Zeros & Poles

Sébastien Boisgérault, Mines ParisTech, under CC BY-NC-SA 4.0

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Preamble

In this chapter, we study the behavior of any holomorphic functions in the neighbourhood of a point that may or may not be in its domain of definition. When the function is not defined at the point of interest – in other words when it is a singularity – we only study the case where it is isolated.

Definition – Isolated Point. A point c of a subset C of $\mathbb{C}$ is *isolated* (in C) if it is in some neighbourhood of c the only point of C:

$$\exists\, r > 0,\ \forall\, z \in C,\ |z - c| < r \;\Rightarrow\; z = c.$$

Remark – Isolated Points in Closed Sets. Note that a point c is isolated in the closed set C if and only if $C \setminus \{c\}$ is still closed. This is directly applicable to singularities of a function defined on an open set Ω, which belongs by definition to the closed set $C = \mathbb{C} \setminus \Omega$.

Definition – Isolated Singularity. A *singularity* of a function $f : \Omega \to \mathbb{C}$ defined on an open subset Ω of $\mathbb{C}$ is a point c of $\mathbb{C} \setminus \Omega$. It is *isolated* if

$$\exists\, r > 0,\ \forall\, z \in \mathbb{C},\ (z \neq c \text{ and } |z - c| < r) \Rightarrow z \in \Omega,$$

in other words if there is a radius $r > 0$ such that the annulus $A(c, 0, r)$ is a subset of Ω. Alternatively, it is isolated if and only if $\Omega \cup \{c\}$ is open.

Zeros of Holomorphic Functions

Definition – Zero & Multiplicity. Let Ω be an open subset of the complex plane. A *zero* (or *root*) c of a function $f : \Omega \to \mathbb{C}$ is a point $c \in \Omega$ such that $f(c) = 0$; it is of *multiplicity* p for some positive number p if

$$\exists\, a^* \in \mathbb{C}^*,\ f(z) \underset{c}{\sim} a^*(z-c)^p$$

or equivalently

$$\exists\, a^* \in \mathbb{C}^*,\ \lim_{z\to c} \frac{f(z)}{(z-c)^p} = a^*.$$

A zero of multiplicity 1 is *simple*; zeros of higher multiplicity (*double*, *triple*, etc.) are *multiple*.

Example – Simple Zero. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If c is a zero of f but not of its derivative f', then

$$\lim_{z\to c} \frac{f(z)}{(z-c)^1} = \lim_{z\to c} \frac{f(z)-f(c)}{z-c} = f'(c) \neq 0,$$

thus $f(z) \underset{c}{\sim} f'(c)(z-c)^1$ and c is a simple zero of f.

Theorem – Characterization of Zero Multiplicity. A zero c of a holomorphic function $f : \Omega \to \mathbb{C}$ is of multiplicity p if and only if one of the equivalent condition holds:

1. The function f and exactly its first $p-1$ derivatives are zero at c:

$$f(c) = 0,\ f'(c) = 0, \ldots,\ f^{(p-1)}(c) = 0 \text{ and } f^{(p)}(c) \neq 0.$$

2. The Taylor expansion of f at c is

$$f(z) = \sum_{n=p}^{+\infty} a_n (z-c)^n \text{ with } a_p \neq 0.$$

3. There is a holomorphic function $a : \Omega \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\ f(z) = a(z)(z-c)^p \text{ and } a(c) \neq 0.$$

Proof. The formula $n! a_n = f^{(n)}(c)$ holds for any $n \in \mathbb{N}$ hence condition 1 and 2 are equivalent. The theorem statement is otherwise a direct consequence of the local behavior of holomorphic functions lemma (refer to the appendix). ■

Zeros of a holomorphic functions have finite multiplicities, except in very specific circumstances:

Lemma – Zero With No (Finite) Multiplicity. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected set Ω. If c is a zero of f but has no finite multiplicity, then f is identically zero.

