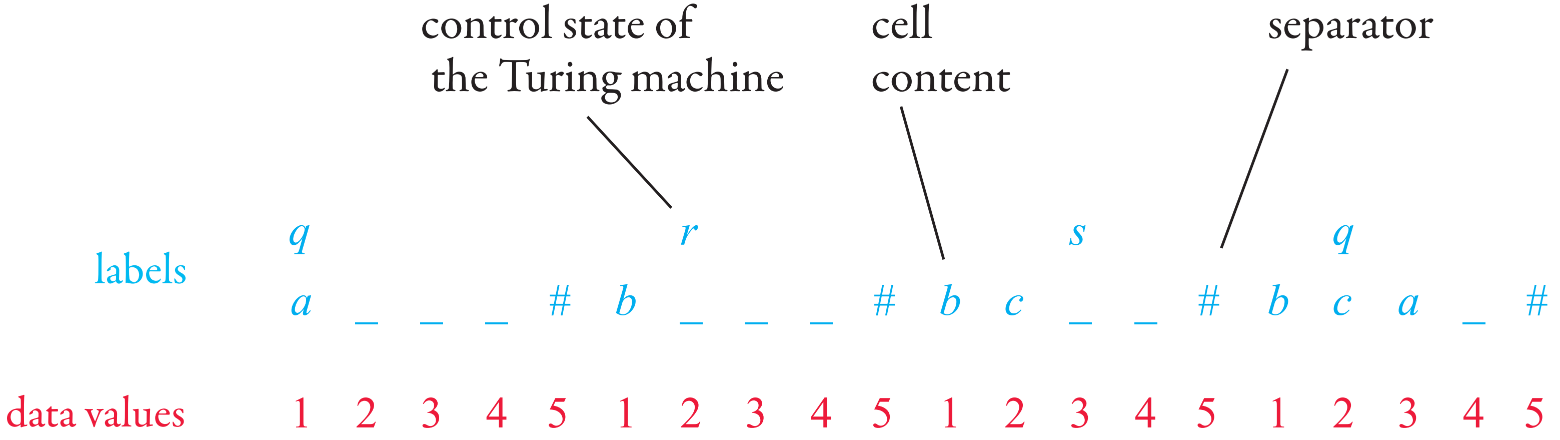
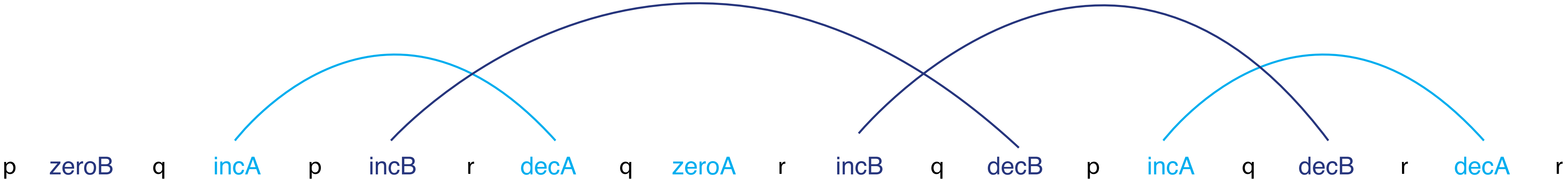
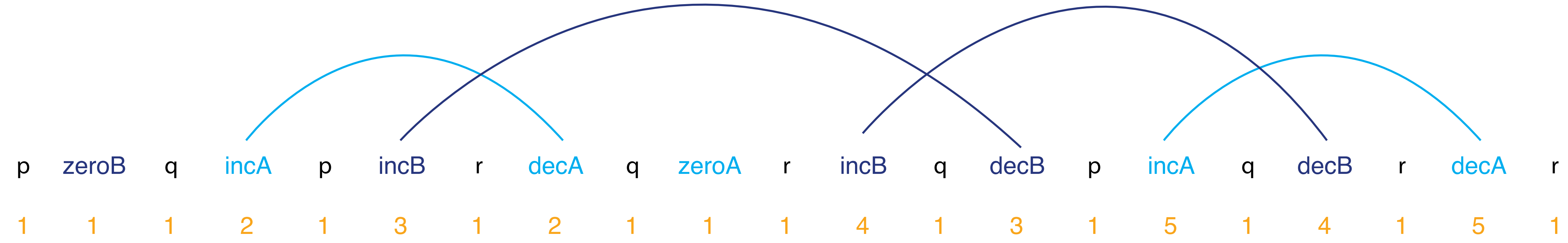
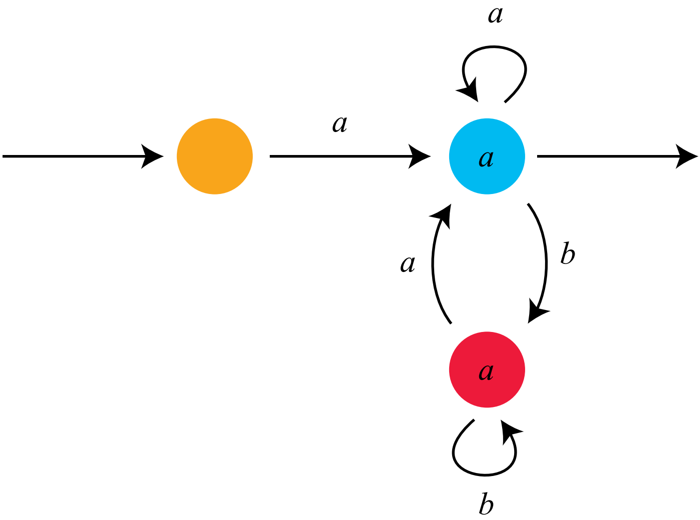
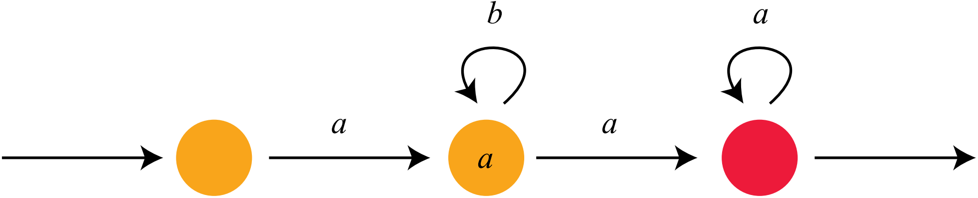

control state of the Turing machine
cell content
separator
labels
q r s q
a _ _ _ # b _ _ _ # b c _ _ # b c a _ #
data values
1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5

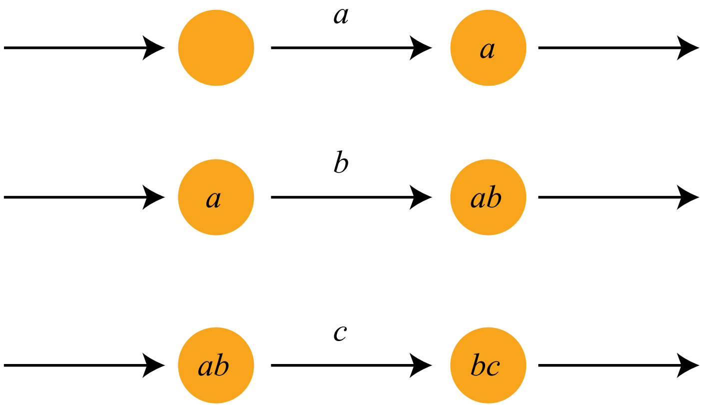

p zeroB q incA p incB r decA q zeroA r incB q decB p incA q decB r decA r

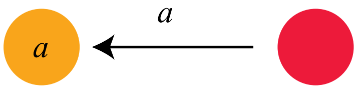

p zeroB q incA p incB r decA q zeroA r incB q decB p incA q decB r decA r
1 1 1 2 1 3 1 2 1 1 1 4 1 3 1 5 1 4 1 5 1

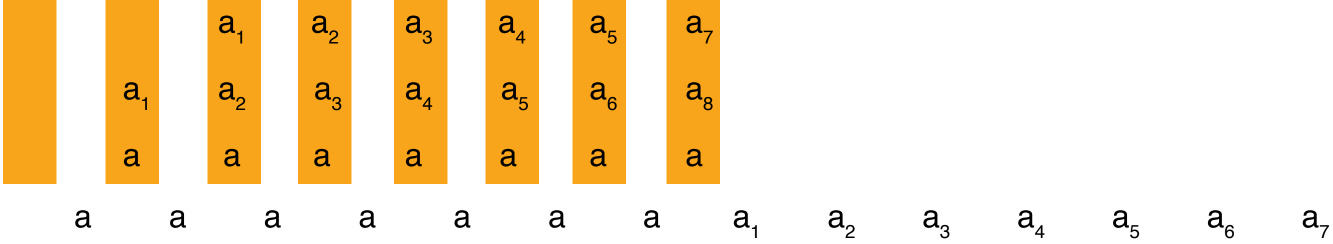

a
a
a
a
b
a
a
b

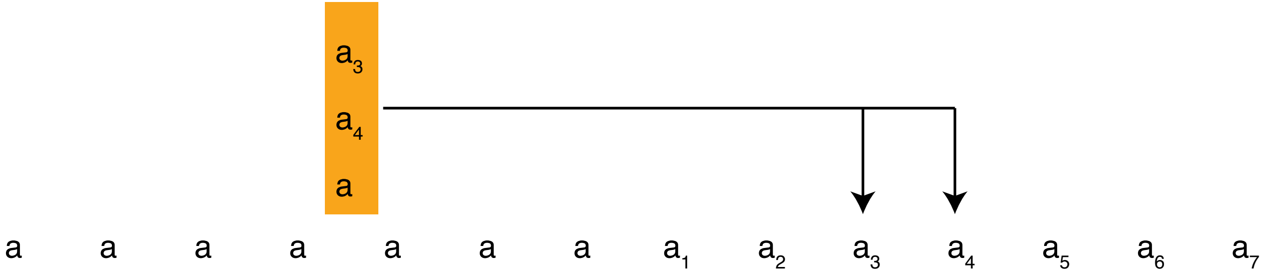


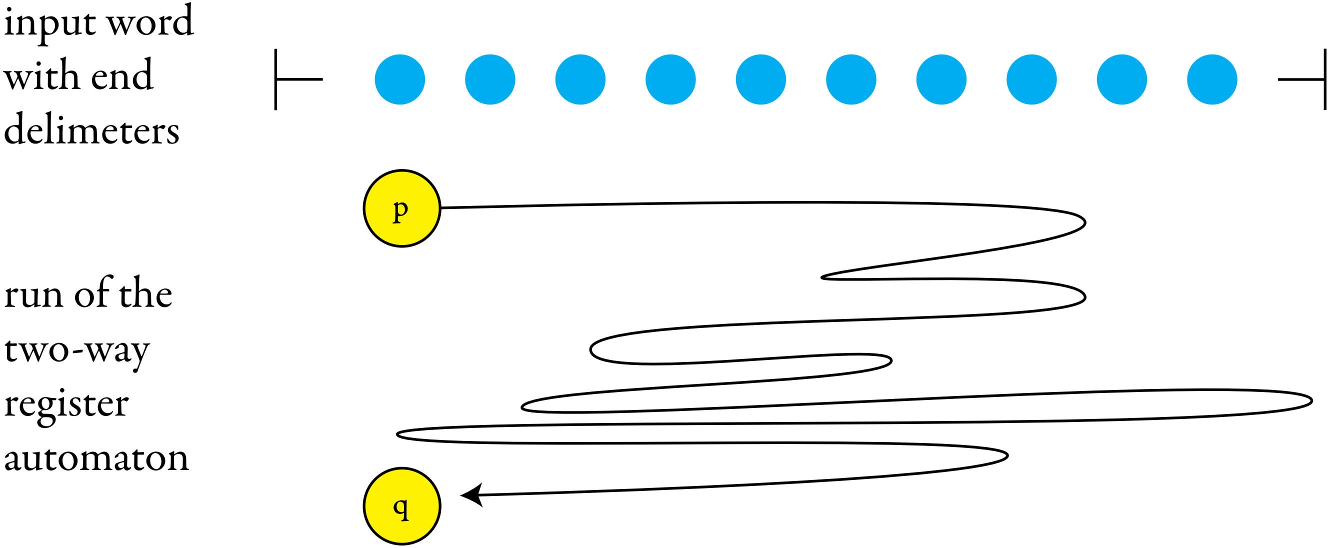
b
a
a
a
a

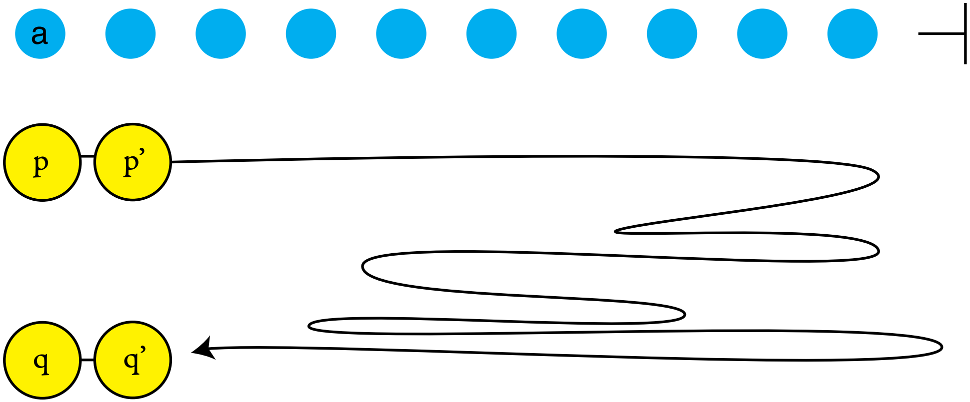

a
a
b
a
ab
c
ab
bc

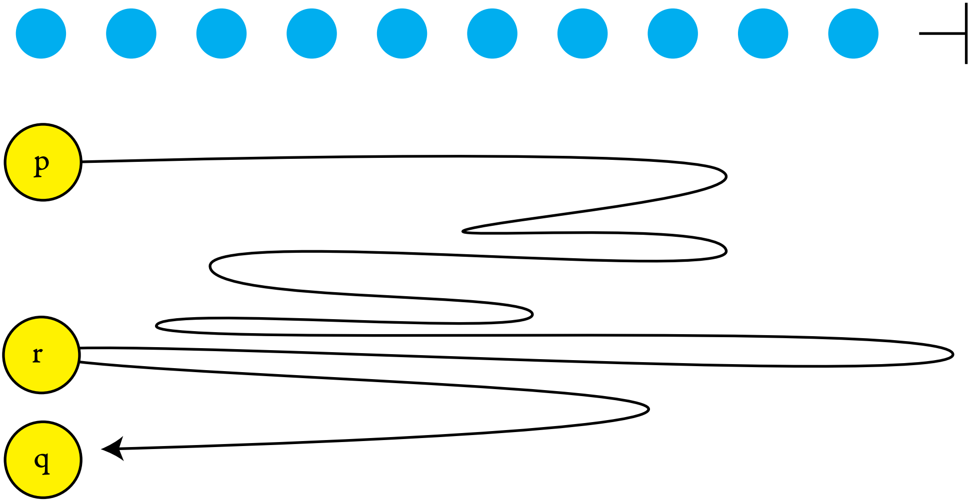

a
a

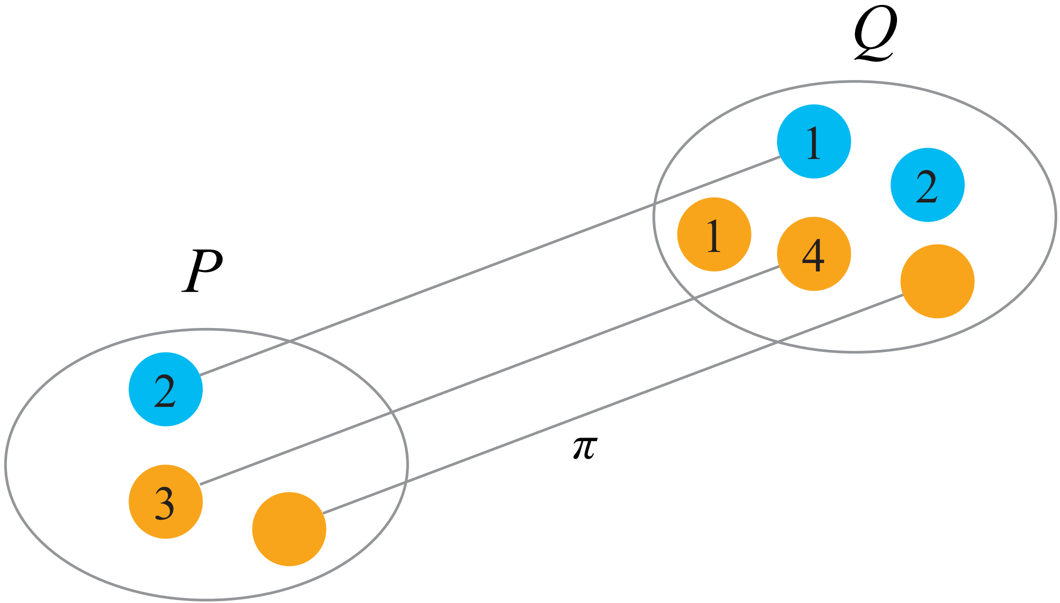

a
a₁
a₂
a₃
a₄
a₅
a₇
a₁
a₂
a₃
a₄
a₅
a₆
a₈
a
a
a
a
a
a
a
a
a
a
a
a
a
a
a₁
a₂
a₃
a₄
a₅
a₆
a₇

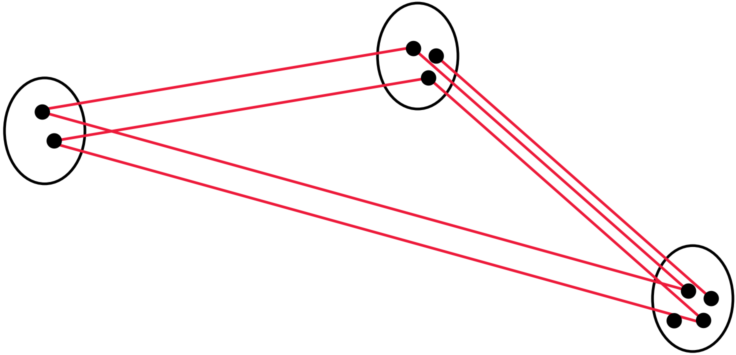

a3
a4
a
a a a a a a a a1 a2 a3 a4 a5 a6 a7

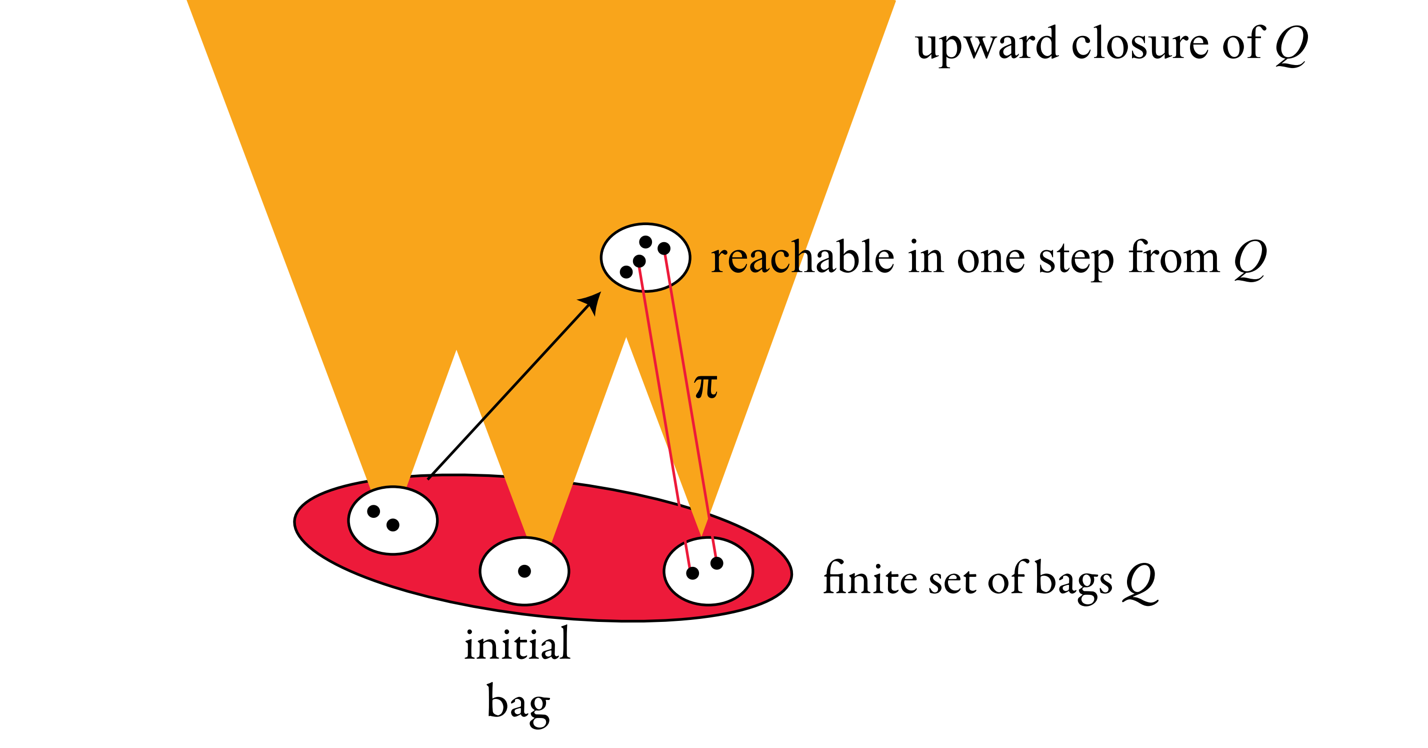

input word with end delimeters
p
run of the two-way register automaton
q

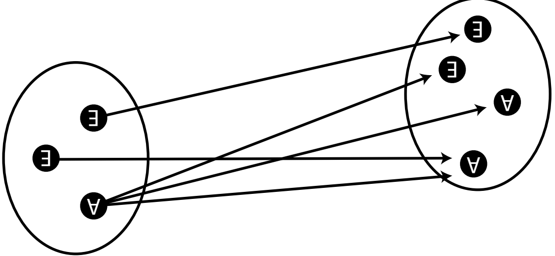

a
p
p'
q
q'

