Cauchy's Integral Theorem – Global Version

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Path Sequences

It is convenient to state Cauchy's integral theorem for finite sequences of paths instead of paths. To this end, we generalize some of the concepts initially defined for paths.

Definition – Opposite & Concatenation. The opposite of the path sequence $\gamma = (\gamma_1, \dots, \gamma_n)$ is the path sequence

$$\gamma^{\leftarrow} = (\gamma_n^{\leftarrow}, \dots, \gamma_1^{\leftarrow}).$$

The concatenation of the path sequences

$$\alpha = (\alpha_1, \dots, \alpha_k) \; \text{ and } \; \beta = (\beta_1, \dots, \beta_l)$$

is the path sequence

$$\alpha \,|\, \beta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l).$$

Definition – Image. The image of the path sequence $\gamma = (\gamma_1, \dots, \gamma_n)$ is the set

$$\gamma([0,1]) = \bigcup_{k=1}^{n} \gamma_k([0,1]).$$

Definition – Winding Number, Exterior, Interior. If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a sequence of closed paths and $a \in \mathbb{C}$ is not on its image, the winding number of γ around a is defined by

$$\operatorname{ind}(\gamma, a) = \sum_{k=1}^{n} \operatorname{ind}(\gamma_k, a).$$

The exterior of γ is the set

$$\operatorname{Ext} \gamma = \{z \in \mathbb{C} \setminus \gamma([0,1]) \mid \operatorname{ind}(\gamma, z) = 0\}$$

and its interior is the set

$$\operatorname{Int} \gamma = \{z \in \mathbb{C} \setminus \gamma([0,1]) \mid \operatorname{ind}(\gamma, z) \neq 0\}$$

or equivalently

$$\operatorname{Int} \gamma = \mathbb{C} \setminus (\gamma([0,1]) \cup \operatorname{Ext} \gamma).$$

Definition – Length & Line Integral. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a sequence of rectifiable paths. The length of γ is defined as

$$\ell(\gamma) = \sum_{k=1}^{n} \ell(\gamma_k).$$

The integral along γ of a complex-valued function f which is defined and continuous on the image of γ is

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\, dz.$$

Cauchy's Theorem & Corollaries

For the global version of Cauchy's integral theorem, the star-shaped assumption is replaced by a weaker geometric requirement:

Theorem – Cauchy's Integral Theorem (Global Version). Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let γ be a sequence of rectifiable closed paths of Ω. If $\operatorname{Int} \gamma \subset \Omega$ then

$$\int_\gamma f(z)\, dz = 0.$$

Remark – One or Two Paths. This version of Cauchy's integral theorem is clearly applicable for a single path γ instead of a path sequence. Now, the next