Proof. By assumption, $f^{(n)}(c) = 0$ for every number n. Consider the function $\chi : \Omega \to \mathbb{C}$ defined by

$$\chi(z) = \left| \begin{array}{ll} 0 & \text{if } \forall\, n \in \mathbb{N},\ f^{(n)}(z) = 0, \\ 1 & \text{otherwise.} \end{array} \right.$$

We can prove that the function χ is locally constant. Indeed, if $\chi(z) = 1$, there is a number n such that $f^{(n)}(z) \neq 0$; by continuity, the function $f^{(n)}$ has no zero in some neighbourgood of z; thus the function χ is equal to 1 in this neighbourghood. If instead $\chi(z) = 0$, then the Taylor expansion of f at z shows that f is zero in a suitable neighbourhood of z, where χ is also zero.

As the set Ω is connected, the function χ is actually constant. On the other hand, $\chi(c) = 0$, hence the function χ is identically zero on Ω, which means that the function f is also identically zero on Ω. ■

Lemma – Zeros of Finite Multiplicity are Isolated. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If c is a zero of multiplicity p of f, it is isolated:

$$\exists\, r > 0,\ \forall\, z \in \Omega,\ (f(z) = 0 \ \text{ and } \ |z - c| < r) \ \Rightarrow\ z = c.$$

Proof. Let c be a zero of multiplicity p of f for some positive number p. There is a holomorphic function $a : \Omega \to \mathbb{C}$ such that $f(z) = a(z)(z - c)^p$ on Ω and $a(c) \neq 0$. By continuity of a at c, there is a $r > 0$ such that the disk $D(c, r)$ is a subset of $\Omega \cup \{c\}$ on which a has no zero. The function f therefore has no zero on $D(c, r) \setminus \{c\}$ either: the point c is an isolated zero of f. ■

Theorem – Isolated Zeros Theorem I. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected open set Ω. Unless f is identically zero, each of its zeros is isolated.

Proof. A direct consequence of the two above lemmas. ■

Remark. More often than not, we leverage the isolated zeros theorem to prove that some holomorphic function is identically zero. In other words, we rely on the contraposition of the theorem. The statement of this contraposition may be slightly rephrased with the introduction of the concept of limit point.

Definition – Limit Point. A point $c \in \mathbb{C}$ is a *limit point* of a set $C \subset \mathbb{C}$ if every open annulus $A(c, 0, r)$ intersects C:

$$\forall\, r > 0,\ A(c, 0, r) \cap C \neq \varnothing$$

or equivalently, if the distance between c and $C \setminus \{c\}$ is zero.

Theorem – Isolated Zeros Theorem II. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected set Ω. If the set of zeros of f has a limit point in Ω, then f is identically zero.

Proof. If the set of zeros of f has a limit point in Ω, then by continuity of f at this point, it is a zero of f, which is clearly not isolated; thus, f is identically zero. ∎

Remark. Despite its apparent simplicity, the importance of the isolated zeros theorem is difficult to overestimate. However, it is a rather low-level tool; the corresponding high-level tool is a permanence principle for functional equations.

Theorem – Principle of Permanence. Let F be a complex-valued function of n complex variables, defined and complex-differentiable on some open subset of $\mathbb{C}^n$. Let Ω be an open connected subset of $\mathbb{C}$ and $(f_1, \dots, f_n) : \Omega \mapsto \mathbb{C}^n$ be a n-uple of holomorphic functions whose image is included in the domain of definition of F. If the set of points $z \in \Omega$ such that $F(f_1(z), \dots, f_n(z)) = 0$ has a limit point in Ω, then

$$\forall\, z \in \Omega,\ F(f_1(z), \dots, f_n(z)) = 0.$$

Proof. Under the assumptions of the theorem, the function

$$z \in \Omega \mapsto F(f_1(z), \dots, f_n(z))$$

is defined and holomorphic on Ω (it is complex-differentiable as the composition of complex-differentiable functions) on Ω. Its set of zeros has a limit point in Ω, which is connected, thus it is identically zero. ∎

Corollary – Uniqueness Principle. Two functions $f_1 : \Omega \to \mathbb{C}$ and $f_2 : \Omega \to \mathbb{C}$ defined and holomorphic on some open connected subset of Ω with the same values on a set with a limit point in Ω are identical.

