Line Integrals & Primitives

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January 2, 2018

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Introduction

The main goal of this chapter is to derive the fundamental theorem of calculus for functions of a complex variable. This theorem characterizes the relation between functions and their primitives with the help of integrals. A version of this theorem for functions of a real variable is the following:

Theorem – Fundamental Theorem of Calculus (Real Analysis). Let I be an open interval of $\mathbb{R}$, $f : I \to \mathbb{R}$ be a continuous function and $a \in I$. A function $g : I \to \mathbb{R}$ is a primitive of f if and only if it satisfies

$$\forall\, x \in I,\ g(x) = g(a) + \int_a^x f(t)\,dt.$$

Proof. Suppose that the function g satisfies the integral equation of the theorem.

For any $x \in I$ and any real number h such that $x + h \in I$,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

$$= \frac{1}{h} \int_x^{x+h} f(x)\,dt + \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\,dt$$

$$= f(x) + \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\,dt,$$

Let $\epsilon > 0$; by continuity of f at x, there is a $\delta > 0$ such that

$$\forall t \in I, \ (|t - x| \leq \delta \Rightarrow |f(t) - f(x)| < \epsilon)$$

thus if $|h| < \delta$,

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| \leq \frac{1}{|h|} |h| \times \epsilon = \epsilon.$$

The difference quotient tends to $f(x)$ when h tends to zero: $g'(x)$ exists and is equal to $f(x)$.

Conversely, suppose that $e : I \to \mathbb{R}$ is a primitive of f. The difference d between e and the function

$$g : x \in I \mapsto e(a) + \int_a^x f(t)\,dt$$

is zero at a and has a zero derivative on I. By the mean value theorem, for any $x \in I$ such that $x \neq a$, there is a $b \in I$ such that

$$\frac{d(x) - d(a)}{x - a} = d'(b) = 0,$$

hence $d(x) = d(a) = 0$ and therefore $e = g$. ■

Paths

Definition – Path. A *path* γ is a continuous function from $[0, 1]$ to $\mathbb{C}$. If A is a subset of the complex plane, γ is a *path of A* if additionally $\gamma([0, 1]) \subset A$.

Definition – Image of a Path. The *image* or *trajectory* or *trace* of the path γ is the image $\gamma([0, 1])$ of the interval $[0, 1]$ by the function γ.

Definition – Path Endpoints. The complex numbers $\gamma(0)$ and $\gamma(1)$ are the *initial point* and *terminal point* of γ – they are its *endpoints*; the path γ *joins* its initial and terminal points. The path is *closed* if the initial and terminal point are the same. The paths $\gamma_1, \ldots, \gamma_n$ are *consecutive* if for $k = 1, \ldots, n - 1$, the terminal point of γ_k is the initial point of γ_{k+1}.

Example – Oriented Line Segment. The oriented line segment (or simply oriented segment) with initial point $a \in \mathbb{C}$ and terminal point $b \in \mathbb{C}$ is denoted $[a \to b]$ and defined as

$$[a \to b] : t \in [0, 1] \mapsto (1 - t)a + tb.$$
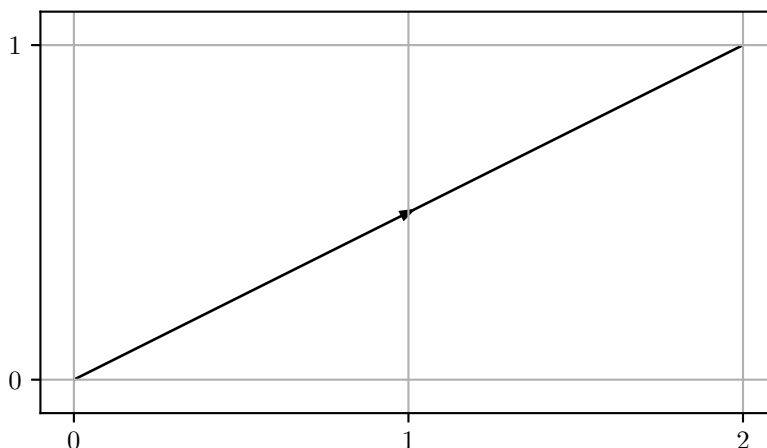
Its image is the line segment $[a, b]$.

Figure 1: Representation of the oriented line segment $[0 \to 2 + i]$

Example – Oriented Circle. The oriented circle of radius one centered at the origin traversed once in the positive sense (counterclockwise) is denoted $[\circlearrowleft]$ and defined as

$$[\circlearrowleft] : t \in [0, 1] \to e^{i2\pi t}.$$

The circle of radius $r \geq 0$ centered at $c \in \mathbb{C}$ traversed $n \in \mathbb{Z}^*$ times in the positive sense is the path

$$c + r[\circlearrowleft]^n : t \in [0, 1] \to c + re^{i2\pi nt}.$$

Its image is the circle centered on c with radius r; its initial and terminal points are both $c + r$, hence it is closed.

Definition – Open (Path-)Connected Sets. An open subset Ω of the complex-plane is *(path-)connected* if for any points x and y of Ω, there is a path of Ω that joins x and y.

