EE2101-Control Systems
Assignment-1
Problem-4

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Question

Solve the following differential equations with the given initial conditions using Laplace transforms.

a. $\frac{dx}{dt} + 7x = 5\cos 2t$ where x(0) = 4, x'(0) = -4 ;

b. $\frac{d^2x}{dt^2} + 6\ \frac{dx}{dt} + 8x = 5\sin 3t$ where x(0) = 4, x'(0) = 1;

c. $\frac{d^2x}{dt^2} + 8\ \frac{dx}{dt} + 25x = 10u(t)$ where x(0) = 2, x'(0) = 3;
where $x'(0) = \frac{dx}{dt}(0)$.
Assume that the forcing functions are zero prior to t = 0-.

Part 'a'

- In the given problem statement, the part 'a' is a differential equation of order=1 and 2 initial conditions are given.
- But only on initial condition x(0) = 4 is needed for a differential equation of order 1.
- So, I am modifying the problem assuming 2 different ways:

  1) Only x(0) = 4 condition is given for the given first order differential equation.

  2) Considering $\frac{d^2x}{dt^2}$ instead of $\frac{dx}{dt}$ with conditions x(0) = 4, x'(0) = -4 because 2 initial conditions are needed for a differential equation of order = 2.

Part 'a' 1st way

$$\frac{dx}{dt} + 7x = 5\cos 2t \text{ where } x(0) = 4$$

Applying Laplace transforms to the differential equation,

$$\mathscr{L}\{x'(t)\} = sX(s) - x(0)$$

$$sX(s) - x(0) + 7X(s) = \frac{5s}{s^2+4}$$

$$sX(s) - 4 + 7X(s) = \frac{5s}{s^2+4}$$

$$(s+7)X(s) = \frac{5s}{s^2+4} + 4$$

$$X(s) = \frac{4s^2+5s+16}{(s^2+4)(s+7)}$$

Now solve it using partial fractions,

$$X(s) = \frac{As+B}{s^2+4} + \frac{C}{s+7}$$

By comparing the numerators,
We get $A = \frac{35}{53}$, $B = \frac{20}{53}$, $C = \frac{177}{53}$
So,

$$X(s) = \frac{35s}{53(s^2+4)} + \frac{20}{53(s^2+4)} + \frac{177}{53(s+7)}$$

Applying inverse Laplace transform, we get x(t).
$\mathscr{L}^{-1}\{\frac{s}{s^2+a^2}\}$ =$\cos at$, $\mathrm{L}^{-1}\{\frac{a}{s^2+a^2}\}$ =$\sin at$, $\mathrm{L}^{-1}\{\frac{1}{s+a}\} = e^{-at}$

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$

$$\boxed{x(t) = \frac{35}{53}\cos 2t + \frac{10}{53}\sin 2t + \frac{177}{53}e^{-7t}}$$

Part 'a' 2nd way

$\frac{d^2x}{dt^2}$ + 7x = 5cos $2t$ where x(0) = 4, x'(0) = -4

Applying Laplace transforms to the differential equation,

$$\mathscr{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0)$$

$$s^2X(s) - sx(0) - x'(0) + 7X(s) = \frac{5s}{s^2+4}$$

$$s^2X(s) - 4s + 4 + 7X(s) = \frac{5s}{s^2+4}$$

$$(s^2 + 7)X(s) = \frac{5s}{s^2+4} + 4s - 4$$

$$\mathrm{X(s)} = \frac{4s^3-4s^2+21s-16}{(s^2+4)(s^2+7)}$$

Now solve it using partial fractions,

$$\mathrm{X(s)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+7}$$

By comparing the numerators,
We get $A = \frac{5}{3}$, $B = 0$, $C = \frac{7}{3}$, $D =$ -4
So,

$$X(s) = \frac{5s}{3(s^2+4)} + \frac{0}{s^2+4} + \frac{7s}{3(s^2+7)} - \frac{4}{s^2+7}$$

Applying inverse Laplace transform, we get x(t).

