The Winding Number

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Definitions

The argument of a non-zero complex number is only defined modulo 2π. A convenient way to describe mathematically this relationship is to associate to any such number the set of admissible values of its argument:

Definition – The Argument Function. The *set-valued* (or *multi-valued*) function Arg, defined on $\mathbb{C}^*$ by

$$\operatorname{Arg} z = \left\{ \theta \in \mathbb{R} \;\middle|\; e^{i\theta} = \frac{z}{|z|} \right\},$$

is called the *argument* function.

If we need a classic *single-valued* function instead, we have for example:

Definition – Principal Value of the Argument. The *principal value of the argument* is the unique continuous function

$$\arg : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{R}$$

such that

$$\arg 1 = 0$$

which is a *choice* of the argument on its domain:

$$\forall\, z \in \mathbb{C} \setminus \mathbb{R}_-,\ \arg z \in \operatorname{Arg} z.$$

Proof (existence and uniqueness). Define arg on $\mathbb{C} \setminus \mathbb{R}_- \to \mathbb{R}$ by:

$$\arg(x+iy) = \left| \begin{array}{cl} \arctan y/x & \text{if } x > 0, \\ +\pi/2 - \arctan x/y & \text{if } y > 0, \\ -\pi/2 - \arctan x/y & \text{if } y < 0. \end{array} \right.$$

This definition is non-ambiguous: if $x > 0$ and $y > 0$, we have

$$\arctan x/y + \arctan y/x = \pi/2$$

and a similar equality holds when $x > 0$ and $y < 0$. As each of the three expressions used to define arg has an open domain and is continuous, the function itself is continuous. It is a choice of the argument thanks to the definition of arctan: for example, if $x > 0$, with $\theta = \arg(x+iy)$, we have

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \tan(\arctan y/x) = \frac{y}{x},$$

hence, as $\cos\theta > 0$ and $x > 0$, there is a $\lambda > 0$ such that

$$x + iy = \lambda(\cos\theta + i\sin\theta) = \lambda e^{i\theta},$$

This equation yields $\arg x + iy \in \operatorname{Arg} x + iy$. The proof for the half-planes $y > 0$ and $y < 0$ is similar.

If f is another continuous choice of the argument on $\mathbb{C} \setminus \mathbb{R}_-$ such that $f(1) = 0$, the image of $\mathbb{C} \setminus \mathbb{R}_-$ by the difference $f - \arg$ is a subset of $2\pi\mathbb{Z}$ that contains 0, and it's also path-connected as the image of a path-connected set by a continuous function. Consequently, it is the singleton $\{0\}$: f and arg are equal. ■

We cannot avoid the introduction of a *cut* in the complex plane when we search for a continuous choice of the argument: there is no continuous choice of the argument on $\mathbb{C}^*$. However, for a continuous choice of the argument along a path of $\mathbb{C}^*$, there is no such restriction:

The following theorem is a special case of the path lifting property (in the context of covering spaces; refer to (Hatcher 2002) for details).

Theorem – Continuous Choice of the Argument. Let $a \in \mathbb{C}$ and γ be a path of $\mathbb{C} \setminus \{a\}$. Let $\theta_0 \in \mathbb{R}$ be a value of the argument of $\gamma(0) - a$:

$$\theta_0 \in \operatorname{Arg}(\gamma(0) - a).$$

There is a unique continous function $\theta : [0,1] \mapsto \mathbb{R}$ such that $\theta(0) = \theta_0$ which is a *choice* of $z \mapsto \operatorname{Arg}(z - a)$ on γ:

$$\forall\, t \in [0,1],\ \theta(t) \in \operatorname{Arg}(\gamma(t) - a).$$

Proof. Let $(x(t), y(t))$ be the cartesian coordinates of $\gamma(t)$ in the system with origin a and basis $(e^{i\theta_0}, ie^{i\theta_0})$. As long as $x(t) > 0$, the function

$$t \mapsto \theta_0 + \arg(x(t) + iy(t))$$

is a continuous choice of the argument of $\gamma(t) - a$. Let d be the distance between a and $\gamma([0,1])$ and let $n \in \mathbb{N}$ such that

$$|t - s| \leq 2^{-n} \Rightarrow |\gamma(t) - \gamma(s)| < d.$$

The condition $x(t) > 0$ is ensured for any t in $[0, 2^{-n}]$. This construction of a continuous choice may be iterated locally on every interval $[k2^{-n}, (k+1)2^{-n}]$ with a new coordinate system to provide a global continuous choice of the argument on $[0,1]$.