Definition – Reverse Path. The *reverse* (or *opposite*) of the path γ is the path $\gamma^{\leftarrow}$ defined by

$$\forall\, t \in [0, 1], \; \gamma^{\leftarrow}(t) = \gamma(1 - t).$$

Definition – Path Concatenation. Let $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ be a partition of the interval $[0, 1]$. The *concatenation* of consecutive paths
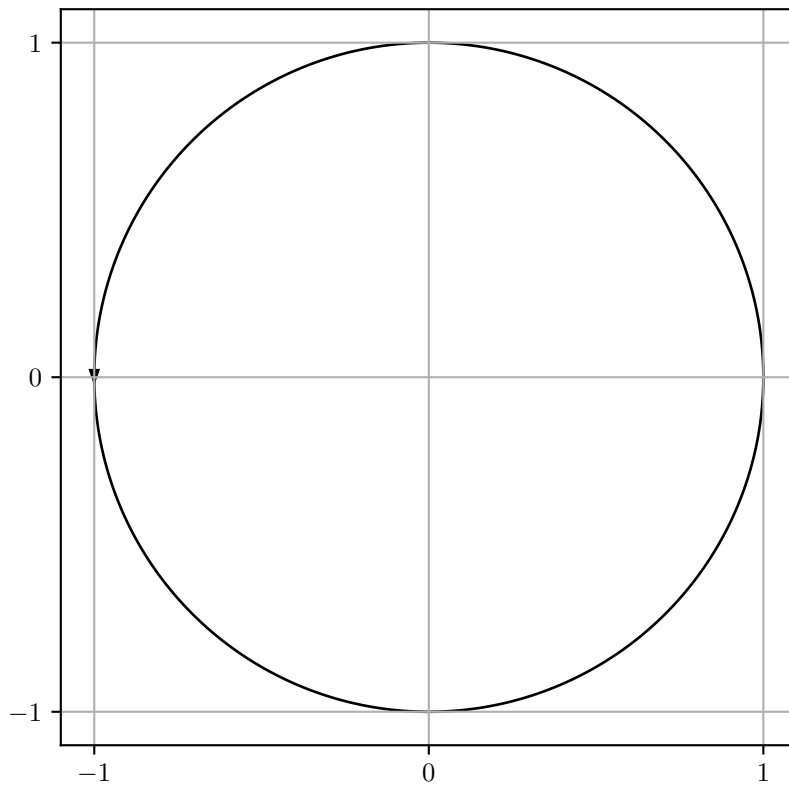
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Figure 2: Representation of the oriented circle [↺]

$\gamma_1, \ldots, \gamma_n$ associated to this partition is the path γ denoted

$$\gamma_1 \mid_{t_1} \cdots \mid_{t_{n-1}} \gamma_n$$

such that

$$\forall\, k \in \{1, \ldots, n\},\ \gamma|_{[t_{k-1}, t_k]} = \gamma_k \left(\frac{t - t_{k-1}}{t_k - t_{k-1}} \right).$$

If the partition of $[0, 1]$ is uniform, that is, if

$$\forall\, k \in \{0, \ldots, n\},\ t_k = k/n,$$

we denote the concatenated path with the simpler notation

$$\gamma_1 \mid \cdots \mid \gamma_n.$$

Example – Oriented Polyline. An oriented polyline (or piecewise linear path) is the concatenation of consecutive oriented line segments. When the associated partition of $[0, 1]$ is uniform, we use the notation

$$[a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n] = [a_0 \to a_1] \mid \cdots \mid [a_{n-1} \to a_n].$$

Definition – Rectifiable Path. A path $\gamma : [0, 1] \to \mathbb{C}$ is *rectifiable* if the function γ is piecewise continuously differentiable.

Given the definition of piecewise continuously differentiable, the following alternate characterization is plain:

Theorem – Continuously Differentiable Decomposition. A path $\gamma : [0, 1] \to \mathbb{C}$ is rectifiable if and only if there are consecutive continuously differentiable paths $\gamma_1, \ldots, \gamma_n$ and a partition $(t_0, \ldots, t_n)$ of the interval $[0, 1]$ such that

$$\gamma = \gamma_1 \mid_{t_1} \cdots \mid_{t_{n-1}} \gamma_n.$$

We characterized initially connected sets via merely continuous paths. However, when such sets are open, we can use rectifiable paths instead:

Lemma – Connectedness & Rectifiable Paths. An open subset Ω of the complex plane is *connected* if and only if every pair of points of Ω may be joined by a rectifiable path of Ω.

