Cauchy's Integral Theorem – Local Version

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Contents

Introduction

We derive in this document a first version of Cauchy's integral theorem:

Theorem – Cauchy's Integral Theorem (Local Version). Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. For any $a \in \Omega$, there is a radius $r > 0$ such that the open disk $D(a, r)$ is included in Ω and for any rectifiable closed path γ of $D(a, r)$,

$$\int_\gamma f(z)\, dz = 0.$$

We will actually state and prove a slightly stronger version – one that does not require the restriction to small disks if Ω is *star-shaped*.

In a subsequent document, we will prove an even more general result, the global version of Cauchy's integral theorem. It will be applicable if Ω is merely *simply connected* (that is "without holes").

Integral Lemma for Polylines

Lemma – Integral Lemma for Triangles. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If Δ is a triangle with vertices a, b and c which is included in Ω

$$\Delta = \{\lambda a + \mu b + \nu c \mid \lambda \geq 0,\, \mu \geq 0,\, \nu \geq 0 \ \text{ and } \ \lambda + \mu + \nu = 1\} \subset \Omega$$

and if $\gamma = [a \to b \to c \to a]$ is an oriented boundary of Δ then

$$\int_\gamma f(z)\, dz = 0.$$

Proof. Let $a_0 = a$, $b_0 = b$, $c_0 = c$; consider the midpoints of the triangle edges:

$$d_0 = \frac{b_0 + c_0}{2},\ e_0 = \frac{a_0 + c_0}{2},\ f_0 = \frac{a_0 + b_0}{2}.$$

The sum of the integrals of f along the four paths $[a_0 \to f_0 \to e_0 \to a_0]$, $[f_0 \to b_0 \to d_0 \to f_0]$, $[e_0 \to d_0 \to c_0 \to e_0]$, $[d_0 \to e_0 \to f_0 \to d_0]$ is equal to the integral of f along γ. By the triangular inequality, there is at least one path in this set, that we denote γ_1, such that

$$\left| \int_{\gamma_1} f(z)\, dz \right| \geq \frac{1}{4} \left| \int_\gamma f(z)\, dz \right|.$$

We can iterate this process and come up with a sequence of paths γ_n such that

$$\left| \int_{\gamma_n} f(z)\, dz \right| \geq \frac{1}{4^n} \left| \int_\gamma f(z)\, dz \right|.$$

Denote Δ_n the triangles associated to the γ_n; they form a sequence of non-empty and nested compact sets. By Cantor's intersection theorem, there is a point w such that $w \in \Delta_n$ for every natural number n. The differentiability of f at w provides a complex-valued function ϵ_w, defined in a neighbourhood of 0, such that $\lim_{h\to 0} \epsilon_w(h) = \epsilon_w(0) = 0$ and

$$f(z) = f(w) + f'(w)(z - w) + \epsilon_w(z - w)|z - w|$$

Consequently, for any $\epsilon > 0$ and for any number n large enough,

$$\left| \int_{\gamma_n} [f(z) - f(w) - f'(w)(z - w)]\, dz \right| \leq \epsilon \operatorname{diam} \Delta_n \times \ell(\gamma_n),$$

where the diameter of a subset A of the complex plane is defined as

$$\operatorname{diam} A = \sup \{|z - w| \mid z \in A,\, w \in A\}.$$

We have $\ell(\gamma_n) = \ell(\gamma)/2^n$ and $\operatorname{diam} \Delta_n = \operatorname{diam} \Delta_0 / 2^n$. Additionally,

$$\int_{\gamma_n} f(w)\, dz = \int_{\gamma_n} f'(w)(z - w)\, dz = 0$$

since the functions $z \in \mathbb{C} \mapsto f(w)$ and $z \in \mathbb{C} \mapsto f'(w)(z-w)$ have primitives. Consequently, for any $\epsilon > 0$, for n large enough,

$$\frac{1}{4^n}\left|\int_\gamma f(z)\,dz\right| \leq \left|\int_{\gamma_n} f(z)\,dz\right| \leq \frac{1}{4^n}\epsilon \operatorname{diam} \Delta_0 \times \ell(\gamma),$$

which is only possible if the integral of f along γ is zero. ■

Definition – Star-Shaped Set. A subset A of the complex plane is *star-shaped* if it contains at least one point c – a *(star-)center*, the set of which is called the *kernel* of A – such that for any z in A, the segment $[c, z]$ is included in A.