Proof. Set $F(w_1, w_2) = w_1 - w_2$ and apply the permanence principle. ∎

Example – A Trigonometric Identity. The identity

$$\sin^2 z + \cos^2 z = 1$$

holds for every $z \in \mathbb{C}$. Indeed, it is satisfied on the real line: every real number is a zero of the holomorphic function $f : z \in \mathbb{C} \mapsto \sin^2 z + \cos^2 z - 1$. Every real number is a limit point of $\mathbb{R}$ in $\mathbb{C}$, hence f is identically zero and the identity may be extended to the whole complex plane. Alternatively, apply the permanence principle with $F(w_1, w_2) = w_1^2 + w_2^2 - 1$, $f_1(z) = \sin z$ and $f_2(z) = \cos z$.

Isolated Singularities of Holomorphic Functions

Definition – Typology of Isolated Singularities. Let Ω be an open subset of the complex plane and $f : \Omega \to \mathbb{C}$ be a holomorphic function. An isolated singularity c of f is:

- a *removable singularity* if there is a holomorphic extension of f over c, that is, a holomorphic function $a : \Omega \cup \{c\} \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\ f(z) = a(z).$$

- a *pole of multiplicity* p for some $p \in \mathbb{N}^*$ if there is a $a^* \in \mathbb{C}^*$ such that

$$f(z) \underset{c}{\sim} \frac{a^*}{(z-c)^p} \ \text{ or equivalently } \ \lim_{z \to c} f(z)(z-c)^p = a^*.$$

- an *essential singularity* otherwise.

Theorem – Characterization of Removable Singularities. An isolated singularity c of a holomorphic function $f : \Omega \to \mathbb{C}$ is removable if and only if one of the following conditions holds:

1. The Laurent expansion of f in some non-empty open annulus $A(c, 0, r)$ is a power series (its coefficients a_n are zero if $n < 0$).
2. The value $f(z)$ has a limit in $\mathbb{C}$ when $z \to c$.
3. The function f is bounded in some non-empty open annulus $A(c, 0, r)$.

Proof. The validity of the criteria 1 and 2 is a direct consequence of the local behavior of holomorphic functions lemma (refer to the appendix). It is also plain that condition 3 is a consequence of condition 2. Conversely, if condition 2 holds and

$$|f(z)| \leq m \ \text{ whenever } \ 0 < |z - c| < r,$$

define $\gamma = c + \rho[\circlearrowleft]$ for $0 < \rho < r$. For any $n < 0$,

$$|a_n| = \left| \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz \right| \leq m\rho^{-n} \to 0 \ \text{ when } \ \rho \to 0$$

thus condition 1 holds. ■

Theorem – Characterization of Pole Multiplicity. An isolated singularity of a holomorphic function $f : \Omega \to \mathbb{C}$ is a pole of multiplicity p if and only if one of the equivalent condition holds:

1. The Laurent expansion of f in some non-empty open annulus $A(c, 0, r)$ is

$$f(z) = \sum_{n=-p}^{+\infty} a_n (z-c)^n \ \text{ with } \ a_{-p} \neq 0.$$

2. There is a holomorphic function $a : \Omega \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\ f(z) = \frac{a(z)}{(z-c)^p} \ \text{ and } \ a(c) \neq 0.$$

Proof. A straightforward consequence of the local behavior of holomorphic functions lemma. ■

Theorem – Characterization of Poles. An isolated singularity c of a holomorphic function $f : \Omega \to \mathbb{C}$ is a pole (of multiplicity p) if and only if the inverse $1/f$ is defined in some open annulus $A(c, 0, r)$, has a holomorphic extension to $D(c, r)$ and c is a zero (of multiplicity p) of this extension. Alternatively, c is a pole of f if and only if

$$|f(z)| \to +\infty \text{ when } z \to c.$$

Proof. If the point c is a pole of order p of f, there is a holomorphic function $a : \Omega \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\ f(z) = \frac{a(z)}{(z-c)^p} \text{ and } a(c) \neq 0.$$

Let $r > 0$ be such that $a(z) \neq 0$ on $D(c, r)$. The function $z \mapsto 1/f(z)$ is holomorphic on $\in D(c, r) \setminus \{c\}$, the function $b : z \mapsto 1/a(z)$ is holomorphic on $D(c, r)$, $b(c) \neq 0$ and

$$\forall\, z \in D(c, r) \setminus \{c\},\ \frac{1}{f(z)} = b(z)(z-c)^p$$

Thus the point c is a removable singularity of $1/f$ and a zero of order p of its holomorphic extension over c. The converse statement may be proved by a similar method.