Proof. If any pair of points of Ω can be joined by a rectifiable path of Ω, then Ω is connected. Conversely, assume that a (merely continuous) path γ of Ω joins x and y. Its image $\gamma([0, 1])$ is a compact subset of Ω – as the image of a compact set by a continuous function – thus the distance r between $\gamma([0, 1])$ and the closed set $\mathbb{C} \setminus \Omega$ is positive. Additionally, the function γ is uniformly continuous – as a continuous function with a compact domain of definition; there is a positive integer n such that

$$\forall\, t \in [0, 1],\ \forall\, s \in [0, 1],\ (|t - s| \leq 1/n \;\Rightarrow\; |\gamma(t) - \gamma(s)| < r).$$

For any $k \in \{0, \dots, n\}$, the point $\gamma(k/n)$ belongs to Ω; the path μ defined as

$$\mu = [\gamma(0) \to \cdots \to \gamma(k/n) \to \cdots \to \gamma(1)]$$

is rectifiable and joins x and y. Now, for any $t \in [0,1]$, let $k \in \{0, \dots, n-1\}$ be such that $t \in [k/n, (k+1)/n]$. We have

$$|\mu(t) - \gamma(k/n)| \leq |\gamma((k+1)/n) - \gamma(k/n)| < r,$$

therefore μ is a path of Ω. ∎

Line Integrals

Definition – Length of a Rectifiable Path. The length of a rectifiable path γ is the nonnegative real number

$$\ell(\gamma) = \int_0^1 |\gamma'(t)| \, dt.$$

Example – Length of an Oriented Segment. The oriented segment $[a \to b]$ is continuously differentiable and thus rectifiable. For any $t \in [0,1]$, $[a \to b]'(t) = b - a$, hence its length is

$$\ell([a \to b]) = \int_0^1 |b - a| \, dt = |b - a|.$$

Example – Length of an Oriented Circle. The oriented circle $c + r[\circlearrowleft]^n$ centered at c with radius $r \geq 0$ traversed n times in the positive sense is continuously differentiable and thus rectifiable. For any $t \in [0,1]$,

$$[c + r[\circlearrowleft]^n]'(t) = (i2\pi n) r e^{i2\pi n t},$$

hence the length of this path is

$$\ell(c + r[\circlearrowleft]^n) = \int_0^1 |(i2\pi n) r e^{i2\pi n t}| \, dt = \int_0^1 |2\pi n r| \, dt = 2\pi r \times |n|.$$

It differs from the length of its circle image – which is $2\pi r$ – unless the circle is traversed exactly once in the positive or negative sense.

Definition – Line Integral. The *line integral* along a rectifiable path γ of a complex-valued function f defined and continuous on the image of γ is the complex number defined by

$$\int_\gamma f(z) \, dz = \int_0^1 f(\gamma(t)) \gamma'(t) \, dt.$$

Remark – Undefined Integrands. In the definitions of the length of γ and of the integral along γ, the integrands

$$|\gamma'(t)| \quad \text{and} \quad f(\gamma(t))\gamma'(t)$$

may be undefined for some values of t if γ is merely rectifiable. However it's not an issue since they are always defined almost everywhere (and integrable).

Remark – Integral Notation. It's sometimes handy to use the notation

$$\int_\gamma f(z)\,|dz| = \int_0^1 f(\gamma(t))|\gamma'(t)|dt.$$

which is similar to the one used for line integrals. With this convention, we have for example

$$\ell(\gamma) = \int_\gamma |dz|.$$

Example – Integration along an Oriented Segment. The line integral of the continuous function $f : [a, b] \mapsto \mathbb{C}$ along the oriented segment $[a \to b]$ is

$$\int_{[a\to b]} f(z)\,dz = \int_0^1 f((1-t)a + tb)(b-a)\,dt$$

$$= (b-a)\int_0^1 f((1-t)a + tb)\,dt.$$

Example – Integration along an Oriented Circle. The line integral of a continuous function $f : \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C}$ on the oriented circle $[\circlearrowleft]$ is

$$\int_{[\circlearrowleft]} f(z)\,dz = \int_0^1 f(e^{i2\pi t})(i2\pi e^{i2\pi t}dt)$$

$$= i\int_0^1 f(e^{i2\pi t})e^{i2\pi t}\,(2\pi dt)$$

$$= i\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta.$$

Theorem – Complex-Linearity of the Line Integral. Let γ be a rectifiable path. For any $\alpha, \beta \in \mathbb{C}$ and any continuous functions f and g defined on the image of γ,

$$\int_\gamma \alpha f(z) + \beta g(z)\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$$

Proof. Since by definition of the line integral

$$\int_\gamma \alpha f(z) + \beta g(z)\,dz = \int_0^1 (\alpha f(\gamma(t)) + \beta g(\gamma(t)))\gamma'(t)dt,$$

the complex-linearity of the integral on $[0, 1]$ provides

$$\int_\gamma \alpha f(z) + \beta g(z) \, dz = \alpha \int_0^1 f(\gamma(t))\gamma'(t) \, dt$$
$$+ \beta \sum_k \int_0^1 g(\gamma(t))\gamma'(t) \, dt$$

■

Theorem – Integration along a Reverse Path. For any rectifiable path γ,

$$\ell(\gamma^\leftarrow) = \ell(\gamma).$$

For any continuous function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of γ,

$$\int_{\gamma^\leftarrow} f(z) \, dz = - \int_\gamma f(z) \, dz.$$

Proof. Since $\gamma^\leftarrow(t) = \gamma(1 - t)$, the length of the opposite of γ satisfies

$$\ell(\gamma^\leftarrow(t)) = \int_0^1 |(\gamma^\leftarrow)'(t)| \, dt = \int_0^1 |-\gamma'(1 - t)| \, dt.$$

