# Line Integrals & Primitives

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## Introduction

The main goal of this chapter is to derive the fundamental theorem of calculus for functions of a complex variable. This theorem characterizes the relation between functions and their primitives with the help of integrals. A version of this theorem for functions of a real variable is the following:

**Theorem – Fundamental Theorem of Calculus (Real Analysis).** Let $I$ be an open interval of $\mathbb{R}$, $f : I \to \mathbb{R}$ be a continuous function and $a \in I$. A function $g : I \to \mathbb{R}$ is a primitive of $f$ if and only if it satisfies

$$\forall\, x \in I,\; g(x) = g(a) + \int_a^x f(t)\, dt.$$

**Proof.** Suppose that the function $g$ satisfies the integral equation of the theorem.

For any $x \in I$ and any real number $h$ such that $x + h \in I$,

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= \frac{1}{h}\int_x^{x+h} f(x)\,dt + \frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,dt \\ &= f(x) + \frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,dt,\end{aligned}$$

Let $\epsilon > 0$; by continuity of $f$ at $x$, there is a $\delta > 0$ such that

$$\forall\, t \in I,\ (|t-x| \leq \delta \,\Rightarrow\, |f(t)-f(x)| < \epsilon)$$

thus if $|h| < \delta$,

$$\left|\frac{g(x+h)-g(x)}{h} - f(x)\right| \leq \frac{1}{|h|}|h| \times \epsilon = \epsilon.$$

The difference quotient tends to $f(x)$ when $h$ tends to zero: $g'(x)$ exists and is equal to $f(x)$.

Conversely, suppose that $e : I \to \mathbb{R}$ is a primitive of $f$. The difference $d$ between $e$ and the function

$$g : x \in I \mapsto e(a) + \int_a^x f(t)\,dt$$

is zero at $a$ and has a zero derivative on $I$. By the mean value theorem, for any $x \in I$ such that $x \neq a$, there is a $b \in I$ such that

$$\frac{d(x)-d(a)}{x-a} = d'(b) = 0,$$

hence $d(x) = d(a) = 0$ and therefore $e = g$. ■

# Paths

**Definition – Path.** A *path* $\gamma$ is a continuous function from $[0,1]$ to $\mathbb{C}$. If $A$ is a subset of the complex plane, $\gamma$ is a *path of* $A$ if additionally $\gamma([0,1]) \subset A$.

**Definition – Image of a Path.** The *image* or *trajectory* or *trace* of the path $\gamma$ is the image $\gamma([0,1])$ of the interval $[0,1]$ by the function $\gamma$.

**Definition – Path Endpoints.** The complex numbers $\gamma(0)$ and $\gamma(1)$ are the *initial point* and *terminal point* of $\gamma$ – they are its *endpoints*; the path $\gamma$ *joins* its initial and terminal points. The path is *closed* if the initial and terminal point are the same. The paths $\gamma_1, \dots, \gamma_n$ are *consecutive* if for $k = 1, \dots, n-1$, the terminal point of $\gamma_k$ is the initial point of $\gamma_{k+1}$.

**Example – Oriented Line Segment.** The oriented line segment (or simply oriented segment) with initial point $a \in \mathbb{C}$ and terminal point $b \in \mathbb{C}$ is denoted $[a \to b]$ and defined as

$$[a \to b] : t \in [0, 1] \mapsto (1 - t)a + tb.$$

Its image is the line segment $[a, b]$.

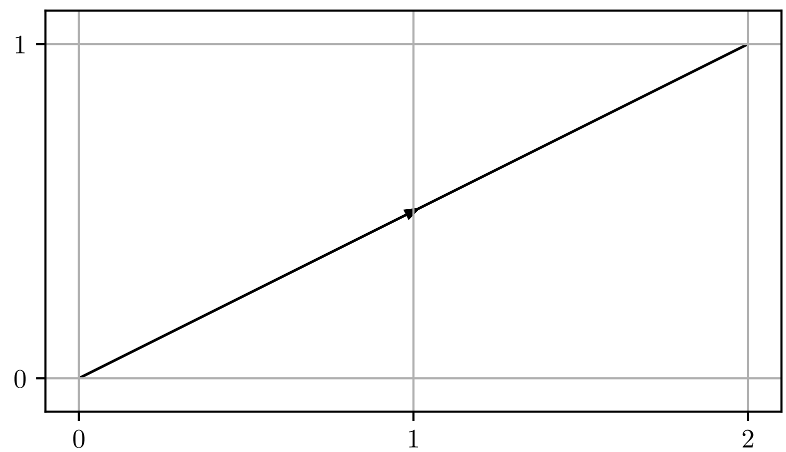


Figure 1: Representation of the oriented line segment $[0 \to 2 + i]$

**Example – Oriented Circle.** The oriented circle of radius one centered at the origin traversed once in the positive sense (counterclockwise) is denoted $[\circlearrowleft]$ and defined as

$$[\circlearrowleft] : t \in [0, 1] \to e^{i2\pi t}.$$

The circle of radius $r \geq 0$ centered at $c \in \mathbb{C}$ traversed $n \in \mathbb{Z}^*$ times in the positive sense is the path

$$c + r[\circlearrowleft]^n : t \in [0, 1] \to c + re^{i2\pi nt}.$$

Its image is the circle centered on $c$ with radius $r$; its initial and terminal points are both $c + r$, hence it is closed.

