Complex-Differentiability

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Exercises

Antiholomorphic Functions

A function $f : \Omega \to \mathbb{C}$ is *antiholomorphic* if its complex conjugate $\overline{f}$ is holomorphic.

Questions

1. Is the complex conjugate function $c : z \in \mathbb{C} \mapsto \overline{z}$ real-linear? complex-linear? Is it real-differentiable? holomorphic? antiholomorphic?

2. Show that any antiholomorphic function f is real-differentiable. Relate the differential of such a function and the differential of its complex conjugate.

3. Find the variant of the Cauchy-Riemann equation applicable to antiholomorphic functions.
4. What property has the composition of two antiholomorphic functions?
5. Let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function; show that the function

$$g : z \in \overline{\Omega} \mapsto \overline{f(\overline{z})}$$

is holomorphic and compute its derivative.

Answers

1. For any $\lambda \in \mathbb{R}$ and $w, z \in \mathbb{C}$, we have

$$\overline{w+z} = \overline{w} + \overline{z} \;\wedge\; \overline{\lambda z} = \lambda \overline{z},$$

therefore the function c is real-linear. However, it is not complex-linear: for example $\overline{i} = -i \neq i \times \overline{1}$. The function c is real-linear and continuous, hence it is real-differentiable and for any $z \in \mathbb{C}$, $dc_z = c$. This differential is not complex-linear, therefore the function is not complex-differentiable (or holomorphic). On the other hand, $\overline{c(z)} = z$, therefore it is antiholomorphic.

2. If the function $\overline{f} : \Omega \to \mathbb{R}$ is holomorphic, it is real-differentiable everywhere on its domain of definition. Hence $f = c \circ \overline{f}$ is real-differentiable as the composition of real-differentiable functions and

$$df_z = d(c \circ \overline{f})_z = c \circ d\overline{f}_z.$$

3. The complex-valued Cauchy-Riemann equation for $\overline{f}$ is

$$\frac{\partial \overline{f}}{\partial x}(z) = \frac{1}{i}\frac{\partial \overline{f}}{\partial y}(z), \quad \text{or } d\overline{f}_z(i) = i \times d\overline{f}_z(1)$$

On the other hand, we have

$$\frac{\partial \overline{f}}{\partial x}(z) = d(c \circ f)_z(1) = (c \circ df_z)(1) = \overline{\frac{\partial f}{\partial x}}$$

and

$$\frac{\partial \overline{f}}{\partial y}(z) = d(c \circ f)_z(i) = (c \circ df_z)(i) = \overline{\frac{\partial f}{\partial y}},$$

hence we can rewrite the Cauchy-Riemann equation for $\overline{f}$ as

$$\frac{\partial f}{\partial x}(z) = -\frac{1}{i}\frac{\partial f}{\partial y}(z), \quad \text{or } df_z(i) = -i \times df_z(1).$$

4. The composition of antiholomorphic functions is holomorphic. Indeed, if f and g are antiholomorphic, they are real-differentiable; their composition – assuming that it is defined – is $f \circ g = (c \circ \overline{f}) \circ (c \circ \overline{g})$; it satisfies

$$d(f \circ g)_z = d(c \circ \overline{f})_{c(\overline{g}(z))} \circ d(c \circ \overline{g})_z = c \circ d\overline{f}_{c(\overline{g}(z))} \circ c \circ d\overline{g}_z.$$

Since for any $h, w \in \mathbb{C}$,

$$c(hw) = \overline{h}c(w),\ d\overline{g}_z(hw) = hd\overline{g}(w),\ d\overline{f}_{c(\overline{g}(z))}(hw) = hd\overline{f}_{c(\overline{g}(z))}(w),$$

we have

$$d(f \circ g)_z(h) = hd(f \circ g)_z(1).$$

The differential of $f \circ g$ is complex-linear: the function is holomorphic.

5. If the point w belongs to $\overline{\Omega}$, then $w = \overline{z}$ for some $z \in \Omega$, thus the complex number $\overline{f(\overline{z})}$ is defined. Additionally, the set

$$\overline{\Omega} = \{\overline{z} \mid z \in \Omega\} = \{w \in \mathbb{C} \mid \overline{w} \in \Omega\}$$

is an open set, as the inverse image of the open set Ω by the continous function c. The function g satisfies

$$g = c \circ f \circ c = (c \circ f) \circ c$$

which is holomorphic as the composition of two antiholomorphic functions. We have

$$dg_z(h) = (c \circ df_{c(z)} \circ c)(h) = \overline{df_{\overline{z}}(\overline{h})} = \overline{f'(\overline{z})\overline{h}} = \overline{f'(\overline{z})}h,$$

hence $g'(z) = \overline{f'(\overline{z})}$.