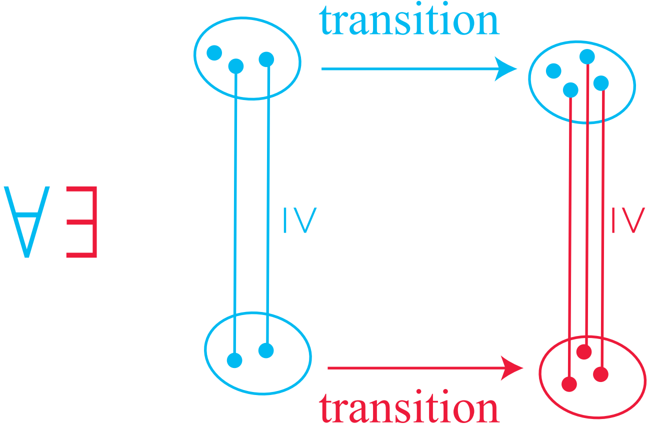

p
r
q


Q
1
2
1
4
P
2
3
π

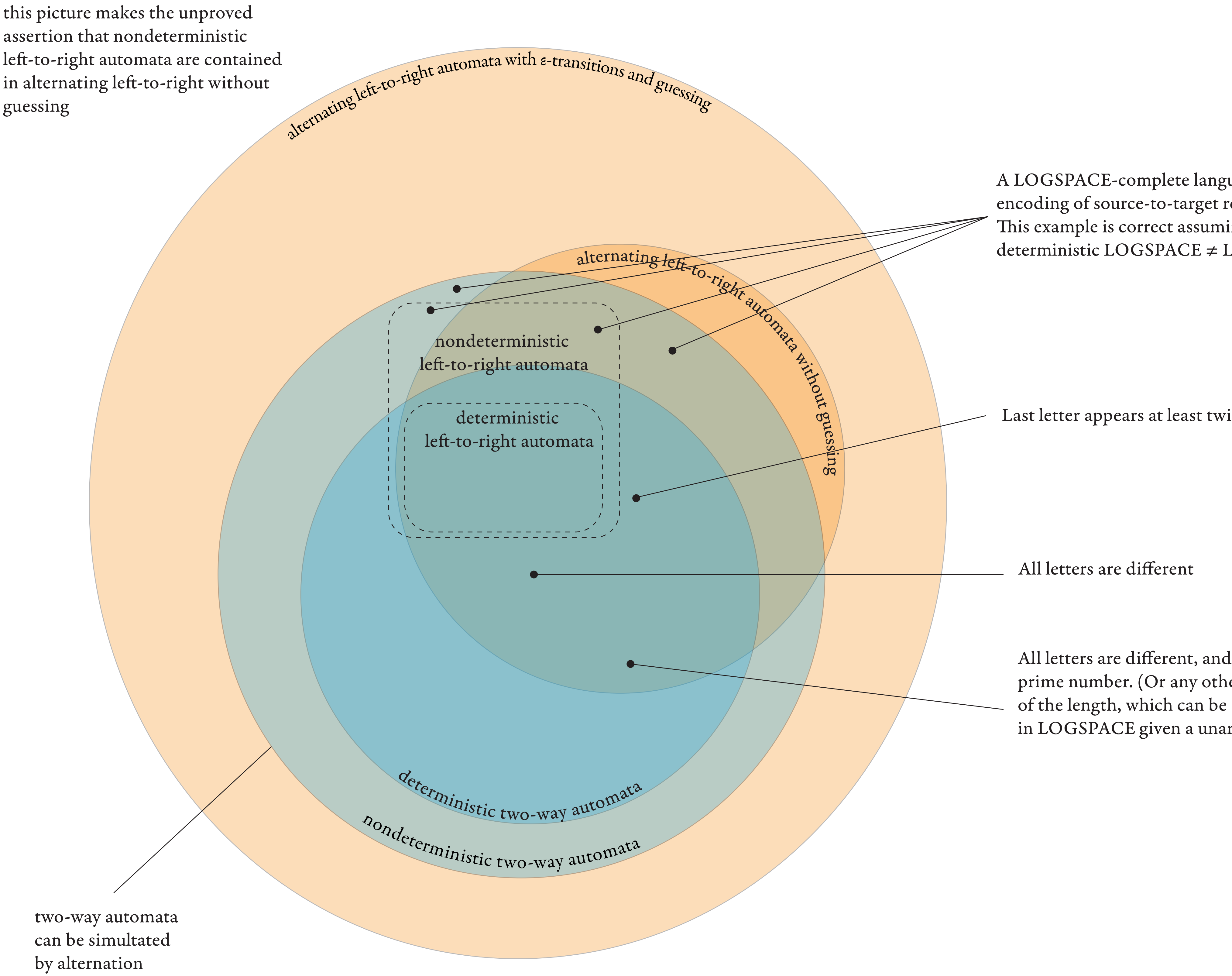


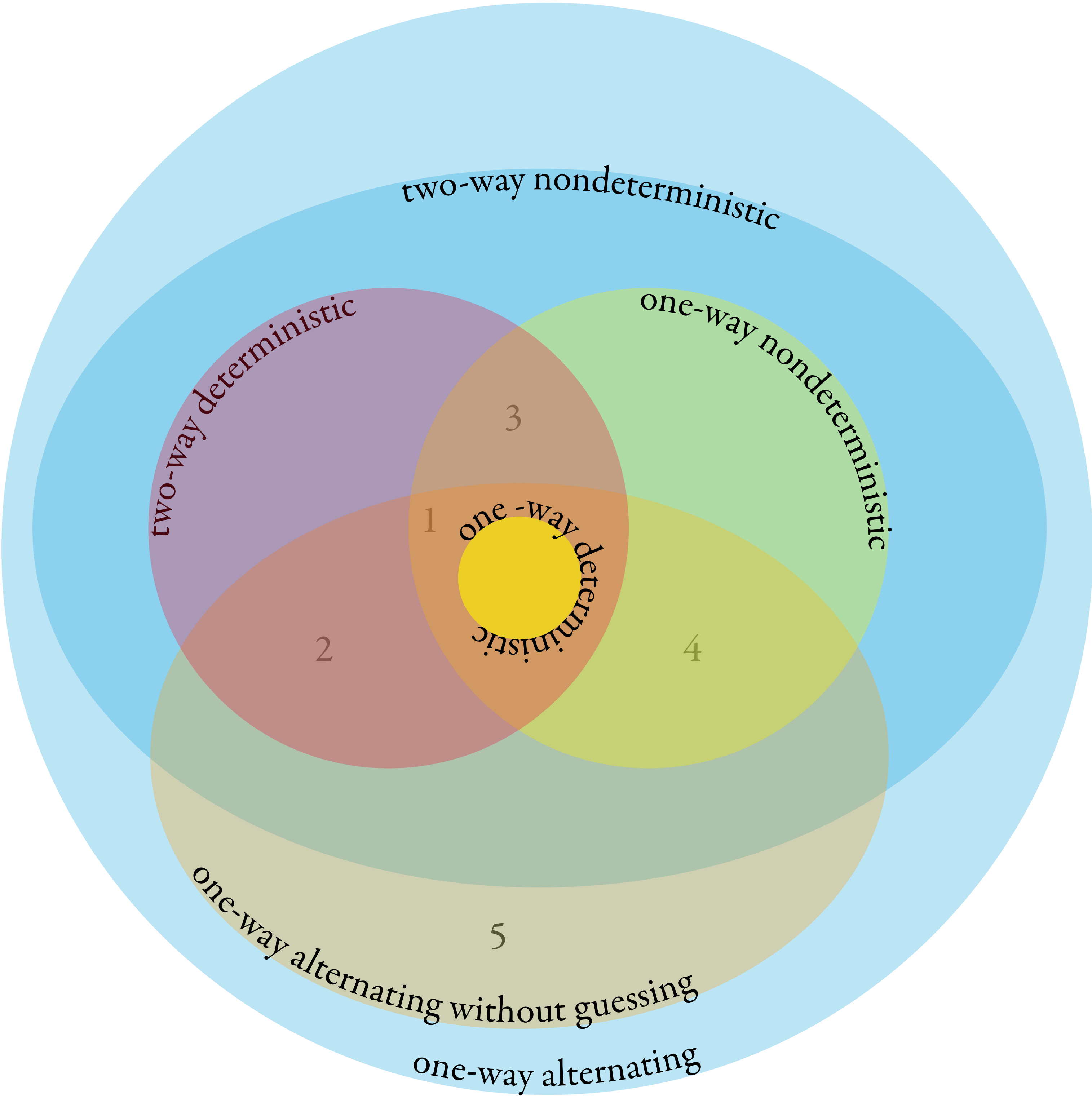

upward closure of Q
reachable in one step from Q
π
finite set of bags Q
initial
bag



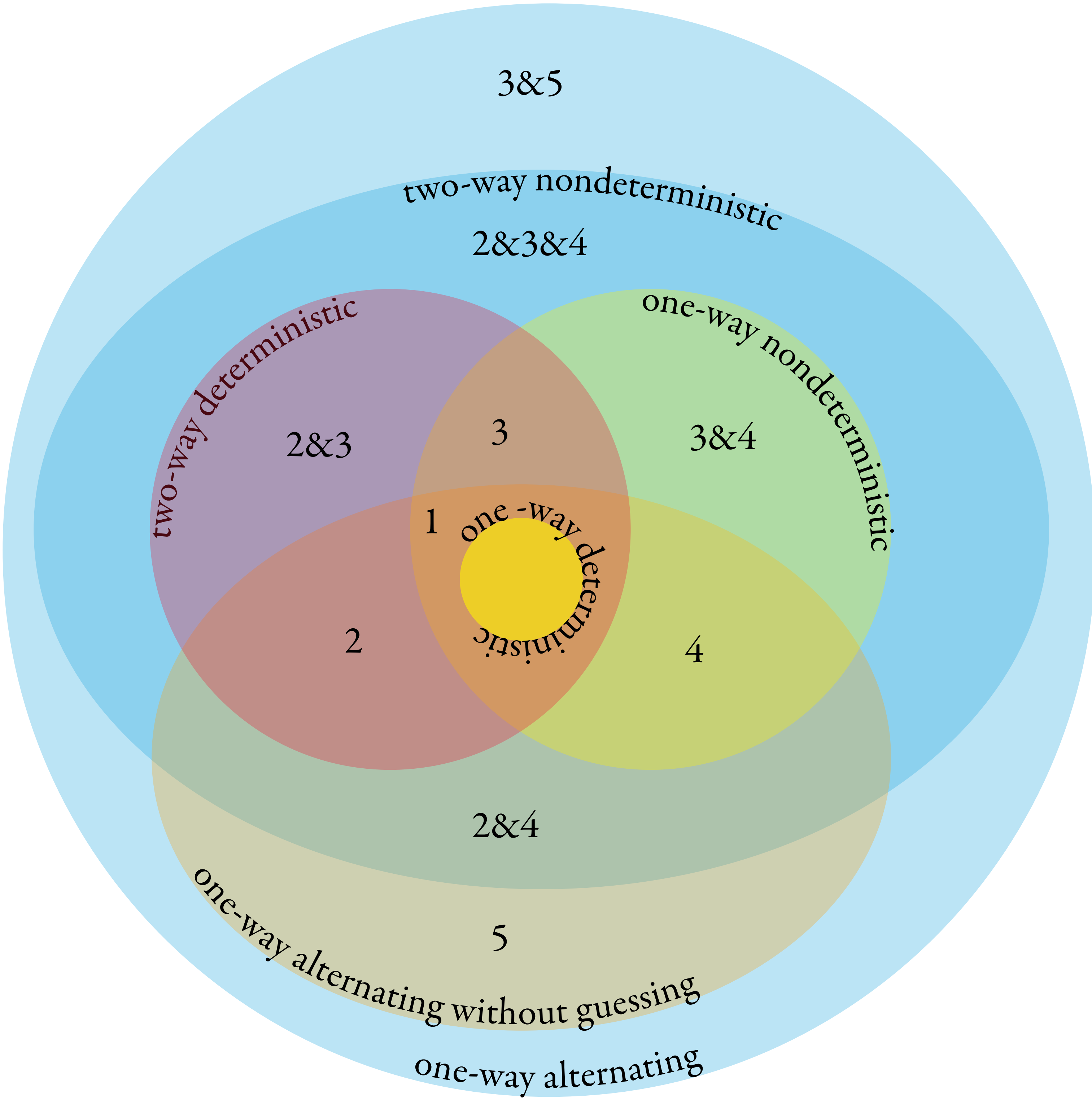


transition
∀
∃
≤
≤
transition

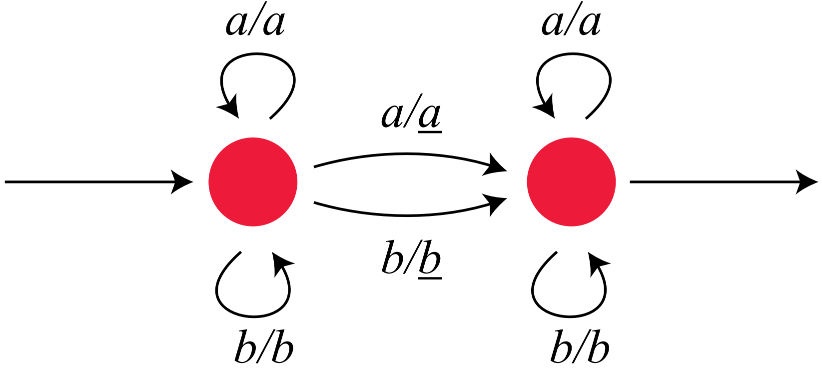

this picture makes the unproved assertion that nondeterministic left-to-right automata are contained in alternating left-to-right without guessing
alternating left-to-right automata with ε-transitions and guessing
alternating left-to-right automata without guessing
nondeterministic left-to-right automata
deterministic left-to-right automata
deterministic two-way automata
nondeterministic two-way automata
A LOGSPACE-complete langu
encoding of source-to-target r
This example is correct assumi
deterministic LOGSPACE ≠ L
Last letter appears at least twi
All letters are different
All letters are different, and
prime number. (Or any othe
of the length, which can be
in LOGSPACE given a unar
two-way automata can be simultated by alternation



two-way nondeterministic
two-way deterministic
one-way nondeterministic
3
1
one -way deterministic
2
4
5
one-way alternating without guessing
one-way alternating



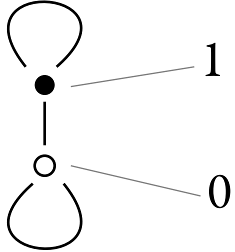
3&5
two-way nondeterministic
2&3&4
two-way deterministic
one-way nondeterministic
2&3
3
3&4
1
one -way deterministic
2
4
2&4
5
one-way alternating without guessing
one-way alternating

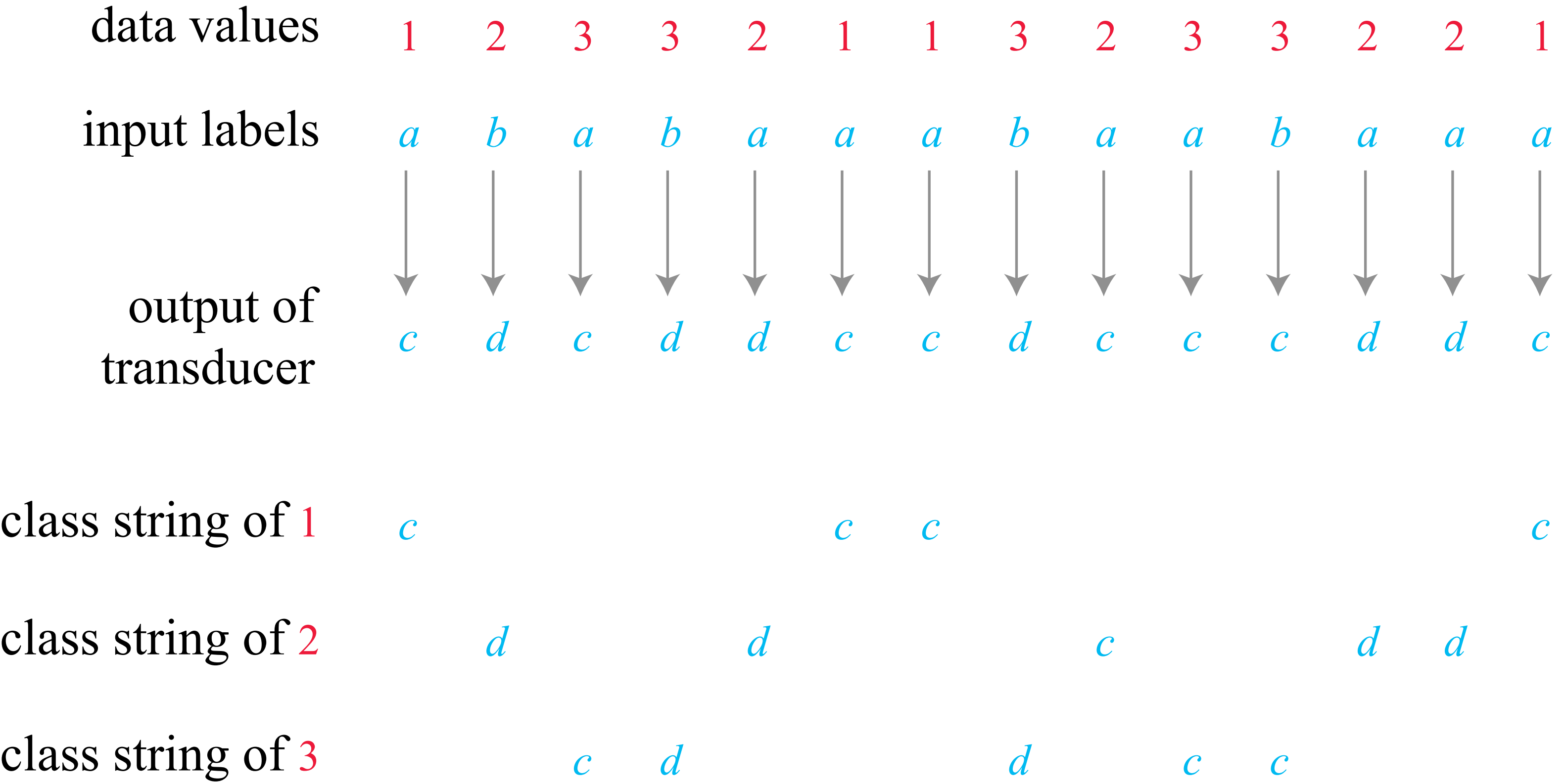


A transition like this

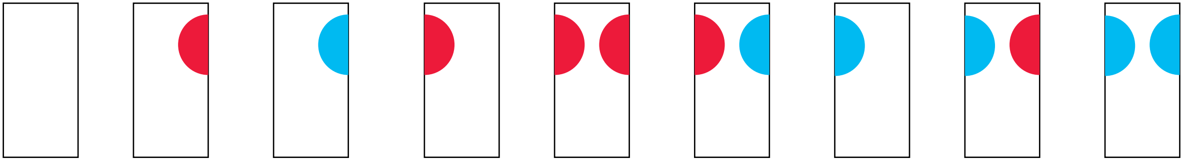
$$\xrightarrow{a/\underline{a}}$$

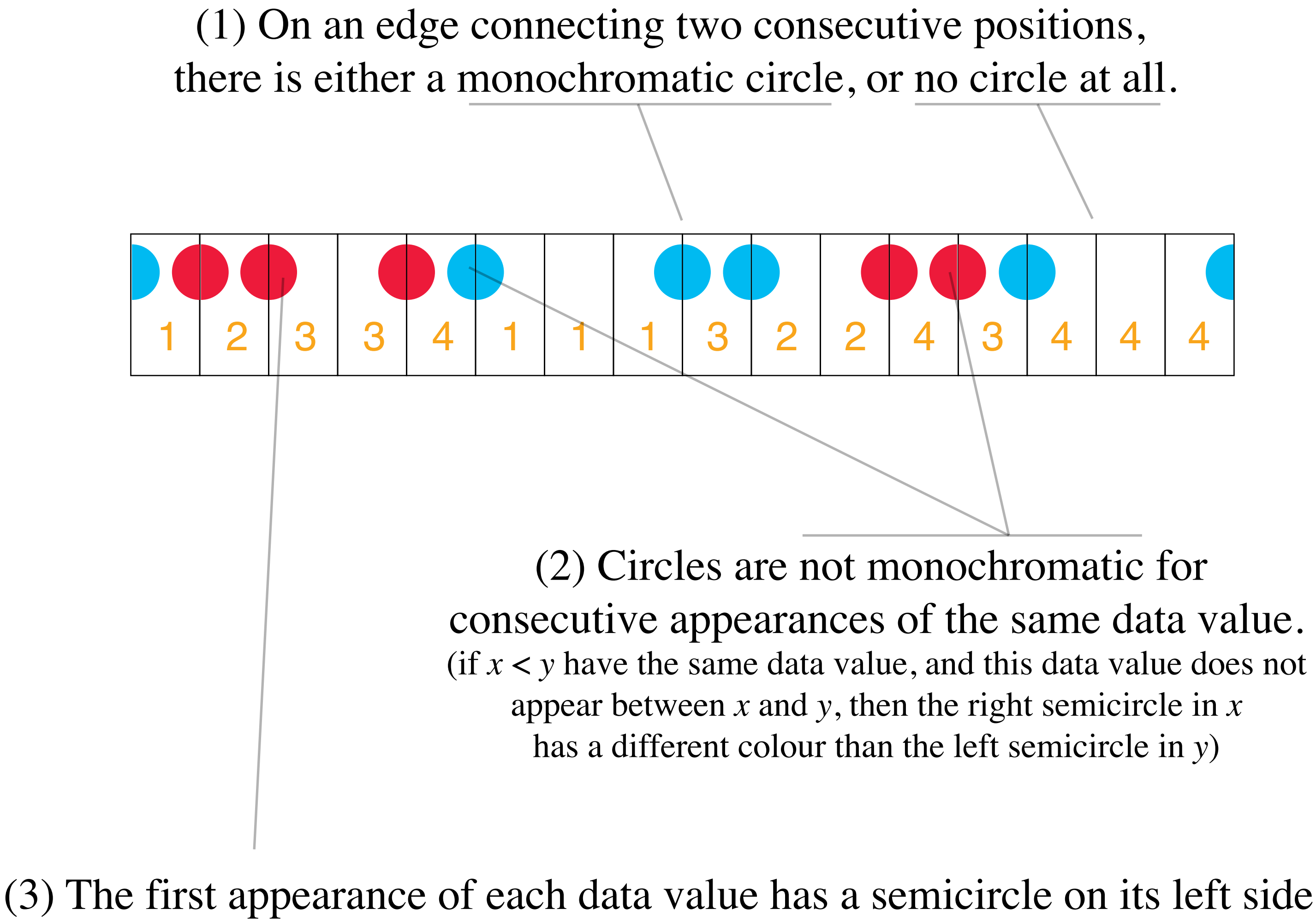
inputs a and outputs $\underline{a}$

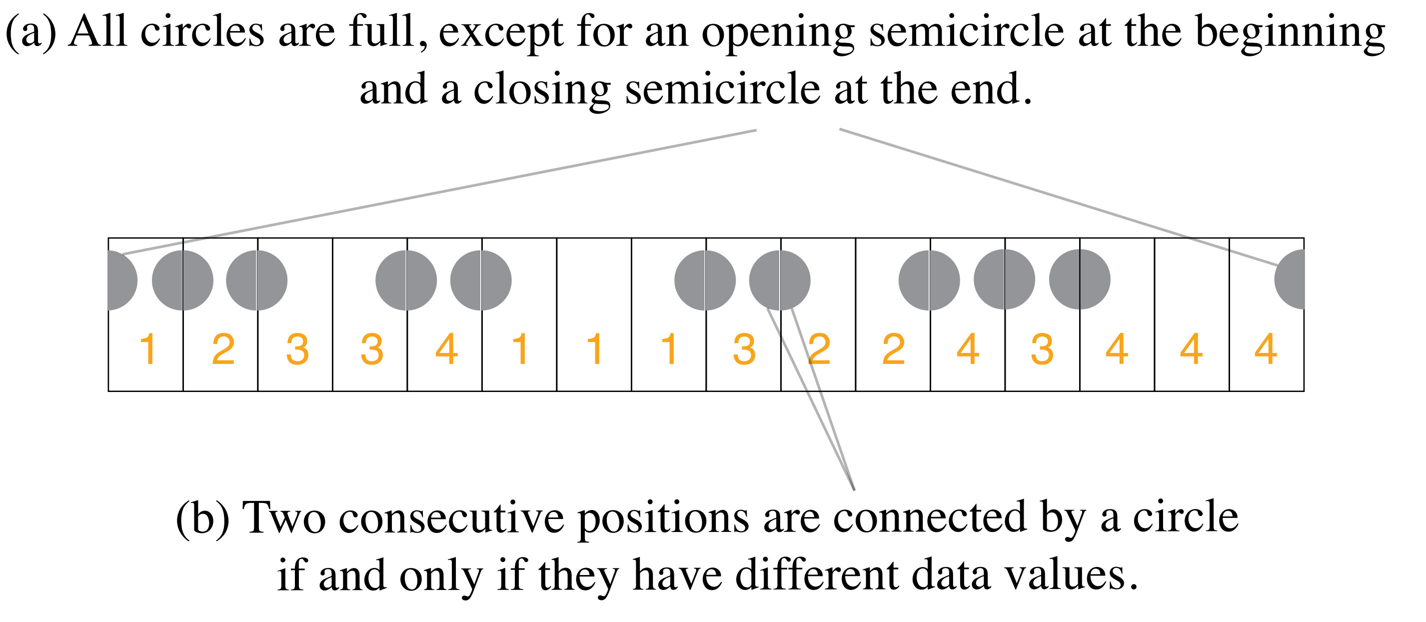

1
0

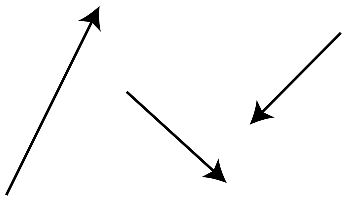


data values 1 2 3 3 2 1 1 3 2 3 3 2 2 1
input labels a b a b a a a b a a b a a a
output of transducer c d c d d c c d c c c d d c
class string of 1 c c c c
class string of 2 d d c d d
class string of 3 c d d c c

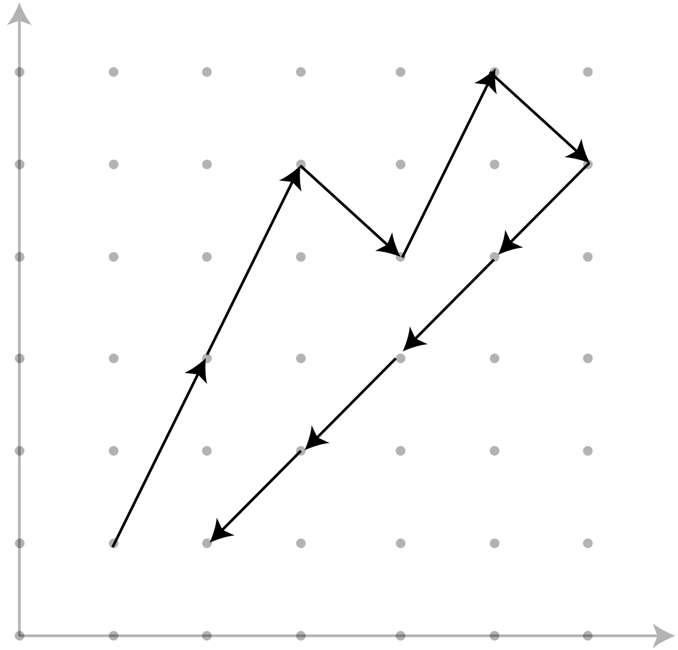



(1) On an edge connecting two consecutive positions,
there is either a monochromatic circle, or no circle at all.
1 2 3 3 4 1 1 1 3 2 2 4 3 4 4 4
(2) Circles are not monochromatic for
consecutive appearances of the same data value.
(if x < y have the same data value, and this data value does not
appear between x and y, then the right semicircle in x
has a different colour than the left semicircle in y)
(3) The first appearance of each data value has a semicircle on its left side

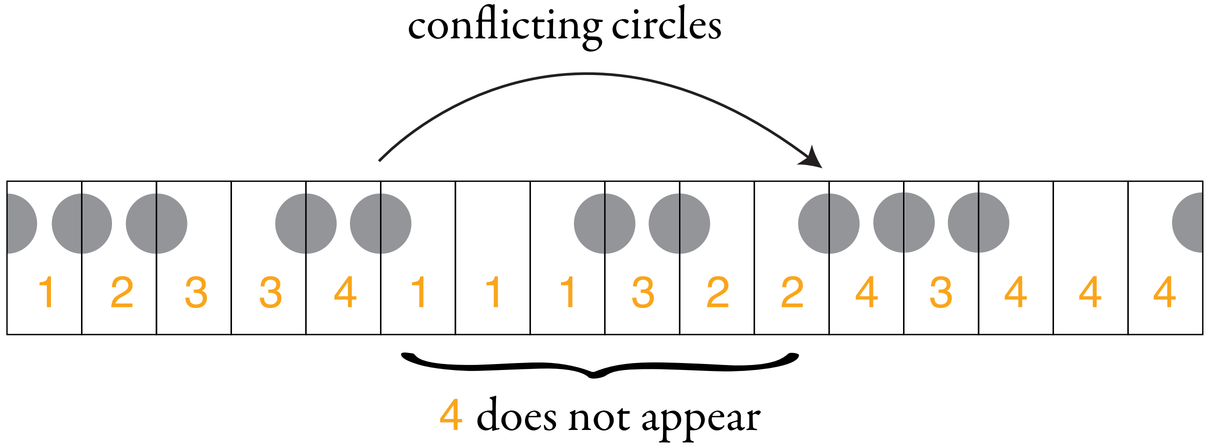

(a) All circles are full, except for an opening semicircle at the beginning and a closing semicircle at the end.
1 2 3 3 4 1 1 1 3 2 2 4 3 4 4 4
(b) Two consecutive positions are connected by a circle if and only if they have different data values.