The condition $|f(z)| \to +\infty$ when $z \to c$ is equivalent to

$$1/f(z) \to 0 \text{ when } z \to c.$$

This property holds if and only if $1/f$ has a holomorphic extension over c and c is a zero of this extension. As $1/f$ is not identically zero locally, this zero has a finite multiplicity p and hence c is a pole of order p of f. ■

Computation of Residues

Theorem – Computation of Residues. Let Ω be an open set of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and c be an isolated singularity of f. If the Laurent series expansion of f in some non-empty annulus $A(c, 0, r) \subset \Omega$ is

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-c)^n$$

then

$$\mathrm{res}(f, c) = a_{-1}.$$

Proof. By definition of the residue, for any $0 < \rho < r$,

$$\operatorname{res}(f, c) = \frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} f(z)dz = \frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} \left[\sum_{n=-\infty}^{+\infty} a_n (z-c)^n \right] dz.$$

The convergence of the Laurent series expansion is uniform on any compact subset of $A(c, 0, r)$, hence

$$\operatorname{res}(f, c) = \sum_{n=-\infty}^{+\infty} \left[a_n \frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} (z-c)^n dz \right].$$

When $n \neq -1$, the function $z \mapsto (z-c)^n$ has a primitive in $\mathbb{C}^*$, hence all the terms but one in the right-hand side of the equation are equal to zero. Finally,

$$\operatorname{res}(f, c) = a_{-1} \left[\frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} \frac{dz}{z-c} \right] = a_{-1},$$

as expected. ■

Corollary – Residue of Poles. Let Ω be an open set of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and c be an isolated singularity of f. If c is a pole of f whose multiplicity is at most p:

$$\exists\, a \in \mathbb{C}, \ \lim_{z \to c} f(z)(z-c)^p = a,$$

then

$$\operatorname{res}(f, c) = \lim_{z \to c} \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \left(f(z)(z-c)^p \right)$$

Proof. Given the assumption on the multiplicity of c, the Laurent series expansion of f on $D(c, r) \setminus \{c\}$ for r small enough is

$$f(z) = \sum_{n=-p}^{+\infty} a_n (z-c)^n, \ 0 < |z-c| < r$$

thus

$$f(z)(z-c)^p = \sum_{m=0}^{+\infty} a_{m-p} (z-c)^m, \ 0 < |z-c| < r.$$

The right-hand side of this equation displays no negative power of z; this series is therefore convergent on the whole disk $D(c, r)$ where the function $z \mapsto f(z)(z-p)^p$ can be extended to a function g which is holomorphic. As the residue a_{-1} of f at c is the coefficient of $(z-c)^{p-1}$ in this Taylor expansion, we have

$$a_{-1} = \frac{g^{(p-1)}(c)}{(p-1)!} = \lim_{z \to c} \frac{g^{(p-1)}(z)}{(p-1)!}$$

which provides the expected formula. ■

Corollary – Residue of Simple Poles I. Let Ω be an open set of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and c be an isolated singularity of f. The point c is a simple pole of f if and only if

$$\exists\, a \in \mathbb{C}^*,\ \lim_{z \to c} f(z)(z-c) = a,$$

and then

$$\mathrm{res}(f, c) = \lim_{z \to c} f(z)(z-c)$$

Proof. Trivial. ■

Corollary – Residue of Simple Poles II. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let c be an isolated singularity of f. If there are two holomorphic functions g and h on Ω such that

$$f = \frac{g}{h} \ \text{ where } \ g(c) \neq 0,\ h(c) = 0,\ h'(c) \neq 0,$$

then c is a simple pole of f and

$$\mathrm{res}(f, c) = \frac{g(c)}{h'(c)}.$$

Proof. Given the assumptions, we have

$$\lim_{z \to c} (z-c) f(z) = \lim_{z \to c} \frac{z-c}{h(z)-h(c)} g(z) = \frac{g(c)}{h'(c)} \neq 0,$$

