The Winding Number

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Star-Shaped Sets

Question

Prove that every open star-shaped subset of $\mathbb{C}$ is simply connected.

Answer

Let Ω be an open star-shaped subset of $\mathbb{C}$ with a center c.

For any $z \in \mathbb{C} \setminus \Omega$ and any $s \geq 0$, the point $w = z + s(z - c)$ belongs to $\mathbb{C} \setminus \Omega$. The ray of all such points w is unbounded and connected, thus it is included in an unbounded component of $\mathbb{C} \setminus \Omega$. All components of $\mathbb{C} \setminus \Omega$ are therefore unbounded: Ω is simply connected.

Alternatively, let γ be a closed path of Ω and let $z = c + re^{i\alpha} \in \mathbb{C} \setminus \Omega$. Since the ray $\{z + se^{i\alpha} \mid s \geq 0\}$ does not intersect Ω, for any $t \in [0, 1]$ and any $s \geq 0$,

$\gamma(t) - z \neq se^{i\alpha}$. Thus $e^{-i(\pi+\alpha)}(\gamma(t) - z) \in \mathbb{C} \setminus \mathbb{R}_-$ and the function

$$\phi : t \in [0, 1] \mapsto e^{i(\pi+\alpha)} \arg(e^{-i(\pi+\alpha)}(\gamma(t) - z))$$

is defined; since it is a continuous choice of the argument $w \mapsto \operatorname{Arg}(w - z)$ along γ,

$$\operatorname{ind}(\gamma, z) = \frac{1}{2\pi}[\phi(1) - \phi(0)] = 0.$$

Therefore, Ω is simply connected.

The Argument Principle for Polynomials

Questions

Let p be the polynomial

$$p(z) = \lambda \times (z - a_1)^{n_1} \times \cdots \times (z - a_m)^{n_m}$$

where λ is a nonzero complex number, $a_1, \ldots, a_m$ are distinct complex numbers (the *zeros* or *roots* of the polynomial) and $n_1, \ldots, n_m$ are positive natural numbers (the roots *orders* or *multiplicities*). Let γ be a closed path whose image contains no root of p:

$$\forall\, t \in [0, 1],\ p(\gamma(t)) \neq 0.$$

The argument principle then states that

$$\operatorname{ind}(p \circ \gamma, 0) = \sum_{k=1}^{m} \operatorname{ind}(\gamma, a_k) \times n_k.$$

1. **Application: Finding the Roots of a Polynomial.**

   Use the figures below to determine – according to the argument principle – the number of roots z of the polynomial $p(z) = z^3 + z + 1$ in the open unit disk centered on the origin.

2. **Argument Principle Proof (Elementary).** For any $k \in \{1, \ldots, m\}$, we denote θ_k a continous choice of $z \mapsto \operatorname{Arg}(z - a_k)$ on γ. Use the functions θ_k to build a continuous choice of $z \mapsto \operatorname{Arg} z$ on $p \circ \gamma$; then, prove the argument principle.

3. **Argument Principle Proof (Complex Analysis).** Assume that γ is rectifiable; write the winding number $\operatorname{ind}(p \circ \gamma, 0)$ as a line integral, then find another way to prove the argument principle in this context.

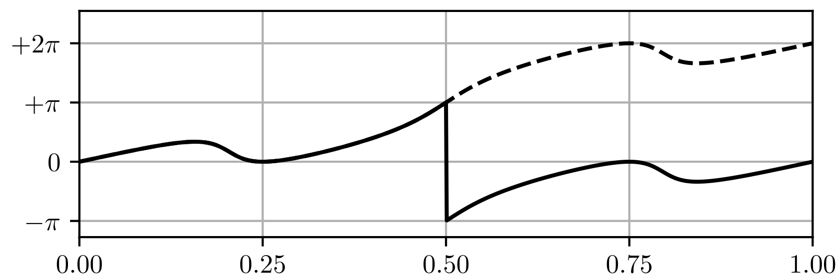

Figure 1: Graph of $t \in [0,1] \mapsto \arg\left[(e^{i2\pi t})^3 + (e^{i2\pi t}) + 1\right]$; this function has a jump of -2π at $t = 0.5$ (where it is undefined). The dashed line represents a continuous choice of the argument of $t \in [0,1] \mapsto (e^{i2\pi t})^3 + (e^{i2\pi t}) + 1$.