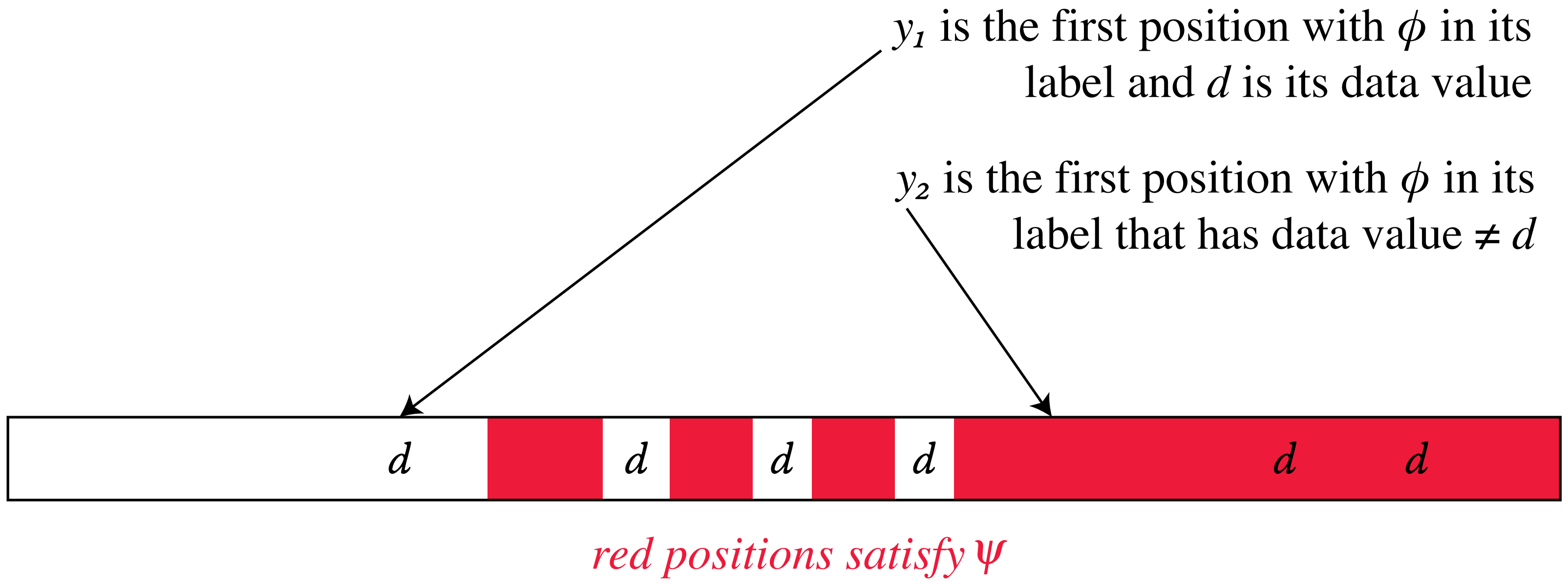
transitions



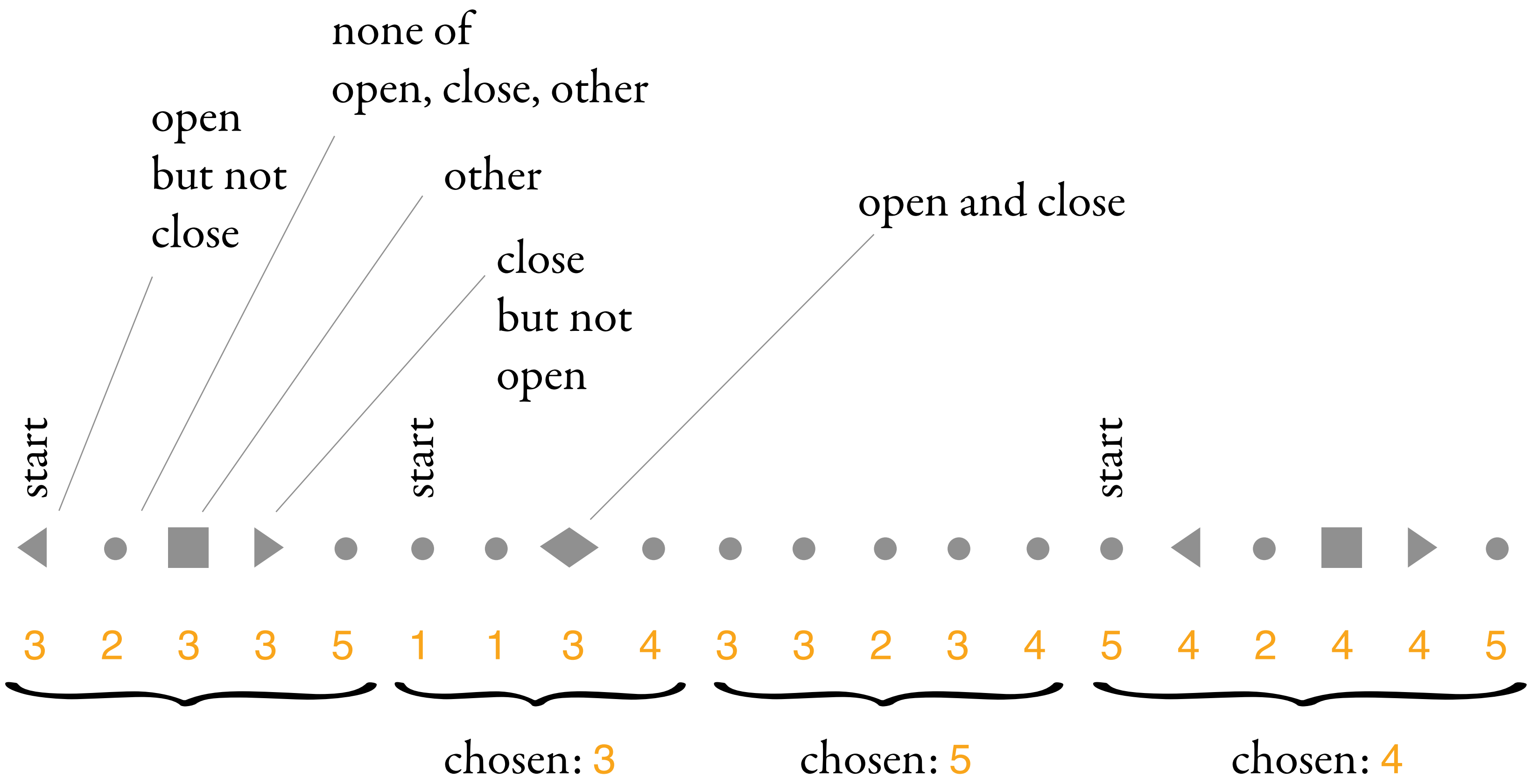
a run



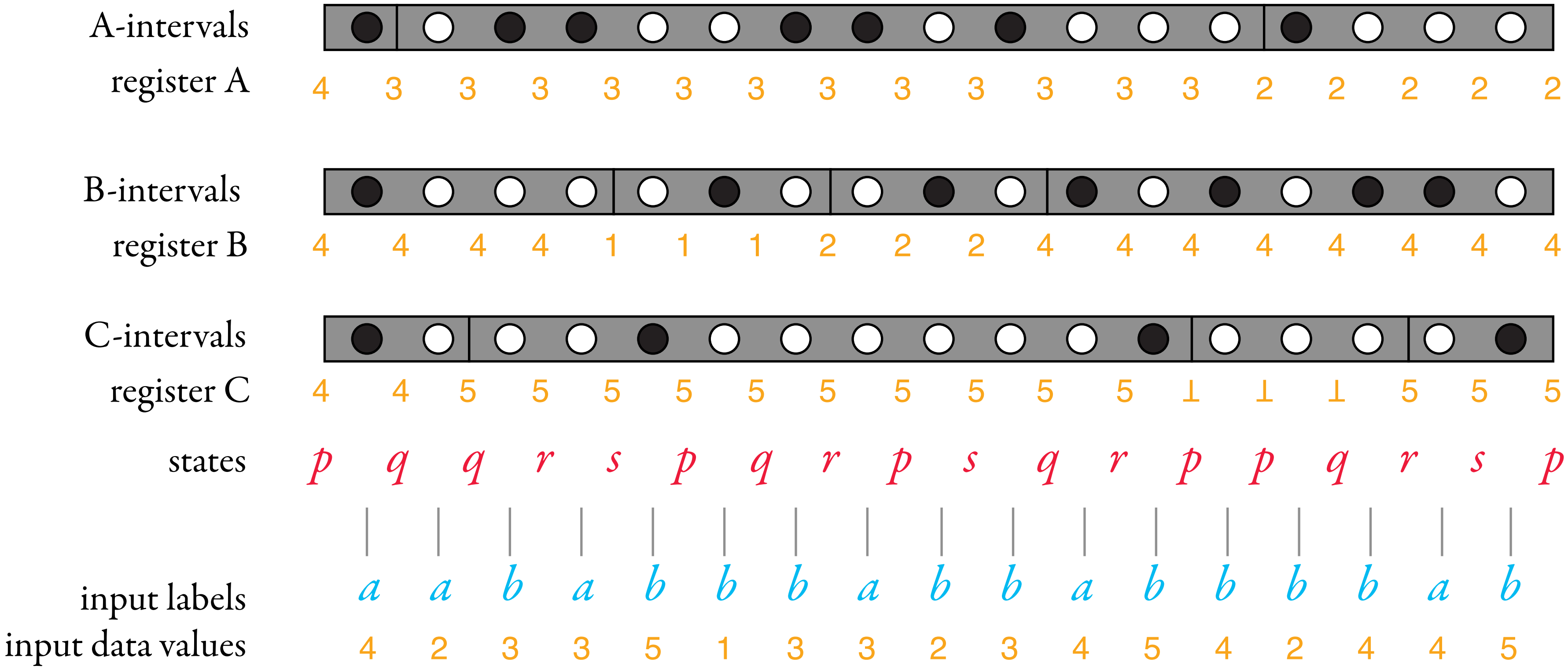
conflicting circles
1 2 3 3 4 1 1 1 3 2 2 4 3 4 4 4
4 does not appear


y_1 is the first position with ϕ in its label and d is its data value
y_2 is the first position with ϕ in its label that has data value $\neq d$
d
d
d
d
d
d
red positions satisfy ψ

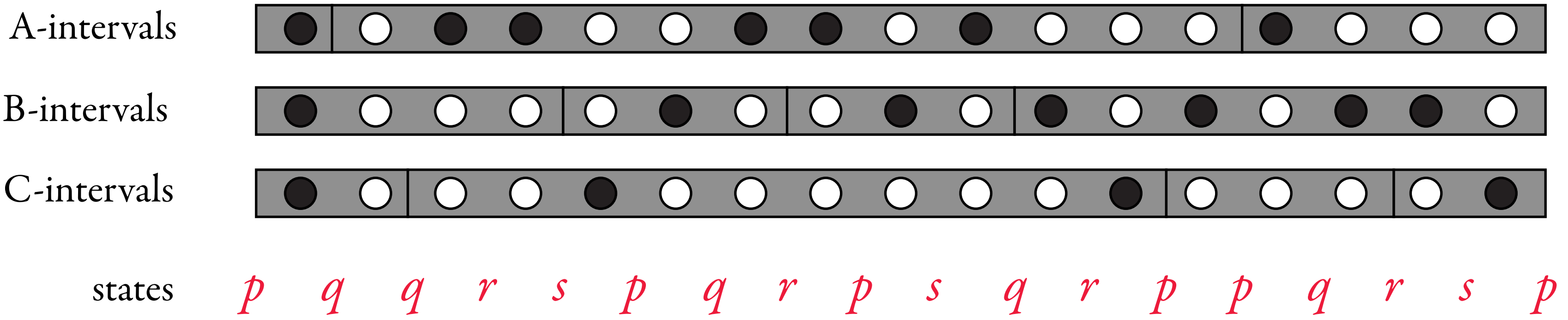
$$\underbrace{3 \quad 2 \quad 3 \quad 3 \quad 5}_{\text{chosen: } 3} \quad \underbrace{1 \quad 1 \quad 3 \quad 4}_{\text{chosen: } 1} \quad \underbrace{3 \quad 3 \quad 2 \quad 3 \quad 4}_{\text{chosen: } 5} \quad \underbrace{5 \quad 4 \quad 2 \quad 4 \quad 4 \quad 5}_{\text{chosen: } 4}$$

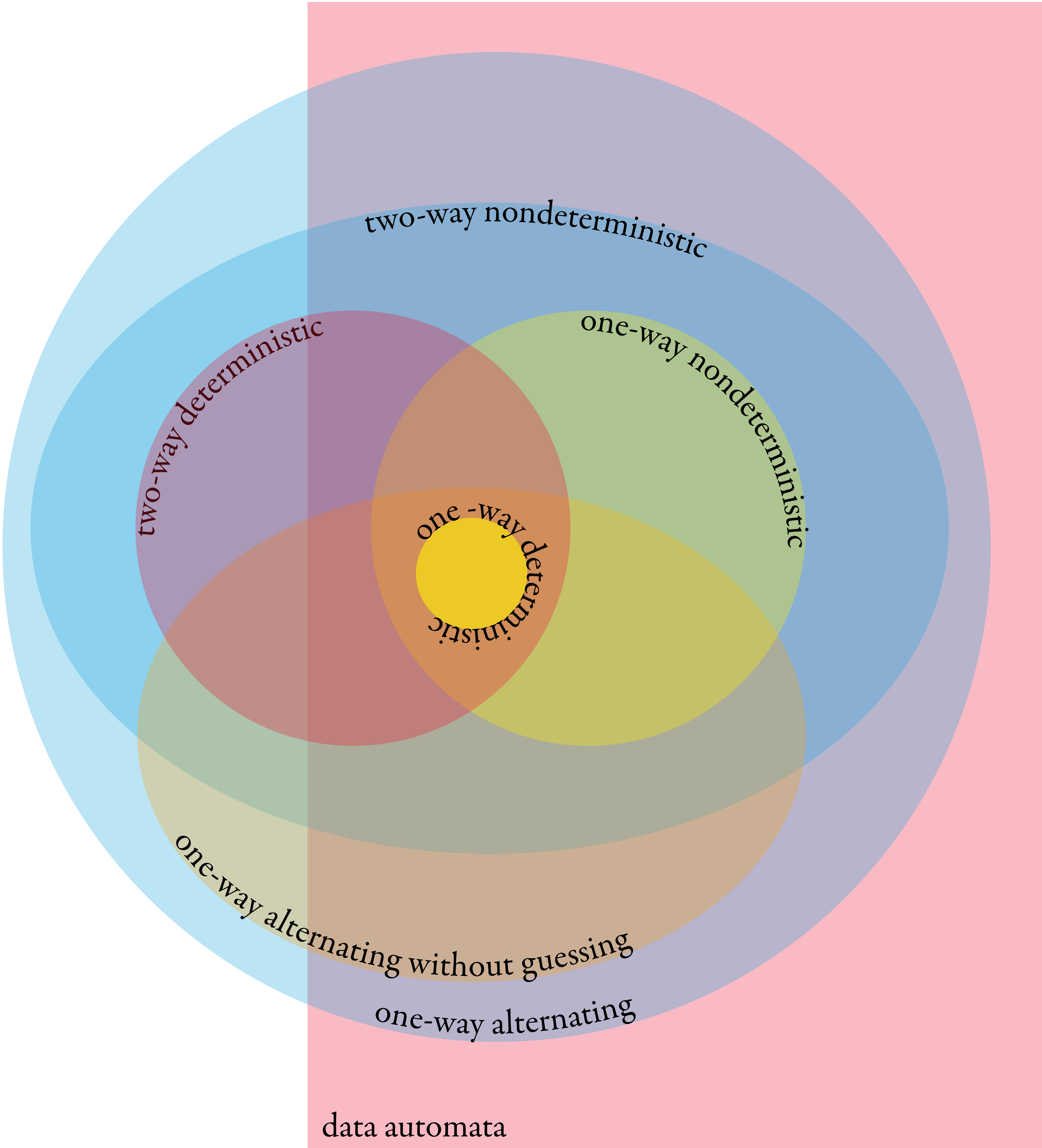

none of
open, close, other
open
but not
close
other
close
but not
open
open and close
start
start
start
3 2 3 3 5 1 1 3 4 3 3 2 3 4 5 4 2 4 4 5
chosen: 3
chosen: 5
chosen: 4

●	◀	■	▶	◆
not the chosen data value	first but not only appearance of chosen data value	appearance of chosen data value that is neither first nor last	last but not only appearance of chosen data value	only appearance of chosen data value



A-intervals
register A
4 3 3 3 3 3 3 3 3 3 3 3 3 2 2 2 2 2
B-intervals
register B
4 4 4 4 1 1 1 2 2 2 4 4 4 4 4 4 4 4
C-intervals
register C
4 4 5 5 5 5 5 5 5 5 5 5 ⊥ ⊥ ⊥ 5 5 5
states
p q q r s p q r p s q r p p q r s p
input labels
a a b a b b b a b b a b b b b a b
input data values
4 2 3 3 5 1 3 3 2 3 4 5 4 2 4 4 5

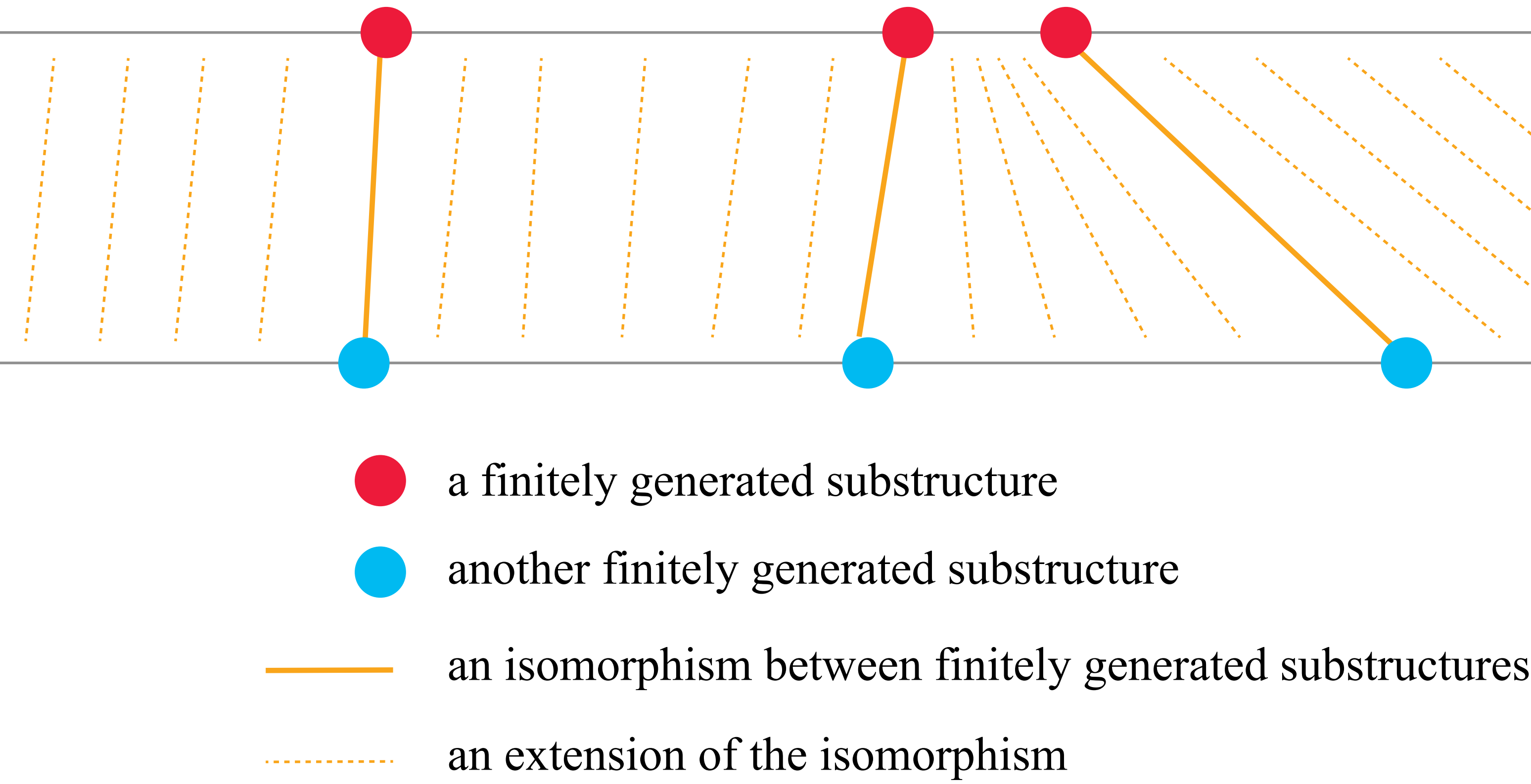

A-intervals
B-intervals
C-intervals
states
p q q r s p q r p s q r p p q r s p

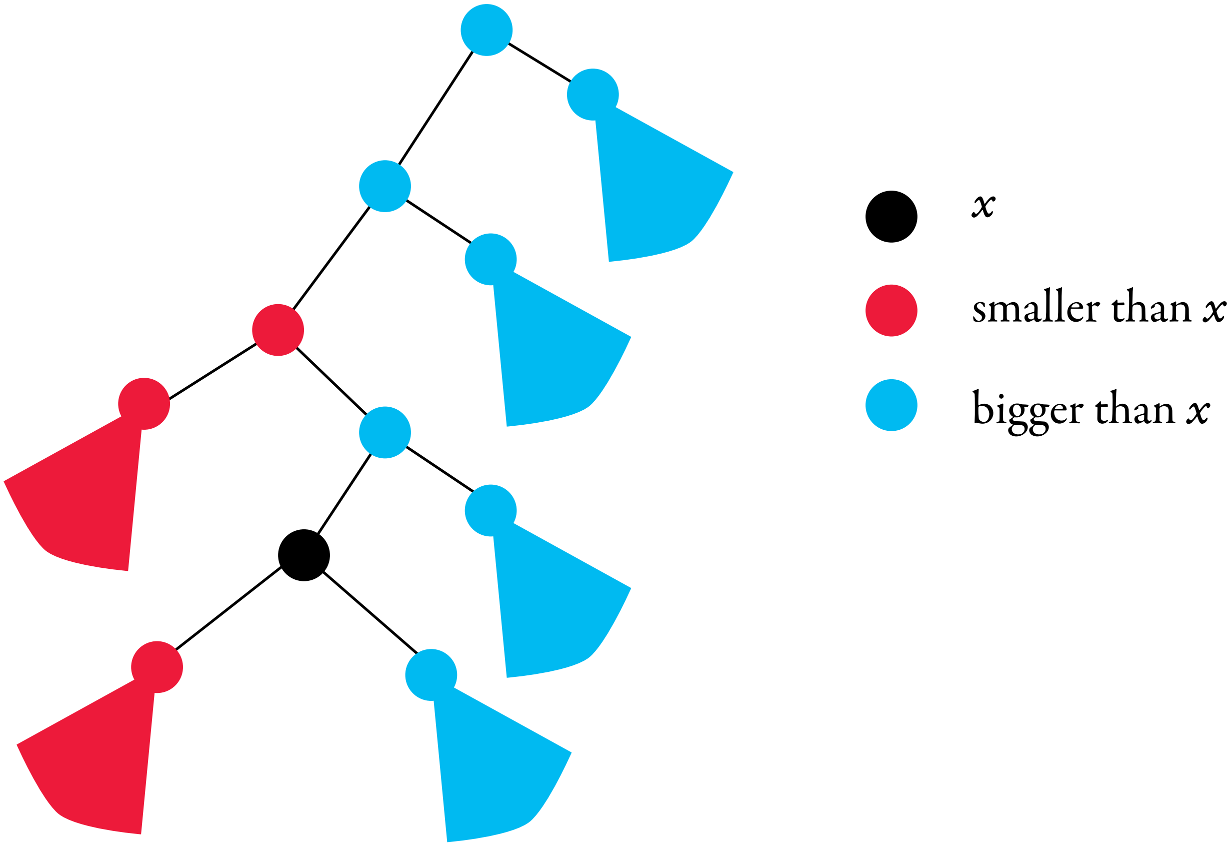

two-way nondeterministic
two-way deterministic
one-way nondeterministic
one -way deterministic
one-way alternating without guessing
one-way alternating
data automata

data values	1	2	3	3	2	1	1	3	2	3	3	2	2	1	1
labels	*a*	*b*	*a*	*b*	*a*	*a*	*a*	*b*	*a*	*a*	*b*	*a*	*a*	*a*	*a*
class string of 1	*a*					*a*	*a*							*a*	*a*

data values	1	2	3	3	2	1	1	3	2	3	3	2	2	1	1
labels	a	b	a	b	a	a	a	b	a	a	b	a	a	a	a
class string of 1	a	b	a	b	a	a	a	b	a	a	b	a	a	a	a

data values	1	2	3	3	2	1	1	3	2	3	3	2	2	1	1
labels	a	b	a	b	a	a	a	b	a	a	b	a	a	a	a
class string of 1	a	$?$	$?$	$?$	$?$	a	a	$?$	$?$	$?$	$?$	$?$	$?$	a	a

