Zeros & Poles

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Contents

Exercises

The Weierstrass-Casorati Theorem

Question

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let $a \in \mathbb{C}$ be an essential singularity of f. Show that the image of f is dense in $\mathbb{C}$:

$$\forall\, w \in \mathbb{C},\ \forall\, \epsilon > 0,\ \exists\, z \in \Omega,\ |f(z) - w| < \epsilon.$$

Hint: assume instead that some complex number w is *not* in the closure of the image of f; study the function $z \mapsto 1/(f(z) - w)$ in a neighbourhood of a.

Answer

Assume that the image of f is not dense in $\mathbb{C}$; let then $w \in \mathbb{C}$ be such that

$$\exists \epsilon > 0, \ \forall z \in \Omega, \ |f(z) - w| \geq \epsilon.$$

The function $z \in \Omega \mapsto 1/(f(z) - w)$ is defined and holomorphic. As it satisfies

$$\forall z \in \Omega, \ \left| \frac{1}{f(z) - w} \right| \leq \frac{1}{\epsilon},$$

it is also bounded. Thus, the point a is a removable singularity of the function, that can be extended as a holomorphic function g on $\Omega \cup \{a\}$:

$$\forall z \in \Omega, \ g(z) = \frac{1}{f(z) - w}$$

By construction, g has no zero in Ω, thus a is either not a zero of g, or a zero of finite multiplicity. Since

$$\forall z \in \Omega, \ f(z) = w + \frac{1}{g(z)}$$

in the first case $f(z) \to w + 1/g(a)$ when $z \to a$ thus a is a removable singularity of a; in the second one, $|f(z)| \to +\infty$ when $z \to a$ thus a is a pole of f.

Note that either way, there is a non-negative integer p and a holomorphic function $h : \Omega \cup \{a\} \to \mathbb{C}$ such that $h(a) \neq 0$ and

$$\forall z \in \Omega \cup \{a\}, \ g(z) = h(z)(z - a)^p.$$

As the function g has no zero on Ω, the function h has no zero on $\Omega \cup \{a\}$; the function $1/h$ is defined and holomorphic on $\Omega \cup \{a\}$, $1/h(a) \neq 0$ and

$$\forall z \in \Omega, \ f(z) = w + \frac{1}{g(z)} = w + \frac{1}{h(z)} \frac{1}{(z - a)^p}.$$

Therefore, the point a is either a removable singularity of f (if $p = 0$), or a pole of order p (if $p \geq 1$).

The Maximum Principle

Question

Let Ω be an open connected subset of the complex plane and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Show that if $|f|$ has a local maximum at some $a \in \Omega$, then f is constant.

Answer

For any holomorphic function $f : \Omega \to \mathbb{C}$ and $a \in \Omega$, the point a is a zero of the holomorphic function $z \mapsto f(z) - f(a)$. We will prove shortly that if a is a zero of finite multiplicity of this function, $|f|$ does not have a local maximum at a. The conclusion of the proof follows by the Isolated Zeros Theorem.

Suppose that there is a positive integer p such that

$$f(z) = f(a) + g(z)(z-a)^p$$

for some holomorphic function $g : \Omega \to \mathbb{C}$ such that $g(a) \neq 0$; there is a function $\epsilon_a : \Omega \to \mathbb{C}$ such that $\epsilon_a(z) \to 0$ when $z \to a$ and

$$f(z) = f(a) + g(a)(z-a)^p + \epsilon_a(z)(z-a)^p$$

Assume that $f(a) \neq 0$ (if $f(a) = 0$, it is plain that $|f(a)| = 0$ cannot be a local maximum of $|f|$ at a). Let α, β and γ be some real numbers such that

$$f(a) = |f(a)|e^{i\alpha},\ g(a) = |g(a)|e^{i\beta},\ \gamma = \frac{\alpha - \beta}{p}.$$

For small enough values $r > 0$, we have

$$|f(a + re^{i\gamma}) - (|f(a)| + |g(a)|r^p)e^{i\alpha}| \leq |\epsilon_a(a + re^{i\gamma})|r^p \leq \frac{|g(a)|}{2} r^p,$$

which yields

$$|f(a + re^{i\gamma})| \geq |f(a)| + |g(a)|r^p - \frac{|g(a)|}{2} r^p > |f(a)|.$$

Therefore f has no maximum at a.