Lemma – Integral Lemma for Polylines. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function where Ω is an open star-shaped subset of $\mathbb{C}$. For any closed path $\gamma = [a_0 \to \cdots \to a_{n-1} \to a_0]$ of Ω,

$$\int_\gamma f(z)\,dz = 0.$$

Proof. Let c be a star-center of Ω and define $a_n = a_0$; for any $k \in \{0, \dots, n-1\}$, the triangle with vertices c, a_k and a_{k+1} is included in Ω. Hence, by the integral lemma for triangles, the integral along the path $\gamma_k = [c \to a_k \to a_{k+1} \to c]$ of f is zero. Now, as

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \int_{\gamma_k} f(z)\,dz,$$

the integral of f along γ is zero as well. ■

Approximations of Rectifiable Paths by Polylines

To extend the integral lemma beyond closed polylines, we prove that polylines provide appropriate approximations of rectifiable paths:

Lemma – Polyline Approximations of Rectifiable Paths. Let γ be a rectifiable path. For any $\epsilon_\ell > 0$ and $\epsilon_\infty > 0$, there is an oriented polyline μ, with the same endpoints as γ, such that

$$\ell(\mu - \gamma) \leq \epsilon_\ell \text{ and } \forall\, t \in [0,1],\ |(\mu - \gamma)(t)| \leq \epsilon_\infty.$$

Proof – Polyline Approximations of Rectifiable Paths. Suppose that the path γ is continuously differentiable. Let $(t_0, \dots, t_n)$ be a partition of the interval $[0,1]$ and let μ be the associated polyline:

$$\mu = [\gamma(t_0) \to \gamma(t_1)] \,|_{t_1} \cdots |_{t_{n-1}} [\gamma(t_{n-1}) \to \gamma(t_n)]$$

The path γ and μ have the same endpoints. The path γ may be considered as the concatenation $\gamma = \gamma_1 \,|_{t_1} \dots |_{t_{n-1}} \gamma_n$ with the paths γ_k defined by

$$\forall\, k \in \{1, \dots, n\},\ \forall t \in [0,1],\ \gamma_k(t) = \gamma\left(t_{k-1} + t(t_k - t_{k-1})\right),$$

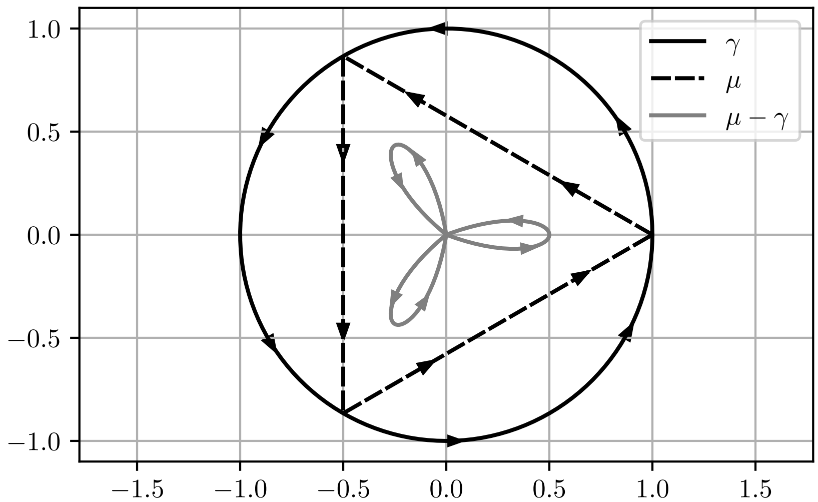

Figure 1: A 3-line approximation of the oriented unit circle.