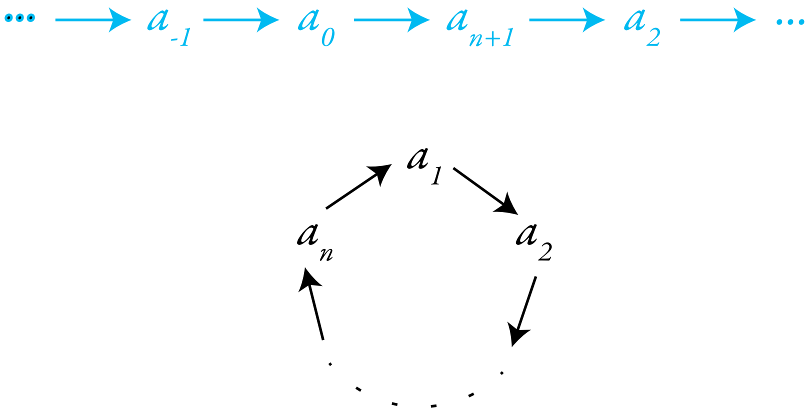

a finitely generated substructure
another finitely generated substructure
an isomorphism between finitely generated substructures
an extension of the isomorphism



x
smaller than x
bigger than x

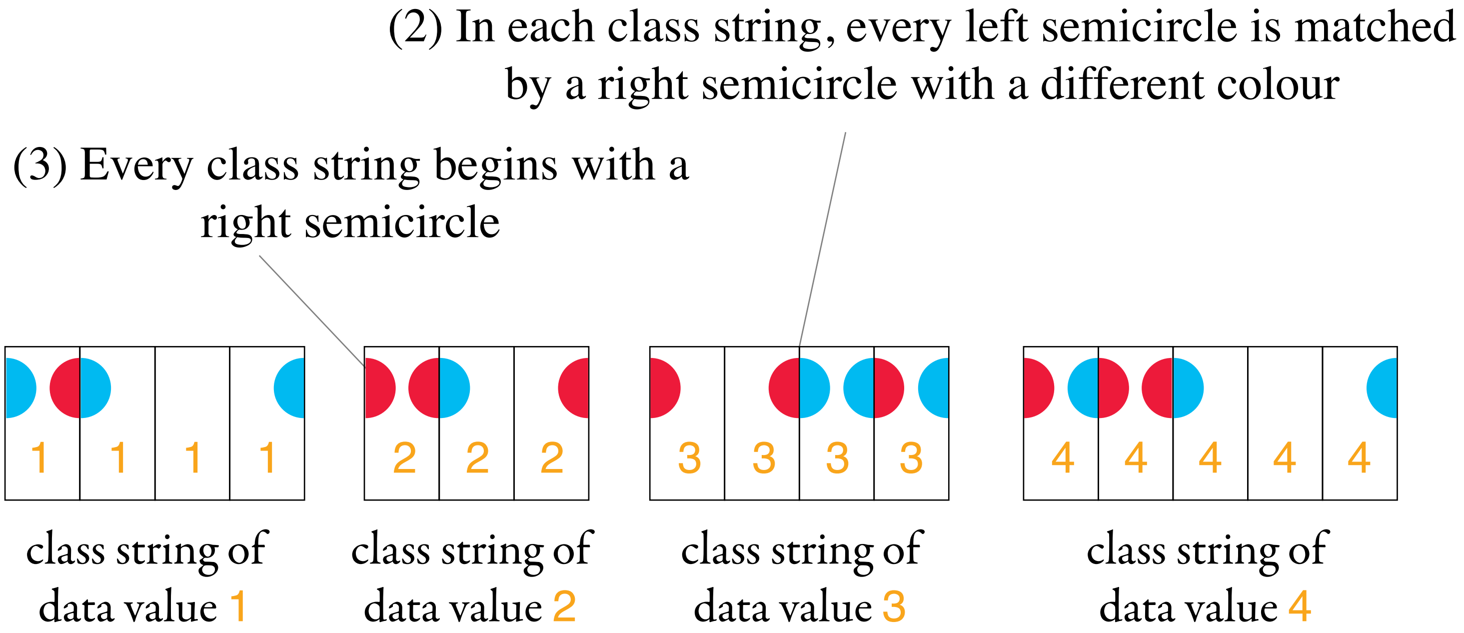
$$\begin{tikzcd} a' \arrow[d, leftrightarrow] & b' \arrow[d, leftrightarrow] \\ a & b \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, leftrightarrow] & b' \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[d, leftrightarrow] & b' \arrow[d, leftrightarrow] \\ a & b \end{tikzcd}$$

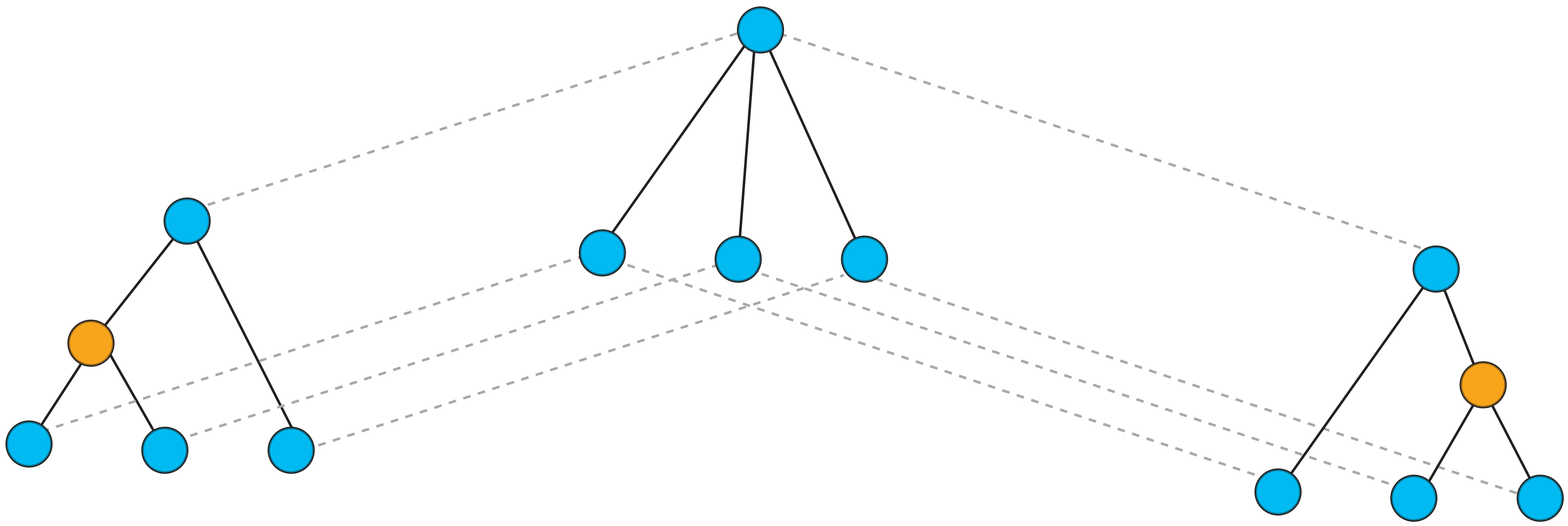
$$\cdots \longrightarrow a_{-1} \longrightarrow a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \cdots$$

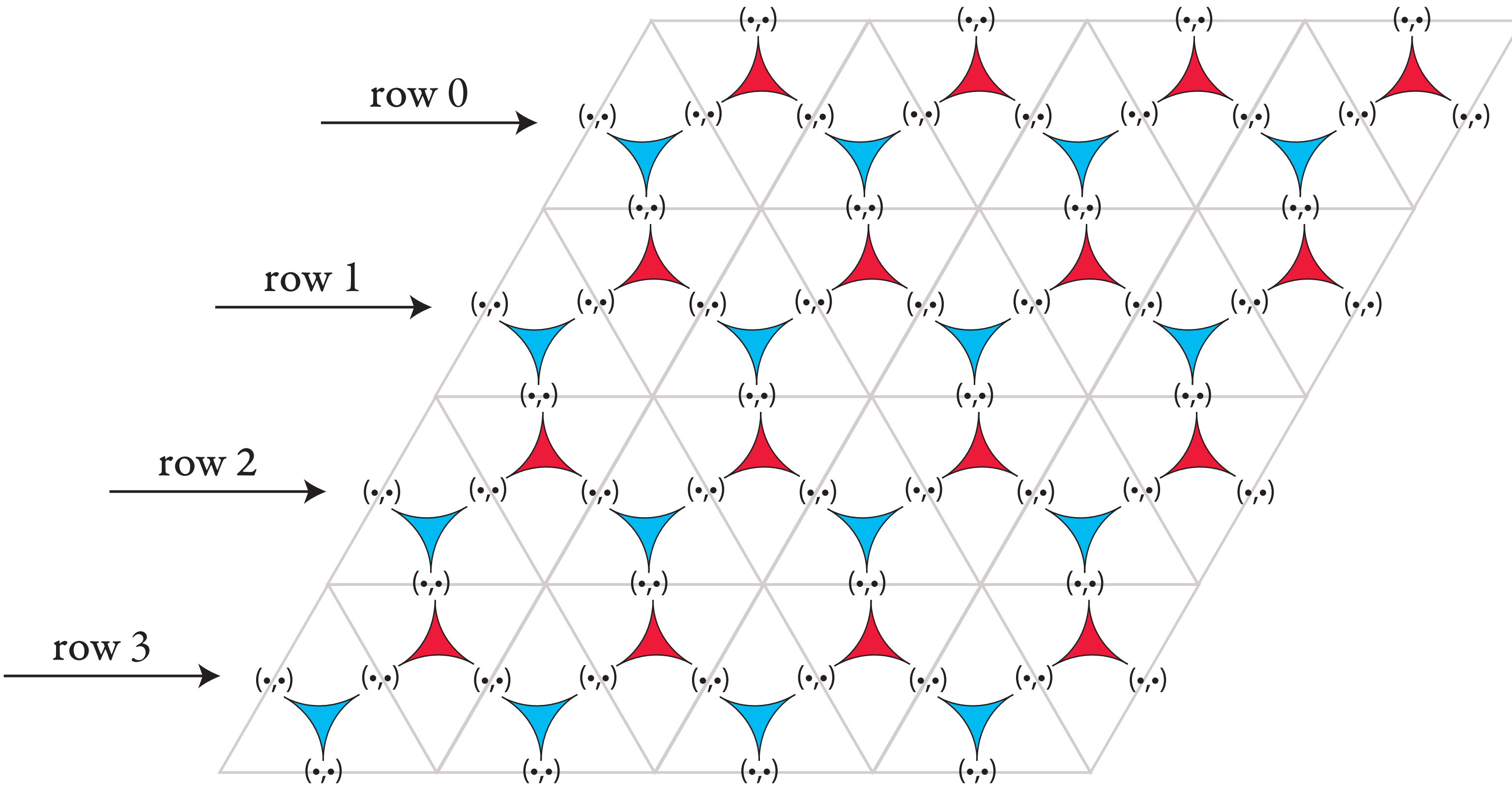


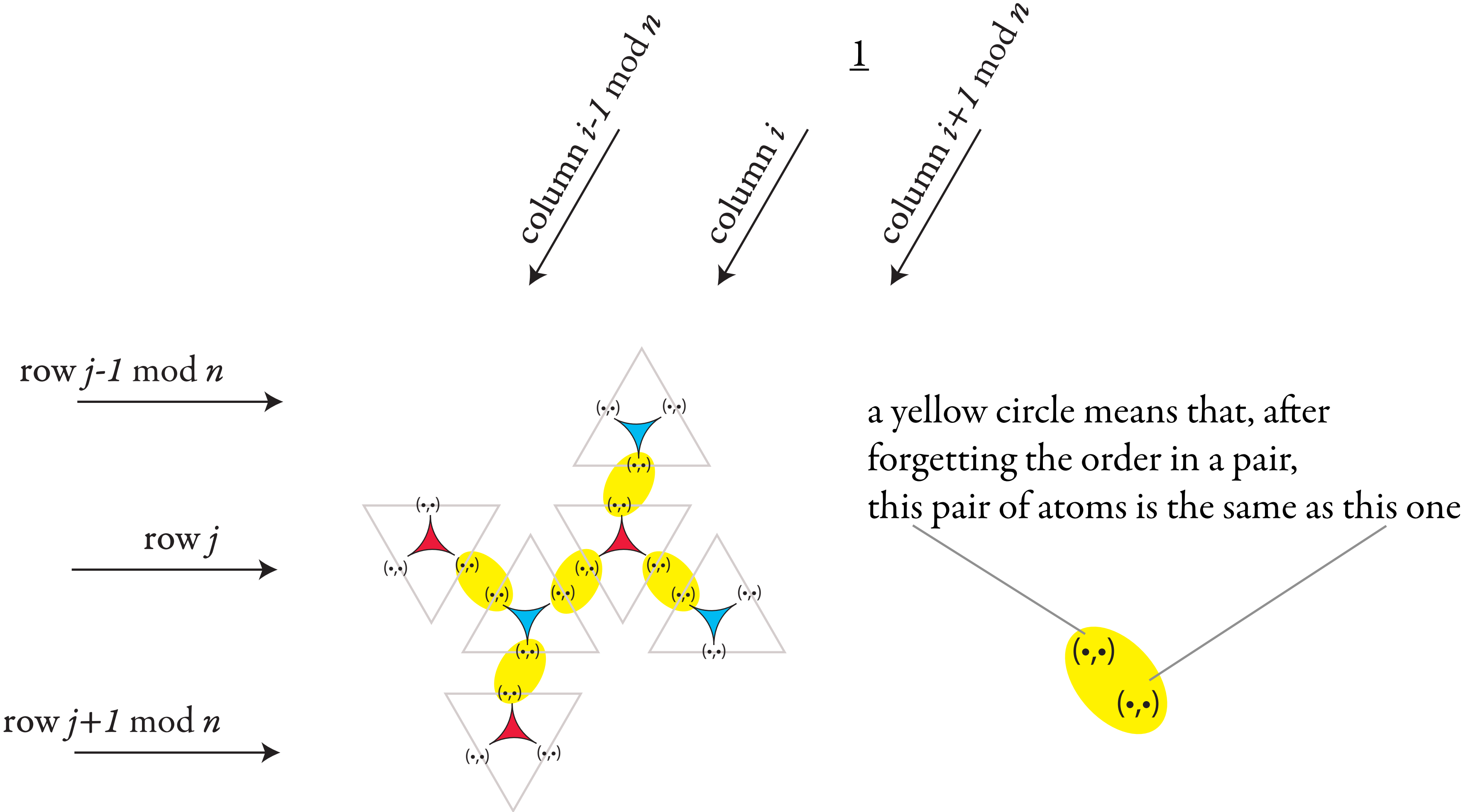
$\cdots \to a_{-1} \to a_0 \to a_{n+1} \to a_2 \to \cdots$
a_1
a_2
a_n

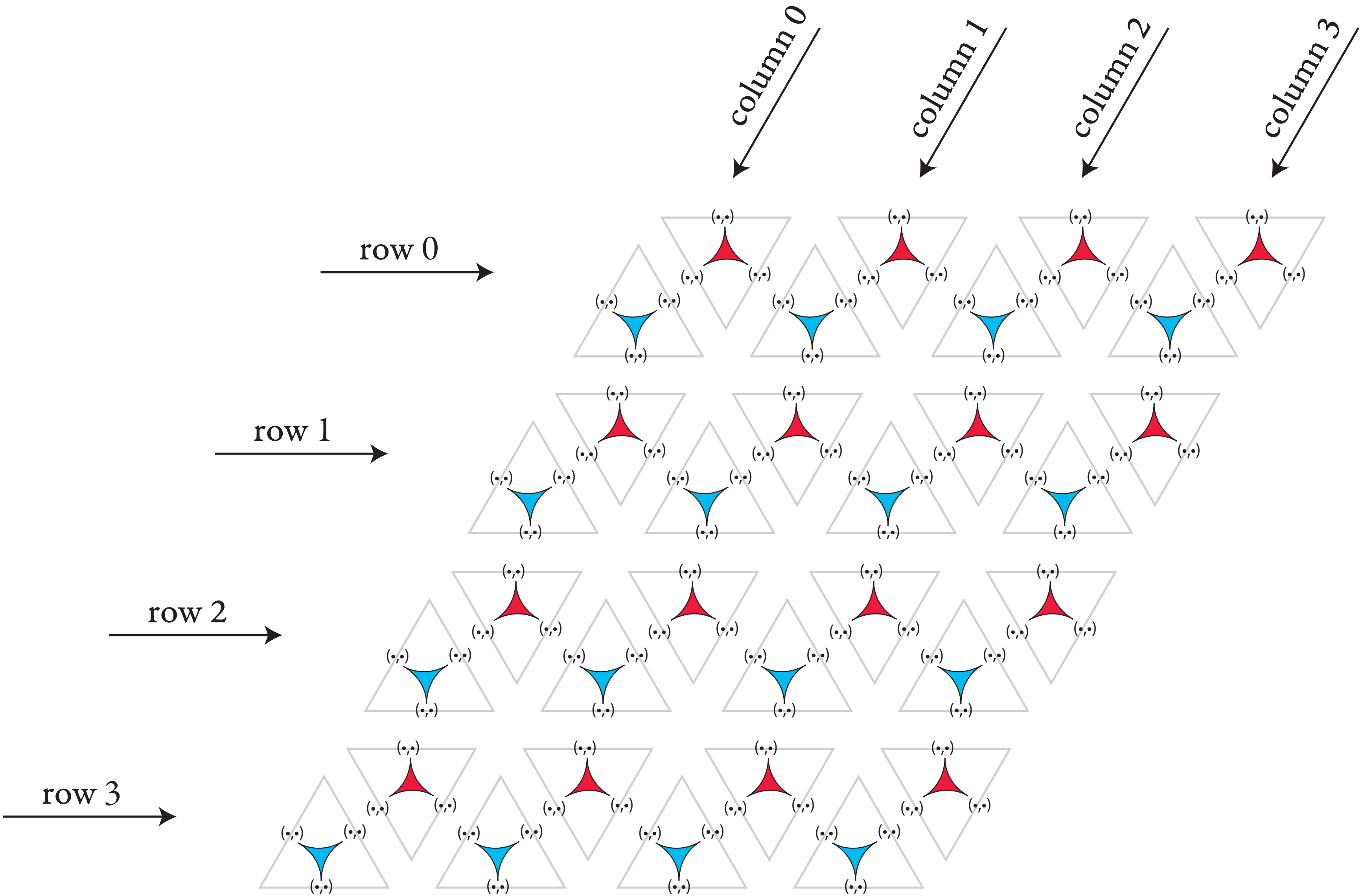
$$a_1 \longleftrightarrow a_{n+1}$$


(2) In each class string, every left semicircle is matched
by a right semicircle with a different colour
(3) Every class string begins with a
right semicircle
1 1 1 1
2 2 2
3 3 3 3
4 4 4 4 4
class string of
data value 1
class string of
data value 2
class string of
data value 3
class string of
data value 4

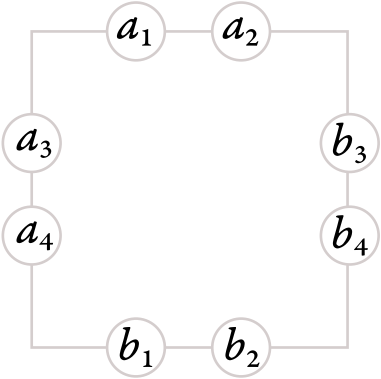



row 0
row 1
row 2
row 3


1
column i-1 mod n
column i
column i+1 mod n
row j-1 mod n
row j
row j+1 mod n
a yellow circle means that, after
forgetting the order in a pair,
this pair of atoms is the same as this one

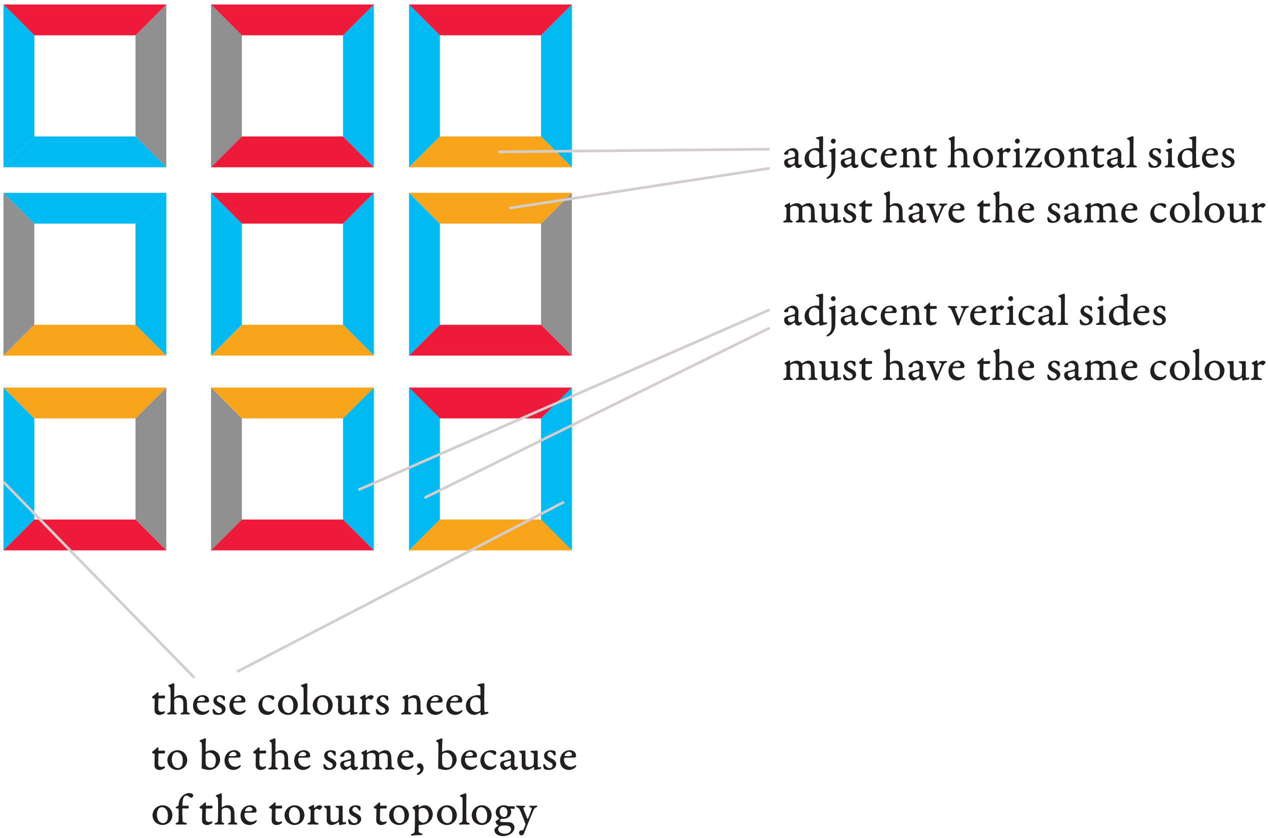


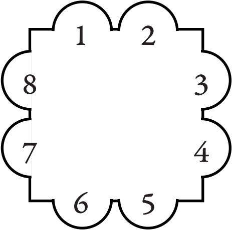
column 0
column 1
column 2
column 3
row 0
row 1
row 2
row 3

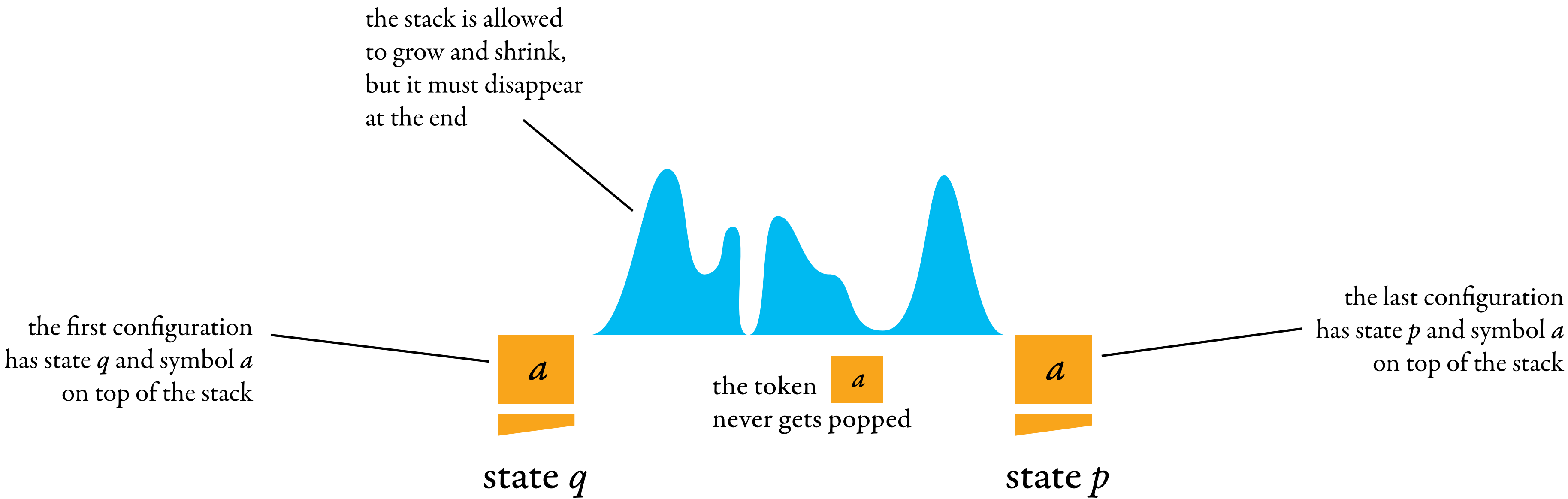

a_1
a_2
a_3
b_3
a_4
b_4
b_1
b_2

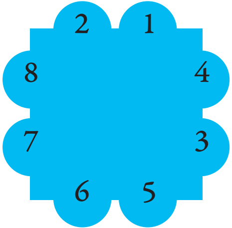
a tile

the colours

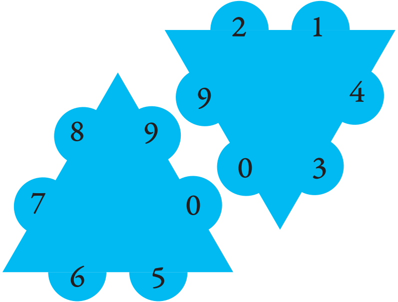

adjacent horizontal sides
must have the same colour
adjacent verical sides
must have the same colour
these colours need
to be the same, because
of the torus topology


