Problem set 5: Generating the wealth distribution

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You should submit your final code. In addition submit a pdf document containing responses to questions including figures and tables.

1 Solve the Aiyagari model

1.1 Standard Aiyagari setting

Firstly solve the Aiyagari model given by:

$$
\begin{aligned}
V(a, y) &= \max_{c, a'} \left\{ u(c) + \beta \mathbb{E} V(a', y') \right\} \\
\text{s.t.} \quad c + a' &= (1+r)a + wy \\
a' &\geq \underline{a}
\end{aligned}
$$

Where factor prices are determined by a Constant Returns to Scale firm operating in competitive markets:

$$
\begin{aligned}
r + \delta &= \partial F(K, L)/\partial K \\
w &= \partial F(K, L)/\partial L
\end{aligned}
$$

1.2 Calibration

Now solve the stationary equilibrium and find the equilibrium interest rate/capital stock using the following parameterisation:

- Annual model. Discount factor $\beta = 0.96$
- Coefficient of relative risk aversion $\gamma = 1.5$
- Borrowing constraint $\underline{a} = -1$
- Log normally distributed productivity process $y_t = \exp(s_t)$
  - $s_{t+1} = \rho s_t + \epsilon_t$ and $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$

- $\rho = 0.94$ and $\sigma_\epsilon^2 = 0.04$
- Use the Floden weighting for the Markov approximation of the productivity process
- Normalize y so that the exogenous labor input in the invariant distribution equals 1

$$L = \sum_{j=1}^{N_y} y_j f(y_j) = 1$$

- Use Cobb-Douglas production $F(K, L) = K^\alpha L^{1-\alpha}$ with $\alpha = 0.3$ and $\delta = 0.1$
- Use 5 grid points in the income dimension and 1,500 uniformly spaced grid points in the asset dimension.
- Set max assets $\overline{a} = 150$
- For the next exercise it will be useful to define the household state as cash on hand (rather than assets) and income (x, y)

$$\begin{aligned} c + a' &= x \\ x' &= (1 + r)a' + wy' \end{aligned}$$

- Set the grid $\mathcal{X} = \{x_1, ..., x_n\}$ with:

$$\begin{aligned} x_1 &= (1 + r)\underline{a} + w\underline{y} \\ x_n &= (1 + r)\bar{a} + w\bar{y} \end{aligned}$$

- Use 1,501 uniformly spaced grid points for the cash on hand dimension.[1]
- Note: the transition matrix is now defined over (X, Y) space with $x \in X, y \in Y$. Some chosen points will not be on the grid. For $\hat{x} \notin \mathcal{X}$ assign a linearly weighted share to the points on the grid $\mathcal{X}$ above and below $x_j < \hat{x} < x_{j+1}$.

1.3 Questions

1. Compute the Gini coefficient for income and wealth
2. Report a table of the share of wealth for the Bottom 50%, Top 10%, Top 1% and Top 0.1%
3. Plot the Lorenz curve for wealth for this economy
4. How does the wealth distribution compare to the data?

[1] Side note: it's always a good idea to use a different number of grid points for different variables as it helps spot errors.

1.4 Adding heterogeneous returns

Now we want to add heterogeneous returns to improve the model's predictions for wealth inequality. The household problem is now:

$$
\begin{aligned}
V(a, y, r_e) &= \max_{c,a'} \left\{ u(c) + \beta \mathbb{E} V(a', y', r_e') \right\} \\
\text{s.t.} \quad c + a' &= (1 + \underline{r} + r^X(a) + r_e)a + wy \\
a' &\geq \underline{a}
\end{aligned}
$$

- $\underline{r}$ is a "risk free" return
- $r^x(a)$ is a deterministic function increasing in a capturing increasing excess returns of the wealthy
- r_e is an idiosyncratic i.i.d. shock to returns $r_e \sim \mathcal{N}(0, \sigma_r^2)$
- Cash in hand is now given by:

$$
x' = (1 + \underline{r} + r^X(a') + r_e)a' + wy'
$$

  - make sure at the minimum cash on hand household can still have positive consumption when $a' = \underline{a}$
- Notice because r_e is i.i.d we don't have to include it as an additional state variable. But we do need to include it in households' expectations over next period's cash on hand x'.
- In the solution we now need to check **two** market clearing conditions (i) capital demand equals capital supply (ii) the (asset weighted) average return on capital equals the aggregate return. We must ensure for a given K^* and $\underline{r}$:

$$
\begin{aligned}
K_d &= \int a_+(x, y) d\lambda(x, y) \\
rK_d &= \int \int (\underline{r} + r^X(a_+(x, y)) + r_e) a_+(x, y) f(r_e) d\lambda(x, y)
\end{aligned}
$$

  - Now we need to iterate on two variables in the household problem $\underline{r}$ and K^* (or equivalently $r = F_k^{-1}(K^*, L) - \delta$)
  - We don't have the same monotonicity. The robust method to solve for this would be to use a non-linear solver.
  - In practice for this problem iterating on $\underline{r}$ while updating r_{gap} where $r = r_{gap} + \underline{r}$ each iteration works quite well.

1.5 Calibration

- Increase max assets $\overline{a} = 500$
- Use $r^x(a) = \chi \max(a, 0)$ with $\chi = 0.035/(\bar{a} - 50)$
- Use 5 grid points for r_e with $\sigma_r = 0.15$

1.6 Questions

1. Plot the returns schedule for different idiosyncratic shocks r_e across the assets holdings distribution a
2. Compute the Gini coefficient for income and wealth
3. Report a table of the share of wealth for the Bottom 50%, Top 10%, Top 1% and Top 0.1%
4. Plot the Lorenz curve for wealth for this economy
5. Has heterogeneous returns improved the match of the wealth distribution?
6. Plot the right tail of the wealth distribution with $\log(a)$ on the x-axis and $\log(\Pr(A > a))$ on the y-axis. What are its properties, how does this compare to the previous model and how does it fit the data?
7. (Bonus) Experiment with just $r^X(a)$ heterogeneity and just r_e heterogeneity. How does each channel contribute?