polynomial sequence a_n and any sequence b_n, eventually $|b_n|$ is dominated by a multiple of $|a_n b_n|$, thus the growth bound of $|b_n|$ is at most the growth bound of $|a_n b_n|$. Reciprocally, the growth bound of $|a_n b_n|$ is at most the product of the growth bound of $|a_n|$ – at most one – and the growth bound of $|b_n|$ and thus at most the growth bound of $|b_n|$. ∎

Theorem – Locally Normal Convergence. The convergence of the power series $\sum_{n=0}^{+\infty} a_n (z-c)^n$ in its open disk of convergence $D(c,r)$ is *locally normal*: for any $z \in D(c,r)$, there is an open neighbourghood U of z in $D(c,r)$ such that

$$\exists\, \kappa > 0,\ \forall\, z \in U,\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| \leq \kappa$$

or equivalently, for every compact subset K of $D(c,r)$,

$$\exists\, \kappa > 0,\ \forall\, z \in K,\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| \leq \kappa.$$

Proof. If K is compact subset of $D(c,r)$ and $\rho = \sup\{|z-c| \mid z \in K\}$,

$$\forall\, z \in K,\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| \leq \sum_{n=0}^{+\infty} |a_n| \rho^n.$$

Since the growth bound of the sequence a_n and $|a_n|$ are identical, the radius of convergence of the series $\sum_{n=0}^{+\infty} |a_n| (z-c)^n$ is r. Given that $\rho < r$, the series $\sum_{n=0}^{+\infty} |a_n| \rho^n$ is convergent; all its terms are non-negative real numbers, thus the sum is finite: there is a $\kappa > 0$ such that $\sum_{n=0}^{+\infty} |a_n| \rho^n \leq \kappa$. ∎

Remark – Other Types of Convergence. The locally normal convergence implies the *absolute convergence*:

$$\forall\, z \in D(c,r),\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| < +\infty.$$

It also provides the *locally uniform convergence*: on any compact subset K of $D(c,r)$, the partial sums $\sum_{n=0}^{p} a_n(z-c)^n$ converge uniformly to the sum $\sum_{n=0}^{+\infty} a_n(z-c)^n$:

$$\lim_{p\to+\infty} \sup_{z\in K} \left| \sum_{n=0}^{p} a_n(z-c)^n - \sum_{n=0}^{+\infty} a_n(z-c)^n \right| = 0.$$

Power Series and Holomorphic Functions

Theorem – Power Series Derivative. A power series and its *formal derivative*

$$\sum_{n=0}^{+\infty} a_n(z-c)^n \;\text{ and }\; \sum_{n=1}^{+\infty} n a_n (z-c)^{n-1}.$$

have the same radius of convergence r. The sum

$$f : z \in D(c,r) \mapsto \sum_{n=0}^{+\infty} a_n(z-c)^n$$

is holomorphic; its derivative is the sum of the formal derivative:

$$\forall\, z \in D(c,r),\; f'(z) = \sum_{n=1}^{+\infty} n a_n (z-c)^{n-1}.$$

More generally, the p-th order derivative of f is defined for any $p \in \mathbb{N}$ and

$$\forall\, z \in D(c,r),\; f^{(p)}(z) = \sum_{n=p}^{+\infty} n(n-1)\cdots(n-p+1) a_n (z-c)^{n-p}.$$

Lemma. For any $z \in \mathbb{C}$, $h \in \mathbb{C}^*$ and $n \geq 2$,

$$\left| \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right| \leq \frac{n(n-1)}{2} (|z| + |h|)^{n-2} |h|.$$

Proof – Lemma. Using the identity $a^n - b^n = (a-b) \sum_{m=0}^{n-1} a^m b^{n-1-m}$ yields

$$(z+h)^n - z^n = h \sum_{m=0}^{n-1} (z+h)^m z^{n-1-m},$$

hence

$$\begin{aligned}
\frac{(z+h)^n - z^n}{h} - n z^{n-1} &= \sum_{m=0}^{n-1} (z+h)^m z^{n-1-m} - \sum_{m=0}^{n-1} z^m z^{n-1-m} \\
&= \sum_{m=0}^{n-1} \left[(z+h)^m - z^m \right] z^{n-1-m}.
\end{aligned}$$