1
2
3
4
5
6
7
8

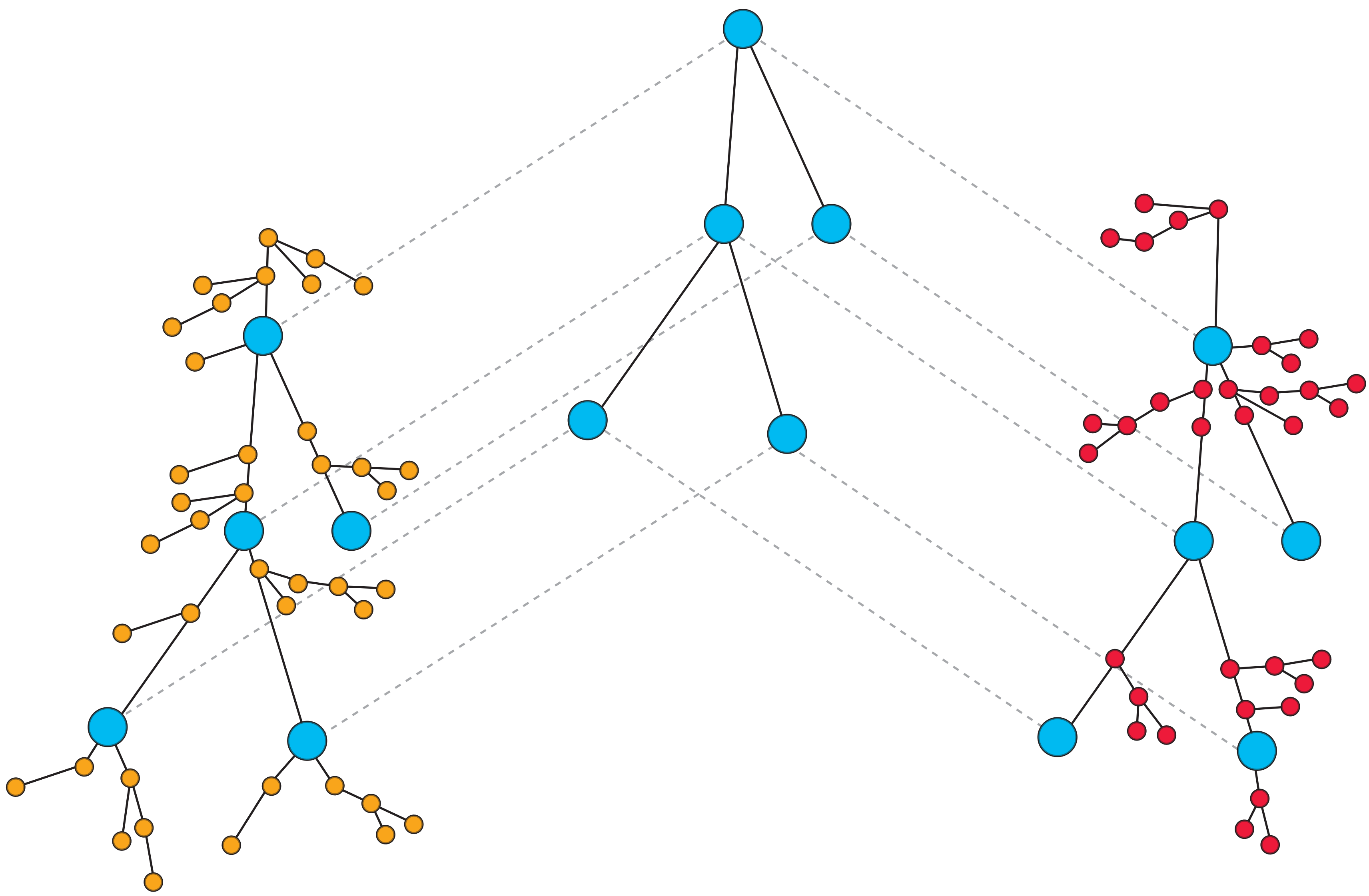

the stack is allowed
to grow and shrink,
but it must disappear
at the end
the first configuration
has state q and symbol a
on top of the stack
a
the token
a
never gets popped
a
the last configuration
has state p and symbol a
on top of the stack
state q
state p



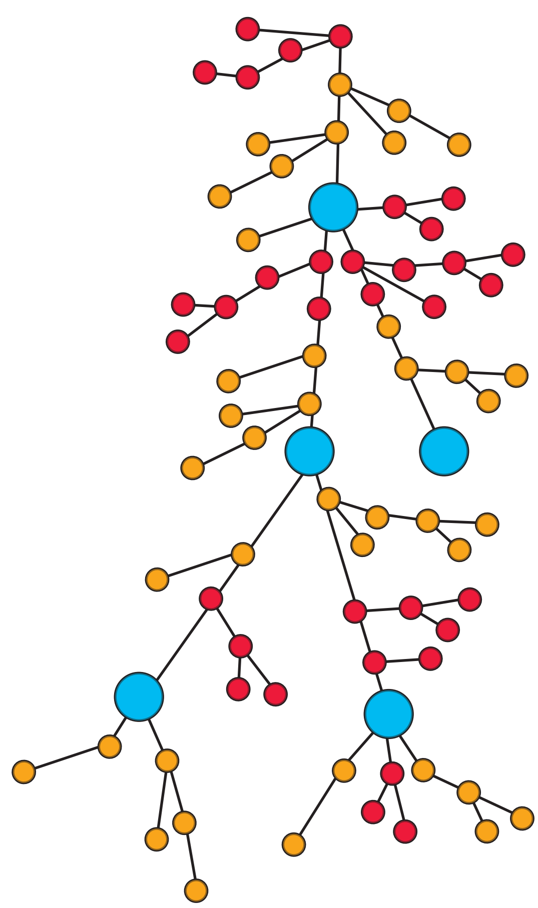
is coded as

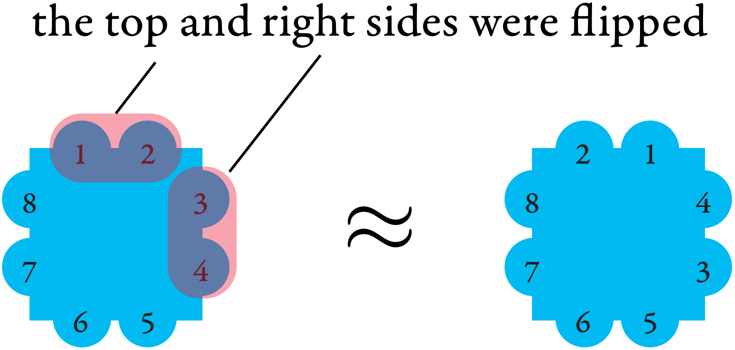


an instance of amalgamation

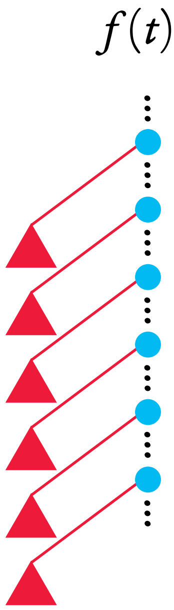


its solution




the top and right sides were flipped
1
2
3
4
5
6
7
8
≈
2
1
8
4
7
3
6
5

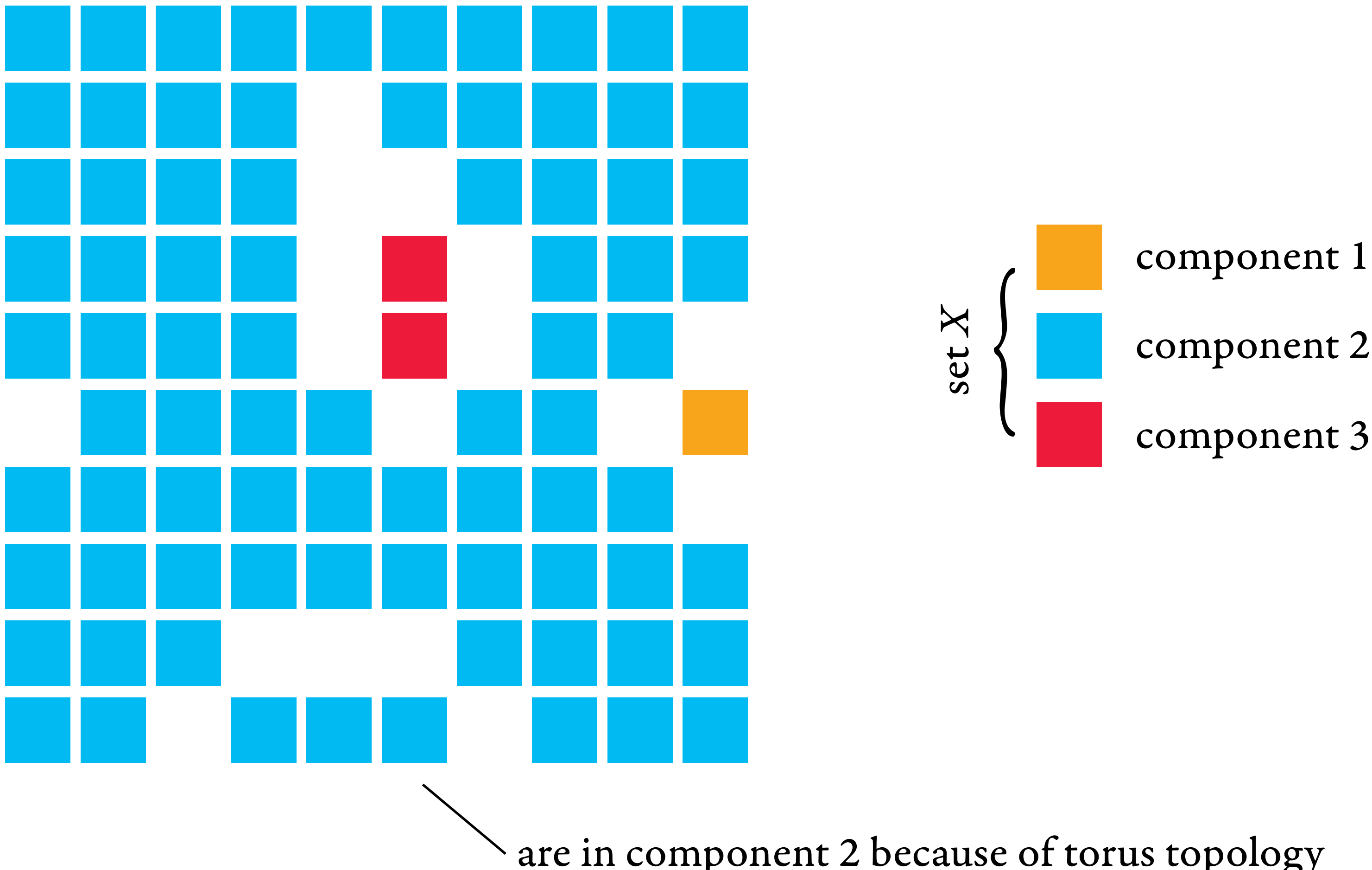
Each blue point gets one child of the form ▲
The notion of child is a bit nonstandard, since ▲ has no minimal elements.

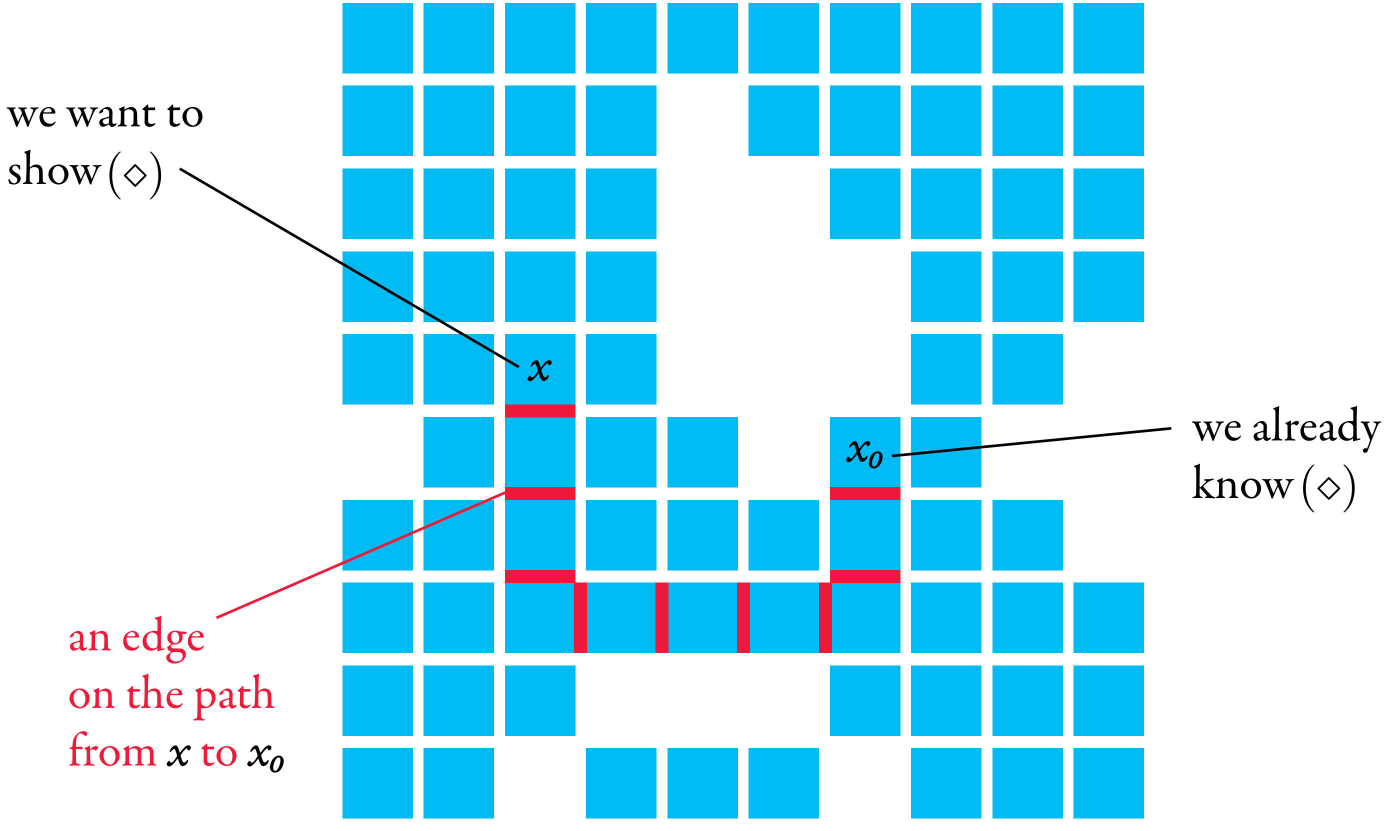


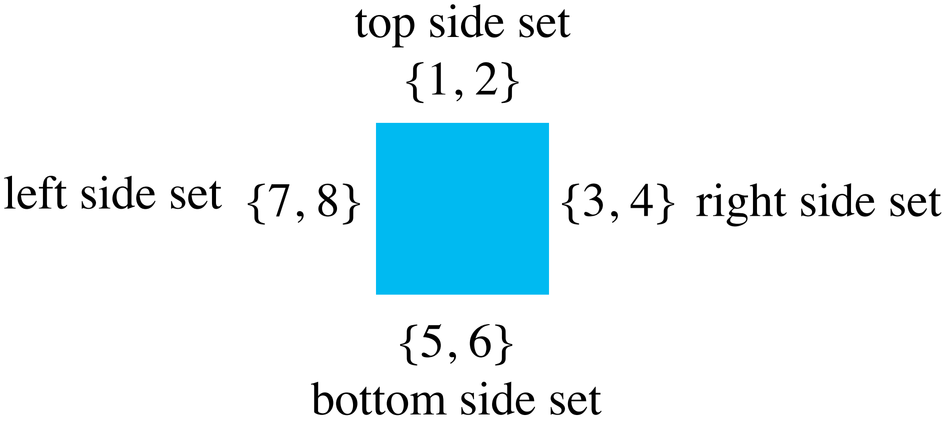
The blue points form a countable dense total order isomorphic to the rational numbers

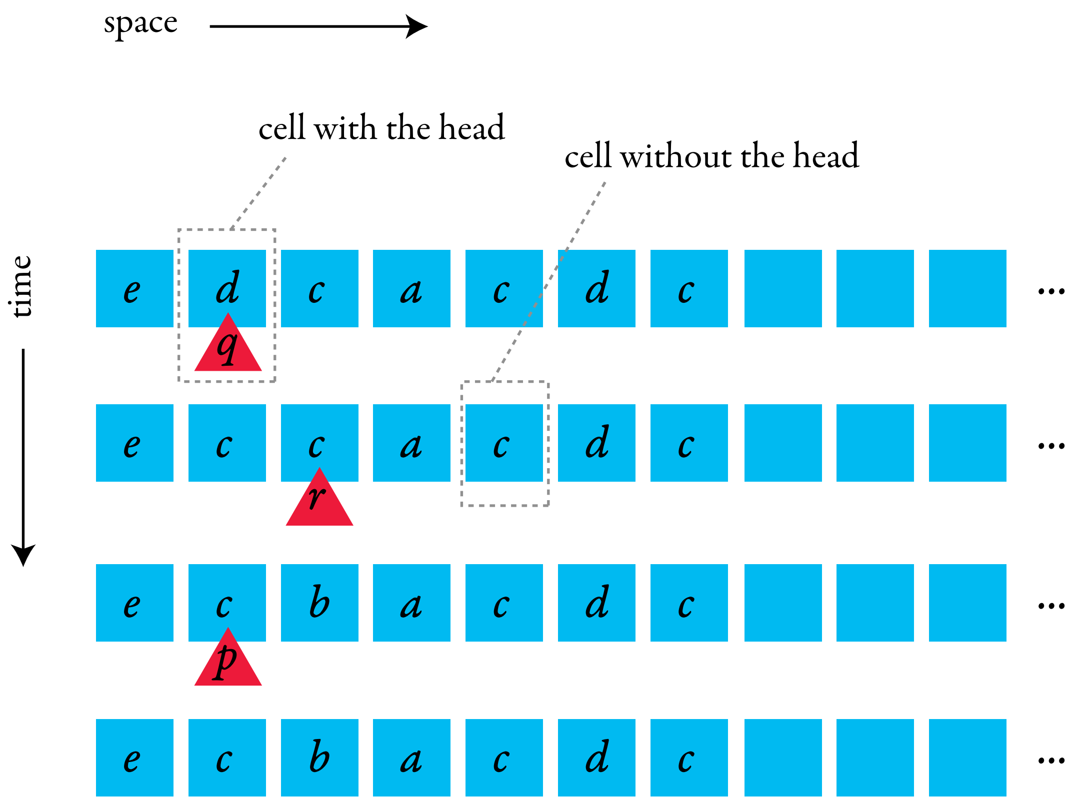
▲ = $t \; t \; t \; t \; \ldots$

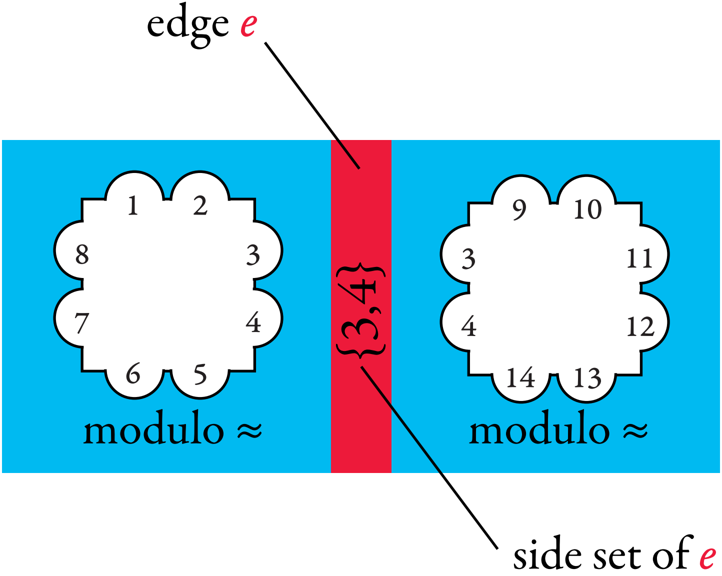
represents a forest consisting of infinitely many copies of t


component 1
set X
component 2
component 3
are in component 2 because of torus topology

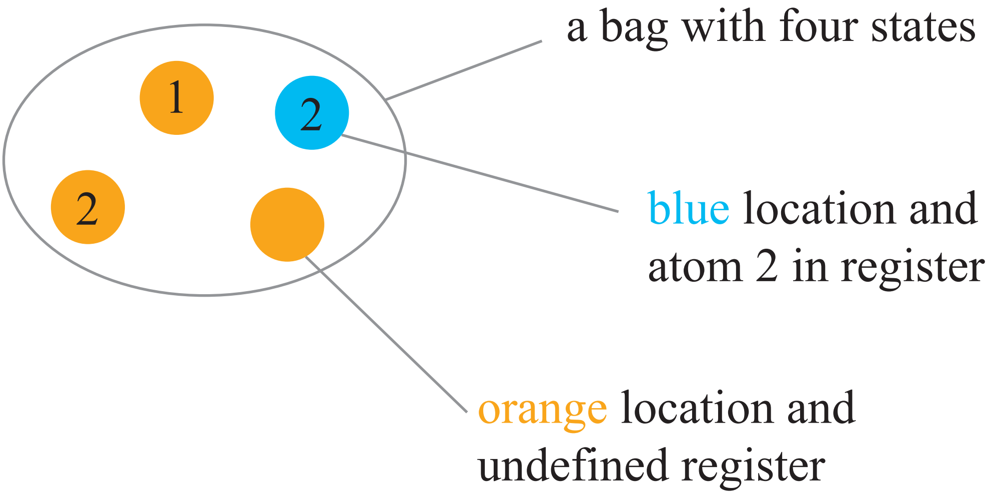

we want to show (◇)
x
x_0
we already know (◇)
an edge on the path from x to x_0

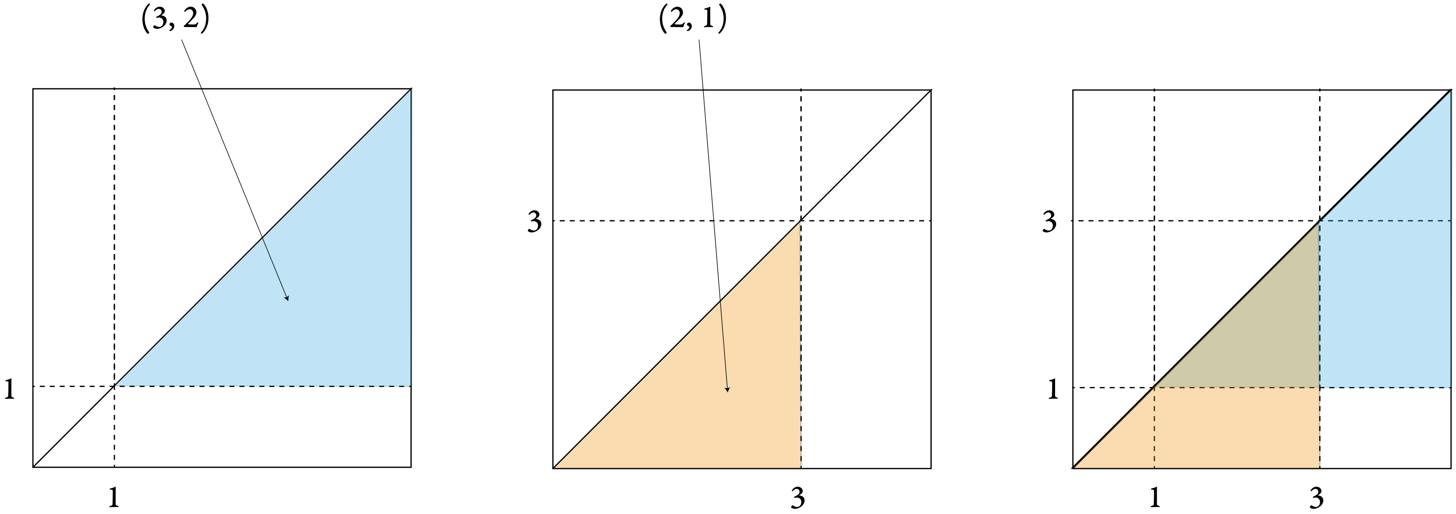

top side set
{1, 2}
left side set
{7, 8}
{3, 4}
right side set
{5, 6}
bottom side set


space
time
cell with the head
cell without the head
e d c a c d c
q
e c c a c d c
r
e c b a c d c
p
e c b a c d c

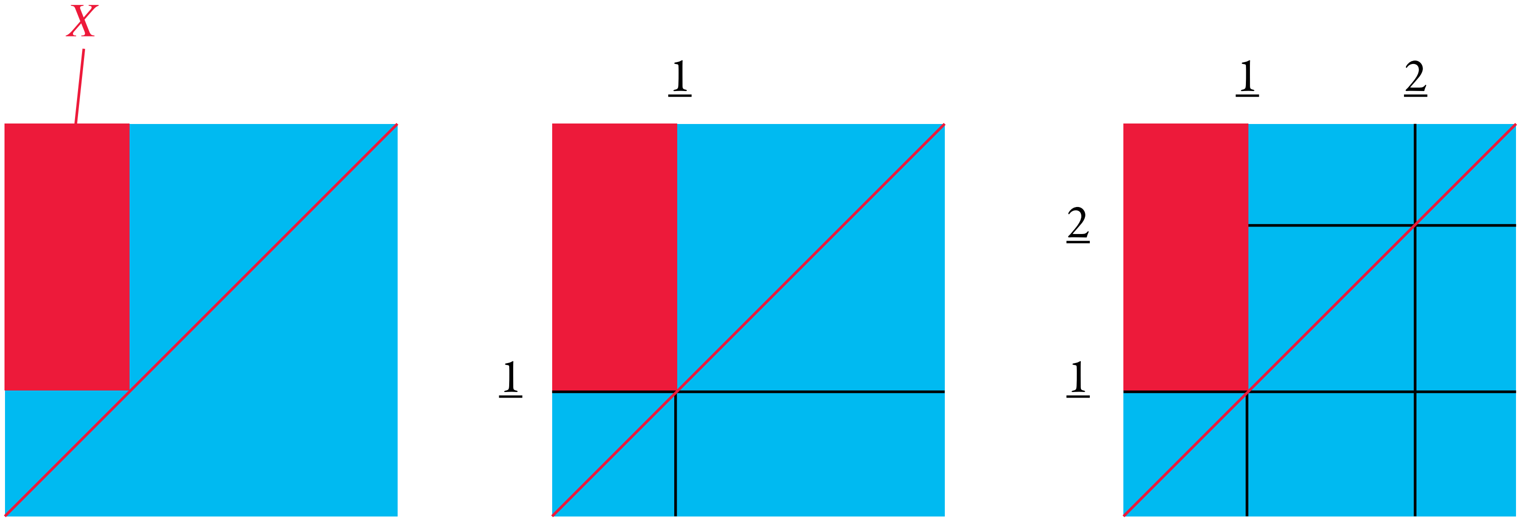

edge e
1
2
3
4
5
6
7
8
modulo ≈
{3,4}
9
10
11
12
13
14
3
4
modulo ≈
side set of e

a sequence of atoms	2		1		1		9		1
position numbers in binary	1		10		11		100		101
deatomisation	1	#	10	#	10	#	100	#	10