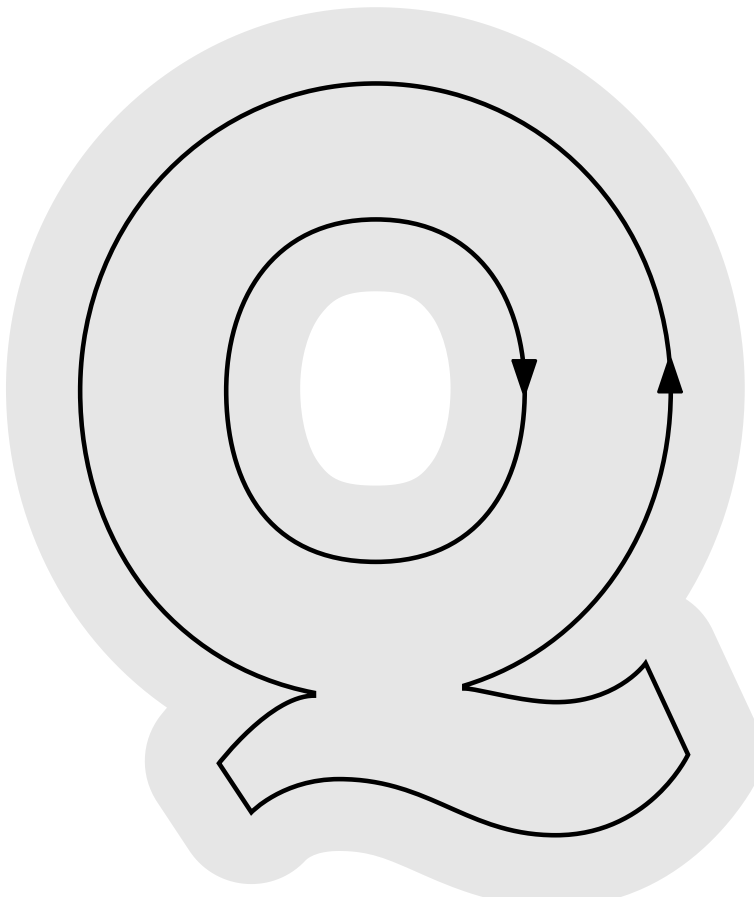

Figure 1: A pair γ of two rectifiable closed paths in an open set Ω represented in light grey. Both paths are concatenations of quadratic Bézier curves.

Figure 2: This path sequence forms the outline of the capital “Q” letter in the League Spartan typeface. The interior of the path sequence is represented in dark grey; as the two paths are oriented in opposite directions, the interior of the path sequence is included in Ω. The interior of the inner path does *not* belong to the interior of the path sequence; a typographist would say that it is *a closed counter* of the letter.

common use case involves two rectifiable closed paths γ and μ of Ω. If they have the same winding number with respect to any point which is not in Ω:

$$\forall\, z \in \mathbb{C} \setminus \Omega,\ \operatorname{ind}(\gamma, z) = \operatorname{ind}(\mu, z),$$

then the interior of the path sequence $(\gamma, \mu^{\leftarrow})$ is included in Ω and Cauchy's integral theorem is applicable. Its conclusion provides

$$\int_\gamma f(z)\, dz = \int_\mu f(z)\, dz.$$

Remark – Simply Connected Sets. If Ω is simply connected, Cauchy's theorem is applicable for any sequence of rectifiable closed paths γ of Ω. Indeed in any such set Ω, for any path γ – and thus for any sequence of paths γ – we have $\operatorname{Int} \gamma \subset \Omega$. Since any star-shaped set is simply connected, the local version of Cauchy's theorem is a special case of the global version.

Cauchy's residue theorem is a generalization of his integral theorem. It covers the case where the interior of the path sequence γ is included in the domain Ω of the holomorphic function f, except for a set of *isolated singularities*. The integral of f along γ in this case can be computed in terms of the *residues* of the function at these singularities.

Definition – Singularity. Let Ω be an open subset of $\mathbb{C}$. A *singularity* of a function $f : \Omega \to \mathbb{C}$ is a point a of $\mathbb{C} \setminus \Omega$. It is *isolated* if its distance to the other singularities of f is positive:

$$\exists\, \epsilon > 0,\ \forall\, z \in \mathbb{C},\ (|z - a| < \epsilon \text{ and } z \neq a\,) \Rightarrow\ z \in \Omega.$$

Definition – Residue. Let a be an isolated singularity of the holomorphic function $f : \Omega \to \mathbb{C}$. Let d be the distance between a and the other singularities of f ($+\infty$ if a is the only singularity of f). The integral of f along $\gamma = a + r[\circlearrowleft]$ is defined and independent of r as long as $0 < r < d$. We define the *residue* of f at a as

$$\operatorname{res}(f, a) = \frac{1}{i2\pi} \int_\gamma f(z)\, dz$$

for any such r.

Examples – Singularity & Residue. Let $a \in \mathbb{C}$ and

$$f : z \in \mathbb{C} \setminus \{a\} \mapsto \frac{1}{z - a}.$$

The point a is the only singularity of f; it is clearly isolated. For any $r > 0$ and $\gamma = a + r[\circlearrowleft]$ we have

$$\int_\gamma f(z)\, dz = \int_\gamma \frac{dz}{z - a} = i2\pi \times \operatorname{ind}(\gamma, a) = i2\pi,$$

thus $\operatorname{res}(f, a) = 1$. Now, let $a \in \mathbb{C}$ and let

$$f : z \in \mathbb{C} \setminus \{a\} \mapsto (z-a)^n \text{ where } n \in \mathbb{Z} \setminus \{-1\}.$$

Since $z \in \mathbb{C} \setminus \{a\} \mapsto (z-a)^{n+1}/(n+1)$ is a primitive of f,

$$\int_\gamma f(z)\, dz = \int_\gamma (z-a)^n \, dz = 0,$$

thus $\operatorname{res}(f, a) = 0$.