The change of variable $t \mapsto 1 - t$ yields

$$\ell(\gamma^\leftarrow(t)) = \int_0^1 |\gamma'(t)| \, dt = \ell(\gamma).$$

Similarly,

$$\int_{\gamma^\leftarrow} f(z) \, dz = \int_0^1 f(\gamma^\leftarrow(t))(\gamma^\leftarrow)'(t) \, dt$$
$$= \int_0^1 f(\gamma(1 - t))(-\gamma'(1 - t)) \, dt$$
$$= \int_0^1 f(\gamma(t))(-\gamma'(t)) \, dt$$
$$= - \int_\gamma f(z) \, dz$$

■

Theorem – Integration along Concatenation of Paths. Let A be a subset of $\mathbb{C}$. Let $\gamma_1, \ldots, \gamma_n$ be consecutive rectifiable paths of A and let γ be their concatenation

$$\gamma = \gamma_1 \big|_{t_1} \cdots \big|_{t_{n-1}} \gamma_n.$$

The length of γ satisfies

$$\ell(\gamma) = \sum_{k=1}^n \ell(\gamma_k).$$

For any continuous function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of γ,

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.$$

Proof. Since by definition $\gamma(t) = \gamma_k((t-t_k)/(t_k - t_{k-1}))$ whenever $t \in [t_k, t_{k+1}]$, the decomposition

$$
\begin{aligned}
\ell(\gamma) &= \int_0^1 |\gamma'(t)|\,dt \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\gamma'(t)|\,dt
\end{aligned}
$$

provides

$$\ell(\gamma) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left| \frac{\gamma_k'\left(\frac{t-t_k}{t_{k+1}-t_k}\right)}{t_k - t_{k-1}} \right|\,dt$$

and the changes of variables $t \in [t_k, t_{k+1}] \mapsto \frac{t-t_k}{t_{k+1}-t_k}$ yield

$$
\begin{aligned}
\ell(\gamma) &= \sum_{k=1}^{n} \int_0^1 |\gamma_k'(t)|\,dt \\
&= \sum_{k=1}^{n} \ell(\gamma_k)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)\,dt \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)\,dt \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f\left(\gamma_k\left(\frac{t-t_k}{t_{k+1}-t_k}\right)\right) \frac{\gamma_k'\left(\frac{t-t_k}{t_{k+1}-t_k}\right)}{t_k - t_{k-1}}\,dt \\
&= \sum_{k=1}^{n} \int_0^1 f(\gamma_k(t))\gamma_k'(t)\,dt \\
&= \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz
\end{aligned}
$$

$\blacksquare$

Theorem – M-L Inequality. For any rectifiable path γ and any continuous function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of γ,

$$\left| \int_\gamma f(z)\, dz \right| \leq \left(\max_{z \in \gamma([0,1])} |f(z)| \right) \times \ell(\gamma).$$

Proof. By definition of the line integral

$$
\begin{aligned}
\left| \int_\gamma f(z)\, dz \right| &= \left| \int_0^1 f(\gamma(t))\gamma'(t)\, dt \right| \\
&\leq \int_0^1 |f(\gamma(t))||\gamma'(t)|\, dt \\
&\leq \left(\max_{t \in [0,1]} |f(\gamma(t))| \right) \times \int_0^1 |\gamma'(t)|\, dt \\
&= \left(\max_{z \in \gamma([0,1])} |f(z)| \right) \times \ell(\gamma).
\end{aligned}
$$

$\blacksquare$

A practical consequence of the M-L inequality:

Corollary – Convergence in Line Integrals. For any rectifiable path γ and any sequence of continuous function $f_n : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of γ which converges uniformly to the function f,

$$\lim_{n \to +\infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz.$$

Proof. The linearity of the line integral and the M-L inequality provide

$$
\begin{aligned}
\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z) \right| &= \left| \int_\gamma (f_n(z) - f(z))\, dz \right| \\
&\leq \left(\max_{z \in \gamma([0,1])} |f_n - f(z)| \right) \times \ell(\gamma)
\end{aligned}
$$

which yields the desired result. $\blacksquare$

Theorem – Invariance By Reparametrization. Let $\gamma : [0,1] \to \mathbb{C}$ be a continuously differentiable path. Let $\phi : [0,1] \to [0,1]$ be an increasing C^1-diffeomorphism – a continuously differentiable function such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi'(t) > 0$ for any $t \in [0,1]$. The following statements hold:

- The path $\mu = \gamma \circ \phi$ is a continuously differentiable path.

- It has the same initial point, terminal point and image as γ.

- The length of μ and γ are identical.

- For any continuous function $f : \gamma([0,1]) \to \mathbb{C}$,

$$\int_\mu f(z)\,dz = \int_\gamma f(z)\,dz.$$

Proof. The function μ is continuously differentiable as the composition of continuously differentiable functions. We have

$$\mu(0) = \gamma(\phi(0)) = \gamma(0),\ \mu(1) = \gamma(\phi(1)) = \gamma(1),$$

hence the endpoints of γ and μ are identical. The function ϕ is a bijection from $[0,1]$ into itself, therefore

$$\mu([0,1]) = \gamma(\phi([0,1])) = \gamma([0,1])$$

and the images of γ and μ are identical.