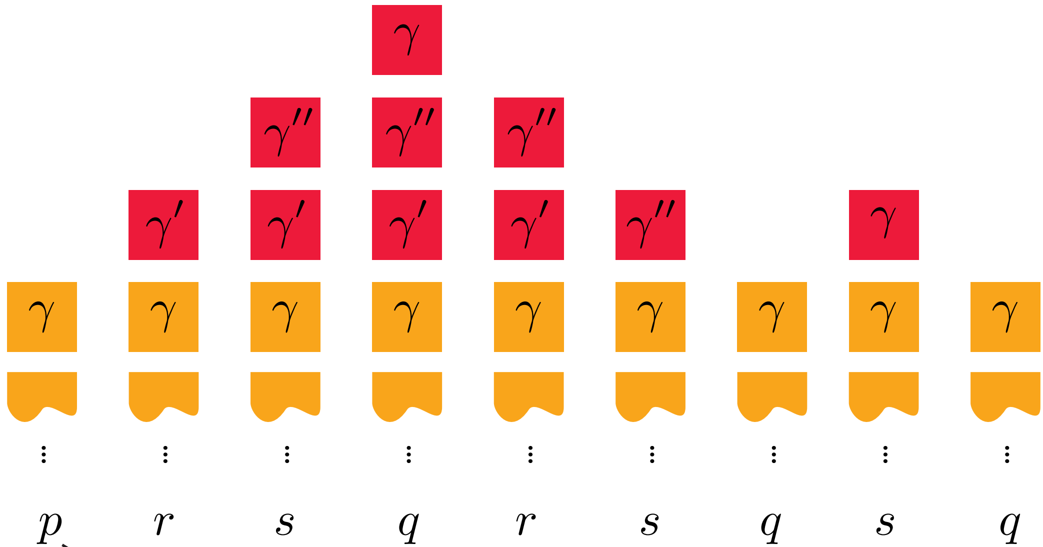

a bag with four states
1
2
2
blue location and atom 2 in register
orange location and undefined register



(3, 2)
1
1
(2, 1)
3
3
3
1
1
3

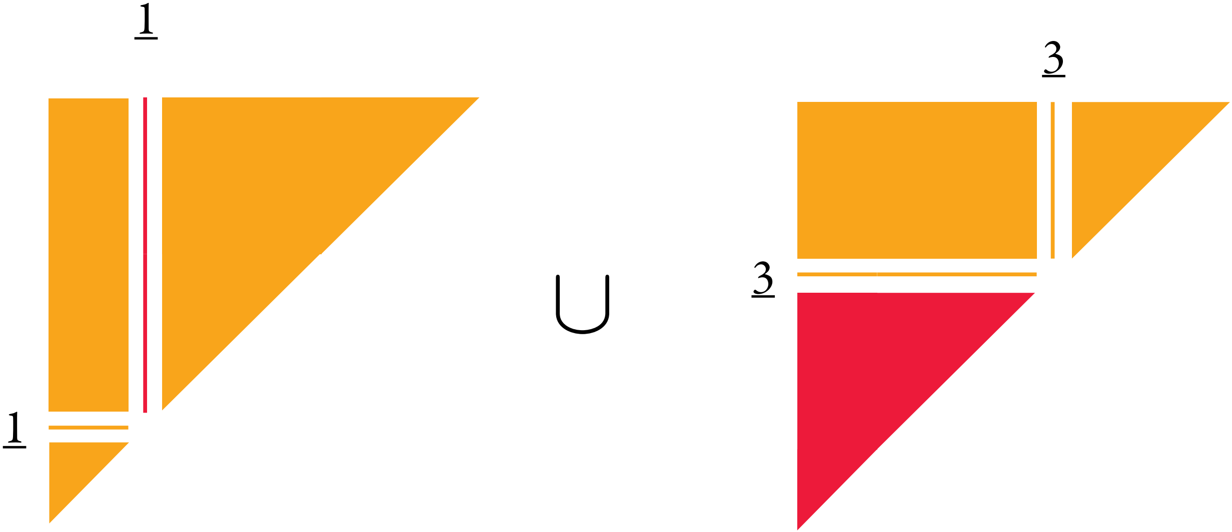

X
1
1
1
2
2
1

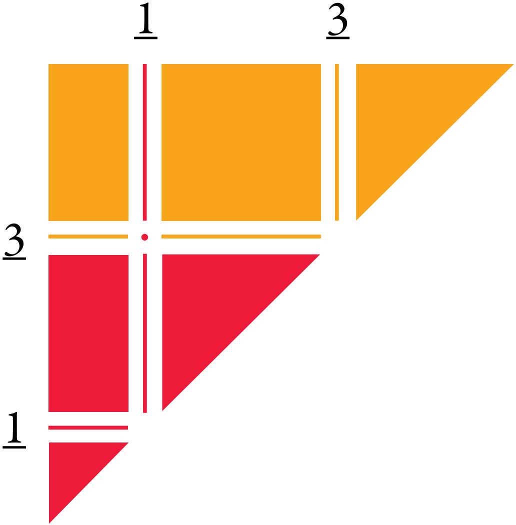
the initial part of the stack remains unchanged through the run



begins with state p and γ on the top of the stack (but possibly other symbols below)

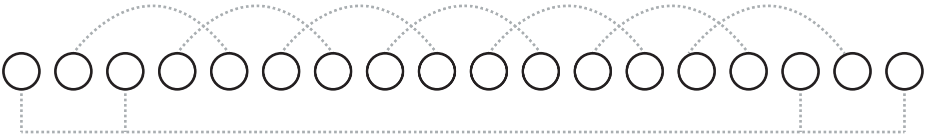
ends with state q and initial stack

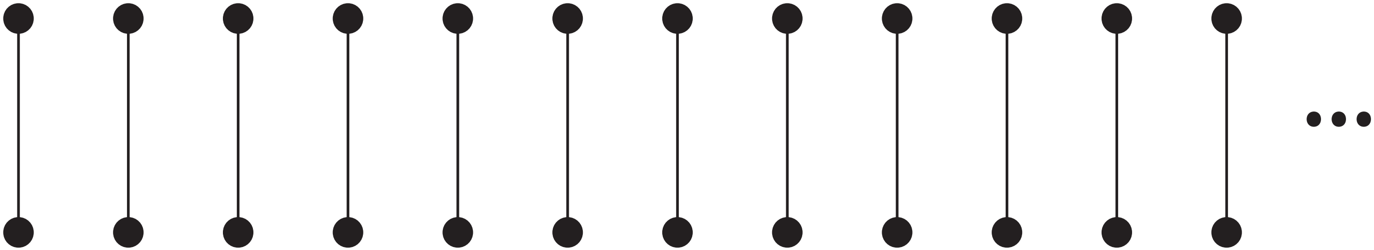

$\underline{1}$
$\underline{1}$
$\cup$
$\underline{3}$
$\underline{3}$

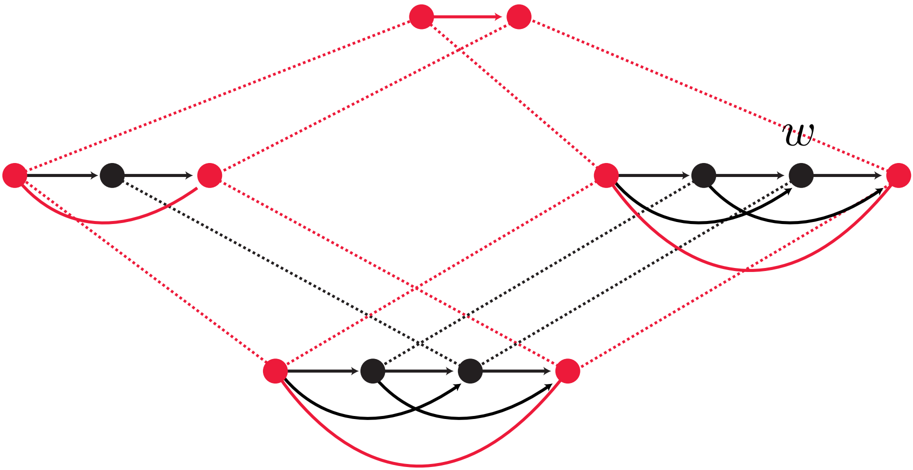


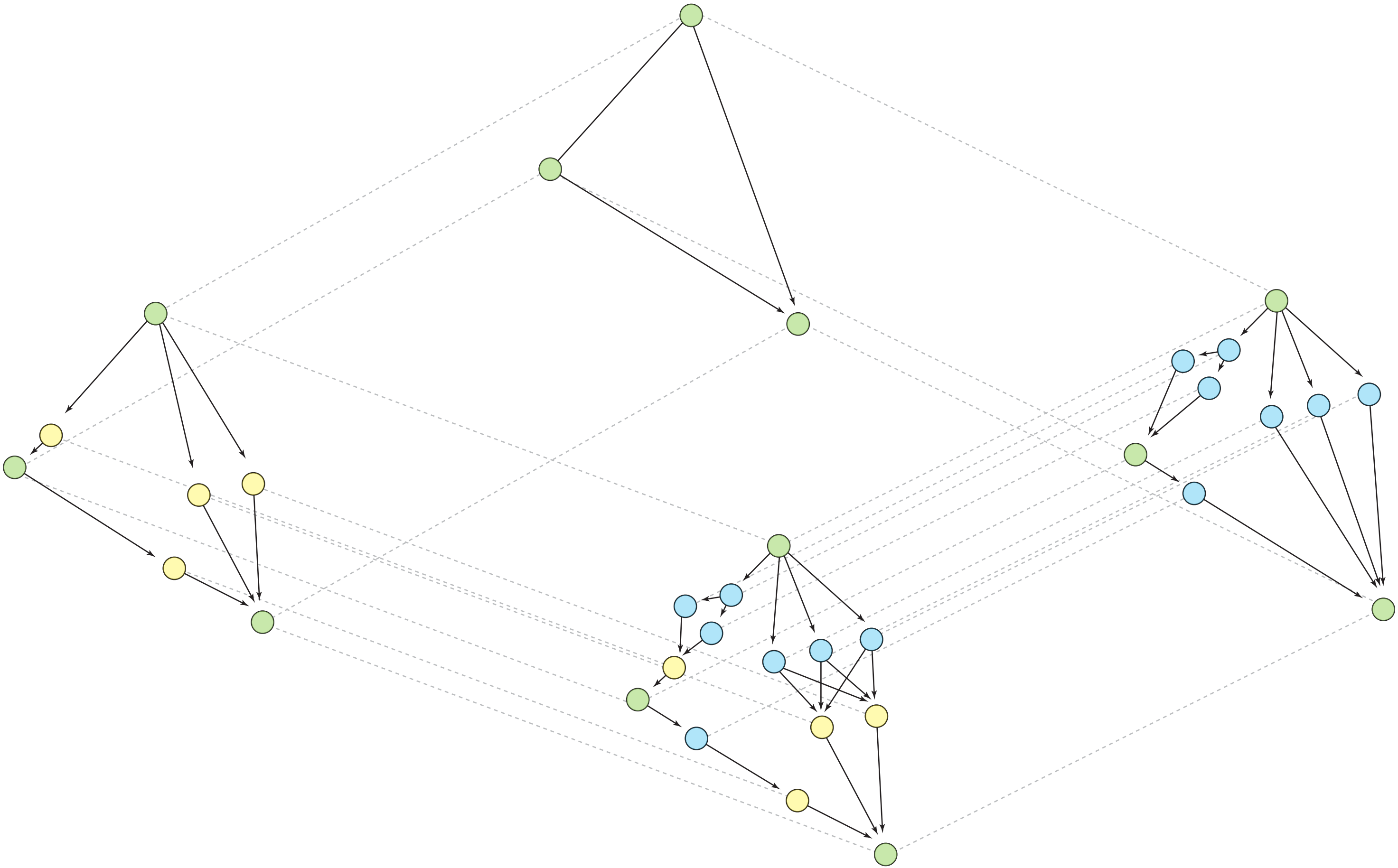
1
3
3
1

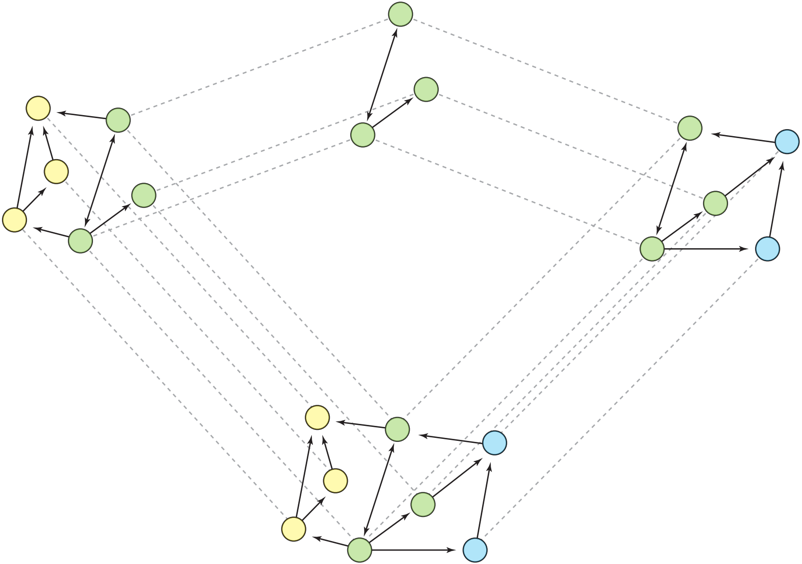
n times

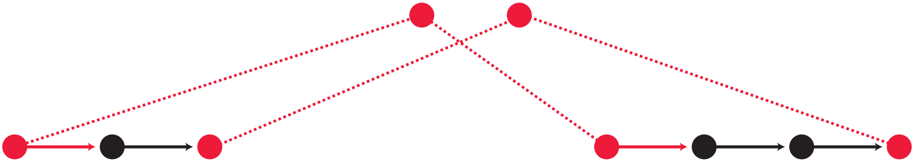


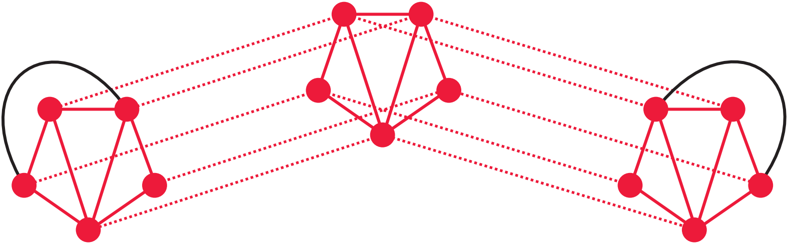


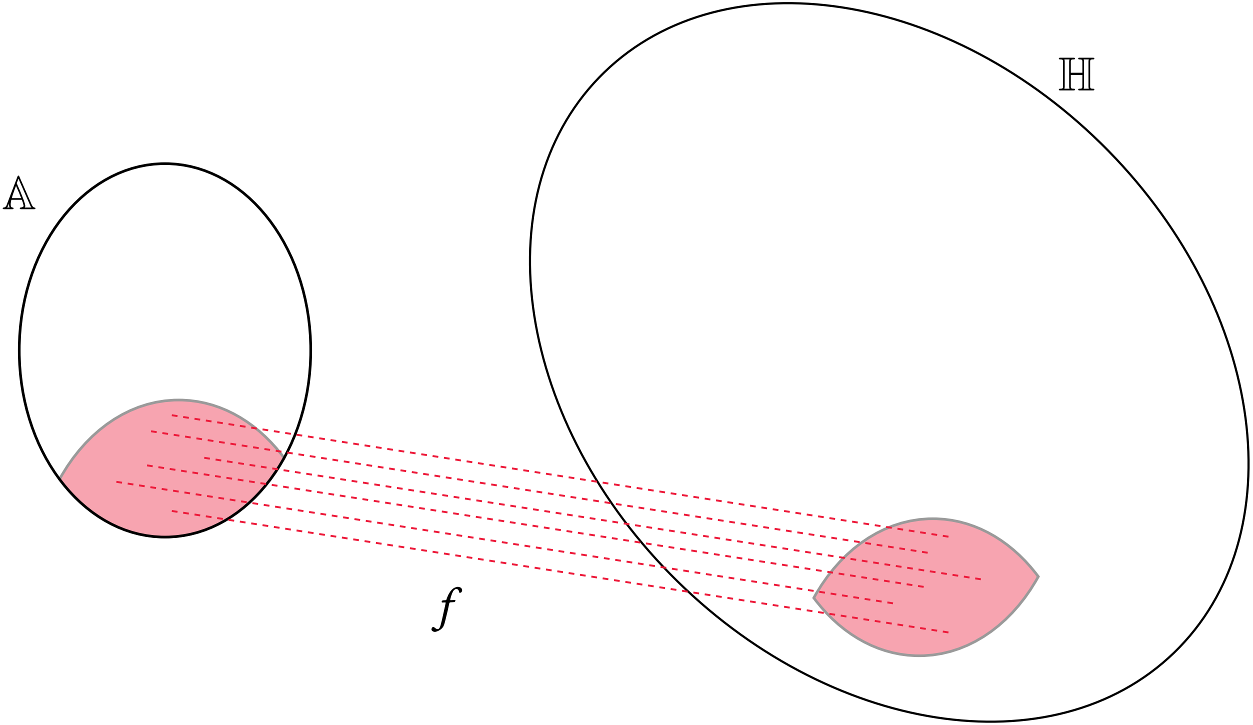

w



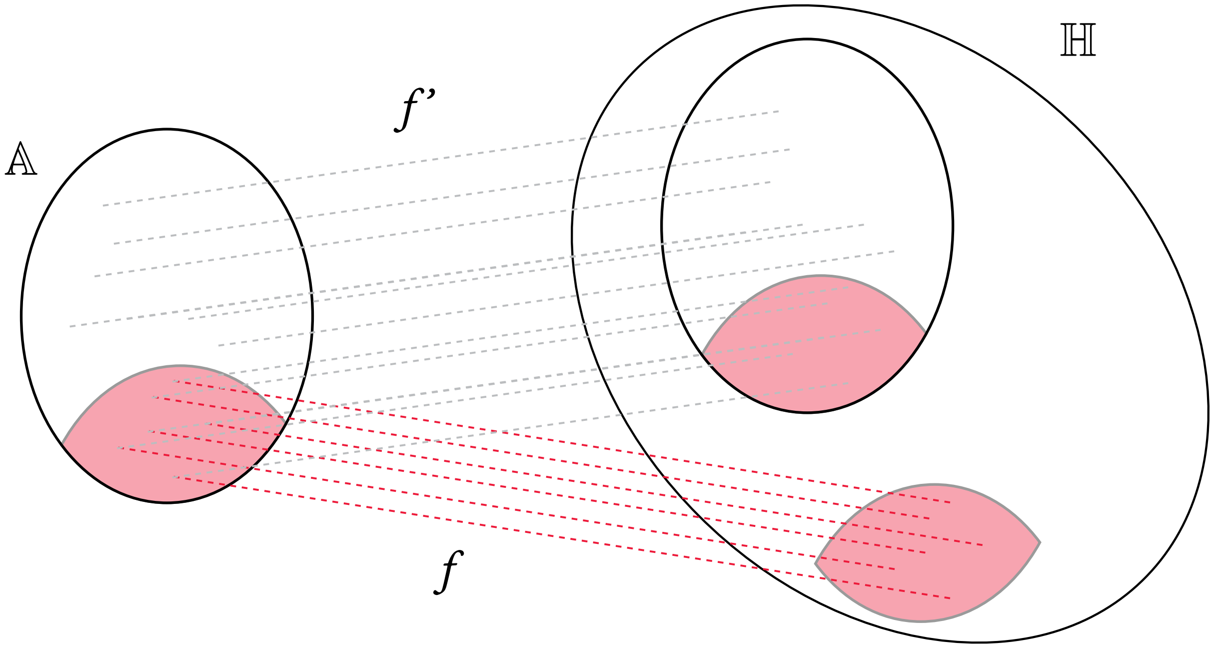








$\mathbb{A}$
$\mathbb{H}$
f

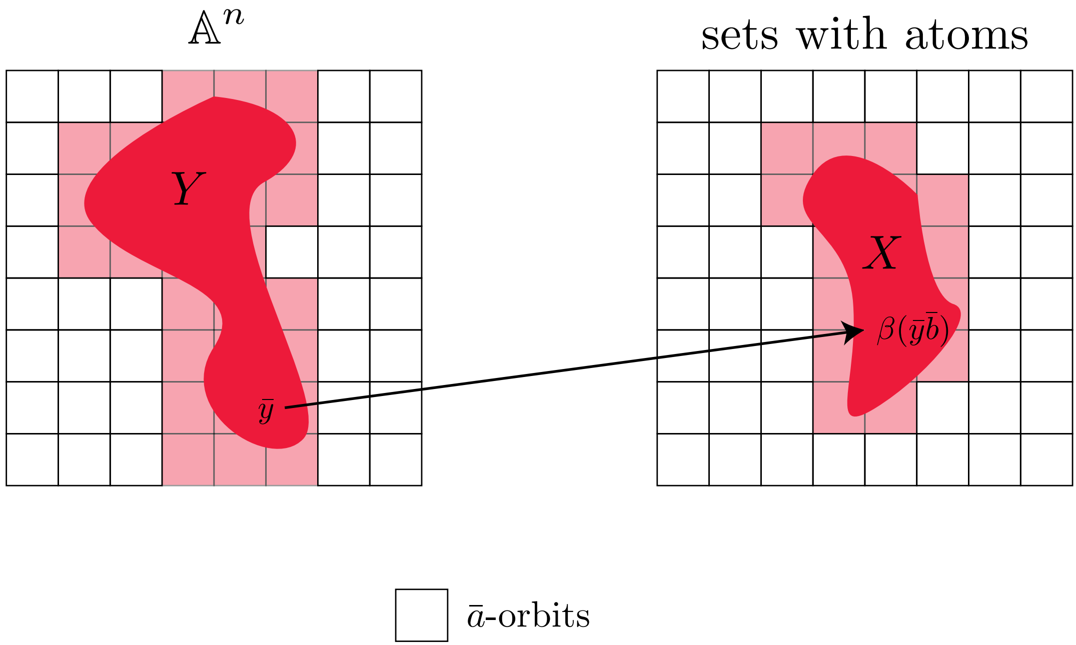


$\mathbb{A}$
f'
$\mathbb{H}$
f

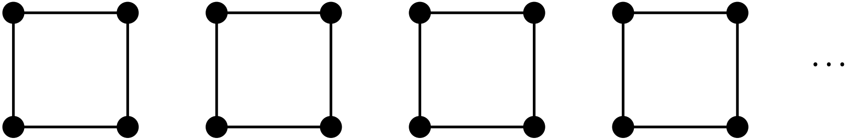
$\forall\exists$

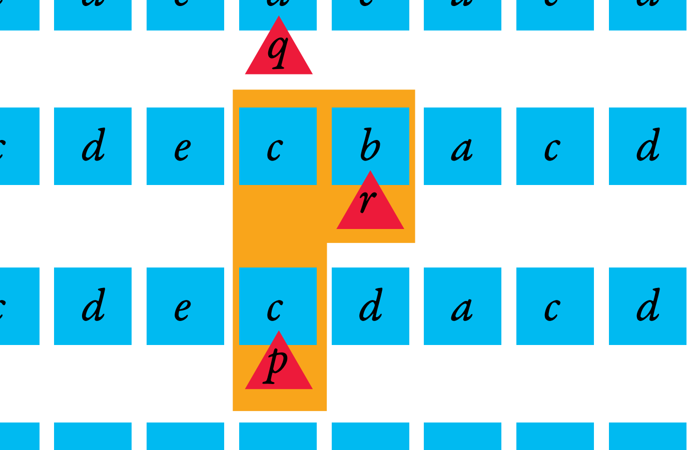
$$
\begin{tikzcd}
\overset{\text{straight}}{A} \arrow[r, "f"] \arrow[dr, two heads, "g"'] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & \overset{\text{straight}}{P} \arrow[d, two heads, "h\ \ \text{reflects supports}"] \\
& Q
\end{tikzcd}
$$

all arrows
and sets
equivariant

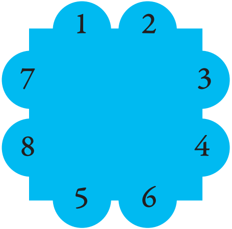


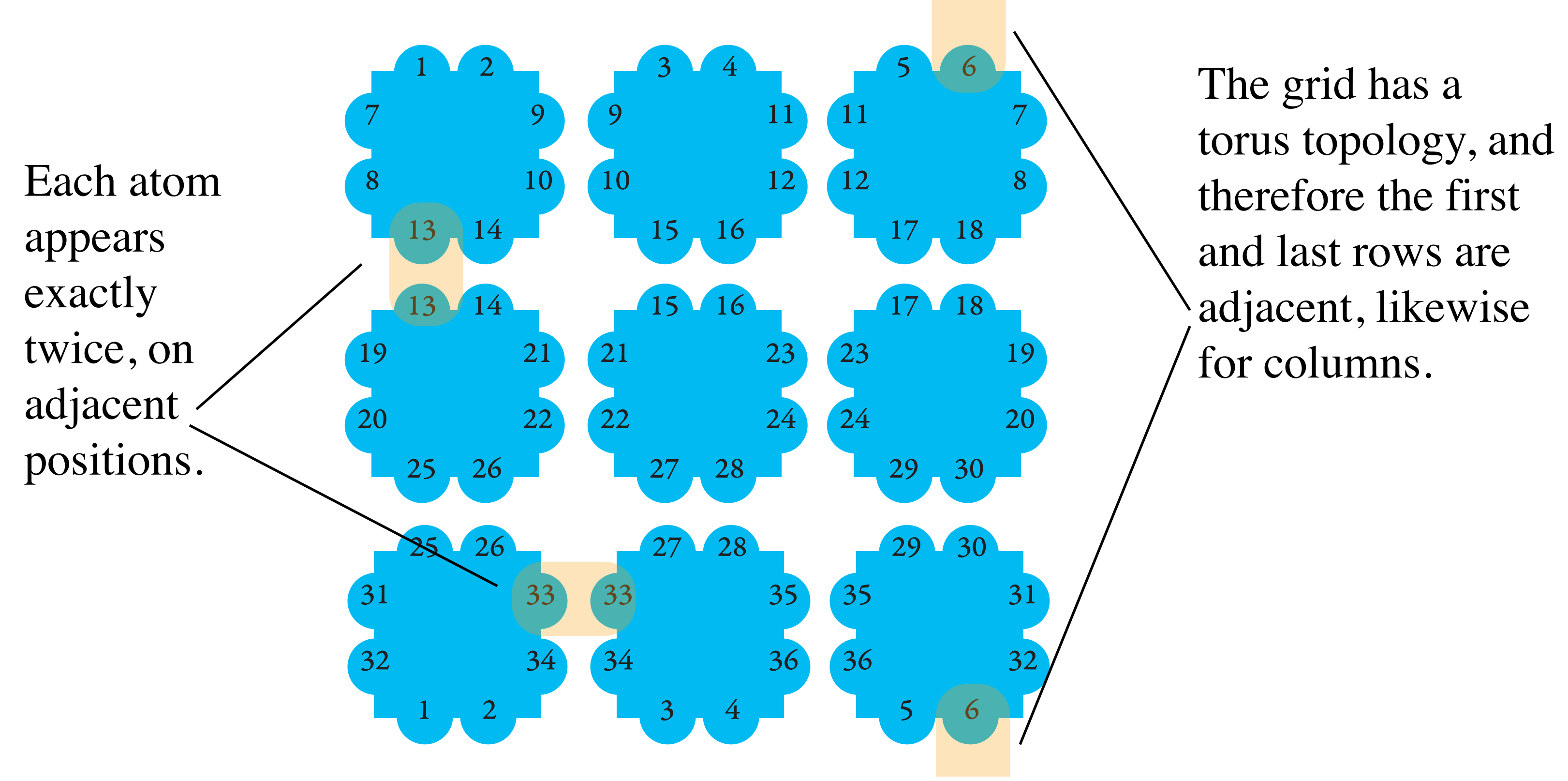
$\mathbb{A}^n$
sets with atoms
Y
$\bar{y}$
X
$\beta(\bar{y}\bar{b})$
$\bar{a}$-orbits