**Definition – Open (Path-)Connected Sets.** An open subset $\Omega$ of the complex-plane is *(path-)connected* if for any points $x$ and $y$ of $\Omega$, there is a path of $\Omega$ that joins $x$ and $y$.

**Definition – Reverse Path.** The *reverse* (or *opposite*) of the path $\gamma$ is the path $\gamma^{\leftarrow}$ defined by

$$\forall\, t \in [0, 1],\ \gamma^{\leftarrow}(t) = \gamma(1 - t).$$

**Definition – Path Concatenation.** Let $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ be a partition of the interval $[0, 1]$. The *concatenation* of consecutive paths

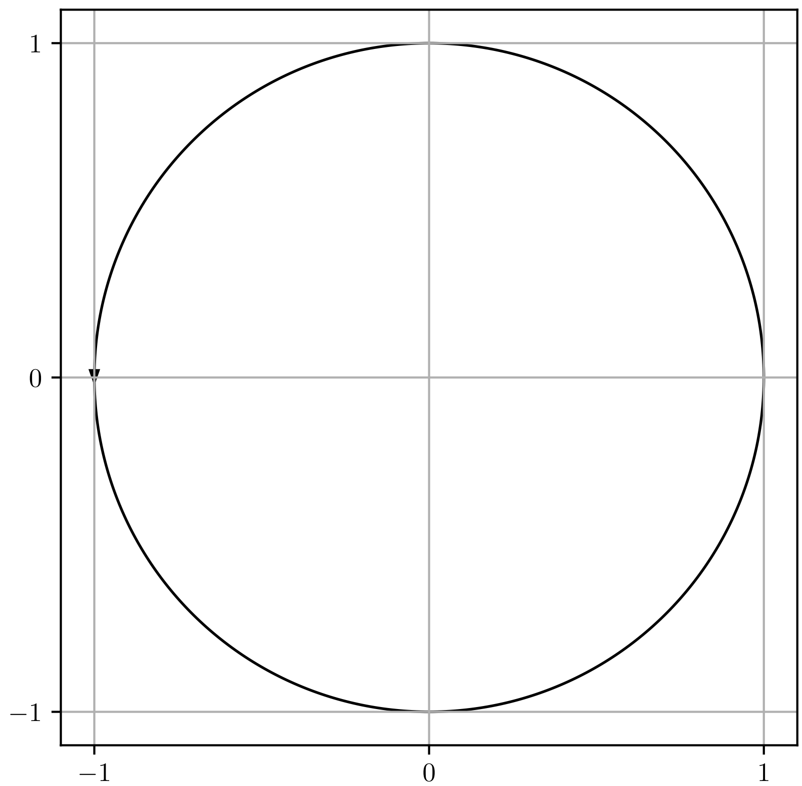


Figure 2: Representation of the oriented circle [↻]

$\gamma_1, \dots, \gamma_n$ associated to this partition is the path $\gamma$ denoted

$$\gamma_1 \,|_{t_1} \cdots |_{t_{n-1}} \gamma_n$$

such that

$$\forall \, k \in \{1, \dots, n\}, \; \gamma|_{[t_{k-1}, t_k]} = \gamma_k \left( \frac{t - t_{k-1}}{t_k - t_{k-1}} \right).$$

If the partition of $[0, 1]$ is uniform, that is, if

$$\forall \, k \in \{0, \dots, n\}, \; t_k = k/n,$$

we denote the concatenated path with the simpler notation

$$\gamma_1 \,|\, \cdots \,|\, \gamma_n.$$

**Example – Oriented Polyline.** An oriented polyline (or piecewise linear path) is the concatenation of consecutive oriented line segments. When the associated partition of $[0, 1]$ is uniform, we use the notation

$$[a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n] = [a_0 \to a_1] \,|\, \cdots \,|\, [a_{n-1} \to a_n].$$

**Definition – Rectifiable Path.** A path $\gamma : [0, 1] \to \mathbb{C}$ is *rectifiable* if the function $\gamma$ is piecewise continuously differentiable.

Given the definition of piecewise continuously differentiable, the following alternate characterization is plain:

**Theorem – Continuously Differentiable Decomposition.** A path $\gamma : [0, 1] \to \mathbb{C}$ is rectifiable if and only if there are consecutive continuously differentiable paths $\gamma_1, \dots, \gamma_n$ and a partition $(t_0, \dots, t_n)$ of the interval $[0, 1]$ such that

$$\gamma = \gamma_1 \,|_{t_1} \cdots |_{t_{n-1}} \gamma_n.$$

We characterized initially connected sets via merely continuous paths. However, when such sets are open, we can use rectifiable paths instead:

**Lemma – Connectedness & Rectifiable Paths.** An open subset $\Omega$ of the complex plane is *connected* if and only if every pair of points of $\Omega$ may be joined by a rectifiable path of $\Omega$.