The Π Function

We introduce the Π function, a holomorphic extension of the factorial.

Questions

1. Find the domain in the complex plane of the function

$$\Pi : z \mapsto \int_0^{+\infty} t^z e^{-t}\, dt$$

and show that it is holomorphic.

2. Prove that whenever $\Pi(z)$ is defined, $\Pi(z+1)$ is also defined and

$$\Pi(z+1) = (z+1)\Pi(z).$$

Compute $\Pi(n)$ for every $n \in \mathbb{N}$.

3. Let Ω be an open connected subset of the complex plane that contains the domain of Π and such that $\Omega + 1 \subset \Omega$. Prove that if Π has a holomorphic extension on Ω (still denoted Π), it is unique and satisfies the functional equation

$$\forall\, z \in \Omega,\ \Pi(z+1) = (z+1)\Pi(z).$$

4. Prove the existence of such an extension Π on

$$\Omega = \mathbb{C} \setminus \{k \in \mathbb{Z} \mid k < 0\}.$$

5. Show that every negative integer is a simple pole of Π; compute the associated residue.

Answers

1. The function $t \in \mathbb{R}_+^* \mapsto t^z e^{-t}$ is continuous and thus measurable. Additionally, for any $t > 0$,

$$|t^z e^{-t}| = |e^{z \ln t} e^{-t}| = e^{(\operatorname{Re} z) \ln t} e^{-t} = t^{\operatorname{Re} z} e^{-t},$$

hence it is integrable if and only if $\operatorname{Re} z > -1$: the domain of Π is

$$\{z \in \mathbb{C} \mid \operatorname{Re} z > -1\}$$

and it is open. Now, let z and h be complex numbers in this domain; the associated difference quotient satisfies

$$\begin{aligned}
\frac{\Pi(z+h) - \Pi(z)}{h} &= \int_0^{+\infty} \frac{t^{z+h} - t^z}{h} e^{-t} dt \\
&= \int_0^{+\infty} \frac{t^h - 1}{h} t^z e^{-t} dt \\
&= \int_0^{+\infty} \frac{e^{h \ln t} - 1}{h} t^z e^{-t} dt \\
&= \int_0^{+\infty} \left[\frac{e^{h \ln t} - 1}{h \ln t}\right] t^z \ln t\, e^{-t} dt
\end{aligned}$$

The integrand converges pointwise when $h \to 0$:

$$\forall\, t > 0,\ \lim_{h \to 0} \left[\frac{e^{h \ln t} - 1}{h \ln t}\right] t^z \ln t\, e^{-t} = t^z \ln t\, e^{-t}.$$

Additionally, we have

$$\forall z \in \mathbb{C}^*, \quad \left|\frac{e^z - 1}{z}\right| \leq e^{|z|};$$

indeed, for any nonzero complex number z, the Taylor expansion of e^z at the origin provides

$$\left|\frac{e^z - 1}{z}\right| = \left|\sum_{n=0}^{+\infty} \frac{1}{(n+1)!} z^n\right| = \sum_{n=0}^{+\infty} \frac{1}{(n+1)!} |z|^n \leq \sum_{n=0}^{+\infty} \frac{1}{n!} |z|^n.$$

Hence,

$$\left|\frac{e^{h \ln t} - 1}{h \ln t}\right| \leq e^{|h||\ln t|} \leq \max(t^{|h|}, t^{-|h|})$$

and our integrand is dominated by

$$\max(t^{z+|h|}, t^{z-|h|}) \ln t \, e^{-t}$$

which is integrable whenever $\operatorname{Re}(z - |h|) > -1$. Finally, Lebesgue's dominated convergence theorem applies and Π is holomorphic.

2. If $\operatorname{Re} z > -1$, then $\operatorname{Re}(z+1) > -1$ and

$$\Pi(z+1) = \int_0^{+\infty} t^{z+1} e^{-t} \, dt.$$

By integration by parts,

$$\begin{aligned} \Pi(z+1) &= [t^{z+1}(-e^{-t})]_0^{+\infty} - \int_0^{+\infty} (z+1) t^z (-e^{-t}) \, dt \\ &= (z+1)\Pi(z). \end{aligned}$$

We have

$$\Pi(0) = \int_0^{+\infty} e^{-t} \, dt = [-e^{-t}]_0^{+\infty} = 1$$

and hence, by induction, $\Pi(n) = n!$ for any $n \in \mathbb{N}$.