Proof – Residue: independence with respect to the radius. Let r_1 and r_2 be two real numbers in $]0, d[$ and let $\gamma_1 = z + r_1[\circlearrowleft]$ and $\gamma_2 = z + r_2[\circlearrowleft]$. If $z \in \mathbb{C} \setminus \Omega$, either $z = a$ and

$$\operatorname{ind}(\gamma_1, z) = \operatorname{ind}(\gamma_2, z) = 1,$$

or $|z - a| \geq d$ and

$$\operatorname{ind}(\gamma_1, z) = \operatorname{ind}(\gamma_2, z) = 0.$$

In any case the winding numbers are equal. The "One or Two Paths" remark therefore provides

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

which concludes the proof. ■

Theorem – Cauchy's Residue Theorem. Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let γ be a sequence of rectifiable closed paths of Ω. If A is a finite set of isolated singularities of f such that

$$\operatorname{Int} \gamma \subset \Omega \cup A$$

then

$$\int_\gamma f(z)\, dz = i2\pi \sum_{a \in A} \operatorname{ind}(\gamma, a) \times \operatorname{res}(f, a).$$

Remark – Infinite Set of Singularities. Note that if we drop the assumption that A is finite, the conclusion of the theorem still holds since only a finite number of singularities of A may be in the interior of γ([1]); the sum in the right-hand side of the theorem equation may then have an infinite number of terms, but only a finite number of them are non-zero.

Proof – Cauchy's Residue Theorem. We may assume that the set A is included in $\operatorname{Int} \gamma$. If this assumption is not satisfied, replace A with $A \cap \operatorname{Int} \gamma$;

[1] Indeed, assume instead that there is a infinite sequence of distincts points of A in $\operatorname{Int} \gamma$; by compactness a subsequence of it converges to some point a in its closure. The singularities of f are a closed set, thus a is itself a singularity. Since the boundary of $\operatorname{Int} \gamma$ is included in the image of γ and hence in Ω, the point a actually belongs to $\operatorname{Int} \gamma$. Now $\operatorname{Int} \gamma \subset \Omega \cup A$, therefore $a \in A$, but by construction it is not isolated, which is a contradiction.

this new set A still satisfies $\operatorname{Int} \gamma \subset \Omega \cup A$ and the conclusion of the theorem for the new set does provide the result for the original set.

Let $\epsilon > 0$ be such that for any $a \in A$, $D(a, \epsilon) \subset \Omega \cup \{a\}$ and let $0 < r < \epsilon$. Define for every a in A the path γ_a by

$$\gamma_a(t) = a + r[\circlearrowleft]^{-\operatorname{ind}(\gamma,a)}.$$

We clearly have $\operatorname{ind}(\gamma_a, a) = -\operatorname{ind}(\gamma, a)$.

Let λ be the concatenation of γ and the sequence of all γ_a for $a \in A$. We now prove that $\operatorname{Int} \lambda \subset \Omega$; we need to establish that $\operatorname{ind}(\lambda, z) = 0$ for every $z \in \mathbb{C} \setminus \Omega$. For such a point z, either

1. $z \in A$.
   In this case, $\operatorname{ind}(\gamma_z, a) = 0$ for any other singularity $a \in A$. Therefore,
   $$\operatorname{ind}(\lambda, z) = \operatorname{ind}(\gamma, z) + \operatorname{ind}(\gamma_z, z) = 0.$$
2. $z \notin \Omega \cup A$.
   We have $\operatorname{ind}(\gamma_a, z) = 0$ for any $a \in A$. Additionally, as $\operatorname{Int} \gamma \subset \Omega \cup A$, $\operatorname{ind}(\gamma, z) = 0$. Finally, $\operatorname{ind}(\lambda, z) = 0$.

Cauchy's integral theorem then provides

$$\int_\lambda f(z) dz = \int_\gamma f(z)\, dz + \sum_{a \in A} \int_{\gamma_a} f(z)\, dz = 0,$$

By construction of the γ_a and the definition of residues, we have

$$\int_{\gamma_a} f(z)\, dz = -\operatorname{ind}(\gamma, a) \times i2\pi \operatorname{res}(f, a).$$

■

There is a third equivalent form of Cauchy's integral theorem: Cauchy's integral formula[2]. It gives the value of f at any point of the interior of γ as a function of its values on the image of γ.