**Proof.** If any pair of points of $\Omega$ can be joined by a rectifiable path of $\Omega$, then $\Omega$ is connected. Conversely, assume that a (merely continuous) path $\gamma$ of $\Omega$ joins $x$ and $y$. Its image $\gamma([0, 1])$ is a compact subset of $\Omega$ – as the image of a compact set by a continuous function – thus the distance $r$ between $\gamma([0, 1])$ and the closed set $\mathbb{C} \setminus \Omega$ is positive. Additionally, the function $\gamma$ is uniformly continuous – as a continuous function with a compact domain of definition; there is a positive integer $n$ such that

$$\forall \, t \in [0, 1], \; \forall \, s \in [0, 1], \; (|t - s| \leq 1/n \; \Rightarrow \; |\gamma(t) - \gamma(s)| < r).$$

For any $k \in \{0, \dots, n\}$, the point $\gamma(k/n)$ belongs to $\Omega$; the path $\mu$ defined as

$$\mu = [\gamma(0) \to \cdots \to \gamma(k/n) \to \cdots \to \gamma(1)]$$

is rectifiable and joins $x$ and $y$. Now, for any $t \in [0,1]$, let $k \in \{0, \dots, n-1\}$ be such that $t \in [k/n, (k+1)/n]$. We have

$$|\mu(t) - \gamma(k/n)| \leq |\gamma((k+1)/n) - \gamma(k/n)| < r,$$

therefore $\mu$ is a path of $\Omega$. ■

## Line Integrals

**Definition – Length of a Rectifiable Path.** The length of a rectifiable path $\gamma$ is the nonnegative real number

$$\ell(\gamma) = \int_0^1 |\gamma'(t)| \, dt.$$

**Example – Length of an Oriented Segment.** The oriented segment $[a \to b]$ is continuously differentiable and thus rectifiable. For any $t \in [0,1]$, $[a \to b]'(t) = b - a$, hence its length is

$$\ell([a \to b]) = \int_0^1 |b - a| \, dt = |b - a|.$$

**Example – Length of an Oriented Circle.** The oriented circle $c + r[\circlearrowleft]^n$ centered at $c$ with radius $r \geq 0$ traversed $n$ times in the positive sense is continuously differentiable and thus rectifiable. For any $t \in [0,1]$,

$$[c + r[\circlearrowleft]^n]'(t) = (i2\pi n) r e^{i2\pi n t},$$

hence the length of this path is

$$\ell(c + r[\circlearrowleft]^n) = \int_0^1 |(i2\pi n) r e^{i2\pi n t}| \, dt = \int_0^1 |2\pi n r| \, dt = 2\pi r \times |n|.$$

It differs from the length of its circle image – which is $2\pi r$ – unless the circle is traversed exactly once in the positive or negative sense.

**Definition – Line Integral.** The *line integral* along a rectifiable path $\gamma$ of a complex-valued function $f$ defined and continuous on the image of $\gamma$ is the complex number defined by

$$\int_\gamma f(z) \, dz = \int_0^1 f(\gamma(t)) \gamma'(t) \, dt.$$

**Remark – Undefined Integrands.** In the definitions of the length of $\gamma$ and of the integral along $\gamma$, the integrands

$$|\gamma'(t)| \ \text{ and } \ f(\gamma(t))\gamma'(t)$$

may be undefined for some values of $t$ if $\gamma$ is merely rectifiable. However it's not an issue since they are always defined almost everywhere (and integrable).

**Remark – Integral Notation.** It's sometimes handy to use the notation

$$\int_\gamma f(z)\,|dz| = \int_0^1 f(\gamma(t))|\gamma'(t)|dt.$$

which is similar to the one used for line integrals. With this convention, we have for example

$$\ell(\gamma) = \int_\gamma |dz|.$$

**Example – Integration along an Oriented Segment.** The line integral of the continuous function $f : [a, b] \mapsto \mathbb{C}$ along the oriented segment $[a \to b]$ is

$$\begin{aligned}\int_{[a\to b]} f(z)\,dz &= \int_0^1 f((1-t)a + tb)(b-a)\,dt \\ &= (b-a)\int_0^1 f((1-t)a + tb)\,dt.\end{aligned}$$

**Example – Integration along an Oriented Circle.** The line integral of a continuous function $f : \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C}$ on the oriented circle $[\circlearrowleft]$ is

$$\begin{aligned}\int_{[\circlearrowleft]} f(z)\,dz &= \int_0^1 f(e^{i2\pi t})(i2\pi e^{i2\pi t}dt) \\ &= i\int_0^1 f(e^{i2\pi t})e^{i2\pi t}\,(2\pi dt) \\ &= i\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta.\end{aligned}$$

**Theorem – Complex-Linearity of the Line Integral.** Let $\gamma$ be a rectifiable path. For any $\alpha, \beta \in \mathbb{C}$ and any continuous functions $f$ and $g$ defined on the image of $\gamma$,

$$\int_\gamma \alpha f(z) + \beta g(z)\,dz = \alpha\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz.$$

**Proof.** Since by definition of the line integral

$$\int_\gamma \alpha f(z) + \beta g(z)\,dz = \int_0^1 (\alpha f(\gamma(t)) + \beta g(\gamma(t)))\gamma'(t)dt,$$

the complex-linearity of the integral on $[0, 1]$ provides

$$\int_\gamma \alpha f(z) + \beta g(z)\, dz = \alpha \int_0^1 f(\gamma(t))\gamma'(t)\, dt \\ + \beta \int_0^1 g(\gamma(t))\gamma'(t)\, dt$$

■

**Theorem – Integration along a Reverse Path.** For any rectifiable path $\gamma$,

$$\ell(\gamma^{\leftarrow}) = \ell(\gamma).$$

For any continuous function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of $\gamma$,

$$\int_{\gamma^{\leftarrow}} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

**Proof.** Since $\gamma^{\leftarrow}(t) = \gamma(1-t)$, the length of the opposite of $\gamma$ satisfies

$$\ell(\gamma^{\leftarrow}(t)) = \int_0^1 |(\gamma^{\leftarrow})'(t)|\, dt = \int_0^1 |-\gamma'(1-t)|\, dt.$$