3. There is at most one holomorphic extension Π of the original function to the connected open set Ω by the isolated zeros theorem (two extensions would be identical on the original domain of Π, which is a non-empty open set: the set of zeros of their difference would not be isolated).

It is plain that the function $z \mapsto \Pi(z+1) - (z+1)\Pi(z)$ is defined and holomorphic on Ω, a connected open set of the plane. Similarly, by the isolated zeros theorem, it is identically zero and hence the functional equation $\Pi(z+1) = (z+1)\Pi(z)$ holds on Ω.

4. We may define the extension $\Pi(z)$ as

$$\Pi(z) = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n)}$$

for any natural number n such $\mathrm{Re}(z+n) > -1$. This definition does not depend on the choice of n: if $m > n$, we have $\mathrm{Re}(z+m) > -1$ and

$$\Pi(z+m) = \Pi(z+n) \times (z+n+1)\cdots(z+m),$$

hence

$$\frac{\Pi(z+m)}{(z+1)(z+2)\cdots(z+m)} = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n)}.$$

It is plain that this extension of the original function Π is holomorphic.

5. Let n be a positive integer. Let z be a complex number such that $|z - (-n)| < 1$; it satisfies $\mathrm{Re}(z+n) > -1$ and thus

$$\Pi(z) = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n)}.$$

Consequently,

$$(z-(-n))\Pi(z) = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n-1)}$$

and

$$\lim_{z\to -n}(z-(-n))\Pi(z) = \frac{\Pi(0)}{(-n-1)(-n-2)\cdots(-1)} = \frac{(-1)^{n-1}}{(n-1)!}.$$

As this number differ from zero, $z = -n$ is a simple pole of Π and

$$\mathrm{res}(\Pi, -n) = \frac{(-1)^{n-1}}{(n-1)!}.$$

Singularities and Residues

Question

Analyze the singularities (location, type, residues) of

$$z \mapsto \frac{\sin \pi z}{\pi z},\ z \mapsto \frac{1}{(\sin \pi z)^2},\ z \mapsto \sin \frac{\pi}{z},\ z \mapsto \frac{1}{\sin \frac{\pi}{z}}.$$

Answer

The function $z \mapsto \sin \pi z$ is defined and holomorphic in $\mathbb{C}$. Its Taylor expansion, valid for any $z \in \mathbb{C}$, is

$$\sin \pi z = \sum_{n=0}^{+\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} z^{2n+1}.$$

The function $z \mapsto \frac{\sin \pi z}{\pi z}$ is therefore defined and holomorphic in $\mathbb{C}^*$ where its Laurent expansion is

$$\frac{\sin \pi z}{\pi z} = \sum_{n=0}^{+\infty} \frac{(-1)^n \pi^{2n}}{(2n+1)!} z^{2n}.$$

The series on the right-hand side of this equation has no negative power of z: it is a power series that converges for any $z \in \mathbb{C}^*$, thus its open disk of convergence is actually $\mathbb{C}$. Its limit is a holomorphic function that extends $z \mapsto \frac{\sin \pi z}{\pi z}$ to $\mathbb{C}$, hence 0 is a removable singularity of this function (and its residue is 0).

The singularities of $z \mapsto 1/(\sin \pi z)^2$ are the zeros of $z \in \mathbb{C} \mapsto \sin \pi z$: the integers. The function is invariant if we substitute $z + k$ to z for any $k \in \mathbb{Z}$, hence we may limit our analysis of the singularities to the origin. If z is not an integer, we have

$$\frac{1}{(\sin \pi z)^2} = \frac{1}{\pi^2 z^2} \left(\frac{\pi z}{\sin \pi z} \right)^2.$$

The function $z \mapsto (\pi z / \sin \pi z)^2$ has a removable singularity at the origin and the value of its holomorphic extension at the origin is nonzero (it is 1), thus the origin is a double pole of the function. We have therefore

$$\operatorname{res}\left(z \mapsto \frac{1}{(\sin \pi z)^2}, 0 \right) = \lim_{z \to 0} \left[z^2 \frac{1}{(\sin \pi z)^2} \right]'.$$

