Power Series

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Exercises

The Fibonacci Sequence

We search for a closed form of the Fibonacci sequence a_n, defined by

$$a_0 = 0,\ a_1 = 1,\ \forall\, n \in \mathbb{N},\, a_{n+2} = a_n + a_{n+1}.$$

Questions

1. Show that the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the largest solution of the equation $x^2 = x+1$ and that the other solution is $\psi = -1/\phi$.

2. Establish that for any $n \in \mathbb{N}$, $a_n \leq \phi^n$.
3. Show that the radius of convergence of the generating function

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is at least $1/\phi$.

4. Compute $f(z)$ when $|z| < 1/\phi$.
5. Find a closed form for a_n, $n \in \mathbb{N}$.

Answers

1. The discriminant Δ of the quadratic equation $x^2 - x - 1 = 0$ is

$$\Delta = (-1)^2 - 4 \times 1 \times (-1) = 5,$$

therefore the solutions are

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

The golden ratio ϕ, equal to $(1+\sqrt{5})/2$, is the largest of the two. The fact that the other root ψ of the equation is equal to $-1/\phi$ can be demonstrated directly; we have indeed

$$\psi = \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{1+\sqrt{5}}\frac{1-\sqrt{5}}{2} = \frac{1^2 - \sqrt{5}^2}{2(1+\sqrt{5})} = -\frac{2}{1+\sqrt{5}}.$$

Alternatively, we know that

$$x^2 - x - 1 = (x-\phi)(x-\psi) = x^2 - (\phi+\psi)x + \phi\psi,$$

hence $\phi\psi = -1$.

2. It is clear that $a_0 = 0 \leq 1 = \phi^0$ and $a_1 = 1 \leq \phi = \phi^1$. If we assume that the inequality $a_n \leq \phi^n$ holds for $n = 0, 1, \ldots, m+1$, the recursive definition of the Fibonacci sequence yields

$$a_{m+2} = a_m + a_{m+1} \leq \phi^m + \phi^{m+1} = \phi^m(1+\phi) = \phi^{m+2}.$$

Hence, by induction, the inequality holds for every $n \in \mathbb{N}$.

3. The inequality $a_n \leq \phi^n$ provides
$$\limsup_{n \to +\infty} \sqrt[n]{|a_n|} \leq \phi,$$
and hence, by the Cauchy-Hadamard formula, the radius of convergence of the series $\sum_{n \geq 0} a_n z^n$ is at least $1/\phi$.

4. If $|z| < 1/\phi$, we can write the expansion of $f(z)$ as
$$f(z) = \sum_{n=0}^{+\infty} a_n z^n = a_0 + a_1 z + \sum_{n=0}^{+\infty} a_{n+2} z^{n+2} = z + \sum_{n=0}^{+\infty} a_{n+2} z^{n+2}.$$
Using $a_{n+2} = a_n + a_{n+1}$, we deduce that
$$f(z) = z + z^2 \sum_{n=0}^{+\infty} a_n z^n + z \sum_{n=0}^{+\infty} a_{n+1} z^{n+1} = z + z^2 f(z) + z f(z),$$
hence
$$f(z) = \frac{z}{1 - z - z^2}.$$

5. The roots of the polynomial $1 - z - z^2$ are $-\phi$ and $-\psi$, hence
$$-z^2 - z + 1 = -(z + \phi)(z + \psi).$$
Thus, for any $|z| < 1/\phi$, we have
$$f(z) = \frac{-z}{(z + \phi)(z + \psi)} = \frac{1}{\phi - \psi} \left[\frac{-\phi}{z + \phi} + \frac{\psi}{z + \psi} \right],$$
or equivalently, using $\psi = -1/\phi$,
$$f(z) = \frac{1}{\phi - \psi} \left[\frac{-1}{1 - \psi z} + \frac{1}{1 - \phi z} \right].$$
If $|z| < 1/\phi$, then $|\phi z| < 1$ and $|\psi z| < 1$ and consequently
$$\frac{1}{1 - \phi z} = \sum_{n=0}^{+\infty} \phi^n z^n, \quad \frac{1}{1 - \psi z} = \sum_{n=0}^{+\infty} \psi^n z^n.$$
Thus, $f(z)$ can be expanded as
$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{\phi - \psi} \left[\phi^n - \psi^n \right] z^n.$$
The power series expansion of $f(z)$ in the disk centered on the origin with radius $1/\phi$ is unique, therefore
$$a_n = \frac{1}{\phi - \psi} \left[\phi^n - \psi^n \right] = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$
for every $n \in \mathbb{N}$.

Entire Functions Dominated By Polynomials

Question

Show that if a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is dominated by a polynomial P of order p

$$\forall\, z \in \mathbb{C},\ |f(z)| \leq |P(z)|$$

then it is a polynomial whose degree is at most p.