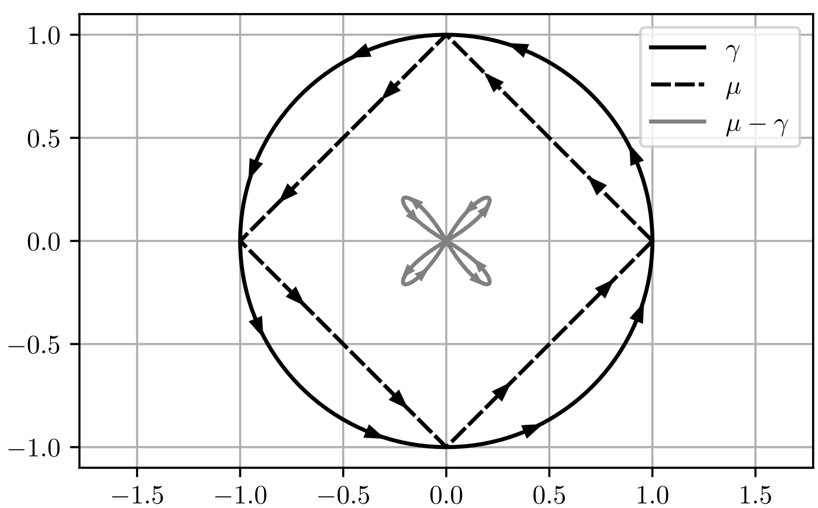

Figure 2: A 4-line approximation of the oriented unit circle.

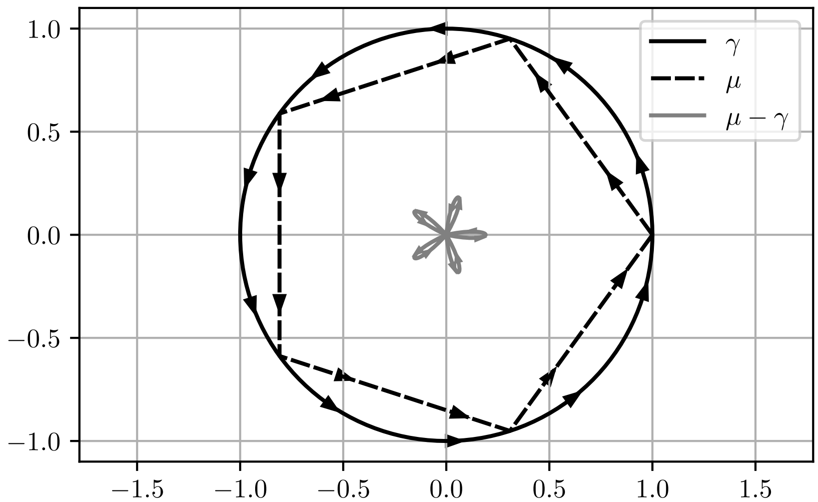

Figure 3: A 5-line approximation of the oriented unit circle.

hence we have

$$\ell(\mu - \gamma) = \sum_{k=1}^{n} \int_0^1 |\gamma(t_k) - \gamma(t_{k-1}) - \gamma_k'(t)| \, dt.$$

As

$$\gamma(t_k) - \gamma(t_{k-1}) = \int_{t_{k-1}}^{t_k} \gamma'(s) \, ds$$

and

$$\begin{aligned} \gamma_k'(t) &= (t_k - t_{k-1}) \gamma' \left(t_{k-1} + t(t_k - t_{k-1})\right) \\ &= \int_{t_{k-1}}^{t_k} \gamma' \left(t_{k-1} + t(t_k - t_{k-1})\right) \, ds, \end{aligned}$$

we have the inequality

$$\ell(\mu - \gamma) \leq \int_0^1 \left[\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(s) - \gamma'(t_{k-1} + t(t_k - t_{k-1}))| \, ds \right] dt$$

The function γ' is by assumption continuous, and hence uniformly continuous, on $[0,1]$, therefore for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that, $|\gamma'(s) - \gamma'(t)| < \epsilon$ whenever $|s - t| < \delta(\epsilon)$. For any $\epsilon_\ell > 0$, for any partition $(t_0, \dots, t_n)$ such that $|t_k - t_{k-1}| < \delta(\epsilon_\ell)$ for any $k \in \{1, \dots, n\}$, we have

$$\ell(\mu - \gamma) \leq \int_0^1 \left[\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \epsilon_\ell \, ds \right] dt = \epsilon_\ell.$$