Principal Value of the Logarithm

According to the definition of log, for any $x + iy \in \mathbb{C} \setminus \mathbb{R}_-$,

$$\log(x + iy) = \ln \sqrt{x^2 + y^2} + i \arg(x + iy),$$

where $\arg : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ is *the principal value of the argument*:

$$\forall\, z \in \mathbb{C} \setminus \mathbb{R}_-,\ \arg z \in \left]-\pi, \pi\right[\ \wedge\ e^{i \arg z} = \frac{z}{|z|}.$$

Questions

1. Show that

$$\arg(x + iy) = \left|\begin{array}{cl} \arctan y/x & \text{if } x > 0, \\ +\pi/2 - \arctan x/y & \text{if } y > 0, \\ -\pi/2 - \arctan x/y & \text{if } y < 0. \end{array}\right.$$

2. Show that the function log is holomorphic and compute its derivative.

Answers

1. Note that the definition of arg is non-ambiguous: for any nonzero real number ϵ,
$$\arctan \epsilon + \arctan 1/\epsilon = \mathrm{sgn}(\epsilon) \times \pi/2,$$
so if $x+iy$ belongs to two of the half-planes $x>0$, $y<0$ and $y>0$, the two relevant expressions which may define $\arg(x+iy)$ are equal.

   As $\arctan(\mathbb{R}) = \left]-\pi/2, \pi/2\right[$, the three expressions that define arg have values in $\left]-\pi, \pi\right[$. Then, if for example $x>0$, with $\theta = \arg(x+iy)$, we have
$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \tan(\arctan y/x) = \frac{y}{x}.$$
Since $\cos\theta > 0$ and $x > 0$, there is a $\lambda > 0$ such that
$$x + iy = \lambda(\sin\theta + i\cos\theta) = \lambda e^{i\theta};$$
this equation yields
$$e^{i \arg(x+iy)} = \frac{x+iy}{|x+iy|}.$$
The proof for the half-planes $y>0$ and $y<0$ is similar.

2. The functions arg, ln and therefore log are continuously real-differentiable. If $x>0$, for example, we have
$$\frac{\partial \arg(x+iy)}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}.$$
and
$$\frac{\partial \arg(x+iy)}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}.$$
On the other hand,
$$\frac{\partial}{\partial x} \ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$
and
$$\frac{\partial}{\partial y} \ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$$
Finally
$$\frac{\partial \log}{\partial x}(x+iy) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{x+iy}$$
and
$$\frac{\partial \log}{\partial y}(x+iy) = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2} = \frac{1}{i}\frac{1}{x+iy}.$$
The computations for $y>0$ and $y<0$ are similar. Conclusion: the function log is complex-differentiable and $\log'(z) = 1/z$.

Conformal Mappings

Questions

A $\mathbb{R}$-linear mapping $L : \mathbb{C} \to \mathbb{C}$ is *angle-preserving* if L is invertible and

$$\forall\, \theta \in \mathbb{R},\ \exists\, \alpha_\theta > 0,\ L(e^{i\theta}) = \alpha_\theta \times e^{i\theta} L(1).$$

A $\mathbb{R}$-differentiable function $f : \Omega \to \mathbb{C}$ *(locally) angle-preserving* – or *conformal* – if its differential is angle-preserving everywhere.

1. Show that an invertible $\mathbb{R}$-linear mapping $L : \mathbb{C} \to \mathbb{C}$ is angle-preserving if and only if it is $\mathbb{C}$-linear.
2. Identify the class of conformal mappings defined on Ω.