hence c is a simple pole of f whose residue is $g(c)/h'(c)$. ■

Appendix – Local Behavior of Holomorphic Functions

Lemma – Local Behavior of Holomorphic Functions. Let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function defined on some open subset Ω of $\mathbb{C}$. Let $c \in \mathbb{C}$ be a point which is either in the domain of definition of f or an isolated singularity of f; in any case, there is a $r > 0$ such that $D(c, r) \subset \Omega \cup \{c\}$.

For any $p \in \mathbb{Z}$ and $a^* \in \mathbb{C}$, the following properties are equivalent:

1. We have
$$\lim_{z \to c} \frac{f(z)}{(z-c)^p} = a^*.$$

2. There is a function $a : \Omega \cup \{c\} \to \mathbb{C}$ such that
$$\forall\, z \in \Omega,\ f(z) = a(z)(z-c)^p \ \text{ and } \ \lim_{z \to c} a(z) = a(c) = a^*.$$

3. There are some $a_n \in \mathbb{C}$ defined for $n \geq p$ such that

$$\forall\, z \in \Omega \cap D(c,r),\ f(z) = \sum_{n=p}^{+\infty} a_n (z-c)^n \ \text{ and } \ a_p = a^*.$$

4. There is a holomorphic function $a : \Omega \cup \{c\} \to \mathbb{C}$, such that

$$\forall\, z \in \Omega,\ f(z) = a(z)(z-c)^p \ \text{ and } \ a(c) = a^*.$$

Proof. If condition 1 holds, the function $a : \Omega \cup \{c\} \to \mathbb{C}$ defined by

$$a(z) = \frac{f(z)}{(z-c)^p} \ \text{ if } \ z \in \Omega \setminus \{c\} \ \text{ and } \ a(c) = a^*$$

satisfies condition 2.

If condition 2 holds, the function a is continuous, thus for any compact set $K \subset D(c,r)$, there is a finite m such that

$$\forall\, z \in K \cap \Omega,\ |a(z)| = \left| \frac{f(z)}{(z-c)^p} \right| \leq m.$$

Hence, if $0 < \rho < r$ and $\gamma = c + \rho[\circlearrowleft]$, the n-th coefficient a_n of the Laurent expansion of f in $D(c,r) \setminus \{c\}$ satisfies

$$a_n = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^p} (z-c)^{p-n-1}\, dz$$

and by the M-L inequality,

$$|a_n| \leq \left[\sup_{|z|=\rho} \left| \frac{f(z)}{(z-c)^p} \right| \right] \rho^{p-n}.$$

If $n < p$, the right-hand side of this inequality tends to zero when $\rho \to 0$, therefore $a_n = 0$. If $n = p$ on the other hand,

$$a_p = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{p+1}}\, dz = \int_0^1 a(c + \rho e^{i2\pi t})\, dt$$

and hence $a_p = \lim_{z \to c} a(z) = a^*$. Now if $c \in \Omega$, the Taylor expansion of f in $D(c,r)$ provides a Laurent expansion of f in $D(c,r) \setminus \{c\}$; this expansion is unique, hence the coefficient sequences are equal and the initial Laurent expansion is valid in $D(c,r)$.

If condition 3 holds, the series

$$\sum_{k=0}^{+\infty} a_{k+p} (z-c)^k$$

is convergent in $D(c,r) \setminus \{c\}$ and hence in $D(c,r)$. Its sum $a_c(z)$ satisfies $f(z) = a_c(z)(z-c)^p$ in $\Omega \cap D(c,r)$. Consequently, the function $a : \Omega \cup \{c\} \to \mathbb{C}$ may be defined unambiguously by

$$a(z) = a_c(z) \text{ if } z \in D(c,r) \text{ and } a(z) = \frac{f(z)}{(z-c)^p} \text{ otherwise}$$

and it is holomorphic.

Finally if condition 4 holds, it is plain that condition 1 holds. ■