For any $\epsilon_\infty > 0$, as

$$\forall\, t \in [0,1],\ |\mu(t) - \gamma(t)| \leq |\mu(0) - \gamma(0)| + \ell(\mu - \gamma) = \ell(\mu - \gamma),$$

any partition $(t_0, \dots, t_n)$ such that $|t_k - t_{k-1}| < \delta(\epsilon_\infty)$ ensures that

$$\forall\ t \in [0,1],\ |(\mu - \gamma)(t)| \leq \epsilon_\infty.$$

If γ is merely rectifiable, the same approximation process, applied to each of its continuously differentiable components provides the result. ■

Cauchy's Integral Theorem

We finally get rid of the polyline assumption:

Theorem – Cauchy's Integral Theorem (Star-Shaped Version). Let $f : \Omega \to \mathbb{C}$ be a holomorphic function where Ω is an open star-shaped subset of $\mathbb{C}$. For any rectifiable closed path γ of Ω,

$$\int_\gamma f(z)\, dz = 0.$$

Proof. Let $\epsilon > 0$. Let $r > 0$ be smaller than the distance between $\gamma([0,1])$ and $\mathbb{C} \setminus \Omega$. The set

$$K = \{z \in \mathbb{C} \,|\, d(z, \gamma([0,1])) \leq r\},$$

is compact and included in Ω. Consequently, the restriction of f to K is bounded and uniformly continuous: there is a $M > 0$ such that

$$\forall\, z \in K,\ |f(z)| \leq M,$$

and a there is a $\eta_\epsilon > 0$ – smaller than or equal to r – such that

$$\forall\, z \in K, \forall\, w \in \gamma([0,1]),\ |z - w| \leq \eta_\epsilon \ \Rightarrow\ |f(z) - f(w)| \leq \frac{\epsilon}{2(\ell(\gamma) + 1)}.$$

Now, let γ_ϵ be a closed polyline approximation of γ such that

$$\ell(\gamma_\epsilon - \gamma) \leq \frac{\epsilon}{2M} \ \text{ and } \ \forall\ t \in [0,1],\ |(\gamma_\epsilon - \gamma)(t)| \leq \eta_\epsilon.$$

By construction, γ_ϵ belongs to K, hence it is a closed path of Ω. Therefore, the integral lemma for polylines provides

$$\int_{\gamma_\epsilon} f(z)\, dz = 0.$$

The rectifiable γ and γ_ϵ have a decomposition into continuously differentiable paths associated to a common partition $(t_0, \dots, t_n)$ of the interval $[0,1]$:

$$\gamma = \gamma_1 \,|_{t_1}\, \cdots |_{t_n}\, \gamma_n \ \text{ and } \ \gamma_\epsilon = \gamma_{1\epsilon} \,|_{t_1}\, \cdots\, |_{t_n}\, \gamma_{n\epsilon}$$

The difference between the integral of f along γ and γ_ϵ satisfies

$$\left|\int_\gamma f(z)\,dz - \int_{\gamma_\epsilon} f(z)\,dz\right| = \left|\sum_{k=1}^n \int_0^1 [(f\circ\gamma_k)\gamma_k' - (f\circ\gamma_{\epsilon k})\gamma_{\epsilon k}'](t)\,dt\right|$$

Since for any $k \in \{1,\dots,n\}$

$$(f\circ\gamma_k)\gamma_k' - (f\circ\gamma_{\epsilon k})\gamma_{\epsilon k}' = (f\circ\gamma_k - f\circ\gamma_{\epsilon k})\gamma_k' + (f\circ\gamma_{\epsilon k})(\gamma_k' - \gamma_{\epsilon k}'),$$

we have

$$\begin{aligned}
&\left|\int_\gamma f(z)\,dz - \int_{\gamma_\epsilon} f(z)\,dz\right| \\
&\leq \left|\sum_{k=1}^n \int_0^1 [(f\circ\gamma_k - f\circ\gamma_{\epsilon k})\gamma_k'](t)\,dt\right| \\
&+ \left|\sum_{k=1}^n \int_0^1 [(f\circ\gamma_{\epsilon k})(\gamma_k' - \gamma_{\epsilon k}')](t)\,dt\right|
\end{aligned}$$

and thus

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &\leq \max_{t\in[0,1]} |f(\gamma(t)) - f(\gamma_\epsilon(t))| \times \ell(\gamma) \\
&\quad + \max_{t\in[0,1]} |f(\gamma_\epsilon(t))| \times \ell(\gamma_\epsilon - \gamma) \\
&\leq \frac{\epsilon}{2(\ell(\gamma)+1)} \times \ell(\gamma) + M \times \frac{\epsilon}{2M} \\
&\leq \epsilon.
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, the integral of f along γ is zero. ■