Theorem – Cauchy's Integral Formula. Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let γ be a sequence of rectifiable closed paths of Ω and $a \in \Omega \setminus \gamma([0, 1])$. If $\operatorname{Int} \gamma \subset \Omega$, then

$$\int_\gamma \frac{f(z)}{z - a}\, dz = i2\pi \times \operatorname{ind}(\gamma, a) \times f(a).$$

Proof. The function

$$g : z \in \Omega \setminus \{a\} \mapsto \frac{f(z)}{z - a}$$

[2]The integral theorem implies the residue theorem which in turn implies the integral formula. Finally, the proof of the integral theorem is straightforward if we assume that the integral formula holds.

is holomorphic. The point a is one of its isolated singularities. For $A = \{a\}$, we have

$$\operatorname{Int} \gamma \subset (\Omega \setminus \{a\}) \cup A = \Omega.$$

Additionally, if $\mu = a + r[\circlearrowleft]$,

$$\operatorname{res}(g, a) = \lim_{r \to 0} \frac{1}{i2\pi} \int_\mu g(z)\, dz = \lim_{r \to 0} \int_0^1 f(a + re^{i2\pi t})\, dt = f(a).$$

Therefore, Cauchy's residue theorem provides

$$\int_\gamma \frac{f(z)}{z-a}\, dz = i2\pi \times \operatorname{ind}(\gamma, a) \times f(a)$$

which is Cauchy's integral formula. ■

The Proof

Definition – Path Sequence Decomposition. A *decomposition* of a sequence of rectifiable paths $\gamma = (\gamma_1, \dots, \gamma_n)$ is a sequence of rectifiable paths

$$\gamma^* = (\gamma_1^*, \dots, \gamma_{p_1}^*, \gamma_{p_1+1}^*, \dots, \gamma_{p_n}^*, \gamma_{p_n+1}^*, \dots, \gamma_p^*)$$

such that for a suitable set of partitions of the unity

$$\begin{array}{rcl} \gamma_1 & = & \gamma_1^* \,|_{t_1^1} \cdots |_{t_{p_1-1}^1} \, \gamma_{p_1}^* \\ \gamma_2 & = & \gamma_{p_1+1}^* \,|_{t_1^2} \cdots |_{t_{p_2-p_1-1}^2} \, \gamma_{p_2}^* \\ & \vdots & \\ \gamma_p & = & \gamma_{p_n+1}^* \,|_{t_1^n} \cdots |_{t_{p-p_n-1}^n} \, \gamma_p^* \end{array}$$

Definition – Equivalent Path Sequences. Let $n_\gamma(\mu)$ be the number of occurences of the path μ in the path sequence γ. Two sequences of rectifiable paths γ and λ are *equivalent* if they have decompositions γ^* and λ^* such that for any path μ

$$n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow}) = n_{\lambda^*}(\mu) - n_{\lambda^*}(\mu^{\leftarrow}).$$

Remark – Integral along Equivalent Paths. If the sequence of rectifiable paths γ has a decomposition into a sequence of rectifiable paths γ^*, then for every continuous and complex-valued function f defined on the image of γ,

$$\int_\gamma f(z)\, dz = \frac{1}{2} \sum_\mu (n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow})) \times \left(\int_\mu f(z)\, dz \right)$$