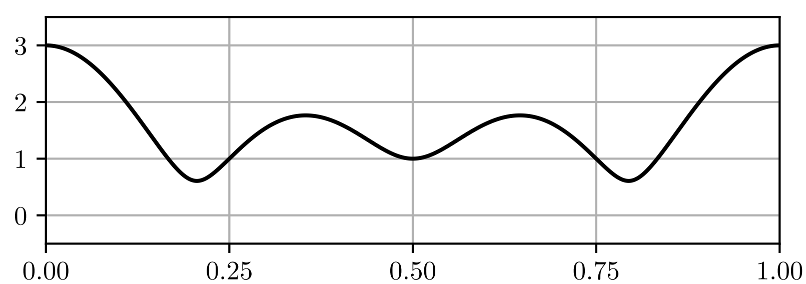

Figure 2: Graph of $t \in [0,1] \mapsto |(e^{i2\pi t})^3 + (e^{i2\pi t}) + 1|$.

Answers

1. Let $\gamma : t \in [0,1] \mapsto e^{i2\pi t}$; we have $(p \circ \gamma)(t) = (e^{i2\pi t})^3 + (e^{i2\pi t}) + 1$. The second figure shows that the graph of $t \mapsto |(p \circ \gamma)(t)|$ does not vanish on $[0,1]$, hence the image of γ contains no root of p. The second figure shows that the variation of the argument of z on the path $p \circ \gamma$ is 2π (a variation of π between $t = 0$ and $t = 0.5$ and also a variation of π between $t = 0.5$ and $t = 1.0$). Accordingly, we have

$$\operatorname{ind}(p \circ \gamma, 0) = 1.$$

   On the other hand, every zero z of p such that $|z| < 1$ satisfies $\operatorname{ind}(\gamma, z) = 1$ and every zero z of p such that $|z| > 1$ satisfies $\operatorname{ind}(\gamma, z) = 0$. Consequently, the expression

$$\sum_{k=1}^{m} \operatorname{ind}(\gamma, a_k) \times n_k$$

   provides the number of roots of p – counted with their multiplicity – within the unit circle. By the argument principle, there is a unique root of p within the unit circle.

2. If θ_0 is an argument of λ, the sum

$$\theta : t \in [0,1] \mapsto \theta_0 + n_1\theta_1(t) \times \cdots + n_m\theta_m(t)$$

   is continuous and

$$\begin{aligned} e^{i\theta(t)} &= e^{i\theta_0} \times e^{in_1\theta_1(t)} \times \cdots \times e^{in_m\theta_m(t)} \\ &= \frac{\lambda}{|\lambda|} \times \frac{(\gamma(t) - a_1)^{n_1}}{|\gamma(t) - a_1|^{n_1}} \times \cdots \times \frac{(\gamma(t) - a_m)^{n_m}}{|\gamma(t) - a_m|^{n_m}} \\ &= \frac{(p \circ \gamma)(t)}{|(p \circ \gamma)(t)|}, \end{aligned}$$

   therefore θ is a choice of the argument of $z \mapsto z$ on $p \circ \gamma$. Consequently,

$$\begin{aligned} [z \mapsto \operatorname{Arg} z]_{p \circ \gamma} &= \theta(1) - \theta(0) \\ &= \theta_0 - \theta_0 + \sum_{k=1}^{m} n_k(\theta_k(1) - \theta_k(0)) \\ &= \sum_{k=1}^{m} n_k \times [z \mapsto \operatorname{Arg}(z - a_k)]_\gamma. \end{aligned}$$

   A division of both sides of this equation by 2π concludes the proof.