Answer

Let $\sum_{n=0}^{+\infty} a_n z^n$ be the power series expansion of f in $\mathbb{C}$. For any $r > 0$, we have

$$a_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz,$$

hence by the M-L estimation lemma,

$$|a_n| \leq \frac{\sup \{|P(re^{i2\pi t})| \mid t \in [0,1]\}}{r^n}.$$

For any $n > p$, letting $r \to +\infty$ provides $a_n = 0$. Hence, the function f is a polynomial of degree at most p.

Existence of Primitives

Question

Show that the function

$$f : z \in \mathbb{C} \setminus [-1, 1] \mapsto \frac{\pi}{z} \frac{1}{\sin \pi/z}$$

has a primitive.

Answer

The function f is defined and holomorphic in $\mathbb{C} \setminus [-1, 1]$ (the zeros of $\sin \pi/z$ are $z = 1/k$ for $k \in \mathbb{N}^*$).

We first consider the restriction of f to the annulus $A(0, 1, +\infty)$. For any z in this annulus, $-z$ also belong to it and $f(-z) = f(z)$. Hence, if $\sum_{n=-\infty}^{+\infty} a_n z^n$ is a Laurent series expansion of f, $\sum_{n=-\infty}^{+\infty} (-1)^n a_n z^n$ is another valid one. The uniqueness of the expansion yields that $a_n = 0$ if n is odd; in particular, $a_{-1} = 0$ and the sum

$$\sum_{p=-\infty}^{+\infty} \frac{a_{2p}}{2p+1} z^{2p+1}$$

provide a primitive of f on the annulus.

Now, let γ be an arbitrary closed rectifiable path of $\mathbb{C} \setminus [-1,1]$. Let $n = \mathrm{ind}(\gamma, 0)$; define the path $\mu : t \in [0,1] \mapsto 2e^{i2\pi nt}$ and the sequence of paths $\nu = (\gamma, \mu^{\leftarrow})$. As $[-1,1]$ is a connected subset of $\mathbb{C} \setminus \nu([0,1])$, for any $z \in [-1,1]$, $\mathrm{ind}(\nu, z) = \mathrm{ind}(\nu, 0) = 0$. Consequently, $\mathrm{Int}\, \nu \subset \mathbb{C} \setminus [-1,1]$ and Cauchy's integral theorem provides

$$\int_\gamma f(z)\, dz = \int_\mu f(z)\, dz.$$

As f has a primitive on the annulus $A(0,1,+\infty)$, the integral in the right-hand side of this equation is equal to zero. The classic criteria therefore proves that primitives of f exist in $\mathbb{C} \setminus [-1,1]$.

A Removable Set

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function which is holomorphic on $\mathbb{C} \setminus \mathbb{U}$ (where $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$).

Question

Prove that f is an entire function.

Answer

Let $\sum_{n=0}^{+\infty} a_n z^n$ be the Taylor series expansion of f in $D(0,1)$; we are going to prove that this expansion is actually a valid expansion of f in $\mathbb{C}$. Consider the Laurent expansion $\sum_{n=-\infty}^{+\infty} b_n z^n$ of f in $A(0,1,+\infty)$. For any $n \in \mathbb{Z}$ and any $r > 1$, we have

$$b_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz,$$

thus, by continuity of f

$$\begin{aligned} b_n &= \lim_{r \to 1^+} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz \\ &= \lim_{r \to 1^-} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz \end{aligned}$$

and consequently, $b_n = a_n$ if n is non-negative and zero otherwise. The sum $\sum_{n=0}^{+\infty} a_n z^n$ is defined for any $|z| > 1$, thus its open disk of convergence is the full complex plane. It is equal to f on $\mathbb{C} \setminus \mathbb{U}$ and both functions are continuous on $\mathbb{C}$, hence they are equal on $\mathbb{C}$: the function f is entire.

Derivative of Power Series

Question

Provide an alternate proof of the existence and value of the derivative of the sum $\sum_{n=0}^{+\infty} a_n (z-c)^n$ in its open disk of convergence.

Hint: a locally uniform limit of a sequence of holomorphic functions is holomorphic.

Answer

Let $f_m(z) = \sum_{n=0}^{m} a_n (z-c)^n$. Every polynomial f_m is holomorphic and the sequence converges locally uniformly to $f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n$ in the open disk of convergence $D(c,r)$ of the series, thus f is holomorphic.

For any holomorphic function ϕ in $D(c,r)$ and any $\rho \in \,]0, r[$

$$\phi'(z) = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{\phi(w)}{(w-z)^2}.$$

Thus, for any $m \in \mathbb{N}$,

$$f_m'(z) = \sum_{n=1}^{m} n a_n (z-c)^{n-1} = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f_m(w)}{(w-z)^2}.$$

The integrand above converges locally uniformly in $D(c,r)$, hence

$$\lim_{m \to +\infty} \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f_m(w)}{(w-z)^2} = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f(w)}{(w-z)^2} = f'(z).$$

Finally,

$$\sum_{n=1}^{+\infty} n a_n (z-c)^{n-1} = \lim_{m \to +\infty} \sum_{n=1}^{m} n a_n (z-c)^{n-1} = f'(z).$$