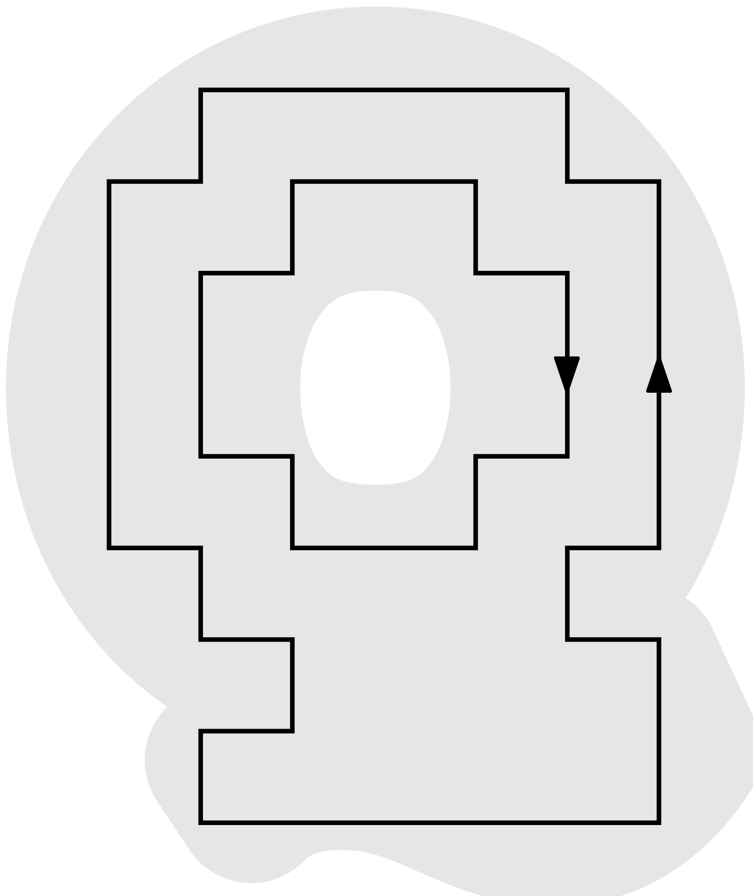

Figure 3: A path sequence made of arrows whose interior is included in Ω.

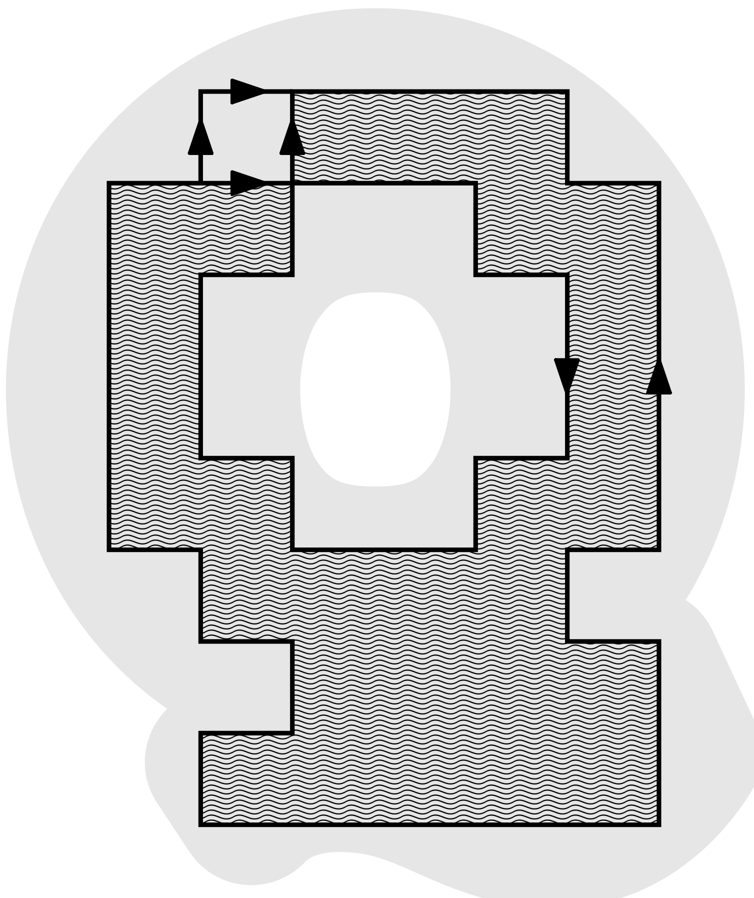

Figure 4: We can apply the local version of Cauchy's integral theorem "cell-by-cell" to such a path to prove the global version of the theorem.