We have

$$\left[z^2 \frac{1}{(\sin \pi z)^2} \right]' = \frac{2}{\pi} \left(\frac{(\pi z) \sin \pi z - (\pi z)^2 \cos \pi z}{(\sin \pi z)^3} \right).$$

The Taylor expansions of the functions sin and cos on $\mathbb{C}$ provide

$$\sin w = w \left(\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} w^{2n} \right) = w - \frac{w^3}{6} + w^5 \left(\sum_{n=2}^{+\infty} \frac{(-1)^n}{(2n+1)!} w^{2n-4} \right)$$

and

$$\cos w = 1 - \frac{w^2}{2} + w^4 \left(\sum_{n=2}^{+\infty} \frac{(-1)^n}{(2n)!} w^{2n-4} \right),$$

thus there are entire functions f and g such that

$$w \sin w - w^2 \cos w = \left(w^2 - \frac{1}{6} w^4 \right) - \left(w^2 - \frac{1}{2} w^4 \right) + w^6 f(w)$$

and

$$(\sin w)^3 = w^3 g(w),\ g(0) = 1.$$

Consequently,

$$\operatorname{res}\left(z \mapsto \frac{1}{(\sin \pi z)^2}, 0\right) = \lim_{w \to 0} \frac{2}{\pi} \frac{w/3 + w^3 f(w)}{g(w)} = 0.$$

Alternatively, to compute the residue, we may notice that if z is not an integer

$$\frac{1}{(\sin \pi z)^2} = \frac{1}{(\sin \pi(-z))^2},$$

thus if $\sum_{n=-\infty}^{+\infty} a_n z^n$ is the Laurent expansion of the right-hand side in $D(0,1) \setminus \{0\}$, the Laurent expansion $\sum_{n=-\infty}^{+\infty} (-1)^n a_n z^n$ is also valid in the same annulus. The uniqueness of the Laurent expansion yields $a_n = 0$ for every odd n, thus the residue of the function at the origin – which is a_{-1} – is zero.

The function $z \mapsto \sin \frac{\pi}{z}$ is defined and holomorphic on $\mathbb{C}^*$. It has a Laurent expansion in this annulus, which is

$$\sin \frac{\pi}{z} = \sum_{n=0}^{+\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} z^{-(2n+1)}.$$

There are an infinite number of nonzero coefficients associated with negative powers of z, thus 0 is an essential singularity of this function. Its residue at 0 is the coefficient of z^{-1}, which is π.

The zeros of $z \in \mathbb{C} \mapsto \sin \pi z$ are the integers, thus $z \mapsto 1/\sin \frac{\pi}{z}$ is defined and holomorphic on the open set $\Omega = \mathbb{C}^* \setminus \{1/k \mid k \in \mathbb{Z}^*\}$. We can write the function as the quotient of $f(z) = 1$ and $g(z) = \sin \frac{\pi}{z}$. The functions f and g are defined and holomorphic in $\mathbb{C}^*$ and

$$g'(z) = \left(\cos \frac{\pi}{z}\right)\left(-\frac{\pi}{z^2}\right).$$

Thus, for any $k \in \mathbb{Z}^*$, $1/k$ is a simple pole of $z \mapsto 1/\sin \frac{\pi}{z}$ and

$$\operatorname{res}\left(z \mapsto \frac{1}{\sin \frac{\pi}{z}}, \frac{1}{k}\right) = \frac{1}{(\cos \frac{\pi}{k^{-1}})(-\frac{\pi}{(k^{-1})^2})} = \frac{(-1)^{k+1}}{\pi k^2}.$$

The origin $z = 0$ is also singularity of $z \mapsto 1/\sin \frac{\pi}{z}$, but it is not isolated, thus its residue is not defined.

Integrals of Functions of a Real Variable

See "Technologie de calcul des intégrales à l'aide de la formule des résidus" (Demailly 2009, chap. III, sec. 4) for a comprehensive analysis of the computation of integrals with the the residue theorem.