The uniqueness of a continuous choice is a consequence of the intermediate value theorem: if we assume that there are two such functions θ_1 and θ_2 with the same initial value θ_0, as $\theta_1(0) - \theta_2(0) = 0$, if $\theta_1(t) - \theta_2(t) \neq 0$ for some $t \in [0,1]$, then either $|\theta_1(t) - \theta_2(t)| < \pi$, or there is a $\tau \in]0, t[$ such that $\theta_1(\tau) - \theta_2(\tau) \neq 0$ and $|\theta_1(\tau) - \theta_2(\tau)| < \pi$. In any case, there is a contradiction since all values of the argument differ of a multiple of 2π. ∎

Definition – Variation of the Argument. Let $a \in \mathbb{C}$ and γ be a path of $\mathbb{C} \setminus \{a\}$. The *variation* of $z \mapsto \operatorname{Arg}(z - a)$ on γ is defined as

$$[z \mapsto \operatorname{Arg}(z - a)]_\gamma = \theta(1) - \theta(0)$$

where θ is a continous choice of $z \mapsto \operatorname{Arg}(z - a)$ on γ.

Proof (unambiguous definition). If θ_1 and θ_2 are two continuous choices of $z \mapsto \operatorname{Arg}(z - a)$ on γ, for any $t \in [0,1]$, they differ of a multiple of 2π. As the function $\theta_1 - \theta_2$ is continuous, by the intermediate value theorem, it is constant. Hence

$$(\theta_1 - \theta_2)(1) = (\theta_1 - \theta_2)(0),$$

and $\theta_1(1) - \theta_1(0) = \theta_2(1) - \theta_2(0)$. ∎

Definition – Winding Number / Index. Let $a \in \mathbb{C}$ and γ be a closed path of $\mathbb{C} \setminus \{a\}$. The *winding number* – or *index* – of γ around a is the integer

$$\operatorname{ind}(\gamma, a) = \frac{1}{2\pi}[z \mapsto \operatorname{Arg}(z - a)]_\gamma.$$

Proof – The Winding Number is an Integer. Let θ be a continuous choice function of $z \mapsto \operatorname{Arg}(z - a)$ on γ; as the path γ is closed, $\theta(0)$ and $\theta(1)$, which are values of the argument of $\gamma(0) - a = \gamma(1) - a$, are equal modulo 2π, hence $(\theta(1) - \theta(0))/2\pi$ is an integer. ∎

Definition – Path Exterior & Interior. The *exterior* and *interior* of a closed path γ are the subsets of the complex plane defined by

$$\operatorname{Ext} \gamma = \{z \in \mathbb{C} \setminus \gamma([0,1]) \mid \operatorname{ind}(\gamma, z) = 0\}.$$

and

$$\operatorname{Int}\gamma = \mathbb{C} \setminus (\gamma([0,1]) \cup \operatorname{Ext}\gamma) = \{z \in \mathbb{C} \setminus \gamma([0,1]) \mid \operatorname{ind}(\gamma, z) \neq 0\}.$$

Properties

Theorem – The Winding Number is Locally Constant. Let $a \in \mathbb{C}$ and γ be a closed path of $\mathbb{C} \setminus \{a\}$. There is a $\epsilon > 0$ such that, for any $b \in \mathbb{C}$ and any closed path β, if

$$|b - a| < \epsilon \ \text{ and } \ (\forall\, t \in [0,1],\ |\beta(t) - \gamma(t)| < \epsilon)$$

then β is a path of $\mathbb{C} \setminus \{b\}$ and

$$\operatorname{ind}(\gamma, a) = \operatorname{ind}(\beta, b).$$

Proof. Let $\epsilon = d(a, \gamma([0,1]))/2$. If $|b-a| < \epsilon$ and for any $t \in [0,1]$, $|\gamma(t) - \beta(t)| < \epsilon$, then clearly $b \in \mathbb{C} \setminus \beta([0,1])$. Additionally, for any $t \in [0,1]$ there are values θ_1 of $\operatorname{Arg}(\gamma(t) - a)$ and θ_2 of $\operatorname{Arg}(\beta(t) - b)$ such that $|\theta_1 - \theta_2| < \pi/2$. If we select some values $\theta_{1,0}$ of $\operatorname{Arg}(\gamma(0) - a)$ and $\theta_{2,0}$ of $\operatorname{Arg}(\beta(0) - b)$ such that $|\theta_{1,0} - \theta_{2,0}| < \pi/2$, then the corresponding continuous choices θ_1 and θ_2 satisfy $|\theta_1(t) - \theta_2(t)| < \pi/2$ for any $t \in [0,1]$([1]). Consequently

$$|\operatorname{ind}(\gamma, a) - \operatorname{ind}(\beta, b)| = \left| \frac{\theta_1(1) - \theta_1(0)}{2\pi} - \frac{\theta_2(1) - \theta_2(0)}{2\pi} \right| < \frac{1}{2}.$$