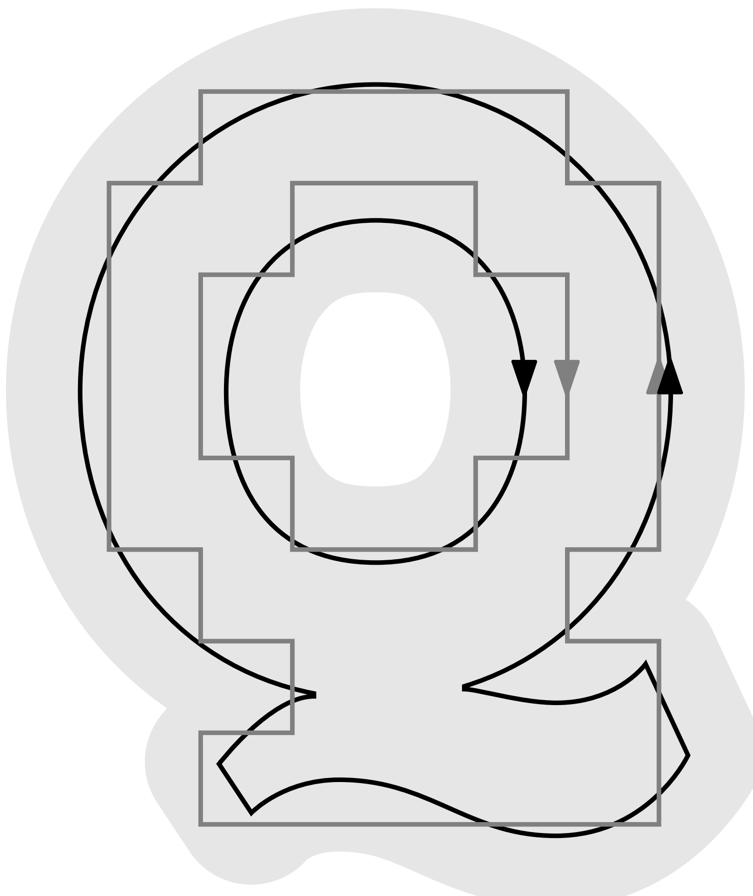

Figure 5: The path sequence made of arrows (in grey) is a suitable approximation of the original outline: the integral on both path sequence of holomorphic functions defined in Ω are equal.

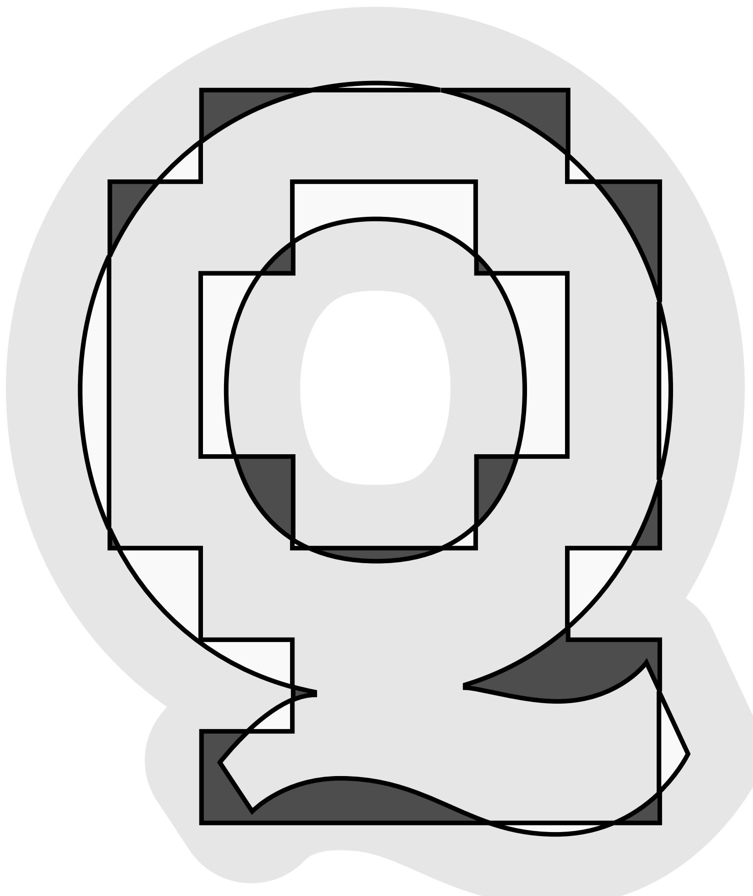

Figure 6: Indeed, the integral on the “difference” between the two paths sequence (the concatenation of the original and the opposite of the grid approximation) is an integral on a collection of “small” closed paths for which the local version of Cauchy’s integral theorem can be applied: all these integrals are equal to zero.

where the sum is taken over the paths μ such that μ or $\mu^{\leftarrow}$ has at least one occurrence in γ^*. Consequently, if γ and λ are equivalent and f is defined and continuous on the images of γ and λ,

$$\int_\gamma f(z)\,dz = \int_\lambda f(z)\,dz.$$

Definition – Path Diameter. The *diameter* of a path $\gamma : [0,1] \to \mathbb{C}$ is defined as the diameter of its image: it is the nonnegative real number

$$\operatorname{diam}(\gamma) = \operatorname{diam}(\gamma([0,1])) = \max\{|z-w| \mid z \in \gamma([0,1]),\, w \in \gamma([0,1])\}.$$

Theorem – Small Closed Paths Theorem. Let Ω be an open subset of $\mathbb{C}$ and γ be a sequence of rectifiable closed paths of Ω such that $\operatorname{Int}\gamma \subset \Omega$. There is a sequence of rectifiable closed paths μ of Ω of arbitrarily small diameter which is equivalent to γ.