The length of μ is

$$\ell(\mu) = \int_0^1 |\mu'(t)|\,dt = \int_0^1 |\gamma'(\phi(t))\phi'(t)|\,dt = \int_0^1 |\gamma'(\phi(t))|\,\phi'(t)dt$$

The change of variable $s = \phi(t)$ provides

$$\int_0^1 |\gamma'(\phi(t))|\,\phi'(t)dt = \int_0^1 |\gamma'(s)|\,ds,$$

hence the lengths of γ and μ are equal. We also have

$$\int_\mu f(z)\,dz = \int_0^1 (f \circ \mu)(t)\mu'(t)\,dt = \int_0^1 (f \circ \gamma)(\phi(t))\gamma'(\phi(t))\,(\phi'(t)dt).$$

The same change of variable leads to

$$\int_\mu f(z)\,dz = \int_0^1 (f \circ \gamma)(s)\gamma'(s)\,ds = \int_\gamma f(z)\,dz,$$

which concludes the proof. ∎

Definition – Image of a Path by a Function. Let $\gamma : [0,1] \to \mathbb{C}$ be a path and $f : A \subset \mathbb{C} \to \mathbb{C}$ be a continuous function defined on the image of γ. The *image of γ by f* is the path $f \circ \gamma$.

Theorem – Change of Variable in Line Integrals. Let Ω be an open subset of $\mathbb{C}$, let γ be a rectifiable path of Ω and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. The path $f \circ \gamma$ is rectifiable and for any continuous function $g : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of $f \circ \gamma$,

$$\int_{f \circ \gamma} g(z)\,dz = \int_\gamma g(f(w))f'(w)\,dw.$$

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Proof. Let $\gamma_1 \,|_{t_1} \,\ldots\, |_{t_{n-1}} \gamma_n$ be a continously differentiable decomposition of γ. We have

$$f \circ \gamma = f \circ \gamma_1 \,|_{t_1} \,\ldots\, |_{t_{n-1}} f \circ \gamma_n,$$

and for any $k \in \{1, \ldots, n\}$, by the chain rule, the function $f \circ \gamma_k$ is continuously differentiable hence the path $f \circ \gamma$ is rectifiable.

Moreover,

$$
\begin{aligned}
\int_\gamma g(f(w))f'(w)\,dw &= \int_0^1 g(f(\gamma(t))f'(\gamma(t))\gamma'(t)\,dt \\
&= \int_0^1 g(f(\gamma(t))(f \circ \gamma)'(t)\,dt \\
&= \int_{f \circ \gamma} g(w)\,dw
\end{aligned}
$$

$\blacksquare$

Primitives

Definition – Primitive. Let $f : \Omega \to \mathbb{C}$ where Ω is an open subset of $\mathbb{C}$. A *primitive* (or *antiderivative)* of f is a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g' = f$.

Theorem – Fundamental Theorem of Calculus (Complex Analysis). Let Ω be an open connected subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a continuous function and let $a \in \Omega$. A function $g : \Omega \to \mathbb{C}$ is a primitive of f if and only if for any $z \in \Omega$ and any rectifiable path γ of Ω that joins a and z,

$$g(z) = g(a) + \int_\gamma f(w)\,dw.$$

Proof. Let g be a primitive of f and γ be a rectifiable path of Ω that joins a and z. Let $\gamma = \gamma_1 \,|_{t_1} \,\ldots\, |_{t_{n-1}} \gamma_n$ be a continuously differentiable decomposition of γ. For any $k \in \{1, \ldots, n\}$, the function

$$\phi : t \in [0, 1] \mapsto g(\gamma_k(t))$$

is differentiable as a composition of real-differentiable functions, with

$$\phi'(t) = dg_{\gamma_k(t)}(\gamma'_k(t)) = g'(\gamma_k(t))\gamma'_k(t).$$

The function ϕ' is continuous hence by the fundamental theorem of calculus (from real analysis) applied to the real and imaginary parts of ϕ' on $]0, 1[$, we have for any positive number ϵ smaller than 1,

$$\phi(1 - \epsilon) - \phi(\epsilon) = \int_\epsilon^{1-\epsilon} \phi'(t)\,dt,$$

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and thus by continuity of ϕ and ϕ'

$$\phi(1) - \phi(0) = \int_0^1 \phi'(t)\, dt,$$

which is equivalent to

$$g(\gamma_k(1)) - g(\gamma_k(0)) = \int_0^1 g'(\gamma_k(t))\gamma_k'(t)\, dt = \int_{\gamma_k} f(w)\, dw.$$

The sum of these equations for all $k \in \{1, \ldots, n\}$ provides

$$g(z) - g(a) = \int_\gamma f(w)\, dw.$$

Conversely, assume that g satisfies the theorem property. Let γ be a rectifiable path of Ω that joins a and z and let $r > 0$ be such that the open disk centered at z with radius r is included in Ω. Consider the concatenation μ of γ and of the oriented segment $[z \to z + h]$ for h such that $|h| < r$. It is a rectifiable path of Ω, hence

$$\begin{aligned}
g(z + h) &= g(a) + \int_\mu f(w)\, dw \\
&= g(a) + \int_\gamma f(w)\, dw + h \int_0^1 f(z + th)\, dt \\
&= g(z) + h \int_0^1 f(z + th)\, dt
\end{aligned}$$

hence

$$\frac{g(z + h) - g(z)}{h} = \int_0^1 f(z + th)\, dt.$$

The right-hand side of this equation converges to $f(z)$ by continuity when h goes to zero, therefore g is a primitive of f. ∎