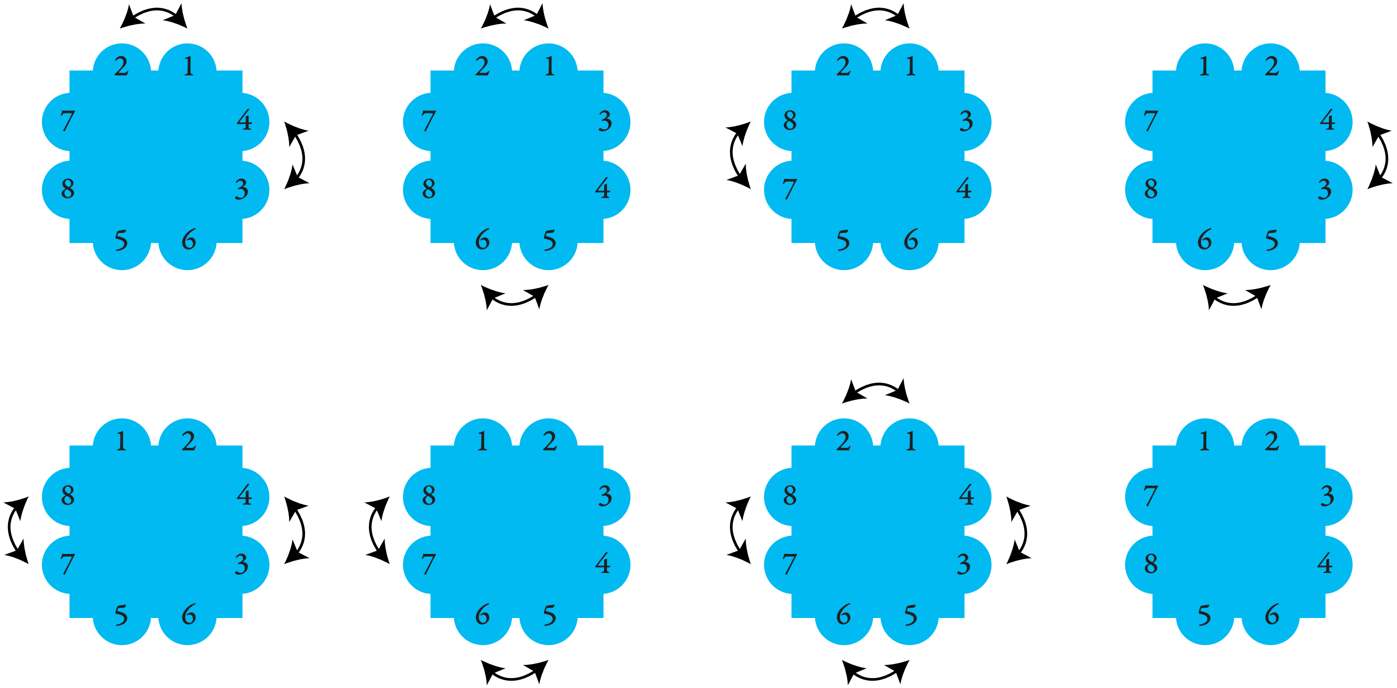




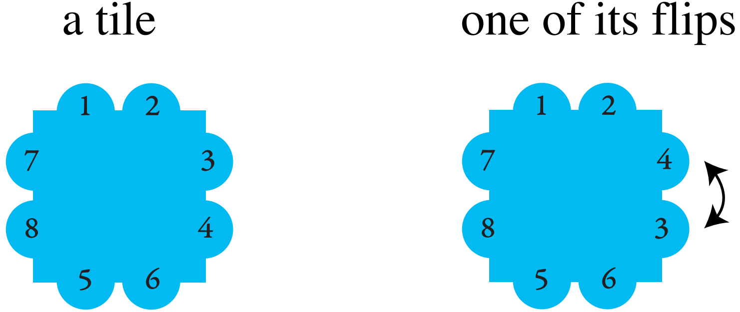
q
d
e
c
b
a
c
d
r
d
e
c
d
a
c
d
p

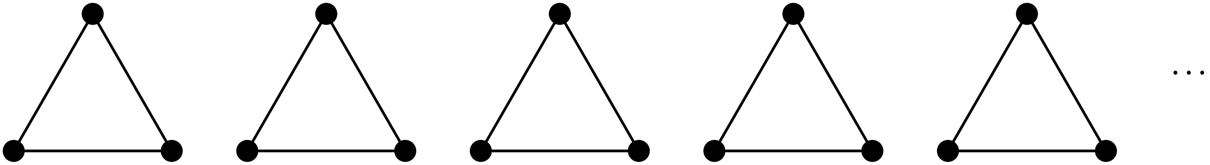
$(1, 2, 3, 4, 5, 6, 7, 8)$ is drawn as 


Each atom appears exactly twice, on adjacent positions.
The grid has a torus topology, and therefore the first and last rows are adjacent, likewise for columns.
1 2 7 9 8 10 13 14
3 4 9 11 10 12 15 16
5 6 11 7 12 8 17 18
13 14 19 21 20 22 25 26
15 16 21 23 22 24 27 28
17 18 23 19 24 20 29 30
25 26 31 33 32 34 1 2
27 28 33 35 34 36 3 4
29 30 35 31 36 32 5 6



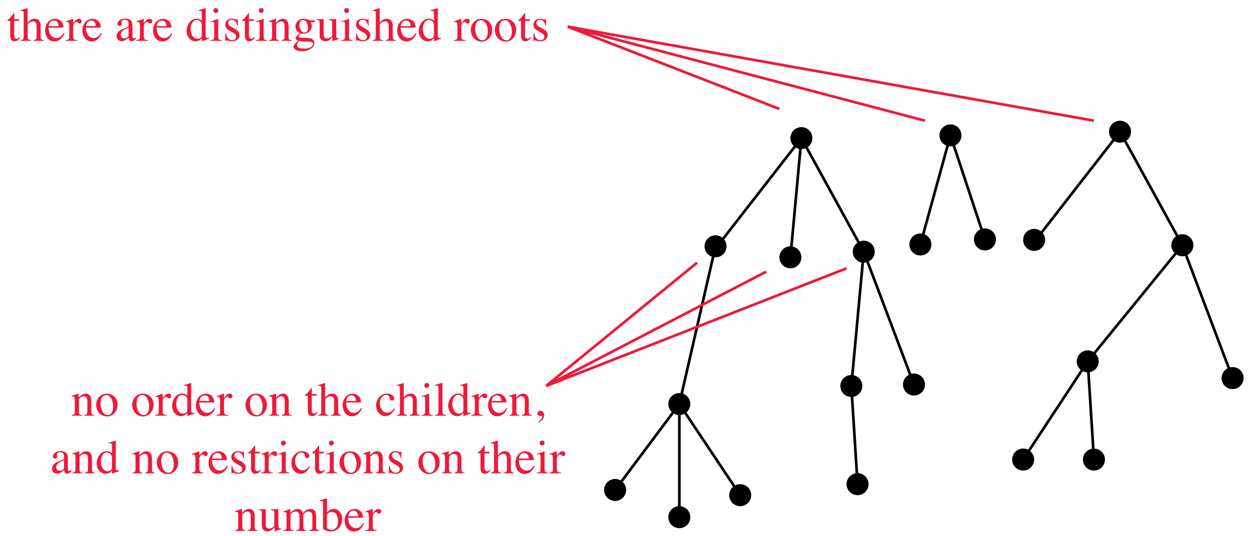
2 1
7 4
8 3
5 6
2 1
7 3
8 4
6 5
2 1
8 3
7 4
5 6
1 2
7 4
8 3
6 5
1 2
8 4
7 3
5 6
1 2
8 3
7 4
6 5
2 1
8 4
7 3
6 5
1 2
7 3
8 4
5 6

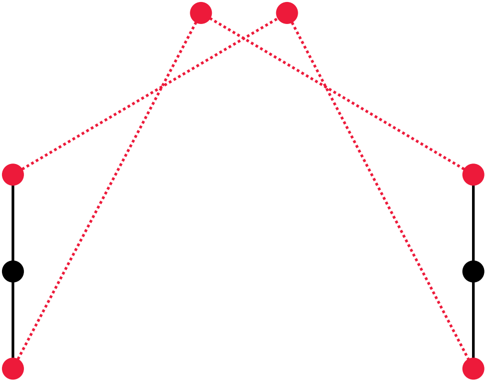

a tile
one of its flips
1
2
7
3
8
4
5
6
1
2
7
4
8
3
5
6

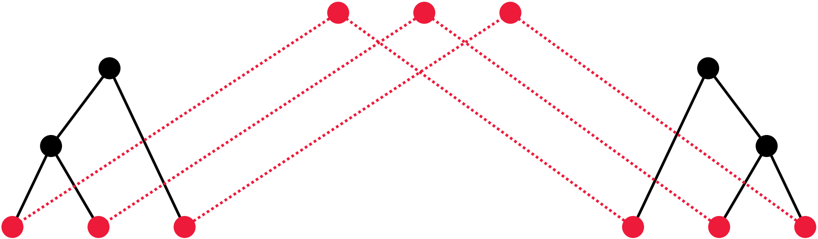


no function f makes this triangle commute

$$
\begin{tikzcd}
\text{atoms} \arrow[r, "f"] \arrow[dr, two heads, "\text{containing triangle}"'] & \text{pairs of distinct atoms in the same triangle} \arrow[d, two heads, "\text{containing triangle}"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \text{triangles}
\end{tikzcd}
$$

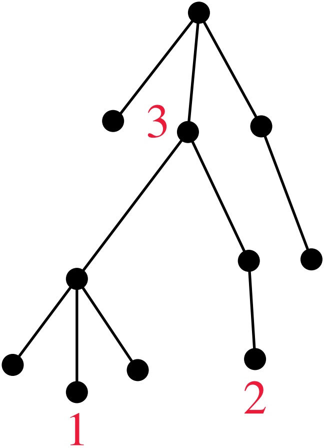

there are distinguished roots
no order on the children,
and no restrictions on their
number





$$cca(1, 2) = 3$$

$$cca(1, 3) = 3$$



variables

$$\{\{y : \text{ for } y \in \mathbb{A} \text{ such that } y \neq x \vee y = \underline{3}\} : \text{ for } x \in \mathbb{A} \text{ such that } x \neq \underline{2}\}$$

parameters

$$\{\{\{a : \text{for } a \in \mathbb{A} \text{ such that } a \neq b\} : \text{for } b \in \mathbb{A} \text{ such that } b \neq a\} : \text{for } a \in \mathbb{A}\}$$

has dimension 2, because variable *a* is reused.

$$\{a : \text{for } a \in \mathbb{A} \text{ such that } \exists b \; a \neq b\}$$

this variable is also counted

$$\{\underline{1}, \{\underline{1}, \underline{5}, \{\underline{6}\}\}, \{\underline{2}, \underline{4}\}\}$$

subexpressions

subformulas

$$\{\{y : \text{ for } y \in \mathbb{A} \text{ such that } y \neq x \vee y = \underline{3}\} : \text{ for } x \in \mathbb{A} \text{ such that } x \neq \underline{2}\}$$

subformulas

$$\{\{b : \text{for } b \in \mathbb{A} \text{ such that } b \neq a \vee b = \underline{3}\} : \text{for } a \in \mathbb{A} \text{ such that } a \neq \underline{2}\}$$

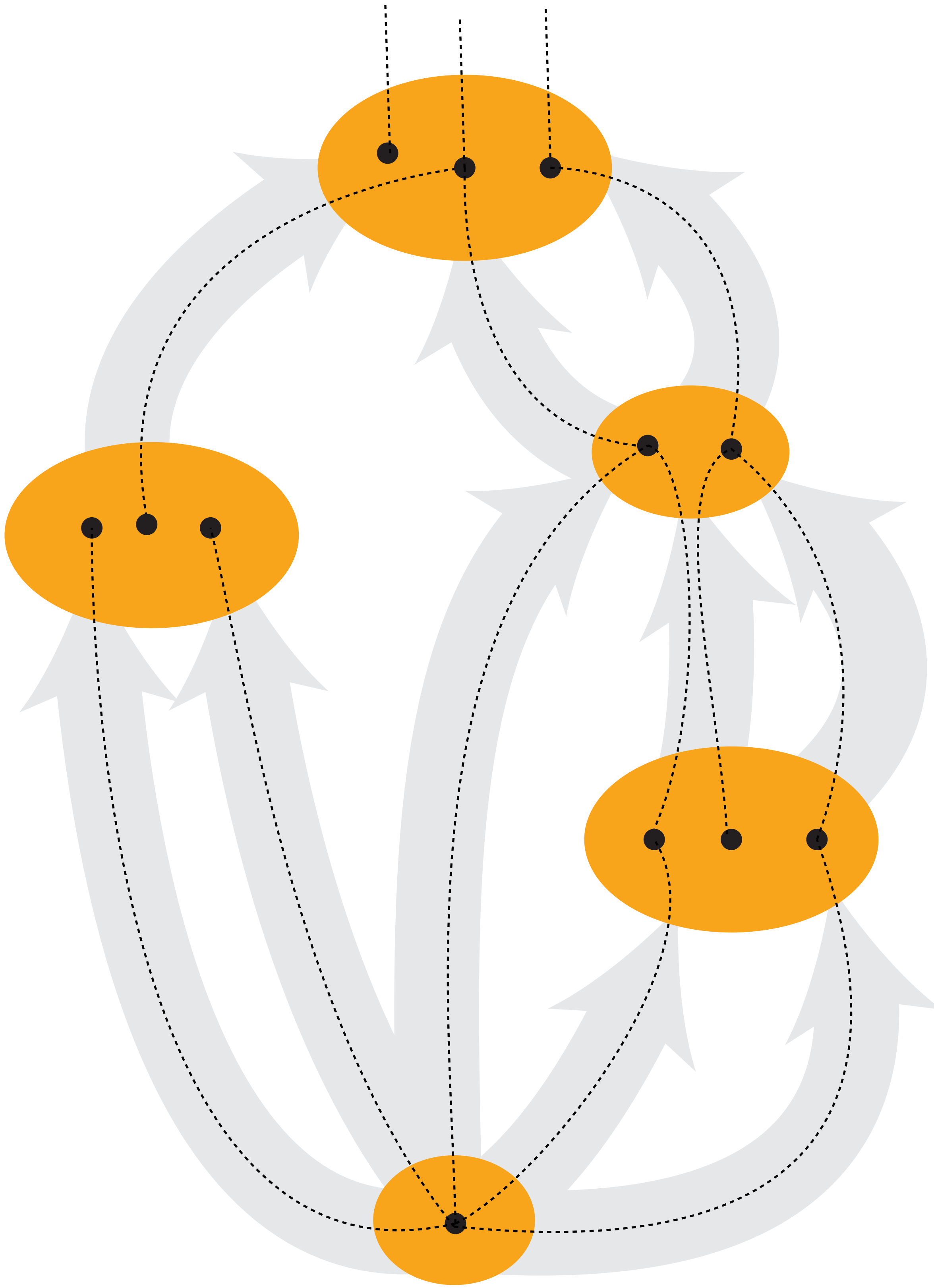
$$\{\alpha\} \cup \{\{\alpha\} \cup \{\beta\}\}$$

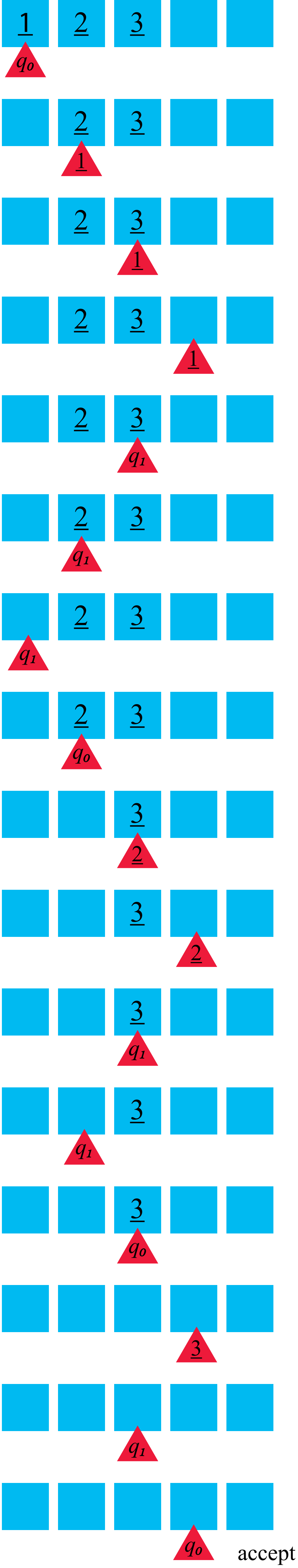
free variable y free variables x, y free variable x

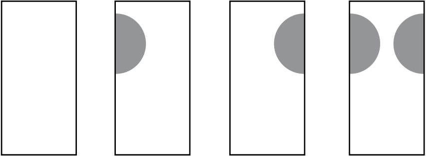
$$\{\{y : \text{ for } y \in \mathbb{A} \text{ such that } y \neq x \vee y = \underline{3}\} : \text{ for } x \in \mathbb{A} \text{ such that } x \neq \underline{2}\}$$

free variable x

no free variables




accept

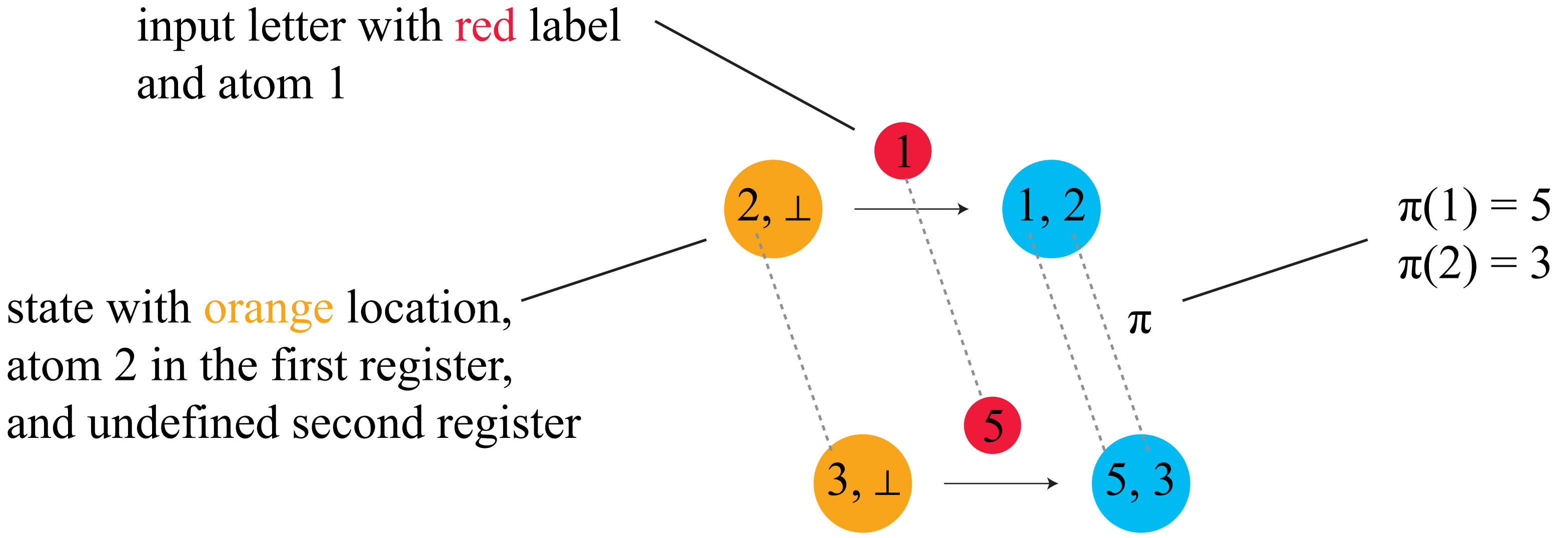


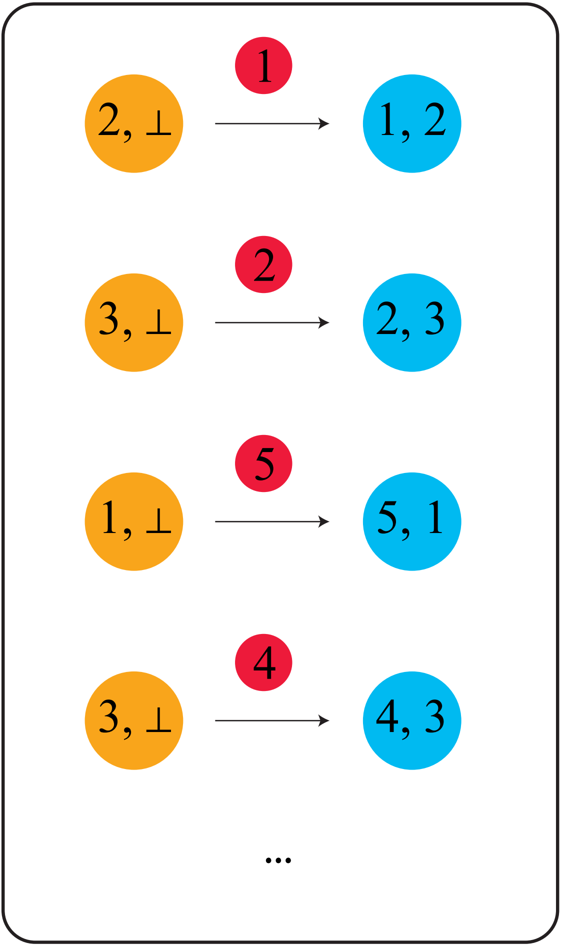
labels of w^{φ}	$\phi_1(x)$, $\phi_2(x)$, $a(y)$	$a(y)$	$b(y)$	$\phi_1(x)$, $\phi_3(x)$, $a(y)$	$\phi_1(x)$, $\phi_2(x)$, $b(y)$	$a(y)$
input labels	a	a	b	a	b	a
data values	1	1	2	3	2	2

labels from $\{a, b\}$

$$\begin{array}{cccccccccccc} a & a & b & a & b & b & a & b & b & b & a & a \\ 1 & 1 & 2 & 4 & 2 & 5 & 3 & 7 & 1 & 8 & 2 & 9 \end{array}$$

data values from $\{1, 2, 3, \ldots\}$


input letter with red label
and atom 1
1
2, ⊥
1, 2
$\pi(1) = 5$
$\pi(2) = 3$
state with orange location,
atom 2 in the first register,
and undefined second register
π
5
3, ⊥
5, 3


1
2, ⊥
1, 2
2
3, ⊥
2, 3
5
1, ⊥
5, 1
4
3, ⊥
4, 3
...