3. The integral expression of the winding number is

$$\operatorname{ind}(p \circ \gamma, 0) = \frac{1}{i2\pi} \int_{p \circ \gamma} \frac{dz}{z}.$$

The polynomial p is holomorphic on $\mathbb{C}$, hence we can perform the change of variable $z = p(w)$, which yields

$$\operatorname{ind}(p \circ \gamma, 0) = \frac{1}{i2\pi} \int_\gamma \frac{p'(w)}{p(w)} dw.$$

If we factor $p(w)$ as $(w - a_k)^{n_k} q(w)$, we see that

$$\frac{p'(w)}{p(w)} = \frac{n_k}{w - a_k} + \frac{q'(w)}{q(w)};$$

applying this process repeatedly for every $k \in \{1, \dots, m\}$, until q is a constant, provides

$$\frac{p'(w)}{p(w)} = \sum_{k=1}^{m} \frac{n_k}{w - a_k}$$

and consequently

$$\begin{aligned}
\operatorname{ind}(p \circ \gamma, 0) &= \frac{1}{i2\pi} \int_\gamma \left[\sum_{k=1}^{m} \frac{n_k}{w - a_k} \right] dw \\
&= \sum_{k=1}^{m} \left[\frac{1}{i2\pi} \int_\gamma \frac{dw}{w - a_k} \right] \times n_k \\
&= \sum_{k=1}^{m} \operatorname{ind}(\gamma, a_k) \times n_k.
\end{aligned}$$

Set Operations & Simply Connected Sets

Questions

Suppose that A, B and $\mathbb{C} \setminus C$ are open subsets of $\mathbb{C}$. For each of the three statements below,

- determine whether or not the statement is true (either prove it or provide a counter-example);
- if the statement is false, find a sensible assumption that makes the new statement true (and provide a proof).

The statements are:

1. **Intersection.** The intersection $A \cap B$ of two simply connected sets A and B is simply connected.
2. **Complement.** The relative complement $A \setminus C$ of a connected set C in a simply connected set A is simply connected.
3. **Union.** The union $A \cup B$ of two connected and simply connected sets A and B is simply connected.

Answers

1. **Intersection.** The statement holds true. Indeed, let γ be a closed path of $A \cap B$; it is a path of A and a path of B. As both sets are simply connected, the interior of γ is included in A and in B, that is in $A \cap B$: this intersection is simply connected.

   Alternatively, let C be a component of

   $$\mathbb{C} \setminus (A \cap B) = (\mathbb{C} \setminus A) \cup (\mathbb{C} \setminus B),$$

   and let $z \in C$; we have $z \in \mathbb{C} \setminus A$ or $z \in \mathbb{C} \setminus B$. If $z \in \mathbb{C} \setminus A$, the component of $\mathbb{C} \setminus A$ that contains z is unbounded; it is a connected set that contains z and is included in $\mathbb{C} \setminus (A \cap B)$, hence, it is also included in C. Consequently, C is unbounded. If instead $z \in \mathbb{C} \setminus B$, a similar argument provides the same result. Consequently, all components of $\mathbb{C} \setminus (A \cap B)$ are unbounded: $A \cap B$ is simply connected.

2. **Complement.** The statement does not hold: consider $A = D(0, 3)$ and $C = \overline{D(0, 1)}$. The set A is open and simply connected and the set C is closed and connected. The set C is actually a component of $A \setminus C$: it is included in $A \setminus C$, connected and maximal.