Corollary – Existence of Primitives [†]**.** Let Ω be an open connected subset of $\mathbb{C}$. The function $f : \Omega \to \mathbb{C}$ has a primitive if and only if it is continuous and for any closed rectifiable path γ

$$\int_\gamma f(z)\, dz = 0.$$

Proof – Existence of Primitives. If the function f has primitives, it is the derivative of a holomorphic function, thus it is continuous. Additionally, for any closed rectifiable path γ of Ω, the fundamental theorem of calculus provides

$$g(\gamma(1)) = g(\gamma(0)) + \int_\gamma f(w)\, dw,$$

hence as $\gamma(1) = \gamma(0)$,

$$\int_\gamma f(w)\, dw = 0.$$

Conversely, assume that any such integral is zero. Select any a in Ω and define for any point z in Ω and any rectifiable path γ of Ω that joins them the function

$$g(z) = g(a) + \int_\gamma f(w)\, dw.$$

This definition is non-ambiguous: if we select a different path μ, the difference between the right-hand sides of the definitions would be

$$\left(g(a) + \int_\gamma f(w)\, dw\right) - \left(g(a) + \int_\mu f(w)\, dw\right) = \int_{\gamma\,|\,\mu^\leftarrow} f(w)\, dw = 0$$

as $\gamma\,|\,\mu^\leftarrow$ is a closed rectifiable path of Ω. Consequently, g is uniquely defined and by the fundamental theorem of calculus, it is a primitive of f. ∎

Corollary – Set of Primitives. Let Ω be an open connected subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$. If $g : \Omega \to \mathbb{C}$ is a primitive of f, the function $h : \Omega \to \mathbb{C}$ is also a primitive of f if and only iff it differs from g by a constant.

Proof. It is clear that a function h that differs from g by a constant is a primitive of f. Conversely, if g and h are both primitives of f, $g - h$ is a primitive of the zero function. The fundamental theorem of calculus shows that for any a and z in Ω and any rectifiable path γ of Ω that joins them,

$$g(z) - h(z) = g(a) - h(a) + \int_\gamma 0\, dw = g(a) - h(a)$$

hence their difference is a constant. ∎

Corollary – Integration by Parts [†]. Let Ω be an open connected subset of $\mathbb{C}$ and let γ be a rectifiable path of Ω. For any pair of holomorphic functions $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$,

$$\int_\gamma f'g(z)\, dz = [fg(\gamma(1)) - fg(\gamma(0))] - \int_\gamma fg'(z)\, dz.$$

Proof. The derivative of the function fg is $f'g + fg'$. It is continuous as a sum and product of continuous functions thus the fundamental theorem of calculus provides

$$fg(\gamma(1)) = fg(\gamma(0)) + \int_\gamma (f'g + fg')(z)\, dz,$$

which is equivalent to the conclusion of the corollary. ∎

Remark & Definition – Variation of a Function on a Path. The difference between the value of a function f at the terminal value and at the initial value of a path γ may be denoted $[f]_\gamma$:

$$[f]_\gamma = f(\gamma(1)) - f(\gamma(0)).$$

With this convention, the formula that relates a function f and its primitive g is

$$[g]_\gamma = \int_\gamma f(z)\, dz$$

and the integration by parts formula becomes

$$\int_\gamma f'g(z)\, dz = [fg]_\gamma - \int_\gamma fg'(z)\, dz.$$

Appendix – A Better Theory of Rectifiability

Rectifiable Paths

The definition we used so far for "rectifiable" is a conservative one. In this section, we come up with a more general definition of the concept that still meets the requirements for the definition of line integrals.

To "rectify" a path (from Latin *rectus* "straight" and *facere* "to make") is to straighten – or by extension to compute its length, which is a trivial operation once a path has been straightened.

The general definition of the length of a path does not require line integrals. Instead, consider any partition $(t_0, \ldots, t_n)$ of the interval $[0, 1]$ and the path $\mu = \mu_1 \mid_{t_1} \ldots \mid_{t_{n-1}} \mu_n$ where

$$\mu_k(t) = (1 - t)\gamma(t_{k-1}) + t\gamma(t_k).$$

We may define the length of such a combination of straight lines as

$$\ell(\mu) = \sum_{k=1}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)|.$$

As the straight line is the shortest path between two points, this number should provide a lower bound of the length of γ. On the other hand, using finer partitions of the interval $[0, 1]$ should also provide better approximations of the length of γ. Following this idea, we may *define* the length of γ as the supremum of the length of μ for all possible partitions of $[0, 1]$:

$$\ell(\gamma) = \sup \left\{ \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \ \middle| \ n \in \mathbb{N}^*, \ t_0 = 0 < \cdots < t_n = 1 \right\}$$

Not every path has a finite length; those who have are by definition *rectifiable*. In general, a function $\gamma : [0, 1] \mapsto \mathbb{C}$ whose length is finite – even if it is not continuous – is of *bounded variation*.