Consequences

Theorem – Cauchy's Integral Formula for Disks. Let Ω be an open subset of the complex plane and $\gamma = c + r[\circlearrowleft]$ be an oriented circle such that the closed disk $\overline{D}(c,r)$ is included in Ω. For any holomorphic function $f : \Omega \to \mathbb{C}$,

$$\forall\, z \in D(c,r),\; f(z) = \frac{1}{i2\pi}\int_\gamma \frac{f(w)}{w-z}dw.$$

Proof. Refer to the answers of exercise "Cauchy's Integral Formula for Disks" ■

Corollary – Derivatives are Complex-Differentiable. The derivative of any holomorphic function is holomorphic.

Proof. Refer to the answers of exercise "Cauchy's Integral Formula for Disks" ■

Theorem – **Morera's Theorem.** Let Ω be an open subset of $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if it is continuous and locally, its line integrals along rectifiable closed paths are zero: for any $c \in \Omega$, there is a $r > 0$ such that $D(c, r) \subset \Omega$ and for any rectifiable closed path γ of $D(c, r)$,

$$\int_\gamma f(z)\, dz = 0.$$

Proof. If f is holomorphic, then it is continuous and by Cauchy's integral theorem, its line integrals along rectifiable closed paths are locally zero. Conversely, if f is continuous and all its line integrals along closed rectifiable paths are zero in some non-empty open disk $D(c, r)$ of Ω, f satisfies the condition for the existence of primitives in $D(c, r)$. Any such primitive is holomorphic; since derivatives are complex-differentiable its derivative is holomorphic too and f is holomorphic in some neighbourhood of c. Since the initial assumption holds for any $c \in \Omega$, we can conclude that f is holomorphic on Ω. ■

Theorem – **Limit of Holomorphic Functions.** Let Ω be an open subset of $\mathbb{C}$. If a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ converges locally uniformly to a function $f : \Omega \to \mathbb{C}$, that is if for any $c \in \Omega$, there is a $r > 0$ such that $D(c, r) \subset \Omega$ and

$$\lim_{n \to +\infty} \sup_{z \in D(c,r)} |f_n(z) - f(z)| = 0,$$

then f is holomorphic.

Proof. The function f is continuous as a locally uniform limit of continuous functions. Now, let $c \in \Omega$ and let $r > 0$ be such that $D(c, r) \subset \Omega$ and the functions f_n converge uniformly to f in $D(c, r)$. By Cauchy's integral theorem, for any rectifiable closed path γ of $D(c, r)$, the integral of f_n along γ is zero. Thus

$$\int_\gamma f(z)\, dz = \lim_{n \to +\infty} \int_\gamma f_n(z)\, dz = 0.$$

By Morera's theorem, f is holomorphic. ■

Theorem – **Liouville's Theorem.** Any holomorphic function defined on $\mathbb{C}$ (any *entire* function) which is bounded is constant.

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq \kappa$ for any $z \in \mathbb{C}$. Since derivatives are complex-differentiable, we may apply Cauchy's integral formula for disks to the function f' and to the oriented circle $\gamma = z + r[\circlearrowleft]$ for $r > 0$ and $z \in \mathbb{C}$. We have

$$f'(z) = \frac{1}{i2\pi} \int_\gamma \frac{f'(w)}{w - z} dw$$

and by integration by parts,

$$f'(z) = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w-z)^2} dw,$$

which yields by the M-L inequality

$$|f'(z)| \leq \frac{\kappa}{r}.$$

This inequality holds for any $r > 0$, thus $f'(z) = 0$. Consequently, the zero function and f are both primitives of f'; since the domain of f' is connected, these two primitives differ by a constant and thus f is constant. ■