Answers

1. If L is $\mathbb{C}$-linear, then for any $\theta \in \mathbb{R}$, $L(e^{i\theta}) = e^{i\theta} L(1)$, hence it is angle-preserving. Reciprocally, if L is angle-preserving, we have on one hand

   $$L(e^{i\theta}) = \alpha_\theta \times e^{i\theta} L(1)$$

   and on the other hand, as $e^{i\theta} = \cos\theta + i \sin\theta$,

   $$L(e^{i\theta}) = \cos\theta \times L(1) + \sin\theta \times L(i) = (\cos\theta + \sin\theta \times \alpha_{\frac{\pi}{2}} i) L(1).$$

   We know that $L(1) \neq 0$, hence these equations provide

   $$\alpha_\theta = \cos\theta e^{-i\theta} + \sin\theta e^{-i\theta} \times \alpha_{\frac{\pi}{2}} i = \frac{1 + \alpha_{\frac{\pi}{2}}}{2} + \frac{1 - \alpha_{\frac{\pi}{2}}}{2} e^{-i2\theta}.$$

   As α_θ is real-valued, $\alpha_{\frac{\pi}{2}} = 1$. Consequently $\alpha_\theta = 1$ and L is $\mathbb{C}$-linear.
2. A mapping $f : \Omega \to \mathbb{C}$ is conformal if it is $\mathbb{R}$-differentiable, df_z is invertible everywhere and is $\mathbb{C}$-linear: this is exactly the class of holomorphic mappings f on Ω such that $f'(z) \neq 0$ everywhere.

Directional Derivative

Source: Mathématiques III, Francis Maisonneuve, Presses des Mines.

Questions

Let f be a complex-valued function defined in a neighbourhood of a point $z_0 \in \mathbb{C}$. Assume that f is $\mathbb{R}$-differentiable at z_0.

1. Let $\alpha \in \mathbb{R}$ and $z_{r,\alpha} = z_0 + re^{i\alpha}$ for $r \in \mathbb{R}$. Show that

$$\ell_\alpha = \lim_{r \to 0} \frac{f(z_{r,\alpha}) - f(z_0)}{z_{r,\alpha} - z_0}$$

exists and determine its value as a function of df_{z_0} and α.

2. What is the geometric structure of the set $A = \{\ell_\alpha \mid \alpha \in \mathbb{R}\}$?

3. For which of these sets A is f $\mathbb{C}$-differentiable at z_0?

Answers

1. The real-differentiability of f at z_0 provides

$$f(z_0 + re^{i\alpha}) = f(z_0) + df_{z_0}(re^{i\alpha}) + \epsilon_{z_0}(re^{i\alpha})|r|$$

where $\lim_{h \to 0} \epsilon_{z_0}(h) = 0$. Therefore,

$$\frac{f(z_{r,\alpha}) - f(z_0)}{z_{r,\alpha} - z_0} = (re^{i\alpha})^{-1}(df_{z_0}(re^{i\alpha}) + \epsilon_{z_0}(re^{i\alpha})|r|).$$

Using the $\mathbb{R}$-linearity of df_{z_0}, we get

$$\frac{f(z_{r,\alpha}) - f(z_0)}{z_{r,\alpha} - z_0} = e^{-i\alpha} df_{z_0}(e^{i\alpha}) + \epsilon_{z_0,\alpha}(r)$$

for some function $\epsilon_{z_0,\alpha}$ such that $\lim_{r \to 0} \epsilon_{z_0,\alpha}(r) = 0$. Hence, the limit that defines ℓ_α exists and

$$\ell_\alpha = e^{-i\alpha} df_{z_0}(e^{i\alpha}).$$

2. For every real number α, we have

$$\ell_\alpha = e^{-i\alpha} \left(\frac{\partial f}{\partial x}(z_0) \cos \alpha + \frac{\partial f}{\partial y}(z_0) \sin \alpha \right).$$

Hence, if we use the equations

$$\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}, \ \sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i},$$

we obtain

$$\ell_\alpha = \frac{1}{2} \left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \right) + \frac{1}{2} \left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \right) e^{-i2\alpha}.$$

Therefore, the set $A = \{\ell_\alpha \mid \alpha \in \mathbb{R}\}$ is a circle centered on

$$c = \frac{\partial f}{\partial x}(z_0) + \frac{1}{i} \frac{\partial f}{\partial y}(z_0)$$

whose radius is

$$r = \left| \frac{\partial f}{\partial x}(z_0) - \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \right|.$$

3. The function f is complex-differentiable at z_0 if and only if the Cauchy-Riemann equation

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

is met, which happens exactly when the radius of the circle A is zero, that is, when A is a single point in the complex plane.