By the same identity, we also have

$$|(z+h)^m - z^m| = \left| h \sum_{l=0}^{m-1} (z+h)^l z^{m-1-l} \right| \leq m(|z|+|h|)^{m-1}|h|.$$

Therefore

$$\begin{aligned} \left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| &\leq \left[\sum_{m=0}^{n-1} m \ (|z|+|h|)^{m-1}(|z|+|h|)^{n-1-m} \right] |h| \\ &\leq \frac{n(n-1)}{2}(|z|+|h|)^{n-2}|h| \end{aligned}$$

as expected. ∎

Proof – Power Series Derivative. Let $D(c,r)$ be the open disk of convergence of the series

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n.$$

The radii of convergence of the series

$$\sum_{n=1}^{+\infty} n a_n (z-c)^{n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} n a_n (z-c)^n$$

are equal. Since the coefficient sequence of the latter series is the product of a_n and a nonzero polynomial sequence, the open radius of convergence of f and of its the formal derivative are identical. For any $z \in D(c,r)$ and any complex number h such that $0 < |h| < r$, define $e(z,h)$ as

$$e(z,h) = \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{+\infty} n a_n (z-c)^{n-1}.$$

A straightforward calculation leads to

$$e(z,h) = \sum_{n=1}^{+\infty} a_n \left[\frac{(z+h-c)^n - (z-c)^n}{h} - n(z-c)^{n-1} \right],$$

hence, using the lemma, we obtain

$$|e(z,h)| \leq \left[\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} |a_n| (|z-c|+|h|)^{n-2} \right] \times |h|.$$

The power series

$$\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} |a_n| z^{n-2}$$

has the same radius of convergence than

$$\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} a_n (z-c)^{n-2}$$

which is the the formal derivative of order 2 of the original series, hence the three series have the same radius of convergence r. Consequently, for any h such that $|z-c| + |h| < r$,

$$\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} |a_n| (|z-c| + |h|)^{n-2} < +\infty$$

and therefore

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \sum_{n=1}^{+\infty} n a_n (z-c)^{n-1}.$$

The statement about the p-th order derivative of f can be obtained by a simple induction on p. ■

Theorem & Definition – Taylor Series. If the complex-valued function f has a power series expansion centered at c inside the non-empty open disk $D(c,r)$, it is the *Taylor series* of f:

$$\forall\, z \in D(c,r),\ f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!} (z-c)^n.$$

Proof. If $f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n$, then for any $p \in \mathbb{N}$, the p-th order derivative of f inside $D(c,r)$ is given by

$$f^{(p)}(z) = \sum_{n=p}^{+\infty} n(n-1)\dots(n-p+1) a_n (z-c)^{n-p}$$

and consequently, $f^{(p)}(c) = p! a_p$. ■

Note that the above theorem is only a uniqueness result; it says nothing about the existence of the power series expansion. This is the role of the following theorem.

Theorem – Power Series Expansion. Let Ω be an open subset of $\mathbb{C}$, let $c \in \Omega$ and $r \in \,]0, +\infty]$ such that the open disk $D(c,r)$ is included in Ω. For any holomorphic function $f : \Omega \to \mathbb{C}$, there is a power series with coefficients a_n such that

$$\forall\, z \in D(c,r),\ f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n.$$

Its coefficients are given by

$$\forall\, \rho \in \,]0, r[\,,\ a_n = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft].$$

Proof – Power Series Expansion. For any $n \in \mathbb{N}$, the complex number

$$a_n = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz \;\text{ with }\; \gamma = c + \rho[\circlearrowleft]$$

is independent of ρ as long as $0 < \rho < r$. Indeed, if ρ_1 and ρ_2 are two such numbers, denote $\gamma_1 = c + \rho_1[\circlearrowleft]$ and $\gamma_2 = c + \rho_2[\circlearrowleft]$. The interior of the sequence of paths $\mu = \gamma_1 \,|\, \gamma_2^{\leftarrow}$ is included in $D(c, r) \setminus \{c\}$ where the function $z \mapsto f(z)/(z-c)^{n+1}$ is holomorphic. Hence, by Cauchy's integral theorem,

$$\int_\mu \frac{f(z)}{(z-c)^{n+1}}\, dz = \int_{\gamma_1} \frac{f(z)}{(z-c)^{n+1}}\, dz - \int_{\gamma_2} \frac{f(z)}{(z-c)^{n+1}}\, dz = 0.$$