The change of variable $t \mapsto 1 - t$ yields

$$\ell(\gamma^{\leftarrow}(t)) = \int_0^1 |\gamma'(t)|\, dt = \ell(\gamma).$$

Similarly,

$$\begin{aligned}
\int_{\gamma^{\leftarrow}} f(z)\, dz &= \int_0^1 f(\gamma^{\leftarrow}(t))(\gamma^{\leftarrow})'(t)\, dt \\
&= \int_0^1 f(\gamma(1-t))(-\gamma'(1-t))\, dt \\
&= \int_0^1 f(\gamma(t))(-\gamma'(t))\, dt \\
&= -\int_\gamma f(z)\, dz
\end{aligned}$$

■

**Theorem – Integration along Concatenation of Paths.** Let $A$ be a subset of $\mathbb{C}$. Let $\gamma_1, \dots, \gamma_n$ be consecutive rectifiable paths of $A$ and let $\gamma$ be their concatenation

$$\gamma = \gamma_1 \,|_{t_1} \, \cdots \, |_{t_{n-1}} \, \gamma_n.$$

The length of $\gamma$ satisfies

$$\ell(\gamma) = \sum_{k=1}^n \ell(\gamma_k).$$

For any continuous function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of $\gamma$,

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.$$

**Proof.** Since by definition $\gamma(t) = \gamma_k((t-t_k)/(t_{k+1}-t_k))$ whenever $t \in [t_k, t_{k+1}]$, the decomposition

$$\begin{aligned}\ell(\gamma) &= \int_0^1 |\gamma'(t)|\,dt \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\gamma'(t)|\,dt\end{aligned}$$

provides

$$\ell(\gamma) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left| \frac{\gamma_k'\left(\frac{t-t_k}{t_{k+1}-t_k}\right)}{t_{k+1}-t_k}\,dt \right|$$

and the changes of variables $t \in [t_k, t_{k+1}] \mapsto \frac{t-t_k}{t_{k+1}-t_k}$ yield

$$\begin{aligned}\ell(\gamma) &= \sum_{k=1}^{n} \int_0^1 |\gamma_k'(t)|\,dt \\ &= \sum_{k=1}^{n} \ell(\gamma_k)\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)\,dt \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)\,dt \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f\left(\gamma_k\left(\frac{t-t_k}{t_{k+1}-t_k}\right)\right) \frac{\gamma_k'\left(\frac{t-t_k}{t_{k+1}-t_k}\right)}{t_k - t_{k-1}}\,dt \\ &= \sum_{k=1}^{n} \int_0^1 f(\gamma_k(t))\gamma_k'(t)\,dt \\ &= \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz\end{aligned}$$

■

**Theorem – M-L Inequality.** For any rectifiable path $\gamma$ and any continuous function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of $\gamma$,

$$\left| \int_\gamma f(z)\, dz \right| \leq \left( \max_{z \in \gamma([0,1])} |f(z)| \right) \times \ell(\gamma).$$

**Proof.** By definition of the line integral

$$\begin{aligned}
\left| \int_\gamma f(z)\, dz \right| &= \left| \int_0^1 f(\gamma(t))\gamma'(t)\, dt \right| \\
&\leq \int_0^1 |f(\gamma(t))||\gamma'(t)|\, dt \\
&\leq \left( \max_{t \in [0,1]} |f(\gamma(t))| \right) \times \int_0^1 |\gamma'(t)|\, dt \\
&= \left( \max_{z \in \gamma([0,1])} |f(z)| \right) \times \ell(\gamma).
\end{aligned}$$

■

A practical consequence of the M-L inequality:

**Corollary – Convergence in Line Integrals.** For any rectifiable path $\gamma$ and any sequence of continuous function $f_n : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of $\gamma$ which converges uniformly to the function $f$,

$$\lim_{n \to +\infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz.$$

**Proof.** The linearity of the line integral and the M-L inequality provide

$$\begin{aligned}
\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z) \right| &= \left| \int_\gamma (f_n(z) - f(z))\, dz \right| \\
&\leq \left( \max_{z \in \gamma([0,1])} |f_n - f(z)| \right) \times \ell(\gamma)
\end{aligned}$$

which yields the desired result. ■

**Theorem – Invariance By Reparametrization.** Let $\gamma : [0,1] \to \mathbb{C}$ be a continuously differentiable path. Let $\phi : [0,1] \to [0,1]$ be an increasing $C^1$-diffeomorphism – a continuously differentiable function such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi'(t) > 0$ for any $t \in [0,1]$. The following statements hold:

- The path $\mu = \gamma \circ \phi$ is a continuously differentiable path.
- It has the same initial point, terminal point and image as $\gamma$.
- The length of $\mu$ and $\gamma$ are identical.

- For any continuous function $f : \gamma([0,1]) \to \mathbb{C}$,

$$\int_\mu f(z)\,dz = \int_\gamma f(z)\,dz.$$

**Proof.** The function $\mu$ is continuously differentiable as the composition of continuously differentiable functions. We have

$$\mu(0) = \gamma(\phi(0)) = \gamma(0),\ \mu(1) = \gamma(\phi(1)) = \gamma(1),$$

hence the endpoints of $\gamma$ and $\mu$ are identical. The function $\phi$ is a bijection from $[0,1]$ into itself, therefore

$$\mu([0,1]) = \gamma(\phi([0,1])) = \gamma([0,1])$$

and the images of $\gamma$ and $\mu$ are identical.