Questions

1. For any $n \geq 2$, compute
$$\int_0^{+\infty} \frac{dx}{1+x^n}.$$

2. Compute
$$\int_0^{+\infty} \frac{\sqrt{x}}{1+x+x^2}\,dx.$$

Answers

1. Let f be the function $z \mapsto 1/(1+z^n)$, defined and holomorphic on
$$\Omega = \mathbb{C} \setminus \left\{ e^{\frac{i(2k+1)\pi}{n}} \;\middle|\; k \in \{0, \dots, n-1\} \right\}.$$
Let $r > 1$ and define the rectifiable paths γ_1, γ_2 and γ_3 as
$$\gamma_1 = [0 \to r],\ \gamma_2 = re^{i[0 \to 2\pi/n]},\ \gamma_3 = [re^{i2\pi/n} \to 0],$$
then set $\gamma = \gamma_1 \mid \gamma_2 \mid \gamma_3$. It is plain that
$$\lim_{r \to +\infty} \int_{\gamma_1} \frac{dz}{1+z^n} = \int_0^{+\infty} \frac{dx}{1+x^n}.$$
Similarly,
$$\int_{\overleftarrow{\gamma_3}} \frac{dz}{1+z^n} = \int_0^1 \frac{re^{i\frac{2\pi}{n}}\,dt}{1+(rt)^n(e^{i\frac{2\pi}{n}})^n} = e^{i\frac{2\pi}{n}} \int_0^r \frac{dx}{1+x^n},$$
thus
$$\lim_{r \to +\infty} \int_{\gamma_3} \frac{dz}{1+z^n} = -e^{i\frac{2\pi}{n}} \int_0^{+\infty} \frac{dx}{1+x^n}.$$
Finally, by the M-L inequality,
$$\left| \int_{\gamma_2} \frac{dz}{1+z^n} \right| \leq \frac{1}{r^n-1} \times \left(\frac{2\pi}{n} r \right),$$
hence
$$\lim_{r \to +\infty} \int_{\gamma_2} \frac{dz}{1+z^n} = 0.$$
On the other hand, the complex number $e^{i\frac{\pi}{n}}$ is the unique singularity of f in the interior of γ; more precisely, we have $\mathrm{ind}(\gamma, e^{i\frac{\pi}{n}}) = 1$. The function f is the quotient of the holomorphic functions $p : z \in \mathbb{C} \mapsto 1$ and $q : z \in \mathbb{C} \mapsto 1+z^n$; the derivative of q at this singularity is
$$q'(e^{i\frac{\pi}{n}}) = n(e^{i\frac{\pi}{n}})^{n-1} = n(e^{i\frac{\pi}{n}})^n e^{-i\frac{\pi}{n}} = -ne^{-i\frac{\pi}{n}},$$

thus

$$\operatorname{res}(f, e^{i\frac{\pi}{n}}) = \frac{p(e^{i\frac{\pi}{n}})}{q'(e^{i\frac{\pi}{n}})} = -\frac{e^{i\frac{\pi}{n}}}{n}$$

Given these results, the residue theorem provides

$$\left(1 - e^{i\frac{2\pi}{n}}\right) \int_0^{+\infty} \frac{dx}{1+x^n} = (i2\pi) \times \left(-\frac{e^{i\frac{\pi}{n}}}{n}\right)$$

or equivalently,

$$\int_0^{+\infty} \frac{dx}{1+x^n} = \frac{\pi}{n} \frac{2i}{e^{i\frac{\pi}{n}} - e^{-i\frac{\pi}{n}}} = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}.$$

2. Let $\log_0$ be the function defined on $\mathbb{C} \setminus \mathbb{R}_+$ by

$$\log_0 z = \log(-z) + i\pi.$$

This function is an analytic choice of the logarithm on $\mathbb{C} \setminus \mathbb{R}_+$: it is holomorphic and $\exp \circ \log_0$ is the identity. It also satisfies

$$\log_0 re^{i\theta} = (\ln r) + i\theta,\ r > 0,\ \theta \in \left]0, 2\pi\right[.$$

We use this function to define

$$f : z \mapsto \frac{e^{\frac{1}{2}\log_0 z}}{1+z+z^2}.$$

The roots of the polynomial $z \mapsto 1+z+z^2$ are j and j^2, where $j = e^{i\frac{2\pi}{3}}$, thus f is defined and holomorphic in $\Omega = \mathbb{C} \setminus \mathbb{R}_+ \setminus \{j, j^2\}$.