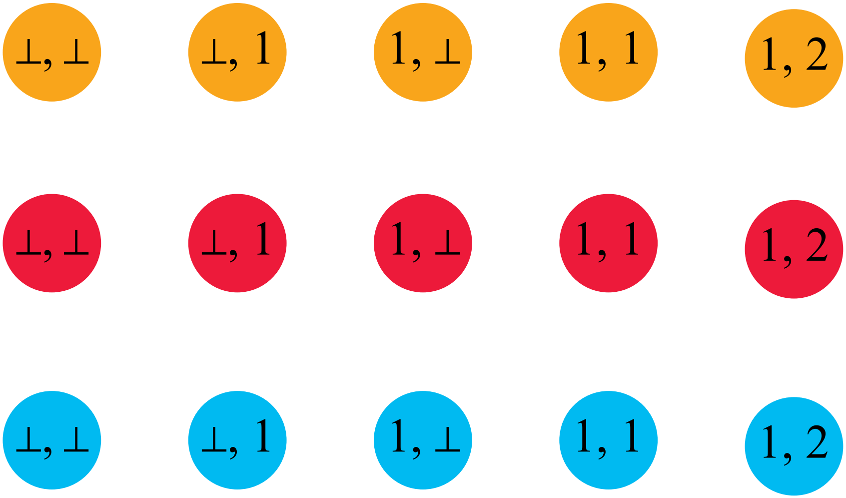
$$q(1, \perp) \xrightarrow{a(2)} p(2, 1)$$

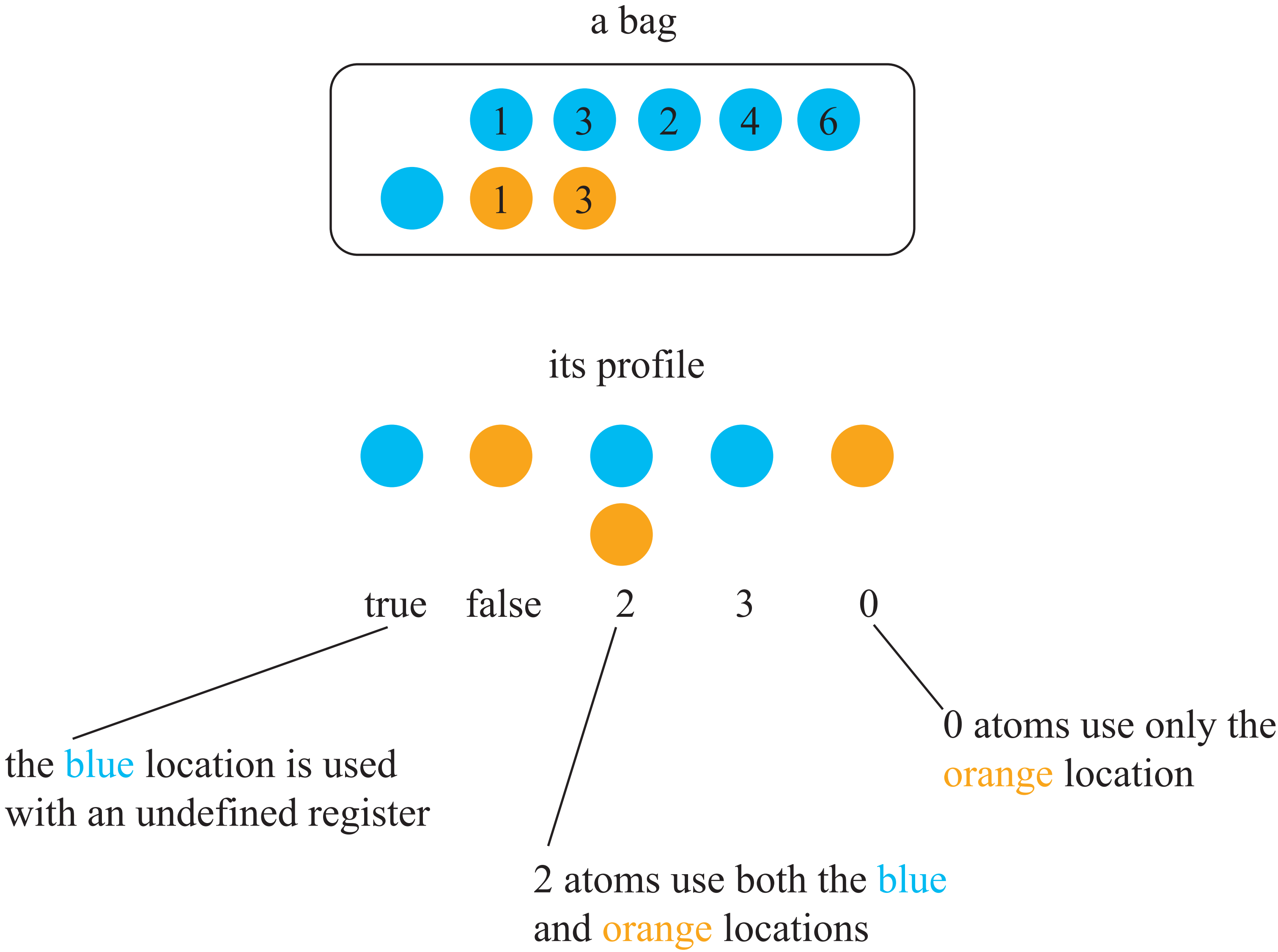
$$q(3, \perp) \xrightarrow{a(4)} p(4, 3)$$

$$q(3, \perp) \xrightarrow{a(2)} p(2, 3)$$

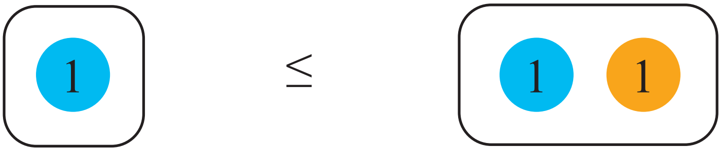
$$q(5, \perp) \xrightarrow{a(1)} p(1, 5)$$

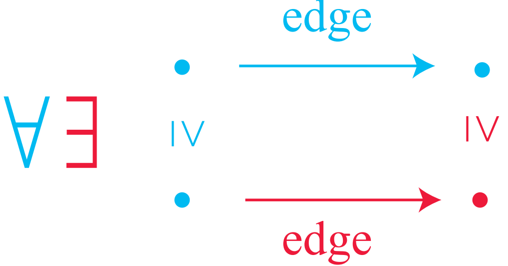
$$\ldots$$

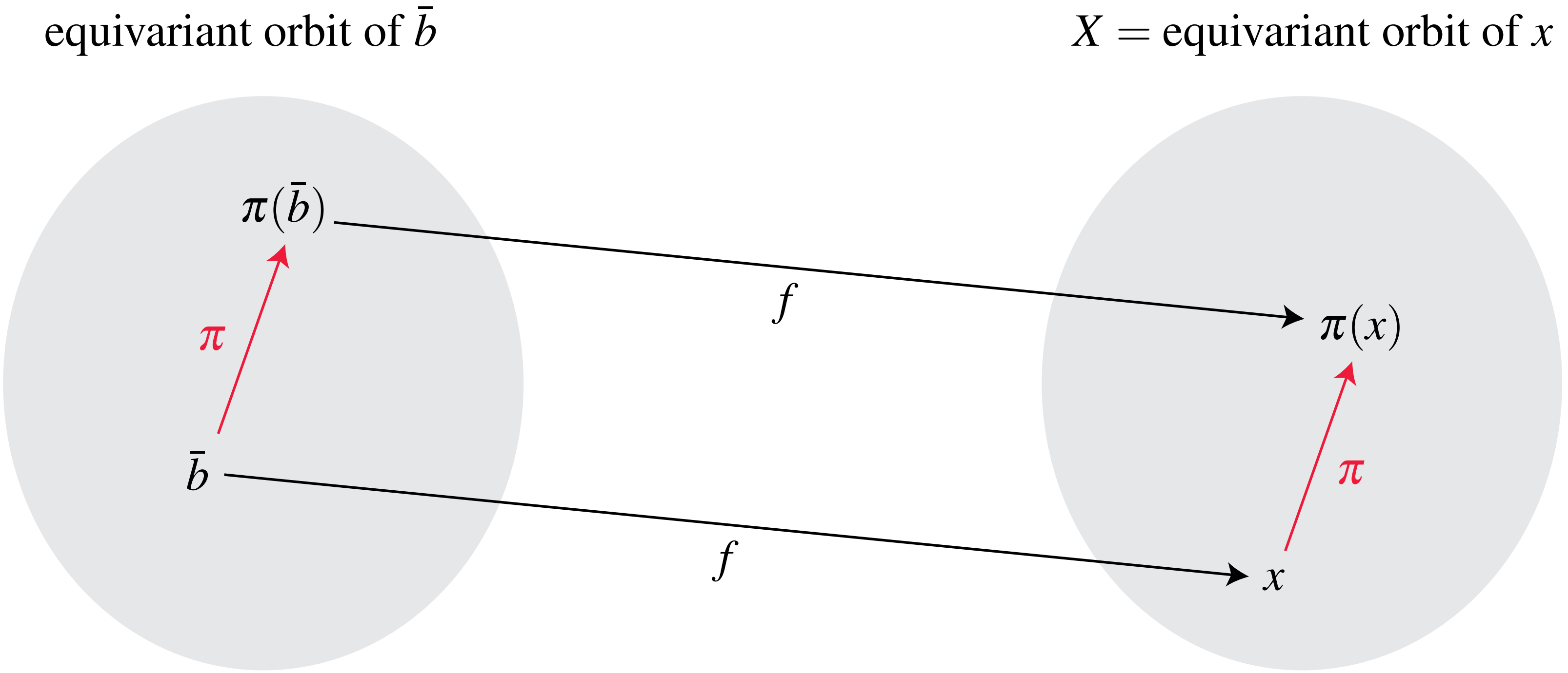

⊥, ⊥
⊥, 1
1, ⊥
1, 1
1, 2
⊥, ⊥
⊥, 1
1, ⊥
1, 1
1, 2
⊥, ⊥
⊥, 1
1, ⊥
1, 1
1, 2

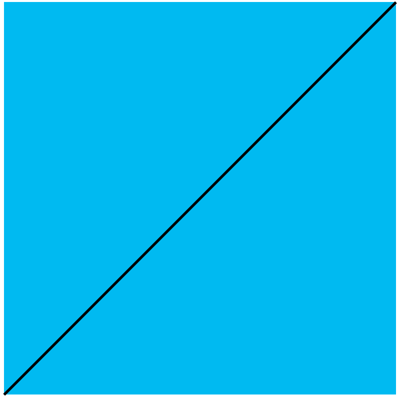


a bag
1
3
2
4
6
1
3
its profile
true
false
2
3
0
the blue location is used
with an undefined register
2 atoms use both the blue
and orange locations
0 atoms use only the
orange location

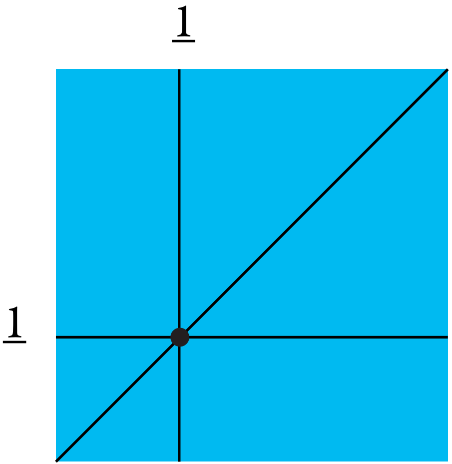

1
≤
1
1


∀ ∃
edge
edge

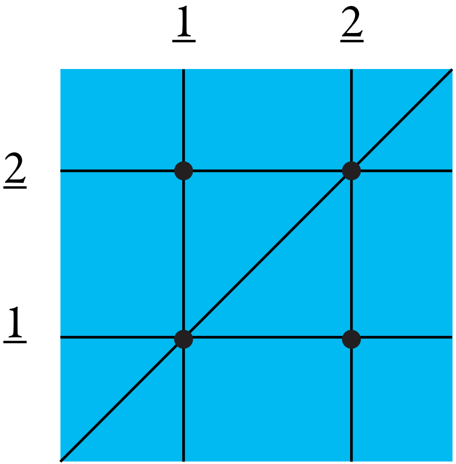

equivariant orbit of $\bar{b}$
$X =$ equivariant orbit of x
$\pi(\bar{b})$
f
$\pi(x)$
π
π
$\bar{b}$
f
x



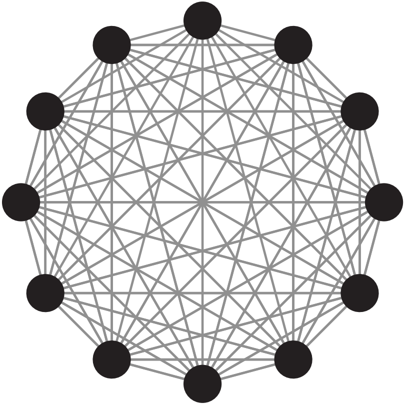
3 equivariant orbits:
2 of dimension 2
1 of dimension 1

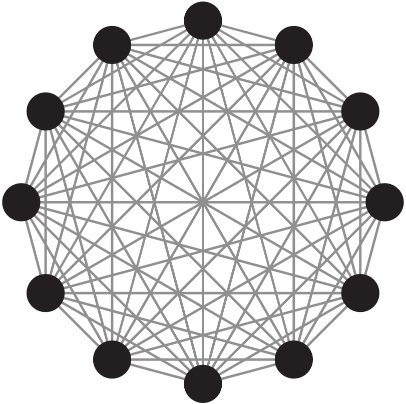


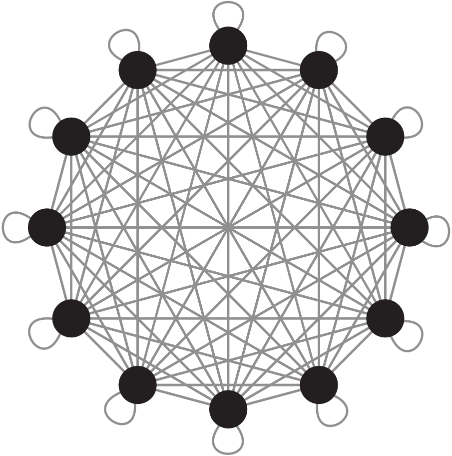
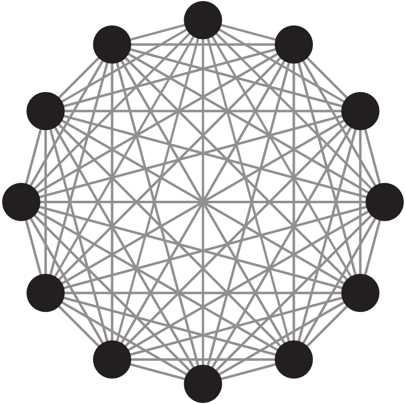
13 $\underline{1}$-orbits:
6 of dimension 2
6 of dimension 1
1 of dimension 0



31 $\underline{12}$-orbits:
12 of dimension 2
15 of dimension 1
4 of dimension 0

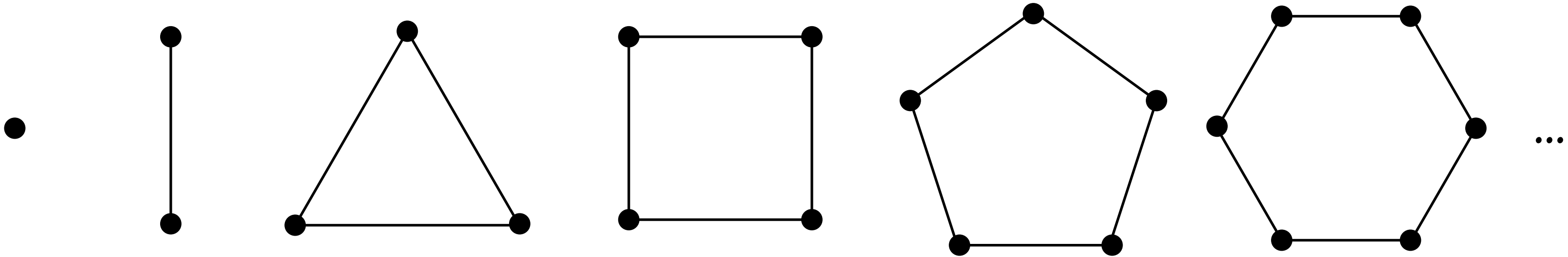


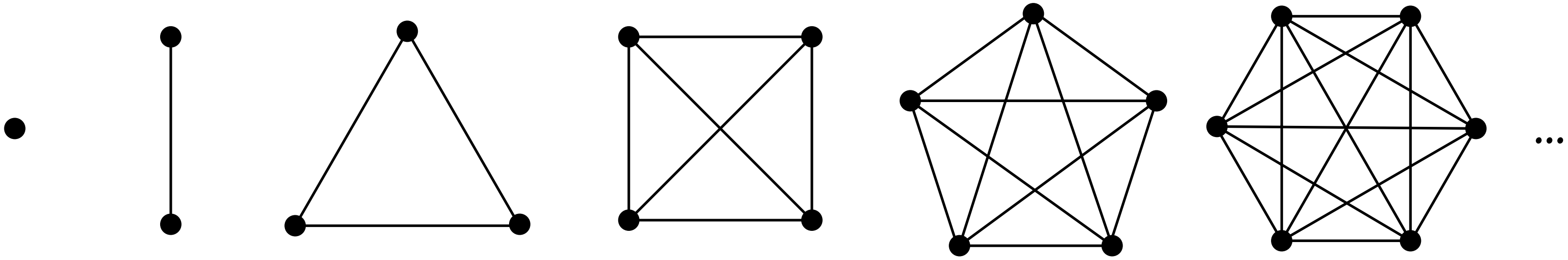


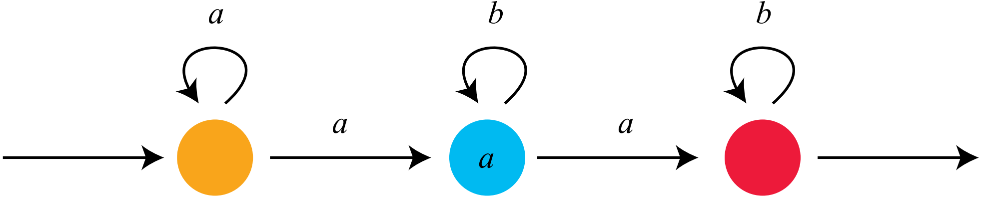



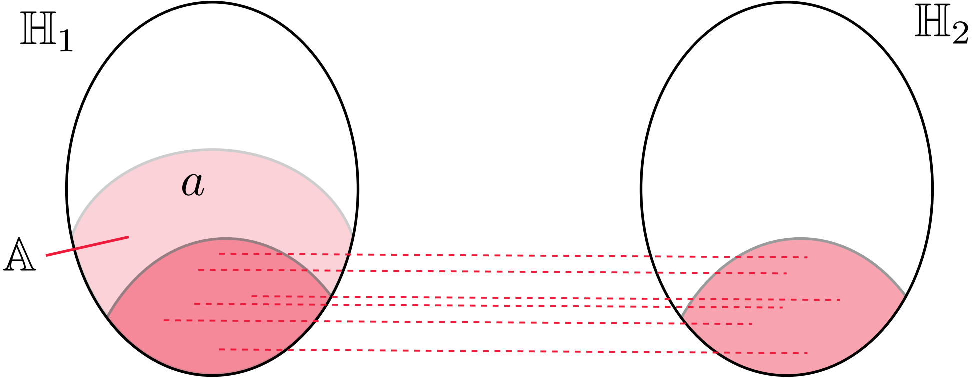
$$\forall \exists \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & P \\ & Q \arrow[u, two heads] \end{tikzcd}$$

all arrows
and sets
equivariant

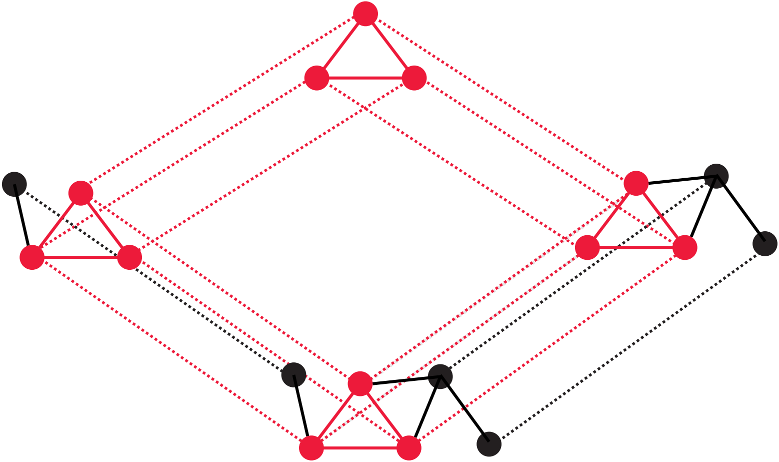


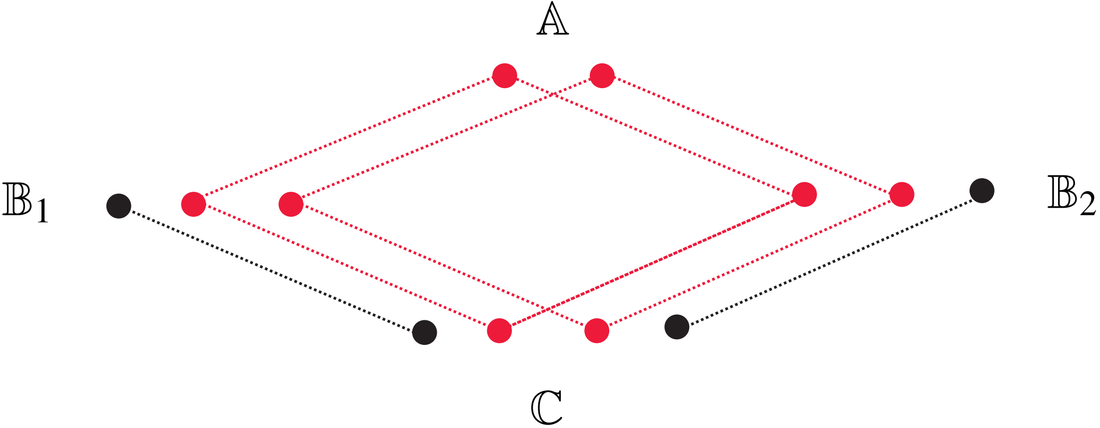


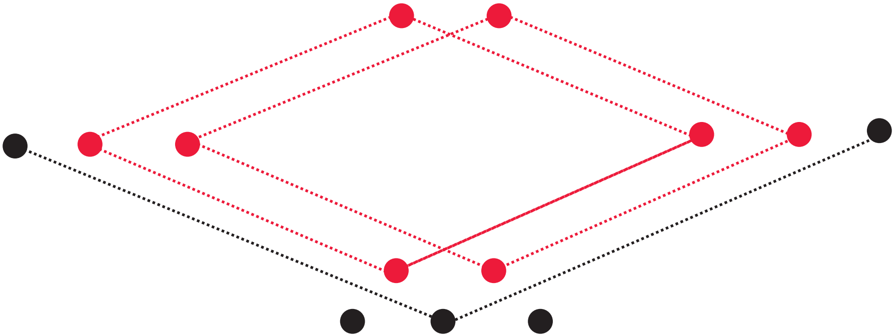

a
b
b
a
a
a

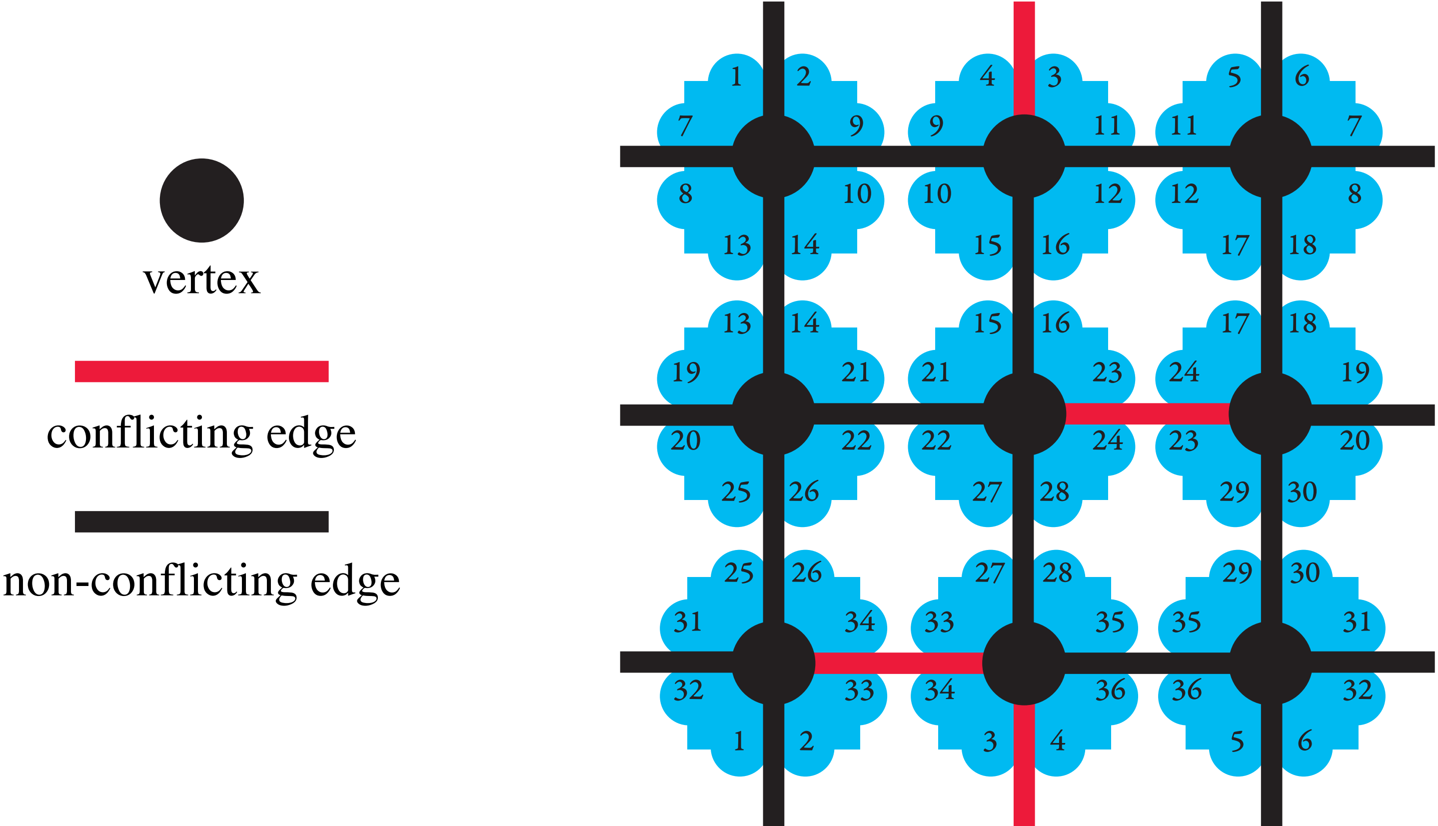


$\mathbb{H}_1$
$\mathbb{H}_2$
a
$\mathbb{A}$




$\mathbb{A}$
$\mathbb{B}_1$
$\mathbb{B}_2$
$\mathbb{C}$




vertex
conflicting edge
non-conflicting edge
1 2 4 3 5 6
7 9 9 11 11 7
8 10 10 12 12 8
13 14 15 16 17 18
13 14 15 16 17 18
19 21 21 23 24 19
20 22 22 24 23 20
25 26 27 28 29 30
25 26 27 28 29 30
31 34 33 35 35 31
32 33 34 36 36 32
1 2 3 4 5 6