The length of $\mu$ is

$$\ell(\mu) = \int_0^1 |\mu'(t)|\,dt = \int_0^1 |\gamma'(\phi(t))\phi'(t)|\,dt = \int_0^1 |\gamma'(\phi(t))|\,\phi'(t)dt$$

The change of variable $s = \phi(t)$ provides

$$\int_0^1 |\gamma'(\phi(t))|\,\phi'(t)dt = \int_0^1 |\gamma'(s)|\,ds,$$

hence the lengths of $\gamma$ and $\mu$ are equal. We also have

$$\int_\mu f(z)\,dz = \int_0^1 (f \circ \mu)(t)\mu'(t)\,dt = \int_0^1 (f \circ \gamma)(\phi(t))\gamma'(\phi(t))\,(\phi'(t)dt).$$

The same change of variable leads to

$$\int_\mu f(z)\,dz = \int_0^1 (f \circ \gamma)(s)\gamma'(s)\,ds = \int_\gamma f(z)\,dz,$$

which concludes the proof. ■

**Definition – Image of a Path by a Function.** Let $\gamma : [0,1] \to \mathbb{C}$ be a path and $f : A \subset \mathbb{C} \to \mathbb{C}$ be a continuous function defined on the image of $\gamma$. The *image of* $\gamma$ *by* $f$ is the path $f \circ \gamma$.

**Theorem – Change of Variable in Line Integrals [†].** Let $\Omega$ be an open subset of $\mathbb{C}$, let $\gamma$ be a rectifiable path of $\Omega$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. The path $f \circ \gamma$ is rectifiable and for any continuous function $g : A \subset \mathbb{C} \to \mathbb{C}$ defined on the image of $f \circ \gamma$,

$$\int_{f \circ \gamma} g(z)\,dz = \int_\gamma g(f(w))f'(w)\,dw.$$

**Proof.** Let $\gamma_1 \,|_{t_1} \, \dots \, |_{t_{n-1}} \, \gamma_n$ be a continously differentiable decomposition of $\gamma$. We have

$$f \circ \gamma = f \circ \gamma_1 \,|_{t_1} \, \dots \, |_{t_{n-1}} \, f \circ \gamma_n,$$

and for any $k \in \{1, \dots, n\}$, by the chain rule, the function $f \circ \gamma_k$ is continuously differentiable ($f'$ being continuous) hence the path $f \circ \gamma$ is rectifiable.

Moreover,

$$\begin{aligned}
\int_\gamma g(f(w))f'(w)\, dw &= \int_0^1 g(f(\gamma(t))f'(\gamma(t))\gamma'(t)\, dt \\
&= \int_0^1 g(f(\gamma(t))(f \circ \gamma)'(t)\, dt \\
&= \int_{f \circ \gamma} g(w)\, dw
\end{aligned}$$

■

## Primitives

**Definition – Primitive.** Let $f : \Omega \to \mathbb{C}$ where $\Omega$ is an open subset of $\mathbb{C}$. A *primitive* (or *antiderivative)* of $f$ is a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g' = f$.

**Theorem – Fundamental Theorem of Calculus (Complex Analysis).** Let $\Omega$ be an open connected subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a continuous function and let $a \in \Omega$. A function $g : \Omega \to \mathbb{C}$ is a primitive of $f$ if and only if for any $z \in \Omega$ and any rectifiable path $\gamma$ of $\Omega$ that joins $a$ and $z$,

$$g(z) = g(a) + \int_\gamma f(w)\, dw.$$

**Proof.** Let $g$ be a primitive of $f$ and $\gamma$ be a rectifiable path of $\Omega$ that joins $a$ and $z$. Let $\gamma = \gamma_1 \,|_{t_1} \, \dots \, |_{t_{n-1}} \, \gamma_n$ be a continuously differentiable decomposition of $\gamma$. For any $k \in \{1, \dots, n\}$, the function

$$\phi : t \in [0, 1] \mapsto g(\gamma_k(t))$$

is differentiable as a composition of real-differentiable functions, with

$$\phi'(t) = dg_{\gamma_k(t)}(\gamma_k'(t)) = g'(\gamma_k(t))\gamma_k'(t).$$

The function $\phi'$ is continuous hence by the fundamental theorem of calculus (from real analysis) applied to the real and imaginary parts of $\phi'$ on $]0, 1[$, we have for any positive number $\epsilon$ smaller than 1,

$$\phi(1 - \epsilon) - \phi(\epsilon) = \int_\epsilon^{1-\epsilon} \phi'(t)\, dt,$$

and thus by continuity of $\phi$ and $\phi'$

$$\phi(1) - \phi(0) = \int_0^1 \phi'(t)\,dt,$$

which is equivalent to

$$g(\gamma_k(1)) - g(\gamma_k(0)) = \int_0^1 g'(\gamma_k(t))\gamma_k'(t)\,dt = \int_{\gamma_k} f(w)\,dw.$$