Now, let $r > 1$ and $0 < \alpha < 2\pi/3$; we define four rectifiable paths that depend on r and α:

$$\begin{aligned}
\gamma_1 &= [r^{-1}e^{i\alpha} \to re^{i\alpha}], \\
\gamma_2 &= re^{i[\alpha \to 2\pi - \alpha]}, \\
\gamma_3 &= [re^{i(2\pi-\alpha)} \to r^{-1}e^{i(2\pi-\alpha)}], \\
\gamma_4 &= r^{-1}e^{i[2\pi-\alpha \to \alpha]}.
\end{aligned}$$

We also consider their concatenation

$$\gamma = \gamma_1 \mid \gamma_2 \mid \gamma_3 \mid \gamma_4.$$

We have

$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz &= \int_{r^{-1}}^{r} \frac{e^{\frac{1}{2}((\ln x)+i\alpha)}}{1+xe^{i\alpha}+x^2e^{i2\alpha}}\, e^{i\alpha}dx \\
&= e^{i3\alpha/2} \int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+xe^{i\alpha}+x^2e^{i2\alpha}}\,dx
\end{aligned}$$

and thus by the dominated convergence theorem[1]

$$\lim_{\alpha\to 0}\int_{\gamma_1} f(z)\,dz = \int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+x+x^2}\,dx.$$

Similarly,

$$\begin{aligned}\int_{\gamma_3^{\leftarrow}} f(z)\,dz &= \int_{r^{-1}}^{r} \frac{e^{\frac{1}{2}((\ln x)+i(2\pi-\alpha))}}{1+xe^{-i\alpha}+x^2e^{-i2\alpha}}\,e^{-i\alpha}dx \\ &= -e^{-i3\alpha/2}\int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+xe^{-i\alpha}+x^2e^{-i2\alpha}}\,dx\end{aligned}$$

and thus by the dominated convergence theorem

$$\lim_{\alpha\to 0}\int_{\gamma_3} f(z)\,dz = \int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+x+x^2}\,dx$$

On the other hand,

$$\left|e^{\frac{1}{2}\log_0 z}\right| = e^{\mathrm{Re}(\frac{1}{2}\log_0 z)} = e^{\frac{1}{2}\ln|z|} = |z|^{\frac{1}{2}};$$

by the M-L inequality, this equality provides

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{r^{\frac{1}{2}}}{-1-r+r^2}\times 2(\pi-\alpha)r$$

and

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{r^{-\frac{1}{2}}}{1-r^{-1}-r^{-2}}\times 2(\pi-\alpha)r^{-1},$$

hence

$$\lim_{r\to+\infty}\left(\lim_{\alpha\to 0}\int_{\gamma_2} f(z)\,dz\right) = \lim_{r\to+\infty}\left(\lim_{\alpha\to 0}\int_{\gamma_4} f(z)\,dz\right) = 0.$$

Now the function f is the quotient of the two functions $z\mapsto e^{\frac{1}{2}\log_0 z}$ and $z\mapsto 1+z+z^2$, defined and holomorphic in a neighbourhood of the singularities j and j^2. The derivative of $z\mapsto 1+z+z^2$ is $z\mapsto 1+2z$, it is nonzero at j and j^2. Thus,

$$\mathrm{res}(f,j) = \frac{e^{\frac{1}{2}\log_0 j}}{1+2j} = \frac{e^{i\frac{\pi}{3}}}{i\sqrt{3}}$$

and

$$\mathrm{res}(f,j^2) = \frac{e^{\frac{1}{2}\log_0 j^2}}{1+2j^2} = \frac{e^{i\frac{2\pi}{3}}}{-i\sqrt{3}}.$$

[1] the function $(\alpha,x)\mapsto\left|\sqrt{x}/(1+xe^{i\alpha}+x^2e^{i2\alpha})\right|$ is defined and continuous in the compact set $[0,\pi/2]\times[r^{-1},r]$, thus it has a finite upper bound.

The winding number of γ around j and j^2 is 1; by the residue theorem,

$$2\int_0^{+\infty} \frac{\sqrt{x}}{1+x+x^2}\,dx = (i2\pi)(\mathrm{res}(f,j) + \mathrm{res}(f,j^2))$$

or equivalently

$$\int_0^{+\infty} \frac{\sqrt{x}}{1+x+x^2}\,dx = \frac{\pi}{\sqrt{3}}(e^{i\frac{\pi}{3}} - e^{i\frac{2\pi}{3}}) = \frac{\pi}{\sqrt{3}}.$$

References

Demailly, Jean-Pierre. 2009. *Fonctions holomorphes et surfaces de riemann.* https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/variable_complexe.pdf.