As both winding numbers are integers, they are equal. ■

Corollary – The Winding Number is Constant on Components. Let γ be a closed path. The function

$$z \in \mathbb{C} \setminus \gamma([0,1]) \mapsto \operatorname{ind}(\gamma, z)$$

is constant on each component of $\mathbb{C} \setminus \gamma([0,1])$. If additionally the component is unbounded, the value of the winding number is zero.

Proof. The mapping $z \mapsto \operatorname{ind}(\gamma, z)$ is locally constant – and hence constant – on every connected component of $\mathbb{C} \setminus \gamma([0,1])$. If a belongs to some unbounded component of this set, there is a b in the same component such that $|b| > r = \max_{t \in [0,1]} |\gamma(t)|$. It is possible to connect b to any point c such that $|c| = r$ by a circular path in $\mathbb{C} \setminus \gamma([0,1])$, thus we may assume that $b \in \mathbb{R}_-$. The function

$$\theta : t \in [0,1] \mapsto \arg(\gamma(t) - b)$$

[1] Otherwise, by the intermediate value theorem, we could find some $t \in]0,1]$ such that $|\theta_1(t) - \theta_2(t)| = \pi/2$, but then, for every value $\theta_{1,t}$ of $\operatorname{Arg}(\gamma(t) - a)$ and $\theta_{2,t}$ of $\operatorname{Arg}(\beta(t) - b)$, we would have

$$\theta_{1,t} - \theta_{2,t} = \theta_1(t) - \theta_2(t) + 2\pi k$$

for some $k \in \mathbb{Z}$. Therefore, the choice of $\theta_{1,t}$ and $\theta_{2,t}$ such that $|\theta_{1,t} - \theta_{2,t}| < \pi/2$ would be impossible.

is a continuous choice of $z \mapsto \operatorname{Arg}(z-b)$ along γ and it satisfies

$$\forall\, t \in [0,1],\ |\theta(t)| = \left|\arctan \frac{\operatorname{Im}(\gamma(t)-b)}{\operatorname{Re}(\gamma(t)-b)}\right| < \arctan \frac{r}{|b|-r} < \frac{\pi}{2}.$$

As γ is a closed path, $\theta(0)$ and $\theta(1)$ – which are equal modulo 2π – are actually equal and

$$\operatorname{ind}(\gamma, a) = \operatorname{ind}(\gamma, b) = \frac{\theta(1)-\theta(0)}{2\pi} = 0$$

as expected. ■

Simply Connected Sets

Definition – Simply/Multiply Connected Set & Holes. Let Ω be an open subset of the plane. A *hole* of Ω is a bounded component of its complement $\mathbb{C} \setminus \Omega$. The set Ω is *simply connected* if it has no hole (if every component of its complement is unbounded) and *multiply connected* otherwise.

Examples.

1. The open set $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x < -1 \text{ or } x > 1\}$ is not connected but it is simply connected: its complement has a unique component which is unbounded, hence it has no holes.

2. The open set $\Omega = \mathbb{C} \setminus \overline{\{2^{-n} \,|\, n \in \mathbb{N}\}}$ is multiply connected: its holes are exactly the singletons of its complement.

Intuitively, we should be able to circle around any hole of Ω without leaving the set; this idea leads to an alternate characterization of simply connected sets.

Theorem – Simply Connected Sets & The Winding Number. An open subset Ω of the complex plane is simply connected if and only if the interior of any closed path γ of Ω is included in Ω:

$$\forall\, z \in \mathbb{C} \setminus \gamma([0,1]),\ \operatorname{ind}(\gamma, z) \neq 0 \ \Rightarrow\ z \in \Omega,$$

or equivalently, if the complement of Ω is included in the exterior of γ:

$$\forall\, z \in \mathbb{C} \setminus \Omega,\ \operatorname{ind}(\gamma, z) = 0.$$

Examples.