This geometric result and the local Cauchy theory yield the global version of Cauchy's integral theorem:

Proof – Cauchy's Integral Theorem. Assume that Ω, f and γ satisfy the assumptions of Cauchy's integral theorem. Since the union of the image of γ and its interior is a compact set, there is a $\epsilon > 0$ such that the open set

$$\Omega' = \{z \in \Omega \mid d(z, \mathbb{C} \setminus \Omega) > \epsilon\}$$

contains the image of γ and its interior. By the small closed paths theorem, there is a sequence of rectifiable closed paths $\mu = (\mu_1, \dots, \mu_n)$ of Ω' of diameter less than ϵ which is equivalent to γ, and therefore such that

$$\int_\gamma f(z)\,dz = \int_\mu f(z)\,dz.$$

The image of every path μ_k of diameter less than ϵ is included in the disk centered on $\mu_k(0)$ and of radius ϵ. This disk belongs to Ω by construction and thus the local version of Cauchy's theorem is applicable. Finally,

$$\int_\mu f(z)\,dz = \sum_{k=1}^{n} \int_{\mu_k} f(z)\,dz = 0.$$

■

The proof of the small closed paths theorem itself requires several lemmas.

Definition – Arrow. An *arrow* is an oriented line segment

$$[(k+il)2^{-n} \to (k'+il')2^{-n}]$$

for some $n \in \mathbb{N}$ and k, l, k', $l' \in \mathbb{Z}$ such that

$$|k'-k| + |l'-l| = 1.$$

Lemma – Small Paths & Path of Arrows. Let Ω be an open subset of $\mathbb{C}$ and γ be a sequence of rectifiable closed paths of Ω. For any $\epsilon > 0$, there are two sequences λ_1 and λ_2 of rectifiable closed paths of Ω such that

1. the path sequences γ and $\lambda_1 \,|\, \lambda_2$ are equivalent.
2. the diameter of every path of λ_1 is smaller than ϵ.
3. λ_2 has a decomposition into arrows.

Proof. We prove the result for a rectifiable closed path γ; the result for path sequences is a simple corollary. Let n be a natural number and let $(\gamma_1, \dots, \gamma_m)$ be a sequence of rectifiable paths such that $\gamma = \gamma_1 \,|\, \cdots \,|\, \gamma_m$ and $\ell(\gamma_k) < 2^{-n}$ for any $k \in \{1, \dots, m\}$. Denote π_n the function defined on $\mathbb{C}$ by

$$\pi_n(z) = [\mathrm{Re}(2^n z)]2^{-n} + i[\mathrm{Im}(2^n z)]2^{-n};$$

where the function $[\,\cdot\,]$ rounds a real number to (one of) the nearest integer(s). For any $z \in \mathbb{C}$, $|\pi_n(z) - z| < 2^{-n}$. The points $\pi_n(\gamma_k(0))$ and $\pi_n(\gamma_k(1))$ are distant by less than 3×2^{-n} and thus may always be joined by a path $\lambda_{2,k}$ which is the concatenation of at most four consecutive arrows of length 2^{-n}.

Define the rectifiable closed path $\lambda_{1,k}$ as the concatenation:

$$\lambda_{1,k} = \gamma_k \,|\, [\gamma_k(1) \to \pi_n(\gamma_k(1))] \,|\, \lambda_{2,k}^{\leftarrow} \,|\, [\pi_n(\gamma_k(0)) \to \gamma_k(0)]$$

The length of the closed path $\lambda_{1,k}$ is smaller than 7×2^{-n}, hence its diameter is smaller than $7/2 \times 2^{-n}$. A suitable choice of n provides $\mathrm{diam}(\lambda_{1,k}) < \epsilon$. The paths $\lambda_1 = (\lambda_{1,1}, \dots, \lambda_{1,m})$ and $\lambda_2 = (\lambda_{2,1}, \dots, \lambda_{2,m})$ satisfy the statement of the lemma. ■

Lemma – Small Closed Paths Theorem (Arrow Version). Let Ω be an open subset of $\mathbb{C}$ and let γ be a sequence of rectifiable closed paths of Ω *with a decomposition into arrows* and such that $\mathrm{Int}\,\gamma \subset \Omega$. There is a sequence of rectifiable closed paths μ of Ω of arbitrarily small diameter which is equivalent to γ.