The sum of these equations for all $k \in \{1, \dots, n\}$ provides

$$g(z) - g(a) = \int_\gamma f(w)\,dw.$$

Conversely, assume that $g$ satisfies the theorem property. Let $\gamma$ be a rectifiable path of $\Omega$ that joins $a$ and $z$ and let $r > 0$ be such that the open disk centered at $z$ with radius $r$ is included in $\Omega$. Consider the concatenation $\mu$ of $\gamma$ and of the oriented segment $[z \to z + h]$ for $h$ such that $|h| < r$. It is a rectifiable path of $\Omega$, hence

$$\begin{aligned} g(z+h) &= g(a) + \int_\mu f(w)\,dw \\ &= g(a) + \int_\gamma f(w)\,dw + h\int_0^1 f(z+th)\,dt \\ &= g(z) + h\int_0^1 f(z+th)\,dt \end{aligned}$$

hence

$$\frac{g(z+h) - g(z)}{h} = \int_0^1 f(z+th)\,dt.$$

The right-hand side of this equation converges to $f(z)$ by continuity when $h$ goes to zero, therefore $g$ is a primitive of $f$. ■

**Corollary – Existence of Primitives [†].** Let $\Omega$ be an open connected subset of $\mathbb{C}$. The function $f : \Omega \to \mathbb{C}$ has a primitive if and only if it is continuous and for any closed rectifiable path $\gamma$

$$\int_\gamma f(z)\,dz = 0.$$

**Proof – Existence of Primitives.** If the function $f$ has primitives, it is the derivative of a holomorphic function, thus it is continuous. Additionally, for any closed rectifiable path $\gamma$ of $\Omega$, the fundamental theorem of calculus provides

$$g(\gamma(1)) = g(\gamma(0)) + \int_\gamma f(w)\,dw,$$

hence as $\gamma(1) = \gamma(0)$,

$$\int_\gamma f(w)\, dw = 0.$$

Conversely, assume that any such integral is zero. Select any $a$ in $\Omega$ and define for any point $z$ in $\Omega$ and any rectifiable path $\gamma$ of $\Omega$ that joins them the function

$$g(z) = g(a) + \int_\gamma f(w)\, dw.$$

This definition is non-ambiguous: if we select a different path $\mu$, the difference between the right-hand sides of the definitions would be

$$\left(g(a) + \int_\gamma f(w)\, dw\right) - \left(g(a) + \int_\mu f(w)\, dw\right) = \int_{\gamma \,|\, \mu^\leftarrow} f(w)\, dw = 0$$

as $\gamma \,|\, \mu^\leftarrow$ is a closed rectifiable path of $\Omega$. Consequently, $g$ is uniquely defined and by the fundamental theorem of calculus, it is a primitive of $f$. ■

**Corollary – Set of Primitives.** Let $\Omega$ be an open connected subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$. If $g : \Omega \to \mathbb{C}$ is a primitive of $f$, the function $h : \Omega \to \mathbb{C}$ is also a primitive of $f$ if and only if it differs from $g$ by a constant.

**Proof.** It is clear that a function $h$ that differs from $g$ by a constant is a primitive of $f$. Conversely, if $g$ and $h$ are both primitives of $f$, $g - h$ is a primitive of the zero function. The fundamental theorem of calculus shows that for any $a$ and $z$ in $\Omega$ and any rectifiable path $\gamma$ of $\Omega$ that joins them,

$$g(z) - h(z) = g(a) - h(a) + \int_\gamma 0\, dw = g(a) - h(a)$$

hence their difference is a constant. ■

**Corollary – Integration by Parts [†].** Let $\Omega$ be an open connected subset of $\mathbb{C}$ and let $\gamma$ be a rectifiable path of $\Omega$. For any pair of holomorphic functions $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$,

$$\int_\gamma f'g(z)\, dz = [fg(\gamma(1)) - fg(\gamma(0))] - \int_\gamma fg'(z)\, dz.$$

**Proof.** The derivative of the function $fg$ is $f'g + fg'$. It is continuous as a sum and product of continuous functions thus the fundamental theorem of calculus provides

$$fg(\gamma(1)) = fg(\gamma(0)) + \int_\gamma (f'g + fg')(z)\, dz,$$

which is equivalent to the conclusion of the corollary. ■

**Remark & Definition – Variation of a Function on a Path.** The difference between the value of a function $f$ at the terminal value and at the initial value of a path $\gamma$ may be denoted $[f]_\gamma$:

$$[f]_\gamma = f(\gamma(1)) - f(\gamma(0)).$$

With this convention, the formula that relates a function $f$ and its primitive $g$ is

$$[g]_\gamma = \int_\gamma f(z)\, dz$$

and the integration by parts formula becomes

$$\int_\gamma f' g(z)\, dz = [fg]_\gamma - \int_\gamma f g'(z)\, dz.$$

# Appendix – A Better Theory of Rectifiability

## Rectifiable Paths

The definition we used so far for "rectifiable" is a conservative one. In this section, we come up with a more general definition of the concept that still meets the requirements for the definition of line integrals.

To "rectify" a path (from Latin *rectus* "straight" and *facere* "to make") is to straighten – or by extension to compute its length, which is a trivial operation once a path has been straightened.

The general definition of the length of a path does not require line integrals. Instead, consider any partition $(t_0, \dots, t_n)$ of the interval $[0, 1]$ and the path $\mu = \mu_1 \,|_{t_1} \,\dots\, |_{t_{n-1}} \mu_n$ where

$$\mu_k(t) = (1-t)\gamma(t_{k-1}) + t\gamma(t_k).$$

We may define the length of such a combination of straight lines as

$$\ell(\mu) = \sum_{k=1}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)|.$$

As the straight line is the shortest path between two points, this number should provide a lower bound of the length of $\gamma$. On the other hand, using finer partitions of the interval $[0, 1]$ should also provide better approximations of the length of $\gamma$. Following this idea, we may *define* the length of $\gamma$ as the supremum of the length of $\mu$ for all possible partitions of $[0, 1]$:

$$\ell(\gamma) = \sup \left\{ \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \;\middle|\; n \in \mathbb{N}^*,\, t_0 = 0 < \dots < t_n = 1 \right\}$$

Not every path has a finite length; those who have are by definition *rectifiable*. In general, a function $\gamma : [0, 1] \mapsto \mathbb{C}$ whose length is finite – even if it is not continuous – is of *bounded variation*.