1. If γ is a closed path of $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x < -1 \text{ or } x > 1\}$ and $z \in \mathbb{C} \setminus \Omega$, since $\mathbb{C} \setminus \Omega$ is connected and unbounded, z belongs to an unbounded component of $\mathbb{C} \setminus \gamma([0,1])$. Thus $\operatorname{ind}(\gamma, z) = 0$ for any $z \in \mathbb{C} \setminus \Omega$.

2. The open set $\Omega = \mathbb{C} \setminus \overline{\{2^{-n} \,|\, n \in \mathbb{N}\}}$ is open and multiply connected: for example $\gamma = 1 + 1/4[\circlearrowleft]$ is a path of Ω, $z = 1$ is a point of $\mathbb{C} \setminus \Omega$ and $\operatorname{ind}(\gamma, 1) = 1$.

Remark. Note that we may not always be able to encircle only one hole at a time. For example, in the case of the set $\Omega = \mathbb{C} \setminus \overline{\{2^{-n} \,|\, n \in \mathbb{N}\}}$, we can find a closed path γ of $\mathbb{C} \setminus \Omega$ such that $\mathrm{ind}(\gamma, 0) = 1$, but then we also have $\mathrm{ind}(\gamma, 2^{-n}) = 1$ for n large enough: we cannot encircle the hole $\{0\}$ of Ω unless we also encircle an infinity of extra holes.

Lemma. If the compact set K is a hole of the open set Ω, there is a compact subset L of $\mathbb{C} \setminus \Omega$ such that $K \subset L$ and $d(L, (\mathbb{C} \setminus \Omega) \setminus L) > 0$.

Proof of the Lemma. The set $C = \mathbb{C} \setminus \Omega$ is closed in $\mathbb{C}$ which is locally compact, thus it is locally compact. Since $\mathbb{C}$ is Hausdorff, its subspace C is also Hausdorff. By the Šura-Bura theorem (Remmert 1998, 304), K has a neighbourhood base in C consisting in open compact subsets of C. Since C is a neighbourhood of K in C, there is a compact set L such that $K \subset L \subset C$ which is open in C. Thus, its complement $(\mathbb{C} \setminus \Omega) \setminus L$ is also closed in C. Since C is closed, both sets are also closed in $\mathbb{C}$. They are also disjoint by construction, and thus $d(L, (\mathbb{C} \setminus \Omega) \setminus L) > 0$. ■

Proof – Simply Connected Sets & The Winding Number. Assume that Ω is simply connected and let γ be a closed path of Ω. Let $z \in \mathbb{C} \setminus \Omega$; this point belongs to an unbounded connected component of $\mathbb{C} \setminus \Omega$ and therefore to an unbounded connected component of $\mathbb{C} \setminus \gamma([0, 1])$, thus $\mathrm{ind}(\gamma, z) = 0$.

Conversely, if Ω is not simply connected, the set $\mathbb{C} \setminus \Omega$ has a hole K which is contained in some compact subset L of $\mathbb{C} \setminus \Omega$ such that the distance ϵ between L and $(\mathbb{C} \setminus \Omega) \setminus L$ is positive. Let $r < \epsilon/\sqrt{2}$; Define for any pair (k, l) of integers the node $n_{k,l} = (k + il)r$ and $S_{k,l}$ as the closed square with vertices $n_{k,l}$, $n_{k+1,l}$, $n_{k+1,l+1}$ and $n_{k,l+1}$. The (positively) oriented boundary of the square $S_{k,l}$ is the polyline

$$[n_{k,l} \to n_{k+1,l} \to n_{k+1,l+1} \to n_{k,l+1} \to n_{k,l}]$$

The collection of squares that intersect L is finite and covers L. Additionally, all of its squares are included in $\Omega \cup L$.

For any square S in the cover of L and any interior point a of S if γ is the oriented boundary of S, then $\mathrm{ind}(\gamma, a) = 1$. Additionally, $\mathrm{ind}(\mu, a) = 0$ for the oriented boundary μ of any other square in the collection. Consequently, if Γ denotes the collection of oriented line segments that composes the oriented boundaries of all squares of the cover of L, we have

$$\sum_{\gamma \in \Gamma} \frac{1}{2\pi} [z \mapsto \mathrm{Arg}(z - a)]_\gamma = 1.$$