The Line Integral

To define line integrals along the path γ, it is enough that γ is of bounded variation. For any such function γ, we may build a (complex-valued, Borel) measure on $[0,1]$ denoted $d\gamma$. This measure is defined by its integral of any continuous function $\phi : [0,1] \to \mathbb{C}$, as a limit of Riemann(-Stieltjes) sums

$$\int_{[0,1]} \phi \, d\gamma = \lim \sum_{m=0}^{n-1} \phi(t_m)(\gamma(t_{m+1}) - \gamma(t_m)).$$

The limit is taken over the partitions of the interval $[0,1]$ with

$$\max \left\{ |t_{m+1} - t_m| \mid m \in \{0, \ldots, n-1\} \right\} \to 0.$$

The line integral of a continuous function $f : \gamma([0,1]) \to \mathbb{C}$ is then defined by

$$\int_\gamma f(z) \, dz = \int_{[0,1]} (f \circ \gamma) \, d\gamma.$$

The *total variation* $|d\gamma|$ of $d\gamma$ is the positive measure defined by

$$|d\gamma|(A) = \sup_{\mathfrak{P}} \sum_{B \in \mathfrak{P}} |d\gamma(B)|$$

where the supremum is taken over all finite partitions $\mathfrak{P}$ of A into measurable sets. This measure provides an integral expression for the length of γ:

$$\ell(\gamma) = \int_{[0,1]} |d\gamma|.$$

A Non-Rectifiable Curve

The Koch snowflake (Koch 1904) is an example of a continuous curve which is is nowhere differentiable; it is also a non-rectifiable closed path. It is defined as the limit of a sequence of polylines γ_n. The first element of this sequence is an oriented equilateral triangle:

$$\gamma_1 = [0 \to 1 \to e^{i\pi 3} \to 0].$$

Then, γ_{n+1} is defined as a transformation of γ_n: every oriented line segment $[a \to a + h]$ that composes γ_n is replaced by the polyline:

$$\left[a \to a + \frac{h}{3} \to a + \left(1 + e^{-i\pi/3}\right)\frac{h}{3} \to a + 2\frac{h}{3} \to a + h \right]$$
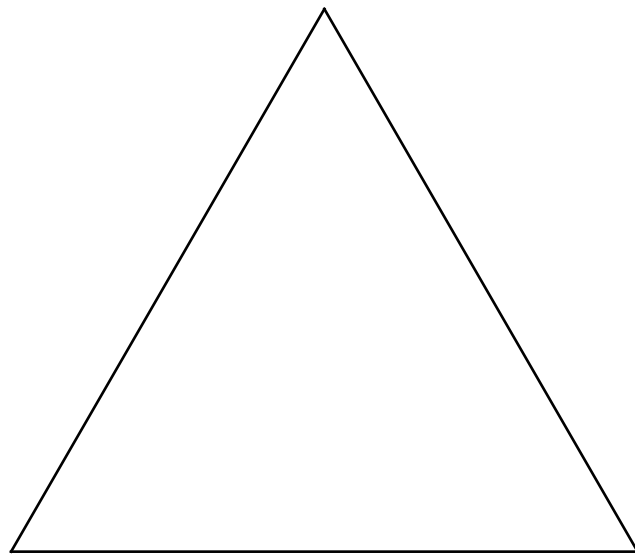
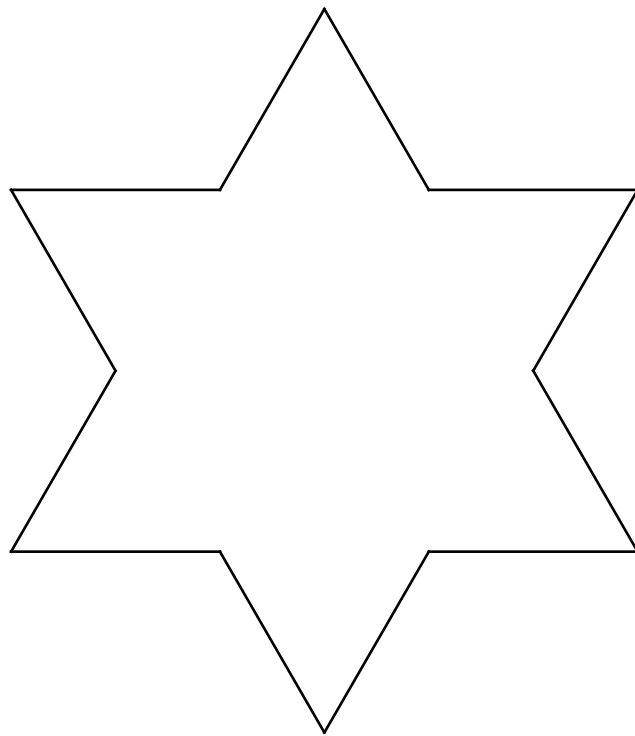Figure 3: Image of the Koch snowflake, first iteration.

Figure 4: Image of the Koch snowflake, second iteration.