   However, the statement holds if additionally the set $C \setminus A$ is not empty. Let γ be a closed path of $A \setminus C$ and let $z \in \mathbb{C} \setminus (A \setminus C)$. If $z \in \mathbb{C} \setminus A$, as A is simply connected, z belongs to the exterior of γ. Otherwise, $z \in A \cap C$; as C is a connected subset that does not intersect the image of γ, the function $w \in C \mapsto \mathrm{ind}(\gamma, w)$ is constant. There is a $w \in C \setminus A$ and $\mathrm{ind}(\gamma, z) = \mathrm{ind}(\gamma, w) = 0$. Therefore z also belongs to the exterior of γ: $A \setminus C$ is simply connected.

   Alternatively, let D be a component of

   $$\mathbb{C} \setminus (A \setminus C) = (\mathbb{C} \setminus A) \cup C.$$

   Some of its elements are in $\mathbb{C} \setminus A$: otherwise, C would be a connected superset of D that is included in $\mathbb{C} \setminus (A \setminus C)$; we would have $C = D$ and therefore $C \setminus A$ would be empty. Now, as D contains at least a point z of $\mathbb{C} \setminus A$, it contains the component of $\mathbb{C} \setminus A$ that contains z; therefore D is unbounded. Consequently, $A \setminus C$ is simply connected.

3. **Union.** The statement doesn't hold: consider

   $$A_s = \{2e^{i2\pi t} \mid t \in [0, 1/2]\}, \ B_s = \{2e^{i2\pi t} \mid t \in [1/2, 1]\}.$$

   and the associated dilations

   $$A = \{z \in \mathbb{C} \mid d(z, A_s) < 1\}, \ B = \{z \in \mathbb{C} \mid d(z, B_s) < 1\}.$$

   They are both open, connected and simply connected (their complement in the plane has a single path-connected component and it is unbounded) but their union $A \cup B$ is the annulus $D(0,3) \setminus D(0,1)$. We already considered this set in question 2: it is not simply connected.

However, the statement holds if additionally, the intersection $A \cap B$ is connected. Let γ be a closed path of $A \cup B$ and let $z \in \mathbb{C} \setminus (A \cap B)$. We have to prove that $\mathrm{ind}(\gamma, z) = 0$.

There exist[1] a sequence $(\gamma_1, \dots, \gamma_n)$ of consecutive paths of $A \cup B$ whose concatenation is γ and such that for any $k \in \{1, \dots, n\}$, $\gamma_k([0,1]) \subset A$ or $\gamma_k([0,1]) \subset B$.

Let a_k be the initial point of γ_k and let $w \in A \cap B$. As A, B and $A \cap B$ are connected, for any $k \in \{1, \dots, n\}$, there is a path β_k from w to a_k such that $\beta_k([0,1]) \subset A$ if $a_k \in A$ and $\beta_k([0,1]) \subset B$ if $a_k \in B$. We denote $\beta_{n+1} = \beta_1$ for convenience; define the paths α_k as the concatenations

$$\alpha_k = \beta_k \,|\, \gamma_k \,|\, \beta_{k+1}.$$

By construction

$$[x \mapsto \mathrm{Arg}(x - z)]_\gamma = \sum_{k=1}^{n} [x \mapsto \mathrm{Arg}(x - z)]_{\alpha_k}.$$

Every path α_k is closed, hence this is equivalent to

$$\mathrm{ind}(\gamma, z) = \sum_{k=1}^{n} \mathrm{ind}(\alpha_k, z),$$

but every α_k belongs either to A or B, which are simply connected, hence the right-hand-side is equal to zero. (This proof was adapted from Ronnie Brown's argument on Math Stack Exchange)

[1]The collection $\{A, B\}$ is an open cover of $\gamma([0,1])$ which is compact. Now, for any positive integer n, consider the sequence $(\gamma_1^n, \dots, \gamma_n^n)$ where

$$\gamma_k^n(t) = \gamma((k-1+t)/n).$$

By uniform continuity of γ, the diameters of the γ_k^n tends uniformly to zero when n tends to $+\infty$. The conclusion follows from Lebesgue's Number Lemma.