Now if the line segment γ belongs to Γ and $\gamma([0, 1]) \cap L \neq \varnothing$, then $\gamma^{\leftarrow}$ also belongs to Γ; if we remove all such pairs from Γ, the resulting collection Γ' also satisfies

$$\sum_{\gamma \in \Gamma'} \frac{1}{2\pi} [z \mapsto \mathrm{Arg}(z - a)]_\gamma = 1.$$

and by construction the image of any γ in Γ' is included in Ω. The original collection Γ is balanced: for any square vertice n, the number of line segments with n as an initial point and with n as a terminal point is the same. The collection Γ' has the same property. Consequently, the line segments of Γ' may be assembled in a finite sequence of closed paths $\gamma_1, \ldots, \gamma_n$ and

$$\sum_{k=1}^{n} \mathrm{ind}(\gamma_k, a) = 1.$$

Every point of L is either an interior point of some square of the collection, or the limit of such point; anyway, that means that

$$\forall\, z \in L,\ \sum_{k=1}^{n} \mathrm{ind}(\gamma_k, z) = 1$$

and thus that there is at least one path γ_k such that $\mathrm{ind}(\gamma_k, z) \neq 0$. ■

A Complex Analytic Approach

If a closed path is rectifiable, we may compute its winding number as a line integral; to prove this, we need the:

Lemma. Let $a \in \mathbb{C}$ and γ be a rectifiable path of $\mathbb{C} \setminus \{a\}$. For any $t \in [0,1]$, let γ_t be the path such that for any $s \in [0,1]$, $\gamma_t(s) = \gamma(ts)$. The function $\mu : [0,1] \to \mathbb{C}$, defined by

$$\mu(t) = \int_{\gamma_t} \frac{dz}{z-a}$$

satisfies

$$\exists\, \lambda \in \mathbb{C}^*,\ \forall\, t \in [0,1],\ e^{\mu(t)} = \lambda \times (\gamma(t) - a).$$

Proof. We only prove the lemma under the assumption that γ is continuously differentiable; the rectifiable case is a straightforward extension.

We have for any $t \in [0,1]$

$$\mu(t) = \int_{\gamma_t} \frac{dz}{z-a} = \int_0^1 \frac{\gamma'(ts) \times t}{\gamma(ts) - a} ds = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} ds,$$

hence

$$\mu'(t) = \frac{\gamma'(t)}{\gamma(t) - a}$$

and the derivative of the quotient $\phi(t) = e^{\mu(t)}/(\gamma(t) - a)$ satisfies

$$\phi'(t) = \mu'(t)\phi(t) - \frac{\gamma'(t)}{\gamma(t) - a}\phi(t) = 0$$

which yields the result. ■

Theorem – The Winding Number as a Line Integral. Let $a \in \mathbb{C}$ and γ be a rectifiable path of $\mathbb{C} \setminus \{a\}$. Then

$$[z \mapsto \operatorname{Arg}(z-a)]_\gamma = \operatorname{Im}\left(\int_\gamma \frac{dz}{z-a}\right).$$

If the path γ is closed, then

$$\operatorname{ind}(\gamma, a) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z-a}.$$

Proof. We use the function μ of the previous lemma. Applying the modulus to both sides of the equation $e^{\mu(t)} = \lambda \times (\gamma(t)-a)$ provides $e^{\operatorname{Re}(\mu(t))} = |\lambda| \times |\gamma(t)-a|$, hence

$$e^{i\operatorname{Im}(\mu(t))} = \frac{\lambda}{|\lambda|} \frac{\gamma(t)-a}{|\gamma(t)-a|}.$$

The function $t \in [0,1] \mapsto \operatorname{Im}(\mu(t))$ is – up to a constant – a continuous choice of $z \mapsto \operatorname{Arg}(z-a)$ on γ. Consequently,

$$[z \mapsto \operatorname{Arg}(z-a)]_\gamma = \operatorname{Im}(\mu(1)) - \operatorname{Im}(\mu(0)) = \operatorname{Im}(\mu(1)),$$

which is the desired result.

If additionally γ is a closed path, the equations

$$\gamma(0) = \gamma(1) \ \text{ and } \ e^{\operatorname{Re}(\mu(t))} = |\lambda| \times |\gamma(t)-a|$$

yield $e^{\operatorname{Re}(\mu(0))} = e^{\operatorname{Re}(\mu(1))}$ and hence $\operatorname{Re}(\mu(1)) = \operatorname{Re}(\mu(0)) = 0$. Thus,

$$\operatorname{ind}(\gamma, a) = \frac{1}{2\pi}\operatorname{Im}(\mu(1)) = \frac{1}{i2\pi}\mu(1),$$

which concludes the proof. ■

References

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