Connected Sets

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Contents

Exercises

Image of Path-Connected/Connected Sets

Let $f : A \subset \mathbb{C} \to \mathbb{C}$ be a continuous function.

Question

Show that if A is path-connected/connected, its image $f(A)$ is path-connected/connected.

Answer

Suppose that A is path-connected. Let $a, b \in f(A)$; there are some $c, d \in A$ such that $f(c) = a$ and $f(d) = b$. As A is path-connected, there is a path γ that joins c and d in A. By continuity of f, it is plain that its image $f \circ \gamma$ is a path of $f(A)$ that joins a and b. Consequently, $f(A)$ is path-connected.

Now suppose that A is connected. Let g be a locally constant function defined on $f(A)$. The function $g \circ f$ is locally constant on A: if $a \in A$, there is a radius $r > 0$ such that g is constant on $D(f(a), r) \cap f(A)$; by continuity of f, there is a $\epsilon > 0$ such that if $b \in D(a, \epsilon) \cap A$, $f(b) \in D(f(a), \epsilon) \cap f(A)$, thus $g \circ f$ is constant on $D(a, \epsilon) \cap A$ and finally, $g \circ f$ is locally constant. Since A is connected, $g \circ f$ is actually constant and g itself is constant: $f(A)$ is connected.

Complement of a Compact Set

Question

Prove that the complement $\mathbb{C} \setminus K$ of a compact subset K of the complex plane has a single unbounded component.

Answer

The compact set K is closed, hence its complement is open. Therefore, the connected and path-connected components of $\mathbb{C} \setminus K$ are the same. The compact set K is also bounded, hence there is a $r > 0$ such that the annulus

$$A = \{z \in \mathbb{C} \mid |z| > r\}$$

is included in $\mathbb{C} \setminus K$. The annulus A is path-connected: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are in A, the path $\gamma = [r_1 \to r_2] e^{i[\theta_1 \to \theta_2]}$, which is defined by

$$\gamma(t) = ((1-t)r_1 + tr_2) e^{i((1-t)\theta_1 + t\theta_2)}$$

belongs to A and joins z_1 and z_2. Hence, A is included in some path-connected component of $\mathbb{C} \setminus K$. The collection of these path-connected components are a partition of $\mathbb{C} \setminus K$, hence every other component C is a subset of $\mathbb{C} \setminus A = \overline{D}(0, r)$: it is bounded.

Union of Separated Sets

Source: "Sur les ensembles connexes" (Knaster and Kuratowski 1921)

Questions

Let A and B be two non-empty subsets of the complex plane.

1. If $A \cap B = \varnothing$, is $A \cup B$ always disconnected ?
2. Assume that $d(A, B) > 0$; show that $A \cup B$ is not connected.
3. Assume that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$; show that $A \cup B$ is not connected.

Answers

1. No. For example, the sets $A = \{z \in \mathbb{C} \mid \Re(z) < 0\}$ and $B = \mathbb{C} \setminus A$ are disjoints, but their union is $\mathbb{C}$, which is connected.
2. Let $r = d(A, B)/2$. Under the assumption, the sets

   $$A' = \cup_{a \in A} D(a, r), \; B' = \cup_{b \in B} D(b, r),$$

   which are both open sets, are disjoints, hence their union is not path-connected. However $A' \cup B'$ is a dilation of $A \cup B$, hence $A \cup B$ is not connected.

   Alternatively, consider the function f equal to 1 on A and 0 on B. It is locally constant – if $z \in A \cup B$, f is constant on $(A \cup B) \cap D(z, r)$ with $r = d(A, B)$ for example – but not constant, hence $A \cup B$ is not connected.
3. Consider again the function f introduced in the previous answer. The assumption yields $A \cap B = \varnothing$; as A and B are non-empty, f is not constant. If this function was not locally constant around some $a \in A$, we could find a sequence of $b_n \in (A \cup B) \setminus A = B$ such that $b_n \to a$. But that would imply that $a \in A \cap \overline{B}$ and would lead to a contradiction. Similarly, if it was not constant around some $b \in B$, that would lead to $b \in \overline{A} \cap B$, another contradiction. Hence, it is locally constant and $A \cup B$ is not connected.

Anchor Set

Questions

1. Prove that if $\mathcal{A}$ is a collection of path-connected/connected sets and there is a set $A^* \in \mathcal{A}$ such that $\forall \, A \in \mathcal{A}$, $A \cap A^* \neq \varnothing$, then the union $\cup \mathcal{A}$ is path-connected/connected.
2. A *deformation retraction* of a subset A of the complex plane onto a subset B of A is a "continuous shrinking process" of A into B; formally, it is a collection of paths γ_a of A, indexed by $a \in A$, such that:
   - $\forall \, a \in A, \; \gamma_a(0) = a \;$ and $\; \gamma_a(1) \in B,$

- $\forall\, a \in B,\ \forall\, t \in [0,1],\ \gamma_a(t) = a$,
- the function $(t, a) \in [0,1] \times A \mapsto \gamma_a(t)$ is continuous.

(see e.g. (Hatcher 2002)). Show that if there is a deformation retraction of A onto B and B is path-connected/connected, then A is also path-connected/connected.

Answers

1. Let $\mathcal{A}'$ be the collection of all the sets $A^* \cup A$ for $A \in \mathcal{A}$. For any $A \in \mathcal{A}$, the collection $\{A, A^*\}$ is composed of two path-connected/connected sets with a non-empty intersection; hence all the sets of $\mathcal{A}'$ in path-connected/connected. Moreover, the unions $\cup\mathcal{A}$ and $\cup\mathcal{A}'$ are identical. By assumption, unless $\mathcal{A}$ is empty, A^* is not empty; hence the intersection $\cap\mathcal{A}'$ that contains A^* is not empty. Therefore, $\cup\mathcal{A} = \cup\mathcal{A}'$ is path-connected/connected.

2. For any $a \in A$, $\gamma_a(0) = a$ and $\gamma_a([0,1]) \subset A$, hence

$$A = \bigcup_{a \in A} \gamma_a([0,1]).$$

For any $a \in A$, the set $\gamma_a([0,1])$ is path-connected (as the image of a path-connected set by a continuous function), and $\gamma_a([0,1]) \cap B$ is non-empty (it contains $\gamma_a(1)$). Consequently, the collection

$$\mathcal{A} = \{B\} \cup \{\gamma_a([0,1]) \,|\, a \in A\}$$

satisfies the assumption of the previous question with $A^* = B$. Consequently, $A = \cup\mathcal{A}$ is path-connected/connected.

References

Hatcher, Allen. 2002. *Algebraic Topology*. Cambridge University Press. `https://www.math.cornell.edu/~hatcher/AT/AT.pdf`.

Knaster, B., and C. Kuratowski. 1921. "Sur les ensembles connexes." *Fundamenta Mathematicae* 2. Polish Academy of Sciences (Polska Akademia Nauk - PAN), Institute of Mathematics (Instytut Matematyczny), Warsaw: 206–55.