HW1 — Number Theory

RYSP

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1. Attempt the formula of
$$1 + 2^4 + \cdots + n^4$$
and use induction to justify your formula.
[Hint: there is a pattern of
$$\sum_{i=1}^{n} i^k,$$
that is a polynomial of degree $n+1$ with zero constant term.]

2. Using induction, show that for all $n \geq 1$,
$$2^n > n.$$

3. Let $S \subset \mathbb{N}$ satisfy $1 \in S$ and whenever $k \in S$ then $k+2 \in S$. Use the Well–Ordering Principle to show $S = \mathbb{N}$.

4. Prove that the Principle of Mathematical Induction implies the Well–Ordering Principle.

5. Conversely, prove that the Well–Ordering Principle implies the Principle of Mathematical Induction.

6. Find all integer solutions to
$$35x + 22y = 1.$$

7. Describe all *primitive* Pythagorean triples: integer solutions to
$$a^2 + b^2 = c^2$$
with $\gcd(a, b, c) = 1$.

8. Prove that the only integer solutions to
$$x^3 + y^3 = z^3$$
are the "trivial" ones with $xyz = 0$.

9. Use the Division Algorithm to find the quotient and remainder when dividing 2025 by 37.

10. Prove that if $a \mid b$ and $b \mid a$, then $|a| = |b|$.

11. Show that for every $n \in \mathbb{N}$, $n \mid n!$.

12. Prove that if $a \mid b$ and $a \mid c$, then for all $x, y \in \mathbb{Z}$,
$$a \mid (bx + cy).$$

13. State and prove the uniqueness of the quotient and remainder in the Division Algorithm.

14. Compute $\gcd(1, a)$ and $\gcd(0, b)$ for arbitrary $a, b \in \mathbb{Z}$ not both zero.

15. Compute $\gcd(1071, 462)$ using the Euclidean Algorithm.

16. Prove in general that
$$\gcd(a, b) \;=\; \gcd\big(b,\; a \bmod b\big).$$

17. Find integers x, y such that
$$240x + 46y \;=\; \gcd(240, 46).$$

18. Prove the corollary: if $\gcd(a, m) = \gcd(b, m) = 1$, then $\gcd(ab, m) = 1$.

19. Prove Bézout's theorem: for $a, b \in \mathbb{Z}$ not both zero, there exist $x_0, y_0 \in \mathbb{Z}$ with
$$ax_0 + by_0 = \gcd(a, b).$$

20. Use Bézout's identity to show that if $c \mid ab$ and $\gcd(c, a) = 1$, then $c \mid b$.

21. Prove Euclid's theorem: there are infinitely many primes.

22. Factor the integer
$$2^{10} - 1$$
completely into primes.

23. Prove the uniqueness part of the Fundamental Theorem of Arithmetic.

24. Show that the sum of the reciprocals of the primes diverges (Euler's proof sketch).

25. Prove the Binomial Theorem:
$$(x + y)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

26. Show combinatorially that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

27. Prove that $\binom{n}{k} = \binom{n}{n-k}$.

28. Establish Pascal's rule:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

29. Prove that the product of any k consecutive integers is divisible by $k!$.

30. Solve the linear congruence

$$14x \equiv 30 \pmod{100}.$$

31. Use the Chinese Remainder Theorem to find all $x \pmod{35}$ satisfying

$$x \equiv 3 \pmod{5}, \quad x \equiv 2 \pmod{7}.$$