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$

$$\boxed{x(t) = \frac{5}{3}\cos 2t + \frac{7}{3}\cos\sqrt{7}t - \frac{4}{\sqrt{7}}\sin\sqrt{7}t}$$

Part 'b'

$$\frac{d^2x}{dt^2} + 6\ \frac{dx}{dt} + 8x = 5\sin 3t \text{ where } x(0) = 4,\ x'(0) = 1$$

Applying Laplace transforms to the differential equation,

$$s^2X(s) - sx(0) - x'(0) + 6(sX(s) - x(0)) + 8X(s) = \frac{15}{s^2+9}$$

$$s^2X(s) - 4s - 1 + 6(sX(s) - 4) + 8X(s) = \frac{15}{s^2+9}$$

$$X(s)(s^2 + 6s + 8) = \frac{15}{s^2+9} + 4s + 25$$

$$X(s) = \frac{4s^3+25s^2+36s+240}{(s^2+9)(s^2+6s+8)}$$

$$X(s) = \frac{4s^3+25s^2+36s+240}{(s^2+9)(s+2)(s+4)}$$

Now solve it using partial fractions,

$$X(s) = \frac{As+B}{s^2+9} + \frac{C}{s+2} + \frac{D}{s+4}$$

By comparing the numerators,
We get $A = $ -$\frac{18}{65}$, $B = $ -$\frac{3}{65}$, $C = \frac{118}{13}$, $D = $ -$\frac{24}{5}$
So,

$$X(s) = \text{-}\frac{18s}{65(s^2+9)}\ \text{-}\frac{3}{65(s^2+9)} + \frac{118}{13(s+2)}\ \text{-}\frac{24}{5(s+4)}$$

Applying inverse Laplace transform, we get x(t).

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$

$$\boxed{x(t) = -\frac{18}{65}\cos 3t - \frac{1}{65}\sin 3t + \frac{118}{13}e^{-2t} - \frac{24}{5}e^{-4t}}$$

Part 'c'

$\frac{d^2x}{dt^2}$ + 8 $\frac{dx}{dt}$ +25x = 10u(t) where x(0) = 2, x'(0) = 3

Applying Laplace transforms to the differential equation,

$$s^2X(s) - sx(0) - x'(0) + 8(sX(s) - x(0)) + 25X(s) = \frac{10}{s}$$
$$s^2X(s) - 2s - 3 + 8(sX(s) - 2) + 25X(s) = \frac{10}{s}$$
$$(s^2 + 8s + 25)X(s) - 2s - 19 = \frac{10}{s}$$
$$(s^2 + 8s + 25)X(s) = \frac{10}{s} + 2s + 19$$
$$\mathrm{X(s)} = \frac{2s^2+19s+10}{s(s^2+8s+25)}$$

Now solve it using partial fractions,

$$\mathrm{X(s)} = \frac{A}{s} + \frac{Bs+C}{s^2+8s+25}$$

By comparing the numerators,
We get $A = \frac{2}{5}$, $B = \frac{8}{5}$, $C = \frac{79}{5}$

So,

$$X(s) = \frac{2}{5s} + \frac{\frac{8}{5}s + \frac{79}{5}}{s^2+8s+25}$$

Now, again seperate $\frac{\frac{8}{5}s + \frac{79}{5}}{s^2+8s+25}$ as partial fractions

$$\frac{\frac{8s+79}{5}}{s^2+8s+25} = \frac{A}{(s+4)^2+9} + \frac{B(s+4)}{(s+4)^2+9}$$

Comparing numerators, we get $A = \frac{47}{5}$, $B = \frac{8}{5}$

$$X(s) = \frac{2}{5s} + \frac{47}{5((s+4)^2+9)} + \frac{8(s+4)}{5((s+4)^2+9)}$$

Applying inverse Laplace transform, we get x(t).

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$

$$\boxed{x(t) = \frac{2}{5} + \frac{47}{15}e^{-4t}\sin 3t + \frac{8}{5}e^{-4t}\cos 3t}$$