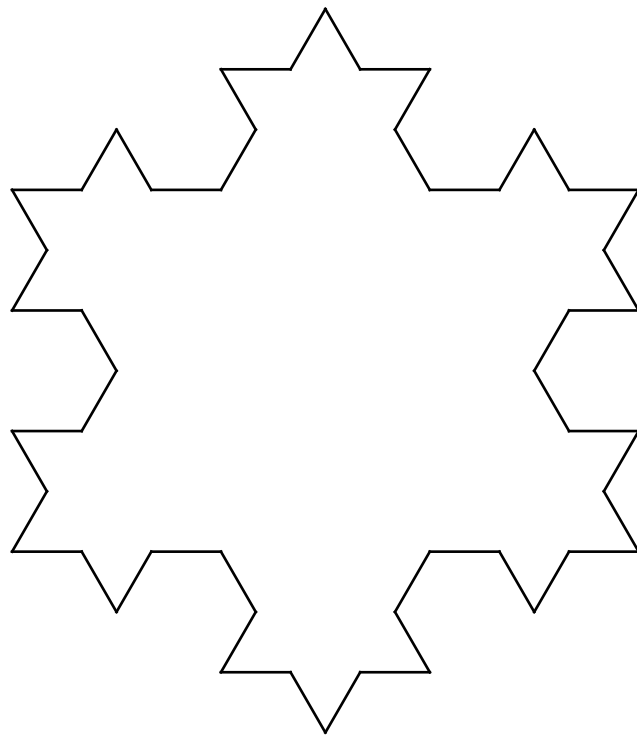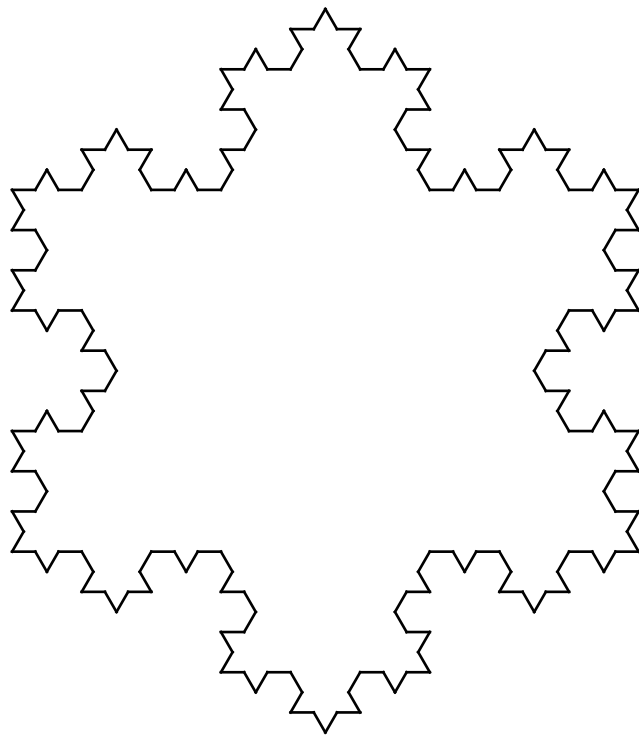Figure 5: Image of the Koch snowflake, third iteration.

Figure 6: Image of the Koch snowflake, fourth iteration

The Koch snowflake γ is defined as the limit of the γ_n sequence. The geometric construction yields that for any n greater than zero,

$$\forall\, t \in [0,1],\ |\gamma_{n+1}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{\sqrt{3}}{2}.$$

As $\sum_{p=0}^{+\infty} \left(\frac{1}{3}\right)^p = \frac{1}{1-1/3} = \frac{3}{2}$, for any positive integer p we have

$$\forall\, t \in [0,1],\ |\gamma_{n+p}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{3}{2}\frac{\sqrt{3}}{2}.$$

The sequence γ_n is a Cauchy sequence in the space of continuous and complex-valued functions defined on $[0,1]$; its uniform limit exists and is also continuous.

On the other hand, the curve is not rectifiable. First, the definition of the sequence γ_n makes it plain that every iteration increases the initial length of the path by one-third:

$$\ell(\gamma_n) = 3 \times \left(\frac{4}{3}\right)^{n-1}.$$

The length of γ_n tends to $+\infty$ when $n \to +\infty$. Now, every point at the junction of the segments of the polyline γ_n also belongs to the Koch snowflake; more precisely

$$\forall\, m \in \{0, \ldots, 3 \times 4^{n-1}\},\ \gamma\left(\frac{m}{3 \times 4^{n-1}}\right) = \gamma_n\left(\frac{m}{3 \times 4^{n-1}}\right).$$

Therefore

$$\ell(\gamma) \geq \sum_{m=0}^{3 \times 4^{n-1} - 1} \left|\gamma\left(\frac{m+1}{3 \times 4^{n-1}}\right) - \gamma\left(\frac{m}{3 \times 4^{n-1}}\right)\right| = \ell(\gamma_n)$$

and thus $\ell(\gamma) = +\infty$: the path γ is not rectifiable.

References

Koch, Helge von. 1904. "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire." *Arkiv för Matematik, Astronomi och Fysik* 1. Kungliga Svenska Vetenskapsakademien.: 681–702.