Now, let $z \in D(c, r)$ and let $\rho \in \,]0, r[$ such that $|z - c| < \rho$. Cauchy's integral formula provides

$$f(z) = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

For any $w \in \gamma([0, 1])$, we have

$$\frac{1}{w-z} = \frac{1}{(w-c)-(z-c)} = \frac{1}{w-c} \frac{1}{1 - \frac{z-c}{w-c}}.$$

Since

$$\left| \frac{z-c}{w-c} \right| = \frac{|z-c|}{\rho} < 1,$$

we may expand $f(w)/(w - z)$ into

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-c} \frac{1}{1 - \frac{z-c}{w-c}} = \sum_{n=0}^{+\infty} \frac{f(w)}{w-c} \left(\frac{z-c}{w-c} \right)^n.$$

The term of this series is dominated by

$$\frac{\sup_{|w-c|=\rho} |f(w)|}{\rho} \left(\frac{|z-c|}{\rho} \right)^n;$$

the convergence of the series is normal – and thus uniform – with respect to the variable w. Finally

$$\begin{aligned}
f(z) &= \frac{1}{i2\pi} \int_\gamma \left[\sum_{n=0}^{+\infty} \frac{f(w)}{(w-c)^{n+1}} (z-c)^n \right] dw \\
&= \sum_{n=0}^{+\infty} \left[\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w-c)^{n+1}} (z-c)^n \, dw \right] \\
&= \sum_{n=0}^{+\infty} \left[\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w-c)^{n+1}}\, dw \right] (z-c)^n
\end{aligned}$$

which is the desired expansion. ■

Laurent Series

Definition – Annulus. Let $c \in \mathbb{C}$ and $r_1, r_2 \in [0, +\infty]$. We denote by

$$A(c, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - c| < r_2\}$$

the *open annulus* with *center* c, *inner radius* r_1 and *outer radius* r_2.

Examples – Annuli.

1. The open annulus $A(0, 0, +\infty)$, centered on the origin, with inner radius 0 and outer radius $+\infty$, is the set $\mathbb{C}^*$.
2. The sets $A(0, 0, 1)$, $A(0, 1, 2)$ and $A(0, 2, +\infty)$ are three open annuli centered on the origin and included in the open set $\Omega = \mathbb{C} \setminus \{i, 2\}$. They are maximal in Ω – if we decrease their inner radius and/or increase their outer radius the resulting annulus is not a subset of Ω anymore.

Definition – Laurent Series. The *Laurent series* centered on $c \in \mathbb{C}$ with coefficients $a_n \in \mathbb{C}$ for every $n \in \mathbb{Z}$ is

$$\sum_{n=-\infty}^{+\infty} a_n (z - c)^n.$$

It is *convergent* for some $z \in \mathbb{C} \setminus \{c\}$ if the series

$$\sum_{n=0}^{+\infty} a_n (z - c)^n \quad \text{and} \quad \sum_{n=1}^{+\infty} a_{-n} (z - c)^{-n}$$

are both convergent – otherwise it is *divergent*. When the Laurent series is convergent its *sum* is defined as

$$\sum_{n=-\infty}^{+\infty} a_n (z - c)^n = \sum_{n=0}^{+\infty} a_n (z - c)^n + \sum_{n=1}^{+\infty} a_{-n} (z - c)^{-n}.$$

Theorem – Convergence of Laurent Series. Let $c \in \mathbb{C}$ and let $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. The *inner radius of convergence* $r_1 \in [0, +\infty]$ and *outer radius of convergence* $r_2 \in [0, +\infty]$ of the Laurent series $\sum_{n=-\infty}^{+\infty} a_n (z - c)^n$ defined by

$$r_1 = \limsup_{n \to +\infty} |a_{-n}|^{1/n} \quad \text{and} \quad r_2 = \frac{1}{\limsup_{n \to +\infty} |a_n|^{1/n}}.$$

are such that the series converges in $A(c, r_1, r_2)$ and diverges if $|z - c| < r_1$ or $|z - c| > r_2$. In this *open annulus of convergence*, the convergence is locally normal.

