$\phi(n) \geq \sqrt{n}$ except for $n = 2, 6$.

RYSP

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Contents

Abstract

We will be discussing the complete proof of an exercise on homework 3, that is

$$\phi(n) \geq \sqrt{n}$$

for $n \neq 2, 6$. This article will break down the proof into **four** sections:

Case I $n = p$, where $p \neq 2$ is a prime.

Case II $n = p^e$, where n is a proper power of prime, i.e. $e \geq 2$.

Case III $n = p_1^{e_1} \cdots p_k^{e_k}$, where p_i's distinct primes, 2^1 cannot appear in the prime factorization of n and $e_i \geq 1$.

Case IV $n = 2p_1^{e_1} \cdots p_k^{e_k}$, where p_i's are distinct primes, and $e_i \geq 1$.

1 CASE I: non-two prime

Consider $n = p \neq 2$, then $\phi(p) = p - 1$. It suffices to argue

$$p - 1 \geq \sqrt{p}$$

for $p \neq 2$.

Proof. Define the sequence $a_n = n - \sqrt{n} - 1$. We want to find where a_n is increasing, so we consider the difference

$$a_{n+1} - a_n > 0.$$

After simplification, this inequality becomes

$$1 > \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

which holds for every integer $n \geq 1$. Hence, the sequence (a_n) is strictly increasing for all n.

Evaluating at $n = 3$,

$$a_3 = 3 - \sqrt{3} - 1 = 2 - \sqrt{3} > 0,$$

which is the minimal value of a_n for $n \geq 3$. Therefore, for all $p > 3$,

$$a_p > a_3 > 0.$$

□

2 CASE II: proper power of prime

Consider $n = p^e$, where p is a prime and $e \geq 2$. Then

$$\frac{\phi(n)}{\sqrt{n}} = p^{\frac{e}{2}-1}(p-1).$$

Proof. Since $p \geq 2$ and $e \geq 2$, we have

$$p^{\frac{e}{2}-1}(p-1) \geq 1,$$

which completes the proof for this case. □

3 CASE III: 2 is exactly not inside n

Proof. We proceed by mathematical induction on the number of distinct prime divisors of n other than 2.

Let

$$P(k) : \text{If } n \text{ has } k \text{ distinct prime divisors (excluding 2), then } \phi(n) \geq \sqrt{n}.$$

The base case $P(1)$ holds by Cases I and II.

Assume $P(k-1)$ is true for some $k \geq 2$. Now consider n with k distinct prime divisors. Write

$$n = p_k^{e_k} \cdot \frac{n}{p_k^{e_k}}.$$

Using multiplicativity of Euler's totient function,

$$\phi(n) = \phi\big(p_k^{e_k}\big) \cdot \phi\left(\frac{n}{p_k^{e_k}}\right).$$

Then,

$$\frac{\phi(n)}{\sqrt{n}} = \frac{\phi\left(p_k^{e_k}\right)}{\sqrt{p_k^{e_k}}} \cdot \frac{\phi\left(\frac{n}{p_k^{e_k}}\right)}{\sqrt{\frac{n}{p_k^{e_k}}}} \geq 1,$$

where the first factor is ≥ 1 by Cases I and II, and the second factor is ≥ 1 by the induction hypothesis.

This completes the proof. $\square$

4 Case IV: 2 is inside n

Proof. Consider

$$n = 2p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k},$$

and assume $n \neq 2 \cdot 3^1$.

If $n = 2p^e$ for some prime p and integer $e \geq 1$, then

$$\frac{\phi(n)}{\sqrt{n}} = \frac{p^{\frac{e}{2}-1}(p-1)}{\sqrt{2}}.$$

We analyze the base cases separately:

- If $e \geq 2$ and $p \geq 3$, then
$$\frac{\phi(n)}{\sqrt{n}} = \frac{p^{\frac{e}{2}-1}(p-1)}{\sqrt{2}} \geq \sqrt{2},$$
which is greater than 1.

- If $e = 1$ and $p \geq 5$, then
$$\frac{\phi(n)}{\sqrt{n}} = \frac{p^{-\frac{1}{2}}(p-1)}{\sqrt{2}} \geq \frac{5^{-\frac{1}{2}} \cdot 4}{\sqrt{2}} = \frac{4}{\sqrt{10}} > 1.$$

Suppose for

$$n = 2p_1^{e_1}p_2^{e_2}\cdots p_{k-1}^{e_{k-1}},$$

the inequality

$$\frac{\phi(n)}{\sqrt{n}} \geq 1$$

holds.

Then for

$$n = 2p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k},$$

we have

$$\frac{\phi(n)}{\sqrt{n}} = \frac{\phi\left(\frac{n}{p_k^{e_k}}\right)}{\sqrt{\frac{n}{p_k^{e_k}}}} \cdot \frac{\phi\left(p_k^{e_k}\right)}{\sqrt{p_k^{e_k}}} \geq 1 \cdot p_k^{\frac{e_k}{2}-1}(p_k-1) > 1,$$

where the last inequality follows from the base cases discussed above.

This completes the proof. $\square$