## The Line Integral

To define line integrals along the path $\gamma$, it is enough that $\gamma$ is of bounded variation. For any such function $\gamma$, we may build a (complex-valued, Borel) measure on $[0,1]$ denoted $d\gamma$. This measure is defined by its integral of any continuous function $\phi : [0,1] \to \mathbb{C}$, as a limit of Riemann(-Stieltjes) sums

$$\int_{[0,1]} \phi \, d\gamma = \lim \sum_{m=0}^{n-1} \phi(t_m)(\gamma(t_{m+1}) - \gamma(t_m)).$$

The limit is taken over the partitions of the interval $[0,1]$ with

$$\max \{|t_{m+1} - t_m| \mid m \in \{0, \ldots, n-1\}\} \to 0.$$

The line integral of a continuous function $f : \gamma([0,1]) \to \mathbb{C}$ is then defined by

$$\int_\gamma f(z) \, dz = \int_{[0,1]} (f \circ \gamma) \, d\gamma.$$

The *total variation* $|d\gamma|$ of $d\gamma$ is the positive measure defined by

$$|d\gamma|(A) = \sup_{\mathfrak{P}} \sum_{B \in \mathfrak{P}} |d\gamma(B)|$$

where the supremum is taken over all finite partitions $\mathfrak{P}$ of $A$ into measurable sets. This measure provides an integral expression for the length of $\gamma$:

$$\ell(\gamma) = \int_{[0,1]} |d\gamma|.$$

## A Non-Rectifiable Curve

The Koch snowflake (Koch 1904) is an example of a continuous curve which is is nowhere differentiable; it is also a non-rectifiable closed path. It is defined as the limit of a sequence of polylines $\gamma_n$. The first element of this sequence is an oriented equilateral triangle:

$$\gamma_1 = [0 \to 1 \to e^{i\pi 3} \to 0].$$

Then, $\gamma_{n+1}$ is defined as a transformation of $\gamma_n$: every oriented line segment $[a \to a+h]$ that composes $\gamma_n$ is replaced by the polyline:

$$\left[a \to a + \frac{h}{3} \to a + \left(1 + e^{-i\pi/3}\right) \frac{h}{3} \to a + 2\frac{h}{3} \to a + h\right]$$

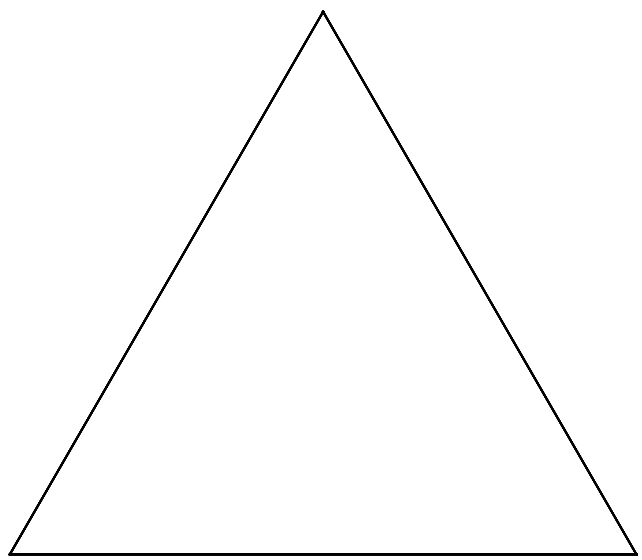

Figure 3: Image of the Koch snowflake, first iteration.

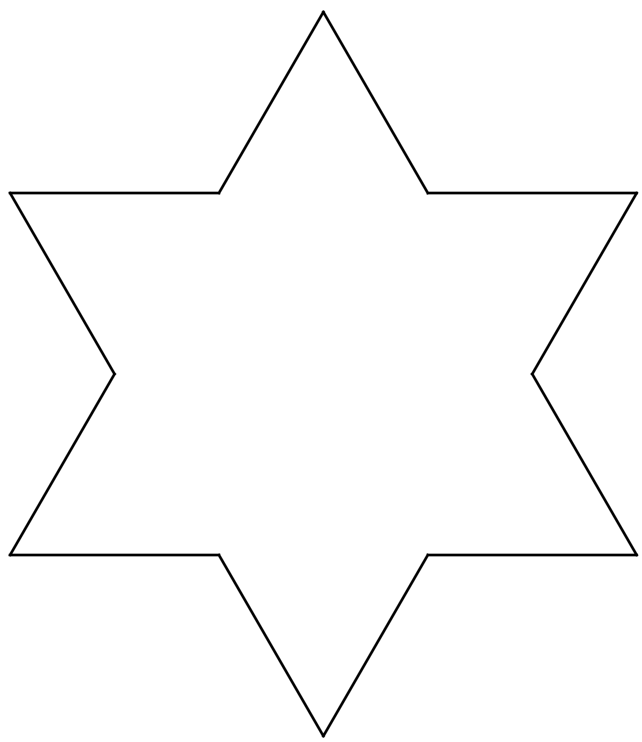

Figure 4: Image of the Koch snowflake, second iteration.