Proof – Small Closed Paths Theorem (Arrow Version). Any sequence γ of rectifiable closed paths made of arrows may be decomposed further into a sequence γ^* of arrows of the same length 2^{-n} for an arbitrary large n. Now, we may associate to this level of decomposition a family indexed by integers k and l of square cells

$$C_{k,l} = \{(k + il + s + it)2^{-n} \mid (s,t) \in [0,1]^2\}.$$

with centers

$$c_{k,l} = k + 0.5 + i(l + 0.5).$$

For every arrow μ of length 2^{-n}, the number $n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow})$ only depends on the numbers $\mathrm{ind}(\gamma, c)$ where c is the center of a cell (or actually with respect to any other point of the cell – the index is constant in each cell) and thus, two path sequences with the same set of winding numbers are equivalent. Consider for example the vertical arrow

$$\mu = [(k+il)2^{-n} \to (k+i(l+1))2^{-n}]$$

and the associated left cell $C_{k-1,l}$ and $C_{k,l}$. If we edit γ^* to replace every occurence of μ with the polyline

$$\mu_2 = [(k+il)2^{-n} \to (k+1+il)2^{-n} \to \\ (k+1+i(l+1))2^{-n} \to (k+i(l+1))2^{-n}]$$

and every occurence of $\mu^{\leftarrow}$ by $\mu_2^{\leftarrow}$, we have increased the index of the right cell (with center $c_{k,l}$) by $n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow})$ and the index of every other cell remains the same. By construction, for this new path sequence γ_2, we have $n_{\gamma_2^*}(\mu) = 0$ and $n^*_{\gamma_2}(\mu^{\leftarrow}) = 0$; the left and right cells belongs to the same component of $\mathbb{C} \setminus \gamma_2([0,1])$. Therefore the index of γ_2 around both cells is the same which means that

$$\mathrm{ind}(\gamma, c_{k-1,l}) = \mathrm{ind}(\gamma, c_{k,l}) + n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow}).$$

The treatment of horizontal arrows is similar.

Now, consider the sequence of centers $(c_1, \dots, c_m)$ such that $\mathrm{ind}(\gamma, c_p) \neq 0$ and the path sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ where λ_p is either the concatenation of $\mathrm{ind}(\gamma, c_p)$ times the boundary of the cell with center c_p oriented counterclockwise if this winding number is positive, or the concatenation of $-\mathrm{ind}(\gamma, c_p)$ times the boundary of the cell of center c_p oriented clockwise if it is negative. Every path λ_p is rectifiable; the corresponding cell with center c_p is included in $\mathrm{Int}\,\gamma$ and therefore $\lambda_p([0,1])$ is included in Ω. Additionally, by construction, for every cell center c, $\mathrm{ind}(\gamma, c) = \mathrm{ind}(\lambda, c)$ and therefore γ and λ are equivalent. The diameter of λ_p is smaller than 2^{-n}; a suitably large choice of n makes the diameter as small as required and this concludes the proof. ■

Proof – Small Closed Paths Theorem. Let $\epsilon > 0$ such that that the open set

$$\Omega' = \{z \in \Omega \mid d(z, \mathbb{C} \setminus \Omega) > \epsilon\}$$

contains the image of γ and its interior. Let λ_1 and λ_2 be the path sequences provided by the small paths & path of arrows lemma with $\Omega = \Omega'$.

The image of every path μ of the sequence λ_1 is included in the disk centered on $\mu(0)$ of radius ϵ which is itself included in Ω. Any point $z \in \mathbb{C} \setminus \Omega$ belongs to the unbounded component of $\mathbb{C} \setminus \mu([0,1])$, thus $\mathrm{ind}(\mu, z) = 0$. Consequently, for any such z,

$$\mathrm{ind}(\lambda_2, z) = \mathrm{ind}(\gamma, z)$$

and thus $\mathrm{Int}\,\lambda_2 \subset \Omega$. The conclusion of the theorem then follows from the application of the arrow version of the small closed paths theorem to the sequence of paths λ_2. ■