Problem Set 2

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Due Date: November 9th at 23.59

Discussion with classmates is encouraged. Please submit your own code and write-up, via email to me.

1 Question 1 (optional): SSJ Toolbox

1 Take a look at underlying code of the SSJ toolbox and do your best to describe its structure.

- **My advice:** briefly try to understand the general structure of the codebase. What are the different blocks? Can you identify the parts of the code that solve the model backward and simulate forward?
- Then look under the hood at some specific algorithms
    - For example, you might want to focus on the heterogenous agent block [het block], and understand the other routines that it calls:
        - [het compiled], [het support], and [utilities]
    - Understanding the het block and sub-routines will help you better understand what the high-level code that we looked at in class is doing under the hood.
    - Adapting some of these routines for your own code might be useful too. Most are good examples of how to write code well.

2 Question 2: Implementing SSJ from Scratch

You should write your own code for this question. You can build on your code from Kieran's problem sets. Do not use the SSJ toolbox.

Our goal is to solve for impulse response functions for the model seen in class. Auclert, Bardóczy, Rognlie and Straub (2021) refer to this as the Krusell and Smith (1998) economy, but importantly it is not an economy with a stochastic steady state. Instead, we are starting from a steady state with constant paths for prices and TFP.[1]

[1] They use the Krusell and Smith (1998) label to refer to a neoclassical economy and distinguish it from the New Keynesian economies that they also discuss in their paper.

Households. Households choose consumption c and savings k to maximize the following optimization

$$V_t(e, k^-) = \max_c \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta \sum_{e'} P(e, e') V_{t+1}(e', k) \right\}$$
$$c + k = (1 + r_t)k^- + w_t e$$
$$k \geq 0,$$

where the productivity endowment e is an AR(1) in logs, and $P(e, e')$ is the discrete transition probabilities.[2] Households have perfect foresight over the aggregate variables r_t and w_t. Households supply labor inelastically, so we can normalize aggregate labor $L_t = \sum_e \pi(e)e$ where $\pi(e)$ is the stationary distribution of the endowment.

Firms. Firms have a Cobb-Douglas production function

$$Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha},$$

and the depreciation rate on capital is δ.

Numbers. Calibrate the economy with numbers that give you a reasonable steady state. You can use the numbers from your previous code, but if you need numbers then I would start with $\beta = 0.98$, $\sigma = 1.5$, $\delta = 0.025$, $\alpha = 0.11$, $Z = 0.85$. For the income process, $\rho_e = 0.966$ and $\sigma_e = 0.5$. For the grids, $nE = 7$, $nA = 500$, $kmin = 0$ and $kmax = 200$ should be a good starting point.

2.1 Preliminaries

1 Explain why the aggregate capital function can be written as a function that depends only on the aggregate sequences

$$\mathcal{K}_t(\{r_s, w_s\}_{s \geq 0}),$$

as opposed to, say, the entire distribution of agents.[3]

2 Derive the first order conditions of the firm problem.

3 Define the equilibrium of this economy.

4 Define the (non-stochastic) steady state of this economy.

5 Explain why this economy can be summarized by the equation $H(\boldsymbol{K}, \boldsymbol{Z})$.

2.2 Steady State

1 Compute the interest rate r that clears the asset market in the stationary equilibrium of this economy.

- Make sure your code uses the (i) endogenous grid method (to solve the household problem) and (ii) Young's method (to simulate the distribution)

2 What is output Y, capital K, labor L in the stationary equilibrium?

[2] You can discretize the using Tauchen's or Rouwenhorst's method.

[3] Btw, in latex you can write curly K using "$\backslash mathcal\{K\}$".

2.3 Transition Dynamics: Brute Force Method

1 Adapt your household problem so that you can feed in a sequence of prices $\{r_s\}_{s=0}^{T}, \{w_s\}_{s=0}^{T}$ for $T = 300$.

- In each period, your household problem will take as **inputs** the value function from the period ahead V_{t+1}, and the current prices w_t and r_t.
- It will give as **outputs**, the choice of capital k and consumption c, with which you can update the distribution.
- The code should **solve backwards** for the policy functions $k_t(e, k^-)$ for the $t = 0, ...300$

  and **simulate forwards** the distribution from $t = 0$ to $T = 300$. Compute the aggregate $\mathcal{K}_t$ at each period.

2 Simulate your steady state for $T = 300$ periods, i.e. feed in a constant r_s and w_s using the market clearing prices for your steady state.

  2 This is your "ghost run". How much does the aggregate savings $\mathcal{K}$ change along the steady state?

3 Compute one column of the Jacobian $J_{t,s}^{\mathcal{K},r} = \frac{\partial \mathcal{K}_t}{\partial r_s}$ for a shock that occurs in the first period $s = 0$.

- i.e. feed in the same interest rate sequence but replacing r_0 to its shocked value $r_0 = r_{ss} \times (1 + \epsilon_r)$.
- Compute the change in capital $\frac{\partial \mathcal{K}_t}{\partial r_s} = (\mathcal{K}_t - \mathcal{K}_{ss})/dr$. Make sure to use your ghost run.
- *Optional:* If using Julia, try using an "automatic differentiation" package.[4] Does it change the speed?

4 Repeat for a shock to the wage rate. Compute one column of the Jacobian $J_{t,s}^{\mathcal{K},w} = \frac{\partial \mathcal{K}_t}{\partial w_s}$ for a shock that occurs in the first period $s = 0$.

5 Repeat for every column from $s = 1$ to $s = T - 1$, for both the interest rate $J_{t,s}^{\mathcal{K},r}$ and the wage rate $J_{t,s}^{\mathcal{K},w}$. Now you have the entire Jacobian.

6 Use your Jacobian and the analytical solution to the firm problem to compute $\boldsymbol{H_K}$ and $\boldsymbol{H_Z}$, and $-\boldsymbol{H_K}^{-1}\boldsymbol{H_Z}$.

7 Simulate the response of capital dK for a shocks to dZ of varying persistence

$$dK = -\boldsymbol{H_K}^{-1}\boldsymbol{H_Z}dZ.$$

8 Compute the response of all the other endogenous variables.

9 Plot and discuss the results.

Comment

You should be able to do all of the above after our class on October 27th. You should wait until after Nov 3rd to try the remaining questions. If you are not finished the above questions by November 3rd, then please keep working on them before attempting question 3. I would prefer that you understand and are able to compute impulse responses with at least one algorithm, rather than rushing to implement three algorithms. That said, I think you will benefit from the remaining questions since the Fake News algorithm is especially helpful.

[4] In Julia, it should be feasible to use auto diff with minimal changes to your code (mostly to make sure functions use the correct types for the auto diff package). You can also try this in other languages but you likely need to re-write all your code to use the relevant tools. In python, you'll use PyTorch/Jax arrays and functions, and similarly Matlab uses deep learning arrays *dlarray*.

3 Question 3: Advanced SSJ

3.1 Fake News

1 Implement the Fake News Algorithm. Compare it's speed to the Brute Force Method above.

3.2 Non-linear impulse responses

1 Discuss the non-linear impulse responses that can be computed with the help of the jacobian. In what sense are they non-linear and what types of non-linearities do they miss out?

   – Hint: it might be helpful to compare to the original Krusell and Smith (1998) paper. Or think about portfolio choice.

2 Solve for the non-linear impulse response. Compare it to the first order approximation. What happens as the size of the shock increases?

References

Auclert, Adrien, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub, "Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models," *Econometrica*, September 2021, *89* (5), 2375–2408.

Krusell, Per and Jr. Smith Anthony A., "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, October 1998, *106* (5), 867–896.