Proof – Convergence of Laurent Series. The first series converges if $|z-c|$ is smaller than the radius of convergence r_2 of this power series and diverges if it is greater. We may rewrite the second series as:

$$\sum_{n=1}^{+\infty} a_{-n}(z-c)^{-n} = \sum_{n=1}^{+\infty} a_{-n}\left(\frac{1}{z-c}\right)^n.$$

Consequently, it converges if $|1/(z-c)|$ is smaller than the radius of convergence $1/r_1$ of the power series $\sum_{n=1}^{+\infty} a_{-n}z^n$, that is if $|z-c| > r_1$, and diverges if $|1/(z-c)|$ is greater than $1/r_1$, that is $|z-c|$ is smaller than r_1.

Now, for any $z \in A(c, r_1, r_2)$, there is an open neighbourhood U of z where $\sum_{n=0}^{+\infty} a_n(z-c)^n$ is normally convergent and an open neighbourhood V of $(z-c)^{-1}$ in $\mathbb{C}^*$ where $\sum_{n=1}^{+\infty} a_{-n}w^n$ is normally convergent. The Laurent series $\sum_{n=-\infty}^{+\infty} a_n(z-c)^n$ is normally convergent in the open neighbourhood $U \cap \{w^{-1} + c \mid w \in V\}$ of z. ■

Theorem – Laurent Series Expansion. Let Ω be an open subset of $\mathbb{C}$, let $c \in \mathbb{C}$ and $r_1, r_2 \in [0, +\infty]$ such that $r_1 < r_2$ and the open annulus $A(c, r_1, r_2)$ is included in Ω. For any holomorphic function $f : \Omega \to \mathbb{C}$, there is a Laurent series with coefficients a_n such that

$$\forall\, z \in A(c, r_1, r_2),\ f(z) = \sum_{n=-\infty}^{+\infty} a_n(z-c)^n.$$

Its coefficients are given by

$$\forall\, \rho \in \,]r_1, r_2[\,,\ a_n = \frac{1}{i2\pi}\int_\gamma \frac{f(z)}{(z-c)^{n+1}}\,dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft].$$

Proof – Laurent Series Expansion. For any integer n, the coefficient

$$a_n = \frac{1}{i2\pi}\int_\gamma \frac{f(z)}{(z-c)^{n+1}}\,dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft]$$

is independent of $\rho \in \,]r_1, r_2[$ – refer to the proof of "Power Series Expansion" for a detailled argument.

Let $z \in A(c, r_1, r_2)$ and $\rho_1, \rho_2 \in \,]r_1, r_2[$ such that $\rho_1 < |z-c| < \rho_2$. Let $\gamma_1 = c + \rho_1[\circlearrowleft]$ and $\gamma_2 = c + \rho_2[\circlearrowleft]$; Cauchy's integral formula provides

$$f(z) = \frac{1}{i2\pi}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{i2\pi}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

As in the proof of "Power Series Expansion", we can establish that

$$\frac{1}{i2\pi}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{+\infty}\left[\frac{1}{i2\pi}\int_{\gamma_2} \frac{f(w)}{(w-c)^{n+1}}\,dw\right](z-c)^n.$$

A similar argument, based on a series expansion of

$$\frac{1}{w-z} = -\frac{1}{(z-c)-(w-c)} = -\frac{1}{z-c}\frac{1}{1-\frac{w-c}{z-c}}$$

yields

$$\frac{1}{i2\pi}\int_{\gamma_1}\frac{f(w)}{w-z}\,dw = -\sum_{n=-1}^{-\infty}\left[\frac{1}{i2\pi}\int_{\gamma_1}\frac{f(w)}{(w-c)^{n+1}}\,dw\right](z-c)^n.$$

The combination of both expansions provides the expected result. ■