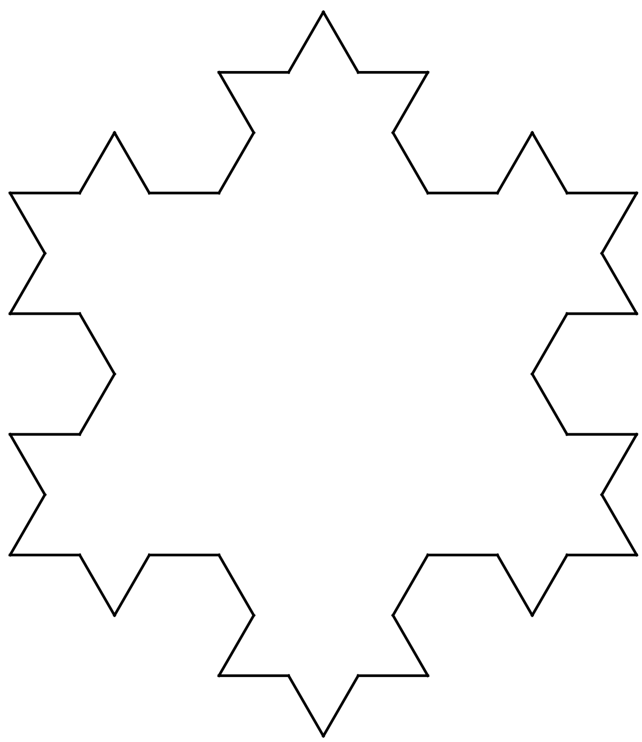

Figure 5: Image of the Koch snowflake, third iteration.

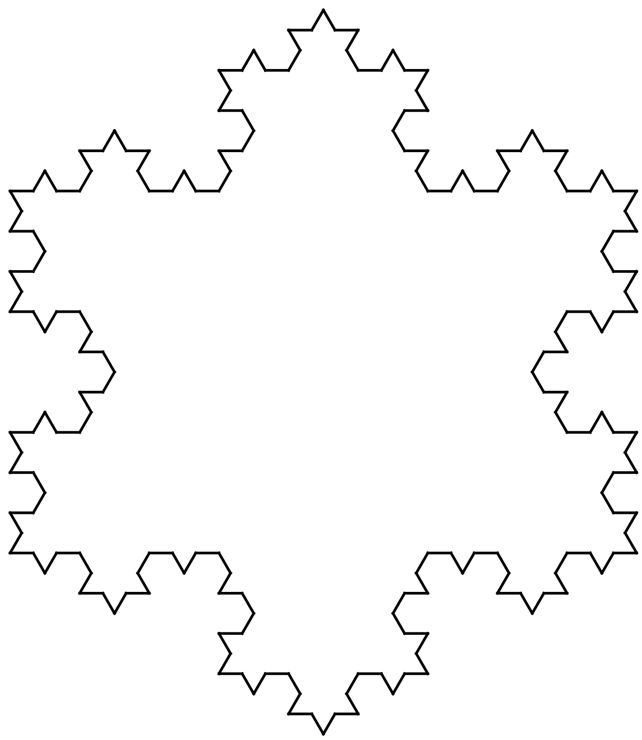

Figure 6: Image of the Koch snowflake, fourth iteration

The Koch snowflake $\gamma$ is defined as the limit of the $\gamma_n$ sequence. The geometric construction yields that for any $n$ greater than zero,

$$\forall\, t \in [0,1],\ |\gamma_{n+1}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{\sqrt{3}}{2}.$$

As $\sum_{p=0}^{+\infty} \left(\frac{1}{3}\right)^p = \frac{1}{1-1/3} = \frac{3}{2}$, for any positive integer $p$ we have

$$\forall\, t \in [0,1],\ |\gamma_{n+p}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{3}{2}\frac{\sqrt{3}}{2}.$$

The sequence $\gamma_n$ is a Cauchy sequence in the space of continuous and complex-valued functions defined on $[0,1]$; its uniform limit exists and is also continuous.

On the other hand, the curve is not rectifiable. First, the definition of the sequence $\gamma_n$ makes it plain that every iteration increases the initial length of the path by one-third:

$$\ell(\gamma_n) = 3 \times \left(\frac{4}{3}\right)^{n-1}.$$

The length of $\gamma_n$ tends to $+\infty$ when $n \to +\infty$. Now, every point at the junction of the segments of the polyline $\gamma_n$ also belongs to the Koch snowflake; more precisely

$$\forall\, m \in \{0, \dots, 3 \times 4^{n-1}\},\ \gamma\left(\frac{m}{3 \times 4^{n-1}}\right) = \gamma_n\left(\frac{m}{3 \times 4^{n-1}}\right).$$

Therefore

$$\ell(\gamma) \geq \sum_{m=0}^{3\times 4^{n-1}-1} \left|\gamma\left(\frac{m+1}{3 \times 4^{n-1}}\right) - \gamma\left(\frac{m}{3 \times 4^{n-1}}\right)\right| = \ell(\gamma_n)$$

and thus $\ell(\gamma) = +\infty$: the path $\gamma$ is not rectifiable.

# References

Koch, Helge von. 1904. "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire." *Arkiv för Matematik, Astronomi och Fysik* 1. Kungliga Svenska Vetenskapsakademien.: 681–702.